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A CLASS OF NONLINEAR INTEGER PROGRAMS SOLVABLE BY A SINGLE LINEAR PROGRAM*

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Abstract. Although the addition of integrality constraints to the existing constraints of an optimization problem will, in general, make the determination of an optimal solution more difficult, we consider here a class of nonlinear programs in which the imposition of integrality constraints on the variables makes it possible to solve the problem by a single, easily-constructed linear program. The class of problems addressed has a separable convex objective function and a totally unimodular constraint matrix. Such problems arise in logistic and personnel assignment applications.

1. Introduction. Nonlinear integer programs of the form

$$\begin{aligned}&\min_x \sum_{i=1}^n f_i(x_i)\\ &\text{subject to } \sum_{i=1}^n x_i = r,\\ &\qquad x=(x_1,\cdots,x_n)^T \geqq 0,\quad x_i \text{ integer } (i=1,\cdots,n)\end{aligned} \tag{1.1}$$

arise in logistic and personnel assignment applications and have been the subject of a number of studies [3], [9], [13], [14], [16]. Here, we consider the broader class of problems of the form

$$\begin{aligned}&\min_x \sum_{i=1}^n f_i(x_i)\\ &\text{subject to } Ax=b,\\ &\qquad x\geqq 0,\quad x \text{ integer},\end{aligned} \tag{1.2}$$

where A is a totally unimodular (T.U.) $m\times n$ matrix and b is integer,[1] (in the following, a vector is said to be *integer* if all its components are integer), and we will show that a solution to the problem (1.2) may be obtained by solving a *single* easily-constructed linear program provided that known bounds exist for the feasible set of (1.2) and each f_i is a *convex* function. This result thus also generalizes the well-known property [8] that, in the case that all the f_i are linear (so that (1.2) is a linear integer program), the solution of (1.2) may be obtained by solving a single linear program.

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[1] Recall that a matrix is said to be *totally unimodular* if the determinant of each of its square submatrices has value 0 or ± 1. Totally unimodular matrices typically arise in optimization problems defined on networks, but may also arise in other contexts such as bounds on sums or differences of subsets of variables. Although we assume here the equality constraints $Ax=b$, analogous results hold if $Ax=b$ is replaced by $Ax\leqq b$ or by *any combination of equations and inequalities* whose aggregate coefficient matrix is totally unimodular, since the conversion of such constraints to a set of equations (by the addition of slack variables) yields a new coefficient matrix that will also be totally unimodular.

Finally, this result may also be thought of as a generalization of a method suggested by Dantzig [5] for a class of transportation problems. (Although single-commodity network problems with convex costs on their arcs can generally be put in the form of (1.2), the study of the converse raises some interesting issues that are discussed in § 4.)

For the case in which bounds for the feasible set are *not* known, a column-generation procedure is developed, and it is shown that, if an optimal solution to (1.2) exists, this procedure will yield an optimal solution (and a proof of its optimality) by the solution of a finite number of linear programs. This column-generation procedure also has computational advantages in the bounded case if the bounds are very large and/or one or more of the f_i are "costly" to evaluate. It differs from parametric procedures proposed by Beale [1] and Hu [7] for certain convex network optimization problems in that it employs "global" rather than "local" cost function approximations.

2. An equivalent linear program. In this section we will establish the equivalence of the nonlinear integer program (1.2) to a linear program ((2.5) below) under the following *hypotheses*:

(A) there exist nonnegative integers l_i, u_i $(i = 1, \cdots, n)$ such that $F \equiv \{x \mid Ax = b, x \geqq 0\} \subseteq \{x \mid l_i \leqq x_i \leqq u_i, i = 1, \cdots, n\}$;

(B) $F \neq \emptyset$;

(C) the matrix A is an $m \times n$ totally unimodular matrix and b is integer;

(D) (for $i = 1, \cdots, n$) f_i is a real-valued convex function on $[l_i, u_i]$.

Note that under hypotheses (A) and (B), the nonlinear integer program (1.2) has an optimal solution since the number of feasible points is finite and nonzero. (The hypotheses (A) and (B) can, in fact, be deleted, as is shown in § 5, but the proofs for the more general case are straightforward extensions of the results of this section.) Let $\tilde{f}_i$ denote the continuous piecewise-linear function defined on $[l_i, u_i]$ that *coincides* with f_i at the integer points in $[l_i, u_i]$ and is *linear* between each pair of adjacent integers in $[l_i, u_i]$. It is easily seen that each $\tilde{f}_i$ is also convex. (In fact, it is convexity of the $\tilde{f}_i$ that is crucial rather than convexity of the f_i.) The problem (1.2) is therefore equivalent[2] to

$$\begin{aligned} &\min_x \sum_{i=1}^n \tilde{f}_i(x_i) \\ &\text{subject to } Ax = b, \\ &\qquad\qquad x \geqq 0, \quad x \text{ integer}, \end{aligned} \tag{2.1}$$

since the objective functions of (1.2) and (2.1) *coincide* over their common feasible set. (Put another way, the values of the objective function terms at noninteger points are completely irrelevant to the optimization problem (1.2), so we can take advantage of this fact by "simplifying" the form of the objective function terms between consecutive integers.) We will now exploit properties of a

[2] If integrality constraints were not present in (1.2), then the $\tilde{f}_i$ would merely be approximations to the f_i; but, given the integrality constraints, no "error" is incurred over the discrete domain by replacing f_i by $\tilde{f}_i$. Thus in this context the $\tilde{f}_i$ should *not* be thought of as "approximations," as is the case when similar substitutions are done in the continuous variable case (see [2], [4]).

particular representation of the $\tilde{f}_i$ in order to get rid of the integrality constraints. (The overall strategy is thus to exploit the integrality constraints to modify the objective function, and then to exploit the modified objective function and total unimodularity to get rid of the integrality constraints.)

It is a well-known result of separable programming (see [5]) that, for $x_i \in [l_i, u_i]$, we have the following representation for the $\tilde{f}_i$ (R_i' denotes the integers in $[l_i, u_i]$):

$$\tilde{f}_i(x_i) = \min_{\lambda_{i,j}} \sum_{j \in R_i'} f_i(j)\lambda_{i,j}$$

$$\text{subject to } \sum_{j \in R_i'} j\lambda_{i,j} = x_i, \tag{2.2}$$

$$\sum_{j \in R_i'} \lambda_{i,j} = 1; \quad \lambda_{i,j} \geqq 0.$$

(In [10], which is an expanded version of this paper, it is shown that the so-called "δ-representation" of $\tilde{f}_i$, namely

$$\tilde{f}_i(x_i) = \min_{\delta_{i,j}} f_i(l_i) + \sum_{j \in R_i} \delta_{i,j}[f_i(j+1) - f_i(j)]$$

$$\text{subject to } l_i + \sum_{j \in R_i} \delta_{i,j} = x_i,$$

$$0 \leqq \delta_{i,j} \leqq 1, \quad j \in R_i,$$

where $R_i = R_i'/\{u_i\}$, may also be used to obtain analogous results. In this paper we will concentrate on the "λ-representation" (2.2), which turns out to be more appropriate for a column-generation method to be discussed below.) Thus, the problem (2.1) is equivalent to the problem

$$\min_{\lambda,x} \sum_{i=1}^{n} c_i\lambda_i$$

$$\text{subject to } Ax = b, \quad x \geqq 0, \quad x \text{ integer}, \tag{2.3}$$

$$D\lambda = x, \quad E\lambda = e, \quad \lambda \geqq 0$$

where $\lambda_i = (\lambda_{i,l_i}, \cdots, \lambda_{i,u_i})^T$, $\lambda = (\lambda_1, \cdots, \lambda_n)$, $c_i = (f_i(l_i), \cdots, f_i(u_i))$, $e = (1, \cdots, 1)^T$, and the constraints $D\lambda = x$, $E\lambda = e$, $\lambda \geqq 0$ represent the constraints of (2.2) as i ranges from 1 to n. The problem (1.2) has thus been transformed into an equivalent *linear* mixed-integer program (2.3). Now if the constraint matrix of (2.3) were totally unimodular, then the integrality constraints of (2.3) could be deleted without affecting the optimal value. However, because D contains integer entries other than 0 or ± 1, the constraint matrix of (2.3) is *not* totally unimodular, and if we consider the linear programming relaxation of (2.3)

$$\min_{\lambda,x} \sum_{i=1}^{n} c_i\lambda_i$$

$$\text{subject to } Ax = b, \quad x \geqq 0, \tag{2.4}$$

$$D\lambda = x, \quad E\lambda = e, \quad \lambda \geqq 0,$$

examples are easily constructed to show that the feasible set of (2.4) may have *noninteger* extreme points. Moreover, if $(\hat{\lambda}, \hat{x})$ is an extreme point of (2.4) then $\hat{x}$ need *not* be an extreme point of F. (This reflects the fact that (1.2) may have a unique optimal solution lying in the *interior* of F.) However, we will show that if $(\hat{\lambda}, \hat{x})$ is an extreme point of (2.4), then the vector $\hat{x}$ must be *integer*, and thus this condition is sufficient to guarantee that the optimal value of (2.4) is equal to the optimal value of (1.2). For notational convenience, we denote the equality constraints of (2.4) as

$$Ax = b, \tag{2.5}$$

$$D\lambda = x, \tag{2.6}$$

$$E\lambda = e. \tag{2.7}$$

THEOREM 2.1. *If* $(\hat{\lambda}, \hat{x})$ *is an extreme point of* (2.4), *then* $\hat{x}$ *is integer.*

Proof. Let λ_B and x_B be the basic variables corresponding to the extreme point $(\hat{\lambda}, \hat{x})$. It is easily seen from (2.6) and (2.7) that at least one and at most two variables from each λ_i must be in λ_B. Let $x'_B \equiv \{x_i | x_i$ is basic and λ_B contains exactly one variable of $\lambda_i\}$ and $x''_B \equiv \{x_i | x_i$ is basic and λ_B contains exactly two variables of $\lambda_i\}$, with corresponding definitions for λ'_B and λ''_B. If x_i is in x'_B, let μ_i denote the basic variable in λ_i, so that (2.6) and (2.7) imply $\hat{\mu}_i = 1$ and $\hat{x}_i = d_i\hat{\mu}_i = d_i$ for some integer d_i in R'_i.

Thus, the variables x'_B are all integer-valued, and we will now show that this is the case for x''_B also. For each variable x_i in x''_B, we let μ_i be one of the corresponding basic variables in λ_i, so that the other basic variables in λ_i can be replaced by $1-\mu_i$ because of (2.7). Denote the coefficient of the variable in (2.6) corresponding to $(1-\mu_i)$ as d_i and the coefficient of the other basic variable μ_i as $d_i + h_i$ (note that h_i is a *nonzero* integer). Using the change of variable $x_i = d_i + x''_i$, we have from (2.6), $d_i + x''_i = d_i(1-\mu_i) + (d_i + h_i)\mu_i$ or $x''_i = h_i\mu_i$, so that each such μ_i is uniquely determined by x''_i. We will now show that the columns of A corresponding to x''_B are linearly independent. For, suppose that they were not, and set all variables other than x''_B and λ''_B to their values in the solution $(\hat{\lambda}, \hat{x})$. If the columns of A corresponding to x''_B were linearly dependent, there would be infinitely many sets of values of x''_B for which (2.5) (with the other variables set to their values in $\hat{x}$) would be satisfied, and for each such set of values, values of μ_i could be determined so that (2.6) and (2.7) were also satisfied. This contradicts the fact that the system (2.5)–(2.7) must have a *unique* solution when the nonbasics are set to 0. Thus having shown that the columns of A corresponding to x''_B are linearly independent, we conclude from the T.U. of A that $\hat{x}''_B$ is integer. □

THEOREM 2.2. *The optimal value of* (2.4) *is equal to the optimal value of* (1.2), *and if* (λ^*, x^*) *is an optimal extreme point of* (2.4), *then* x^* *solves* (1.2).

Proof. Since the feasible set of the linear program [LP] (2.4) is nonempty and bounded, then (2.4) has an optimal solution. Thus, there exists an extreme point of (2.4) that is optimal, and the x-coordinates of this extreme point must be optimal for (1.2). □

Thus, the original nonlinear integer program (1.2) can be solved by computing the values of each f_i at the integer points in $[l_i, u_i]$ and solving the linear program (2.4) by the simplex method, which will generate an optimal extreme point.

It should also be noted that Theorem 2.2 also implies that (1.2) and (2.4) have the *same* optimal value as the problem obtained from (2.1) by deleting its integrality constraints, namely

$$\begin{aligned} &\min \sum_{i=1}^{n} \tilde{f}_i(x_i) \\ &\text{subject to } Ax = b, \quad x \geqq 0. \end{aligned} \tag{2.8}$$

The next two results show that optimal solutions can be obtained when each f_i is replaced by a piecewise-linear convex function that coincides with f_i at *some* rather than *all* of the integer points in the interval $[l_i, u_i]$. These more general results suggest the use of "column-generation" strategies in the event that evaluation of the f_i at *all* integer points in the intervals $[l_i, u_i]$ would be "costly". (Details of these "column-generation" procedures are given in § 3).

COROLLARY 2.3. *Let the functions $\hat{f}_i$ $(i = 1, \cdots, n)$ be convex piecewise-linear functions of the form*

$$\begin{aligned} \hat{f}_i(x_i) = &\min_{\lambda_{i,j}} \sum_{j \in R_i''} f_i(j)\lambda_{i,j} \\ &\textit{subject to} \sum_{j \in R_i''} j\lambda_{i,j} = x_i, \\ &\sum_{j \in R_i''} \lambda_{i,j} = 1, \quad \lambda_{i,j} \geqq 0, \end{aligned} \tag{2.9}$$

where each R_i'' is a finite, nonempty subset of the integers. If the optimal value of the problem

$$\begin{aligned} &\min \sum_{i=1}^{n} \hat{f}_i(x_i) \\ &\textit{subject to } Ax = b, \quad x \geqq 0, \quad x \textit{ integer} \end{aligned} \tag{2.10}$$

exists, then it is equal to the optimal value of

$$\begin{aligned} &\min \sum_{i=1}^{n} \hat{f}_i(x_i) \\ &\textit{subject to } Ax = b, \quad x \geqq 0. \end{aligned} \tag{2.11}$$

Proof. Since (2.10) is assumed to have an optimal solution, it is easily seen that (2.11) must also have an optimal solution, and by an argument analogous to the proof of Theorem 2.1, the LP equivalent to (2.11) must have an optimal solution with x integer-valued. □

In the case that the f_i are affine, the equivalence of (2.10) and (2.11) follows from the integrality of the extreme points of F, but note that the Corollary 2.3 cannot be based on this fact since the optimal solutions of (2.11) need not be extreme points of F. It should be recognized, however, that the conclusions of Corollary 2.3 need *not* hold if the $\hat{f}_i$ are general convex functions or if the $\hat{f}_i$ are even piecewise-linear convex functions with "breakpoints" at noninteger points. This is easily seen by letting the constraints $Ax = b$ be given by $x_1 + x_2 = 1$ $(n = 2)$ and letting $\hat{f}_i(x_i) = (x_i - \frac{1}{2})^2$ or $|x_i - \frac{1}{2}|$. Such convex functions must be replaced by "equivalent" piecewise-linear functions with integral breakpoints before the integrality constraints may be deleted.

We will now show that the optimal value of the problem (2.11) coincides with the optimal value of the nonlinear problem (1.2) if the index sets R_i'' are sufficiently "fine" near an integer optimal solution of (2.11). This result is essentially equivalent to the fact that a local solution of (2.8) must also be a global solution of (2.8).

THEOREM 2.4. *If x^{**} is an integer optimal solution of* (2.11) *and if $R_i'' \supseteq \{x_i^{**}-1, x_i^{**}, x_i^{**}+1\} \cap [l_i, u_i]$ for $i=1, \cdots, n$, then x^{**} is an optimal solution of the nonlinear integer program* (1.2).

Proof. Since the feasible sets of (2.8) and (2.11) coincide, x^{**} is feasible for (2.8). Moreover, $\hat{f}_i(y)=\tilde{f}_i(y)$ for $y \in [x_i^{**}-1, x_i^{**}+1] \cap [l_i, u_i]$, so x^{**} must be a local minimum of (2.8). Because of convexity, x^{**} is also a global minimum of (2.8), and the conclusion follows from Theorem 2.2 and the equivalence of (2.4) and (2.8). □

It should be noted that it is *not* sufficient for optimality to simply have $R_i'' \supseteq \{x_i^{**}\}$ for all i, as may be seen from following example: consider the following problem of the form (1.2)

$$\min\ (x_1-1)^2+(x_2-1)^2$$
$$\text{subject to } x_i + x_2 = 2,$$
$$x_i \geqq 0 \text{ and integer},$$

and let $l_i=0$, $u_i=2$, $R_i''=\{0, 2\}$ for $i=1, 2$; then $\hat{f}_i \equiv 1$ on $[0, 2]$, so that optimal solutions for the corresponding approximating problem occur at $x_1=0$, $x_2=2$ and $x_1=2$, $x_1=0$, but the unique optimal solution of the original problem is $x_1=1$, $x_2=1$.

Finally, note that these results do *not* generalize to the case in which the x_i are *vector* variables rather than the scalar components of x. This may be seen from the following example.

Example. Let $f(x_1, x_2, x_3, x_4)=f_1(x_1, x_2)+f_2(x_3, x_4)$, where f_1 is any convex function such that $f_1(0,0)=f_1(1,1)=1$ and $f_1(0,1)=f_1(1,0)=-1$ (for example, f_1 could be taken as $2(x_1+x_2-1)^2-1$ or as a convex piecewise-linear function with those values), and f_2 is any convex function such that $f_2(0,0)=f_2(1,1)=-1$ and $f_2(0,1)=f_2(1,0)=1$ (for example, f_2 could be taken as $2(x_3-x_4)^2-1$ or as a convex piecewise-linear function with those values). Consider the problem:

$$\begin{aligned} &\min f_1(x_1, x_2)+f_2(x_3, x_4)\\ &\text{subject to } x_1 \qquad -x_3 \qquad = 0,\\ &\qquad\qquad x_2 \qquad -x_4 = 0,\\ &\qquad\qquad 0 \leqq x_i \leqq 1,\\ &\qquad\qquad x_i \text{ integer } (i=1, \cdots, 4).\end{aligned}$$

It is easily seen that the constraint matrix is totally unimodular. The four feasible points $(0,0,0,0)$, $(1,0,1,0)$, $(0,1,0,1)$, $(1,1,1,1)$ all have objective function values of 0, and hence are all *optimal solutions*. Suppose, however, we replace the f_i by piecewise-linear *convex* functions $\bar{f}_i$ that agree with the f_i at (0, 0), (0, 1), (1, 0), (1, 1), and delete the integrality constraints to obtain the problem

$$\begin{aligned} &\min \bar{f}_1(x_1, x_2)+\bar{f}_2(x_3, x_4)\\ &\text{subject to } x_1 \quad -x_3 \quad =0,\\ &\qquad x_2 \quad -x_4=0,\\ &\qquad 0\leqq x_i\leqq 1 \quad (i=1,\cdots,4). \end{aligned}$$

It is easily seen that, as a result of convexity, the objective function of this new problem has a value no greater than -2 at the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, so that the deletion of the integrality constraints results in a change in the optimal solution and the optimal value. □

3. Computational considerations. Theorem 2.4 establishes the validity of the following *column-generation* procedure for solving the nonlinear integer program (1.2) under the hypotheses (A)–(D) of § 2: [3]

(3.1) Set the iteration index $k=0$, and select an initial set of breakpoints
$$R_i^0 \supseteq \{l_i, u_i\} \, (i=1,\cdots,n).$$

(3.2) Solve the LP (2.11) with $R_i''=R_i^k$.

(3.3) If the optimal solution obtained for (2.11) satisfies the optimality conditions of Theorem 2.4, then it also solves (1.2), and the algorithm terminates; otherwise, increase k by 1 and add the breakpoints that would have been required to satisfy the breakpoint hypotheses of Theorem 2.4 for the solution obtained in (3.2) (thereby obtaining "finer" index sets R_i^k) and return to (3.2).

Since the maximum possible number of breakpoints is finite, and at least one new breakpoint is added at each iteration, this procedure must terminate in a *finite* number of iterations with an optimal solution of (1.2). As with other column-generation procedures, each succeeding iteration can be started with the optimal basis from the previous iteration. If function evaluations are much more "expensive" than pivot operations, the procedure could be modified by selecting the initial breakpoints close to an estimate of the optimal solution and adding only some of the "missing" breakpoints in step (3.3).

Linear programming can also be used to establish a *lower* bound on the optimal value of (1.2). (A lower bound may be useful if convergence is slow and one is content to have a feasible solution whose objective value is "close" to optimal.) To compute a lower bound, the f_i are replaced by convex, piecewise-linear functions f_i^* satisfying $f_i^*(x_i)\leqq f_i(x_i)$ for $x_i \geqq 0$ and integer. Such f_i^* may be

[3] This procedure may also be used if hypothesis (B) is violated (i.e., $F=\varnothing$), since $F=\varnothing$ if and only if the feasible set of (1.2) is empty, and the first attempt to analyze an LP in (3.2) will establish $F=\varnothing$ if this is the case. The case in which hypothesis (A) is violated (i.e., F is unbounded) is dealt with in § 5, where it is shown that an analogous procedure will converge in a finite number of iterations if (1.2) has an optimal solution.

obtained from a finite, nonempty set $R_i^{(k)}$ of integers ($R_i^{(k)} \subseteq [l_i, u_i - 1]$) by defining

$$f_i^*(x_i) \equiv \min_{z_i, \lambda_{i,j}} z_i$$

$$\text{subject to } z_i \geqq f_i(j) + \lambda_{i,j}(f_i(j+1) - f_i(j)),$$
$$x_i = j + \lambda_{i,j} \quad (j \in R_i^{(k)}).$$

Replacing the f_i by f_i^* and deleting the integrality constraints yields a linear program whose optimal value is a lower bound on the optimal value of (1.2). Conditions guaranteeing finiteness of this lower bound and further details and refinements may be found in [11].

In the case of the problem (1.1), it should be noted that, when the δ-form representation is used, the equivalent LP has such a simple structure that its solution is obvious. In fact, rather than dealing with (1.1), we can consider the more general class of problems of the form

$$\min \sum_{i=1}^{n} f_i(x_i)$$

$$\text{subject to } \sum_{i=1}^{n} x_i = r, \tag{3.4}$$
$$l_i \leqq x_i \leqq u_i \quad (i = 1, \cdots, n),$$
$$x_i \text{ integer} \quad (i = 1, \cdots, n).$$

(Note that (1.1) is equivalent to a problem of the form (3.4) with $l_i = 0$ and $u_i = r$ $(i = 1, \cdots, n)$.) Figure 1 shows the network that results after the arcs corresponding to flows fixed at lower bounds have been accounted for by reducing the

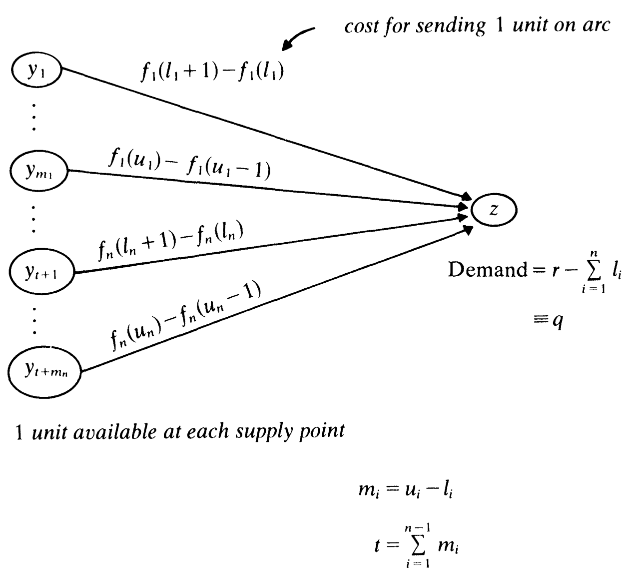

FIG. 1. *A network equivalent of* (3.4)

"demand" by $\sum_{i=1}^{n} l_i$. (We assume that the condition $\sum_{i=1}^{n} l_i \leqq r \leqq \sum_{i=1}^{n} u_i$, which is necessary and sufficient for feasibility, has been verified.) This network has $t+m_n$ supply points, each of which can ship at most 1 unit to the demand point z, at which the demand is $q \equiv r-\sum_{i=1}^{n} l_i$. If $q=0$, the optimal solution is obtained by setting $x_i = l_i$ for all i; otherwise, the optimal solution is obtained by sending one unit along each of the q "least expensive" arcs. Because of convexity, for each i we have $f_i(j)-f_i(j-1) \leqq f_i(j+1)-f_i(j)$ for j satisfying $l_i \leqq j-1 \leqq j+1 \leqq u_i$, and using this property it may be shown that the determination of the q least expensive arcs requires at most $2n+(r-\sum_{i=1}^{n} l_i)-1$ function evaluations. (For further details on (1.1) see [9], [10], [13].)

4. Total unimodularity and networks. On the one hand, it is well-known that the standard minimum cost single-commodity network problem gives rise to a constraint matrix that is totally unimodular. In fact, in dealing with separable convex functions defined on the arcs of a network, it is possible to mimic the δ-formulation within the network context by replacing each arc by a set of arcs in a manner similar to that of Fig. 1 (for details, see [10] or [11]). Given the efficiency of special techniques for network optimization a network formulation will generally be more efficiently solved by such techniques than by using the ordinary simplex algorithm.

On the other hand, it is unclear whether, in all cases, network formulations can be constructed for problems of the form (1.2) with A totally unimodular. (Specifically, we would like to be able to construct from A and b a directed graph with node-arc incidence matrix B and a (possibly unbounded) hyper-rectangle R such that a vector x is feasible for (2.8) if and only if there exists a y such that the pair $\binom{x}{y}$ satisfies $B\binom{x}{y}=0$, $\binom{x}{y} \in R$. Certain results in matroid theory (see, for example [12], [15]), while not addressing questions of quite this generality, suggest that such a construction is not always possible.) In any event, the linear programming approach of the previous sections makes conversion to a network unnecessary, since it specifies the *algebraic* transformation of the original problem that will yield an equivalent LP.

5. The unbounded case. The results of this section show that the hypotheses regarding boundedness of the feasible set and finiteness of the number of breakpoints can be deleted, provided that the conclusions are appropriately generalized.

THEOREM 5.1. *Let the function $\bar{f}_i$ $(i=1,\cdots,n)$ be continuous piecewise-linear convex functions on* $[0,+\infty)$ *whose derivatives are also continuous except possibly on subsets of the positive* integers. *Let A be an $m \times n$ totally unimodular matrix, and b be an integer vector. Then the values*

$$\inf \sum_{i=1}^{n} \bar{f}_i(x_i) \tag{5.1}$$

$$\textit{subject to } Ax=b, \quad x \geqq 0, \quad x \textit{ integer}$$

and

$$\inf \sum_{i=1}^{n} \bar{f}_i(x_i) \tag{5.2}$$
$$\text{subject to } Ax = b, \quad x \geqq 0$$

coincide, and (5.1) *has an optimal solution if and only if* (5.2) *has an optimal solution.* (*The value is taken to be* $+\infty$ *when the constraints are infeasible.*)

Proof. If (5.2) has a feasible solution, then the feasible set of (5.2) has an extreme point, which is thus integer and therefore a feasible solution of (5.1).

The conclusions of the theorem are then easily proved by considering feasible or optimal solutions of (5.1) and (5.2), adding appropriate bounds to both problems, and applying the results of § 2. □

It should be noted that it is possible that (5.1) and (5.2) may have finite infima that are not attained, and the theorem shows that if this is the case for one of these problems, it must be true for the other also.

We now state the analogue of Theorem 2.4 in the absence of upper bounds.

THEOREM 5.2. *If the hypotheses of Theorem* 5.1 *are satisfied, and* x^{**} *is an integer optimal solution of* (5.2), *then* x^{**} *is an optimal solution of*

$$\inf \sum_{i=1}^{n} f_i(x_i) \tag{5.3}$$
$$\text{subject to } Ax = b, \quad x \geqq 0, \quad x \text{ integer}$$

where, for $i = 1, \cdots, n$, f_i *is any convex function that agrees with* $\bar{f}_i$ *on the set* $\{x_i^{**}-1, x_i^{**}, x_i^{**}+1\} \cap \{y \mid y \geqq 0\}$.

Proof. Suppose that there exists an $\bar{x}$ feasible for (5.3) such that $\sum_{i=1}^n f_i(\bar{x}_i) < \sum_{i=1}^n f_i(x_i^{**})$. Generate bounded variants of (5.2) and (5.3) by adding the constraints $\min\{\bar{x}_i, x_i^{**}\} \leqq x_i \leqq \max\{\bar{x}_i, x_i^{**}\}$ $(i = 1, \cdots, n)$. The bounded variant of (5.3) must have an optimal solution x^* such that $\sum_{i=1}^n f_i(x_i^*) < \sum_{i=1}^n f_i(x^{**})$. However, by applying Theorem 3.4 to the bounded variants of (5.2) and (5.3), we obtain a contradiction. □

Finite convergence of an extension to the unbounded case of the column-generation procedure of § 3 follows in a straightforward fashion from the next theorem, which applies to a broad class of integer programs, since total unimodularity of the constraint matrix A is *not* required for the result. (For notational convenience we define

$$f(x) \equiv \sum_{i=1}^{n} f_i(x_i) \quad \text{and} \quad F_I \equiv \{x \mid Ax = b, x \geqq 0, x \text{ integer}\}.$$

Details of an algorithm for the unbounded case are discussed in [11].)

THEOREM 5.3. *If* f_i $(i = 1, \cdots, n)$ *are convex functions on* $[0, +\infty)$ *and if the problem*

$$\inf \sum_{i=1}^{n} f_i(x_i) \tag{5.4}$$
$$\text{subject to } Ax = b, \quad x \geqq 0, \quad x \text{ integer}$$

has an optimal solution, then, for each real number M, the set $\{f(x)|x \in F_I\}$ contains a finite number (possibly 0) of distinct values in the range $(-\infty, M]$.

Proof. Suppose the result is false, for some M and let $\{x^{(k)}\}$ be a sequence contained in F_I with the property that $\{f(x^{(k)})\}$ is a sequence of distinct values in $(-\infty, M]$. Using the nonnegativity of the $x^{(k)}$, we shall construct an increasing subsequence I_n of integers and a partition J', J'' of the index set $\{1, \cdots, n\}$ such that if $r, s, t \in I_n$ with $r<s<t$, then $0 \leqq x_i^{(s)} - x_i^{(r)} \leqq x_i^{(t)} - x_i^{(s)}$, with strict inequalities holding for $i \in J''$. If $\{x_1^{(k)}\}$ is bounded, then there exists an integer $\bar{x}_1$ such that $x_1^{(k)} = \bar{x}_1$ for infinitely many k; in this case $1 \in J'$ and we denote by I_1 an increasing infinite subsequence of $\{1, 2, \cdots\}$ such that $x_1^{(k)} = \bar{x}_1$. If $\{x_1^{(k)}\}$ is not bounded, $1 \in J''$ and I_1 is taken to be an increasing infinite subsequence of $\{1, 2, \cdots\}$ such that $r, s, t \in I_1$ implies $0 < x_1^{(s)} - x_1^{(r)} < x_1^{(t)} - x_1^{(s)}$. Proceeding in an analogous fashion with the sequence $\{x_2^{(k)} \mid k \in I_1\}$, we place the index 2 in J' or J'' and extract from I_1 a subsequence I_2, and continue this process until all indices have been placed in J' or J'' and I_n has been extracted from I_{n-1}. Clearly J', J'' and I_n have the required properties, and note that $\{x_i^{(k)}|k \in I_n\}$ is constant for $i \in J'$. J' may be empty, but $J'' \neq \emptyset$ since otherwise $x^{(k)}$ would be constant for $k \in I_n$, contradicting the assumed distinctness of the elements of $\{f(x^{(k)})\}$.

Now $\{f(x^{(k)})|k \in I_n\}$ contains either a decreasing subsequence or an increasing subsequence. If it contains a decreasing subsequence, choose p such that $p \in I_n$ and $x_i^* \leqq x_i^{(p)}$ for $i \in J''$. Let $\Delta \equiv x^{(p+1)} - x^{(p)}$, and note that $\Delta \geqq 0$ and $A\Delta = 0$, so that $(x^* + \Delta) \in F_I$. However, using the convexity of f we have

$$
\begin{aligned}
f(x^*) - f(x^* + \Delta) &= \sum_{i=1}^{n} [f_i(x_i^*) - f_i(x_i^* + \Delta_i)] \\
&= \sum_{i \in J''} [f_i(x_i^*) - f_i(x_i^* + \Delta_i)] \\
&\geqq \sum_{i \in J''} [f_i(x_i^{(p)}) - f_i(x_i^{(p)} + \Delta_i)] \\
&= f(x^{(p)}) - f(x^{(p+1)}) > 0,
\end{aligned}
$$

which contradicts the assumed optimality of x^*. In the remaining case, $\{f(x^{(k)})|k \in I_n\}$ contains an increasing sequence, and since this sequence is bounded from above, there exist $r, s, t \in I_n$ such that $r<s<t$ and $f(x^{(t)}) - f(x^{(s)}) < f(x^{(s)}) - f(x^{(r)})$. However, $x^{(t)} - x^{(s)} \geqq x^{(s)} - x^{(r)} \geqq 0$ and the convexity and separability of f imply $f(x^{(t)}) - f(x^{(s)}) \geqq f(x^{(s)}) - f(x^{(r)})$, a contradiction. □

It might be noted that a straightforward extension of this result to the convex, *nonseparable* case is not possible, since it is easily seen that taking $f(x_1, x_2) = (x_1 - \sqrt{2}\, x_2)^2$, $A = 0$, $b = 0$ satisfies all of the hypotheses of the theorem except separability, but violates the conclusion of the theorem.

The following corollary, an immediate consequence of Theorem 5.3, establishes *finite convergence* for any "primal, nondegenerate" method for the class of problems considered in that theorem.

COROLLARY 5.4. *If the hypotheses of Theorem 5.3 hold, then any algorithm for the problem (5.4) that yields feasible iterates $x^{(0)}, x^{(1)}, \cdots$ satisfying $f(x^0) > f(x^1) > \cdots$ will generate an optimal solution for (5.4) in a finite number of iterations.*

6. Conclusions. We have shown how optimal solutions to bounded nonlinear integer programs of the form (1.2) (with f_i convex, A totally unimodular, and b integer) may be obtained by solving an easily-generated linear programming problem. These results generalize certain results in (linear) integer programming dealing with totally unimodular constraint matrices as well as results for nonlinear integer programs of the form (1.1), and provide a rigorous and finite approach for obtaining optimal solutions. Furthermore, in the case that known bounds are not available for the variables in (1.2), it is shown that an appropriate linear programming "column-generation" algorithm will yield an optimal solution in a finite number of iterations if (1.2) actually has an optimal solution.

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