MTH 101

24 ELEMENTARY MATHEMATICS I

PARTIAL FRACTIONS

$\frac{1}{3} + \frac{1}{2}$] → Parts

$\frac{2+3}{6}$

$= \frac{5}{6}$] → Fraction

$\frac{5}{6}$, ~~when~~ dissolved will result in $\frac{1}{3} + \frac{1}{2}$

Ex! Resolve $\frac{3x+1}{(x+1)(x+2)}$ into partial fraction

$$\frac{3x+1}{(x+1)(x+2)} \equiv \frac{A}{(x+1)} + \frac{B}{(x+2)}$$

$$\frac{3x+1}{(x+1)(x+2)} \equiv \frac{A(x+2) + B(x+1)}{(x+1)(x+2)}$$

$$3x+1 \equiv A(x+2) + B(x+1)$$

Put $x = -1$; so $B = 0$

$3(-1)+1 = A(-1+2)$

$-2 = A$; $A = -2$

Put $x = -2$; so $A = 0$

$3(-2)+1 = B(-2+1)$

$-5 = -B$

$B = 5$

$\therefore \frac{3x+1}{(x+1)(x+2)} = \frac{-2}{(x+1)} + \frac{5}{x+2}$

Resolved ~~fo dissolved~~

2: Resolve $\frac{5}{(x-1)(x+1)}$ into partial fraction

$$\frac{5}{(x-1)(x+1)} \equiv \frac{A}{x-1} + \frac{B}{x+1}$$

$$\frac{5}{(x-1)(x+1)} \equiv \frac{A(x+1) + B(x-1)}{(x-1)(x+1)}$$

$$5 = A(x+1) + B(x-1)$$

$x = 1$

$5 = A(1+1) + B(0)$

$5 = 2A$; $A = 5/2$

$x = -1$

$5 = A(0) + B(-2)$

$5 = -2B$ | $\therefore B = -5/2$

$$\therefore \frac{5}{(x-1)(x+1)} \equiv \frac{5}{2(x-1)} - \frac{5}{2(x+1)}$$

3. $$\frac{2x+3}{(x+2)(x+1)} \equiv \frac{C}{(x+2)} + \frac{D}{(x+1)}$$

$$\frac{2x+3}{(x+2)(x+1)} \equiv \frac{C(x+1) + D(x+2)}{(x+2)(x+1)}$$

$$2x+3 = C(x+1) + D(x+2)$$

$x = -1$

$2(-1)+3 = C(0) + D(-1+2)$

$1 = D$

$x = -2$

$2(-2)+3 = C(-2+1) + D(0)$

$-1 = -C$; $C = 1$

$$\therefore \frac{2x+3}{(x+2)(x+1)} \equiv \frac{1}{(x+2)} + \frac{1}{(x+1)}$$

MONDAY 1st JULY, 2024

COMPLEX NUMBERS

If a number is not real, it is complex.

Real numbers: Any number that can be expressed on the number line. They include whole numbers, decimals, rational (fractions), irrational numbers (surd).

Number line: $-4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$

Complex numbers: They are denoted as Z and are written as

$$Z = a + ib$$

where a & b are real numbers. NB: $\boxed{i^2 = -1}$

e.g. $Z = 2 - 3i$; $Z = 2 + 3i$ $\quad \boxed{i = \sqrt{-1}}$

$Z = -5 + 2i$

Parts of complex number:

$Z = 2 + 3i$

① Real part ② Imaginary part

e.g. $Z = 3i \Rightarrow$ A complex number

$\boxed{Z = 0 + 3i}$

CONJUGATE OF A COMPLEX NUMBER

In real numbers, the conjugate is usually a change in sign:

$3+\sqrt{2} \Rightarrow 3-\sqrt{2}$; $3-\sqrt{2} \rightarrow 3+\sqrt{2}$

However, conjugate of a complex number is written as:

$z = 3-2i$

$z^* = 3+2i$ or $\bar{z} = 3+2i$

ADDITION & SUBTRACTION OF COMPLEX NUMBER

ADDITION: The real parts are added together and imaginary parts are also added together $\Rightarrow$ Collect Like Terms.

Ex: Add $z_1 = 4+6i$ and to $z_2 = 3+5i$

$z_1 + z_2 = 7 + 11i$

Subtraction: $z_1 - z_2 = 1 - 1i$

MULTIPLICATION:

$z_1 = 3+5i$; $z_2 = -2-6i$

$z_1 z_2 = (3+5i)(-2-6i)$

$z_1 z_2 = 3(-2-6i) + 5i(-2+6i)$

$z_1 z_2 = -6 - 18i + 10i - 30i^2$

$z_1 z_2 = -6 - 28i - 30i^2$

NOTE!

Similar to multiplication of Algebraic expressions

Substitute −1 for i^2 Ans ≠ [illegible]

$z_1 z_2 = -6 - 18i - 30(-1)$

$z_1 z_2 = -6 - 28i + 30$

$z_1 z_2 = 24 - 28i$

Division: Similar to real numbers, find the conjugate

$$\frac{z_1}{z_2} = \frac{3+5i}{-2-6i}$$

$$\frac{z_1}{z_2} = \frac{3+5i}{-2-6i} \times \frac{-2+6i}{-2+6i}$$

$$\frac{z_1}{z_2} = \frac{3(-2+6i) + 5i(-2+6i)}{-2(-2+6i) - 6i(-2+6i)}$$

$$\frac{z_1}{z_2} = \frac{-6+18i-10i+30i^2}{4 \cancel{-12i} \cancel{+12i} - 36i^2}$$

$$\frac{z_1}{z_2} = \frac{-6+8i+30i^2}{4-36i^2}$$

a = 8

Subst for i^2

$$\frac{z_1}{z_2} = \frac{-6+8i+30(-1)}{4-36(-1)}$$

$$\frac{z_1}{z_2} = \frac{-6+8i-30}{4+36}$$

$$\frac{z_1}{z_2} = \frac{-36+8i}{40}$$

$$\frac{z_1}{z_2} = \frac{-36}{40} + \frac{8i}{40}$$

$$\boxed{\frac{z_1}{z_2} = \frac{-9}{10} + \frac{1}{5}i}$$

$\left|\frac{-6}{3}\right| = \left|\frac{-6}{3}\right|$

MODULUS OF A COMPLEX NUMBER

It is denoted by $|z|$. $\Rightarrow$

$z = x + iy$

$$\boxed{|z| = \sqrt{x^2 + y^2}}$$ → Pythagorean's Theorem

where $x^2 + y^2 > 0$

Ex! $z = 4 - 6i$

$|z| = \sqrt{4^2 + (-6)^2}$

$|z| = \sqrt{16 + 36}$

$|z| = \sqrt{52}$

$|z| = 13\sqrt{2}$

$| \, |$ → absolute

PROPERTIES OF MODULUS

1. $|z| = |-z| = |\bar{z}|$

$|z| = |-z|$

$-z = z$

2. $|z_1 \pm z_2| \geq |z_1| - |z_2|$

~~(6 - 2)~~ $(6) - (2)$

3. $|z|^2 = |z^2| = z\bar{z}$

4. $|z_1 z_2| = |z_1| |z_2|$

5. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ $\boxed{\text{where } z_2 \neq 0 \quad z_2 \mathrel{!}= 0}$

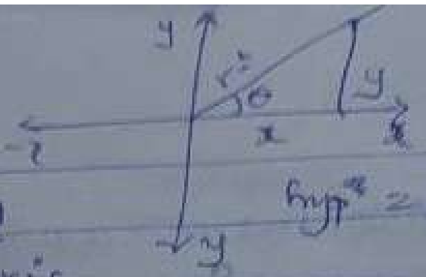
$9 = 3^2$

$(3^2) = (3)^2$

$9 = 9 = (-3)^2$

$(-3^2) \neq 9$

-9



ARGANDS ~~ADAM's~~ DIAGRAM

$hyp^2 = opp^2 + adj^2$

Similar to coordinate axis

$z = x + yi$

Resultant:

Angle:

$\tan\theta = y/x$

$\Rightarrow \boxed{\theta = \tan^{-1}(y/x)}$

Pythagorean's Theorem

$r^2 = x^2 + y^2$

$\boxed{r = \sqrt{x^2 + y^2}}$

$\tan\theta = \frac{y}{x}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ is called the argument

To express a complex number in its polar form these are used:

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$Z = a + ib$

$Z = r(\cos\theta + i\sin\theta) \rightarrow$ Polar form of a complex number

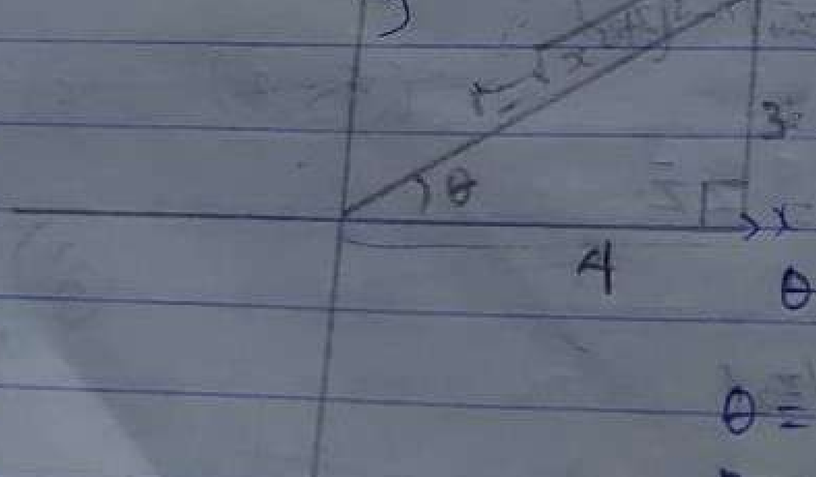
$\rightarrow \boxed{Z = re^{i\theta}} \Rightarrow$ Euler's formular

Ex: $z = 4 + 3i$ in polar form

$r = \sqrt{3^2 + 4^2}$

$r = \sqrt{25}$

$r = 5$



$\theta = \tan^{-1}(y/x)$

$\theta = \tan^{-1}(3/4)$

$\cancel{\theta = 53.13}$ $\theta = 36.87$

$Z = 5(\cos 36.87 + i\sin 36.87)$

Polar expression: $z = 5(\cos 36.9^\circ + i\sin 36.9^\circ)$

De-Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ and $\boxed{z^n = r^n(\cos n\theta + i\sin n\theta)}$

then ($n=2$)

$$\boxed{z^2 = r^2(\cos 2\theta + i\sin 2\theta)}$$

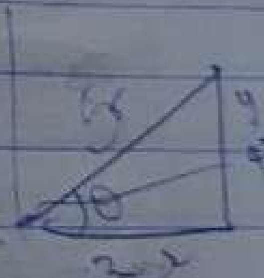
$z^3 = r^3$, $5^3(\cos 3 \ldots)$

$\boxed{\text{n-th roots of a complex number}}$ N-th root of a complex number

$$z^{1/n} = z_k = r^{1/n}\left[\cos\left(\frac{2\pi k+\theta}{n}\right) + i\sin\left(\frac{2\pi k+\theta}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots, n-1$.

Ex: Find all the fourth roots of $z = 3+4i$



$n = 4$; $k = 0, 1, 2, \ldots, (4-1)$

$\hookrightarrow k = 0, 1, 2, 3$

$r = \sqrt{3^2 + 4^2}$ $\qquad \theta = \tan^{-1}(y/x)$

$r = \sqrt{25}$ $\qquad \theta = \tan^{-1}\left(\frac{4}{3}\right)$

$r = 5$ $\qquad \theta = 53.13^\circ$

$2\pi \rightarrow 360^\circ$

$\pi = 180^\circ$

$2\pi = 360^\circ$

When $k = 0$:

$$Z_0 = 5^{1/4}\left[\cos\left(\frac{2\pi(0)+53.13}{4}\right) + i\sin\left(\frac{2\pi(0)+53.13}{4}\right)\right]$$

$$= 1.25(\cos 13.2825 + i\sin 13.2825)$$

$z_0 = 1.25\,(0.9732 + 0.2298i)$ $\boxed{2\pi = 360^\circ}$

$\boxed{z_0 = 1.217 + 0.287i}$

$$z_1 = 5^{1/4}\left[\cos\left(\frac{2\pi(1)+53.13^\circ}{4}\right) + i\sin\left(\frac{2\pi(1)+53.13}{4}\right)\right]$$

$$= 5^{1/4}\left[\cos 103.28^\circ + i\sin 103.28^\circ\right]$$

$$= 1.25\,(-0.2297 + 0.9733i)$$

$$\boxed{z_1 = 0.287 + 1.217i}$$

$$z_2 = 5^{1/4}\left[\cos\left(\frac{2\pi(2)+53.13}{4}\right) + i\sin\left(\frac{2\pi(2)+53.13}{4}\right)\right]$$

$$= 1.25\,(\cos 193.2825 + i\sin 193.2825)$$

$z_2 =$

$$z_3 = 5^{1/4}\left[\cos\left(\frac{2\pi(3)+53.13}{4}\right) + i\sin\left(\frac{2\pi(3)+53.13}{4}\right)\right]$$

$z_3 = 1.25\,(\cos \qquad + i\sin \qquad)$

$$\cancel{z_4 = 5^{1/4}\left[\cos\left(\frac{2\pi(4)+53.13}{4}\right) + i\sin\left(\frac{2\pi(4)+53.13}{4}\right)\right]}$$

$\therefore$ The four first roots of $z = 3+4i$ are

$x = \text{'oyin'}$

July 2024: MATHEMATICAL INDUCTION

1. Prove that:

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

Assume $n=1$ is true

$$L.H.S = \sum_{r=1}^{1} r^2 = 1^2 = 1$$

$$R.H.S. = \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{1(1+1)(2(1)+1)}{6}$$

$$= \frac{1(2)(3)}{6}$$

$$= 1$$

L.H.S. = R.H.S.

$\therefore n=1$ is true #

Assume $n=2$ is true

$$\sum_{r=1}^{2} r^2$$

$$1^2 + 2^2 = 1+4$$

$$= 5$$

$$RHS = \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{2(2+1)(2(2)+1)}{6}$$

$$= \frac{2(3)(5)}{6}$$

$$= 5$$

LHS = R.H.S

$\therefore n=2$ is true

Assume $n=k$ is true

$$\sum_{r=1}^{k} r^2 = \frac{k(k+1)(2k+1)}{6}$$

$$\sum_{r=1}^{k} r^2 = 1^2+2^2+3^2+\ldots+k^2 \cancel{+(k+1)^2}$$

Show that $n=k+1$ is true

$$\sum_{r=1}^{k+1} r^2 = 1^2+2^2+3^2+\ldots k^2+(k+1)^2$$

$$\sum_{r=1}^{k+1} r^2 = \sum_{1}^{k} r^2 + (k+1)^2$$

$$\sum_{r=1}^{k+1} r^2 = \frac{k(k+1)(2k+1)}{6} + \frac{(k+1)^2}{1}$$

$$= (k+1)\left(\frac{k\cancel{(k+1)}(2k+1)}{6} + \frac{(k+1)}{1}\right)$$

$$= k+1\left(\frac{k\cancel{(k+1)}(2k+1)}{6} + \frac{6(k+1)}{1}\right)$$

$$\frac{(k+1)}{6}\left(\frac{2k^2+k+6k+6}{\cancel{6}}\right)$$

$$\frac{k+1}{6}\left(\frac{\cancel{3k^2}\ 2k^2+7k+6}{\cancel{6}}\right)$$

$$\left(\frac{k+1}{6}\right)\left((2k^2+4k)+(3k+6)\right)$$

$$\frac{k+1}{6}\left[\cancel{1}2k(k+2)+3(k+2)\right]$$

$$\frac{k+1}{6}\left[(2k+3)+(k+2)\right]$$

$$\boxed{\frac{(k+1)(k+2)(2k+3)}{6}}$$ Proof:

Subst for n with (k+1)

$$\frac{n(n+1)(2n+1)}{6}$$

$$\frac{(k+1)(k+1)(2(k+1)+1)}{6}$$

$$\boxed{\frac{(k+1)(k+2)(2k+3)}{6}}$$

Successfully Proven!!!

$\forall$ = for all

Conclusion: Since $n=1, n=2$ is true and $n=k$ is true, then $n=k+1$ is true ~~$\forall$~~ for all ($\forall$) value of n.

By mathematical induction! Prove that:

(i) $2^1+2^2+2^3+\dots+2^n=2(2^n-1)$

(ii) If $A=\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}$ then $A^n=\begin{pmatrix}1+2n & -4n\\ n & 1-2n\end{pmatrix}$

Solution

(i) $2^1+2^2+2^3+\dots+2^n = 2(2^{n-1})$

Assume that $n=1$ is true

$\sum_{r=1}^{1} 2^r = 2^1+2^2+2^3$

R.H.S

$2(2^{n-1})$

$2(2^{1-1}) = 2(2^0)$

$= 2$

L.H.S

(ii) $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ then $A^n = \begin{pmatrix} 1+2n & -4n \\ n & 1-2n \end{pmatrix}$

Assume $n=1$ is true

R.H.S.

$A^1 = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$

$A^1 = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$

L.H.S

$A^1 = \begin{pmatrix} 1+2(1) & -4(1) \\ 1 & 1-2(1) \end{pmatrix}$

$A^1 = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$

Assume $n=2$ is true

$A^n = A^2 = A \times A = \cancel{\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}}$

$\Rightarrow \ln 2x-5$

$$\frac{\delta y}{\delta x} = \frac{2}{2x-5}$$

$\Rightarrow$ $y = e^x \ln x$

Using product rule:

$u = e^x$, $v = \ln x$; $\delta u/\delta x = e^x$; $\delta v/\delta x = \frac{1}{x}$

$$\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x}$$

$$\frac{\delta y}{\delta x} = e^x\left(\frac{1}{x}\right) + \ln x\left(e^x\right)$$

$$\frac{\delta y}{\delta x} = \frac{e^x}{x} + e^x \ln x$$

$\Rightarrow$ $y = 2x^2 e^{x^2-3}$

$u = 2x^2$; $v = e^{x^2-3}$; $\delta u/\delta x = 4x$; $\delta v/\delta x = 2xe^{x^2-3}$

$$\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x}$$

$$\frac{\delta y}{\delta x} = 2x^2\left(2xe^{x^2-3}\right) + e^{x^2-3}\left(4x\right)$$

$$\frac{\delta y}{\delta x} = 4x^3 e^{x^2-3} + 4xe^{x^2-3}$$

$$\therefore \delta y/\delta x = 4xe^{x^2-3}\left(x^2+1\right)$$

$$y = \frac{e^x}{x}$$ Quotient Rule

Let $u = e^x$; $v = x$; $\delta u/\delta x = e^x$; $\delta v/\delta x = 1$

$$\frac{\delta y}{\delta x} = \frac{v\frac{\delta u}{\delta x} - u\frac{\delta v}{\delta x}}{v^2}$$

$$\frac{\delta y}{\delta x} = \frac{x(e^x) - e^x(1)}{x^2}$$

$$\frac{\delta y}{\delta x} = \frac{xe^x - e^x}{x^2}$$

$$\frac{\delta y}{\delta x} = \frac{e^x(x-1)}{x^2}$$

Differentiation of Trigonometric Function

RATIOS: $\sin\theta$ $\qquad$ $\mathrm{cosec}\,\theta = 1/\sin\theta$

$\cos\theta$ $\qquad$ $\sec\theta = 1/\cos\theta$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\qquad$ $\cot\theta = 1/\tan\theta = \frac{\cos\theta}{\sin\theta}$

IDENTITIES:

$\sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

$$\boxed{\begin{aligned} y &= \sin\theta \\ \frac{\delta y}{\delta\theta} &= \cos\theta \end{aligned}}$$

$$\boxed{\begin{aligned} y &= \cos\theta \\ \frac{\delta y}{\delta\theta} &= -\sin\theta \end{aligned}}$$

$\cancel{y = \tan\theta}$

$\cancel{\frac{\delta y}{\delta\theta} =}$

$y = \tan\theta$

$y = \frac{\sin\theta}{\cos\theta}$

$$\frac{\delta y}{\delta x} = \frac{v\frac{\delta u}{\delta x} - u\frac{\delta v}{\delta x}}{v^2}$$

$u = \sin\theta$; $v = \cos\theta$; $\delta u/\delta\theta = \cos\theta$; $\delta v/\delta\theta = -\sin\theta$

$$\frac{\delta y}{\delta\theta} = \frac{\cos\theta(\cos\theta) - \sin\theta(-\sin\theta)}{(\cos\theta)^2}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}$$

$$\Rightarrow \frac{\delta y}{\delta\theta} = \frac{1}{\cos^2\theta}$$

The differentiation of $\tan\theta$ is $\sec^2\theta$

$\therefore \delta y/\delta\theta = \sec^2\theta$

Which is also: $1 + \tan^2\theta$

Wednesday 18th ...

Ex: $y = \sin 5x$

$$\frac{\delta y}{\delta x} = 5\cos 5x$$

2. $y = \sin(7x^2 - 3x + 1)$

$$\frac{\delta y}{\delta x} = (14x - 3)\cos(7x^2 - 3x + 1)$$

3. $y = x\sin 3x$

Using Product Rule

$$\frac{\delta y}{\delta x} = \frac{u\,dv}{\delta x} + \frac{v\,\delta u}{\delta x}$$

$$\frac{\delta y}{\delta x} = x(3\cos 3x) + \sin 3x(1)$$

$$= 3x\cos 3x + \sin 3x$$

4. $y = \sin^2 x^2$

Using Chain rule: $\rightarrow (\sin u)^2$

Let $u = x^2$; $y = \sin^2 u$; $\delta u/\delta x = 2x$; $\delta y/\delta u = 2\sin u\cos u$

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$$

$$\frac{\delta y}{\delta x} = 2\sin u\cos u \times 2x$$

$$\frac{\delta y}{\delta x} = 4x\sin x^2\cos x^2.$$

FURTHER DERIVATIVES $\left(\frac{d^2y}{dx^2}\right)$ n = no of differentiation

1f $y = 7x^5 - 6x^4 + 5x^3$; find $\frac{\delta^6 y}{\delta x^6}$

$$\frac{\delta y}{\delta x} = 35x^4 - 24x^3 + 15x^2$$

$$\frac{\delta^2 y}{\delta x^2} = 140x^3 - 72x^2 + 30x$$

$$\frac{\delta^3 y}{\delta x^3} = 420x^2 - 144x + 30$$

$$\frac{\delta^4 y}{\delta x^4} = 840x - 144$$

$$\frac{\delta^5 y}{\delta x^5} = 840$$

$$\frac{\delta^6 y}{\delta x^6} = 0$$

NOTE: $\frac{d^2y}{\delta x^2} \neq \left(\frac{\delta x}{\delta y}\right)^2$

STATIONARY POINTS (Turning Point / Critical pt.)

conditions: When $\boxed{\frac{\delta y}{\delta x} = 0} \Rightarrow x = a$

Ex. $y = 2x^3 - 1.5x^2 - 45x + 1$

At stationary point, $\frac{dy}{dx} = 0$

$$\frac{\delta y}{\delta x} = 6x^2 - 3x - 45$$

$$6x^2 - 3x - 45 = 0$$

$$2x^2 - x - 15 = 0$$

$$(2x^2 - 6x) + (5x - 15) = 0$$

$$2x(x-3) + 5(x-3) = 0$$

$$(2x+5)(x-3) = 0$$

$$x = -5/2 \text{ or } 3$$

$\therefore x = -5/2, 3$ are the stationary point.

Tuesday 23rd July, 2024

Maximum & Minimum Points and Values of a function

STEP ONE! $y = f(x)$ STEP TRE: $\frac{\delta y}{\delta x} = 0$ (Critical point/Turning...)

STEP TWO $\frac{\delta y}{\delta x}$

If $\frac{\delta y}{\delta x} = 2x + 1 = 0$
$\Rightarrow x = a$

elif $\frac{\delta y}{\delta x} = x^2 + 5x + 6 = 0$
$x = a, x = b$

$$\frac{\delta^2 y}{\delta x^2} = f(x)$$

$\frac{\delta^2 y}{\delta x^2} > 0$ (min. point)

$\frac{d^2 y}{dx^2} < 0$ (~~minimum~~ maximum point)

The equation of a curve is given by:

$y = 2/3x^3 + 1/2x^2 - x + 5$; (i) determine the critical points of this curve and discriminate between them.

(ii) Determine the max. & min. values of y corresponding to the points of inflection

$\frac{dy}{dx} = 2x^2 + x - 1$

At critical point:

$\boxed{dy/dx = 0}$

(i) $(2x^2 + 2x)(-x - 1) = 0$

$= 2x(x+1) - 1(x+1) = 0$

$(2x-1)(x+1)$

$2x - 1 = 0$ or $x + 1 = 0$

$x = 1/2, -1$ are the critical points.

$\frac{d^2y}{dx^2} = 4x + 1$

$= 4(1/2) + 1$

$2 + 1 = 3$ (min. point)

$\frac{d^2y}{dx^2} = 4(-1) + 1$

$-4 + 1$

$= -3$ (max. point)

Therefore, $x = -1$ is the maximum point, while $x = 1/2$ is the minimum point

(ii) $y = 2/3x^3 + 1/2x^2 - x + 5$ Subst. x_{min} into y_{min}

$y_{min} = 2/3(1/2)^3 + 1/2(1/2)^2 - 1/2 + 5$

$= 2/3(1/8) + 1/2(1/4) - 1/2 + 5$

$= 1/12 + 1/8 - 1/2 + 5/1$

$$y_{min} = \frac{2+3+6+100}{24}$$

$$= \frac{1\,\cancel{1}3}{24}$$

$$y_{max} = \tfrac{2}{3}(-1)^3 + \tfrac{1}{2}(-1)^2 - (-1) + 5$$

$$= -\tfrac{2}{3} + \tfrac{1}{2} + 1 + 5$$

$$= \frac{-4+3+6+30}{6}$$

$$y_{max} = \frac{35}{6}$$

(iii) Value of x at point of inflection

$\frac{\partial^2 y}{\partial x^2} = 0$; $\frac{\partial^2 y}{\partial x^2} = 4x+1 = 0$

$\therefore \boxed{x = -1/4}$ At point of inflection

$$y_{\text{point of inflection}} = \tfrac{2}{3}(-1/4)^3 + \tfrac{1}{2}(-1/4)^2 - (-1/4) + 5$$

$$y = \frac{2}{3}\left(-\tfrac{1}{64}\right) + \tfrac{1}{2}\left(\frac{1}{16}\right) + \tfrac{1}{4} + 5$$

$$y = -\tfrac{1}{96} + \frac{1}{32} + \tfrac{1}{4} + 5$$

$$y = \frac{-1+3+24+480}{96}$$

$$y = \frac{506}{96} \quad y = \boxed{\frac{253}{48}}$$

Wednesday 24th July, 2024

LIMIT OF A FUNCTION

$\lim_{x \to a} f(x)$

where : a is a constant

number = constant

alphabets = variable

Ex: $\lim_{x \to -1} 2x = 2(-1) = -2$

$\frac{0}{0}$ or $\frac{\infty}{\infty}$ = ~~undefined~~ Indeterminants

L'hospital's Rule . . .

Properties of Limits

1. $\lim_{x \to a} C = C$ where a and c are constants

The limit of a constant is that constant.

2. $\lim_{x \to a} c\,f(x) = C \lim_{x \to a} f(x)$

Ex: $\lim_{x \to 2} 5x^2 = 5 \lim_{x \to 2} x^2$

$= 5(2)^2 \Rightarrow 20$

3. ~~Log~~ $\lim_{x \to a} [f(x) \pm g(x)] = \left[\lim_{x \to a} f(x)\right] \pm \left[\lim_{x \to a} g(x)\right]$

Ex: $\lim_{x \to 3} 2x^2 + 5x + 7 = \lim_{x \to 3} 2x^2 + \lim_{x \to 3} 5x + \lim_{x \to 3} 7$

$2 \lim_{x \to 3} x^2 + 5 \lim_{x \to 3} x + 7$

$2(3)^2 + 5(3) + 7$

$18 + 15 + 7 = 40$

4. $\lim_{x \to a} [f(x) \cdot g(x)] = [\lim_{x \to a} f(x)] \cdot [\lim_{x \to a} g(x)]$

5. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$

6. $\lim_{x \to a} P(x) = P(a)$, where p is a polynomial

7. $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$, where n is an integer

$\lim_{x \to 1} x^5 = [\lim_{x \to 1} x]^5$

$1^5 = 1$

(ii) $2^1+2^2+2^3 \ldots +2^n = 2(2^n-1)$

RHS: Assume $n=1$ is true

$2(2^n-1)$

$2(2^1-1) = 2(2-1)$

$2(1) = 2$

LHS: $2^1+2^2+2^3 \ldots +2^n$

$2^1 = 2^1(2^1-1)$

$\cancel{2} = 2(1)$

$\cancel{2} = \cancel{2}\ 2$

Assume $n=2$ is true

LHS

$2^2 = 2(2^2-1)$

$= 2(4-1)$

$= 2(3)$

$= 6$

RHS

$= 2(2^n-1)$

$2(2^2-1)$

$2(4-1)$

$2(3)$

$= 6$

Assume $n=k$ is true

LHS: $2^k = 2(2^k-1)$

$= 4^k - 2$

$= \cancel{2^{k}-2}$

$2(2^k-1)$

$2^{k+1}-2$

RHS $= 2(2^k-1)$

$= 4^k - 2$

$= \boxed{2^{k+1}-2}$

$(2^{k+1}-2)\times 2$ | $2^{k+2}-4$

$4^3-4=60$ | $16-4=12$

show that

~~Assume~~ $k+1$ is true

LHS: 2^{k+1} | $2^{k+1}=2^k\cdot 2^1$

$2^k\cdot 2^1$ | $(2^{k+1}-2)\times 2$

$2^k=4^k-2$ | $\boxed{2^{k+2}-4}$

$2^1=2$

$(4^k-2)\cdot(2)$

8^k-4.

RHS: $2^{k+1}=2(2^{k+1}-1)$

$=4^{k+1}-2$ | $2^{k+1+1}-2$

$=4^k\cdot 4^1-2$ | $\boxed{2^{k+2}-2}$

$=4^k\times 4-2$

$=16^k-2$

9th of July, 2024

Differentiation ($\delta y/\delta x$)

δ = difference

x, y: axes of difference

δx: with respect with x (the denominator)

A function is used (An equation) to determine what would be differentiated and with respect to what/which.

$A = \pi r^2$

$\frac{dA}{\delta r} = 2\pi r$

NOTE:
The differentiation of a constant is zero (0) because when it has a coefficient ~~that~~ is raised to the power of zero.

$$y = c$$
$$\frac{\delta y}{\delta x} = 0$$

formula: $\boxed{y = x^n}$ ⇒ Function (Equation

$\boxed{\frac{dy}{dx} = nx^{n-1}}$ Differentiation (Derivative

Ex!

$y = x^6$

$\delta y/\delta x = 6x^{6-1}$

$= 6x^5$

$y = 5$

$\delta y/\delta x = 0$

Because: $y = 5x^0$

$\frac{\delta y}{\delta x} = 0$

$y = 3x^4$

$\delta y/\delta x = 12x^3$

$y = x^5 + 3x^4 + 2x^3$

$\delta y/\delta x = 5x^4 + 12x^3 + 6x^2$

Types of Differentiation

1. First Principle
2. Chain Rule
3. Product Rule
4. Quotient Rule

CHAIN RULE:

- "Function of a function"

$$\boxed{\frac{dy}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}}$$

CHAIN RULE!!!

EX! $y = (x-2)^3$; $u = x-2$; $\boxed{y = u^3}$ → $dy/\delta u = 3u^2$; $\delta u/\delta x = 1$

$$\frac{\delta y}{\delta x} = 3u^2 \times 1$$

$$= 3u^2 \Rightarrow 3(x-2)^2$$

$y = (2x^2+5x)^4$

$u = 2x^2+5x$; $y = u^4$; $\delta y/\delta u = 4u^3$; $\delta u/dx = 4x+5$

$$\frac{dy}{dx} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$$

$$\frac{\delta y}{\delta x} = 4u^3 \times 4x+5$$

$$= 4(2x^2+5x)^3(4x+5)$$

PRODUCT RULE

- When a function multiplies another function

$$\boxed{\frac{dy}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x}}$$

$$y = \overset{u}{(2x+5)}\,\overset{v}{(3x^2+7x)}$$

$$\frac{\delta u}{\delta x} = 2 \;;\; \frac{\delta v}{\delta x} = 6x+7$$

$$\frac{\delta y}{\delta x} = (2x+5)(6x+7) + (3x^2+7x)\,2$$

$$= (2x+5)(6x+7) + 6x^2 + 14x$$

$$= 12x^2 + 14x + 30x + 35 + 6x^2 + 14x$$

$$= 18x^2 + 58x + 35$$

QUOTIENT RULE:

$$\boxed{\frac{\delta y}{\delta x} = \frac{v\frac{\delta u}{\delta x} - u\frac{\delta v}{\delta x}}{v^2}}$$

$$y = \frac{2x+5 \;\; u}{x-2 \;\; v} \qquad \delta u/\delta x = 2 \qquad \delta v/dx = 1$$

$$\frac{\delta y}{\delta x} = \frac{(x-2)(2) - (2x+5)(1)}{(x-2)^2}$$

$$\frac{\cancel{2x} - 4 - \cancel{2x} + 5}{(x-2)^2} = \frac{-9}{(x-2)^2}$$

$$\rightarrow \quad y = \overset{u}{(2x+3)^2}\,\overset{v}{(3x+5)^4} \qquad \delta y/\delta x = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$$

Chain rule!

a1 $\quad u = 2x+3 \;;\; y = u^2$

$$\frac{\delta y}{\delta x} = 2 \qquad \frac{\delta y}{\delta u} = 2u$$

$$\delta y/\delta x = 2u \times 2$$

$$= 4u$$

$$= 4(2x+3)\frac{\delta u}{\delta x}$$

b: $u = 3x+5$; $\delta u / \delta x = 3$ $\quad \frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$

$y = u^4$; $\delta y / \delta u = 4u^3$

$$\frac{\delta y}{\delta x} = 4u^3 \times 3$$
$$= 12u^3$$
$$= 12(3x+5)^3 \quad \left(\frac{\delta v}{\delta x}\right)$$

PRODUCT RULE

$$\frac{\delta u}{\delta x} / = \frac{\delta y}{\delta x} = u\frac{dv}{\delta x} + v\frac{\delta u}{\delta x}$$

$$\frac{\delta y}{\delta x} = (2x+3)^2 \times 12(3x+5)^3 + (3x+5)^4 \times 4(2x+3)$$

$$= 12(2x+3)^2(3x+5)^3 + 4(3x+5)^4(2x+3)$$
$$= 4(2x+3)(3x+5)^3\left(3(2x+3) + 3x+5\right)$$
$$= 4(2x+3)(3x+5)^3(6x+9+3x+5)$$
$$= \underbrace{4(2x+3)(3x+5)^3(9x+14)}_{\text{differentiated!}}$$

MONDAY 15TH JULY, 2024

IMPLICIT FUNCTIONS ~~Product~~

$xy = 1$

$x^2 + y^2 = 2xy$

$$y' = \frac{dy}{\delta x}$$

PRODUCT RULES: $\frac{\delta y}{\delta x} = u\frac{dv}{\delta x} + v\frac{\delta u}{\delta x}$

1. $xy' + yx' = 0$

$x\frac{dy}{\delta x} + y \cdot 1 = 0$

$x\frac{dy}{\delta x} + y = 0$

$x\frac{\delta y}{\delta x} = -y$

$\frac{\delta y}{\delta x} = \frac{-y}{x}$

2. $x^2 + y^2 = 5$

$2x + 2y\frac{\delta y}{\delta x} = 0$

$2y\frac{\delta y}{\delta x} = -2x$

$\frac{\delta y}{\delta x} = \frac{-2x}{2y}$

$\frac{\delta y}{\delta x} = -x/y$

3. $x^2y - 5x + 3$

$x^2\frac{\delta y}{\delta x} + 2xy - 5 = 0$

$x^2\frac{\delta y}{\delta x} = 5 - 2xy$

$\frac{\delta y}{\delta x} = \frac{5-2xy}{x^2}$

Assignment

1. $x^2 + 2x^2y - 2xy^2 + y^2 = 0$
2. $x^3 + y^3 = 3xy$
3. $7x^3 + 5xy = 10y^2$

DIFFERENTIATION OF EXPONENTIAL FUNCTIONS

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$y = e^x$

$$\boxed{\frac{dy}{dx} = e^x}$$

$y = e^u$

$$\frac{dy}{du} = e^u$$

$y = e^{\frac{2x+5}{3x}}$

SOLVED ON LATER PAGES OF THIS CHRONICLE.

Examples:

$y = e^{2x}$

$$\frac{dy}{dx} = 2e^{2x}$$

$y = e^{2x+5}$

$$dy/dx = 2e^{2x+5}$$

$y = e^{x^2}$

$$\frac{dy}{dx} = 2x e^{x^2}$$

DIFFERENTIATION OF GENERAL EXPONENTIAL FUNCTION.

$y = a^x$

Assignment

1. $x^2 + 2x^2y - 2xy^2 + y^2 = 0$

$$2x + 2x^2\frac{dy}{dx} + 4xy - 4x\frac{dy}{dx} - 2xy^2 + 2y = 0$$

$$2x^2\frac{dy}{dx} - 4x\frac{dy}{dx} = 2xy^2 - 4xy - 2x - 2y$$

$$\frac{dy}{dx} = \frac{2xy^2 - 4xy - 2x - 2y}{2x^2 - 4x}$$

$$\frac{dy}{dx} = \frac{xy^2 - 2xy - x - y}{x^2 - 2x}$$

2. $x^3 + y^3 = 3xy$

$$3x^2 + 3y^2\frac{\delta y}{\delta x} = 3x\frac{\delta y}{\delta x} + 3y$$

$$3y^2\frac{\delta y}{\delta x} - 3x\frac{\delta y}{\delta x} = 3y - 3x^2$$

$$\frac{\delta y}{\delta x} = \frac{3y - 3x^2}{3y^2 - 3x}$$

$$\frac{\delta y}{\delta x} = \frac{y - x^2}{y^2 - x}$$

~~8~~.

3. $7x^3 + 5xy = 10y^2$

$$21x^2 + 5x\frac{\delta y}{\delta x} + 5y = 20y\frac{\delta y}{\delta x}$$

$$5x\frac{\delta y}{\delta x} - 20y\frac{\delta y}{\delta x} = -21x^2 - 5y$$

$$\frac{\delta y}{\delta x} = \frac{-21x^2 - 5y}{5x - 20y}$$

1. $x^2 + 2x^2y - 2xy^2 + y^2 = 0$

$$\checkmark \quad 2x + 2x^2\frac{\delta y}{\delta x} + 4xy - 4x\frac{\delta y}{\delta x} - 2y^2 + 2y\frac{\delta y}{\delta x} = 0$$

$$2x^2\frac{\delta y}{\delta x} - 4x\frac{\delta y}{\delta x} + 2y\frac{\delta y}{\delta x} = 2y^2 - 4xy - 2x$$

$$\frac{\partial y}{\partial x} = \frac{2y^2-4xy-2x}{2x^2-4x+2y}$$

$$\frac{\partial y}{\partial x} = \frac{y^2-2xy-x}{x^2-2x+y}$$

Differentiation of General Exponential Functions

$y = a^x$

$\frac{\partial y}{\partial x} = a^x \log_e a \Rightarrow \boxed{a^x \ln a}$

$y = 2^x$

$\frac{\partial y}{\partial x} = 2^x \log_e 2 \Rightarrow \boxed{2^x \ln 2}$

Differentiation of Logarithmic Functions

$y = \log_e x$ $\quad \log_e = \ln \quad \log_e = \ln$

$y = \ln x$

$\frac{\partial y}{\partial x} = \frac{1}{x}$

$y = \ln x^2$

$\partial y/\partial y = \frac{2x}{x^2} \Rightarrow 2/x$