STA 112

PROBABILITY

PROBABILITY

Course Outline

- Permutation and combination
- Concept of probability
- Random variable
- Probability distribution
- Sample distribution
- Exploratory data analysis

FACTORIALS

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$$

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Examples:

1.

$$\frac{30!}{28!} = \frac{30 \times 29 \times 28!}{28!} = 30 \times 29 = 870$$

2.

$$\frac{(n-1)!}{(n-2)!} = \frac{(n-1) \times (n-2) \times (n-2)!}{(n-2)!} = (n-1)(n-2)$$

PERMUTATION

Given:
ABC → AB, AC, BC

General formula:

$$P = \frac{n!}{(n-r)!}$$

Example:

$$^4P_2 = \frac{4!}{(4-2)!} = \frac{4!}{2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1} = 12$$

$$^5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 60$$

Combined example:

$$^4P_2 + ^5P_3 = \frac{4!}{2!} + \frac{5!}{2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1} + \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 12 + 60 = 72$$

PERMUTATION OF IDENTICAL OBJECTS

General formula:

$$\frac{n!}{n_1! \times n_2! \times n_3! \times \cdots}$$

Example: How many ways can "NOSHOESHOES" be written?

$$\text{Ways} = \frac{10!}{3! \times 2! \times 1! \times 2! \times 1! \times 1!} = 75,600 \text{ ways}$$

EXCELL

$$= \frac{6!}{2! \times 2!} = \frac{720}{2 \times 2} = \frac{720}{4} = 180$$

Round Table Arrangement:
5 people sit around a circular table:

(5 - 1)! = 4! = 24 ways

Conditional Permutation:
With conditions attached:

a) Two O3s must be together
b) Two O3s must not be together.

Solution:

(a) SCHOOL = 5! = 120 ways

Or 5 × 4! = 120 ways

(b) SCHOOL
(5 - 1)! = 4!
$= \frac{5 \times 4 \times 4!}{2} = 240$

COMBINATION

Formulas

- **Combination (selection without order):**

$$^nC_r = \frac{n!}{(n-r)! \cdot r!}$$

- **Permutation (selection with order):**

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

Evaluate: $^5C_3 + ^4P_3$

Break it down step by step:

$$^{5}C_{3} = \frac{5!}{(5-3)! \cdot 3!} = \frac{5!}{2! \cdot 3!}$$

$$= \frac{5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1)(2 \times 1)} = \frac{120}{6 \cdot 2} = \frac{120}{12} = 10$$

$$^{4}P_{3} = \frac{4!}{(4-3)!} = \frac{4!}{1!} = \frac{4 \times 3 \times 2 \times 1}{1} = 24$$

Solution:

$$^{5}C_{3} + ^{4}P_{3} = 10 + 24 = \boxed{34}$$

Example Problem:

Question:
A committee has 7 members. A subcommittee of 5 members is to be selected.
In how many ways can this be done?

$$^{7}C_{5} = \frac{7!}{(7-5)! \cdot 5!} = \frac{7!}{2! \cdot 5!}$$

$$= \frac{7 \times 6 \times 5!}{2 \times 1 \cdot 5!} = \frac{42 \cdot 5!}{2 \cdot 5!} = \frac{42}{2} = \boxed{21 \text{ ways}}$$

Example Problem:
How many 3-letter words can be formed from the letters A, B, C, D, and E if repetition is not allowed?

Solution:
Since order matters and repetition are not allowed, we use permutations:

$$^5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 5 \times 4 \times 3 = \boxed{60 \text{ words}}$$

Example Problem:
How many ways can 4 students be selected from a class of 10?

Solution:
Since order does not matter, we use combinations:

$$^{10}C_4 = \frac{10!}{(10-4)! \cdot 4!} = \frac{10!}{6! \cdot 4!}$$

Simplify the numerator and denominator:

$$= \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = \frac{5040}{24} = \boxed{210 \text{ ways}}$$

Concepts and Principles of Probability

- **Probability** is a measure of the **likelihood** of an event occurring.
- It is often described as a **game of chance**.
- Probability values always lie between **0 and 1**:

$$0 \leq P(A) \leq 1$$

- Probability is **never negative** and **never greater than 1**.
- The **total probability** of all possible events equals **1**.
- The probability of an event that **never occurs** is **0**.

Key Terms

- **Random Experiment:** An experiment with an **uncertain outcome**.
- **Outcome:** The **result** of a random experiment (element of the **sample space**).
    - Example: Rolling a die
      Outcomes: {1, 2, 3, 4, 5, 6}
- **Event:** A **subset** of the sample space (a group of outcomes).

Probability Notation

- P(A∩B): Probability of **both** events A **and** B occurring.
- P(A∪B): Probability of **either** event A **or** B occurring.

Example Problem

Question:
Find the probability of getting heads or tails when tossing **two coins**.

Possible outcomes:
HH, HT, TH, TT

Probability of getting at least one head:3/4

(Only TT has no heads.)

Another Example

A box contains **5 blue (B)** balls and **4 red (R)** balls.
If a ball is picked at random, find the probabilities:

- **Red:**

$$P(R) = \frac{4}{9}$$

- **Blue or Red:**

$$P(\text{Not R}) = \frac{5}{9}$$

- **Both Blue and Red:** (If interpreted as choosing one ball, then both can't occur at the same time)

$$P(B \cap R) = 0 \quad \text{(if one ball is picked)}$$

- **Not Red:**

Events

- **Independent Events:**
  Two events are independent if the occurrence of one **does not affect** the occurrence of the other.
- **Dependent Events:**
  Two events are dependent if the occurrence of one affects the probability of the other.
  If

$$P(A \mid B) = P(A)$$

and vice versa, then the events are **independent**, not dependent.
So for **dependent events**,

$$P(A \mid B) \neq P(A)$$

2. Multiplication Rule:

$$P(A \cap B) = P(A) \times P(B) \quad \text{(if A and B are independent)}$$

3. Mutually Exclusive Events:
Events that cannot occur at the same time:

$P(A\cap B) = 0$

So, the **Addition Rule** becomes:

$$P(A\cup B) = P(A)+P(B)$$

In general (not necessarily mutually exclusive):

$$P(A\cup B) = P(A)+P(B)-P(A\cap B)$$

4. Conditional Probability:

$$P(A \mid B) = \frac{P(A\cap B)}{P(B)}$$

Using the multiplication rule:

$$P(A \mid B) = \frac{P(A)\times P(B)}{P(B)} = P(A) \quad \text{(only if A and B are independent)}$$

Given Values:

$P(A)=0.5$, $P(B)=0.4$, $P(A\cap B) = 0.2$

Find P(A∪B)

Using the general addition formula:

$$P(A\cup B) = P(A)+P(B)-P(A\cap B)$$

Substitute the values:

$$P(A\cup B) = 0.5+0.4-0.2=0.7$$

Example 1
A die is rolled once. what is the probability of getting a number less than 5?
Possible outcomes: {1, 2, 3, 4, 5, 6}
Probability: 4/6 = 2/3

Example 2

A box contains 5 red and 4 blue balls. Two balls are picked at random, one after the other.

If the balls are picked **with replacement**, find the probability that:

- Both are of the same color
- Both are of different colors
- If the balls are picked **without replacement**

Solution

- With Replacements

Probability that both are the same color

P(Red then Blue) = $\frac{5}{9} \times \frac{4}{8} = \frac{20}{72}$

P(Blue then Red) = $\frac{4}{9} \times \frac{5}{8} = \frac{20}{72}$

Total P(same color) =

$$\frac{25+16}{81} = \frac{41}{81} \approx \boxed{0.506}$$

Probability that both are different colors

P(Red then Blue) = $\frac{5}{9} \times \frac{4}{9} = \frac{20}{81}$

P(Blue then Red) = $\frac{4}{9} \times \frac{5}{9} = \frac{20}{81}$

Total P(different colors) =

$$\frac{20+20}{81} = \frac{40}{81} \approx \boxed{0.494}$$

- Without Replacement

Probability that **both are the same color**

P(Red then Red) = $\frac{5}{9} \times \frac{4}{8} = \frac{20}{72}$

P(Blue then Blue) = $\frac{4}{9} \times \frac{3}{8} = \frac{12}{72}$

Total P(same color) =

$$\frac{20+12}{72} = \frac{32}{72} = \frac{8}{18} = \boxed{\frac{4}{9}} \approx 0.444$$

Probability that **both are different colors**

P(Red then Blue) = $\frac{5}{9} \times \frac{4}{8} = \frac{20}{72}$

P(Blue then Red) = $\frac{4}{9} \times \frac{5}{8} = \frac{20}{72}$

Total P(different colors) =

$$\frac{20+20}{72} = \frac{40}{72} = \frac{10}{18} = \boxed{\frac{5}{9}} \approx 0.556$$

Example 3

The probability that a seed will germinate is 2/5. Of two seeds A and B, find the probability that:

- Both germinate
- None germinates
- Exactly one germinates
- Of 3 seeds X, Y, and Z, find the probability that exactly 2 germinate.

Given:

Probability that **a seed germinates** = $\frac{2}{5}$

Therefore, probability that **a seed does not germinate** =

$$1 - \frac{2}{5} = \frac{3}{5}$$

Part 1: **Probability that both A and B germinate**

$$P(A \text{ and } B \text{ germinate}) = \frac{2}{5} \times \frac{2}{5} = \frac{4}{25}$$

Answer: $\boxed{\frac{4}{25}}$

Part 2: **Probability that none germinates**

$$P(\text{Neither A nor B germinates}) = \frac{3}{5} \times \frac{3}{5} = \frac{9}{25}$$

Answer: $\boxed{\frac{9}{25}}$

Part 3: **Probability that exactly one germinates**

This means **one germinates and the other does not.** Two ways this can happen:

- A germinates, B does not: $\frac{2}{5} \times \frac{3}{5} = \frac{6}{25}$
- A does not, B germinates: $\frac{3}{5} \times \frac{2}{5} = \frac{6}{25}$

So total:

$$P(\text{Exactly one}) = \frac{6}{25} + \frac{6}{25} = \frac{12}{25}$$

Answer: $\boxed{\frac{12}{25}}$

A random variable is defined as

$$f(x) = 2x, \quad \text{for } 0 < x < 4$$

Find:

(i) $E(x)$

(ii) $E(x^2)$

Solution:

$$E(x) = \int_0^4 x \cdot f(x)\,dx$$
$$= \int_0^4 x \cdot 2x\,dx$$
$$= \int_0^4 2x^2\,dx$$
$$= \left[\frac{2x^3}{3}\right]_0^4$$
$$= \frac{2(4)^3}{3} - 0 = \frac{128}{3}$$

(ii)

$$E(x^2) = \int_0^4 x^2 \cdot f(x)\,dx$$
$$= \int_0^4 x^2 \cdot 2x\,dx$$
$$= \int_0^4 2x^3\,dx$$
$$= \left[\frac{2x^4}{4}\right]_0^4$$
$$= \frac{2(4)^4}{4} - 0 = 2(4)^3 = 128$$

Variance:

$$V(x) = E(x^2) - [E(x)]^2$$

Given:

x	0	1	2	3
$P(x)$	0.3	a	0.2	0.3

Find:
The value of a
E(x)
$E(x^2)$
V(x)

Solution:
Since total probability must be 1:

$$E(P(x)) = 1$$

$$0.3 + a + 0.2 + 0.3 = 1$$

$$a + 0.8 = 1$$

$$a = 1 - 0.8 = 0.2$$

(i)

$$E(x) = \sum x \cdot P(x)$$

$$= 0 \cdot 0.3 + 1 \cdot a + 2 \cdot 0.2 + 3 \cdot 0.3$$

$$= 0 + 0.2 + 0.4 + 0.9 = 1.5$$

(ii)

$$E(x^2) = \sum x^2 \cdot P(x)$$

$$= 0^2 \cdot 0.3 + 1^2 \cdot 0.2 + 2^2 \cdot 0.2 + 3^2 \cdot 0.3$$

$$= 0 + 0.2 + 0.8 + 2.7 = 3.7$$

(iii)

$$V(x) = E(x^2) - [E(x)]^2$$

$$= 3.7 - (1.5)^2 = 3.7 - 2.25 = 1.45$$

(iv) $V(x) = E(x^2) - (E(x))^2$

$V(x)=3.7 - (1.45)^2=2.25$

PROPERTIES OF EXPECTATION

Given a random variable X, the k-th moment about the origin is given by:

$$E(X^k) = \sum_{x=0}^{\infty} x^k P(x)$$

The second moment about the origin ($k = 2$) is:

$$E(X^2) = \sum_{x=0}^{\infty} x^2 P(x)$$

Linearity of Expectation:

$$E(kX) = kE(X)$$

$$E(kX) = \sum_{x=0}^{\infty} kx \cdot P(x) = k \sum_{x=0}^{\infty} x \cdot P(x) = kE(X)$$

If X and Y are two **independent** random variables:

$$E(X + Y) = E(X) + E(Y)$$

$$E(kX + \alpha Y) = kE(X) + \alpha E(Y) \quad \text{where } k \text{ and } \alpha \text{ are scalars}$$

DISCRETE RANDOM VARIABLE

Assume X is a **continuous** random variable. Then the k-th moment about the origin is:

$$E(X^k) = \int_{\infty}^{\infty} x^k f(x)\, dx$$

- For $k = 1$ (i.e., the expectation or mean):

$$E(X) = \int_{\infty}^{\infty} x f(x)\, dx$$

- For $k = 2$:

$$E(X^2) = \int_{\infty}^{\infty} x^2 f(x)\, dx$$

- For $k = 4$:

$$E(X^4) = \int_{\infty}^{\infty} x^4 f(x)\, dx$$

Also, using scalar multiplication:

$$E(kX) = k \int_{\infty}^{\infty} xf(x)\,dx = kE(X)$$

Where $f(x)$ is the **probability density function** (PDF) for continuous random variables, and $P(x)$ is the **probability mass function** (PMF) for discrete random variables.

For **independent random variables:**

$$E(X + Y) = E(X) + E(Y)$$

And also for continuous variables:

$$E(Y) = \int_{\infty}^{\infty} yf(y)\,dy$$

PROBABILITY DISTRIBUTION

- Discrete
- Continuous

$$E((x-\mu)^2) = \sum (x-\mu)^2 p(x)$$

$$\sum z^2 = \sum z p(x)$$

$$E(z^2) = \int z^2 f(x)\,dx$$

$$E(z^r) = \sum z^r p(x) = \sum z^r p(x)$$

$$z \int z^2 f(x)\,dx = z \int z^2 f(x)\,dx$$

Continuous random variable p(x) probability density function

$$\bar{x}_c = \frac{\sum x}{n}$$

$$p(x) = \frac{1}{n}$$

$$\bar{x}_c = E(x) = \sum x p(x)$$

$$z \frac{\sum x}{n} z = \frac{\sum x}{n}$$

Discrete Probability Distributions

1. Bernoulli distribution

2. Binomial distribution

3. Poisson distribution

4. Geometric distribution

5. Hypergeometric distribution

6. Negative Binomial distribution

Continuous Probability Distribution

1. Uniform distribution

2. Exponential distribution

3. Gamma distribution

4. Normal distribution

Bernoulli Distribution

Let x be a discrete random variable with constant probability p. Then x is said to follow a **Bernoulli distribution** if and only if the probability mass function is given by:

$$P(x = r) = p^x(1-p)^{1-x}, \quad x = 0, 1$$

$$p(x) = P(x = r) = p^r(1-p)^{1-r}$$

$$x \sim \text{Bern}(p)$$

Mean of Bernoulli

$$E(x) = \sum xp(x)$$

$$E(x) = \sum g \cdot p(x) = (1-p)^r$$

$$\sum_{0} (1-p)(1-p)^{1-r}$$

Variance of Binomial Distribution

$$\mathrm{Var}(x) = E(x^2) - [E(x)]^2$$

By definition:

$$E(x^2) = \sum x^2 P(x)$$

For the geometric distribution (possibly referenced in error):

$$E(x^2) = \sum_{x=0}^{\infty} x^2 P(1-P)^x = 0 \cdot P(1-P)^0 + 1 \cdot P(1-P)^1 + \ldots$$

But for a **Binomial Distribution**, where x~Bin(n,p)

- Mean: E(x)=np
- Variance: Var(x)=np(1−p)

A *photon detector* has a 90% chance of detecting a photon.
What is the probability that it **fails** to detect the photon in a single trial?

SOLUTION

If a photon detector has a **90% chance of detecting a photon**, that means the **probability of detection is:**

P(detection)=0.90

Since failure to detect is the complement of detection, the **probability that it fails to detect** the photon is:

P(failure)=1−P(detection)=1−0.90=0.10

Final Answer: 0.10 or 10%

Binomial Distribution
Let x be a discrete random variable representing the number of **successes** in n

independent trials, each with a constant probability of success p. Then:

$$x \sim \text{Bin}(n, p)$$

The **probability mass function (PMF)** is:

$$P(x = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad \text{where } k = 0, 1, 2, ..., n$$

Binomial Expansion

The binomial theorem expands:

$$(a + b)^r = \sum_{k=0}^{r} \binom{r}{k} a^{r-k} b^k$$

$$(a + b)^r = \binom{r}{0} a^r + \binom{r}{1} a^{r-1} b + \binom{r}{2} a^{r-2} b^2 + \ldots + \binom{r}{r} b^r$$

Mean of a Binomial Distribution

If x∼Bin(n,p), then:

- **Mean**:
  E(x)=np
- **Variance**:
  Var(x)=np(1−p)

$$E(x) = \sum_{k=0}^{n} k \cdot P(x = k) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1 - p)^{n-k}$$

$$\sum_{x=0}^{n} \frac{n!}{(n - x)!\, x!} p^x q^{n-x} = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$$

$$E(x) = \sum_{x=0}^{n} x \cdot \binom{n}{x} p^x q^{n-x}$$

$$E(x) = np(p+q)^{n-1} = np(1)^{n-1} = np$$

$$E(x) = np$$

$$V(x) = \mathrm{Var}(x) = E(x^2) - [E(x)]^2$$

$$\sigma = \sqrt{\mathrm{Var}(x)} = \sqrt{np(1-p)}$$

Example 1:
A factory produces light bulbs with a 95% success rate. If 10 bulbs are randomly selected, what is the probability that exactly 8 are not effective? E(x), V(x)?

Example 2:
A student guesses answers on a 5-question true/false quiz. What is the probability that at least 3 answers are correct?

Solution:

$$P(X=8) = \binom{10}{8} \cdot \left(\frac{95}{100}\right)^8 \cdot \left(\frac{5}{100}\right)^2$$

$$= \frac{10!}{8! \cdot 2!} \cdot (0.95)^8 \cdot (0.05)^2$$

$$= \frac{10 \times 9}{2!} \cdot 0.663 \cdot 0.0025$$

$$= 45 \cdot 0.663 \cdot 0.0025 = 0.075$$

2. Binomial Probability

$$P(x) = \binom{n}{x} \cdot p^x \cdot q^{n-x}$$

Given:

$$n = 5, \quad p = 0.5, \quad q = 0.5, \quad x = 3$$

$$P(x=3) = \binom{5}{3} \cdot (0.5)^3 \cdot (0.5)^2 = \frac{5!}{3!(5-3)!} \cdot 0.125 \cdot 0.25 = \frac{5 \times 4}{2 \times 1} \cdot 0.125 \cdot 0.25 = 10 \cdot 0.125 \cdot 0.25 = 0.3125$$

At Least 3 Successes:

$$P(x \geq 3) = P(x = 3) + P(x = 4) + P(x = 5)$$

1. $P(x = 3) = 0.3125$ (already calculated)

2. $P(x = 4)$

$$= \binom{5}{4} \cdot (0.5)^4 \cdot (0.5)^1 = 5 \cdot 0.0625 \cdot 0.5 = 5 \cdot 0.03125 = 0.15625$$

3. $P(x = 5)$

$$= \binom{5}{5} \cdot (0.5)^5 \cdot (0.5)^0 = 1 \cdot 0.03125 \cdot 1 = 0.03125$$

Total:

P(x≥3) =0.3125+0.15625+0.03125=0.5

POISSON DISTRIBUTION

Let X be a discrete random variable with parameter λ. X is said to follow a Poisson distribution if and only if its probability mass function is given by:

$$P(X = x) = \frac{\lambda^x \cdot e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots$$

X ~ Poisson(λ)

The mean and variance are equal for λ.
e.g. **E(X) = λ**
Var(X) = λ

Example 3:
The number of emails a person receives per hour follows a Poisson distribution with an average of 3 emails/hour. What is the probability that they receive exactly 2 emails in one hour?

$$P(X = 2) = \frac{e^{-3} \cdot 3^2}{2!}$$

$$= \frac{e^{-3} \cdot 9}{2}$$

$$P(X = 2) \approx \frac{0.0498 \cdot 9}{2} = \frac{0.4482}{2} = 0.2241$$

Example 4:
A call centre receives an average of 5 calls per minute. What's the probability that no calls come in at a randomly selected minute?

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$P(X = 0) = \frac{e^{-5} \cdot 5^0}{0!} = e^{-5} \cdot 1 = e^{-5}$$

$$e^{-5} \approx 0.0067$$

Geometric Distribution (Success)
Let X be a discrete random variable with parameter P.
X is said to follow a geometric distribution if and only if its probability PMF is given by:

$$P(X = x) = (1 - P)^{x-1} \cdot P, \quad x = 1, 2, 3, \ldots$$

X ~ Geometric(P)

- Mean: $E(X) = \frac{1}{P}$
- Variance: $\mathrm{Var}(X) = \frac{1-P}{P^2}$

Example 5: A football player has a 20% chance of scoring a free kick. What is the probability that they score their free kick on the 4th attempt?

Soln:

Let $X \sim \mathrm{Geometric}(0.2)$, and $x = 4$

$$P(X = 4) = (1 - 0.2)^{4-1} \cdot 0.2 = (0.8)^3 \cdot 0.2 = 0.512 \cdot 0.2 = \boxed{0.1024}$$

Example 6: A software tester is testing for the first bug in a randomly chosen line of code where each line has a 1% chance of having a bug. What is the probability that the first bug is found on the 10th line of code checked?"

Let $X \sim \text{Geometric}(0.01)$, and $x = 10$

$$P(X = 10) = (1 - 0.01)^9 \cdot 0.01 = 0.99^9 \cdot 0.01 = 0.9135172 \cdot 0.01 = \boxed{0.00914}$$

Assignments

Derive the Expectation and Variance of Geometric and Poisson Distribution.