Political Redistricting by Computer

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The problems of political redistricting are considered and a computer method for redistricting is presented. Criteria for acceptable redistricting are discussed, including population equality, compactness, contiguity, and preservation of natural and/or political boundaries. Only nonpartisan criteria are considered. Using 1970 Bureau of Census population data, specific results are given for the ten Congressional Districts in the state of Missouri and for the seven St. Louis County Council seats. Results from the use of the algorithm indicate the feasibility of political redistricting with the aid of a computer.

Key Words and Phrases: political redistricting, reapportionment, compactness, equal population, contiguity, transportation algorithm, legislative districts, population units

CR Categories: 2.12, 3.35, 5.41

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Introduction

Political redistricting is the means by which a geographic area is divided into districts. In the past, redistricting has been done by state legislatures, but this tends to introduce partisan political considerations into the final district plans. In many cases, redistricting becomes a battle to protect incumbents, or an attempt to reorganize districts to benefit a specific party or special interest group. Political expediency is a dominant factor, and the result is often a gerrymander. This in turn can lead to legal intervention and eventually to an imposed solution, like that established in Baker vs. Carr [12, 13, and 14]. Thus rigorous and well-defined approaches to redistricting need to be formulated and documented for future use.

We attempted a nonpartisan computer-based algorithm to systematically eliminate the inequities of present redistricting practices. We then applied the algorithm to the specific problems of redistricting the state of Missouri into ten congressional districts and St. Louis County into seven County Council seats.

Initial Considerations

Ideally, a formal method of redistricting would consider each dwelling (e.g. house or apartment building) separately. Since this is not practicable, dwellings are grouped into larger units called *population units*. Population units are assigned to one, and only one, district.

If the population units chosen have too large a population, it may be impossible to formulate a plan which assigns all units successfully; or the algorithm may be forced to overlook solutions which could be superior. Even if a solution falls within an acceptable population tolerance, it may well be possible to form another, more desirable, plan which would produce a further subdivision of the population units. Thus for most algorithms a major consideration is the number of population units that can be handled by the algorithm in a reasonable amount of computer time and storage.

A major drawback of the redistricting process is the means of grouping by which population units are formed. A partisan political bias can be introduced (either consciously or unconsciously) in this initial phase. This is of particularly great concern in large urban areas where, for instance, the method of assigning a ghetto area may deprive a minority group of a representative.

Data made available by the U.S. Census Bureau group population in several ways, including block groups or enumeration districts (the smallest units available), census tracts, townships, and counties. Population units are chosen from these data in such a way that acceptable solutions may be obtained without incurring prohibitive costs (e.g. in computer time and storage). This choice is based on the relative population size and density given in the various population data available, as well as on the particular redistricting algorithm.

Criteria

The following criteria could be considered in the development of nonpartisan computer-based methods for political redistricting.

1. *Population Equality.* This criterion is based on Westberry vs. Sanders [22], "... one man's vote in a congressional election is to be worth as much as another's." Population equality should be the prime concern of any redistricting plan. There are three common measures of equality. One is the ratio of the largest district to the smallest. Specifically, in Drum vs. Seawell [15], ".... the aim should be to come as close as possible to a one-to-one ratio." Another measure of equality is the smallest percentage of voters controlling the majority of representatives. Thus for the ten Missouri congressional districts this is the percentage of voters living in the five smallest districts. (This figure should be nearly 50 percent.) A third measure of population equality is the deviation of the size of the population of each district from that of an ideal district. For the current congressional districts for Missouri (Map 1) these values are given in Table I.

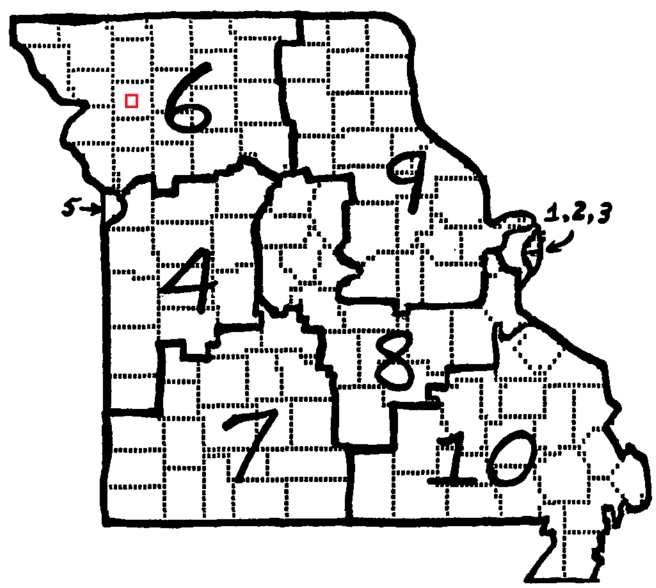
2. *Compactness.* A district is said to be geographically compact if it is nearly convex. A district is said to be compact in terms of population if the population center is close to the geographic center of the district. Hess [6] suggests that a measure which takes into account the distance of the population from the center of the district (e.g. population compactness) is more reasonable. However, no quantitative definition of compactness has yet been accepted.

Table I. Statistics for the Current Congressional Districts for the State of Missouri (1970 Census Figures)

District number	Population	Deviation from ideal (467,650)
1	382,673	−84,977
2	510,683	+43,033
3	390,606	−77,044
4	515,497	+47,847
5	378,874	−88,776
6	452,764	−14,886
7	484,495	+16,845
8	613,491	+145,841
9	546,058	+78,408
10	401,380	−66,270
	4,676,501	

Ratio of largest to smallest district: 1.62.
Minimum percent to control the majority: 42.90%.

Map 1. The ten current Congressional Districts for the state of Missouri.



3. *Contiguity.* A district is said to be contiguous if its area is not divided into two (or more) geographically separate entities.

4. *Preservation of Existing Boundaries.* It is desired that districts cross predefined or predetermined boundaries (e.g. rivers, major highways, county lines, etc.) as infrequently as possible. Maintenance of these boundaries simplifies the mechanics of the election procedure by providing a uniform ballot within each political unit. Maintenance of such boundaries also tends to reduce the number of alternative solutions produced by an algorithm although it could preclude solutions that are potentially superior in terms of population deviation, compactness, or other criteria.

The redistricting solutions presented below should not be interpreted as "the only possible" solutions satisfying the above criteria. Rather these plans are solutions satisfying the criteria imposed by the method. Obviously, with the same criteria weighted differently, different solutions could result.

Our objective was to obtain solutions that would satisfy Supreme Court criteria for redistricting plans; this objective can be achieved by considering the minimal set of criteria mentioned above. Strong arguments can be given for and against the inclusion of other criteria. For example, one might consider the heterogeneity or homogeneity of cultural backgrounds of the population. Additional considerations could include voting histories, projected population shifts, party affiliations, protection of incumbents, and others of a more partisan nature. Obviously there is no such thing as *a* "best solution" to a particular redistricting problem.

Transportation Approach

This approach treats political redistricting like a classical linear programming transportation problem. The transportation problem seeks to minimize the cost of transporting quantities of a given product from m different origins to n different destinations. Constraints exist for the availability of the product at each origin and the demand for the product at each destination. Our approach is to consider population units as the product. We intend to ship these units to the districts while minimizing an objective function designed to achieve population compactness.

Let D_j be the jth district ($j = 1, \ldots, n$) and U_i be the ith population unit ($i = 1, \ldots, m$). Let (u_i, v_i) be the cartesian coordinates of the population centroid of the ith population unit. For units small enough in area, this centroid can be approximated effectively by the geographic center of the unit. Let p_i be the population of the ith population unit. Our iterative scheme is to improve compactness with each pass through the transportation algorithm. We continue to generate redistricting plans until successive plans differ insignificantly. Following is an outline of the method.

1. Guess the centroids, (a_j, b_j), of D_j ($j = 1, \ldots, n$). To avoid undesirable results, some care should be taken in their selection. A possible initial solution would be the centers of the current districts. It may be possible to generate different redistricting plans by considering different initial solutions. Thus several alternative sets of these starting points should be tried.

2. Compute

$$d_{ij} = ((u_i - a_j)^2 + (v_i - b_j)^2)^{\frac{1}{2}}, \qquad i = 1, \ldots, m; \quad j = 1, \ldots, n,$$

the distances from the centers of the D_j to the centers of the U_i.

[1] An enclave is a region which is too small to constitute a district by itself and which cannot be combined into a complete district.

3. Using the transportation algorithm, solve the following subproblem for the x_{ij}'s to obtain a new redistricting plan. Define

$$\begin{aligned} x_{ij} &= 1 \text{ if } U_i \text{ is in } D_j \\ &= 0 \text{ if } U_i \text{ is not in } D_j . \end{aligned}$$

Then we minimize

$$z = \sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} d_{ij} p_i , \tag{1}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, \ldots, m, \tag{2}$$

$$\sum_{i=1}^{m} x_{ij} = g_j, \quad j = 1, \ldots, n. \tag{3}$$

Equation (2) is used to guarantee that each population unit is assigned to one, and only one, district. Since $\sum_i \sum_j x_{ij} = m$, we are guaranteed that each population unit will be assigned to a district. Therefore, it is not possible for enclaves[1] to be formed.

Equation (3) determines the number of population units g_j to be assigned to district j. If the population units are relatively equal in number of people, then we can choose $g_j = m/n$ for all j. Therefore, the same number of population units would be shipped to each legislative district and the condition of population equality would be reasonably satisfied. Since we required each initial x_{ij} to be either 0 or 1, the elements of the solution matrix (x_{ij}) will be integers if m, the number of population units chosen, is a multiple of n, the number of districts to be formed [5].

In general, it is impractical to find population units that are relatively equal in population so that it is not necessarily true that each $g_j = m/n$, an integer. Even if m equal population units can be found, it may not be true that m is a multiple of n. Therefore, the g_j are computed iteratively. Any choice for the g_j could be used to start the procedure, although some thought should be given to selection so that computer time will not be wasted. The method we found to be effective for iteratively modifying the g_j is outlined in step 6.

If eq. (1) is minimized, reasonably compact districts will result. Although no explicit provisions are made for contiguity, it is believed that a reasonably compact district will be contiguous. In our application of this procedure noncontiguous districts have not resulted.

4. Calculate the centroids, $(\bar{a}_j, \bar{b}_j)$, of the districts generated in step 3 as follows:

$$\bar{a}_j = \sum_{i=1}^{m} u_i p_i x_{ij} \Big/ \sum_{i=1}^{m} p_i x_{ij} , \quad j = 1, \ldots, n,$$

$$\bar{b}_j = \sum_{i=1}^{m} v_i p_i x_{ij} \Big/ \sum_{i=1}^{m} p_i x_{ij} , \quad j = 1, \ldots, n.$$

5. Check to see if the new solution is significantly different by computing $| \bar{a}_j - a_j |$ and $| \bar{b}_j - b_j |$. If these differences are less than a predefined tolerance

Table II. Statistics for the Plan Generated by the Transportation Method for Ten Congressional Districts for the State of Missouri

District number	Population	Deviation from ideal (467,650)
1	467,960	+310
2	468,324	+674
3	467,823	+173
4	468,324	+674
5	467,542	−108
6	467,649	−1
7	467,065	−585
8	467,468	−182
9	467,173	−477
10	467,173	−477
	4,676,501	

Ratio of largest to smallest district: 1.002.
Percent to control the majority: 49.96%.

(e.g. $|\bar{a}_j - a_j| < \epsilon \wedge |\bar{b}_j - b_j| < \epsilon$) proceed to step 6. If not, go to step 2 and repeat the above process using $(\bar{a}_j, \bar{b}_j)$ as the new values for (a_j, b_j), $j = 1, ..., n$. The redistricting plan just generated is represented by the x_{ij}. When $x_{ij} = 1$, then U_i is in D_j.

6. Modify the g_j ($j = 1, ..., n$) as follows:

a. Find the district with the largest population and call this district l. Likewise, find the district with the smallest population and call this district s.

b. Calculate the ratio of largest to smallest districts (ratio $= p_l/p_s$).

c. Compare this new ratio with the lowest ratio obtained thus far. The first time step 6 is executed, this comparison is ignored. Instead proceed with the modification outlined in (i) below.

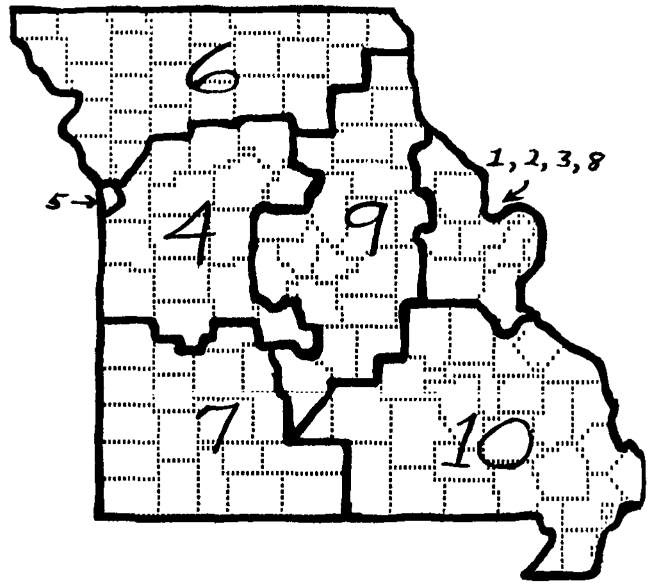
(i) If the new ratio is less than the previous ratio, then subtract one from g_l and add one to g_s. All other g_j will remain the same. Retain this new ratio as the best ratio obtained thus far and set $r = 0$. Then proceed to step 2.

(ii) Otherwise add one to r. If this new value of r is equal to h (h is a predefined constant that specifies the number of consecutive times this part is allowed to be executed) then terminate. Otherwise subtract one from g_l and add one to g_s. All other g_js will remain the same. Proceed to step 2.

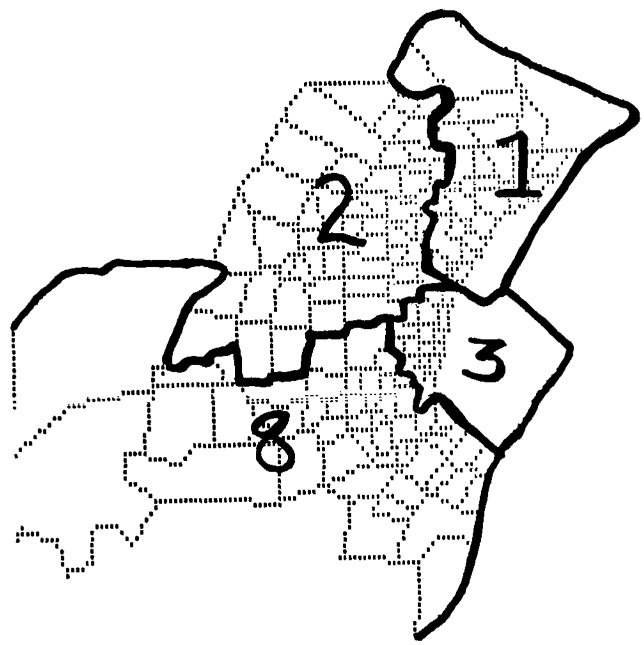
Although one may consider a more complex scheme for modifying the g_j (e.g. changing more than two of the g_js at a time) we have found the above method sufficient.

Hess [7] also considers solving the redistricting problem by using mathematical programming techniques. He formulates this problem as a warehouse location problem that assigns people to a specific warehouse (political district). Solving the problem in this manner

Map 2. Districting plan generated by the transportation method for ten Congressional Districts for the state of Missouri.



Map 3. Detail of congressional-districting plan in St. Louis area generated by the transportation method. District 2 includes the counties of Warren, Lincoln, Pike, and St. Charles. District 8 includes the counties of Franklin and Jefferson.



Map 4. Districting plan generated by the transportation method for the seven County Council seats for St. Louis County.

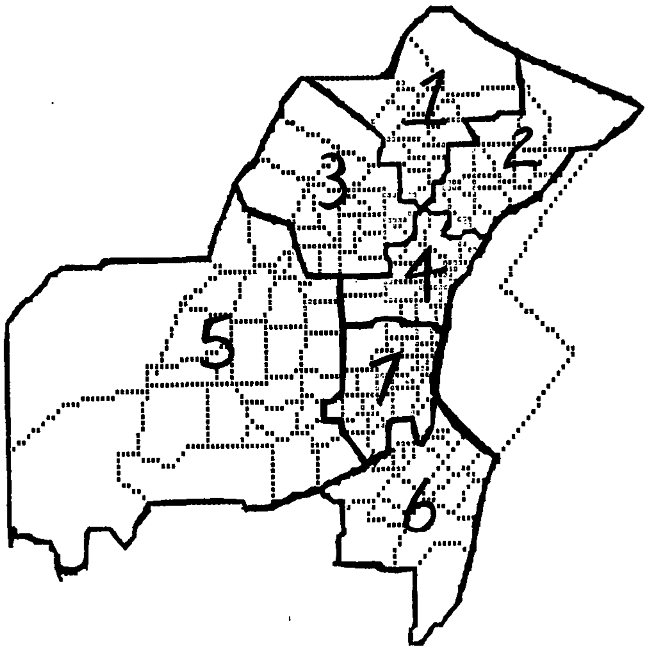


Table III. Statistics for the Plan Generated by the Transportation Method for the Seven County Council Seats for St. Louis County

Seat Number	Population	Deviation from ideal (135,907)
1	136,486	+579
2	137,347	+1,440
3	136,820	+913
4	136,534	+627
5	133,551	−2,356
6	135,841	−66
7	134,774	−1,133
	955,353	

Ratio of largest to smallest district: 1.0284
Percent to control majority: 56.82%.
(Based on four smallest divisions—ideally 57.14%)

forces each resulting district to have the same ideal population. In general, this will require population units to be split between political districts. This solution is acceptable only if political or natural boundaries are not to be preserved. Even when such solutions are acceptable, one does not know the precise boundaries of split population units. In an extreme case, households may be split between districts. To avoid these problems, Hess regroups so that no population units are split between districts. He uses the solution obtained from regrouping as a starting point for continuing as before. However, there is no guarantee that these solutions are monotonically converging.

Hess [7] also formulates the political redistricting problem as an integer programming problem, one that he admits cannot be solved at present. Our formulation is also an integer programming problem, but it is one that can be solved efficiently using the transportation algorithm [5]. In our formulation we chose to assign population units rather than individual people as in the warehouse approach. This choice allowed us to preserve the integrity of the population units and consequently avoid the regrouping problem.

Results

We applied our procedure to the problem of generating ten congressional districts in the state of Missouri. Our choice of population units was counties and census tracts. Because of differing population densities, counties were used in rural areas of the state while census tracts were used in metropolitan areas. Since the average population of a rural county in Missouri is approximately 18,000 and the population of an average urban census tract is 6,500, it was necessary to combine urban census tracts to form population units of approximately equal population. Similarly, smaller rural counties were combined. This produced 200 population units for the entire state.

Initial attempts were made to determine ten districts in a single application of the algorithm. Problems arose in finding a complete solution because of the large differences in population density between outstate and metropolitan areas. Results indicated the following sequential development of districts:

1. The St. Louis metropolitan area should have four districts, Kansas City (within Jackson County) one district, and the rural counties five districts. The precise boundary of the combined four-district St. Louis area was also defined.
2. The transportation procedure above was applied to the rural population units to determine the five rural districts.
3. The same procedure was subsequently applied to the metropolitan St. Louis population units to obtain the four St. Louis area districts.

These results appear on Maps 2 and 3 and in Table II. The districting plans were generated from the following starting points:

1. Rural districts—four corners of the state and Jackson County.
2. Metropolitan St. Louis districts—as far north, south, east, and west as possible within the area considered.

Note that in Map 2, two rural counties (Wright and Clay) were each split between two districts. These divisions were necessary in order to obtain a desirable ratio of largest to smallest district. Other sets of starting values were tried, but the ratios of largest to smallest district in these attempts did not yield as low a value as the plans presented. The ratio for this plan (1.002) is well within the bounds specified in U.S. Supreme Court decisions.

Seven St. Louis County Council seats were also determined by the above procedure. The 147 census tracts for St. Louis County were used as the population units. It was not necessary to sequentially develop these districts, as it was for state districts, since the population is more uniformly distributed. The starting points were generated randomly. These results appear on Map 4 and Table III.

The redistricting algorithm that produced the above results was written in PL/I and run on an IBM 360/50 under OS. The program required on the average 65K bytes and ran from 3 to 7 minutes of CPU time in 2.5 μs core. The times and storage requirements varied with the initial solution.

Other Approaches

Thoreson and Liittschwager [10] present a grid-type approach to the political redistricting problem. A grid is placed over a map of the region to be redistricted to obtain the population units. Legislative districts are then formed by successively grouping the cells of the grid. With a few modifications we implemented Method 2 of Thoreson and Liittschwager [10]. Results from using this method can be found in Maps 5, 6, and 7, and Tables IV and V. We found this procedure to be a fast (approximately 30 seconds CPU time) method for obtaining a good initial solution for our transportation algorithm.

Another method is the implicit enumeration technique presented by Garfinkel [2, 3]. This method generates all possible districts based on his constraints. From this set he determines acceptable redistricting plans. With the data for St. Louis County, this program[2] required excessive computer time.

Conclusions

A suit is now pending to allow the courts to reapportion the ten congressional districts for the state of Missouri [21]. We have submitted our solutions to the court for consideration [4]. The Office of the Attorney General of the State of Missouri has recommended to the federal panel that it use computers in accomplishing the task of redistricting the state of Missouri. An editorial in *The St. Louis Post-Dispatch* [8] has made public our efforts to redraw Missouri congressional districts. How-

[2] The program for Garfinkel's algorithm was obtained from Dr. Nelson Heller of the St. Louis Police Department.

Map 5. Districting plan generated by the modified grid approach for ten Congressional Districts for the state of Missouri.

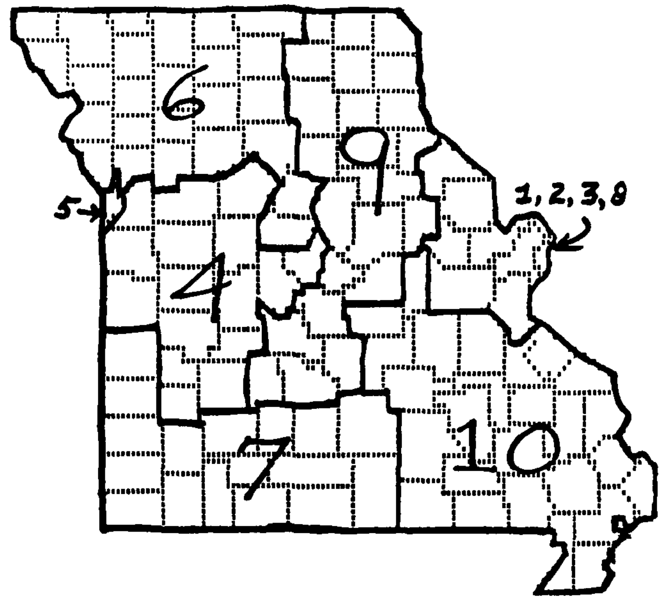


Table IV. Statistics for the Plan Generated by the Modified Grid Approach for Ten Congressional Districts for the State of Missouri

District number	Population	Deviation from ideal (467,650)
1	466,677	−973
2	466,505	−1,145
3	468,790	+1,140
4	454,293	−13,357
5	467,542	−108
6	478,057	+10,407
7	476,349	+8,699
8	469,603	+1,953
9	461,245	−6,405
10	467,440	−210
	4,676,501	

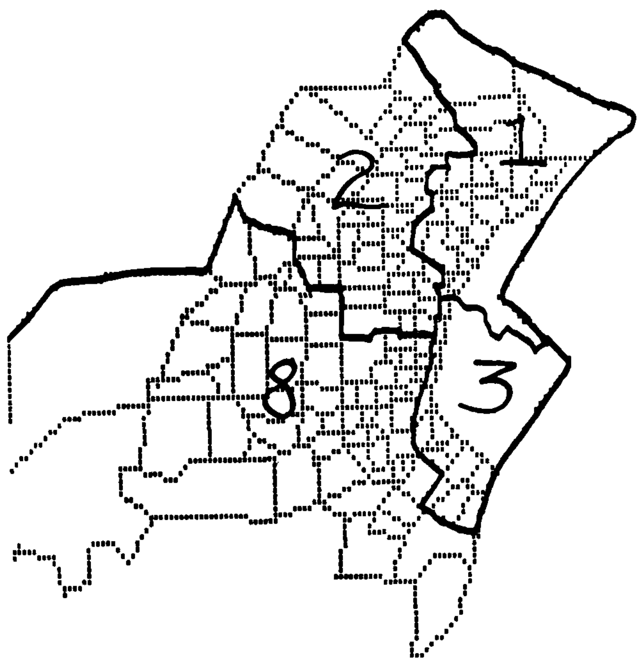
Ratio of largest to smallest district: 1.05.
Minimum percent to control the majority: 49.52%.

Table V. Statistics for the Plan Generated by the Modified Grid Approach for the Seven County Council Seats for St. Louis County

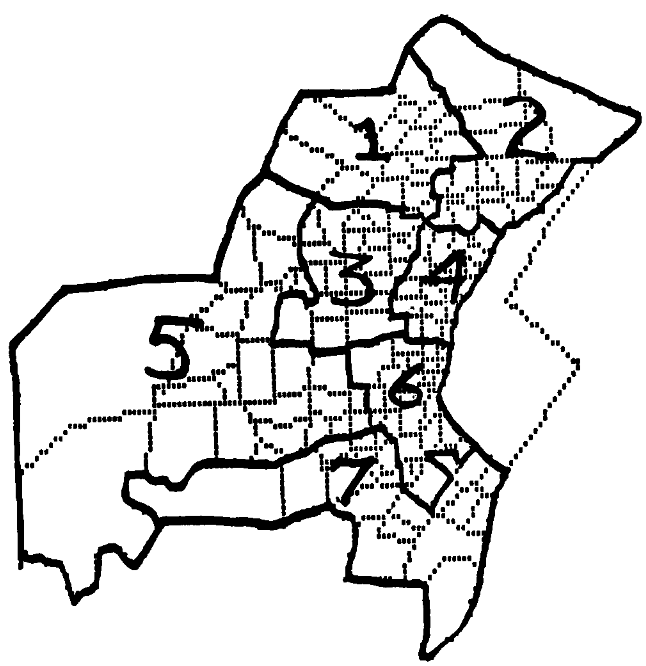
County Council Number	Population	Deviation from ideal (135,907)
1	140,223	+4,316
2	141,429	+5,522
3	135,039	−868
4	134,444	−1,463
5	132,033	−3,874
6	137,537	+1,630
7	130,648	+5,259
	951,353	

Ratio of largest to smallest district: 1.08.
Smallest percent of people controlling the majority: 56.66%.
(Based on four smallest divisions—ideally 57.14%)

Map 6. Detail of congressional-districting plan in St. Louis area generated by the modified grid approach (subdivisions within St. Louis County correspond to census tracts). District 2 inludes the counties of Warren, Lincoln, Pike and St. Charles. District 8 includes the counties of Franklin and Jefferson.



Map 7. Districting plan generated by the modified grid approach for the seven County Council seats for St. Louis County.



ever, it is not our intent to try to impose these solutions or to actively lobby for their adoption, but rather to demonstrate the feasibility of accomplishing political redistricting with the aid of a computer.

Political redistricting, and its consequences, should not be taken lightly. For a political system such as that in the United States to function properly, popular representation must truly be as equal as possible, for, "No state legislature can unnecessarily abridge one's right to an equal vote in a congressional election any more than it may constitutionally deny his right to trial by jury, his right of free speech, his right to freedom of religion, his right to peaceful assemblage, or any one of his many other rights guaranteed by the Constitution of the United States." [17]

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