Assignment 2: CS 215

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Ans 1.

$$Z = XY$$

$$\text{We know that } f_Z(z) = \frac{\mathrm{d}}{\mathrm{d}z}(F_Z(z))$$

$$F_Z(z) = P(XY \leq z) = \int_0^{\infty}\int_{-\infty}^{\frac{z}{x}} f_{XY}(x,y)dydx + \int_{-\infty}^{0}\int_{\frac{z}{x}}^{\infty} f_{XY}(x,y)dydx$$

$$\implies f_Z(z) = \frac{\mathrm{d}}{\mathrm{d}z}(F_Z(z)) = \frac{\mathrm{d}}{\mathrm{d}z}(\int_0^{\infty}\int_{-\infty}^{\frac{z}{x}} f_{XY}(x,y)dydx) + \frac{\mathrm{d}}{\mathrm{d}z}(\int_{-\infty}^{0}\int_{\frac{z}{x}}^{\infty} f_{XY}(x,y)dydx)$$

$$= \int_0^{\infty}\frac{\mathrm{d}}{\mathrm{d}z}(\int_{-\infty}^{\frac{z}{x}} f_{XY}(x,y)dy)dx + \int_{-\infty}^{0}\frac{\mathrm{d}}{\mathrm{d}z}(\int_{\frac{z}{x}}^{\infty} f_{XY}(x,y)dy)dx$$

$$= \int_0^{\infty}\frac{f_{XY}(x,\frac{z}{x}}{x}dx + \int_{-\infty}^{0}(-1)\frac{f_{XY}(x,\frac{z}{x}}{x}dx$$

$$Now\ P(X \leq Y) = \int_{-\infty}^{\infty}\int_{x}^{\infty} f_{XY}(x,y)dydx$$

Ans 2.

$$Y_1 = max(X_1, X_2, ..., X_n)\ and\ Y_2 = min(X_1, X_2, ..., X_n)$$

Since $X_1, X_2, ..., X_n$ are identically distributed, their cdf($= F_X(x)$) and pdf($= f_X(x) = F'_X(x)$)

$$P(X_i \leq x) = F_X(x) \forall iin[1,n]$$

$$For\ Y_1 \leq x,\ each\ of\ X_1, X_2, ..., X_n \leq x,$$

$$Thus\ P(Y_1 \leq x) = P(X_1 \leq x, X_2 \leq x, ..., X_n \leq x)$$

Since $X_1, X_2, ..., X_n$ are independent,

$$cdf\ of\ Y_1 = F_{Y_1}(x) = P(Y_1 \leq x) = P(X_1 \leq x)P(X_2 \leq x).....P(X_n \leq x)$$

$$= (F_X(x))^n$$

$$pdf\ of\ Y_1 = F'_{Y_1}(x) = ((F_X(x))^n)' = n(F_X(x))^{n-1}F'_X(x)$$

$$Now\ P(X_i > x) = 1 - P(X_i \leq x) = 1 - F_X(x)\ \forall i\ in\ [1, n]$$
$$For\ Y_2 > x,\ each\ of\ X_1, X_2, ..., X_n > x,$$
$$Thus\ P(Y_2 > x) = P(X_1 > x, X_2 > x, ..., X_n > x)$$
$$\text{Since } X_1, X_2, ..., X_n \text{ are independent,}$$
$$P(Y_2 > x) = P(X_1 > x)P(X_2 > x).....P(X_n > x)$$
$$= (1 - F_X(x))^n$$
$$P(Y_2 > x) = 1 - P(Y_2 \leq x) = (1 - F_X(x))^n$$
$$cdf\ of\ Y_2 = F_{Y_2}(x) = P(Y_2 \leq x) = 1 - (1 - F_X(x))^n$$
$$pdf\ of\ Y_2 = F'_{Y_2}(x) = ((1 - F_X(x))^n)' = n(1 - F_X(x))^{n-1}F'_X(x)$$

Ans 3.

Since the total amount doubles in each trial, the amount bet in the n^{th} trial $= 2^{n-1}x$
Amount won in the n^{th} trial $= 2^{n-1}x$ Total amount lost in the previous n-1 trials $= \sum_{i=1}^{n-1} 2^{i-1}x$ Net amount won

$$= 2^{n-1}x - \sum_{i=1}^{n-1} 2^{i-1}x$$
$$= 2^{n-1}x - \frac{2^{n-1} - 1}{2 - 1}x$$
$$= 2^{n-1}x - (2^{n-1} - 1)x$$
$$= x$$

Ans 4.

$$Covariance = E((x - \mu_x)(y - \mu_y))$$
$$= E(XY - X\mu_y - Y\mu_X + \mu_x\mu_y)$$
$$= E(XY) - \mu_y E(X) - \mu_x E(Y) + \mu_x\mu_y (\because\ E()\ is\ a\ linear\ operator.)$$
$$= E(XY) - \mu_y\mu_x - \mu_x\mu_y + \mu_x\mu_y$$
$$= E(XY) - \mu_x\mu_y$$
$$= E(XY) - E(X)E(Y)$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_XY(x, y)dxdy$$

If X and Y are independent $f_XY(x,y) = f_X(x)f_Y(y)$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_XY(x,y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x)f_Y(y)dxdy$$

$$= \int_{-\infty}^{\infty} xf_X(x)dx \int_{-\infty}^{\infty} yf_Y(y)dy$$

$$= E(X)E(Y)$$

$$\implies E(XY) - E(X)E(Y) = 0$$

Thus independence implies Covariance is 0. But the converse may not always be true.

Counter Example: Consider the set 1,2,3 with equal probability of selecting any one of the elements. Now we select one element from the set.

Consider the random variables X_1, X_2 such that,

$$X_1 = \begin{cases} 1 & Selected\ number\ is\ even \\ 0 & Selected\ number\ is\ odd \end{cases}$$

$$X_2 = \text{Selected number}$$

$$Now\ E(X_1) = 1 * \frac{1}{3} + 0 * \frac{2}{3} = \frac{1}{3}$$

$$E(X_2) = 1 * \frac{1}{3} + 2 * \frac{1}{3} + 3 * \frac{1}{3} = 2$$

$$E(X_1X_2) = 1 * 2 * \frac{1}{3} + 0 = \frac{2}{3} = E(X_1)E(X_2)$$

Hence Covariance of X_1, X_2 is 0.

Now when $X_2 = 1,\ then X_1 must be 0$(since 1 is odd). Hence,

$$P(X_1 = 1, X_2 = 1) = 0$$

$$But\ P(X_1 = 1) = 1 * \frac{1}{3}\ and\ P(X_2 = 1) = \frac{1}{3}$$

$$Thus\ P(X_1 = 1, X_2 = 1) \neq P(X_1 = 1)P(X_2 = 1)$$

Hence $X_1\ and\ X_2$ are not independent. Thus $Covariance = 0 \not\Rightarrow Independence$

Ans 5.

$$Var(X) = E((x - \mu)^2)$$

$$We\ know\ that\ E((x - c)^2)\ is\ minimum\ when\ c = \mu$$

$$Let c = \frac{a+b}{2}$$

$$Thus\ Var(X) \leq E((x - \frac{a+b}{2})^2)$$

$$= \int_a^b (x - \frac{a+b}{2})^2 f_X(x)dx$$

$\frac{a+b}{2}$ is the mid-point of the interval [a,b]. Now to maximise $(x - \frac{a+b}{2})^2$ we need to take the value farthest from $\frac{a+b}{2}$ in the interval [a,b]. These values are x=a and x=b which give

$$(x - \frac{a+b}{2})^2 = (a - \frac{a+b}{2})^2 = (b - \frac{a+b}{2})^2 = (\frac{b-a}{2})^2$$

$$Hence \int_a^b (x - \frac{a+b}{2})^2 f_X(x)dx \leq (\frac{b-a}{2})^2 \int_a^b f_X(x)dx$$

$= (\frac{b-a}{2})^2$ ($\because$ the total probability over the entire range in 1)

$$Hence\ Var(X) \leq (\frac{b-a}{2})^2 = \frac{(b-a)^2}{4}$$

Ans 6.

The equation of tangent at $x = x_o$ is

$$\frac{y - g(x_o)}{x - x_o} = g'(x)$$

$$\implies y = (x - x_o)g'(x_o) + g(x_o)$$

As g(x) is a convex function

$$g(x) \geq y = (x - x_o)g'(x_o) + g(x_o)$$

$$\implies g(x) \geq (x - x_o)g'(x_o) + g(x_o)$$

$$\implies \int_{-\infty}^{\infty} g(x)f_X(x)dx \geq \int_{-\infty}^{\infty} ((x - x_o)g'(x_o) + g(x_o))f_X(x)dx$$

$$E(g(x)) \geq (E(x) - x_o)g'(x_o) + g(x_o)$$

This holds for all $x = x_o$ Let $x_o = E(x)$

$$E(g(x)) \geq g(E(x))$$

Hence proved.