Ans 4 Assuming we are given the sample data $x_1, x_2, \dots x_n$ drawn from a uniform distribution over $(0, \theta)$, X_i's are independent.

$$P(x_1, x_2, \dots x_n | \theta) = \frac{1}{\theta^n} \quad \text{if } 0 < x_i < \theta \ \forall i \in [1, n], i \in \mathbb{Z}$$

$$= 0 \quad \text{otherwise}$$

To maximize this likelihood, we have to minimize θ.

The minimum value that θ can take such that probability $\neq 0$ is $\max(x_1, x_2, \dots x_n)$

The $\hat{\theta}_{ML} = \max(x_1, x_2, \dots, x_n)$

For MAP estimator:

Prior $P(\theta) \propto \left(\frac{\theta_m}{\theta}\right)^\alpha \quad \forall \theta \geq \theta_m, \ \theta_m > 0, \ \alpha > 1$

0 otherwise.

The posterior distribution $P(\theta | x_1, x_2, \dots x_n)$ is

$$\frac{P(x_1, x_2, \dots x_n | \theta) \ P(\theta)}{P(x_1, x_2, x_3 \dots x_n)}$$

$$P(x_1, x_2, \dots x_n | \theta) = \frac{1}{\theta^n} \quad \text{if } 0 < x_i < \theta \ \forall i \in [1, n], i \in \mathbb{Z}$$

$$P(\theta) = k\left(\frac{\theta_m}{\theta}\right)^\alpha \quad \text{if } \theta \geq \theta_m$$

$$0 \quad \text{otherwise.}$$

The denominator of the posterior is a constant w.r.t θ

The numerator of the posterior is $\left(\frac{1}{\theta^n}\right) k \left(\frac{\theta_m}{\theta}\right)^\alpha$

For maximum posterior, θ has to be minimum.

The minimum value that θ can take such that posterior $\neq 0$ is $\max(\max(x_1, x_2, \dots x_n), \theta_m)$

$$\hat{\theta}_{Map} = \max(\max(x_1, x_2 \dots x_n), \theta_m)$$

(ii) When $n \to \infty$, as $x_1, x_2, \dots x_n$ are uniformly distributed $\max(\{x_i\}_{i=1}^n) \to \theta_{true}$

Thus $\hat{\theta}_{ML} \to \theta_{true}$ as $n \to \infty$

If $\theta_m < \theta_{true}$ then $\hat{\theta}_{map} = \max(\max\{x_i\}_{i=1}^n, \theta_m) \to \theta_{true}$ as $n \to \infty$ because $\max\{x_i\}_{i=1}^n \to \theta_{true} > \theta_m$ as $n \to \infty$

Thus $\hat{\theta}_{map} \to \hat{\theta}_{ML}$ as $n \to \infty$ when $\theta_m < \theta_{true}$

This is a desirable case.

(iii) $$P(\theta \mid x_1, x_2 \dots x_n) = \frac{P(x_1, x_2 \dots, x_n \mid \theta)\, P(\theta)}{P(x_1, x_2, \dots x_n)}$$

$$P(x_1, x_2, \dots x_n \mid \theta) = \begin{cases} \frac{1}{\theta^n} & \text{if } 0 < x_i \leq \theta \quad \forall i \in [1, n],\ i \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

$$P(\theta) = \begin{cases} k\left(\frac{\theta_m}{\theta}\right)^{\alpha}, & \theta_m > 0, \alpha > 1 \\ 0 & \text{otherwise} \end{cases}$$

$$P(x_1, x_2, \dots x_n) = \int_{-\infty}^{\infty} P(x_1, x_2, \dots x_n \mid \theta)\, P(\theta)\, d\theta$$

$$= \int_{-\infty}^{\max(\max\{x_i\}, \theta_m)} 0 \cdot d\theta + \int_{\max(\max\{x_i\}_{i=1}^{n}, \theta_m)}^{\infty} P(x_1, x_2 \dots x_n \mid \theta)\, P(\theta)\, d\theta$$

$$= \int_{\theta'}^{\infty} \left(\frac{1}{\theta^n}\right) k \left(\frac{\theta_m}{\theta}\right)^{\alpha} \cdot d\theta \quad \text{where } \theta' = \max\left(\max\{x_i\}_{i=1}^{n}, \theta_m\right)$$

$$= k\,\theta_m^{\alpha} \left. \frac{\theta^{1-n-\alpha}}{1-n-\alpha} \right|_{\theta'}^{\infty} = k\theta_m^{\alpha}\left(0 - \frac{(\theta')^{1-n-\alpha}}{1-n-\alpha}\right)$$

because $\alpha > 1$ Limit at ∞ is 0.

$$= \frac{k\,\theta_m^{\alpha}\,(\theta')^{1-n-\alpha}}{n+\alpha-1}$$

$$P(\theta \mid x_1, x_2 \dots x_n) = \frac{\cancel{k\,\theta_m^{\alpha}}\,(\theta')^{n+\alpha-1}(n+\alpha-1)}{\theta^{n+\alpha}\,\cancel{k\,\theta_m^{\alpha}}} = \frac{(\theta')^{n+\alpha-1}(n+\alpha-1)}{\theta^{n+\alpha}}$$

if $\theta > \theta'$

0 otherwise.

$$E_{P(\theta \mid x_1, x_2 \dots x_n)}(\theta) = \int_{\theta'}^{\infty} \frac{\theta\,(\theta')^{n+\alpha-1}(n+\alpha-1)}{\theta^{n+\alpha}} \cdot d\theta$$

$$= (\theta')^{n+\alpha-1}(n+\alpha-1)\left. \left(\frac{\theta^{2-n-\alpha}}{2-n-\alpha}\right)\right|_{\theta'}^{\infty}$$

$$= (\theta')^{n+\alpha-1}(n+\alpha-1)\left(0 - \frac{(\theta')^{2-n-\alpha}}{2-n-\alpha}\right) = \theta' * \frac{n+\alpha-1}{n+\alpha-2}$$

Thus $$\hat{\theta}_{\text{Posterior mean}} = \theta' * \frac{n+\alpha-1}{n+\alpha-2}$$

(iv) As $n \to \infty$, $\hat{\theta}_{posterior\ mean} \to \theta' = \max(\max(\{x_i\}_{i=1}^{n}), \theta_m)$

and $\theta' \to \theta_{true}$ if $\theta_m < \theta_{true}$

Thus $\hat{\theta}_{posterior mean} \to \hat{\theta}^{ML}$ if $\theta_m < \theta_{true}$.

This case is desirable.