Assignment 2: CS 215

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Ans 1.

$$Z = XY$$

$$\text{We know that } f_Z(z) = \frac{\mathrm{d}}{\mathrm{d}z}(F_Z(z))$$

$$F_Z(z) = P(XY \le z) = \int_0^\infty \int_{-\infty}^{\frac{z}{x}} f_{XY}(x,y)dydx + \int_{-\infty}^0 \int_{\frac{z}{x}}^\infty f_{XY}(x,y)dydx$$

$$\implies f_Z(z) = \frac{\mathrm{d}}{\mathrm{d}z}(F_Z(z)) = \frac{\mathrm{d}}{\mathrm{d}z}(\int_0^\infty \int_{-\infty}^{\frac{z}{x}} f_{XY}(x,y)dydx) + \frac{\mathrm{d}}{\mathrm{d}z}(\int_{-\infty}^0 \int_{\frac{z}{x}}^\infty f_{XY}(x,y)dydx)$$

$$= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}z}(\int_{-\infty}^{\frac{z}{x}} f_{XY}(x,y)dy)dx + \int_{-\infty}^0 \frac{\mathrm{d}}{\mathrm{d}z}(\int_{\frac{z}{x}}^\infty f_{XY}(x,y)dy)dx$$

$$= \int_0^\infty \frac{f_{XY}(x,\frac{z}{x})}{x}dx + \int_{-\infty}^0 (-1)\frac{f_{XY}(x,\frac{z}{x})}{x}dx$$

$$Now\ P(X \le Y) = \int_{-\infty}^\infty \int_x^\infty f_{XY}(x,y)dydx$$

If X and Y are independent

$$f_Z(z) = \int_0^\infty \frac{f_X(x)f_Y(\frac{z}{x})}{x}dx + \int_{-\infty}^0 (-1)\frac{f_X(x)f_Y(\frac{z}{x})}{x}dx$$

$$P(X \le Y) = \int_{-\infty}^\infty f_X(x)(\int_x^\infty f_Y(y)dy)dx$$

$$(SinceY\ lies\ above\ the\ line\ y = x,\ we\ integrate\ the\ area\ above\ the\ line\ y = x.)$$

$$= \int_{-\infty}^\infty f_X(x)(\int_{-\infty}^\infty f_Y(y)dy - \int_{-\infty}^x f_Y(y)dy)dx$$

$$= \int_{-\infty}^\infty f_X(x)(1 - F_Y(x))dx$$

Ans 2.

$$Y_1 = max(X_1, X_2, ..., X_n) \; and \; Y_2 = min(X_1, X_2, ..., X_n)$$

Since $X_1, X_2, ..., X_n$ are identically distributed, their cdf($= F_X(x)$) and pdf($= f_X(x) = F'_X(x)$)

$$P(X_i \leq x) = F_X(x) \forall iin[1, n]$$

$$For \; Y_1 \leq x, \; each \; of \; X_1, X_2, ..., X_n \leq x,$$

$$Thus \; P(Y_1 \leq x) = P(X_1 \leq x, X_2 \leq x, ..., X_n \leq x)$$

Since $X_1, X_2, ..., X_n$ are independent,

$$cdf \; of \; Y_1 = F_{Y_1}(x) = P(Y_1 \leq x) = P(X_1 \leq x)P(X_2 \leq x).....P(X_n \leq x)$$

$$= (F_X(x))^n$$

$$pdf \; of \; Y_1 = F'_{Y_1}(x) = ((F_X(x))^n)' = n(F_X(x))^{n-1}F'_X(x)$$

$$Now \; P(X_i > x) = 1 - P(X_i \leq x) = 1 - F_X(x) \; \forall i \; in \; [1, n]$$

$$For \; Y_2 > x, \; each \; of \; X_1, X_2, ..., X_n > x,$$

$$Thus \; P(Y_2 > x) = P(X_1 > x, X_2 > x, ..., X_n > x)$$

Since $X_1, X_2, ..., X_n$ are independent,

$$P(Y_2 > x) = P(X_1 > x)P(X_2 > x).....P(X_n > x)$$

$$= (1 - F_X(x))^n$$

$$P(Y_2 > x) = 1 - P(Y_2 \leq x) = (1 - F_X(x))^n$$

$$cdf \; of \; Y_2 = F_{Y_2}(x) = P(Y_2 \leq x) = 1 - (1 - F_X(x))^n$$

$$pdf \; of \; Y_2 = F'_{Y_2}(x) = ((1 - F_X(x))^n)' = n(1 - F_X(x))^{n-1}F'_X(x)$$

Ans 3.

Since the total amount doubles in each trial, the amount bet in the n^{th} trial $= 2^{n-1}x$

Amount won in the n^{th} trial $= 2^{n-1}x$ Total amount lost in the previous n-1 trials $= \sum_{i=1}^{n-1} 2^{i-1}x$ Net amount won

$$= 2^{n-1}x - \sum_{i=1}^{n-1} 2^{i-1}x$$

$$= 2^{n-1}x - \frac{2^{n-1} - 1}{2 - 1}x$$

$$= 2^{n-1}x - (2^{n-1} - 1)x$$

$$= x$$

Ans 4.

$$Covariance = E((x-\mu_x)(y-\mu_y))$$

$$= E(XY - X\mu_y - Y\mu_X + \mu_x\mu_y)$$

$$= E(XY) - \mu_y E(X) - \mu_x E(Y) + \mu_x\mu_y (\because\ E()\ is\ a\ linear\ operator.)$$

$$= E(XY) - \mu_y\mu_x - \mu_x\mu_y + \mu_x\mu_y$$

$$= E(XY) - \mu_x\mu_y$$

$$= E(XY) - E(X)E(Y)$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_XY(x,y)dxdy$$

If X and Y are independent $f_XY(x,y) = f_X(x)f_Y(y)$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_XY(x,y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x)f_Y(y)dxdy$$

$$= \int_{-\infty}^{\infty} xf_X(x)dx \int_{-\infty}^{\infty} yf_Y(y)dy$$

$$= E(X)E(Y)$$

$$\implies E(XY) - E(X)E(Y) = 0$$

Thus independence implies Covariance is 0. But the converse may not always be true.

Counter Example: Consider the set 1,2,3 with equal probability of selecting any one of the elements.
Now we select one element from the set.
Consider the random variables X_1, X_2 such that,

$$X_1 = \begin{cases} 1 & Selected\ number\ is\ even \\ 0 & Selected\ number\ is\ odd \end{cases}$$

$$X_2 = \text{Selected number}$$

$$Now\ E(X_1) = 1 * \frac{1}{3} + 0 * \frac{2}{3} = \frac{1}{3}$$

$$E(X_2) = 1 * \frac{1}{3} + 2 * \frac{1}{3} + 3 * \frac{1}{3} = 2$$

$$E(X_1X_2) = 1 * 2 * \frac{1}{3} + 0 = \frac{2}{3} = E(X_1)E(X_2)$$

Hence Covariance of X_1, X_2 is 0.
Now when $X_2 = 1,\ then X_1 must be 0$(since 1 is odd). Hence,

$$P(X_1 = 1, X_2 = 1) = 0$$

$$But\ P(X_1 = 1) = 1 * \frac{1}{3}\ and\ P(X_2 = 1) = \frac{1}{3}$$

$$Thus\ P(X_1 = 1, X_2 = 1) \neq P(X_1 = 1)P(X_2 = 1)$$

Hence X_1 *and* X_2 are not independent. Thus $Covariance = 0 \not\Rightarrow Independence$

Ans 5.

$$Var(X) = E((x-\mu)^2)$$

$$We\ know\ that\ E((x-c)^2)\ is\ minimum\ when\ c = \mu$$

$$Letc = \frac{a+b}{2}$$

$$Thus\ Var(X) \leq E((x - \frac{a+b}{2})^2)$$

$$= \int_a^b (x - \frac{a+b}{2})^2 f_X(x)dx$$

$\frac{a+b}{2}$ is the mid-point of the interval [a,b]. Now to maximise $(x - \frac{a+b}{2})^2$ we need to take the value farthest from $\frac{a+b}{2}$ in the interval [a,b]. These values are x=a and x=b which give

$$(x - \frac{a+b}{2})^2 = (a - \frac{a+b}{2})^2 = (b - \frac{a+b}{2})^2 = (\frac{b-a}{2})^2$$

$$Hence \int_a^b (x - \frac{a+b}{2})^2 f_X(x)dx \leq (\frac{b-a}{2})^2 \int_a^b f_X(x)dx$$

$= (\frac{b-a}{2})^2$ ($\because$ the total probability over the entire range in 1)

$$Hence\ Var(X) \leq (\frac{b-a}{2})^2 = \frac{(b-a)^2}{4}$$

Ans 6.

The equation of tangent at $x = x_o$ is

$$\frac{y - g(x_o)}{x - x_o} = g'(x)$$

$$\implies y = (x - x_o)g'(x_o) + g(x_o)$$

As g(x) is a convex function

$$g(x) \geq y = (x - x_o)g'(x_o) + g(x_o)$$

$$\implies g(x) \geq (x - x_o)g'(x_o) + g(x_o)$$

$$\implies \int_{-\infty}^{\infty} g(x)f_X(x)dx \geq \int_{-\infty}^{\infty} ((x - x_o)g'(x_o) + g(x_o))f_X(x)dx$$

$$E(g(x)) \geq (E(x) - x_o)g'(x_o) + g(x_o)$$

This holds for all $x = x_o$ Let $x_o = E(x)$

$$E(g(x)) \geq g(E(x))$$

Hence proved.

Ans 7.

	Corr. Coeff.	**QMI**	**Measure (3)**
Normal Case	0.9844	0.0359	1.4735
Scrambled case	0.0029	2.5222e-06	0.0242

The correlation coefficient decreases because when we scramble the image, the intensity no more varies gradually. Its all very random. So there is no dependence between neighbours. So correlation coefficient decreases. The $2^{nd} and 3^{rd}$ measures also decrease because since the neighbours have independent intensities, the joint probability function comes closer to the product of individual probability functions. Hence the dependence decreases.

The minimum value of measure (3) occurs when all the pixel value are of same intensity(say a completely black image). So the minimum value is 0.

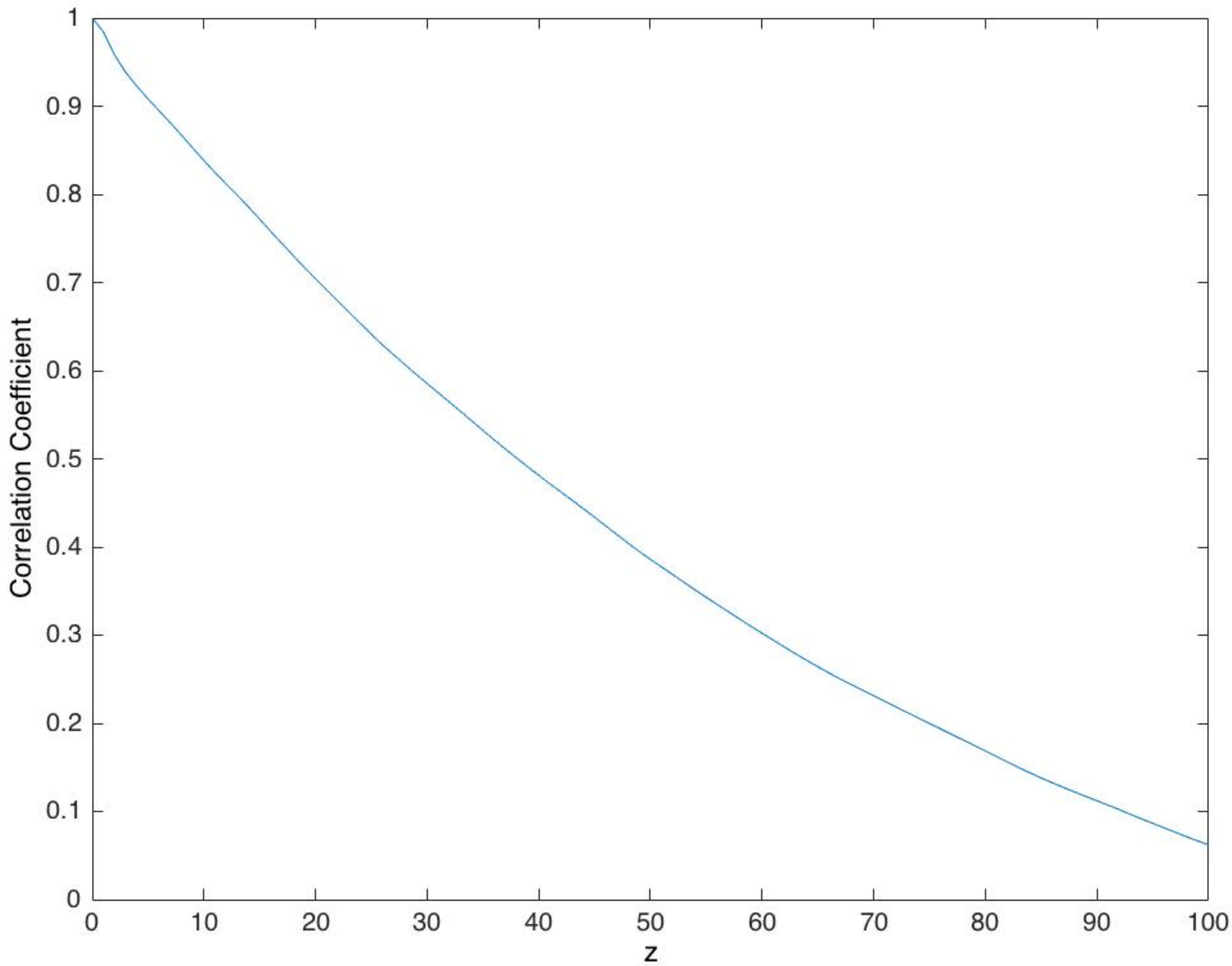

Figure 1: Task 7b

Correlation coefficient decreases as z increases. This is because when z is small the pixels are closeby and since the intensity varies gradually for most part of the image(except the boundaries), so the corresponding X1 and X2 values are close to each other. Thus correlation is high for such neighbours. As z increases the intensities of neighbours become more and more independent of each other. Thus correlation coefficient decreases.