3) Since the points are uniformly distributed,
(a) consider a random variable θ, such that $P(\theta) = \frac{1}{2\pi} \quad \forall \theta \in (0, 2\pi)$

(θ represents angle on the circle)

Now we have a 2-dimentional data sample X, such that

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

(x, y) is a point on the circle. (x, y are RV)

To fit a gaussian in it we assume it as a bivariate Gaussian with $\mu = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}$ and covariance matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$b = c = Cov(x, y)$$
$$= E[(x - E(x))(y - E(y))]$$

$E[x] = E[r\cos\theta] = rE[\cos\theta] = 0$ (since θ is uniformly distributed & $\cos\theta$ takes both equal negative & positive values) Also N is very large)

Similarly $E[y] = 0$

$$\Rightarrow b = c = E[xy]$$
$$= r^2 E[\cos\theta \sin\theta] = \frac{r^2}{2} E[\sin 2\theta] = 0$$

(same reason as above)

Thus $b = c = 0$

Now $a = Cov(x, x) = \sigma_x^2$

$d = Cov(y, y) = \sigma_y^2$

Now for any value of α in $(0, 1)$, the probability of $\sin\theta$ and $\cos\theta$ taking that value is same when θ is uniformly distributed in $(0, 2\pi)$.

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Since n is very large, the assumption of uniform holds

$a \sim d$

Thus covariance matrix $= \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = C \qquad (a = \sigma_x^2 = \sigma_y^2)$

$|C| = a^2$

$$P(x_1, x_2 \ldots x_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi} \times \sqrt{|C|}} \exp\left\{ -0.5 \left(\begin{bmatrix} x_i \\ y_i \end{bmatrix} - \mu \right)^T C^{-1} \left(\begin{bmatrix} x_i \\ y_i \end{bmatrix} - \mu \right) \right\}$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi} \times a} \exp\left\{ -\frac{1}{2a} \left((x_i - \mu_x)^2 + (y_i - \mu_y)^2 \right) \right\}$$

$$= \frac{1}{(2\pi a^2)^{n/2}} \exp\left\{ -\frac{1}{2a} \Sigma \left((x_i - \mu_x)^2 + (y_i - \mu_y)^2 \right) \right\}$$

Now to maximise likelihood,

we try to maximise $\exp\left\{ \frac{-1}{2a} \Sigma \left((x_i - \mu_x)^2 + (y_i - \mu_y)^2 \right) \right\}$

For this we minimize $\Sigma (x_i - \mu_x)^2$ and $\Sigma (y_i - \mu_y)^2$,

We know these are minimised at sample mean,

Hence $\mu_x = \frac{\Sigma x_i}{n}$, $\mu_y = \frac{\Sigma y_i}{n}$

$\Rightarrow \mu = \begin{bmatrix} \frac{\Sigma x_i}{n} \\ \frac{\Sigma y_i}{n} \end{bmatrix}$

Since x, y are uniformly distributed, we will have $(-x, -y)$ for every (x, y)

Since N is large, the assumption of uniformity almost holds,

$\Rightarrow \Sigma x_i = \Sigma y_i = 0$

$\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow P(x_1, x_2 \ldots x_n) = \frac{1}{(2\pi a^2)^{n/2}} \exp\left\{-\frac{1}{2a} \Sigma (x_i^2 + y_i^2)\right\}$$

$$= \frac{1}{(2\pi a^2)^{n/2}} \exp\left\{-\frac{1}{2a} \times \sum_{i=1}^{n} (r^2)\right\} \quad \left(\because x_i^2 + y_i^2 = r^2 \text{ (circle) } \forall i\right)$$

$$P = \frac{1}{(2\pi a^2)^{n/2}} \exp\left\{-\frac{nr^2}{2a}\right\}$$

To maximise P, we maximise log P w.r.t. variance a.

$$\log P = -\frac{nr^2}{2a} - \frac{n}{2}\log(2\pi a^2)$$

Differentiating both sides,

$$\frac{d}{da}\log P = \frac{nr^2}{2a^2} - \frac{n}{a} = 0$$

$$\Rightarrow a = \frac{r^2}{2}$$

Double differentiating

$$\frac{d^2 \log P}{da^2} = \frac{n}{a^2}\left(1 - \frac{r^2}{a}\right) < 0 \text{ at } a = \frac{r^2}{2}$$

Hence it is a maxima

$$\Rightarrow C = \begin{bmatrix} r^2/2 & 0 \\ 0 & r^2/2 \end{bmatrix}$$

(b) Yes Gaussian fits the data well. It is a good model because we see that the calculated values (in MATLAB) are very close to the theoretical values.

(e) $N = 10^6$ $\qquad \frac{\sigma^2}{2} = 50$

$\sigma = 10$

$$\mu = \begin{bmatrix} 0.0108 \\ -0.0060 \end{bmatrix}$$

$$Cov = C = \begin{bmatrix} 50.0423 & -0.0467 \\ -0.0467 & 49.9576 \end{bmatrix}$$

Since N is very large, our calculated values are very close to theoretical values. So our model fits well.