Ans 2- (i) Analytic form for distribution of y

$$y = -\frac{1}{\lambda}\log(x)$$

$$x = e^{-\lambda y} = g^{-1}(y)$$

$$P(x) = \text{PDF of } x = 1 \quad \forall x \in (0,1)$$

$$f(y) = \text{PDF of } y = P(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$= 1 \times \left| \frac{d}{dy} e^{-\lambda y} \right| \quad (\because e^{-\lambda y} \in (0,1))$$

$$\Rightarrow \boxed{f(y) = \lambda e^{-\lambda y}}$$

(ii) Formula for posterior mean

Let there be n samples $x_1, x_2 \dots x_n$ from which we get $y_1, y_2 \cdots y_n$

$$P(\lambda \mid y_1, y_2 \cdots y_n) = \frac{P(y_1, y_2, \cdots y_n \mid \lambda) \times P(\lambda)}{P(y_1, y_2 \cdots y_n)}$$

$$P(\lambda) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} \quad \text{(Given - Gamma prior)}$$

$$P(y_1, y_2 \cdots y_n \mid \lambda) = P(y_1 \mid \lambda)\, P(y_2 \mid \lambda) \cdots P(y_n \mid \lambda)$$

($\because$ Independently picked samples)

$$= (\lambda)^n e^{-\lambda \Sigma y_i}$$

$$\Rightarrow P(\lambda \mid y_1, y_2 \cdots y_n) = \frac{\lambda^n e^{-\lambda \Sigma(y_i)} \times \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}}{\int_0^\infty \cancel{\lambda e^{-\lambda y}}\, P(y_1, y_2 \cdots y_n, \lambda)\, d\lambda}$$

($\because \lambda > 0$)

$$= k \times \left( \lambda^{n+\alpha-1} + e^{-\lambda(\beta + \Sigma y_i)} \right)$$

where $k$ is constant

The posterior is also a gamma distribution with parameters

$$\alpha' = n + \alpha \quad \beta' = \beta + \Sigma y_i$$

We know the mean of Gamma distribution is $\alpha/\beta$

Thus $$\boxed{\hat{\lambda}^{\text{Posterior Mean}} = \frac{N + \alpha}{\beta + \Sigma y_i}}$$

(iii) Interpretation of Graph

(i) As $n$ increases, ~~both~~ the median of the errors and the spread around the median decrease in case of both Maximum Likelihood Estimate and Posterior Mean.
The median of the errors tend towards zero in both cases.

(ii) We will prefer the Posterior Mean since for almost all values of N, the median of errors and spread around the median is smaller for Posterior Mean. It is only in case of very large N that sometimes Maximum Likelihood errors are lesser than Posterior Mean error, but they are still very comparable.