1)

a) Given $X = (X_1, X_2)$ is a random vector where X_1, X_2 are both independent and have uniform distribution over $(-1, 1)$

$$P(\|X\| < 1) = \int_{-1}^{1} P(X_1 = x_1)\, P(|x_2| \le \sqrt{1 - x_1^2})\, dx_1$$

$\Downarrow$

$X_1^2 + X_2^2 \le 1$

$X_2^2 \le 1 - X_1^2$

$|X_2| \le \sqrt{1 - X_1^2}$

Now $x_1 \in [-1, 1]$

and $\sqrt{1 - x_1^2} \in [-1, 1]$

Thus $P(X = X_1) = 1/2$

$$P(\sqrt{1-x_1^2} \le X_2 \le \sqrt{1 - x_1^2}) = \frac{1}{2} \times 2\sqrt{1 - x_1^2}$$

$$= \int_{-1}^{1} \frac{1}{2} \times 2\sqrt{1 - x_1^2}\, dx_1$$

$$= \frac{\pi}{4}$$

b) Let $S = (\|X\| \le 1)$..

$S = 0$ when $\|X\| > 1$

$S = 1$ when $\|X\| \le 1$

Thus S is a bernoulli Random variable with parameter $\pi/4$

In simulation we generate data samples $X = (x_1\ x_2)^T$

We find the number of X for which $\|X\| \le 1$.

$S = 1$ for these data points

The maximum likelihood estimate of the probability $\hat{p}_{ML} = \dfrac{\sum_{i=1}^{N} S_i}{N}$

For large N

$\hat{p}_{ML} \to p$

Thus $\hat{p}_{ML} = p = \pi/4$

Thus $\pi = 4 \cdot \hat{p}_{ML}$ as $n \to \infty$

c) Estimates of π:-

N	Estimate of π
10^1	2.0000
10^2	3.2400
10^3	3.2440
10^4	3.1228
10^5	3.1397
10^6	3.1421
10^7	3.1418
10^8	3.1417

It is not possible to handle 10^9 amont of memory in MATLAB. We can divide the sample into 10 sets of size 10^8 and then run a simulation to find the number of x_i's such $\|x_i\| \leq 1$. For each set we calculate a maximum likelihood estimates and then take the average of all of them.

(d) As $N \to \infty$

$$\hat{\pi} = \frac{4\sum_{i=1}^{N} S_i}{N} \sim 4\,G\left(\mu, \frac{\sigma}{N}\right) = G\left(4p, \frac{16p(1-p)}{N}\right)$$

where $\mu = E(s) = p$

$\sigma = Var(s) = p(1-p)$

For a gaussian distribution we find that data lies in the range $(\mu - 1.96\sigma, \mu + 1.96\sigma)$ with probability 0.95 using lookup tables.

In case we ~~did~~ want data to lie in the range $(\mu - 0.01, \mu + 0.01)$

Thus -

$$0.01 = \left(\sqrt{\frac{16\, p(1-p)}{N}}\right) 1.96$$

$$p = \pi/4$$

$$\sqrt{16\left(\frac{\pi}{4}\right)\left(1-\frac{\pi}{4}\right) * \frac{1}{N}} = \frac{1}{196}$$

$$\frac{(\pi)(4-\pi)}{N} = \frac{1}{(196)^2}$$

$$\Rightarrow N = (\pi)(4-\pi)(196)^2$$

$$\boxed{N \sim 103600}$$