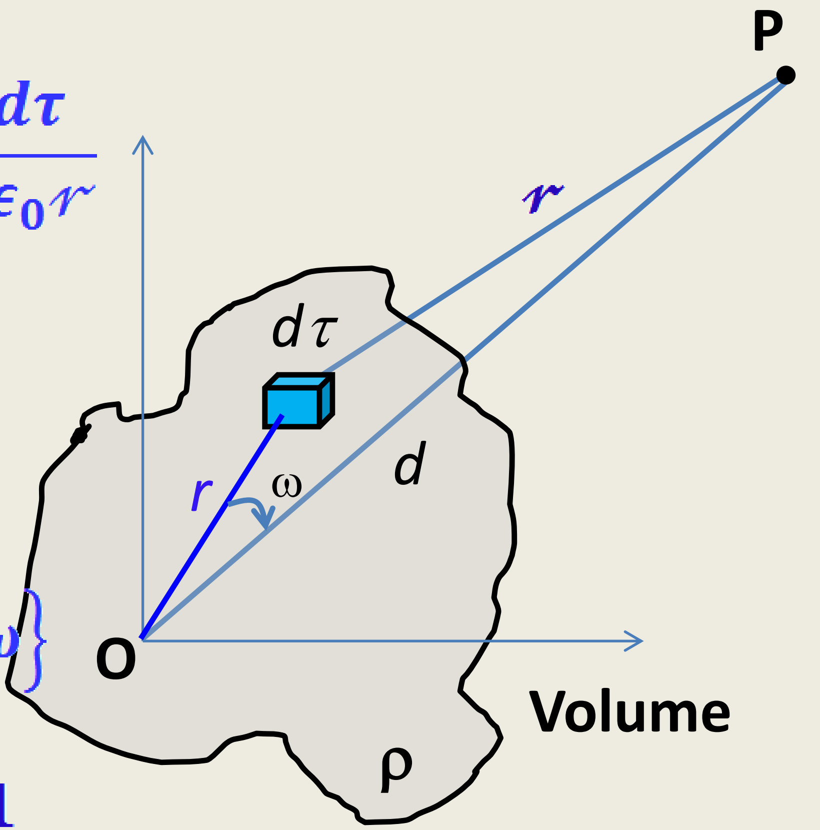
Calculating potentials :
Multipole Expansion

Multipole Expansion

$$dV(P) = \frac{\rho\, d\tau}{4\pi\epsilon_0 \mathscr{r}} \longrightarrow V(P) = \int_V \frac{\rho\, d\tau}{4\pi\epsilon_0 \mathscr{r}}$$



$$\mathscr{r}^2 = r^2 + d^2 - 2rd\cos\omega$$

$$= d^2\left\{1 - \frac{2r}{d}\cos\omega + \frac{r^2}{d^2}\right\}$$

$$= d^2(1+\delta) \quad \text{where} \quad \delta = \frac{r}{d}\left\{\frac{r}{d} - 2\cos\omega\right\}$$

$$\frac{1}{\mathscr{r}} = \frac{1}{d}(1+\delta)^{-1/2} \qquad d \gg r \Longrightarrow \delta \ll 1$$

$$\frac{1}{\mathscr{r}} = \frac{1}{d}\left(1 - \frac{1}{2}\delta + \frac{3}{8}\delta^2 + \ldots\ldots..\right) = \frac{1}{d}\left(1 - \frac{1}{2}\left[\frac{r}{d}\left\{\frac{r}{d} - 2\cos\omega\right\}\right]\right.$$

$$\left. + \frac{3}{8}\frac{r^2}{d^2}\left\{\frac{r}{d} - 2\cos\omega\right\}^2 + \ldots\right)$$

Multipole Expansion

$$\frac{1}{r} = \frac{1}{d} + \frac{r}{d^2}\cos\omega + \frac{r^2}{d^3}\left(\frac{3}{2}\cos^2\omega - \frac{1}{2}\right) + \dots\dots$$

$$V(P) = \underbrace{\frac{1}{4\pi\epsilon_0 d}\int_V \rho\, d\tau}_{1} + \overbrace{\frac{1}{4\pi\epsilon_0 d^2}\int_V (r\cos\omega)\rho\, d\tau}^{2} + \underbrace{\frac{1}{4\pi\epsilon_0 d^3}\int_V \left(r^2\left[\frac{3}{2}\cos^2\omega - \frac{1}{2}\right]\right)\rho\, d\tau}_{3} + \cdots.$$

(1) Monopole term

$$V(P) = \frac{1}{4\pi\epsilon_0 d}\int_V \rho\, d\tau = \frac{Q_{\text{tot}}}{4\pi\epsilon_0 d}$$

- Potential due to charge at the origin
- exact potential
- Higher multipole terms vanish

(2) dipole term : If $Q_{tot} = 0$, then the dominant term

$$V(P) = \frac{1}{4\pi\epsilon_0 d^2}\int_V (r\cos\omega)\rho\, d\tau \;=\; \frac{1}{4\pi\epsilon_0 d^2}\int_V (\vec{r}\cdot\hat{n})\rho\, d\tau$$

$$= \frac{1}{4\pi\epsilon_0 d^2}\,\hat{n}\cdot\int_V (\vec{r})\rho\, d\tau$$

$\hat{n}$ unit vector pointing towards point P: constant

$$V_{\text{dipole}}(P) = \frac{1}{4\pi\epsilon_0 d^2}\,\hat{n}\cdot\vec{p}$$

p is the dipole moment – vector quantity

$$\vec{p} = \int_V (\vec{r})\rho\, d\tau$$

$$= \int_S (\vec{r})\sigma\, dS = \int_L (\vec{r})\lambda\, dl = \sum_i q_i\vec{r}_i$$

Dipole moment is determined by the size, shape and charge density

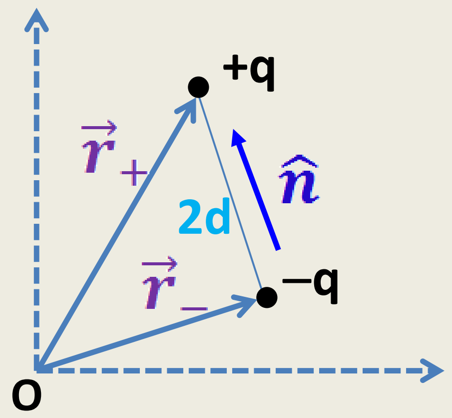
dipole moment

$Q_{tot} = 0$: dominant term : dipole

$$\vec{p} = \sum_i q_i \vec{r}_i = +q\vec{r}_+ + (-q)\vec{r}_-$$

$$= q(\vec{r}_+ - \vec{r}_-)$$

$$= q\, 2d\, \hat{n}$$



dipole moment

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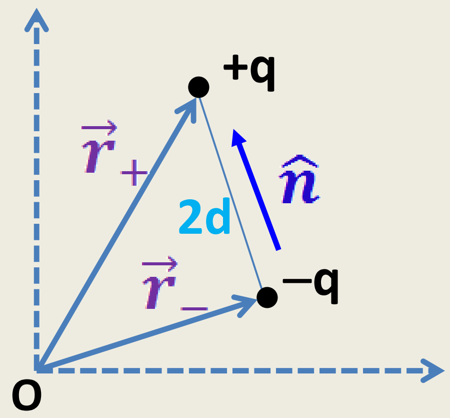
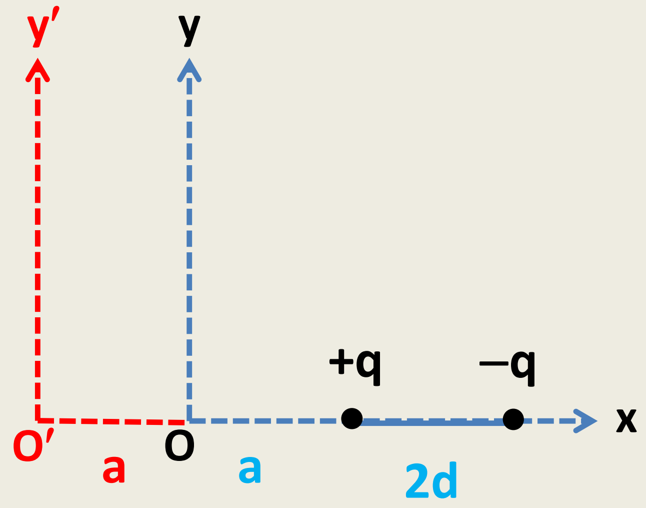




$$\vec{p} = \sum_i q_i \vec{r}_i = +qa\,\hat{\imath} + (-q)(a + 2d)\hat{\imath}$$

$$= 2qd(-\hat{\imath})$$

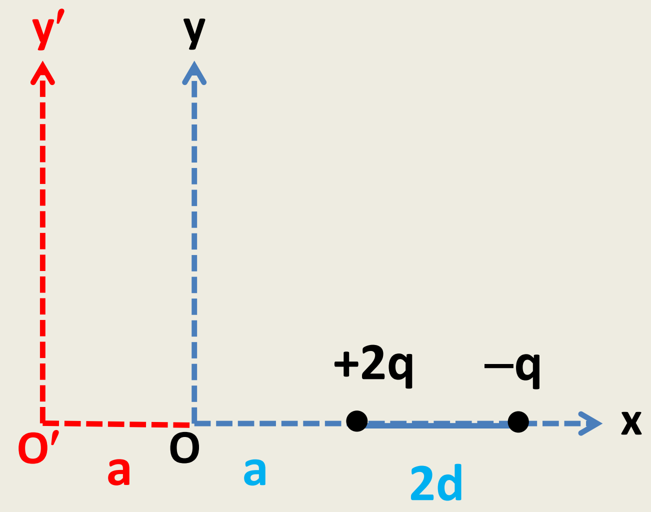
$$\vec{p} = \sum_i q_i \vec{r}_i = +q2a\,\hat{\imath} + (-q)(2a + 2d)\hat{\imath}$$

$$= 2qd(-\hat{\imath})$$

Here dipole moment is independent of origin

$$\vec{p} = \sum_i q_i \vec{r}_i = +2qa\,\hat{\imath} + (-q)(a + 2d)\hat{\imath}$$

$$= qa\hat{\imath} + 2qd(-\hat{\imath})$$



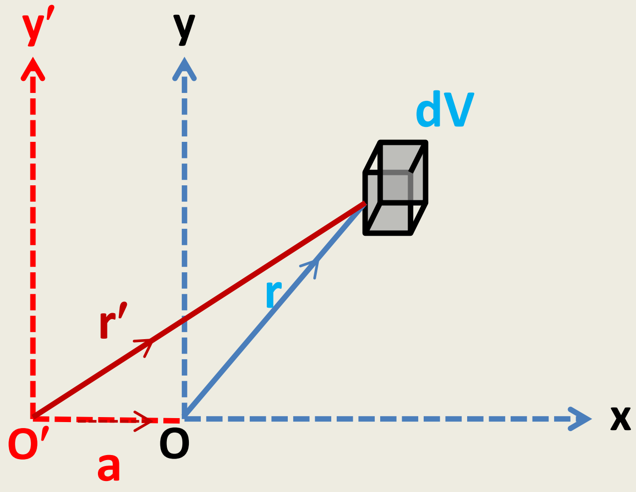
$$\vec{p} = +2q(2a)\,\hat{\imath} + (-q)(2a + 2d)\hat{\imath}$$

$$= 2qa\hat{\imath} + 2qd(-\hat{\imath})$$

Here dipole moment is dependent of origin

$$\vec{p}\,' = \int \vec{r}\,'\,\rho\,dV = \int (a\hat{\imath} + \vec{r})\,\rho\,dV$$

$$= a\hat{\imath}\int \rho\,dV + \int \vec{r}\;\;\rho\,dV$$



If Q_{tot} = 0, then

$$\vec{p}\,' = \vec{p}$$

Quadrupole moment

$$V(P) = \frac{1}{4\pi\epsilon_0 d}\int_V \rho\, d\tau + \frac{1}{4\pi\epsilon_0 d^2}\int_V (r\cos\omega)\rho\, d\tau + \underbrace{\frac{1}{4\pi\epsilon_0 d^3}\int_V \left(r^2\left[\frac{3}{2}\cos^2\omega - \frac{1}{2}\right]\right)\rho\, d\tau}_{3} + \cdots .$$

Quadrupole moment

Eg: spherical shell with charge density $\sigma = \sigma_0 \cos\theta$

$$V_{\text{out}}(r,\theta) = \frac{\sigma_0 R^3 \cos\theta}{3\varepsilon_0 r^2}$$

Monopole moment :

$$Q = \int \sigma\, dS = \int (\sigma_0 \cos\theta)(R^2 \sin\theta\, d\theta\, d\phi) = 0$$

Hence monopole potential is zero

Dipole moment :

$$\vec{p} = \int \vec{r}\,\sigma\, dS = \int (R\sin\theta\cos\phi\,\hat{i} + R\sin\theta\,\sin\phi\,\hat{j} + R\cos\theta\,\hat{k})(\sigma_0\cos\theta)(R^2\sin\theta\, d\theta\, d\phi)$$

x and y components will vanish due to integration of cosϕ (sinϕ) OR

if the charge distribution does not have a ϕ dependence, then dipole moment can have only z component!

$$\vec{p} = \hat{k}\int (R\cos\theta)\ (\sigma_0\cos\theta)(R^2\sin\theta\, d\theta\, d\phi) = \hat{k}\,\frac{4}{3}\pi R^3\sigma_0$$

$$V_{\text{dipole}} = (\hat{k}\cdot\hat{n})\frac{\frac{4}{3}\pi R^3\sigma_0}{4\pi\varepsilon_0 d^2}$$, where $\hat{n}$ is the unit vector along d

$$V_{\text{dipole}} = \frac{R^3\sigma_0}{3\varepsilon_0 d^2},$$ along z-axis

$$V_{\text{dipole}} = \frac{R^3\sigma_0\cos\theta}{3\varepsilon_0 d^2},$$ anywhere else where θ is the polar angle

Quadrupole potential $$= \frac{1}{4\pi\epsilon_0 d^3}\int_V \left(r^2\left[\frac{3}{2}\cos^2\omega - \frac{1}{2}\right]\right)\sigma\, dS$$

where r is the distance from origin to the area element dS (= R here)

Difficult to evaluate since ω is NOT a coordinate variable. Hence we will evaluate Quadrupole potential along z-axis where $\omega = \theta$

Quadrupole moment $$= \int_V \left(R^2\left[\frac{3}{2}\cos^2\theta - \frac{1}{2}\right]\right)(\sigma_0\cos\theta)\;(R^2\sin\theta\, d\theta\, d\phi)$$

$$= 0$$

Hence quadrupole potential is zero

DIELECTRICS

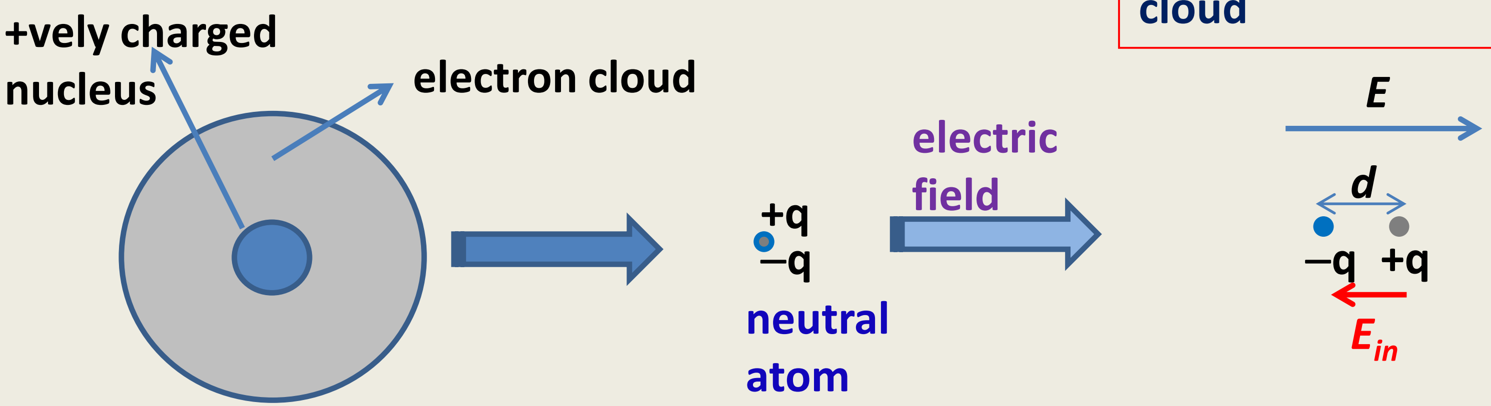
Dielectrics

Insulators – NO FREE electrons

What happens if a dielectric is placed in an e.f.?

Becomes polarised:
$\boldsymbol{p} = \alpha\, \boldsymbol{E}$ (approx.)

electron cloud retains the shape
nucleus is displaced by *d*
E_{in} – e.f. produced by the electron cloud



In equilibrium,
$E = E_{in}$

$$E_{\text{in}} = \frac{\rho d}{3\varepsilon_0} = \frac{qd}{4\pi\varepsilon_0 a^3}$$

in equilibrium

$$E = E_{\text{in}} = \frac{qd}{4\pi\varepsilon_0 a^3}$$

Each atom becomes a tiny dipole, moment $p = qd = 4\pi\varepsilon_0 a^3 E$

α : atomic polarizibility

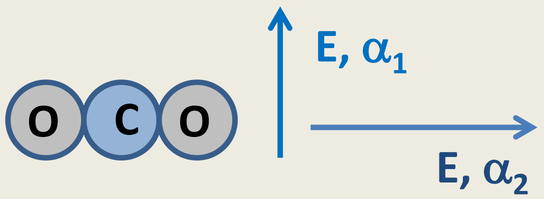
$$4\pi\varepsilon_0 a^3 = \alpha = 3\varepsilon_0 v$$

where v is the volume of the atom.

if a dielectric is placed in an e.f., neutral atoms get polarized and becomes tiny dipoles
All dipoles align along the direction of the applied e.f.
Dielectric gets POLARIZED. Polarization can be defined as
Polarisation = dipole moment per unit volume of the dielectric

Polarization, $\vec{P} = \dfrac{\vec{p}}{\tau}$

Molecules: behavior in electric field is not very simple



$\alpha_2 = 2.25\,\alpha_1$

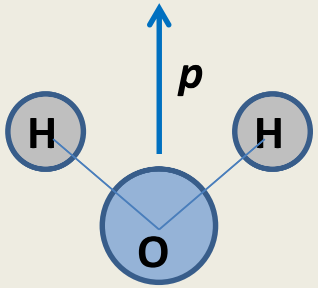
$$p = \alpha E$$

$$p_x = \alpha_{xx}E_x + \alpha_{xy}E_y + \alpha_{xz}E_z$$

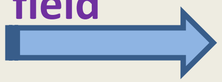
$$p_y = \alpha_{yx}E_x + \alpha_{yy}E_y + \alpha_{yz}E_z$$

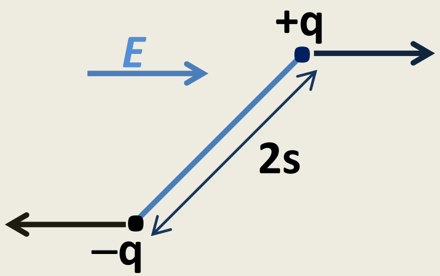
$$p_z = \alpha_{zx}E_x + \alpha_{zy}E_y + \alpha_{zz}E_z$$

Polar Molecules : built-in dipole moment even in the absence of e.f.



Uniform electric field





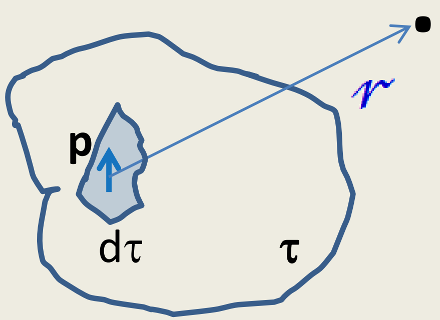
Torque $\vec{N} = \{\vec{s} \times q\vec{E} + (-\vec{s}) \times -q\vec{E}\}$

$= q2\vec{s} \times E \quad = \vec{p} \times \vec{E}$

This torque produces a rotation of the molecule

Field of a polarized object

Polarization, $\vec{P} = \dfrac{\vec{p}}{\tau}$



Potential due to one dipole in the dielectric

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{\hat{r}\cdot\vec{p}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\hat{r}\cdot\vec{P}}{r^2}\,d\tau \quad \text{in volume } d\tau$$

$$\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$$

$$V = \frac{1}{4\pi\varepsilon_0}\int_\tau \frac{\hat{r}\cdot\vec{P}}{r^2}\,d\tau = \frac{1}{4\pi\varepsilon_0}\int_\tau \vec{P}\cdot\vec{\nabla}\left(\frac{1}{r}\right)d\tau$$

$$\text{Here, } \vec{\nabla}\left(\frac{1}{r}\right) = \frac{\hat{r}}{r^2}$$

$$= \frac{1}{4\pi\varepsilon_0}\int_\tau \vec{\nabla}\cdot\left(\frac{1}{r}\vec{P}\right)d\tau - \frac{1}{4\pi\varepsilon_0}\int_\tau \frac{1}{r}(\vec{\nabla}\cdot\vec{P})\,d\tau$$

$$d\vec{S} = dS\,\hat{n}$$

$$= \frac{1}{4\pi\varepsilon_0}\int_S \frac{1}{r}\vec{P}.d\vec{S} - \frac{1}{4\pi\varepsilon_0}\int_\tau \frac{1}{r}(\vec{\nabla}\cdot\vec{P})\,d\tau$$

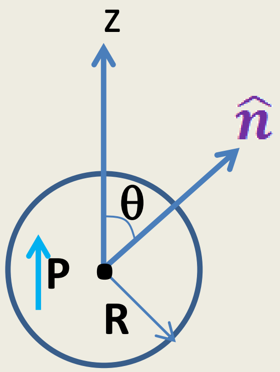
$$\sigma_b = \vec{P}.\hat{n}$$

$$\rho_b = -\vec{\nabla}\cdot\vec{P}$$

$$= \int_S \frac{\sigma_b\,dS}{4\pi\varepsilon_0 r} + \int_\tau \frac{\rho_b}{4\pi\varepsilon_0 r}\,d\tau$$

Potential due to bound surface charge density and bound volume charge density

E due to uniformly polarized sphere, $\vec{P} = P_0 \hat{k}$



$$\sigma_b = \vec{P}.\hat{n} = P_0 \cos\theta$$

$$\rho_b = \vec{\nabla} \cdot \vec{P} = 0$$

Equivalent to a spherical shell with surface charge density $\sigma = P_0 \cos\theta$

$$V(r,\theta) = \frac{P_0}{3\varepsilon_0} r\cos\theta = \frac{P_0}{3\varepsilon_0} z \qquad r \le R$$

$$V(r,\theta) = \frac{P_0}{3\varepsilon_0}\frac{R^3}{r^2}\cos\theta \qquad r \ge R$$

$$V_{out} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$ where $$\vec{p} = \frac{4}{3}\pi R^3 \vec{P}$$

Potential due to a dipole at the origin

$$E_{in} = -\frac{P_0}{3\varepsilon_0}\hat{k} = -\frac{\vec{P}}{3\varepsilon_0} \qquad r \le R$$

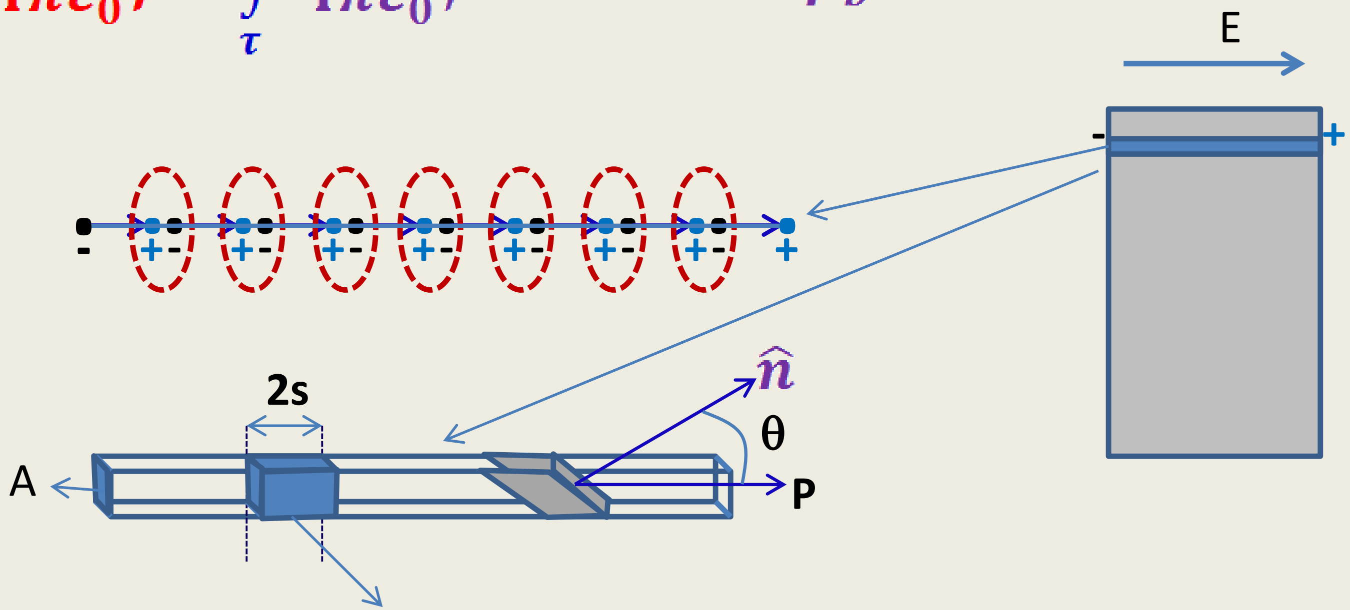
Physical interpretation of bound charges

Uniform polarization P

Potential due to a polarized object

$$V = \int_S \frac{\sigma_b dS}{4\pi\varepsilon_0 r} + \int_\tau \frac{\rho_b}{4\pi\varepsilon_0 r} d\tau$$

$$d\vec{S} = dS\,\hat{n}$$
$$\sigma_b = \vec{P}.\hat{n}$$
$$\rho_b = -\vec{\nabla}\cdot\vec{P}$$



Dipole moment = P (A 2s) = q (2s)

Bound charge at the end of the tube, $q = P A$

Surface charge density, $\sigma_b = q/A = P$

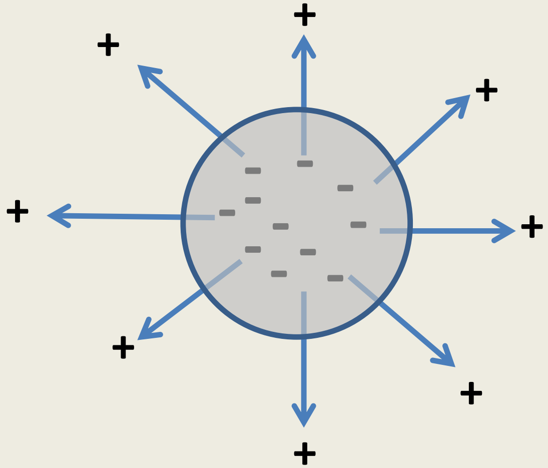
Surface charge density, $\sigma_b = P\cos\theta = \vec{P}.\hat{n}$

Physical interpretation of bound charges

NON - Uniform polarization *P*

accumulation of bound charges within the volume

Net bound charge in a given volume = amount that has been pushed out through the surface



$$\int_{\tau} \rho_b \, d\tau = -\int_{S} \vec{P} . d\vec{S} = -\int_{\tau} (\vec{\nabla} \cdot \vec{P}) d\tau$$

Since it is true for any volume

$$\rho_b = -(\vec{\nabla} \cdot \vec{P})$$

Uniform Polarization : ONLY surface bound charges (σ_b)

Non-uniform Polarization : surface & volume bound charges (σ_b & ρ_b)