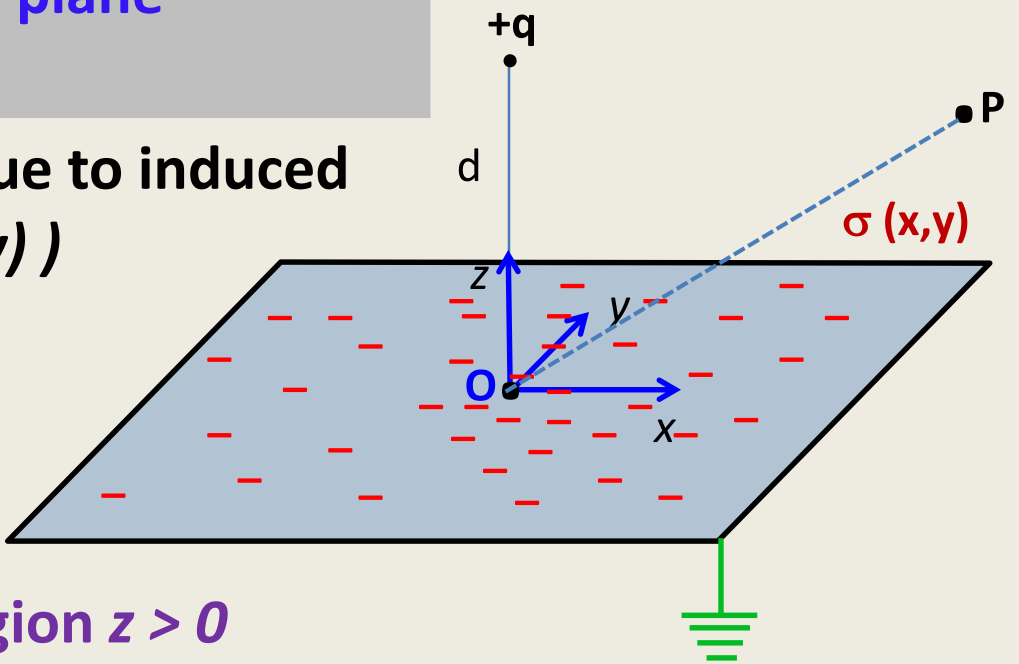
Method of Images

Point charge +q near an infinite plane CONDUCTOR

V(P) = (V(P) due to q) + (V(P) due to induced charges on the conductor $\sigma(x,y)$)



$\sigma(x,y)$ is NOT known

Solve Poisson's eqn. in a region $z > 0$ with a single point charge at $(0,0,d)$ and conductor at origin in the x-y plane

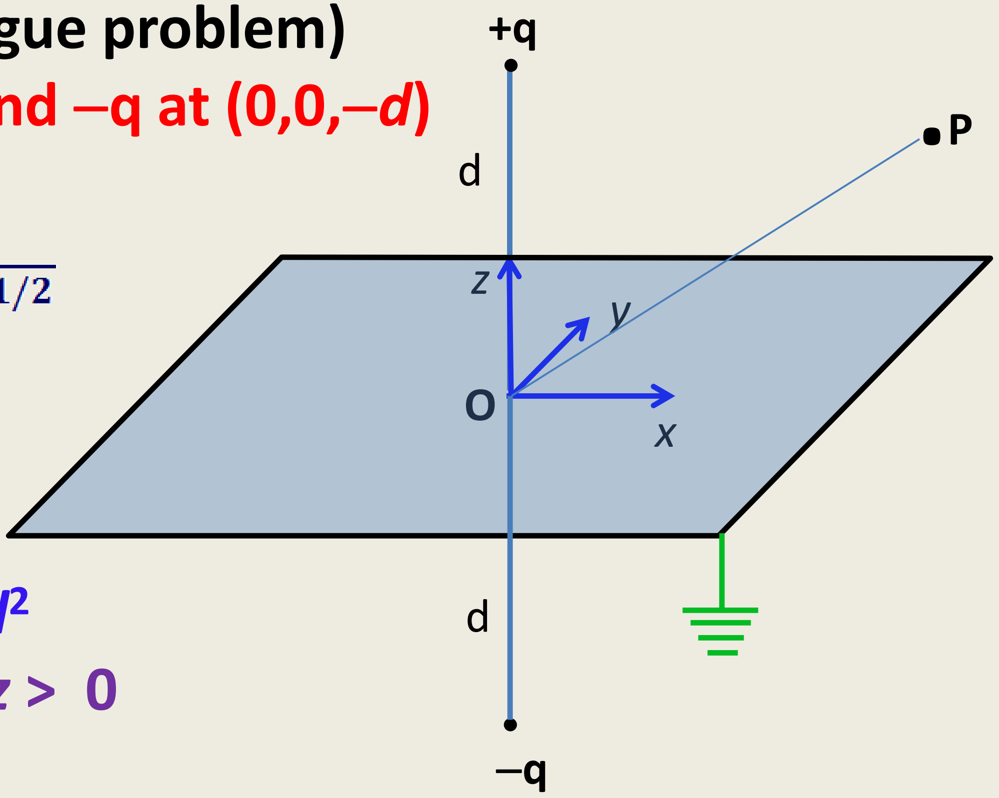
Boundary conditions:

(1) $V = 0$ at $z = 0$ (conductor is grounded)

(2) $V \to 0$ as $(x^2 + y^2 + z^2) >> d^2$

Solve a different problem (Analogue problem)

Two point charges +q at (0,0,d) and –q at (0,0,–d)



$$V(P) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{[x^2+y^2+(z-d)^2]^{1/2}} + \frac{-q}{[x^2+y^2+(z+d)^2]^{1/2}}\right\}$$

(i) $V = 0$ when $z = 0$

(ii) $V \to 0$ as $(x^2 + y^2 + z^2) >> d^2$

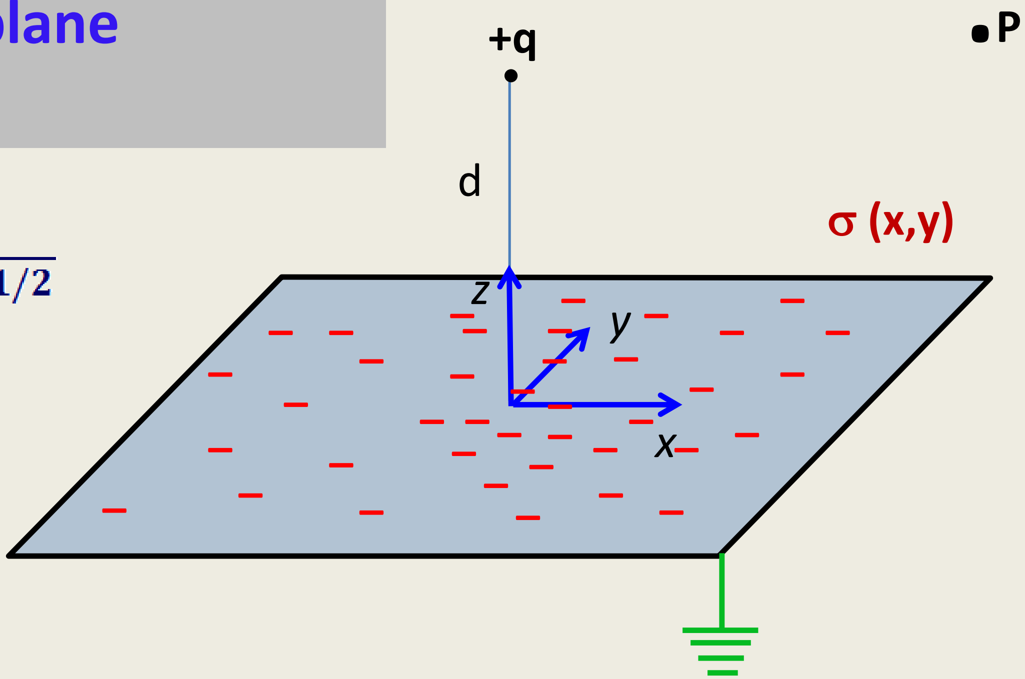
(iii) There is only +q at (0,0,d), $z > 0$

Now bring the infinite conducting sheet and keep in the x-y plane.
Adjust potential to be zero (grounding)

Top configuration has the same potential as the original problem.

Then this is the only solution (Uniqueness theorem)

Point charge +q near an infinite plane CONDUCTOR



$$V(P) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{[x^2+y^2+(z-d)^2]^{1/2}} + \frac{-q}{[x^2+y^2+(z+d)^2]^{1/2}}\right\}$$

Induced charge density $\sigma\,(x, y)$

$$\sigma(x,y) = -\varepsilon_0\frac{\partial V}{\partial n} = -\varepsilon_0\frac{\partial V}{\partial z}\bigg|_{z=0} = -\frac{qd}{2\pi}\frac{1}{(d^2+x^2+y^2)^{3/2}}$$

Total Induced charge q_{ind}

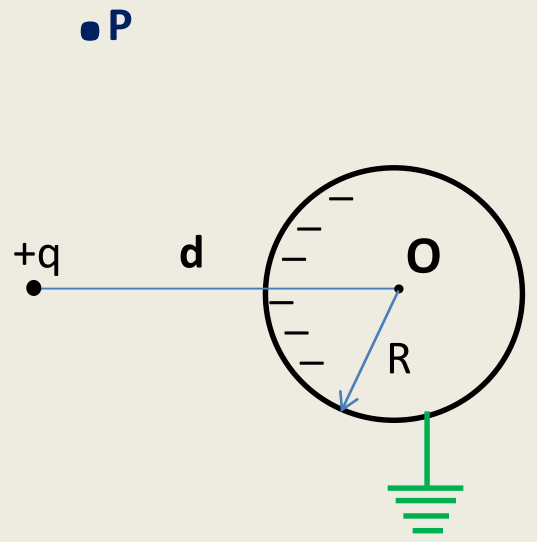
$$q_{\text{ind}} = \int \sigma dS = -\frac{qd}{2\pi}\int\frac{1}{(d^2+x^2+y^2)^{3/2}}dx\,dy = -q$$

Force of attraction

$$F = \frac{q\,(-q)}{4\pi\varepsilon_0(2d)^2}\ \text{(analogue problem)}$$

Point charge +q near an infinite plane CONDUCTING SPHERE

***V*(P) = *V*(P) due to q + *V*(P) due to induced charges on the conductor $\sigma(\theta, \phi)$**



$\sigma(\theta, \phi)$ **is NOT known**

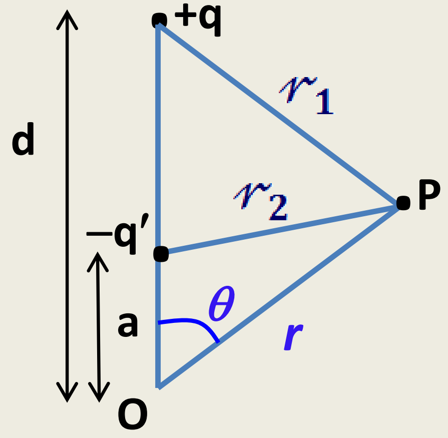
Solve Poisson's eqn. in a region $r > R$ with a single point charge at ($r = d$) and conductor with origin at the centre of the coordinate system

Boundary conditions:

(1) $V = 0$ at $r = R$ (conductor is grounded)

(2) $V \rightarrow 0$ as $r \rightarrow \infty$

Analogue problem



$$V(P) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_1} - \frac{q'}{r_2}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{(r^2 + d^2 - 2rd\cos\theta)^{1/2}} - \frac{q'}{(r^2 + a^2 - 2ra\cos\theta)^{1/2}}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{d\left(1 + \frac{r^2}{d^2} - 2\frac{r}{d}\cos\theta\right)^{1/2}} - \frac{q'}{r\left(1 + \frac{a^2}{r^2} - 2\frac{a}{r}\cos\theta\right)^{1/2}}\right)$$

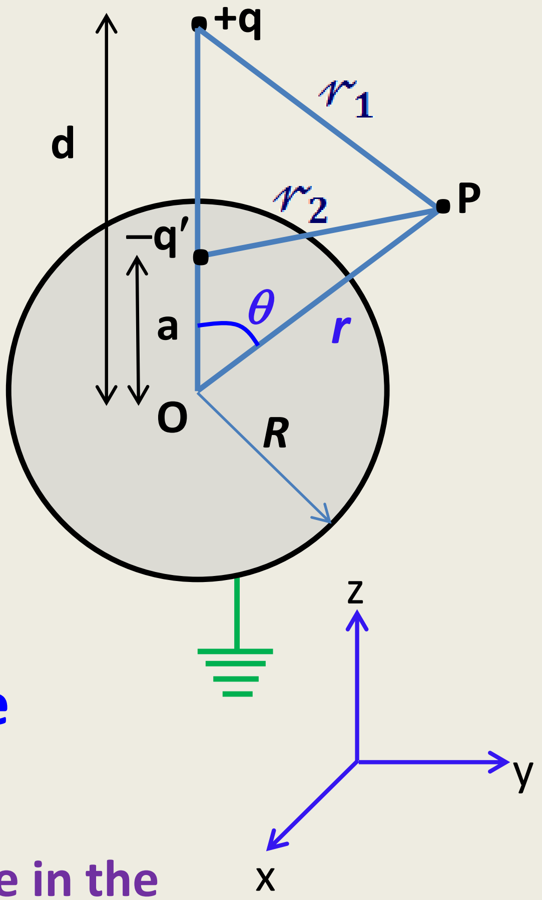
Let us make $V = 0$ at some point $r = R$

$$\frac{q}{d\left(1 + \frac{R^2}{d^2} - 2\frac{R}{d}\cos\theta\right)^{1/2}} = \frac{q'}{R\left(1 + \frac{a^2}{R^2} - 2\frac{a}{R}\cos\theta\right)^{1/2}}$$

Solve for *q'*

$$q' = \left\{q\frac{R}{d}\right\}\frac{\left(1+\frac{a^2}{R^2}-2\frac{a}{R}\cos\theta\right)^{1/2}}{\left(1+\frac{R^2}{d^2}-2\frac{R}{d}\cos\theta\right)^{1/2}}$$

For this to be true for any θ,
$q' = qR/d$ and $a = R^2/d$



$$V(r,\theta,\phi) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{(r^2+d^2-2rd\cos\theta)^{1/2}} - \frac{q'}{(r^2+a^2-2ra\cos\theta)^{1/2}}\right)$$

where

$$q' = \left\{q\frac{R}{d}\right\} \quad \text{and} \quad a = \frac{R^2}{d}$$

Image charge $q' = -q\,R/d$ is formed at a distance $a = R^2/d$ from the origin

Since the image charge is inside the sphere (*r* < *R*), no change in the conditions for the solution of Poisson's equation

image charge cannot be placed in a region where *V* is to be calculated

Induced charge σ

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n}\bigg|_{\text{surface}}$$

$$V(r,\theta,\phi) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{(r^2+d^2-2rd\cos\theta)^{1/2}} - \frac{q'}{(r^2+a^2-2ra\cos\theta)^{1/2}}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{(r^2+d^2-2rd\cos\theta)^{1/2}} - \frac{1}{\left(\frac{d^2r^2}{R^2}+R^2-2rd\cos\theta\right)^{1/2}}\right)$$

$$\frac{\partial V}{\partial r}\bigg|_{r=R} = \frac{-q}{4\pi\varepsilon_0}\left(\frac{(R-d\cos\theta)}{(R^2+d^2-2Rd\cos\theta)^{3/2}} - \frac{\left(\frac{d^2}{R}-d\cos\theta\right)}{(d^2+R^2-2rd\cos\theta)^{3/2}}\right)$$

$$\sigma = \frac{-q(d^2-R^2)}{4\pi R(R^2+d^2-2Rd\cos\theta)^{3/2}} = \sigma(\theta)$$

$$q_{\text{induced}} = \int \sigma\, dS = \frac{-qR}{d}$$

Force of attraction $$F = \frac{1}{4\pi\varepsilon_0}\frac{q(-q')}{(d-a)^2} = \frac{-q^2Rd}{4\pi\varepsilon_0(d^2-R^2)^2}$$

Point charge +q near a floating CONDUCTING SPHERE

Image charge $q' = -q\,R/d$ forms at distance $a = R^2/d$ from the centre, inside the sphere

Since the sphere is NOT grounded, the potential has to be raised to a value of $q/4\pi\varepsilon_0 d$

This has to be achieved by the left over +ve charges since +q and –q′ makes the potential on the surface to zero

This can be achieved by putting the left over charges $q'' = +q\,R/d$ at the centre of the sphere

$$\text{Then, } V = \frac{q''}{4\pi\varepsilon_0 R} = \frac{+q\,R/d}{4\pi\varepsilon_0 R} = \frac{+q}{4\pi\varepsilon_0 d}$$