Potential V due to a spherical shell

Electric field due to a spherical shell $\vec{E} = \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$ outside, E = 0 inside

Potential V outside

$$V(r) = -\int_{\infty}^{r} E.\,dr = \frac{\sigma R^2}{\varepsilon_0 r} = \frac{Q}{4\pi\varepsilon_0 r}$$

Total charge is centered at the origin

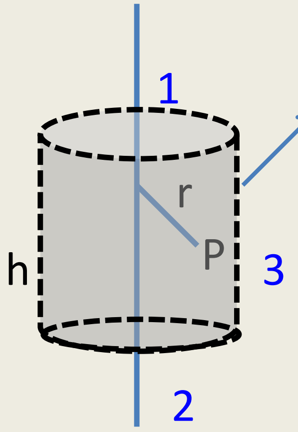
Potential V inside (depends on what the electric field outside is!)

$$V(r) = -\int_{\infty}^{R} E_{\text{out}}.\,dr - \int_{R}^{r} E_{\text{in}}.\,dr = \frac{\sigma R^2}{\varepsilon_0 R} = \frac{Q}{4\pi\varepsilon_0 R}$$

Potential is constant inside

Potential V due to an infinite wire (λ), cylinder (ρ)

(A). Electric field at P at a distance *r* from the wire

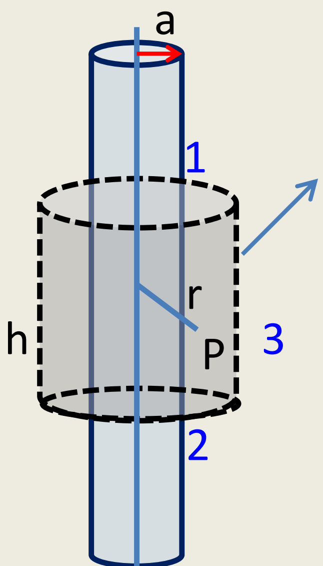


Draw Gaussian surface (cylinder of radius *r* and height *h*, enclosing the wire) such that the P is on the surface.

From symmetry, $\vec{E} = E\hat{r}$ ⇒ **flux through flat surfaces (1 & 2) = 0, E ⊥ dS**

$$\oiint \vec{E}\cdot\vec{ds} = E\,2\pi r h = \frac{\lambda h}{\varepsilon_0} \Longrightarrow \vec{E} = \pm\hat{r}\,\frac{\sigma}{2\pi\varepsilon_0 r}$$

(B). E due to infinite cylinder, volume charge density ρ



(i) E outside

$$\oiint \vec{E}\cdot\vec{ds} = E\,2\pi r h = \frac{\rho\pi a^2 h}{\varepsilon_0} \Longrightarrow \vec{E}_{\text{out}} = \hat{r}\,\frac{\rho a^2}{2\varepsilon_0 r}$$

(ii) E inside

$$\oiint \vec{E}\cdot\vec{ds} = E\,2\pi r h = \frac{\rho\pi r^2 h}{\varepsilon_0} \Longrightarrow \vec{E}_{\text{in}} = \hat{r}\,\frac{\rho r}{2\varepsilon_0}$$

Potential V due to an infinite wire λ

$$\vec{E}(r) = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$$

$$V(r) = -\int_{\infty}^{r} E.\,dr = -\frac{\lambda}{2\pi\varepsilon_0}\ln r|_{\infty}^{r}$$

Infinity as a reference point is NOT good;
only a potential difference can be found in this case

$$V(r) = -\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{r}{a}$$ **where *r = a* is a point where the potential is well defined**

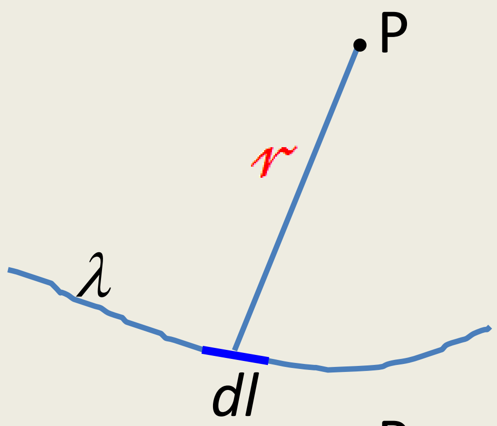
with the condition that

$$\vec{E} = -\vec{\nabla}V = \frac{\lambda}{2\pi\varepsilon_0}\frac{\partial}{\partial r}\left(\ln\frac{r}{a}\right)\hat{r} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$$

Potential due to CONTINUOUS CHARGE DISTRIBUTIONS

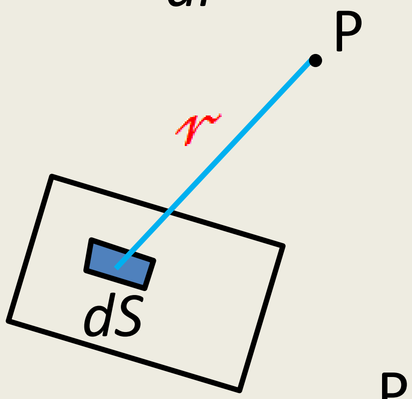
LINE CHARGE

$$V(P) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda dl}{r}$$



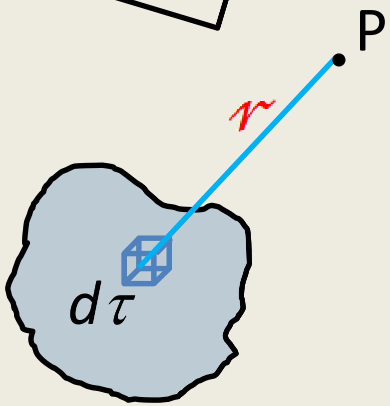
SURFACE CHARGE

$$V(P) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma dS}{r}$$

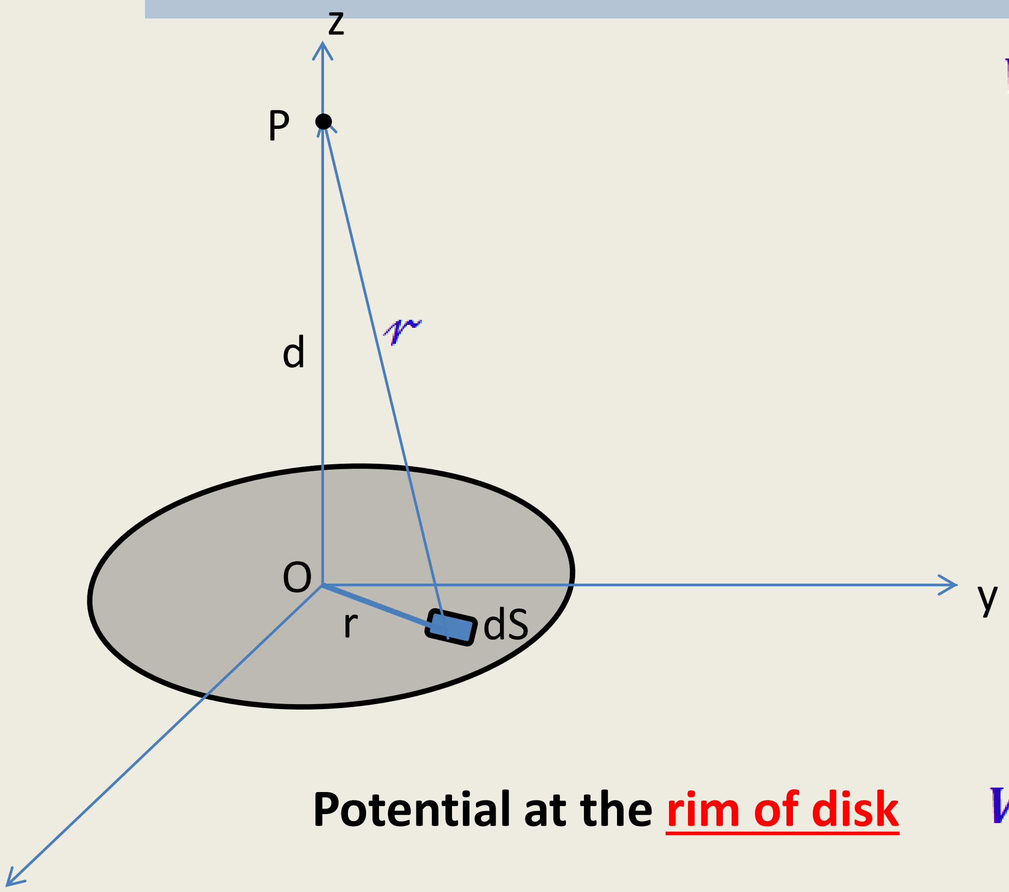


VOLUME CHARGE

$$V(P) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho d\tau}{r}$$



Potential due to uniformly charged disk (σ), radius *a*, along the axis



$$V(P) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma dS}{r}$$

$$= \frac{\sigma}{4\pi\varepsilon_0}\iint \frac{r dr d\phi}{\sqrt{(r^2+d^2)}}$$

$$= \frac{\sigma}{2\varepsilon_0}\left\{\sqrt{a^2+d^2} - d\right\}$$

Potential at the centre of disk, *d = 0*

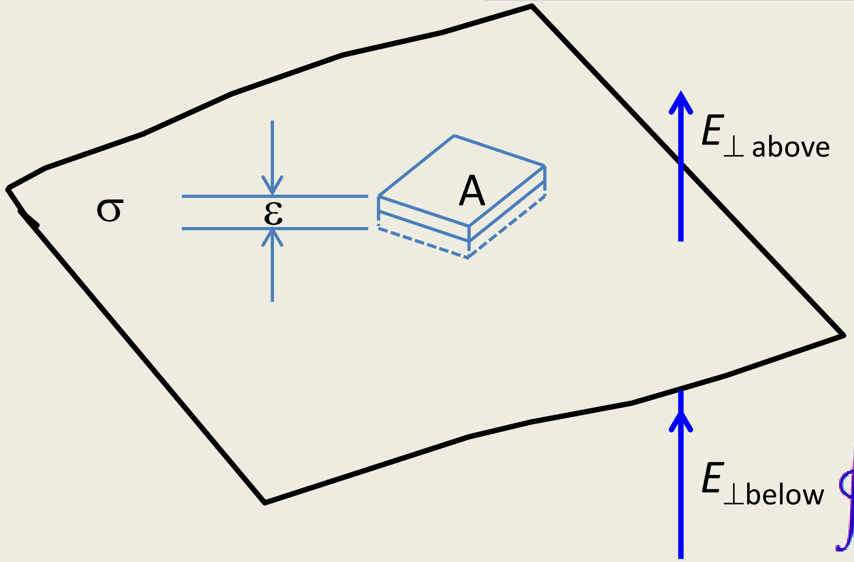
$$V(O) = \frac{\sigma a}{2\varepsilon_0} = \text{constant}$$

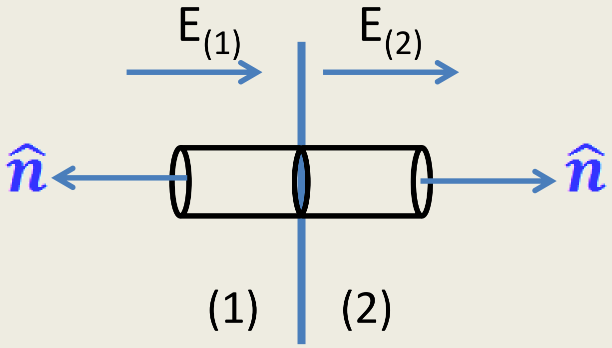
Potential at the rim of disk $V = \frac{\sigma a}{\pi\varepsilon_0}$ *Check This!! (Hint: Use a suitable co-ordinate system)*

Potential is max. at the centre of disk and decreases towards the rim! DISK is NOT an EQUIPotential surface

If d >> a $V = \frac{\sigma a^2}{4\varepsilon_0 d} = \frac{Q}{4\pi\varepsilon_0 d} = \text{potentail due to pt.charge}$

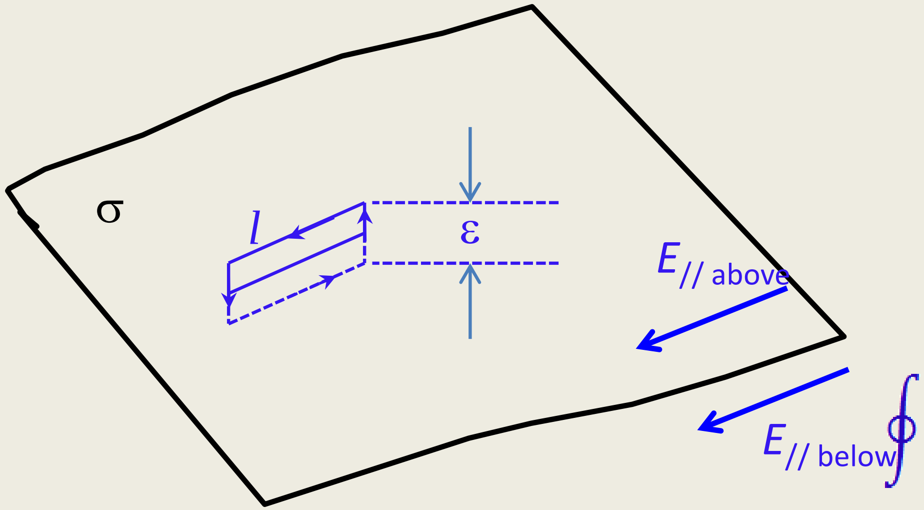
Electrostatic boundary conditions

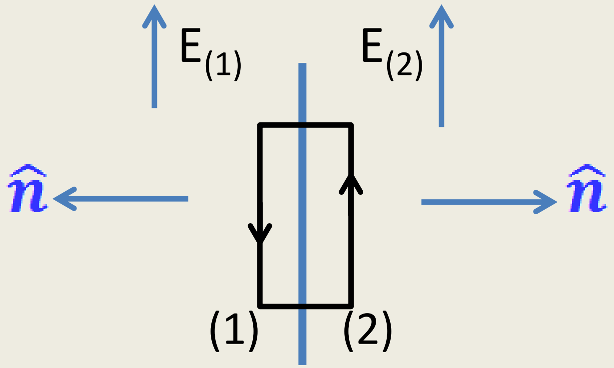




$$\oiint \vec{E}\cdot\vec{ds} = (E_{2\perp} - E_{1\perp})A = \frac{Q}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

NORMAL component of E is discontinuous by an amount σ/ε_0 across a surface charge





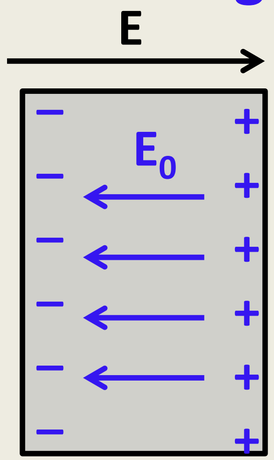
$$\oint \vec{E}.\vec{dl} = 0 \implies (E_{2\parallel} - E_{1\parallel})L = 0 \implies E_{2\parallel} = E_{1\parallel}$$

PARALLEL component of E is continuous

CONDUCTORS

Electrical conductor : large number of free e^- s; $10^{28}/m^3$; but uncharged

What happens when a conductor is placed in an electric field



- e^- s move to one side
- Only e^- s are mobile; +ve ions are left at other end
- Charges pile up; +ve on one side, –ve on other side
- Movements of charges occur till $E_0 = E$
- E_0 exactly cancels the field inside the conductor

Properties of conductors

- (i) E = 0 inside (material) of a conductor
- (ii) V = const in the volume of the conductor : EQUIPOTENTIAL surface

$$\vec{E} = -\vec{\nabla}V = 0 \implies V = \text{const}$$

$$-\int_a^b \vec{E}\cdot\vec{dl} = V(a) - V(b) = 0 \implies V(a) = V(b)$$

- (iii) Charge density $\rho = 0$ inside a conductor $\quad \nabla\cdot E = \dfrac{\rho}{\varepsilon_0} = 0 \implies \rho = 0$

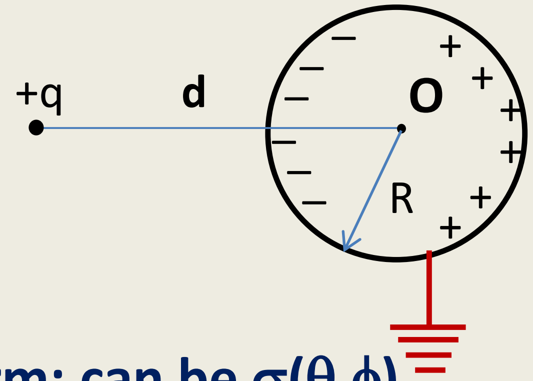
No net charge; NO free charges can reside inside volume of a metal

• (iv) **Any added charge can reside only on the surface :** **only surface charge density**

• **(v) E should be ⊥ surface just outside :** **any tangential component will make the charges flow.**

$$\vec{E} = \frac{\sigma}{\varepsilon_0}\,\hat{n}$$

Point charge near a conductor



- **conductor will move charges around such that E = 0 inside**
- **Since –ve charges are closer to +q, net attraction**
- **Charge density on the conductor will be non-uniform; can be σ(θ,ϕ)**

Potential on the conductor

- $$V(O) = \frac{q}{4\pi\varepsilon_0 d} + \iint \frac{\sigma(\theta,\phi)}{4\pi\varepsilon_0 R}\,dS = \frac{q}{4\pi\varepsilon_0 d} + \frac{1}{4\pi\varepsilon_0 R}\iint \sigma(\theta,\phi)\,dS$$

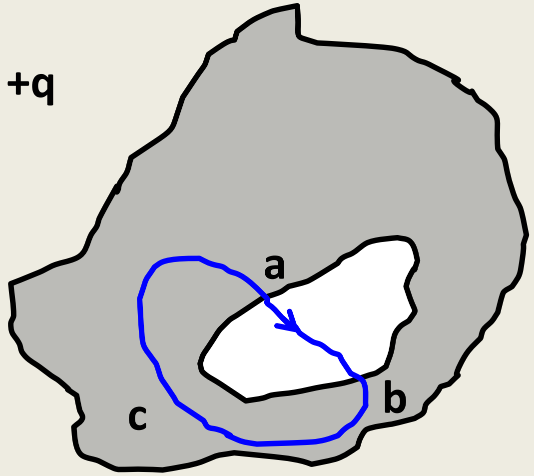
$$= \frac{q}{4\pi\varepsilon_0 d}$$

Earthing (grounding) the conductor

Cavity inside a conductor

(A) Charge outside the conductor:

- Since E = 0 inside the conductor, E is cancelled at the outer surface
- E = 0 inside the conductor ; inside the cavity
- For electrostatic field, $\oint \vec{E} \cdot \vec{dl} = 0$



$$\oint \vec{E} \cdot \vec{dl} = \int_{a \to b} \vec{E} \cdot \vec{dl} + \int_{b \to c \to a} \vec{E} \cdot \vec{dl} = 0$$

$$\int_{b \to c \to a} \vec{E} \cdot \vec{dl} = 0 \text{ since } E = 0 \text{ inside conductor}$$

Hence to make $\int_{a \to b} \vec{E} \cdot \vec{dl} = 0,$ $\vec{E}$ has to be zero

$$V(r_b) - V(r_a) = -\int_{r_a}^{r_b} \vec{E} . \vec{dl}$$

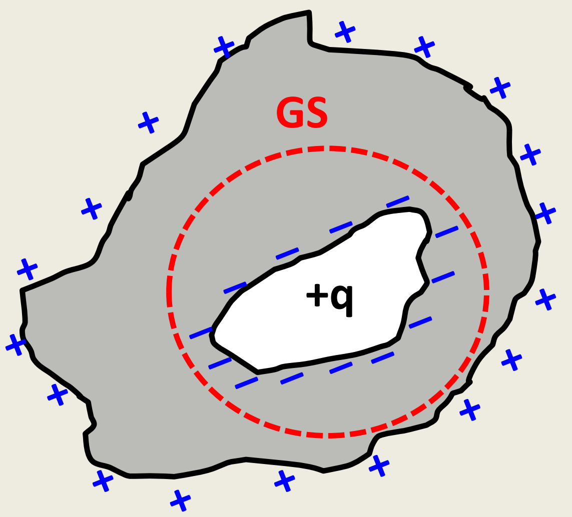
Since $V(r_b) = V(r_a)$ for a conductor (equipotential surface)

$$\int_{a \to b} \vec{E} \cdot \vec{dl} = 0$$

- cavity and contents are electrically shielded from outside electric field

(B) Charge INSIDE cavity:

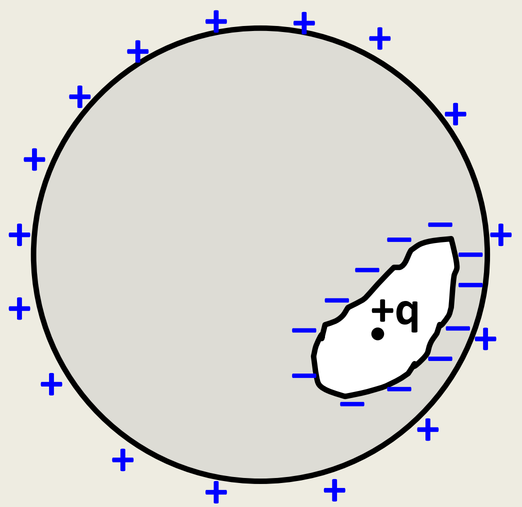
- $\mathrm{E} \neq 0$ Inside the cavity!
- E = 0 inside the conductor; E is cancelled at the cavity walls
- Conductor will move charges around such that E = 0 inside
- –q will be induced on the Cavity wall
- +q will be distributed at the outer surface



Cavity inside a spherical conductor

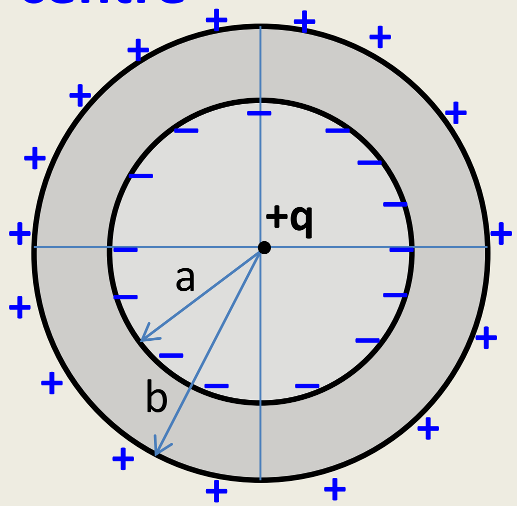
- +q inside cavity
- −q induced on the cavity wall
- +q induced on the outer surface : uniform
- +q inside cavity and −q induced

completely kills off the electric field inside the metal



$$E_{\text{out}} = \frac{q}{4\pi\varepsilon_0 r^2}; \quad V_{\text{out}} = \frac{q}{4\pi\varepsilon_0 r}$$

Spherical shell (conductor) with charge +q at the centre



Surface charge density

$$\sigma_a = \frac{-q}{4\pi a^2} \qquad \sigma_b = \frac{+q}{4\pi b^2}$$

$$\vec{E}_{\mathrm{r<a}} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$$

$$\vec{E}_{\mathrm{a<r<b}} = 0$$

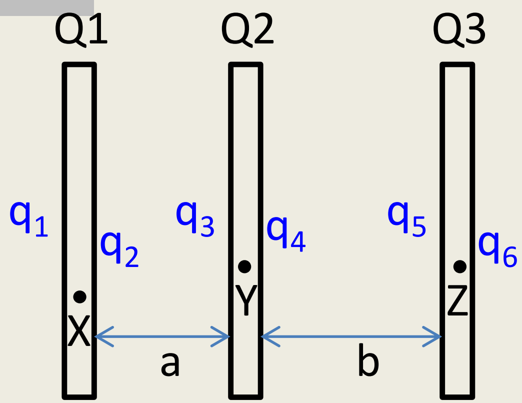
$$\vec{E}_{\mathrm{r>b}} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$$

$$V_{\mathrm{r>b}} = \frac{q}{4\pi\varepsilon_0 r}$$

$$V_{\mathrm{a<r<b}} = \frac{q}{4\pi\varepsilon_0 b} = \mathrm{const}$$

$$V_{\mathrm{r<a}} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{b} + \frac{1}{r} - \frac{1}{a}\right)$$

Parallel plate CONDCUTORS

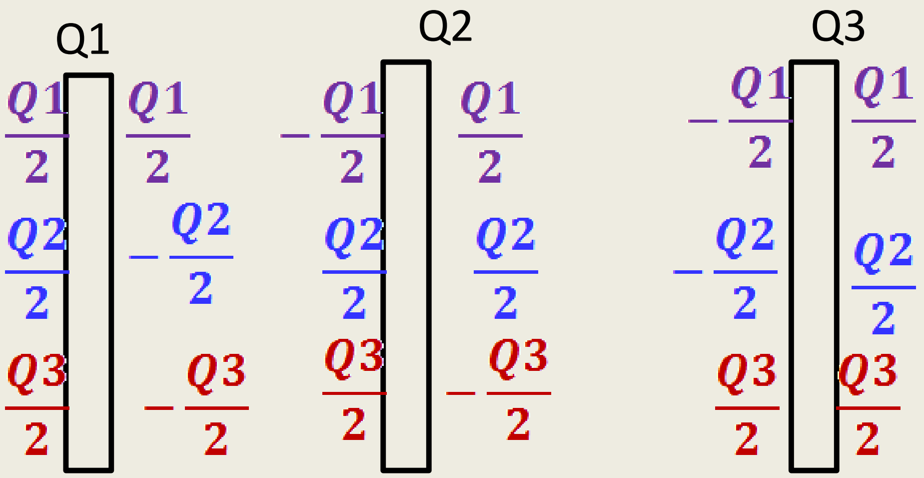


$$E(X) = \frac{1}{A\varepsilon_0}(q_1 - q_2 - q_3 - q_4 - q_5 - q_6) = 0$$

$$(q_1 - [Q1 - q_1] - q_3 - [Q2 - q_3] - q_5 - [Q3 - q_5]) = 0$$

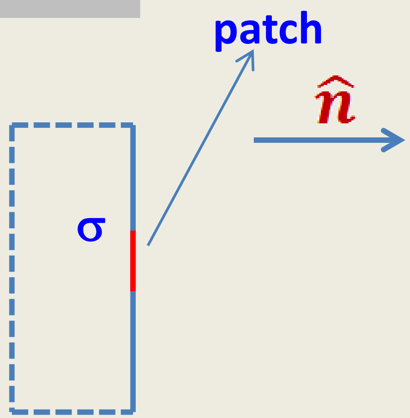
$$q_1 = \frac{1}{2}(Q1 + Q2 + Q3) \qquad q_2 = \frac{1}{2}(Q1 - Q2 - Q3)$$

Similarly, for E(Y), we can get q_3 and q_4 & E(Z), we can get q_5 and q_6



- **Outermost surfaces divide the whole charge equally among them**
- **Inner surfaces carry equal and opposite charges and cancel each other**

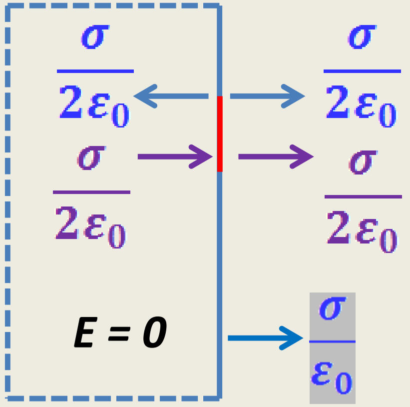
Force on the surface charge of a CONDUCTOR



Consider the patch as a charge sheet

$$E_{\text{patch}} = +\frac{\sigma}{2\varepsilon_0} \text{ right} \qquad E_{\text{patch}} = -\frac{\sigma}{2\varepsilon_0} \text{ left}$$

Then how do we get the asymmetric electric field of a conducting surface? i.e., zero immediately inside and non-zero immediately outside ?



We have to add an extra e.f. (which is not generated locally by the patch). The required field is

$$E' = \frac{\sigma}{2\varepsilon_0} \text{ right and left}$$

Since the e.f. generated locally by the Patch cannot exert a force on itself, the force on the patch is due to E′

$$f = \frac{\sigma^2}{2\varepsilon_0}\hat{n}$$ acts outwards!

Electrostatic pressure (acting outwards) on any charged conductor : $P = \frac{\varepsilon_0}{2}E^2$

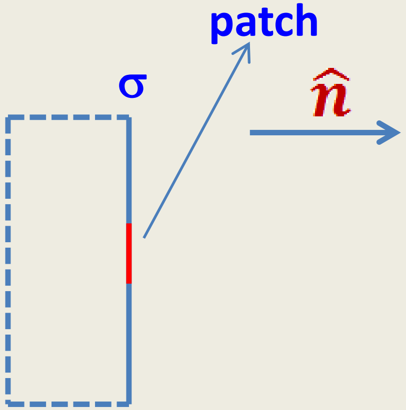
That is why a charged soap bubble bursts?

Force on the surface charge of a CONDCUTOR

Electric field just outside a conductor , $\vec{E} = \dfrac{\sigma}{\varepsilon_0}\,\hat{n}$

Hence $\dfrac{\partial V}{\partial n} = -\dfrac{\sigma}{\varepsilon_0}$ $\qquad \sigma = -\varepsilon_0 \dfrac{\partial V}{\partial n}\Big|_{\text{surface}}$

Force on the surface charge of a CONDCUTOR



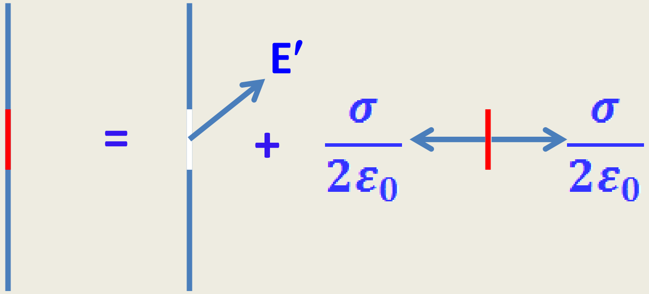
When electric field is present, surface charge will experience a force, F = QE

Force per unit area f = σE

Let us calculate the force on a small patch

But E is discontinuous at the surface charge. ie, E = 0 inside the conductor and σ/ε_0 outside . So which field?

$E_{tot} = E_{patch} + E'$, where E′ is field at the patch due to everything else



Patch cannot exert a force on itself. Hence the force on the patch is due to E′

$$\vec{E}_{\text{right}} = E' + \frac{\sigma}{2\varepsilon_0}\hat{n} \qquad \vec{E}_{\text{left}} = E' - \frac{\sigma}{2\varepsilon_0}\hat{n}$$

$$\vec{E}' = \frac{1}{2}\left(\vec{E}_{\text{right}} + \vec{E}_{\text{left}}\right) = \vec{E}_{\text{average}} = \frac{\sigma}{2\varepsilon_0}\hat{n}$$

$$f = \frac{\sigma^2}{2\varepsilon_0}\hat{n}$$ acts outwards!

Electrostatic pressure : $P = \frac{\varepsilon_0}{2}E^2$

Special techniques for calculating potentials

Uniqueness theorem for potential

$$V = \int \frac{1}{4\pi\varepsilon_0 r}\sigma\, dS$$

For a conductor σ is complicated

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}, \quad \vec{E} = -\vec{\nabla}V \text{ and hence } \nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

Poisson's Equation

$$\nabla^2 V = 0 \text{ in a charge free region}$$

Laplace's Equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\text{In 1-D,} \quad \frac{d^2V}{dx^2} = 0 \quad \Rightarrow \quad V = mx + c$$

Boundary conditions (which?)

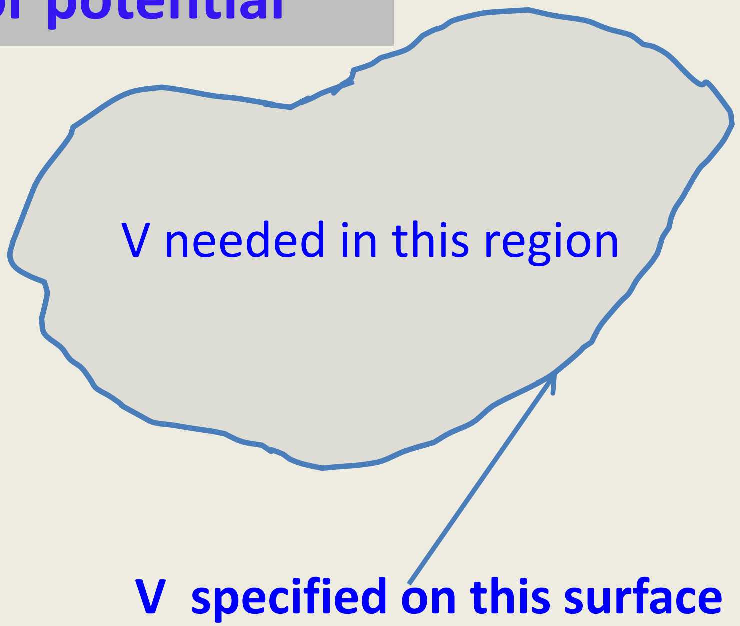
- V(x) is the average of V(x+R) and V(x-R)
- Laplace's Equation implies no local maxima or minima; extreme values of V must occur at end points only

Uniqueness theorem for potential

Uniqueness theorem:

(i) If a solution can be found (it does not matter which way) that satisfies Laplace's equation (OR Posisson's eqn)

(ii) If it has the correct value on the boundary



Then the solution is UNIQUE

Let V_1 and V_2 are 2 solutions. Then $\nabla^2 V_1 = 0 = \nabla^2 V_2$

Let $V_3 = V_1 - V_2$ Then $\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0$

Also, $V_3 = 0$ at the boundary, since $V_1 = V_2$ at the boundary. But Laplace's equation permits no local extrema, it can occur only at the boundaries.

Hence V_3 is zero everywhere.

Special techniques of calculating potentials : Separation of variables

Solving Laplace's equations

(i) Cartesian coordinates

Laplace's Equation in Cartesian coordinates is given by (2-D)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Look for solutions of the type $V(x, y) = X(x)\, Y(y)$

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$ divide by $V(x,y)$ $\rightarrow$ $$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} = 0$$

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -\frac{1}{Y}\frac{d^2 Y}{dy^2}$$

$$\frac{d^2 X}{dx^2} = kX, \quad \frac{d^2 Y}{dy^2} = -kY$$

Take the constant as k^2

$$\frac{d^2X}{dx^2} - k^2X = 0, \qquad \frac{d^2Y}{dy^2} + k^2Y = 0$$

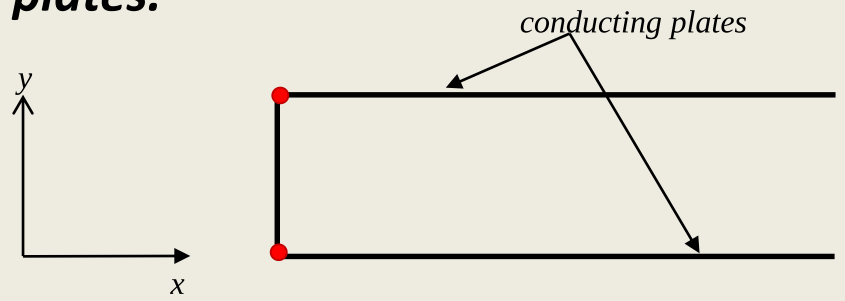
Solutions are

$$X(x) = Ae^{kx} + Be^{-kx}, \qquad Y = C\sin(ky) + D\cos(ky)$$

$$V(x,y) = X(x)\,Y(y) = \{Ae^{kx} + Be^{-kx}\}\{C\sin(ky) + D\cos(ky)\}$$

where A, B, C, D are constants to be determined from the boundary conditions of the given problem.

Example : Consider two semi-infinite, grounded, conducting plates lying parallel to the x-z plane, one at $y = 0$, and the other at $y = \pi$, as shown in the figure. The left end, at $x = 0$, is closed off by an infinite strip insulated from the two plates, and maintained at a specified potential $V_0(y)$. We have to find the potential in the region between the plates.



Solution : V is z-independent; 2-D Laplace's solution

Boundary conditions are

$$(1).\ V(x,0) = 0, \qquad (2).\ V(x,\pi) = 0$$

$$(3).\ V(0,y) = V_0(y), \text{ for } 0 \le y \le \pi$$

$$(4).\ V(x,y) \to 0 \text{ as } x \to \infty$$

$$V(x,y) = \{Ae^{kx} + Be^{-kx}\}\{C\sin(ky) + D\cos(ky)\}$$

$$(1).\ V(x,0) = 0, \qquad (2).\ V(x,\pi) = 0$$

$$(3).\ V(0,y) = V_0(y),\ \text{for}\ \ 0 \le y \le \pi$$

$$(4).\ V(x,y) \to 0\ as\ \ x\ \to \infty$$

$$V(x,y) = \{Ae^{kx} + Be^{-kx}\}\{C\sin(ky) + D\cos(ky)\}$$

(4) $\Rightarrow$ A = 0 (1) $\Rightarrow$ D = 0

(2) $\Rightarrow$ sin(kπ) = 0 $\Rightarrow$ k is an integer, say *n*

$$V(x,y) = \{Be^{-nx}\}\{C\sin(ny)\}$$

General solution:

$$V(x,y) = \sum_{n=1}^{\infty} E_n\, e^{-nx} \sin(ny)$$

satisfies boundary conditions 1, 2 and 4

Apply boundary conditions 3

$$V(0,y) = V_0(y) = \sum_{n=1}^{\infty} E_n \sin(ny)$$

Fourier sine series

Now we have to choose E_n that fits an arbitrary function $V_0(y)$

Multiply both sides by $\sin(n'y)$ and integrate over *y* from 0 to π

$$\int_0^{\pi} V_0(y)\sin(n'y)\,dy = \sum_{n=1}^{\infty} E_n \int_0^{\pi} \sin(ny)\sin(n'y)\,dy$$

From the orthogonality property of sine functions,

$$\int_0^{\pi} \sin(ny)\sin(n'y)\,dy = \frac{\pi}{2}\delta_{nn'}$$

$$\delta_{nn'} = 1 \text{ if } n = n'$$
$$= 0 \text{ if } n \neq n'$$

$$E_n = \frac{2}{\pi}\int_0^{\pi} V_0(y)\sin(ny)\,dy$$

Now we have to get functional form of $V_0(y)$

As __application__ of the above formalism: *Take $V_0(y)$ as a constant potential V_0*

$$E_n = \frac{2}{\pi} V_0 \int_0^{\pi} \sin(ny)\, dy = \frac{2}{\pi} V_0 \frac{1}{n} \{1 - \cos(n\pi)\}$$

$$E_n = 0, \text{ if } n \text{ is even}$$

$$= \frac{4V_0}{n\pi}, \text{ if } n \text{ is odd}$$

General solution:

$$V(x, y) = \sum_{n=1}^{\infty} E_n\, e^{-nx} \sin(ny)$$

$$\boxed{V(x, y) = \frac{4V_0}{\pi} \sum_{n=1,3,5\ldots}^{\infty} \frac{e^{-nx} \sin(ny)}{n}}$$