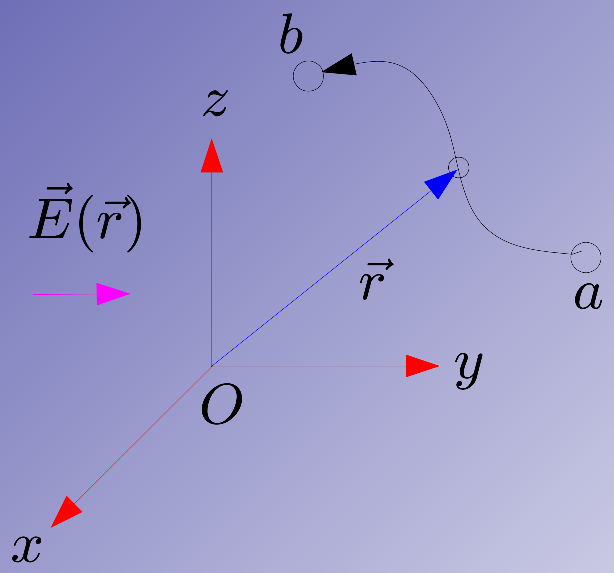
Work and Energy in Electrostatics



Work and Energy in Electrostatics

Outline

I. Work done in moving a charge in an electric field

II. Electrostatic energy of assemblies of discrete and continuous charge distributions

III. Electrostatic energy of a charged solid sphere

Work and Energy in Electrostatics

Learning Objectives

I. To learn about work done in moving a charge in an electric field.

II. To learn to calculate the work done in assembling a discrete or continuous charge distribution and the corresponding electrostatic energy.

Work and Energy in Electrostatics

Learning Outcomes

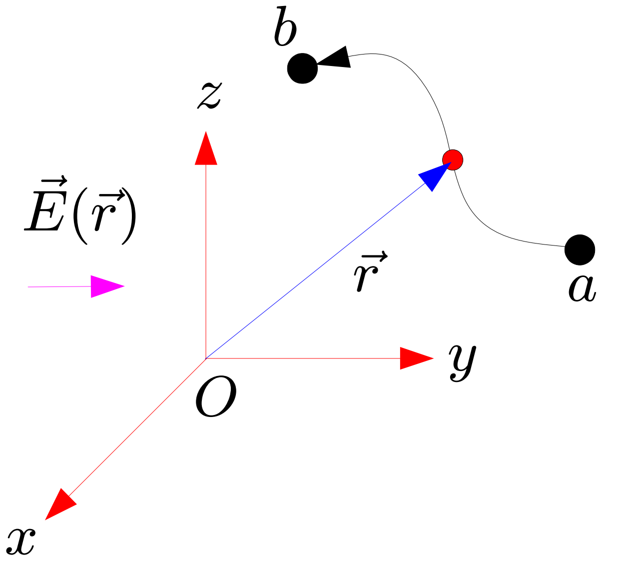
I. To be able to calculate work done in moving a charge in a given electric field.

II. To be able to calculate the electrostatic energy of a charged configuration- discrete or continuous.

Work and Energy in Electrostatics

Consider moving a test charge *q* from point $a \to b$ in the presence of an electric field $\vec{E}$. The mechanical work done on the charge *q* is

$$W = \int_{\mathrm{a}}^{\mathrm{b}} \vec{F}_{\mathrm{m\ ech}}(\vec{r}) \cdot d\vec{l}$$



where $\vec{F}_{\mathrm{m\ ech}}(\vec{r}) = -q\vec{E}(\vec{r})$

So that, the work done on *q* is equal to

$$W = -q \int_{\mathrm{a}}^{\mathrm{b}} \vec{E}(\vec{r}) \cdot d\vec{l}$$

Since, $\triangle V_{\mathrm{ab}} = V_{\mathrm{b}} - V_{\mathrm{a}} = -\int_{\mathrm{a}}^{\mathrm{b}} \vec{E}(\vec{r}) \cdot d\vec{l}$

$$\therefore W = q[V_{\mathrm{b}} - V_{\mathrm{a}}] = q\triangle V_{\mathrm{ab}}$$

Work and Energy in Electrostatics

Taking $\vec{r}_\mathrm{a} = \infty$ and $V(\vec{r}_\mathrm{a}) = V(\infty) = 0$, the work done on q becomes

$W = qV(\vec{r}),$ where we have taken $\vec{r}_\mathrm{b} = r$.

Thus, the work done in moving a charge q from ∞ to r is

$$\boxed{W = qV(\vec{r})}$$

which is also equal to the potential energy of the charge q.

SI units of work, potential energy and potential difference

Work: Joules (J)

Potential energy: Coulomb-Volts=Joules

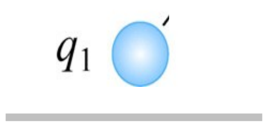
Potential difference: Volts=Joules/Coulomb

$$1\,eV = 1.602 \times 10^{-19}\,Joules$$

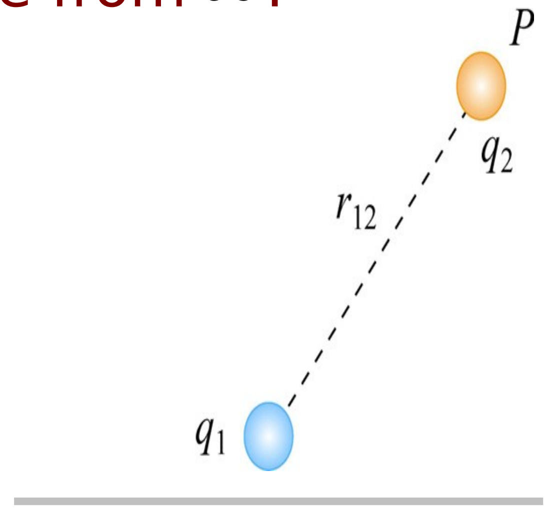
Work and Energy in Electrostatics

Electrostatic energy of assembly of a point charge distribution:

Consider moving charges from ∞ to $\vec{r}_i$, one by one, as shown in the figure. What is the work done in bringing in each charge from ∞?

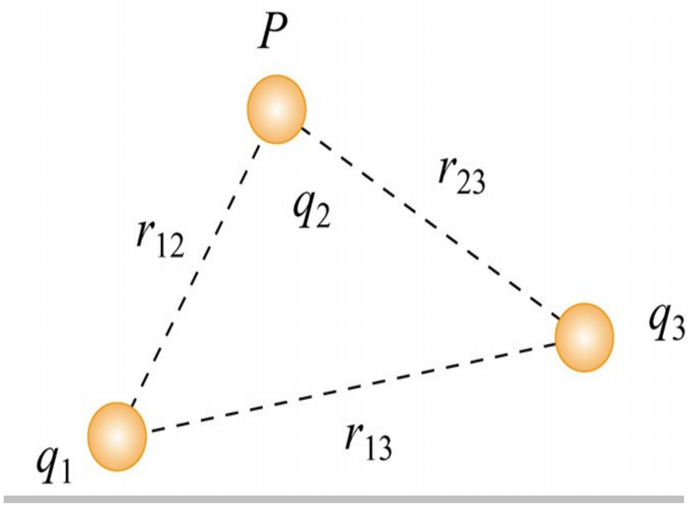


$$W_1 = 0$$



$$W_2 = q_2 \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{r_{12}}\right)$$

$$= q_2 V_1(\vec{r}_2)$$

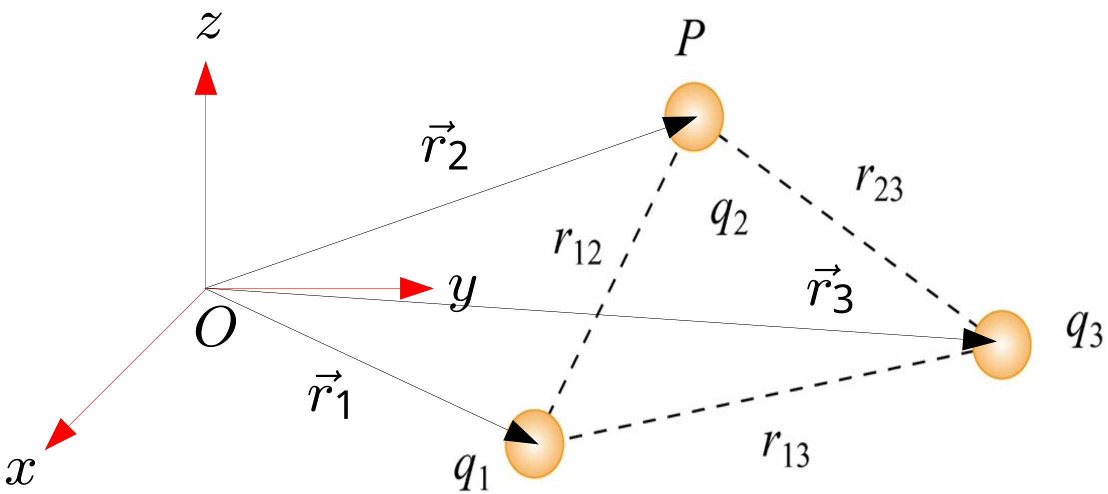


$$W_3 = q_3 \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right)$$

$$= q_3 V_1(\vec{r}_3) + q_3 V_2(\vec{r}_3)$$

Work and Energy in Electrostatics

Consider assembling three charges as shown in the figure.



The position and the relative vectors are related by

$$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1 , \qquad r_{12} = |\vec{r}_{12}|$$

The work done in assembling two charges,

$$W_2 = q_2 \;\; \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{r_{12}}\right)\Bigg] = q_2 V_1(\vec{r}_2)\,, \text{ where } \; V_1(\vec{r}_2) = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{r_{12}}\right)$$

Similarly, for adding the third charge,

$$W_3 = q_3 \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right) \; = q_3 V_1(\vec{r}_3) + q_3 V_2(\vec{r}_3)$$

Work and Energy in Electrostatics

The total work done in assembling three charges,

$$W_{TOT} = W_1 + W_2 + W_3 = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$

For assembling *N* charges, the total work done can be written as

$$W_{TOT} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j>i}}^{N}\left(\frac{q_i q_j}{r_{ij}}\right)$$

To avoid writing *j*>*i*, we can count each pair twice and then divide by 2, so that the total work done becomes

$$W_{TOT} = \frac{1}{2}\sum_{i=1}^{N} q_i \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{1}{4\pi\epsilon_0}\left(\frac{q_j}{r_{ij}}\right) = \frac{1}{2}\sum_{i=1}^{N} q_i V(\vec{r}_i)$$

Note that the second sum represents the potential at $\vec{r}_i$ due to all the charges.

Work and Energy in Electrostatics

Instead of point charges, if we have a continuous charge distribution, then using

$$\sum_{i=1}^{N} q_i V(\vec{r}_i) \Longrightarrow \int_V dq V(\vec{r}) = \int_V \rho(\vec{r}) V(\vec{r}) d\tau$$

We get for the total work done in assembling a continuous volume charge distribution,

$$W_{TOT} = \frac{1}{2} \int_V \rho(\vec{r}) V(\vec{r}) d\tau$$

It is known as the electrostatic energy of a continuous charge distribution.

Work and Energy in Electrostatics

Electrostatic energy of a continuous charge distribution:

$$\boxed{W_{\mathsf{TOT}} = \frac{1}{2}\int_{\mathsf{V}} \rho(\vec{r})V(\vec{r})d\tau}$$

The above expression for electrostatic energy can be reformulated in terms of the electric field in all space as follows. Use Gauss' law to write the energy as

$$W_{\mathsf{TOT}} = \frac{\epsilon_0}{2}\int_{\mathsf{V}} \left(\vec{\nabla}\cdot\vec{E}\right) V(\vec{r})d\tau$$

Use the divergence theorem to transfer the derivative from $\vec{E}$ to *V* using $\vec{\nabla}\cdot(f\,\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot(\vec{\nabla}f)$ and integrating over a volume *V* enclosed by a surface *S*,

$$\int_{\mathsf{V}} \left(\vec{\nabla}\cdot f\,\vec{A}\right) d\tau = \int_{\mathsf{V}} f\left(\vec{\nabla}\cdot\vec{A}\right) d\tau + \int_{\mathsf{V}} A\cdot\left(\vec{\nabla}f\right) d\tau = \oint_{\mathsf{S}} f\,\vec{A}\cdot d\vec{S}$$

or, $$\int_{\mathsf{V}} f\left(\vec{\nabla}\cdot\vec{A}\right) d\tau = -\int_{\mathsf{V}} \vec{A}\cdot\left(\vec{\nabla}f\right) d\tau + \oint_{\mathsf{S}} f\,\vec{A}\cdot d\vec{S}$$

Work and Energy in Electrostatics

We have,

$$W_{TOT} = \frac{\epsilon_0}{2}\left[-\int_V \vec{E}\cdot\left(\vec{\nabla}V\right)d\tau + \oint_S V\vec{E}\cdot d\vec{S}\right]$$

$$= \frac{\epsilon_0}{2}\left[\int_V \vec{E}\cdot\vec{E}d\tau + \oint_S V\vec{E}\cdot d\vec{S}\right]$$

$$\text{or,}\quad W_{TOT} = \frac{\epsilon_0}{2}\left[\int_V E^2 d\tau + \oint_S V\vec{E}\cdot d\vec{S}\right]$$

If we let $r \to \infty$, then since

$$r \to \infty \implies \oint_S V\vec{E}\cdot d\vec{S} = 0$$

The total electrostatic energy due to a localized charged distribution producing the electric field $\vec{E}$ is given by

$$W_{TOT} = \frac{\epsilon_0}{2}\int_{V\,\substack{all\\space}} E^2 d\tau$$

Calculating Electrostatic Energy of Charged Distributions

We have the following four equations for calculating the electrostatic energy of a given charge distribution $\rho(\vec{r})$.

1. $$W_{TOT} = \frac{1}{2}\int_V \rho(\vec{r})V(\vec{r})d\tau$$

2. $$W_{TOT} = \frac{\epsilon_0}{2}\int_{\substack{V\\ all\\ space}} E^2 d\tau$$

3. $$W_{TOT} = \frac{\epsilon_0}{2}\left[\int_V E^2 d\tau + \oint_S V\vec{E}\cdot d\vec{S}\right]$$

4. $$W_{TOT} = \int dqV(\vec{r})$$

Based on efficiency and convenience, one can use any one of these equations to calculate the electrostatic energy.

Electrostatic Energy of a Charged Sphere

Consider a uniform volume charge density $\rho(\vec{r})$ in the form of a sphere of radius *R* as shown in the figure. Calculate the work done in assembling the charged sphere.

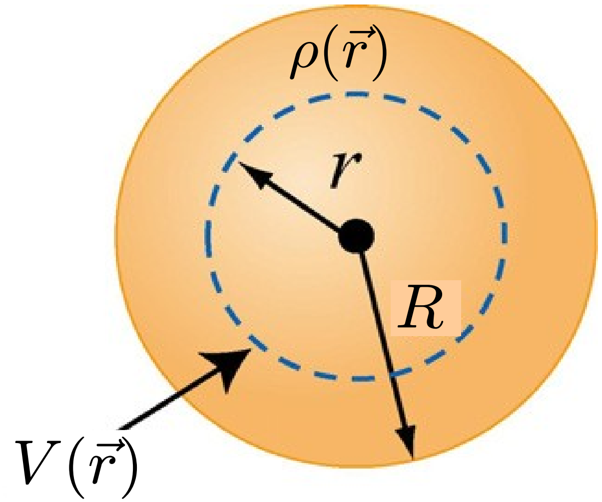
For using Eq. (1), $W = \frac{1}{2}\int_V \rho(\vec{r})V(\vec{r})d\tau$

to calculate the work done in assembling the charge *Q*, we have to know $\rho(\vec{r})$ and $V(\vec{r})$ inside the sphere.

We have, $\rho = \frac{3Q}{4\pi R^3}$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{2R}\left(3 - \frac{r^2}{R^2}\right)$$

Substituting these values in Eq. (1),



Electrostatic Energy of a Charged Sphere

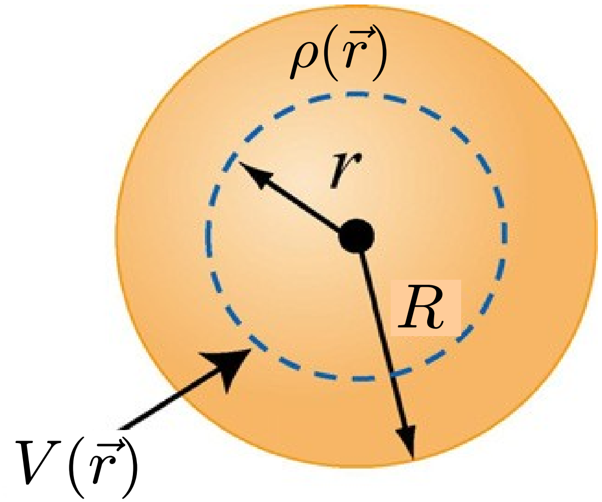
we get,

$$W = \frac{1}{2}\rho\frac{1}{4\pi\epsilon_0}\frac{Q}{2R}\int_0^R \left(3 - \frac{r^2}{R^2}\right)4\pi r^2 dr$$

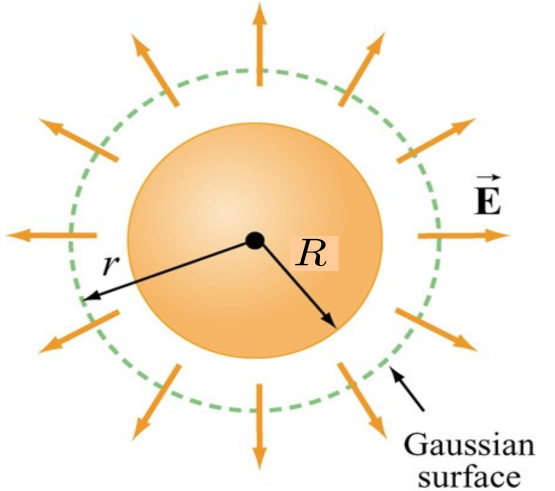
$$= \frac{1}{2}\rho\frac{1}{\epsilon_0}\frac{Q}{2R}\left(R^3 - \frac{R^3}{5}\right) = \frac{\rho Q}{\epsilon_0}\frac{R^2}{5} = \frac{Q^2}{\epsilon_0(4/3)\pi R^3}\frac{R^2}{5}$$

The work done in assembling the sphere is

$$W = \frac{1}{4\pi\epsilon_0}\frac{3Q^2}{5R}$$



Electrostatic Energy of a Charged Sphere



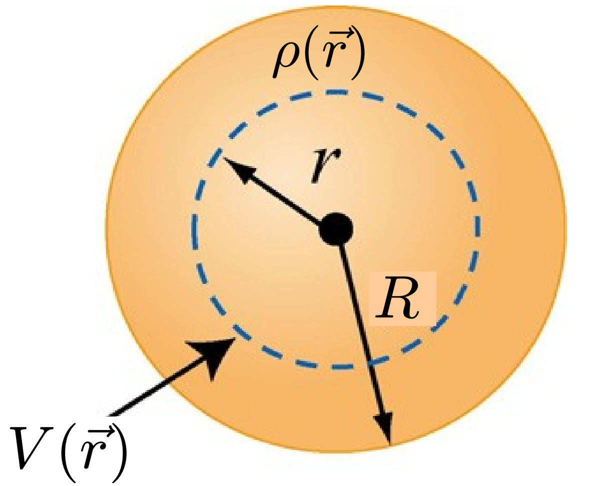
For using Eq. (2),

$$W = \frac{\epsilon_0}{2}\int_{\substack{V \\ all \\ space}} E^2 d\tau$$

to calculate the work done, we have to know the electric field $\vec{E}(\vec{r})$ in all space.

All space can be divided into two regions with the electric fields in the two regions given by

$$\vec{E} = \begin{cases} \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}, & r > R \\ \frac{Qr}{4\pi\epsilon_0 R^3}\hat{r}, & r < R \end{cases}$$



Substituting for $\vec{E}(\vec{r})$ in Eq. (2),

Electrostatic Energy of a Charged Sphere

we get,

$$W = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2\left[\frac{1}{R^6}\int_0^R \left(r^2\right)4\pi r^2 dr + \int_R^\infty \left(\frac{1}{r^4}\right)4\pi r^2 dr\right]$$

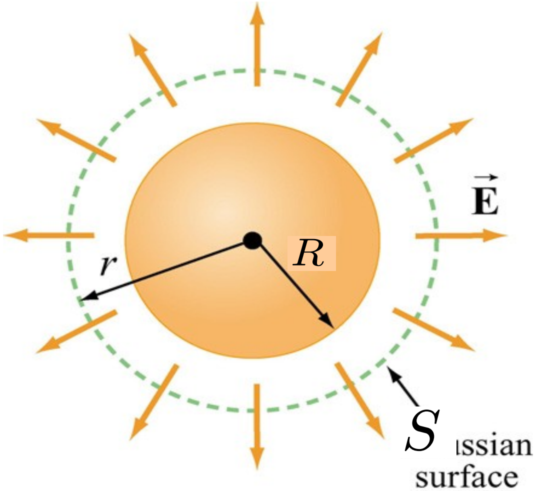
$$= \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 4\pi\left[\frac{1}{R^6}\frac{R^5}{5} - \left(\frac{1}{r}\right)\Big|_R^\infty\right]$$

$$= \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 4\pi\frac{6}{5R}$$

The work done in assembling the sphere is

$$W = \frac{1}{4\pi\epsilon_0}\frac{3Q^2}{5R}$$

Electrostatic Energy of a Charged Sphere



For using Eq. (3),

$$W_{TOT} = \frac{\epsilon_0}{2}\left[\int_V E^2 d\tau + \oint_S V\vec{E}\cdot d\vec{S}\right]$$

to calculate the work done, we have to know the electric field $\vec{E}(\vec{r})$ and $V(\vec{r})$ inside the volume *V* enclosed by the surface *S*.

Substituting for $\vec{E}(\vec{r})$ and $V(\vec{r})$ in Eq. (2), we get

$$W = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2\left[\frac{1}{R^6}\int_0^R \left(r^2\right)4\pi r^2 dr + \int_R^a \left(\frac{1}{r^4}\right)4\pi r^2 dr\right]$$

$$+\frac{\epsilon_0}{2}\left(\frac{1}{4\pi\epsilon_0}\right)^2\int_{r=a}\left(\frac{Q}{r}\right)\left(\frac{Q}{r^2}\right)r^2\sin\theta d\theta d\phi$$

Electrostatic Energy of a Charged Sphere

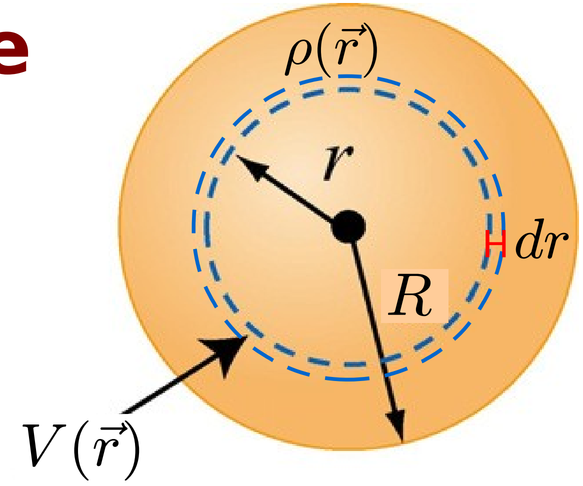
or, $W = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 4\pi\left[\frac{1}{R^6}\frac{R^5}{5} - \left(\frac{1}{r}\right)\Big|_R^a\right] + \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2\left(\frac{1}{a}\right)(4\pi)$

$$= \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 4\pi\left[\frac{1}{5R} - \frac{1}{a} + \frac{1}{R} + \frac{1}{a}\right]$$

We see that as $a \to \infty,$ the work done in assembling the sphere becomes (as expected)

$$W = \frac{1}{4\pi\epsilon_0}\frac{3Q^2}{5R}$$

Electrostatic Energy of a Charged Sphere



For using Eq. (4),

$$W = \int dq V(\vec{r})$$

to calculate the work done, we consider a sphere of radius *r*, and then bring in a layer of charge *dq* from infinity to *r*. The charge *dq* is distributed uniformly over the sphere of radius r such that

$$dq = 4\pi r^2 dr \rho$$

Since *V*(*r*) is equal to $V(r) = \frac{1}{4\pi\epsilon_0}\frac{4\pi r^3\rho}{3r}$

The work done in bringing *dq* from infinity to *r* is

$$dW = dqV(\vec{r}) = 4\pi r^2 dr\rho\frac{r^2\rho}{3\epsilon_0} = \frac{4\pi\rho^2}{3\epsilon_0}r^4 dr$$

Electrostatic Energy of a Charged Sphere

Then the total work done is obtained by integrating from 0 to *R*.

$$W = \int dq V(\vec{r}) = \frac{4\pi\rho^2}{3\epsilon_0}\int_0^R r^4 dr = \frac{4\pi\rho^2}{3\epsilon_0}\frac{R^5}{5}$$

Substituting for ρ in terms of *Q*, we get,

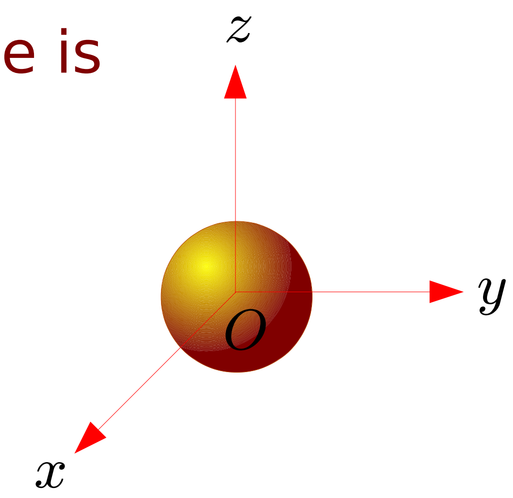
$$W = \frac{1}{4\pi\epsilon_0}\frac{3Q^2}{5R}$$

the work done in assembling the sphere.

Electrostatic Energy of a Point Charge

Explain the infinite electrostatic energy of a point charge q.

The energy associated with the point charge is



$$W^q = \frac{q^2}{(4\pi\epsilon_0)^2}\int_{V \; all \; space} \frac{1}{r^4} r^2 dr \sin\theta d\theta d\phi$$

$$= \frac{q^2(2\pi)}{(4\pi\epsilon_0)^2}\int_0^\infty \frac{1}{r^2} dr = \infty$$

The energy associated with the point charge is infinite.

The problem is resolved by realizing that classical physics breaks down at length scales below the reduced Compton wavelength of the electron

$$\boxed{r \leq \frac{\lambda_e}{2\pi} = \frac{\hbar c}{m_e c^2} = 385.5 \times 10^{-15} m}$$

Electrostatic Potential Energy and Superposition Principle

We have seen that the electric forces, fields and potentials follow *linear superposition principle.* Does electrostatic potential energy follow linear superposition principle?

Consider an electric field obtained by linearly superposing two fields,

$$\vec{E}_{TOT}(\vec{r}) = \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r})$$

The total electrostatic potential energy is

$$W_{TOT} = \frac{\epsilon_0}{2} \int_{\substack{V \\ all \\ space}} \vec{E}_{TOT} \cdot \vec{E}_{TOT} \, d\tau$$

$$= \frac{\epsilon_0}{2} \int_{\substack{V \\ all \\ space}} \{\vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r})\} \cdot \{\vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r})\} d\tau$$

Electrostatic Potential Energy and Superposition Principle

or, $$W_{TOT} = \frac{\epsilon_0}{2}\int_{\substack{V\\ all\\ space}} \{\vec{E}_1^2(\vec{r}) + \vec{E}_2^2(\vec{r}) + 2\vec{E}_1(\vec{r})\cdot\vec{E}_2(\vec{r})\}d\tau$$

$$= \frac{\epsilon_0}{2}\left[\int_{\substack{V\\ all\\ space}} \vec{E}_1^2(\vec{r})d\tau + \int_{\substack{V\\ all\\ space}} \vec{E}_2^2(\vec{r})d\tau + \int_{\substack{V\\ all\\ space}} 2\vec{E}_1(\vec{r})\cdot\vec{E}_2(\vec{r})d\tau\right]$$

or, $$W_{TOT} = W_1 + W_2 + \frac{\epsilon_0}{2}\int_{\substack{V\\ all\\ space}} 2\vec{E}_1(\vec{r})\cdot\vec{E}_2(\vec{r})d\tau$$

$$\boxed{\neq W_1 + W_2 \Longrightarrow \text{No linear superposition principle}}$$

Electrostatic potential energy doesn't follow linear superposition principle!

Work and Energy in Electrostatics

Summary

I. We learned about the work done in moving a charge in an electric field (or potential)

$$W = -q\int_a^b \vec{E}(\vec{r})\cdot d\vec{l}\ , \qquad W = qV(\vec{r})$$

II. For *N* charges, we showed that the work done is

$$W_{TOT} = \frac{1}{2}\sum_{i=1}^{N} q_i V(\vec{r}_i)$$

III. The electrostatic energy of a charge distribution can be calculated as

1. $$W_{TOT} = \frac{1}{2}\int_V \rho(\vec{r})V(\vec{r})d\tau$$

2. $$W_{TOT} = \frac{\epsilon_0}{2}\int_V E^2 d\tau$$

3. $$W_{TOT} = \frac{\epsilon_0}{2}\left[\int_V E^2 d\tau + \oint_S V\vec{E}\cdot d\vec{S}\right]$$

4. $$W_{TOT} = \int dqV(\vec{r})$$