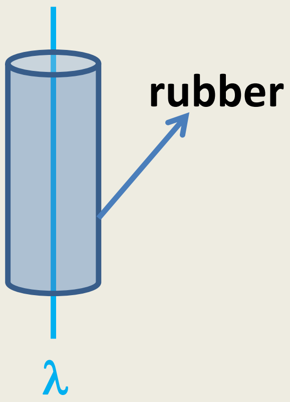
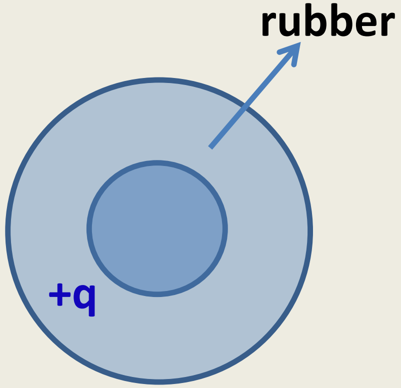
Gauss's Law in Dielectrics

$$Polarisation \Rightarrow \sigma_b = \vec{P} \cdot \hat{n}$$
$$\rho_b = -\vec{\nabla} \cdot \vec{P}$$



Line charge surrounded by a cylindrical dielectric



Charged conducting sphere surrounded by a spherical dielectric

In both cases, we have free charges and bound charges

$$\Rightarrow \quad \rho = \rho_b + \rho_f$$

Gauss's Law

$$\epsilon_0 \vec{\nabla} \cdot \vec{E} = \rho = \rho_b + \rho_f = \rho_f - \vec{\nabla} \cdot \vec{P}$$

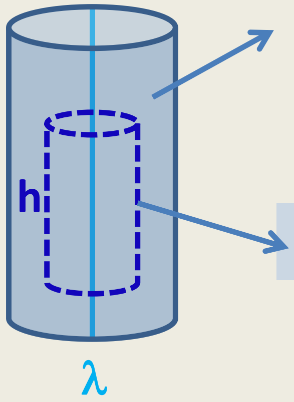
$$\vec{\nabla} \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho_f \qquad \boxed{\vec{\nabla} \cdot \vec{D} = \rho_f}$$

$$\oiint \vec{D} \cdot \vec{dS} = Q_{f,\mathrm{encl}}$$

Electric displacement

$$\vec{D} = (\epsilon_0 \vec{E} + \vec{P})$$

Gauss's Law in Dielectrics : Applications



Line charge surrounded by a long cylindrical dielectric

Electric field due to λ, $\quad E = \dfrac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$

$$\oiint \vec{D}\cdot\vec{dS} = Q_{f,\text{encl}}$$

$$D\,2\pi r h = \lambda h \quad OR \quad \boxed{\vec{D} = \frac{\lambda}{2\pi r}\hat{r}}$$

in the dielectric

$$\vec{D} = \frac{\lambda}{2\pi r}\hat{r}$$ in the air (outside the dielectric)

$$\vec{E}_{\text{diel}} = \frac{1}{\epsilon_0}(\vec{D}-\vec{P})$$ we have to know *P* to calculate *E* in the dielectric

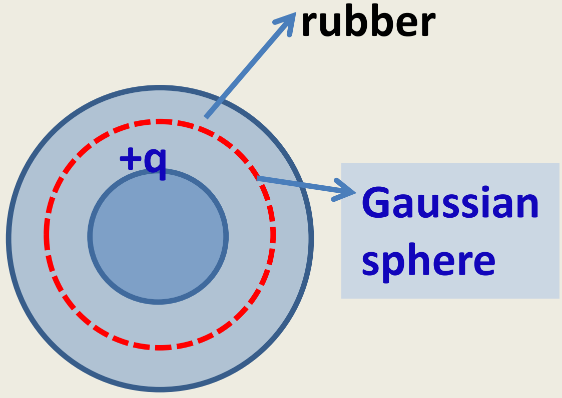
$$\vec{E}_{\text{air}} = \frac{1}{\epsilon_0}(\vec{D}-\vec{P}) = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$$ since *P* = 0 in the air

Dielectric surrounding the wire does not affect electric field outside

Gauss's Law in Dielectrics : Applications

Electric field due to charged sphere,

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$$



Charged conducting sphere surrounded by a spherical dielectric

$$D\,4\pi r^2 = q \quad OR \quad \boxed{\vec{D} = \frac{q}{4\pi r^2}\hat{r}}$$

in the dielectric

$$\vec{D} = \frac{q}{4\pi r^2}\hat{r}$$ in vacuum (outside the dielectric)

we have to know *P* to calculate *E* in the dielectric

$$\vec{E}_{\text{air}} = \frac{1}{\epsilon_0}\left(\vec{D} - \vec{P}\right) = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$$

Dielectric surrounding the sphere does not affect electric field outside

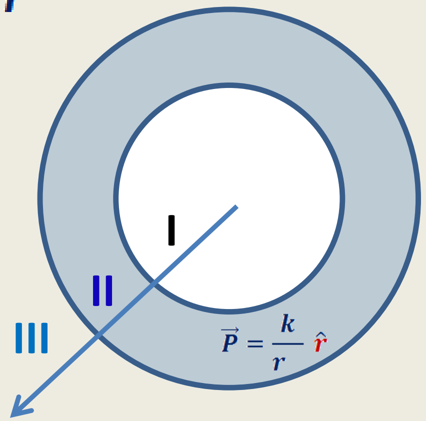
Gauss's Law in Dielectrics : Applications

Dielectric shell, inner radius a, outer radius b, polarization $\vec{P} = \frac{k}{r}\,\hat{r}$

D = 0 everywhere (no free charges)

$$\vec{E} = \frac{1}{\epsilon_0}(\vec{D} - \vec{P})$$

$$\vec{E}_I = 0 \qquad \vec{E}_{II} = \frac{1}{\epsilon_0}(-\vec{P}) = \frac{-k}{\epsilon_0 r}\hat{r} \qquad \vec{E}_{III} = 0$$



$$\sigma_b = \vec{P}\cdot\hat{n} = \vec{P}\cdot -\hat{r}\Big|_{r=a} = \frac{-k}{a} \qquad \sigma_b = \vec{P}\cdot\hat{r}\Big|_{r=b} = \frac{k}{b}$$

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = \frac{-k}{r^2}$$

$$Q_{b,\,\text{total}} = -\frac{k}{a}4\pi a^2 + \frac{k}{b}4\pi b^2 + \int \rho_b\, d\tau$$

$$\int \rho_b\, d\tau = \int_a^b \frac{-k}{r^2}\, 4\pi r^2 dr = 4\pi a - 4\pi b$$

$$Q_{b,\,\text{total}} = 0$$

Linear Dielectrics

Polarization in a dielectric results from an electric field which aligns the atomic dipoles.

This implies that $P \sim E$; *P* and *E* are related.

IF *P* is directly proportional to *E* , then the dielectric is LINEAR

$P \propto E; \;\; P = \varepsilon_0 \chi_e E$ $\quad E$ is TOTAL electric field, not just the part due to Polarization

χ_e is the electrical susceptibility; a dimension-less quantity

For a linear dielectric,

$$\vec{D} = \left(\epsilon_0 \vec{E} + \vec{P}\right) = \epsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E}$$

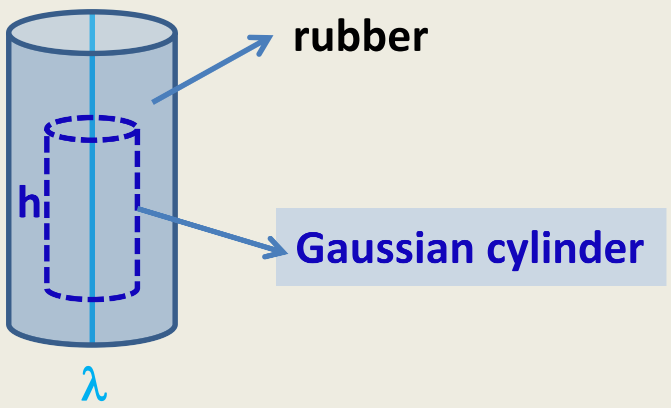
$$= \epsilon_0 (1 + \chi_e) \vec{E} = \varepsilon \vec{E}$$

ε - permitivity of medium

$$(1 + \chi_e) = \kappa = \frac{\varepsilon}{\varepsilon_0}$$

Dielectric constant of medium

Gauss's Law in Dielectrics : Applications



Line charge surrounded by a long cylindrical dielectric

Dielectric constant κ

$$\vec{E} = \frac{\vec{D}}{\varepsilon} = \frac{\lambda}{2\pi\varepsilon r}\,\hat{r} = \frac{\lambda}{2\pi\varepsilon_0 \kappa r}\,\hat{r}$$

e.f. is reduced by a factor κ

$$\vec{D} = \frac{\lambda}{2\pi r}\,\hat{r}$$

$$\vec{E}_{\text{diel}} = \frac{1}{\epsilon_0}(\vec{D} - \vec{P})$$

we have to know *P* to calculate *E* in the dielectric

$$\vec{E}_{\text{air}} = \frac{1}{\epsilon_0}(\vec{D} - \vec{P}) = \frac{\lambda}{2\pi\varepsilon_0 r}\,\hat{r}$$

since *P* = 0 in vacuum

Dielectric surrounding the wire does not affect electric field outside

Gauss's Law in Dielectrics : Applications

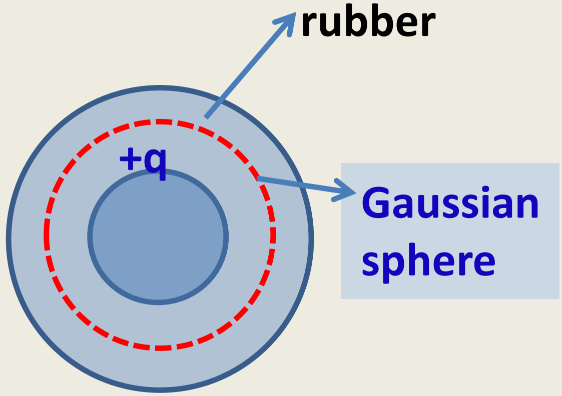
$$\vec{D} = \frac{q}{4\pi r^2}\,\hat{r}$$

$$\vec{E}_{diel} = \frac{\vec{D}}{\varepsilon} = \frac{q}{4\pi\varepsilon r^2}\,\hat{r} = \frac{q}{4\pi\varepsilon_0\kappa r^2}\,\hat{r}$$

$$\vec{P} = \varepsilon_0\chi_e\vec{E} = \frac{\varepsilon_0(\kappa-1)q}{4\pi\varepsilon_0\kappa r^2}\,\hat{r}$$

$$\sigma_b = \vec{P}\cdot\hat{n} = \vec{P}\cdot -\hat{r}\Big|_{r=a} = -\frac{\varepsilon_0(\kappa-1)q}{4\pi\varepsilon_0\kappa a^2}$$

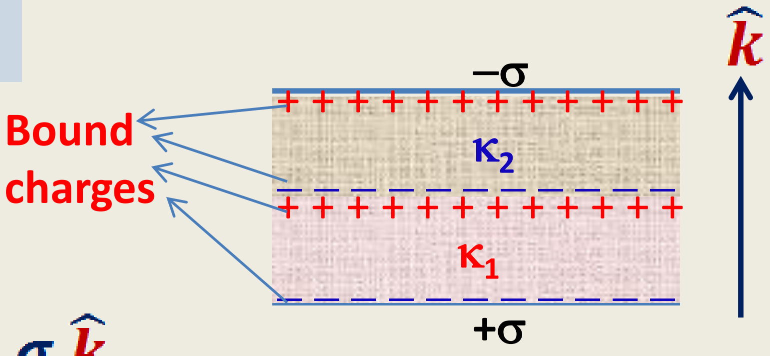
$$\sigma_b = \vec{P}\cdot\hat{n} = \vec{P}\cdot\hat{r}\Big|_{r=b} = -\frac{\varepsilon_0(\kappa-1)q}{4\pi\varepsilon_0\kappa b^2}$$



Charged conducting sphere surrounded by a spherical dielectric

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = 0$$

Gauss's Law in Dielectrics : Applications



D = 0 outside

$\vec{D} = \sigma\,\hat{k}$ in both dielectrics

$$\vec{E}_1 = \frac{\vec{D}}{\varepsilon_0\kappa_1} = \frac{\sigma\,\hat{k}}{\varepsilon_0\kappa_1} \qquad \vec{E}_2 = \frac{\vec{D}}{\varepsilon_0\kappa_2} = \frac{\sigma\,\hat{k}}{\varepsilon_0\kappa_2}$$

$$\vec{P}_1 = \varepsilon_0\chi_e\vec{E}_1 = \frac{\varepsilon_0(\kappa_1-1)\,\sigma}{\varepsilon_0\kappa_1}\hat{k} = \frac{(\kappa_1-1)\,\sigma}{\kappa_1}\hat{k}$$

$$\vec{P}_2 = \frac{(\kappa_2-1)\,\sigma}{\kappa_2}\hat{k}$$

$$\boxed{\sigma_{b,1} = -\frac{\varepsilon_0(\kappa_1-1)\,\sigma}{\varepsilon_0\kappa_1}}$$

bottom of the slab 1

$$\sigma_{b,1} = +\frac{\varepsilon_0(\kappa_1-1)\,\sigma}{\varepsilon_0\kappa_1}$$

top of the slab 1

$$\sigma_{b,2} = +\frac{\varepsilon_0(\kappa_2-1)\,\sigma}{\varepsilon_0\kappa_1}$$

bottom of the slab 2

$$\sigma_{b,2} = -\frac{\varepsilon_0(\kappa_2-1)\,\sigma}{\varepsilon_0\kappa_1}$$

top of the slab 2

Difference between *D* and *E*

Divergence and curl are needed to define a vector

$$\boxed{\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{AND} \quad \vec{\nabla}\times\vec{E} = 0}$$

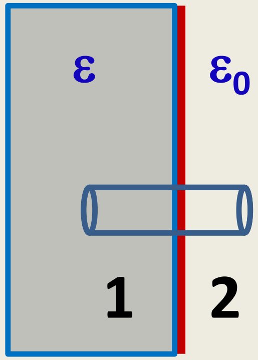
$$\vec{D} = \left(\epsilon_0\,\vec{E} + \vec{P}\right)$$

$$\boxed{\vec{\nabla}\cdot\vec{D} = \rho_f}$$

$$\boxed{\vec{\nabla}\times\vec{D} = \epsilon_0\,\vec{\nabla}\times\vec{E} + \vec{\nabla}\times\vec{P} = 0 + \vec{\nabla}\times\vec{P}}$$

D can be calculated like E from the integral form of Gauss's law only in the case of symmetries : Here Curl P automatically goes to zero

Boundary conditions



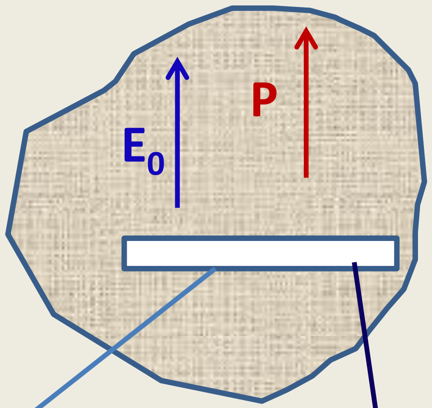
$$\boxed{\vec{\nabla}\cdot\vec{D} = \rho_f} \Longrightarrow \begin{array}{l} D_{2\perp} - D_{1\perp} = \sigma_f \\ D_{2\perp} = D_{1\perp},\ \text{if}\ \sigma_f = 0 \end{array}$$

No such conditions for D-parallel !

However, E-parallel is still continuous !!!!

Boundary conditions : Applications

$$\vec{D}_0 = \left(\epsilon_0 \vec{E}_0 + \vec{P}\right)$$



D and E inside the wafer?

$\rho_{\text{free}} = 0 \;\Rightarrow\; D_\perp$ is continuous

Wafer-shaped cavity $\perp$ P

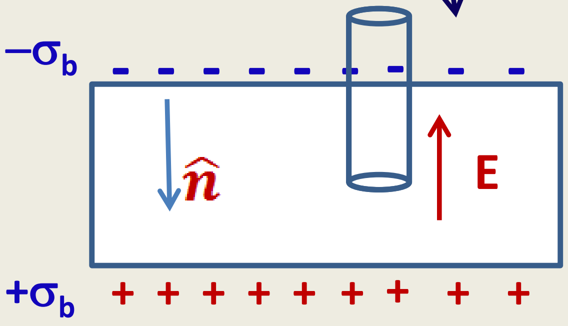
$$\vec{D} = \vec{D}_0$$

$$\varepsilon_0 \vec{E} = \varepsilon_0 \vec{E}_0 + \vec{P}$$

$$\vec{E} = \vec{E}_0 + \frac{\vec{P}}{\varepsilon_0}$$

$$E_0 A - E A = \frac{-\sigma_b A}{\varepsilon_0}$$

$$E_0 - E = \frac{-\vec{P}}{\varepsilon_0}$$

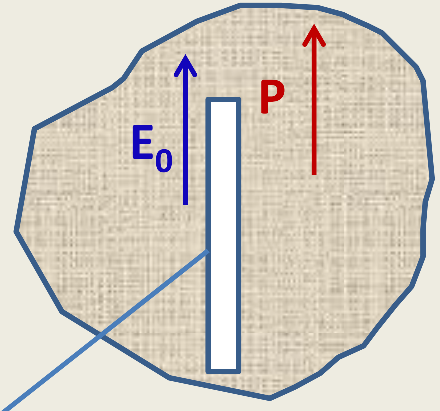


Wafer-shaped cavity is equivalent to wafer-shaped dielectric with polarization -P

Boundary conditions : Applications

E-parallel is continuous

$$\vec{E} = \vec{E}_0 \qquad \frac{\vec{D}}{\varepsilon_0} = \frac{\vec{D}_0}{\varepsilon} \quad \Rightarrow \quad \vec{D} = \vec{D}_0 \frac{\varepsilon_0}{\varepsilon} = \frac{\vec{D}_0}{\kappa}$$



$$\vec{E} = \vec{E}_0 = \frac{1}{\varepsilon_0}\left(\vec{D}_0 - \vec{P}\right)$$

$$\varepsilon_0 \vec{E} = \left(\vec{D}_0 - \vec{P}\right) \quad OR \quad \vec{D} = \vec{D}_0 - \vec{P}$$

needle-shaped cavity || P

Electrostatic Energy in Dielectrics

We have to put some extra charges in order to bring the potential since e.f. inside dielectric cancels some of the original electric field

For parallel plate capacitor,

$$W_0 = \frac{1}{2} C_0 V^2$$ C_0 is the capacitance in vacuum

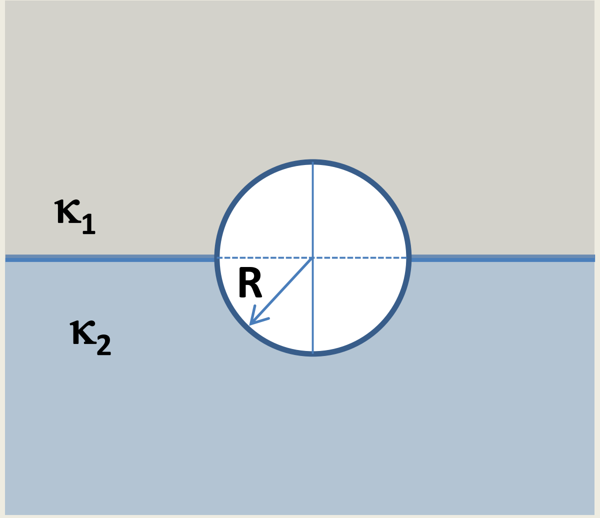
If a dielectric is inserted, *E* decreases by a factor κ; *V* also decreases by a factor κ

$Q = CV \Longrightarrow C = \kappa C_0 \Longrightarrow W = \kappa W_0$

In general, for a dielectric-filled capacitor,

$$W = \kappa \frac{\varepsilon_0}{2} \int E^2 \, d\tau = \frac{1}{2} \int \left(\vec{D} \cdot \vec{E}\right) d\tau$$

Two semi-infinite linear dielectrics
Conducting sphere with charge +Q



***E* should be radial and should have the same value in both dielectrics (*E*-parallel is continuous)**

This implies *D* is different in the two dielectrics!

$$\oiint D.dS = Q_{free} \quad \Rightarrow \quad 2\pi r^2 D_1 + 2\pi r^2 D_2 = Q$$

$$2\pi r^2 \varepsilon_0 \kappa_1 E + 2\pi r^2 \varepsilon_0 \kappa_2 E = Q$$

$$\vec{E} = \frac{Q}{2\pi\varepsilon_0(\kappa_1 + \kappa_2)r^2}\hat{r}$$

$$\vec{D}_1 = \varepsilon_0 \kappa_1 \vec{E} = \frac{Q}{2\pi r^2}\frac{\kappa_1}{(\kappa_1 + \kappa_2)}\hat{r}$$

$$\vec{D}_2 = \varepsilon_0 \kappa_2 \vec{E} = \frac{Q}{2\pi r^2}\frac{\kappa_2}{(\kappa_1 + \kappa_2)}\hat{r}$$

Electrostatic Energy in Dielectrics : charging a capacitor

Fix the dielectric in position, give a charge density ρ_f. $$W = \int \rho_f \, d\tau \, V$$

Increase ρ_f by $\delta\rho_f$ $$\delta W = \int (\delta\rho_f) \, d\tau \, V = \int \vec{\nabla} \cdot \left(\delta\vec{D}\right) V d\tau$$

$$= \int \vec{\nabla} \cdot \left(\delta\vec{D}\, V\right) d\tau - \int (\delta\vec{D} \cdot \vec{\nabla} V) d\tau = \int_V \vec{\nabla} \cdot \left(\delta\vec{D}\, V\right) d\tau - \int_V (\delta\vec{D} \cdot (-\vec{E})) d\tau$$

$$= \int_{all\ space} (\delta\vec{D} \cdot \vec{E}) d\tau$$

For a linear dielectric

$$\vec{D} = \varepsilon\,\vec{E} \Rightarrow \frac{1}{2}\,\delta\left(\vec{D} \cdot \vec{E}\right) = \vec{E} \cdot \delta\left(\varepsilon\vec{E}\right) = (\delta\vec{D}) \cdot \vec{E}$$

$$\delta W = \int_{all\ space} \left(\delta\vec{D} \cdot \vec{E}\right) d\tau = \frac{1}{2} \int_{all\ space} \delta\left(\vec{D} \cdot \vec{E}\right) d\tau$$

$$W = \frac{1}{2} \int_{all\ space} \left(\vec{D} \cdot \vec{E}\right) d\tau$$