1. Basics of Group Theory : (for non-mathematics majors)

Group G : A set with an operation $\circ$ satisfying

- closure : $g, h \in G \Rightarrow g \circ h \in G$
- Associativity : $(g \circ h) \circ k = g \circ (h \circ k)$ $\forall g$
- identity : There exists $e \in G$ s.t. $e \circ g = g \circ e = g$
- Inverses : for all $g \in G$, there exists $g^{-1} \in G$ with $g \circ g^{-1} = e$

- Group Action on a set :

$X : G \times X \to X \; (g, x) \to g \cdot x$ satisfying

- $e \cdot x = x$
- $(g \circ h) \cdot x = g \cdot (h \cdot x)$

2. Equivariance and invariance :

- A function $f : X \to Y$ is equivariant w.r.t. group G if for all $g \in G$

$f(g \cdot x) = \rho_Y(g) f(x)$

where ρ_Y is a representation of G on output space Y.

- invariance is a special case where

$\rho_Y(g) = id$ so

$f(g \cdot x) = f(x)$

3. Group Equivalent CNNs :

Traditional CNNs are $\underbrace{\text{translational equivariant}}$

shifting the input to a network, the output will also shift in the same direction and by the same amount.

equivalence mapping

↳ preserves the algebraic structure of transformation.

Suppose the existence of new neural network layer given as an operator ϕ on images / feature maps f and it has the following properties.

↳ linear $\phi(\alpha f + \beta f') = \alpha \phi(f) + \beta \phi(f')$

↳ shift-equivariant $\phi({}^{m,n}S(f)) = {}^{m,n}S(\phi(f))$

↳ impulse response h (not a constraint)

when operator is applied to a Dirac impulse $0,0p$ centred at the origin $(0,0)$:

$$h = \phi({}^{0,0}p)$$

Expressing the image as a linear combination of Dirac impulses p at different locations.

$$[\phi(f)](x,y) = \left[\phi \sum_{m=-M/2}^{M/2} \sum_{n=-N/2}^{N/2} f(m,n)\, {}^{m,n}p\right](x,y)$$

This was an essence for showing how traditional CNNs are translationally equivariant.

Coming Back to (G-CNNs)

[Traditional CNNs : $[L_t f](x) = f(x-t)$]

$$(f * \psi)(x) = \int f(y)\, \psi(x-y)\, dy$$

satisfies : $L_t(f * \psi) = (L_t f) * \psi$

Group Convolution

generalises this for Group G :

$$[f * \psi](g) = \int_G f(h)\, \psi(g^{-1}h)\, d\mu(h)$$

$d\mu$ is the Haar measure of G

discrete groups (like rotations / reflections), the integral is replaced by Σ (trivial)

• ~~Lie~~ Group Equivariant networks

Lie groups ⟶ continuous groups with smooth manifold structure.

group convolution uses the Haar measure on the Lie group

$$[f * \psi](g) = \int_G f(h)\, \psi(g^{-1}h)\, d\mu(h)$$

$f(g)$ → infinite dimensional (spherical harmonics)

ψ → steerable filter

$$\psi(g \cdot x) = \rho(g)\, \psi(x)$$

Lie algebra $\mathfrak{g}$ is the tangent space at identity (used to parametrize infinitesimal transformations)

5. Gauge Equivariants CNNs :

Gauge symmetry → local symmetry

group transformations vary smoothly over a manifold M.

Data f is a section of fiber bundle over M with fibers carrying a representation of group G.

(#) gauge transformation

$g(x)$ acts locally on fibers

$$f(x) \mapsto \rho(g(x))\, f(x)$$

Convolution must be modified with a gauge ~~connect~~ connection A to parallely transport filters

$$[f * \psi](x) = \int_M \rho(U_{x \leftarrow y})\, f(y)\, \psi(d(x,y))\, dy$$

$U_{x \leftarrow y}$: parallel transport operator along a path (y to x)

$$[\rho(g(x)) f] * \psi = \rho(g(x)) [f * \psi]$$

* Terminologies I used :

Haar Measure = Locally compact topological group is a Borel measure invariant under translations, finite on compact sets.

[Translational invariance for Borel sets]

fiber = set of all possible values or "states" attached to a point in space (or manifold)

↓

fiber bundle theory →
Base space : M
fiber . a space F
Total space (E) all combinations (x, v) $x \in M$ $v \in F_x$
Projection : π : E ...