# Design and Performance of Visible Light Communication Systems[1]

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[1] My PhD student Mr. GVSS Praneeth Varma's Thesis

- What is Visible Light Communication (VLC) ?
- How light is used for communication ?
- Why light-emitting diodes (LED) are used ?

# Introduction

## Literature

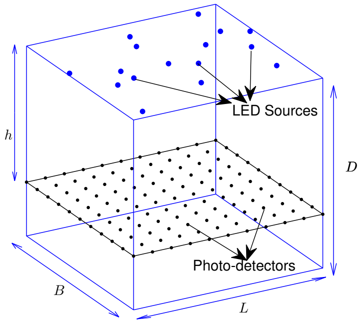


Figure 1: System model

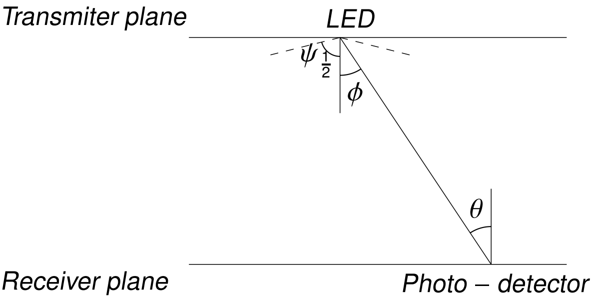


Figure 2: Propagation model

- Using the Lambertian radiation pattern to model the LED radiant intensity,

$$\mathcal{R}(\phi) = \frac{(m+1)\cos^{m}(\phi)}{2\pi},$$

$m = \frac{\ln \frac{1}{2}}{\ln\left(\cos\left(\psi_{\frac{1}{2}}\right)\right)}$ is the order of Lambertian emission, with $\psi_{\frac{1}{2}}$ being the LED semi-angle at half power, provided by the manufacturer.

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- The channel direct current (DC) gain can then be expressed as

$$H = \frac{\mathcal{R}(\phi)\cos(\theta)A}{d^2} = \frac{(m+1)\cos^{m}(\phi)\,A\cos(\theta)}{2\pi d^2}$$

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$$y_j = RP_{r_j} + n_j,$$

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where,

$$H_{ij} = \frac{(m+1)\, Ah^{m+1}}{2\pi d_{ij}^{m+3}}$$

and $n_j$ is additive white Gaussian noise (AWGN) with $n_j \sim \mathcal{N}\left(0, \sigma_j^2\right)$.

# Introduction

## Literature

The AWGN noise $n_j$ at the photo-detector is the sum of the contributions from shot noise and thermal noise, and expressed as

$$\sigma_j^2 = \sigma_{shot}^2 + \sigma_{thermal}^2$$

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$$\sigma_{shot}^2 = 2qRP_{r_j}B + 2qI_{bg}I_2B$$

$$\sigma_{thermal}^2 = \frac{8\pi kT_k}{G}\eta A I_2 B^2 + \frac{16\pi^2 kT_k \Gamma}{g_m}\eta^2 A^2 I_3 B^3$$

Signal to Noise Ratio (SNR) at photo-detector $j$ is defined as

$$\Lambda_j = \frac{\left(RP_{r_j}\right)^2}{\sigma_j^2}$$

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Signal to Noise Ratio (SNR) at photo-detector $j$ is defined as

$$\Lambda_j = \frac{\left(RP_{r_j}\right)^2}{\sigma_j^2}$$

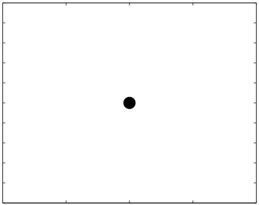
The quality factor, for measuring the performance of the light source, can be expressed as

$$F_\Lambda = \frac{\overline{\Lambda}}{2\sqrt{\mathrm{var}(\Lambda)}},$$

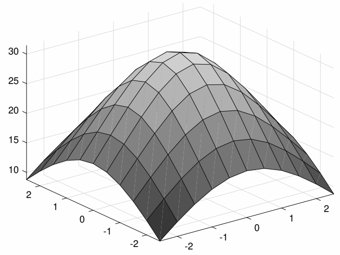
where $\overline{\Lambda}$ and $\mathrm{var}(\Lambda)$ are the mean and variance of $\left\{\Lambda_j\right\}_{j=1}^{K}$, where $K$ is the number of photodetectors.

# Introduction

## Literature



(a) Single point source




(b) SNR profile

# Introduction

## Motivation

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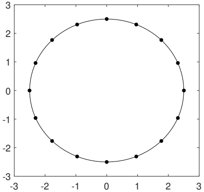
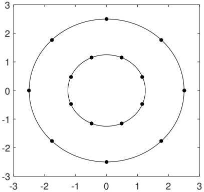
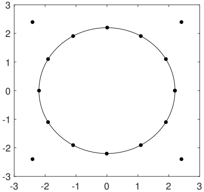


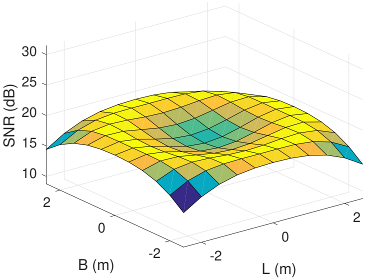

(c) Circular geometry



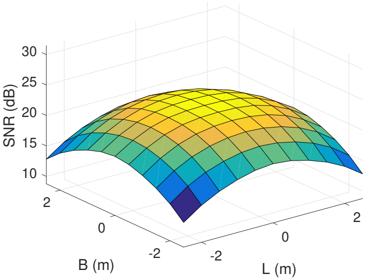

(d) Concentric circular geometry




(e) Circle-square geometry




(a) Circular geometry

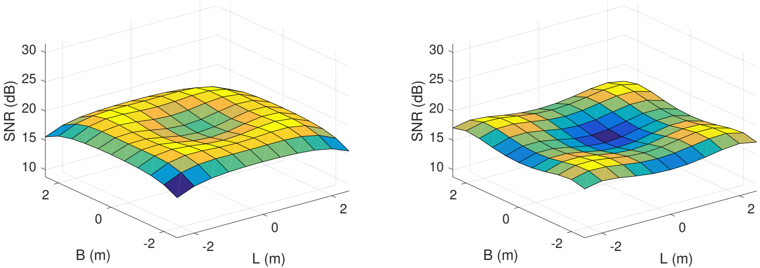



(b) Concentric Circular

Figure 4: SNR distribution with equal power allocation

# Introduction

## Motivation




(a) With equal power allocation and optimal location

(b) With optimal power allocation and optimal location

Figure 5: SNR distribution for circle-square geometry

Power Allocation for Uniform Illumination with Stochastic LED Arrays

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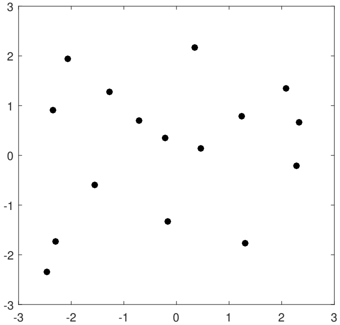
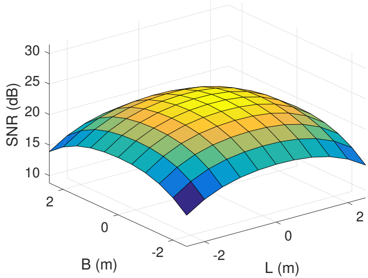



(a) A realization of BPP




(b) Average SNR profile with equal power allocation

## Heuristic power allocation:

$$P_{t_i} = \frac{r_i^\alpha}{\sum_{n=1}^{N} r_n^\alpha} P,$$

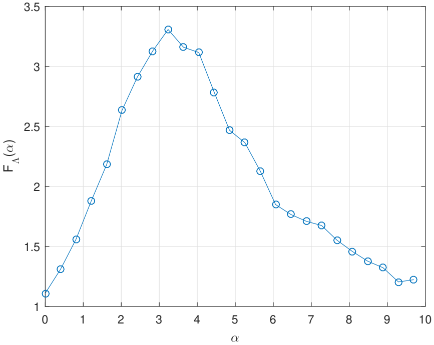
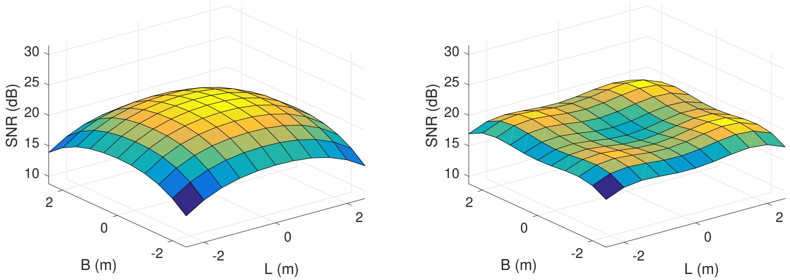



Figure 6: $F_\Lambda(\alpha)$ has a maximum.

Optimum value of $\alpha$ is obtained from golden section search algorithm

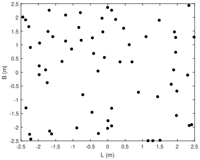



(a) Equal power allocation

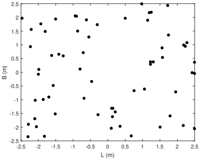
(b) Suboptimal power allocation

Figure 7: Average SNR for a BPP. $N = 16$

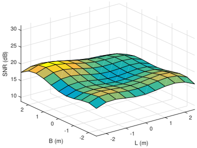
# Power Allocation for Uniform Illumination with Stochastic LED Arrays



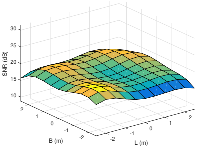

(a) $N = 64$, BPP realization 1




(b) $N = 64$, BPP realization 2




(c) SNR profile for realization 1




(d) SNR profile for realization 2

Uniform illumination possible with random distribution.

Performance of Stochastic LED Arrays based VLC with Uniform Irradiance

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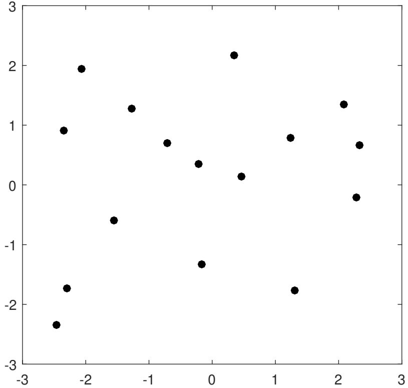


Figure 8: Realization of a BPP

The optical signal transmitted by the *i*th LED of a VLC is given by

$$p_i(t) = P_{t_i}\left[1 + M_I x_i(t)\right],$$

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After removing DC component, the signal received at *j*th photo-detector from all the source LEDs is

$$y_j = RP_{r_j} + n_j$$

where,

$$P_{r_j} = \sum_{i=1}^{N} H_{ij} P_{t_i} M_I x_i.$$

The AWGN noise $n_j$ at the photo-detector is the sum of the contributions from shot noise and thermal noise, and expressed as

$$\sigma_j^2 = \sigma_{shot}^2 + \sigma_{thermal}^2$$

where

$$\sigma_{shot}^2 = 2qRP_{r_j}B + 2qI_{bg}I_2B$$

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**Heuristic power allocation:**

The transmit power at the $i$th transmitter is given by the heuristic

$$P_{t_i} = \frac{r_i^\alpha}{\sum_{j=1}^N r_j^\alpha}P,$$

The received symbol at the central photodetector is given by

$$y = RP_{r_0} + n_0$$

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The received symbol at the central photodetector is given by

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and,

$$P_{r_0} = C_1 \sum_{i=1}^{N} V_i x_i$$

where $x_i$ is the modulating bipolar OOK signal. All LEDs are assumed to transmit the same message signal.

$$C_1 = \frac{PM_l\,(m+1)\,Ah^{m+1}}{2\pi} \quad \text{and}$$

$$V_i = \frac{r_i^{\alpha}}{\left(\sum_{j=1}^{N} r_j^{\alpha}\right)\left(\sqrt{h^2 + r_i^2}\right)^{m+3}}.$$

## Performance of Stochastic LED Arrays based VLC with Uniform Irradiance

$$y = \left( RC_1 \sum_{i=1}^{N} V_i \right) x + n_0$$

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BER for this system is given by

$$P_e = \mathbb{E}_{\Phi} \left[ Q \left( \frac{RC_1 \sum_{i=1}^{N} V_i}{\sigma_0} \right) \right]$$

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Using first order taylor approximation, Schwarz's inequality, and Jensen's inequality the above expresseion is reduced to

$$P_e \gtrapprox Q \left( \frac{RC_1 \sum_{i=1}^{N} \mathbb{E}_{\Phi} \left[ V_i \right]}{\sqrt{\mathbb{E}_{\Phi} \left[ \sigma_0^2 \right]}} \right)$$

For a BPP distributed over a square region of area $W$, the probability density function (PDF) of the distance to the $i^{th}$ nearest LED from the origin is

$$f_{r_i} = \begin{cases} \frac{2\pi r}{W} \frac{(1-p)^{N-i} p^{i-1}}{\mathcal{B}(N-i+1,i)} & 0 < r < R_{in} \\ \frac{2(\pi-4\theta)r}{W} \frac{(1-q)^{N-i} q^{i-1}}{\mathcal{B}(N-i+1,i)} & R_{in} < r < R_c \\ 0 & R_c < r \end{cases}$$

where $\theta = \cos^{-1}(R_{in}/r)$, $p = \frac{\pi r^2}{W}$, $q = \frac{\pi r^2 - 4r^2\theta + 2r^2 \sin(2\theta)}{W}$, $R_{in}$ and $R_c$ are the radii of the incircle and circumcircle of $W$.

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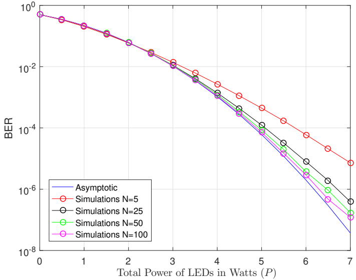
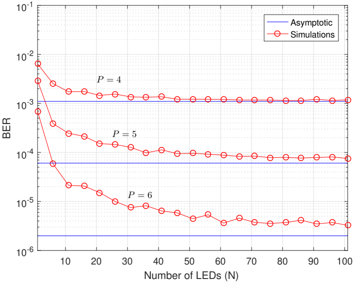
$$\mathbb{E}_\Phi[V_i] = \frac{1}{\sum_{j=1}^{N} \mathbb{E}_\Phi\left[r_j^\alpha\right]} \left[ \frac{R_{in}^\alpha}{h^{m+3}\mathcal{B}(N-i+1,i)} \times \right.$$

$$\sum_{k=0}^{N-i} \frac{\binom{N-i}{k}(-1)^k}{(k+i+\alpha/2)} \left(\frac{\pi R_{in}^2}{W}\right)^{k+i} {}_2F_1\left(\frac{m+3}{2}, k+i+\alpha/2; k+i+\alpha/2+1; -\frac{R_{in}^2}{h^2}\right)$$

$$\left. + \frac{R_{in}^\alpha}{\mathcal{B}(N-i+1,i)} \sum_{k=0}^{N-i} \binom{N-i}{k}(-1)^k \left(\frac{R_{in}^2}{W}\right)^{k+i} \int_0^{\frac{\pi}{4}} \frac{2(\pi-4\theta)\sin(\theta)(\pi-4\theta+2\sin(2\theta))^{k+i-1}}{\cos^{2(k+i-1)+\alpha-m}(\theta)\left(\sqrt{R_{in}^2 + h^2\cos^2(\theta)}\right)^{m+3}} d\theta \right]$$

## Performance of Stochastic LED Arrays based VLC with Uniform Irradiance




(a) BER performance with respect to source power for different number of LEDs

(b) Convergence of BER with respect to number of LEDs

Closed-form expressions of BER for BPP in circular region

Source LEDs are distributed uniformly in circular region of radius $R_c$(circumference of dimension of room).

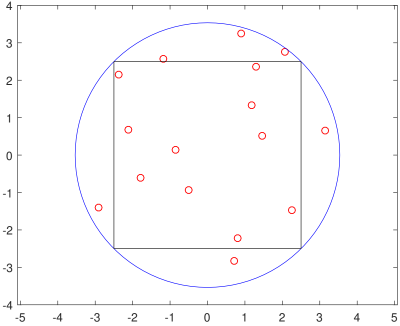



Figure 9: Realization of a circular BPP

The PDF of the distance of $i^{th}$ nearest LED location from origin for circular BPP for $0 \leq r \leq R_c$ is given by

$$f_{r_i} = \frac{2r}{R_c^2 B\left(i, N-i+1\right)} \left(\frac{r^2}{R_c^2}\right)^{i-1} \left(1 - \frac{r^2}{R_c^2}\right)^{N-i}$$

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$$\mathbb{E}_{\Phi}\left[r_i^{\alpha}\right] = \frac{R_c^{\alpha}}{\mathcal{B}\left(N-i+1, i\right)} \sum_{k=0}^{N-i} \frac{\binom{N-i}{k}\left(-1\right)^k}{\left(i+k+\frac{\alpha}{2}\right)}$$

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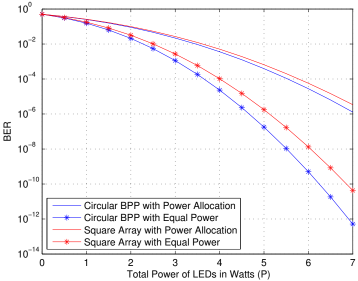
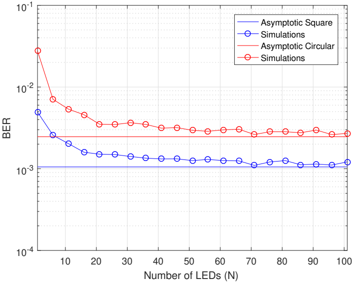
$$\mathbb{E}_\Phi\left[V_i\right] = \frac{R_c^\alpha}{h^{m+3}\mathcal{B}\left(N-i+1, i\right)\sum_{j=1}^{N}\mathbb{E}_\Phi\left[r_j^\alpha\right]} \sum_{k=0}^{N-i} \frac{\binom{N-i}{k}(-1)^k}{\left(i+k+\frac{\alpha}{2}\right)} \times {}_2F_1\left(\frac{m+3}{2}, i+k+\frac{\alpha}{2}; i+k+\frac{\alpha}{2}+1; -\frac{R_c^2}{h^2}\right)$$

# Closed-form expressions of BER for BPP in circular region




BER can be used to estimate the cost of the system in terms of the number of LEDs

Optimal Power Allocation for Uniform Illumination in VLC Systems

Given a distribution of source LEDs, How to find the optimum power allocation for uniform illuminance ?

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Minimize the variance of the received power

$$\underset{P_{t_i}}{\text{minimize}} \quad \mathbb{E}\left[\left(P_{r_j} - \mathbb{E}\left[P_{r_j}\right]\right)^2\right]$$

Given a distribution of source LEDs, How to find the optimum power allocation for uniform illuminance ?

Minimize the variance of the received power

$$\underset{P_{t_i}}{\text{minimize}} \quad \mathbb{E}\left[\left(P_{r_j} - \mathbb{E}\left[P_{r_j}\right]\right)^2\right]$$

$$\begin{aligned} var(P_{r_j}) = {} & \frac{1}{2}\sum_{i=1}^{N}\left(\frac{2\sum_{j=1}^{K} H_{ij}^2}{K} - \frac{2\left(\sum_{j=1}^{K} H_{ij}\right)^2}{K^2}\right) P_{t_i}^2 \\ & + 2\sum_{u=1}^{N}\sum_{v=i+1}^{N}\left(\frac{2\sum_{j=1}^{K} H_{uj}H_{vj}}{K}\right. \\ & \left. - \frac{2\left(\sum_{j=1}^{K} H_{uj}\right)\left(\sum_{j=1}^{K} H_{vj}\right)}{K^2}\right) P_{t_u} P_{t_v} \end{aligned}$$

The objective function is expressed in quadratic form as

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \frac{1}{2}\mathbf{x}^T \mathcal{P} \mathbf{x} \\ \text{subject to} \quad & \mathcal{G}\mathbf{x} \geq \mathbf{0} \\ & \mathcal{A}\mathbf{x} = P \end{aligned}$$

where $\mathcal{G} = diag\,(-1, \cdots, -1)$, $\mathcal{A} = \ [1, \cdots, 1]$ and elements $\beta_{ij}$ of matrix $\mathcal{P}$ is given by

$$\beta_{ij} = \begin{cases} \frac{2\sum_{p=1}^{K} a_{ip}^2}{K} - \frac{2\left(\sum_{p=1}^{K} a_{ip}\right)^2}{K^2}, & i = j \\ \frac{2\sum_{p=1}^{K} a_{ip} a_{jp}}{K} - \frac{2\left(\sum_{p=1}^{K} a_i p\right)\left(\sum_{p=1}^{K} a_j p\right)}{K^2}, & i \neq j \end{cases}$$

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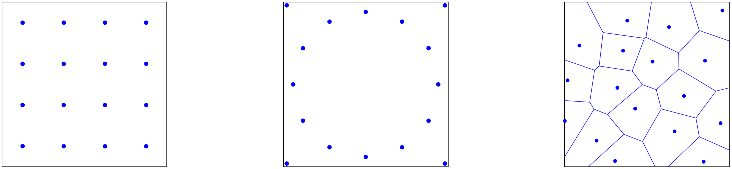
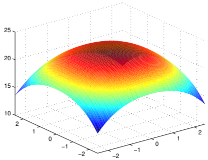
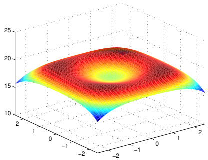
Numerically solved using quadratic programming (QP) through the solvers.qp command in Python using CVXOPT solver.



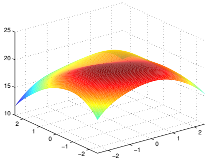
(a) Square Array (b) Circle Square (c) Realization of HCPP



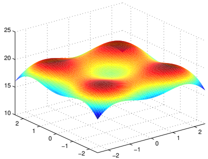

(d) Square array with equal power



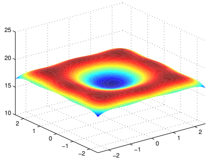

(e) Circle square with equal power



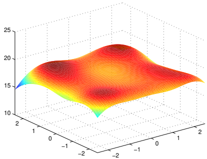

(f) HCPP with equal power




(g) Square array with optimum power




(h) Circle square with optimum power




(i) HCPP with optimum power

Table 0: Variance of received power

| | Equal power | Heuristic power | Optimum power |
|---|---|---|---|
| Square Array | 6.4414e-15 | 6.5287e-16 | 6.3785e-16 |
| Circle Square | 3.0134e-15 | 1.1267e-15 | 8.0382e-16 |
| BPP | 5.1986e-14 | 3.9660e-14 | 9.8679e-15 |
| HCPP | 2.2504e-14 | 1.2721e-14 | 2.2736e-15 |

Table 1: Quality factor

| | Equal power | Heuristic power | Optimum power |
|---|---|---|---|
| Square Array | 1.34 | 3.30 | 3.28 |
| Circle Square | 3.79 | 4.48 | 5.06 |
| BPP | 1.27 | 2.80 | 3.49 |
| HCPP | 1.49 | 3.27 | 4.73 |

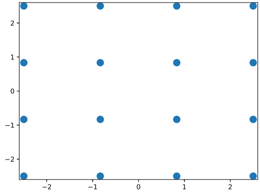
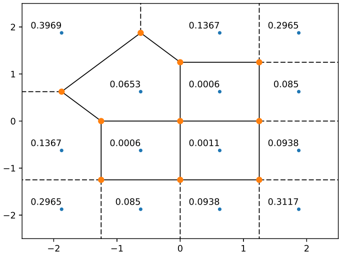
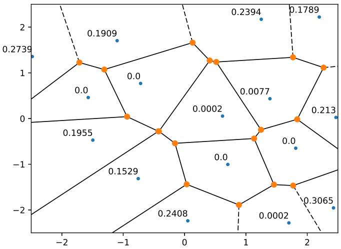



Table 2: Quality factor

| | Equal power | Heuristic power | Optimum power |
|---|---|---|---|
| Square Array | 1.34 | 3.30 | 3.28 |
| Circle Square | 3.79 | 4.48 | 5.06 |
| BPP | 1.27 | 2.80 | 3.49 |
| HCPP | 1.49 | 3.27 | 4.73 |
| Square Array *optimumlocation* | 2.62 | 10.78 | 11.55 |




(a) Square array with optimum power allocation




(b) HCPP realization with optimum power allocation

# Publications


1. G. V. S. S. Praneeth Varma, Rayapati Sushma, Vandana Sharma, Abhinav Kumar, and G. V. V. Sharma, “Power allocation for uniform illumination with stochastic LED arrays,” Optics Express **25**(8), 8659–8669 (2017).
2. G. V. S. S. Praneeth Varma, Abhinav Kumar, and G. V. V. Sharma, “Resource Allocation for Visible Light Communication using Stochastic Geometry,” (Submitted).
3. G. V. S. S. Praneeth Varma, Abhinav Kumar, and G. V. V. Sharma, “Resource Allocation for Visible Light Communication using Stochastic Geometry,” (Submitted).

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# Thank You

## Proof for asymptotic BER

$$P_e = \mathbb{E}_\Phi\left[Q\left(\frac{RC_1\sum_{i=1}^{N} V_i}{\sigma_0}\right)\right] \tag{1}$$

*Jensens Inequality:* If $X$ is a random variable (RV) and $f$ is a convex function, then ,

$$f\left(\mathbb{E}\left[X\right]\right) \leq \mathbb{E}\left[f\left(X\right)\right] \tag{2}$$

$$P_e \geq Q\left(\mathbb{E}_\Phi\left[\frac{RC_1\sum_{i=1}^{N} V_i}{\sigma_0}\right]\right) \tag{3}$$

since $Q\left(\cdot\right)$ is convex.

## Lemma

*Consider random variables X and Y where Y either has no mass at 0 (discrete) or has support* $[0, \infty)$. *Then*

$$\mathbb{E}\left[f\left(X, Y\right)\right] \approx f\left(\mathbb{E}\left[X\right], \mathbb{E}\left[Y\right]\right) \tag{4}$$

## Corollary

$$\mathbb{E}\left[\frac{X}{Y}\right] \approx \frac{\mathbb{E}\left[X\right]}{\mathbb{E}\left[Y\right]} \tag{5}$$

$$\mathbb{E}\left[X^2\right] \approx \left(\mathbb{E}\left[X\right]\right)^2 \tag{6}$$

Since $\sigma_0 > 0$, using (5) and (6) in (3),

$$Q\left(\mathbb{E}_\Phi\left[\frac{RC_1 \sum_{i=1}^{N} V_i}{\sigma_0}\right]\right) \approx Q\left(\frac{RC_1 \sum_{i=1}^{N} \mathbb{E}_\Phi\left[V_i\right]}{\mathbb{E}_\Phi\left[\sigma_0\right]}\right)$$

$$= Q\left(\frac{RC_1 \sum_{i=1}^{N} \mathbb{E}_\Phi\left[V_i\right]}{\sqrt{\mathbb{E}_\Phi\left[\sigma_0^2\right]}}\right) \tag{7}$$

resulting in

$$P_e \gtrapprox Q\left(\frac{RC_1 \sum_{i=1}^{N} \mathbb{E}_\Phi\left[V_i\right]}{\sqrt{\mathbb{E}_\Phi\left[\sigma_0^2\right]}}\right) \tag{8}$$

## BER Analysis for Square BPP

$$\begin{aligned}
\mathbb{E}_{\Phi}\left[r_i^{\alpha}\right] &= \int_{-\infty}^{\infty} r^{\alpha} f_{r_i}(r) dr \\
&= \int_{0}^{R_i} r^{\alpha} \frac{2\pi r}{W} \frac{(1-p)^{N-i} p^{i-1}}{\mathcal{B}(N-i+1, i)} dr \\
&\quad + \int_{R_i}^{R_c} r^{\alpha} \frac{2(\pi - 4\theta) r}{W} \frac{(1-q)^{N-i} q^{i-1}}{\mathcal{B}(N-i+1, i)} dr \\
&= \mathcal{I}_1 + \mathcal{I}_2
\end{aligned}$$

where

$$\mathcal{I}_1 = \int_0^{R_i} r^\alpha \frac{2\pi r}{W} \frac{(1-p)^{N-i} p^{i-1}}{\mathcal{B}(N-i+1,i)} dr$$

$$= \frac{2\pi}{W\mathcal{B}(N-i+1,i)} \sum_{k=0}^{N-i} \binom{N-i}{k} (-1)^k \left(\frac{\pi}{W}\right)^{k+i-1}$$

$$\times \int_0^{R_i} r^{2(k+i+\alpha/2+1/2)} dr$$

$$= \frac{1}{\mathcal{B}(N-i+1,i)} \sum_{k=0}^{N-i} \frac{\binom{N-i}{k}(-1)^k}{k+i+\alpha/2} \left(\frac{\pi}{W}\right)^{k+i} R_i^{2(k+i)+\alpha} \qquad (9)$$

$$\mathcal{I}_2 = \int_{R_i}^{R_c} r^\alpha \frac{2(\pi-4\theta) r}{W} \frac{(1-q)^{N-i} q^{i-1}}{\mathcal{B}(N-i+1,i)} dr$$

$$= \frac{1}{\mathcal{B}(N-i+1,i)} \sum_{k=0}^{N-i} \frac{\binom{N-i}{k}(-1)^k}{W^{k+i}} R_i^{2(k+i)+\alpha-1}$$

$$\times \int_0^{\frac{\pi}{4}} \frac{2(\pi-4\theta)(\pi-4\theta+2\sin(2\theta))^{k+i-1}}{\cos^{2(k+i)+\alpha-1}(\theta)} d\theta \qquad (10)$$

For simplifying the analysis, using the approximation

$$\sum_{j=1}^{N} \mathbb{E}_{\Phi}\left[r_j^{\alpha}\right] \approx \sum_{j=1}^{N} r_j^{\alpha} \tag{11}$$

in (1),

$$\mathbb{E}_{\Phi}\left[V_i\right] = \frac{1}{\sum_{j=1}^{N} \mathbb{E}_{\Phi}\left[r_j^{\alpha}\right]} \mathbb{E}_{\Phi}\left[\frac{r^{\alpha}}{\left(\sqrt{r^2+h^2}\right)^{m+3}}\right] \tag{12}$$

$$= \frac{\mathcal{J}_1 + \mathcal{J}_2}{\sum_{j=1}^{N} \mathbb{E}_{\Phi}\left[r_j^{\alpha}\right]} \tag{13}$$

where

$$
\begin{aligned}
\mathcal{J}_1 &= \int_0^{R_i} \frac{r^\alpha}{\left(\sqrt{r^2+h^2}\right)^{m+3}} \frac{2\pi r}{W} \frac{(1-p)^{N-i} p^{i-1}}{\mathcal{B}(N-i+1,i)} dr \\
&= \frac{R_i^\alpha}{h^{m+3}\mathcal{B}(N-i+1,i)} \sum_{k=0}^{N-i} \binom{N-i}{k} (-1)^k \left(\frac{\pi R_i^2}{W}\right)^{k+i} \\
&\quad \times \int_0^{R_i^2/h^2} \frac{t^{(k+i+\alpha/2)-1}}{(1+t)^{\frac{m+3}{2}}} dt
\end{aligned} \tag{14}
$$

after some algebra.

$$\int_0^u \frac{x^{\mu-1}}{(1+\beta x)^{\nu}} dx = \frac{u^{\mu}}{\mu} \, {}_2F_1\left(\nu, \mu; 1+\mu; -\beta u\right) \qquad \left[\left|arg\left(1+\beta u\right)\right| < \pi, Re\mu > 0\right] \tag{15}$$

Substituting the above in (14),

$$\mathcal{J}_1 = \frac{h^{\alpha}}{h^{m+3}\mathcal{B}\left(N-i+1, i\right)} \sum_{k=0}^{N-i} \binom{N-i}{k} (-1)^k \left(\frac{\pi h^2}{W}\right)^{k+i}$$
$$\times \, {}_2F_1\left(\frac{m+3}{2}, \left(k+i+\alpha/2\right), k+i+\alpha/2+1; -\frac{R_i^2}{h^2}\right) \tag{16}$$

Similarly,

$$\mathcal{J}_2 = \int_{R_i}^{R_c} \frac{r^{\alpha}}{\left(\sqrt{r^2+h^2}\right)^{m+3}} \frac{2\left(\pi-4\theta\right) r}{W} \frac{\left(1-q\right)^{N-i} q^{i-1}}{\mathcal{B}\left(N-i+1, i\right)} dr$$
$$= \frac{1}{\mathcal{B}\left(N-i+1, i\right)} \sum_{k=0}^{N-i} \frac{\binom{N-i}{k}(-1)^k}{W^{k+i}}$$
$$\times \int_0^{\frac{\pi}{4}} \frac{2\left(\pi-4\theta\right) R_i^{2(k+i)+\alpha-1}}{\cos^{2(k+i-2)+\alpha-m}\left(\theta\right)} \frac{\left(\pi-4\theta+2\sin\left(2\theta\right)\right)^{k+i-1}}{\left(\sqrt{R_i^2+h^2\cos^2\left(\theta\right)}\right)^{m+3}} d\theta \tag{17}$$

The probability density function (PDF) of the distance of $i^{th}$ nearest LED location from origin for circular BPP for $0 \leq r \leq R_c$ is given by

$$
\begin{aligned}
f_{r_i} &= \frac{2r}{R_c^2 B\left(i, N-i+1\right)} \left(\frac{r^2}{R_c^2}\right)^{i-1} \left(1 - \frac{r^2}{R_c^2}\right)^{N-i} \\
\Rightarrow \mathbb{E}_{\Phi}\left[r_i^{\alpha}\right] &= \int_0^{\infty} r^{\alpha} f_{r_i}(r) dr \\
&= \int_0^{R_c} \frac{2rr^{\alpha}}{R_c^2 B\left(N-i+1\right), i} \left(\frac{r^2}{R_c^2}\right)^{i-1} \left(1 - \frac{r^2}{R_c^2}\right)^{N-i} dr \\
&= \frac{1}{B\left(N-i+1, i\right)} \sum_{k=0}^{N-i} \binom{N-i}{k} (-1)^k \int_0^{R_c^2} \frac{t^{i+k+\alpha/2-1}}{R_c^{2(i+k)}} dt \\
&= \frac{R_c^{\alpha}}{B\left(N-i+1, i\right)} \sum_{k=0}^{N-i} \frac{\binom{N-i}{k} (-1)^k}{\left(i+k+\frac{\alpha}{2}\right)} \qquad (18)
\end{aligned}
$$

through a change of variables

$$
\begin{aligned}
\mathbb{E}_{\Phi}\left[V_i\right] &= \frac{1}{\sum_{j=1}^{N}\mathbb{E}_{\Phi}\left[r_j^{\alpha}\right]}\mathbb{E}_{\Phi}\left[\frac{r^{\alpha}}{\left(\sqrt{r^2+h^2}\right)^{m+3}}\right] \\
&= \frac{1}{\sum_{j=1}^{N}\mathbb{E}_{\Phi}\left[r_j^{\alpha}\right]}\int_0^{R_c}\frac{2r}{\mathcal{B}\left(N-i+1,i\right)}\sum_{k=0}^{N-i}\binom{N-i}{k}\frac{(-1)^k r^{2(i+k+\alpha/2-1)}}{R_c^{2(i+k)}\left(\sqrt{r^2+h^2}\right)^{m+3}}dr \\
&= \frac{1}{\mathcal{B}\left(N-i+1,i\right)\sum_{j=1}^{N}\mathbb{E}_{\Phi}\left[r_j^{\alpha}\right]}\int_0^{R_c^2/h^2}\sum_{k=0}^{N-i}\frac{\binom{N-i}{k}(-1)^k h^2\left(h^2 t\right)^{(i+k+\alpha/2-1)}}{R_c^{2(i+k)}h^{m+3}\left(\sqrt{1+t}\right)^{m+3}}dt \\
&= \frac{1}{B\left(N-i+1,i\right)\sum_{j=1}^{N}\mathbb{E}_{\Phi}\left[r_j^{\alpha}\right]}\sum_{k=0}^{N-i}\binom{N-i}{k}\frac{(-1)^k h^{2(i+k+\alpha/2)}}{h^{m+3}R_c^{2(i+k)}} \\
&\quad\times\int_0^{R_c^2/h^2}\frac{t^{(i+k+\alpha/2-1)}}{(1+t)^{(m+3)/2}}dt \\
&= \frac{R_c^{\alpha}h^{-(m+3)}}{B\left(N-i+1,i\right)\sum_{j=1}^{N}\mathbb{E}_{\Phi}\left[r_j^{\alpha}\right]} \\
&\quad\times\sum_{k=0}^{N-i}\frac{\binom{N-i}{k}(-1)^k}{\left(i+k+\frac{\alpha}{2}\right)}\,{}_2F_1\left(\frac{m+3}{2},i+k+\frac{\alpha}{2};i+k+\frac{\alpha}{2}+1;-\frac{R_c^2}{h^2}\right) \qquad (19)
\end{aligned}
$$

## Formulation of optimization problem

$$P_{r_j} = \sum_{i=1}^{N} H_{ij} P_{t_i} \tag{20}$$

$$var(P_{r_j}) = \mathbb{E}\left[\left(P_{r_j}\right)^2\right] - \left(\mathbb{E}\left[P_{r_j}\right]\right)^2$$

$$\begin{aligned}
\left(\mathbb{E}\left[P_{r_j}\right]\right)^2 &= \left(\frac{\sum_{j=1}^{K} P_{r_j}}{K}\right)^2 \\
&= \left(\frac{\sum_{j=1}^{K} \sum_{i=1}^{N} H_{ij} P_{t_i}}{K}\right)^2 \\
&= \left(\frac{\sum_{i=1}^{N} \gamma_i P_{t_i}}{K}\right)^2 \text{ where } \gamma_i = \sum_{j=1}^{K} H_{ij} \\
&= \frac{\sum_{i=1}^{N} \gamma_i^2 P_{t_i}^2 + 2\sum_{i=1}^{N} \sum_{p=i+1}^{N} \gamma_i \gamma_p P_{t_i} P_{t_p}}{K^2}
\end{aligned} \tag{21}$$

$$
\begin{aligned}
\mathbb{E}\left[\left(P_{r_j}\right)^2\right] &= \mathbb{E}\left[\left(\sum_{i=1}^{N} H_{ij} P_{t_i}\right)^2\right] && (22)\\
&= \mathbb{E}\left[\sum_{i=1}^{N} H_{ij}^2 P_{t_i}^2 + 2\sum_{i=1}^{N}\sum_{q=i+1}^{N} H_{ij} H_{qj} P_{t_i} P_{t_q}\right]\\
&= \frac{\sum_{j=1}^{K}\left[\sum_{i=1}^{N} H_{ij}^2 P_{t_i}^2 + 2\sum_{i=1}^{N}\sum_{q=i+1}^{N} H_{ij} H_{qj} P_{t_i} P_{t_q}\right]}{K}\\
&= \frac{\sum_{i=1}^{N} \mu_{ii} P_{t_i}^2 + 2\sum_{i=1}^{N}\sum_{q=i+1}^{N} \mu_{iq} P_{t_i} P_{t_q}}{K}\\
&\quad \text{where } \mu_{iq} = \sum_{j=1}^{K} H_{ij} H_{qj} && (23)
\end{aligned}
$$

$$
\begin{aligned}
var(P_{r_j}) &= \frac{\sum_{i=1}^{N} \mu_{ii} P_{t_i}^2 + 2\sum_{i=1}^{N}\sum_{q=i+1}^{N} \mu_{iq} P_{t_i} P_{t_q}}{K} \\
&\quad - \frac{\sum_{i=1}^{N} \gamma_i^2 P_{t_i}^2 + 2\sum_{i=1}^{N}\sum_{p=i+1}^{N} \gamma_i \gamma_p P_{t_i} P_{t_p}}{K^2} \\
&= \sum_{i=1}^{N} \frac{\mu_{ii}}{K} - \frac{\gamma_i^2}{K^2} P_{t_i}^2 + 2\sum_{i=1}^{N}\sum_{q=i+1}^{N} \frac{\mu_{iq}}{K} - \frac{\gamma_i\gamma_p}{K^2} P_{t_i} P_{t_q} \\
&= \frac{1}{2}\sum_{i=1}^{N}\left(\frac{2\sum_{j=1}^{K} H_{ij}^2}{K} - \frac{2\left(\sum_{j=1}^{K} H_{ij}\right)^2}{K^2}\right) P_{t_i}^2 \\
&\quad + 2\sum_{u=1}^{N}\sum_{v=i+1}^{N}\left(\frac{2\sum_{j=1}^{K} H_{uj}H_{vj}}{K} - \frac{2\left(\sum_{j=1}^{K} H_{uj}\right)\left(\sum_{j=1}^{K} H_{vj}\right)}{K^2}\right) P_{t_u} P_{t_v} \\
&= \frac{1}{2}\left[P_{t_1}, \cdots, P_{t_N}\right]\begin{bmatrix} \beta_{11} & \cdots & \beta_{1N} \\ \vdots & \ddots & \vdots \\ \beta_{N1} & \cdots & \beta_{NN} \end{bmatrix}\begin{bmatrix} P_{t_1} \\ \vdots \\ P_{t_N} \end{bmatrix} \\
&= \frac{1}{2} x^T \mathcal{P} x
\end{aligned}
\tag{24}
$$