Problem Set: Calculus

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I. Continuity

1) Show that the following functions are continuous by using $\epsilon - \delta$ definition.

   (i) x^n
   (ii) $\sin x$
   (iii) e^{2x}
   (iv) $\ln x$
   (v) $\sqrt{x}$
   (vi) $\sin^{-1}(x)$
   (vii) $x^2 + x$
   (viii) $\frac{x+1}{x-2}, x \neq 2$
   (ix) $\sinh 4x$
   (x) $\ln(\sin x)$
   (xi) $\sin(2x) + x^2$
   (xii) $(x+2)^3$

2) Investigate the continuity of each of the following functions at the indicated point x_0.

   (i) $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0; \\ 0, & x = 0 \end{cases}, x_0 = 0.$
   (ii) $f(x) = x - |x|, x_0 = 0.$
   (iii) $f(x) = \begin{cases} \frac{x^3-8}{x^2-4}, & x \neq 2; \\ 3, & x = 2 \end{cases}, x_0 = 2.$
   (iv) $f(x) = \begin{cases} \sin \pi x, & 0 < x < 1 \\ \ln x, & 1 < x < 2 \end{cases}, x_0 = 1.$

3) Give examples of functions with the following properties:
   (a) A function f which is continuous at *only* a finite number of points.
   (b) A function f which is discontinuous at *only* a finite number of points.
   (c) An f which is nowhere continuous on the real line.
   (d) An f which is continuous at every rational number.
   (e) An f which is discontinuous at every rational and continuous at every irrational on $(0, \infty)$.
   (f) A $c \in \mathbb{R}$ and two functions f, g that are discontinuous at c but such that $f + g$ and fg are continuous at c.

4) Prove that any polynomial of finite degree over $\mathbb{R}$ is continuous.Hence show that rational functions are continuous over their admissible domain.

5) Let f,g be two continuous functions on (a, b) such that $f(x) = g(x)$ for every rational $x \in (a, b)$.Prove that $f(x) = g(x)$ for all $x \in (a, b)$.

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6) Determine the domain of continuity of the following functions.

(i) $\sqrt{1-x^2}$

(ii) $\sin(e^{-x^2})$

(iii) $\ln(1+\sin x)$

(iv) $\frac{1+\cos x}{3+\sin x}$

(v) $\frac{1}{\sqrt{10+x}}$

(vi) $\sin \frac{1}{(x-1)^2}$

(vii) $\sin\left(\frac{1}{\cos x}\right)$

(viii) $\sqrt{(x-3)(6-x)}$

(ix) $\begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

(x) $\sqrt{x+\sqrt{x}}$

(xi) $\cos(\sqrt{1+x^2})$

(xii) $\frac{\sqrt{1+|\sin|x}}{x}$

7) (a) Prove that $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & x \neq 0 \\ 5, & x = 0 \end{cases}$, is not continuous at $x = 0$.

(b) Can you redefine $f(0)$ so that f is continuous at $x = 0$.

8) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$.
(a) If $f(r) = 0$ for every rational number r , then prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

(b) If $f(r) = 0$ for every irrational number r , then is it true that $f(x) = 0$ on whole of $\mathbb{R}$.

9) Show that every polynomial of odd degree with real coefficients has atleast one real root.

10) Prove that $\lim\limits_{x\to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{\sin x} = 0$.

11) Let I = $[a, b]$ and $f : I \to \mathbb{R}$ be a continuous function.
(a) If $f(x) > 0$ for all $x \in I$,show that there exists an $\alpha > 0$ such that $f(x) \geqslant \alpha$ for all $x \in I$.

(b) If for each $x \in I$ there exists a $y \in I$ such that $|f(y)| \leqslant \frac{1}{2}f(x)$,then show that there exists a $c \in I$ such that $f(c) = 0$.

(c) Show that f is bounded on I, i.e., there exists an $M > 0$ such that $|f(x)| < M$ for all $x \in I$. What if the interval $I = (-\infty, b]$ or $I = [a, \infty)$,will f still be bounded?

(d) Show that f attains both maximum and minimum values over I, i.e., there exists $p, q \in I$ such that $f(p) \leqslant f(x) \leqslant f(q)$ for all $x \in I$.What happens if either $a = -\infty$ or $b = \infty$?

(e) Let I =$[0, 1]$ and $f(0) = f(1)$. Prove that there exists a point $c \in [0, \frac{1}{2}]$ such that $f(c) = f\left(c + \frac{1}{2}\right)$.

(f) Let I =$[0,1]$ and let $0 \leqslant f(x) \leqslant 1$ for all $x \in I$. Show that there exists a point $c \in I$ such that $f(c) = c$, i.e., f has a fixed point.

(g) Show that if $f(a) \leqslant a$ and $f(b) \geqslant b$ then f has a fixed point in I, i.e., there exists a point $c \in I$ such that $f(c) = c$.

12) Let $K > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy the condition $|f(x) - f(y)| \leqslant K|x - y|$ for all $x, y \in \mathbb{R}$. Show that f is continuous at every point $c \in \mathbb{R}$.

13) Define $g : \mathbb{R} \to \mathbb{R}$ as follows. $g(x) = \begin{cases} 2x, & x \in \mathbb{Q} \\ x+3, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Find all the points at which g is continuous.

14) Let $A \subseteq B \subseteq \mathbb{R}$, $f : \mathbb{B} \to \mathbb{R}$ and g be the restriction of f to A, i.e., $g(x) = f(x)$ for $x \in A$.
- If f is continuous at some $c \in A$, then show that g is also continuous at c.
- Show by example that g is continuous at some $c \in A$ does not necessarily imply that f is continuous at that c.

15) Let A, $B \subseteq \mathbb{R}$, $f : A \to B$ and $g : B \to \mathbb{R}$. If f is continuous on A and g is continuous on B, show that $g \circ f$ is continuous on $f(A)$.

16) While the intermediate value theorem assures you of a root of a continuous function which assumes both negative and positive values on a bounded interval,how do you actually find the roots? Read up on some such methods, viz., Bisection Method,Newton-Raphson method, etc. Investigate the additional requirements on f for such methods to be applicable.

17) Let $f : [a,b] \to \mathbb{R}$ be integrable on $[a,x]$ for every $x \in [a,b]$ and let the indefinite integral $A(x)$ be defined as

$$A(x) = \int_a^x f(t)dt.$$

Show that A is continuous on whole of $[a,b]$.(At each end point we have one-sided continuity.)

II. Differentiation

18) Find the derivatives of the following functions from the definition.

(i) $\frac{3+x}{3-x}, x \neq 3$

(ii) $\sqrt{2x-1}$

(iii) $\ln(1+\sin x)$

(iv) $\frac{1+\cos x}{3+\sin x}$

(v) $\frac{1}{\sqrt{10+x}}$

(vi) $\sin\frac{1}{(x-1)^2}$

(vii) $\sin\left(\frac{1}{\cos x}\right)$

(viii) $\sqrt{(x-3)(6-x)}$, $3 \leqslant x \leqslant 6$

(ix) $x^2 \sin\left(\frac{1}{x}\right), x \neq 0; f(0) = 0$

(x) $\sqrt{x + \sqrt{x}}$

(xi) $\cos(\sqrt{1+x^2})$

(xii) $\frac{\sqrt{1+|\sin x|}}{x}$

19) Let $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

(a). Is f differentiable at $x = 0$?

(b). Is f' is continuous at $x = 0$?

20) Use L'Hospital's rule to evaluate the following limits.

(i) $\lim\limits_{x\to 0} \frac{e^{2x}-1}{x}$

(ii) $\lim\limits_{x\to 0} \frac{1+\cos \pi x}{x^2-2x+1}$

(iii) $\lim\limits_{x\to \infty} \frac{3x^2-x+5}{5x^2+6x-3}$

(iv) $\lim\limits_{x\to \infty} x^2 e^{-x}$

(v) $\lim\limits_{x\to 0} (\cos x)^{\frac{1}{x^2}}$

(vi) $\lim\limits_{x\to 1} \frac{(2x-x^4)^{\frac{1}{2}} - x^{\frac{1}{3}}}{1-x^{\frac{3}{4}}}$

(vii) $\lim\limits_{x\to \infty} \frac{5x^2-3x}{7x^2+1}$

(viii) $\lim\limits_{x\to \infty} \left(x - \sqrt{x+x^2}\right)$

(ix) $\lim\limits_{x\to \infty} \frac{\sqrt{x+2}}{\sqrt{x+1}}$