6. Strategic-form Games

6.1 Strategic-form games with cardinal payoffs

At the end of Chapter 4 we discussed the possibility of incorporating random events in extensive-form games by means of chance moves. The introduction of chance moves gives rise to probabilistic outcomes and thus to the issue of how a player might rank such outcomes. Random events can also occur in strategic-form games, as shown in Figure 6.1, which represents the simple first-price auction of Example 6.1 below.

		Player 2	
		bid \$100	bid \$200
Player 1	bid \$100	$\begin{pmatrix} o_1 & o_2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$	o_3
	bid \$200	o_4	$\begin{pmatrix} o_3 & o_4 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

o_1 : Player 1 wins and pays \$100

o_2 : Player 2 wins and pays \$100

o_3 : Player 2 wins and pays \$200

o_4 : Player 1 wins and pays \$200

Figure 6.1: A game-frame in strategic form representing Example 6.1

■ **Example 6.1** Two players simultaneously submit a bid for a painting. Only two bids are possible: \$100 and \$200. If one player bids \$200 and the other \$100 then the high bidder wins the painting and has to pay her own bid. If the two players bid the same amount then a fair coin is tossed and if the outcome is Heads the winner is Player 1 (who then has to pay her own bid) while if the outcome is Tails the winner is Player 2 (who then has to pay his own bid). ■

Suppose that Player 1 ranks the basic outcomes as follows: $o_1 \succ_1 o_4 \succ_1 o_2 \sim_1 o_3$, that is, she prefers winning to not winning; conditional on winning, she prefers to pay less and, conditional on not winning, she is indifferent as to how much Player 2 pays. Suppose also that Player 1 believes that Player 2 is going to submit a bid of \$100 (perhaps she has been informed of this by somebody spying on Player 2). What should we expect Player 1 to do? Knowing her ranking of the basic outcomes is of no help, because we need to know how she ranks the probabilistic outcome $\begin{pmatrix} o_1 & o_2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ relative to the basic outcome o_4.

The theory of expected utility introduced in Chapter 5 provides one possible answer to the question of how players rank probabilistic outcomes. With the aid of expected utility theory we can now generalize the definition of strategic-form game. First we generalize the notion of game-frame in strategic form (Definition 2.1.1, Chapter 2) by allowing probabilistic outcomes, or lotteries, to be associated with strategy profiles. In the following definition, the bulleted items coincide with the first three items of Definition 2.1.1 (Chapter 2); the modified item is the last one, preceded by the symbol ★.

Definition 6.1.1 A *game-frame in strategic form* is a quadruple $\langle I, (S_i)_{i\in I}, O, f\rangle$ where:

- $I = \{1, \ldots, n\}$ is a set of *players* ($n \geq 2$).
- For every Player $i \in I$, S_i is the set of *strategies* (or choices) of Player i. As before, we denote by $S = S_1 \times \cdots \times S_n$ the set of *strategy profiles*.
- O is a set of *basic outcomes*.

★ $f : S \to \mathscr{L}(O)$ is a function that associates with every strategy profile s a lottery over the set of basic outcomes O (as in Chapter 5, we denote by $\mathscr{L}(O)$ the set of lotteries, or probability distributions, over O).

If, for every $s \in S$, $f(s)$ is a degenerate lottery (that is, a basic outcome) then we are back to Definition 2.1.1 (Chapter 2).

From a game-frame one obtains a game by adding, for every player $i \in I$, a von Neumann-Morgenstern ranking $\succsim_i$ of the elements of $\mathscr{L}(O)$. It is more convenient to represent such a ranking by means of a von Neumann-Morgenstern utility function $U_i : O \to \mathbb{R}$. We denote by $\mathbb{E}[U_i(f(s))]$ the expected utility of lottery $f(s) \in \mathscr{L}(O)$ for Player i. The following definition mirrors Definition 2.1.2 of Chapter 2.

Definition 6.1.2 A *game in strategic form with cardinal payoffs* is a quintuple $\langle I, (S_i)_{i\in I}, O, f, (\succsim_i)_{i\in I}\rangle$ where:

- $\langle I, (S_i)_{i\in I}, O, f\rangle$ is a game-frame in strategic form (Definition 6.1.1) and
- for every Player $i \in I$, $\succsim_i$ is a von Neumann-Morgenstern ranking of the set of lotteries $\mathscr{L}(O)$.

If we represent each ranking $\succsim_i$ by means of a von Neumann-Morgenstern utility function U_i and define $\pi_i : S \to \mathbb{R}$ by $\pi_i(s) = \mathbb{E}[U_i(f(s))]$, then $\langle I, (S_1, \ldots, S_n), (\pi_1, \ldots, \pi_n)\rangle$ is called a *reduced-form game in strategic form with cardinal payoffs* ('reduced-form' because some information is lost, namely the specification of the possible outcomes). The function $\pi_i : S \to \mathbb{R}$ is called the *von Neumann-Morgenstern payoff function of Player i*.

For example, consider the first-price auction of Example 6.1 whose game-frame in strategic form was shown in Figure 6.1. Let $O = \{o_1, o_2, o_3, o_4\}$ and suppose that Player 1 has a von Neumann-Morgenstern ranking of $\mathscr{L}(O)$ that is represented by the following von Neumann-Morgenstern utility function U_1 (note that the implied ordinal ranking of the basic outcomes is indeed $o_1 \succ_1 o_4 \succ_1 o_2 \sim_1 o_3$):

outcome:	o_1	o_2	o_3	o_4
U_1 :	4	1	1	2

Then, for Player 1, the expected utility of lottery $\begin{pmatrix} o_1 & o_2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is 2.5 and the expected utility of lottery $\begin{pmatrix} o_3 & o_4 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is 1.5.

Suppose also that Player 2 has (somewhat spiteful) preferences represented by the following von Neumann-Morgenstern utility function U_2:

outcome:	o_1	o_2	o_3	o_4
U_2 :	1	6	4	5

Thus, for Player 2, the expected utility of lottery $\begin{pmatrix} o_1 & o_2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is 3.5 and the expected utility of lottery $\begin{pmatrix} o_3 & o_4 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is 4.5. Then we can represent the game in reduced form as shown in Figure 6.2.

		Player 2	
		\$100	*\$200*
Player 1	*\$100*	2.5 , 3.5	1 , 4
	\$200	2 , 5	1.5 , 4.5

Figure 6.2: A cardinal game in reduced form based on the game-frame of 6.1

The game of Figure 6.2 does not have any Nash equilibria. However, we will show in the next section that if we extend the notion of strategy, by allowing players to choose randomly, then the game of Figure 6.2 does have a Nash equilibrium.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 6.5.1 at the end of this chapter.

6.2 Mixed strategies

Definition 6.2.1 Consider a game in strategic form with cardinal payoffs and recall that S_i denotes the set of strategies of Player i. From now on, we shall call S_i the set of *pure strategies* of Player i. We assume that S_i is a finite set (for every $i \in I$). A *mixed strategy* of Player i is a probability distribution over the set of pure strategies S_i. The set of mixed strategies of Player i is denoted by Σ_i.

R Since among the mixed strategies of Player i there are the degenerate strategies that assign probability 1 to a pure strategy, the set of mixed strategies includes the set of pure strategies (viewed as degenerate probability distributions).

For example, one possible mixed strategy for Player 1 in the game of Figure 6.2 is $\begin{pmatrix} \$100 & \$200 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$. The traditional interpretation of a mixed strategy is in terms of objective randomization: the player, instead of choosing a pure strategy herself, delegates the choice to a random device.[1] For example, Player 1 choosing the mixed strategy $\begin{pmatrix} \$100 & \$200 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ is interpreted as a decision to let, say, a die determine whether she will bid \$100 or \$200: Player 1 will roll a die and if the outcome is 1 or 2 then she will bid \$100, while if the outcome is 3, 4, 5 or 6 then she will bid \$200. Suppose that Player 1 chooses this mixed strategy and Player 2 chooses the mixed strategy $\begin{pmatrix} \$100 & \$200 \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}$. Since the players rely on independent random devices, this pair of mixed strategies gives rise to the following probabilistic outcome:

$$\begin{pmatrix} \text{strategy profile} & (\$100,\$100) & (\$100,\$200) & (\$200,\$100) & (\$200,\$200) \\ \text{outcome} & \begin{pmatrix} o_1 & o_2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} & o_3 & o_4 & \begin{pmatrix} o_3 & o_4 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \\ \text{probability} & \frac{1}{3}\left(\frac{3}{5}\right)=\frac{3}{15} & \frac{1}{3}\left(\frac{2}{5}\right)=\frac{2}{15} & \frac{2}{3}\left(\frac{3}{5}\right)=\frac{6}{15} & \frac{2}{3}\left(\frac{2}{5}\right)=\frac{4}{15} \end{pmatrix}$$

If the two players have von Neumann-Morgenstern preferences, then – by the Compound Lottery Axiom (Chapter 5) – they will view the above as the following lottery:

$$\begin{pmatrix} \text{outcome} & o_1 & o_2 & o_3 & o_4 \\ \text{probability} & \frac{3}{30} & \frac{3}{30} & \frac{8}{30} & \frac{16}{30} \end{pmatrix}.$$

[1] An alternative interpretation of mixed strategies in terms of beliefs will be discussed in Chapter 10 and in Part V (Chapters 14-16).

Using the von Neumann-Morgenstern utility functions postulated in the previous section, namely

$$\begin{array}{lcccc} outcome: & o_1 & o_2 & o_3 & o_4 \\ U_1: & 4 & 1 & 1 & 2 \end{array} \quad \text{and} \quad \begin{array}{lcccc} outcome: & o_1 & o_2 & o_3 & o_4 \\ U_2: & 1 & 6 & 4 & 5 \end{array}$$

the lottery $\begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ \frac{3}{30} & \frac{3}{30} & \frac{8}{30} & \frac{16}{30} \end{pmatrix}$ has an expected utility of

For Player 1: $\frac{3}{30}(4) + \frac{3}{30}(1) + \frac{8}{30}(1) + \frac{16}{30}(2) = \frac{55}{30}$

For Player 2: $\frac{3}{30}(1) + \frac{3}{30}(6) + \frac{8}{30}(4) + \frac{16}{30}(5) = \frac{133}{30}$.

Thus we can define the payoffs of the two players from this mixed strategy profile by

$$\Pi_1\left[\begin{pmatrix} \$100 & \$200 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \$100 & \$200 \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}\right] = \frac{55}{30}$$

$$\Pi_2\left[\begin{pmatrix} \$100 & \$200 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \$100 & \$200 \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}\right] = \frac{133}{30}$$

Note that we can calculate these payoffs in a different – but equivalent – way by using the reduced-form game of Figure 6.2, as follows:

$$\begin{pmatrix} \text{strategy profile} & (\$100,\$100) & (\$100,\$200) & (\$200,\$100) & (\$200,\$200) \\ \text{expected utilities} & (2.5, 3.5) & (1, 4) & (2, 5) & (1.5, 4.5) \\ \text{probability} & \frac{1}{3}\left(\frac{3}{5}\right) = \frac{3}{15} & \frac{1}{3}\left(\frac{2}{5}\right) = \frac{2}{15} & \frac{2}{3}\left(\frac{3}{5}\right) = \frac{6}{15} & \frac{2}{3}\left(\frac{2}{5}\right) = \frac{4}{15} \end{pmatrix}$$

so that the expected payoff of Player 1 is

$$\frac{3}{15}(2.5) + \frac{2}{15}(1) + \frac{6}{15}(2) + \frac{4}{15}(1.5) = \frac{55}{30}$$

and the expected payoff of Player 2 is

$$\frac{3}{15}(3.5) + \frac{2}{15}(4) + \frac{6}{15}(5) + \frac{4}{15}(4.5) = \frac{133}{30}.$$

The previous example provides the rationale for the following definition. First some notation.

- Let $\sigma_i \in \Sigma_i$ be a mixed strategy of Player i; then, for every pure strategy $s_i \in S_i$ of Player i, we denote by $\sigma_i(s_i)$ the probability that σ_i assigns to s_i.[2]
- Let Σ be the set of mixed-strategy profiles, that is, $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$.
- Consider a mixed-strategy profile $\sigma = (\sigma_1, ..., \sigma_n) \in \Sigma$ and a pure-strategy profile $s = (s_1, \ldots, s_n) \in S$; then we denote by $\sigma(s)$ the product of the probabilities $\sigma_i(s_i)$, that is, $\sigma(s) = \prod_{i=1}^{n} \sigma_i(s_i) = \sigma_1(s_1) \times ... \times \sigma_n(s_n)$.[3]

Definition 6.2.2 Consider a reduced-form game in strategic form with cardinal payoffs $G = \langle I, (S_1, ..., S_n), (\pi_1, ..., \pi_n) \rangle$ (Definition 6.1.2), where, for every Player $i \in I$, the set of pure strategies S_i is finite. Then the *mixed-strategy extension of* G is the reduced-form game in strategic form $\langle I, (\Sigma_1, ..., \Sigma_n), (\Pi_1, ..., \Pi_n) \rangle$ where, for every Player $i \in I$,

- Σ_i is the set of mixed strategies of Player i in G (that is, Σ_i is the set of probability distributions over S_i).
- The payoff function $\Pi_i : \Sigma \to \mathbb{R}$ is defined by $\Pi_i(\sigma) = \sum_{s \in S} \sigma(s)\, \pi_i(s)$.[a]

[a] In the above example, if $\sigma_1 = \begin{pmatrix} \$100 & \$200 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ and $\sigma_2 = \begin{pmatrix} \$100 & \$200 \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}$

then $\Pi_1(\sigma_1, \sigma_2) = \frac{3}{15}(2.5) + \frac{2}{15}(1) + \frac{6}{15}(2) + \frac{4}{15}(1.5) = \frac{55}{30}$.

Definition 6.2.3 Fix a reduced-form game in strategic form with cardinal payoffs $G = \langle I, (S_1, \ldots, S_n), (\pi_1, \ldots, \pi_n) \rangle$ (Definition 6.1.2), where, for every player $i \in I$, the set of pure strategies S_i is finite. A *Nash equilibrium in mixed-strategies of* G is a Nash equilibrium of the mixed-strategy extension of G.

[2] In the above example, if $\sigma_1 = \begin{pmatrix} \$100 & \$200 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ then $\sigma_1(\$200) = \frac{2}{3}$.

[3] In the above example, if $\sigma = (\sigma_1, \sigma_2)$ with $\sigma_1 = \begin{pmatrix} \$100 & \$200 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ and $\sigma_2 = \begin{pmatrix} \$100 & \$200 \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}$
then $\sigma_1(\$200) = \frac{2}{3}$, $\sigma_2(\$100) = \frac{3}{5}$ and thus $\sigma((\$200, \$100)) = \frac{2}{3}\left(\frac{3}{5}\right) = \frac{6}{15}$.

For example, consider the reduced-form game of Figure 6.3 (which reproduces Figure 6.2: with all the payoffs multiplied by 10; this corresponds to representing the preferences of the players with different utility functions that are a obtained from the ones used above by multiplying them by 10).

Is $\sigma = (\sigma_1, \sigma_2)$ with $\sigma_1 = \begin{pmatrix} \$100 & \$200 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ and $\sigma_2 = \begin{pmatrix} \$100 & \$200 \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}$ a mixed-strategy Nash equilibrium of this game?

Player 2

		$100	$200
Player 1	$100	25 35	10 40
	$200	20 50	15 45

Figure 6.3: The game of Figure 6.2 with the payoffs multiplied by 10

The payoff of Player 1 is

$$\Pi_1(\sigma_1, \sigma_2) = \tfrac{3}{15}(25) + \tfrac{2}{15}(10) + \tfrac{6}{15}(20) + \tfrac{4}{15}(15) = \tfrac{55}{3}.$$

If Player 1 switched from $\sigma_1 = \begin{pmatrix} \$100 & \$200 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ to $\hat{\sigma}_1 = \begin{pmatrix} \$100 & \$200 \\ 1 & 0 \end{pmatrix}$, that is, to the pure strategy \$100, then Player 1's payoff would be larger:

$$\Pi_1(\hat{\sigma}_1, \sigma_2) = \tfrac{3}{5}(25) + \tfrac{2}{5}(10) = 19.$$

Thus, since $19 > \frac{55}{3}$, it is not a Nash equilibrium.

John Nash (who shared the 1994 Nobel Memorial prize in economics with John Harsanyi and Reinhard Selten), proved the following theorem.

Theorem 6.2.1 — Nash, 1951. Every reduced-form game in strategic form with cardinal payoffs $\langle I, (S_1, \ldots, S_n), (\pi_1, \ldots, \pi_n) \rangle$ (Definition 6.1.2), where, for every Player $i \in I$, the set of pure strategies S_i is finite, has at least one Nash equilibrium in mixed-strategies.

We will not give the proof of this theorem, since it is rather complex (it requires the use of fixed-point theorems).

		Player 2	
		\$100	\$200
Player 1	\$100	25 35	10 40
	\$200	20 50	15 45

Going back to the game of Figure 6.3 reproduced above, let us verify that, on the other hand, $\sigma^* = (\sigma_1^*, \sigma_2^*)$ with $\sigma_1^* = \sigma_2^* = \begin{pmatrix} \$100 & \$200 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is a Nash equilibrium in mixed strategies. The payoff of Player 1 is

$$\Pi_1(\sigma_1^*, \sigma_2^*) = \tfrac{1}{4}(25) + \tfrac{1}{4}(10) + \tfrac{1}{4}(20) + \tfrac{1}{4}(15) = \tfrac{70}{4} = 17.5.$$

Could Player 1 obtain a larger payoff with some other mixed strategy $\sigma_1 = \begin{pmatrix} \$100 & \$200 \\ p & 1-p \end{pmatrix}$ for some $p \neq \frac{1}{2}$?

Fix an arbitrary $p \in [0,1]$ and let us compute Player 1's payoff if she uses the strategy $\sigma_1 = \begin{pmatrix} \$100 & \$200 \\ p & 1-p \end{pmatrix}$ against Player 2's mixed strategy $\sigma_2^* = \begin{pmatrix} \$100 & \$200 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$:

$$\Pi_1\left[\begin{pmatrix} \$100 & \$200 \\ p & 1-p \end{pmatrix}, \begin{pmatrix} \$100 & \$200 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right] = \tfrac{1}{2}p(25) + \tfrac{1}{2}p(10) + \tfrac{1}{2}(1-p)(20) + \tfrac{1}{2}(1-p)(15)$$

$$= p\left(\tfrac{1}{2}25 + \tfrac{1}{2}10\right) + (1-p)\left(\tfrac{1}{2}20 + \tfrac{1}{2}15\right) = \tfrac{35}{2} = 17.5.$$

Thus if Player 2 uses the mixed strategy $\sigma_2^* = \begin{pmatrix} \$100 & \$200 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, then *Player 1 gets the same payoff no matter what mixed strategy she employs.*

It follows that any mixed strategy of Player 1 is a best reply to $\sigma_2^* = \begin{pmatrix} \$100 & \$200 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$; in particular, $\sigma_1^* = \begin{pmatrix} \$100 & \$200 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is a best reply to $\sigma_2^* = \begin{pmatrix} \$100 & \$200 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

It is easy to verify that the same applies to Player 2: any mixed strategy of Player 2 is a best reply to Player 1's mixed strategy $\sigma_1^* = \begin{pmatrix} \$100 & \$200 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Hence $\sigma^* = (\sigma_1^*, \sigma_2^*)$ is indeed a Nash equilibrium in mixed strategies.

We will see in the next section that this "indifference" phenomenon is true in general.

R Since, among the mixed strategies of Player i there are the degenerate strategies that assign probability 1 to a pure strategy, every Nash equilibrium in pure strategies is also a Nash equilibrium in mixed strategies. That is, the set of mixed-strategy Nash equilibria includes the set of pure-strategy Nash equilibria.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 6.5.2 at the end of this chapter.

6.3 Computing the mixed-strategy Nash equilibria

How can we find the mixed-strategy equilibria of a given game? The first important observation is that if a pure strategy is strictly dominated by another pure strategy then it cannot be played with positive probability at a Nash equilibrium. Thus, for the purpose of finding Nash equilibria, one can delete all the strictly dominated strategies and focus on the resulting game. But then the same reasoning applies to the resulting game and one can delete all the strictly dominated strategies in that game, and so on. Thus we have the following observation.

R In order to find the mixed-strategy Nash equilibria of a game one can first apply the iterated deletion of strictly dominated strategies (IDSDS: Chapter 2) and then find the Nash equilibria of the resulting game (which can then be viewed as Nash equilibria of the original game where all the pure strategies that were deleted are assigned zero probability). Note, however, that – as we will see in Section 6.4 – one can perform more deletions than allowed by the IDSDS procedure.

For example, consider the game of Figure 6.4.

		Player 2					
		E		F		G	
Player 1	A	2	4	3	3	6	0
	B	4	0	2	4	4	2
	C	3	3	4	2	3	1
	D	3	6	1	1	2	6

Figure 6.4: A reduced-form game with cardinal payoffs

In this game there are no pure-strategy Nash equilibria; however, by Nash's theorem there will be at least one mixed-strategy equilibrium. To find it we can first note that, for Player 1, D is strictly dominated by B; deleting D we get a smaller game where, for Player 2, G is strictly dominated by F. Deleting G we are left with a smaller game where A is strictly dominated by C. Deleting A we are left with the game shown in Figure 6.5.

		Player 2			
		E		F	
Player 1	B	4	0	2	4
	C	3	3	4	2

Figure 6.5: The result of applying the IDSDS procedure to the game of Figure 6.4

We will see that the game of Figure 6.5 has a unique Nash equilibrium in mixed strategies given by $\left[\begin{pmatrix} B & C \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}, \begin{pmatrix} E & F \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}\right]$.

Thus the game of Figure 6.4 has a unique Nash equilibrium in mixed strategies given by

$$\left[\begin{pmatrix} A & B & C & D \\ 0 & \frac{1}{5} & \frac{4}{5} & 0 \end{pmatrix}, \begin{pmatrix} E & F & G \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}\right].$$

Once we have simplified the game by applying the IDSDS procedure, in order to find the mixed-strategy Nash equilibria we can use the following result.

First we recall some notation that was introduced in Chapter 2. Given a mixed-strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ and a Player i, we denote by σ_{-i} the profile of strategies of the players other than i and use (σ_i, σ_{-i}) as an alternative notation for σ; furthermore, (τ_i, σ_{-i}) denotes the result of replacing σ_i with τ_i in σ, that is, $(\tau_i, \sigma_{-i}) = (\sigma_1, ..., \sigma_{i-1}, \tau_i, \sigma_{i+1}, \ldots, \sigma_n)$.

Theorem 6.3.1 Consider a reduced-form game in strategic form with cardinal payoffs.
- Suppose that $\sigma^* = (\sigma_1^*, ..., \sigma_n^*)$ is a Nash equilibrium in mixed strategies.
- Consider an arbitrary Player i.
- Let $\pi_i^* = \Pi_i(\sigma^*)$ be the payoff of Player i at this Nash equilibrium and let $s_{ij}, s_{ik} \in S_i$ be two pure strategies of Player i such that $\sigma_i^*(s_{ij}) > 0$ and $\sigma_i^*(s_{ik}) > 0$, that is, s_{ij} and s_{ik} are two pure strategies to which the mixed strategy σ_i^* of Player i assigns positive probability.
- Then $\Pi_i\left(s_{ij}, \sigma_{-i}^*\right) = \Pi_i\left(s_{ik}, \sigma_{-i}^*\right) = \pi_i^*$.

In other words, when the other players use the mixed-strategy profile σ_{-i}^*, Player i gets the same payoff no matter whether she plays the mixed strategy σ_i^* or the pure strategy s_{ij} or the pure strategy s_{ik}.

The details of the proof of Theorem 6.3.1 will be omitted, but the idea is simple: if s_{ij} and s_{ik} are two pure strategies to which the mixed strategy σ_i^* of Player i assigns positive probability and $\Pi_i\left((s_{ij}, \sigma_{-i}^*)\right) > \Pi_i\left((s_{ik}, \sigma_{-i}^*)\right)$, then Player i can increase her payoff from $\pi_i^* = \Pi_i(\sigma^*)$ to a larger number by reducing the probability of s_{ik} to zero and adding that probability to $\sigma_i^*(s_{ij})$, that is, by switching from σ_i^* to the mixed strategy $\hat{\sigma}_i$ obtained as follows: $\hat{\sigma}_i(s_{ik}) = 0$, $\hat{\sigma}_i(s_{ij}) = \sigma_i^*(s_{ij}) + \sigma_i^*(s_{ik})$ and, for every other $s_i \in S_i$, $\hat{\sigma}_i(s_i) = \sigma_i^*(s_i)$. But this would contradict the hypothesis that $\sigma^* = (\sigma_1^*, ..., \sigma_n^*)$ is a Nash equilibrium.

Let us now go back to the game of Figure 6.5, which is reproduced in Figure 6.5, and see how we can use Theorem 6.3.1 to find the Nash equilibrium in mixed strategies.

		Player 2	
		E	F
Player 1	B	4 0	2 4
	C	3 3	4 2

Figure 6.6: Copy of Figure 6.5

We want to find values of p and q, strictly between 0 and 1, such that

$$\left[\begin{pmatrix} B & C \\ p & 1-p \end{pmatrix}, \begin{pmatrix} E & F \\ q & 1-q \end{pmatrix}\right]$$

is a Nash equilibrium.

By Theorem 6.3.1, if Player 1 played the pure strategy B against $\begin{pmatrix} E & F \\ q & 1-q \end{pmatrix}$ she should get the same payoff as if she were to play the pure strategy C.

The former would give her a payoff of $4q+2(1-q)$ and the latter a payoff of $3q+4(1-q)$.

Thus we need q to be such that $4q+2(1-q) = 3q+4(1-q)$, that is, $q = \frac{2}{3}$.

When $q = \frac{2}{3}$, both B and C give Player 1 a payoff of $\frac{10}{3}$ and thus any mixture of B and C would also give the same payoff of $\frac{10}{3}$.

In other words, Player 1 is indifferent among all her mixed strategies and thus any mixed strategy is a best response to $\begin{pmatrix} E & F \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$.

Similar reasoning for Player 2 reveals that, by Theorem 6.3.1, we need p to be such that $0p+3(1-p) = 4p+2(1-p)$, that is, $p = \frac{1}{5}$.

Against $\begin{pmatrix} B & C \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}$ any mixed strategy of Player 2 gives him the same payoff of $\frac{12}{5}$; thus any mixed strategy of Player 2 is a best reply to $\begin{pmatrix} B & C \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}$.

It follows that $\left[\begin{pmatrix} B & C \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}, \begin{pmatrix} E & F \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}\right]$ is a Nash equilibrium.

R It follows from Theorem 6.3.1, and was illustrated in the above example, that at a mixed strategy Nash equilibrium where Player i plays two or more pure strategies with positive probability, Player i does not have an incentive to use that mixed strategy: she would get the same payoff if, instead of randomizing, she played one of the pure strategies in the support of her mixed strategy (that is, if she increased the probability of any pure strategy from a positive number to 1).[4] *The only purpose of randomizing is to make the other player indifferent among two or more of his own pure strategies.*

Player 2

		D		E	
Player 1	A	3	0	0	2
	B	0	2	3	0
	C	2	0	2	1

Figure 6.7: A reduced-form game with cardinal payoffs

The "indifference" condition explained above provides a necessary, *but not sufficient*, condition for a mixed-strategy profile to be a Nash equilibrium. To see that the condition is not sufficient, consider the game of Figure 6.7 and the mixed-strategy profile $\left[\begin{pmatrix} A & B & C \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} D & E \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right]$. Given that Player 2 plays the mixed strategy $\begin{pmatrix} D & E \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, Player 1 is indifferent between the two pure strategies that are in the support of her own mixed strategy, namely A and B: the payoff from playing A is 1.5 and so is the payoff from playing B (and 1.5 is also the payoff associated with the mixed strategy under consideration). However, the profile $\left[\begin{pmatrix} A & B & C \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} D & E \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right]$ is not a Nash equilibrium, because Player 1 could get a payoff of 2 by switching to the pure strategy C.

We know from Theorem 6.2.1 that this game does have a mixed-strategy Nash equilibrium. How can we find it? Let us calculate the best response of Player 1 to every possible mixed strategy $\begin{pmatrix} D & E \\ q & 1-q \end{pmatrix}$ of Player 2 (with $q \in [0,1]$).

For Player 1 the payoff from playing A against $\begin{pmatrix} D & E \\ q & 1-q \end{pmatrix}$ is $3q$, the payoff from playing B is $3-3q$ and the payoff from playing C is constant and equal to 2. These functions are shown in Figure 6.8.

[4] The support of a mixed strategy is the set of pure strategies that are assigned positive probability by that mixed strategy.

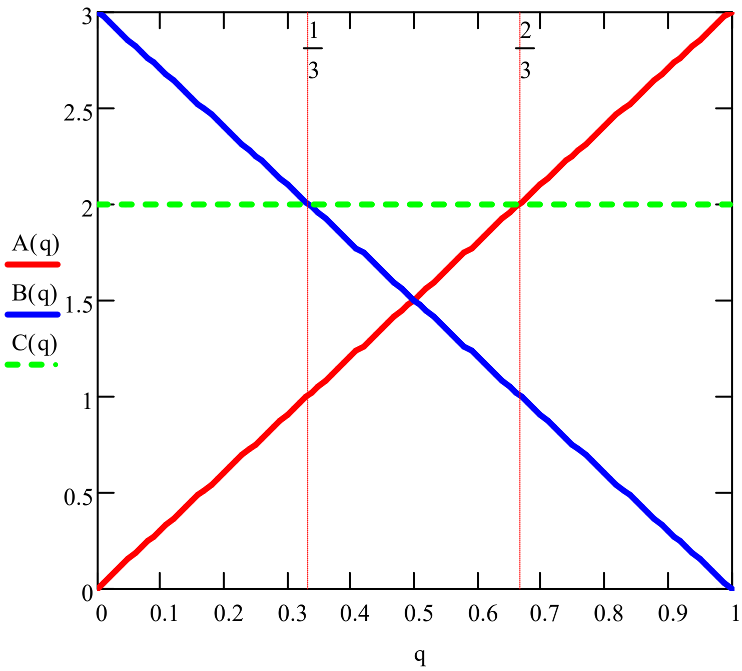

Figure 6.8: Player 1's payoff from each pure strategy against an arbitrary mixed strategy of Player 2

The upward-sloping line plots the function $A(q) = 3q$, the downward-sloping line plots the function $B(q) = 3 - 3q$ and the horizontal dashed line the function $C(q) = 2$.

The downward-sloping and horizontal lines intersect when $q = \frac{1}{3}$ and the upward-sloping and horizontal lines intersect when $q = \frac{2}{3}$.

The maximum payoff is given by the downward-sloping line up to $q = \frac{1}{3}$, then by the horizontal line up to $q = \frac{2}{3}$ and then by the upward-sloping line.

Thus the best reply function of Player 1 is as follows:

$$\text{Player1's best reply} = \begin{cases} B & \text{if } 0 \le q < \frac{1}{3} \\ \begin{pmatrix} B & C \\ p & 1-p \end{pmatrix} \text{ for any } p \in [0,1] & \text{if } q = \frac{1}{3} \\ C & \text{if } \frac{1}{3} < q < \frac{2}{3} \\ \begin{pmatrix} A & C \\ p & 1-p \end{pmatrix} \text{ for any } p \in [0,1] & \text{if } q = \frac{2}{3} \\ A & \text{if } \frac{2}{3} < q \le 1 \end{cases}$$

Hence if there is a mixed-strategy Nash equilibrium it is either of the form

$$\left[\begin{pmatrix} A & B & C \\ 0 & p & 1-p \end{pmatrix}, \begin{pmatrix} D & E \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}\right] \quad \text{or of the form} \quad \left[\begin{pmatrix} A & B & C \\ p & 0 & 1-p \end{pmatrix}, \begin{pmatrix} D & E \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}\right].$$

The latter cannot be a Nash equilibrium for any p, because when Player 1 plays B with probability 0, E strictly dominates D for Player 2 and thus Player 2's mixed strategy is not a best reply (E is the unique best reply). Thus the only candidate for a Nash equilibrium is of the form

$$\left[\begin{pmatrix} A & B & C \\ 0 & p & 1-p \end{pmatrix}, \begin{pmatrix} D & E \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}\right].$$

In this case, by Theorem 6.3.1, we need p to be such that Player 2 is indifferent between D and E: we need $2p = 1-p$, that is, $p = \frac{1}{3}$. Hence the Nash equilibrium is

$$\left[\begin{pmatrix} A & B & C \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}, \begin{pmatrix} D & E \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}\right].$$

In games where the number of strategies or the number of players are larger than in the examples we have considered so far, finding the Nash equilibria involves lengthier calculations. However, computer programs have been developed that can be used to compute all the Nash equilibria of a finite game in a very short time.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 6.5.3 at the end of this chapter.

6.4 Strict dominance and rationalizability

We remarked in the previous section that a pure strategy that is strictly dominated by another pure strategy cannot be played with positive probability at a Nash equilibrium. Thus, when looking for a Nash equilibrium, one can first simplify the game by applying the IDSDS procedure (Chapter 2). When payoffs are cardinal (von Neumann-Morgenstern payoffs) it turns out that, *in a two-person game*, a pure strategy cannot be a best response to any mixed-strategy of the opponent not only when it is strictly dominated by another pure strategy but also when it is strictly dominated *by a mixed strategy*. To see this, consider the game of Figure 6.9.

		Player 2	
		D	E
Player 1	A	0 1	4 0
	B	1 2	1 4
	C	2 0	0 1

Figure 6.9: A strategic-form game with cardinal payoffs

The pure strategy B of Player 1 is not strictly dominated by another pure strategy and yet it cannot be a best reply to any mixed strategy of Player 2.

To see this, consider an arbitrary mixed strategy $\begin{pmatrix} D & E \\ q & 1-q \end{pmatrix}$ of Player 2 with $q \in [0,1]$. If Player 1 plays B against it, she gets a payoff of 1; if, instead, she plays the mixed strategy $\begin{pmatrix} A & B & C \\ \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}$ then her payoff is $\frac{1}{3}4(1-q)+\frac{2}{3}2q=\frac{4}{3}>1$.

Theorem 6.4.1 — Pearce, 1984. Consider a two-player reduced-form game in strategic form with cardinal payoffs, an arbitrary Player i and a pure strategy s_i of Player i. Then there is no mixed-strategy of the opponent to which s_i is a best response, if and only if s_i is strictly dominated by a mixed strategy σ_i of Player i (that is, there is a $\sigma_i \in \Sigma_i$ such that $\Pi_i(\sigma_i,\sigma_j) > \Pi_i(s_i,\sigma_j)$, for every $\sigma_j \in \Sigma_j$).

Note that, since the set of mixed strategies includes the set of pure strategies, strict dominance by a mixed strategy includes as a sub-case strict dominance by a pure strategy.

When the number of players is 3 or more, the generalization of Theorem 6.4.1 raises some subtle issues: see Exercise 6.14. However, we can appeal to the intuition behind Theorem 6.4.1 (see the remark below) to refine the IDSDS procedure for general n-player games with cardinal payoffs as follows.

Definition 6.4.1 — Cardinal IDSDS. The *Cardinal Iterated Deletion of Strictly Dominated Strategies* is the following algorithm. Given a finite n-player ($n \geq 2$) strategic-form game with cardinal payoffs G, let G^1 be the game obtained by removing from G, for every Player i, those pure strategies of Player i (if any) that are strictly dominated in G by some *mixed* strategy of Player i; let G^2 be the game obtained by removing from G^1, for every Player i, those pure strategies of Player i (if any) that are strictly dominated in G^1 by some mixed strategy of Player i, and so on. Let G^∞ be the output of this procedure. Since the initial game G is finite, G^∞ will be obtained in a finite number of steps. For every Player i, the pure strategies of Player i in G^∞ are called her *rationalizable strategies*.

Figure 6.10 illustrates this procedure as applied to the game in Panel (i).

Player 2

Player 1	D		E		F	
A	3	4	2	1	1	2
B	0	0	1	3	4	1
C	1	4	1	4	2	6

(i) The game $G^0 = G$

Player 2

Player 1	D		E	
A	3	4	2	1
B	0	0	1	3

(iii) The game G^2 after Step 2

Player 2

Player 1	D		E		F	
A	3	4	2	1	1	2
B	0	0	1	3	4	1

(ii) The game G^1 after Step 1

Player 2

Player 1	D		E	
A	3	4	2	1

Player 2

Player 1	D	
A	3	4

(iv) The game G^3 at the top.
The game $G^4 = G^\infty$ at the bottom

Figure 6.10: Application of the cardinal IDSDS procedure

In the first step, the pure strategy C of Player 1 is deleted, because it is strictly dominated by the mixed strategy $\begin{pmatrix} A & B \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ thus yielding game G^1 shown in Panel (ii).

In the second step, the pure strategy F of Player 2 is deleted, because it is strictly dominated by the mixed strategy $\begin{pmatrix} D & E \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ thus yielding game G^2 shown in Panel (iii).

In the third step, B is deleted because it is strictly dominated by A thus yielding game G^3 shown in the top part of Panel (iv).

In the final step, E is deleted because it is strictly dominated by D so that the final output is the strategy profile (A,D).

Hence the only rationalizable strategies are A for Player 1 and D for Player 2.

Note that, in the game of Figure 6.10 Panel (i), since the only rationalizable strategy profile is (A,D), it follows that (A,D) is also the unique Nash equilibrium.

As noted in Chapter 2 the significance of the output of the IDSDS procedure is as follows. Consider game G in Panel (i) of Figure 5.9. Since, for Player 1, C is strictly dominated, if Player 1 is rational she will not play C. Thus, if Player 2 believes that Player 1 is rational then he believes that Player 1 will not play C, that is, he restricts attention to game G^1; since, in G^1, F is strictly dominated for Player 2, if Player 2 is rational he will not play F. It follows that if Player 1 believes that Player 2 is rational and that Player 2 believes that Player 1 is rational, then Player 1 restricts attention to game G^2 where rationality requires that Player 1 not play B, etc.

R Define a player to be rational if her chosen pure strategy is a best reply to her belief about what the opponent will do. In a two-player game a belief of Player 1 about what Player 2 will do can be expressed as a probability distribution over the set of pure strategies of Player 2; but this is the same object as a mixed strategy of Player 2. Thus, by Theorem 6.4.1, a rational Player 1 cannot choose a pure strategy that is strictly dominated by one of her own mixed strategies. The iterated reasoning outlined above can be captured by means of the notion of common knowledge of rationality. Indeed, it will be shown in Chapter 10 that if there is common knowledge of rationality then only rationalizable strategy profiles can be played. In a game with more than two players a belief of Player i about her opponents is no longer the same object as a mixed-strategy profile of the opponents, because a belief can allow for correlation in the behavior of the opponents, while the notion of mixed-strategy profile rules out such correlation (see Exercise 6.14).

R The iterated reasoning outlined above *requires that the von Neumann-Morgenstern preferences of both players be common knowledge between them.* For example, if Player 2 believes that Player 1 is rational but only knows her ordinal ranking of the outcomes, then Player 2 will not be able to deduce that it is irrational for Player 1 to play C and thus it cannot be irrational for him to play F. Expecting a player to know the von Neumann-Morgenstern preferences of another player is often (almost always?) very unrealistic! Thus one should be aware of the implicit assumptions that one makes (and one should question the assumptions made by others in their analyses).

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 6.5.4 at the end of this chapter.

6.5 Exercises

6.5.1 Exercises for Section 6.1: Strategic-form games with cardinal payoffs

The answers to the following exercises are in Section 6.6 at the end of this chapter.

Exercise 6.1 Consider the following game-frame in strategic form, where o_1, o_2, o_3 and o_4 are basic outcomes:

		Player 2	
		c	d
Player 1	a	o_1	o_2
	b	o_3	o_4

Both players satisfy the axioms of expected utility.
- The best outcome for Player 1 is o_3; she is indifferent between outcomes o_1 and o_4 and ranks them both as worst; she considers o_2 to be worse than o_3 and better than o_4; she is indifferent between o_2 with certainty and the lottery $\begin{pmatrix} o_3 & o_1 \\ 0.25 & 0.75 \end{pmatrix}$.
- The best outcome for Player 2 is o_4, which he considers to be just as good as o_1; he considers o_2 to be worse than o_1 and better than o_3; he is indifferent between o_2 with certainty and the lottery $\begin{pmatrix} o_1 & o_3 \\ 0.4 & 0.6 \end{pmatrix}$.

Find the normalized von Neumann-Morgenstern utility functions for the two players and write the corresponding reduced-form game. ■

Exercise 6.2 Consider the game-frame shown in Figure 6.11, where $o_1, \dots, o_4$ are basic outcomes. Both players have von Neumann-Morgenstern rankings of the basic outcomes. The ranking of Player 1 can be represented by the following von Neumann-Morgenstern utility function:

outcome:	o_1	o_2	o_3	o_4
U_1 :	12	10	6	16

and the ranking of Player 2 can be represented by the following von Neumann-Morgenstern utility function:

outcome:	o_1	o_2	o_3	o_4
U_2 :	6	14	8	10

Write the corresponding reduced-form game. ■

		Player 2	
		C	D
Player 1	A	$\begin{pmatrix} o_1 & o_4 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$	$\begin{pmatrix} o_1 & o_2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$
	B	o_3	$\begin{pmatrix} o_3 & o_4 \\ \frac{2}{5} & \frac{3}{5} \end{pmatrix}$

Figure 6.11: A game-frame in strategic form

6.5.2 Exercises for Section 6.2: Mixed strategies

The answers to the following exercises are in Section 6.6 at the end of this chapter.

		Player 2			
		D		E	
Player 1	A	0	1	6	3
	B	4	4	2	0
	C	3	0	4	2

Figure 6.12: A strategic-form game with cardinal payoffs

Exercise 6.3 Consider the reduced-form game with cardinal payoffs shown in Figure 6.12.

(a) Calculate the players' payoffs from the mixed strategy profile

$$\left[\begin{pmatrix} A & B & C \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix}\begin{pmatrix} D & E \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right].$$

(b) Is $\left[\begin{pmatrix} A & B & C \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix}\begin{pmatrix} D & E \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right]$ a Nash equilibrium? ■

Exercise 6.4 Consider the following reduced-form game with cardinal payoffs:

		Player 2	
		D	E
Player 1	A	2 , 3	8 , 5
	B	6 , 6	4 , 2

Prove that $\left[\begin{pmatrix} A & B \\ \frac{2}{3} & \frac{1}{3}\end{pmatrix}\begin{pmatrix} D & E \\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\right]$ is a Nash equilibrium. ■

6.5.3 Exercises for Section 6.3: Computing the mixed-strategy Nash equilibria

The answers to the following exercises are in Section 6.6 at the end of this chapter.

Exercise 6.5 Consider again the game of Exercise 6.1.

(a) Find the mixed-strategy Nash equilibrium.

(b) Calculate the payoffs of both players at the Nash equilibrium. ■

Exercise 6.6 Find the Nash equilibrium of the game of Exercise 6.2. ■

Exercise 6.7 Find all the mixed-strategy Nash equilibria of the game of Exercise 6.4 and calculate the payoffs of both players at every Nash equilibrium. ■

Exercise 6.8 Find the mixed-strategy Nash equilibria of the following game:

		Player 2	
		L	R
	T	1 , 4	4 , 3
Player 1	C	2 , 0	1 , 2
	B	1 , 5	0 , 6

■

Exercise 6.9 Consider the following two-player game, where $o_1, o_2, \ldots, o_6$ are basic outcomes.

		Player 2	
		d	e
	a	o_1	o_2
Player 1	b	o_3	o_4
	c	o_5	o_6

The players rank the outcomes as indicated below (as usual, if outcome o is above outcome o' then o is strictly preferred to o' and if o and o' are on the same row then the player is indifferent between the two):

$$\text{Player 1 :} \begin{pmatrix} o_1 \\ o_6 \\ o_4, o_2 \\ o_5 \\ o_3 \end{pmatrix} \qquad \text{Player 2 :} \begin{pmatrix} o_3, o_4 \\ o_2 \\ o_1, o_5 \\ o_6 \end{pmatrix}$$

(a) One player has a strategy that is strictly dominated. Identify the player and the strategy.

[Note: in order to answer the following questions, you can make your life a lot easier if you simplify the game on the basis of your answer to part **(a)**.]

Player 1 satisfies the axioms of Expected Utility Theory and is indifferent between o_6 and the lottery $\begin{pmatrix} o_1 & o_5 \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}$ and is indifferent between o_2 and the lottery $\begin{pmatrix} o_6 & o_5 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

(b) Suppose that Player 1 believes that Player 2 is going to play d with probability $\frac{1}{2}$ and e with probability $\frac{1}{2}$. Which strategy should he play?

Player 2 satisfies the axioms of Expected Utility Theory and is indifferent between o_5 and the lottery $\begin{pmatrix} o_2 & o_6 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$.

(c) Suppose that Player 2 believes that Player 1 is going to play a with probability $\frac{1}{4}$ and c with probability $\frac{3}{4}$. Which strategy should she play?

(d) Find all the (pure- and mixed-strategy) Nash equilibria of this game.

■

Exercise 6.10 Consider the following game (where the payoffs are von Neumann-Morgenstern payoffs):

		Player 2	
		C	D
Player 1	A	x , y	3 , 0
	B	6 , 2	0 , 4

(a) Suppose that $x = 2$ and $y = 2$. Find the mixed-strategy Nash equilibrium and calculate the payoffs of both players at the Nash equilibrium.

(b) (b) For what values of x and y is $\left[\begin{pmatrix} A & B \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}, \begin{pmatrix} C & D \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}\right]$ a Nash equilibrium?

■

Exercise 6.11 Find the mixed-strategy Nash equilibria of the game of Exercise 6.3. Calculate the payoffs of both players at every Nash equilibrium that you find. ■

6.5.4 Exercises for Section 6.4: Strict dominance and rationalizability

The answers to the following exercises are in Section 6.6 at the end of this chapter.

Exercise 6.12 In the following game, for each player, find all the rationalizable pure strategies (that is, apply the cardinal IDSDS procedure).

		Player 2		
		L	M	R
Player 1	A	3 , 5	2 , 0	2 , 2
	B	5 , 2	1 , 2	2 , 1
	C	9 , 0	1 , 5	3 , 2

■

Note: The next three exercises are more difficult than the previous ones.

Exercise 6.13 Is the following statement true or false? Either prove that it is true or give a counterexample.

> "Consider a two-player strategic-form game with cardinal payoffs.
> - Let A and B be two pure strategies of Player 1.
> - Suppose that both A and B are rationalizable (that is, they survive the cardinal IDSDS procedure).
> - Then any mixed strategy that attaches positive probability to both A and B and zero to every other strategy is a best reply to some mixed strategy of Player 2."

■

Exercise 6.14 Consider the three-player game shown in Figure 6.13, where only the payoffs of Player 1 are recorded.

(a) Show that if Player 1 assigns probability $\frac{1}{2}$ to the event "Player 2 will play E and Player 3 will play G" and probability $\frac{1}{2}$ to the event "Player 2 will play F and Player will play H", then playing D is a best reply.

Next we want to show that there is no mixed-strategy profile

$$\sigma_{-1} = \left(\begin{pmatrix} E & F \\ p & 1-p \end{pmatrix}, \begin{pmatrix} G & H \\ q & 1-q \end{pmatrix} \right)$$

of Players 2 and 3 against which D is a best reply for Player 1.
Define the following functions: $A(p,q) = \Pi_1(A, \sigma_{-1})$ (that is, $A(p,q)$ is Player 1's expected payoff if she plays the pure strategy A against σ_{-1}), $B(p,q) = \Pi_1(B, \sigma_{-1})$, $C(p,q) = \Pi_1(C, \sigma_{-1})$ and $D(p,q) = \Pi_1(D, \sigma_{-1})$.

(b) In the (p,q) plane (with $0 \le p \le 1$ and $0 \le q \le 1$) draw the curve corresponding to the equation $A(p,q) = D(p,q)$ and identify the region where $A(p,q) > D(p,q)$.

(c) In the (p,q) plane draw the curve corresponding to the equation $C(p,q) = D(p,q)$ and identify the region where $C(p,q) > D(p,q)$.

(d) In the (p,q) plane draw the two curves corresponding to the equation $B(p,q) = D(p,q)$ and identify the region where $B(p,q) > D(p,q)$.

(e) Infer from parts **(b)-(c)** that there is no mixed-strategy profile of Players 2 and 3 against which D is a best reply for Player 1.

■

Player 3: G

Player 1 \ Player 2	E	F
A	3	0
B	0	3
C	0	0
D	2	0

Player 3: H

Player 1 \ Player 2	E	F
A	0	0
B	3	0
C	0	3
D	0	2

Figure 6.13: A three-player game where only the payoffs of Player 1 are shown

Exercise 6.15 — ⋆⋆⋆ Challenging Question ⋆⋆⋆. A team of n professional swimmers ($n \geq 2$) – from now on called players – are partying on the bank of the Sacramento river on a cold day in January. Suddenly a passerby shouts "Help! My dog fell into the water!" Each of the swimmers has to decide whether or not to jump into the icy cold water to rescue the dog. One rescuer is sufficient: the dog will be saved if at least one player jumps into the water; if nobody does, then the dog will die. Each player prefers somebody else to jump in, but each player prefers to jump in himself if nobody else does.

Let us formulate this as a game. The strategy set of each player $i = 1,\ldots,n$ is $S_i = \{J, \neg J\}$, where J stands for 'jump in' and $\neg J$ for 'not jump in'.

The possible basic outcomes can be expressed as subsets of the set $I = \{1,...,n\}$ of players: outcome $N \subseteq I$ is interpreted as 'the players in the set N jump into the water'; if $N = \emptyset$ the dog dies, while if $N \neq \emptyset$ the dog is saved.

Player i has the following ordinal ranking of the outcomes:

(1) $N \sim N'$, for every $N \neq \emptyset$, $N' \neq \emptyset$ with $i \notin N$ and $i \notin N'$,

(2) $N \succ N'$ for every $N \neq \emptyset$, $N' \neq \emptyset$ with $i \notin N$ and $i \in N'$,

(3) $\{i\} \succ \emptyset$.

(a) Find all the pure-strategy Nash equilibria.

(b) Suppose that each player i has the following von Neumann-Morgenstern payoff function (which is consistent with the above ordinal ranking):

$$\pi_i(N) = \begin{cases} v & \text{if } N \neq \emptyset \text{ and } i \notin N \\ v - c & \text{if } N \neq \emptyset \text{ and } i \in N \\ 0 & \text{if } N = \emptyset \end{cases} \quad \text{with } \; 0 < c < v.$$

Find the symmetric mixed-strategy Nash equilibrium (symmetric means that all the players use the same mixed strategy).

(c) Assuming that the players behave according to the symmetric mixed-strategy Nash equilibrium of Part **(b)**, is it better for the dog if n (the number of players) is large or if n is small? Calculate the probability that the dog is saved at the mixed-strategy Nash equilibrium as a function of n, for all possible values of c and v (subject to $0 < c < v$), and plot it for the case where $c = 10$ and $v = 12$. ■

6.6 Solutions to exercises

Solution to Exercise 6.1. The normalized von Neumann-Morgenstern utility functions are:

Player 1:

outcome	U_1
o_3	1
o_2	0.25
o_1, o_4	0

Player 2:

outcome	U_2
o_1, o_4	1
o_2	0.4
o_3	0

The reduced-form game is shown in Figure 6.14. □

		Player 2	
		c	d
Player 1	a	0, 1	0.25, 0.4
	b	1, 0	0, 1

Figure 6.14: The reduced-form game for Exercise 6.1

Solution to Exercise 6.2.

The expected utility of the lottery $\begin{pmatrix} o_1 & o_4 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$ is $\frac{1}{4}(12)+\frac{3}{4}(16)=15$ for Player 1 and $\frac{1}{4}(6)+\frac{3}{4}(10)=9$ for Player 2.

The expected utility of the lottery $\begin{pmatrix} o_1 & o_2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is 11 for Player 1 and 10 for Player 2.

The expected utility of the lottery $\begin{pmatrix} o_3 & o_4 \\ \frac{2}{5} & \frac{3}{5} \end{pmatrix}$ is 12 for Player 1 and 9.2 for Player 2.

The reduced-form game is shown in Figure 6.15. □

		Player 2	
		C	D
Player 1	A	15, 9	11, 10
	B	6, 8	12, 9.2

Figure 6.15: The reduced-form game for Exercise 6.2

Solution to Exercise 6.3.

(a) $\Pi_1=\frac{1}{8}(0)+\frac{1}{8}(6)+\frac{3}{8}(4)+\frac{3}{8}(2)=3$ and $\Pi_2=\frac{1}{8}(1)+\frac{1}{8}(3)+\frac{3}{8}(4)+\frac{3}{8}(0)=2$.

(b) No, because if Player 1 switched to the pure strategy C then her payoff would be $\frac{1}{2}(3)+\frac{1}{2}(4)=3.5>3$. □

Solution to Exercise 6.4. Player 1's payoff is $\Pi_1 = \frac{2}{6}(2) + \frac{2}{6}(8) + \frac{1}{6}(6) + \frac{1}{6}(4) = 5$.

If Player 1 switches to any other mixed strategy $\begin{pmatrix} A & B \\ p & 1-p \end{pmatrix}$, while Player 2's strategy is kept fixed at $\begin{pmatrix} C & D \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, then her payoff is $\Pi_1 = \frac{1}{2}(p)(2) + \frac{1}{2}(p)(8) + \frac{1}{2}(1-p)(6) + \frac{1}{2}(1-p)(4) = 5$.

Thus any mixed strategy of Player 1 is a best response to $\begin{pmatrix} C & D \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

Similarly, Player 2's payoff is $\Pi_2 = \frac{2}{6}3 + \frac{2}{6}(5) + \frac{1}{6}(6) + \frac{1}{6}(2) = 4$. If Player 2 switches to any other mixed strategy $\begin{pmatrix} C & D \\ q & 1-q \end{pmatrix}$, while Player 1's strategy is kept fixed at $\begin{pmatrix} A & B \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$, then her payoff is $\Pi_2 = \frac{2}{3}(q)(3) + \frac{2}{3}(1-q)(5) + \frac{1}{3}(q)(6) + \frac{1}{3}(1-q)(2) = 4$.

Thus any mixed strategy of Player 2 is a best response to $\begin{pmatrix} A & B \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$.

Hence $\begin{pmatrix} A & B \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$ is a best reply to $\begin{pmatrix} C & D \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ and $\begin{pmatrix} C & D \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is a best reply to $\begin{pmatrix} A & B \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$, that is, $\left[\begin{pmatrix} A & B \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \begin{pmatrix} C & D \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \right]$ is a Nash equilibrium. □

Solution to Exercise 6.5.

(a) We have to find the Nash equilibrium of the following game:

		Player 2	
		c	d
Player 1	a	0 , 1	0.25 , 0.4
	b	1 , 0	0 , 1

To make calculations easier, let us multiply all the payoffs by 100 (that is, we re-scale the von Neumann-Morgenstern utility functions by a factor of 100):

		Player 2	
		c	d
Player 1	a	0 , 100	25 , 40
	b	100 , 0	0 , 100

There are no pure-strategy Nash equilibria. To find the mixed-strategy Nash equilibrium, let p be the probability with which Player 1 chooses a and q be the probability with which Player 2 chooses c.

Then, for Player 1, the payoff from playing a against $\begin{pmatrix} c & d \\ q & 1-q \end{pmatrix}$ must be equal to the payoff from playing b against $\begin{pmatrix} c & d \\ q & 1-q \end{pmatrix}$. That is, it must be that $25\,(1-q) = 100q$, which yields $q = \frac{1}{5}$. Similarly, for Player 2, the payoff from playing c against $\begin{pmatrix} a & b \\ p & 1-p \end{pmatrix}$ must be equal to the payoff from playing d against $\begin{pmatrix} a & b \\ p & 1-p \end{pmatrix}$. This requires $100p = 40p + 100(1-p)$, that is, $p = \frac{5}{8}$. Thus the Nash equilibrium is $\left[\begin{pmatrix} a & b \\ \frac{5}{8} & \frac{3}{8} \end{pmatrix}, \begin{pmatrix} c & d \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}\right]$.

(b) At the Nash equilibrium the payoffs are 20 for Player 1 and 62.5 for Player 2. (If you worked with the original payoffs, then the Nash equilibrium payoffs would be 0.2 for Player 1 and 0.625 for Player 2.) □

Solution to Exercise 6.6. We have to find the Nash equilibria of the following game.

		Player 2	
		C	D
Player 1	A	15 , 9	11 , 10
	B	6 , 8	12 , 9.2

For Player 2 D is a strictly dominant strategy, thus at a Nash equilibrium Player 2 must play D with probability 1. For Player 1, the unique best reply to D is B. Thus the pure-strategy profile (B,D) is the only Nash equilibrium. □

Solution to Exercise 6.7. We have to find the Nash equilibria of the following game.

		Player 2	
		D	E
Player 1	A	2 , 3	8 , 5
	B	6 , 6	4 , 2

(B,D) (with payoffs (6,6)) and (A,E) (with payoffs (8,5)) are both Nash equilibria. To see if there is also a mixed-strategy equilibrium we need to solve the following equations, where p is the probability of A and q is the probability of D: $2q + 8(1-q) = 6q + 4(1-q)$ and $3p + 6(1-p) = 5p + 2(1-p)$. The solution is $p = \frac{2}{3}$ and $q = \frac{1}{2}$ so that

$$\left[\begin{pmatrix} A & B \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \begin{pmatrix} D & E \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right]$$

is a Nash equilibrium. The payoffs at this Nash equilibrium are 5 for Player 1 and 4 for Player 2. □

Solution to Exercise 6.8. Since B is strictly dominated (by C), it cannot be assigned positive probability at a Nash equilibrium. Let p be the probability of T and q the probability of L. Then p must be such that $4p+0(1-p)=3p+2(1-p)$ and q must be such that $q+4(1-q)=2q+(1-q)$. Thus $p=\frac{2}{3}$ and $q=\frac{3}{4}$. Hence there is only one mixed-strategy equilibrium, namely

$$\left[\begin{pmatrix} T & C & B \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}, \begin{pmatrix} L & R \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}\right].$$

□

Solution to Exercise 6.9.

(a) Since Player 1 prefers o_5 to o_3 and prefers o_6 to o_4, strategy b is strictly dominated by strategy c.

Thus, at a Nash equilibrium, Player 1 will not play b with positive probability and we can simplify the game to

		Player 2	
		d	e
Player 1	a	o_1	o_2
	c	o_5	o_6

Of the remaining outcomes, for Player 1 o_1 is the best outcome (we can assign utility 1 to it) and o_5 is the worst (we can assign utility 0 to it). Since he is indifferent between o_6 and the lottery $\begin{pmatrix} o_1 & o_5 \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}$, the utility of o_6 is $\frac{4}{5}$. Hence the expected utility of $\begin{pmatrix} o_5 & o_6 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is $\frac{1}{2}(0)+\frac{1}{2}\left(\frac{4}{5}\right)=\frac{2}{5}$ and thus the utility of o_2 is also $\frac{2}{5}$.

(b) If Player 2 plays d with probability $\frac{1}{2}$ and e with probability $\frac{1}{2}$, then for Player 1 playing a gives a payoff of $\frac{1}{2}(1)+\frac{1}{2}\left(\frac{2}{5}\right)=\frac{7}{10}$, while playing c gives a payoff of $\frac{1}{2}(0)+\frac{1}{2}\left(\frac{4}{5}\right)=\frac{4}{10}$. Hence he should play a.

If you did not follow the suggestion to simplify the analysis as was done above, then you can still reach the same answer, although in a lengthier way. You would still set $U(o_1)=1$. Then the expected payoff from playing a is

$$\Pi_1(a)=\tfrac{1}{2}U(o_1)+\tfrac{1}{2}U(o_2)=\tfrac{1}{2}+\tfrac{1}{2}U(o_2) \qquad (\star)$$

Since o_2 is as good as $\begin{pmatrix} o_5 & o_6 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$,

$$U(o_2)=\tfrac{1}{2}U(o_5)+\tfrac{1}{2}U(o_6). \qquad (\diamond)$$

Since o_6 is as good as $\begin{pmatrix} o_1 & o_5 \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}$,

$$U(o_6) = \tfrac{4}{5} + \tfrac{1}{5}U(o_5). \qquad (\dagger)$$

Replacing ($\dagger$) in ($\diamond$)(we get $U(o_2) = \frac{2}{5} + \frac{3}{5}U(o_5)$ and replacing this expression in ($\star$) we get $\Pi_1(a) = \frac{7}{10} + \frac{3}{10}U(o_5)$. Similarly,

$$\Pi_1(c) = \tfrac{1}{2}U(o_5) + \tfrac{1}{2}U(o_6) = \tfrac{1}{2}U(o_5) + \tfrac{1}{2}\left(\tfrac{4}{5} + \tfrac{1}{5}U(o_5)\right) = \tfrac{4}{10} + \tfrac{6}{10}U(o_5)$$

Now, $\Pi_1(a) > \Pi_1(c)$ if and only if $\frac{7}{10} + \frac{3}{10}U(z_5) > \frac{4}{10} + \frac{6}{10}U(z_5)$ if and only if $3 > 3U(o_5)$ if and only if $U(o_5) < 1$, which is the case because o_5 is worse than o_1 and $U(o_1) = 1$. Similar steps would be taken to answer parts **(c)** and **(d)**.

(c) In the reduced game, for Player 2 o_2 is the best outcome (we can assign utility 1 to it) and o_6 is the worst (we can assign utility 0 to it). Thus, since she is indifferent between o_5 and the lottery $\begin{pmatrix} o_2 & o_6 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$, the utility of o_5 is $\frac{1}{4}$ and so is the utility of o_1. Thus playing d gives an expected payoff of $\frac{1}{4}(\frac{1}{4}) + \frac{3}{4}(\frac{1}{4}) = \frac{1}{4}$ and playing e gives an expected utility of $\frac{1}{4}(1) + \frac{3}{4}(0) = \frac{1}{4}$. Thus she is indifferent between playing d and playing e (and any mixture of d and e).

(d) Using the calculations of parts **(b)** and **(c)** the game is as follows:

		Player 2	
		d	e
Player	a	1 , $\frac{1}{4}$	$\frac{2}{5}$, 1
1	c	0 , $\frac{1}{4}$	$\frac{4}{5}$, 0

There is no pure-strategy Nash equilibrium. At a mixed-strategy Nash equilibrium, each player must be indifferent between his/her two strategies. From part **(c)** we already know that Player 2 is indifferent if Player 1 plays a with probability $\frac{1}{4}$ and c with probability $\frac{3}{4}$. Now let q be the probability with which Player 2 plays d. Then we need $q + \frac{2}{5}(1-q) = \frac{4}{5}(1-q)$, hence $q = \frac{2}{7}$. Thus the Nash equilibrium is $\left(\begin{array}{ccc|cc} a & b & c & d & e \\ \frac{1}{4} & 0 & \frac{3}{4} & \frac{2}{7} & \frac{5}{7} \end{array}\right)$ which can be written more succinctly as $\left(\begin{array}{cc|cc} a & c & d & e \\ \frac{1}{4} & \frac{3}{4} & \frac{2}{7} & \frac{5}{7} \end{array}\right)$. $\square$

Solution to Exercise 6.10.

(a) Let p be the probability of A and q the probability of B. Player 1 must be indifferent between playing A and playing B: $2q+3(1-q)=6q$; this gives $q=\frac{3}{7}$. Similarly, Player 2 must be indifferent between playing C and playing D: $2=4(1-p)$; this gives $p=\frac{1}{2}$. Thus the Nash equilibrium is given by

$$\left[\begin{pmatrix} A & B \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} C & D \\ \frac{3}{7} & \frac{4}{7} \end{pmatrix}\right]$$

The equilibrium payoffs are $\frac{18}{7}=2.57$ for Player 1 and 2 for Player 2.

(b) Player 1 must be indifferent between playing A and playing B: $\frac{3}{4}(x)+\frac{1}{4}(3)=\frac{3}{4}(6)$. Thus $x=5$. Similarly, Player 2 must be indifferent between playing C and playing D: $\frac{1}{5}(y)+\frac{4}{5}(2)=\frac{4}{5}(4)$. Thus $y=8$. $\square$

Solution to Exercise 6.11. We have to find the Nash equilibria of the following game:

		Player 2	
		D	E
	A	0 , 1	6 , 3
Player 1	B	4 , 4	2 , 0
	C	3 , 0	4 , 2

There are two pure-strategy equilibria, namely (B,D) and (A,E). To see if there is a mixed-strategy equilibrium we calculate the best response of Player 1 to every possible mixed strategy $\begin{pmatrix} D & E \\ q & 1-q \end{pmatrix}$ of Player 2 (with $q\in[0,1]$). For Player 1 the payoff from playing A against $\begin{pmatrix} D & E \\ q & 1-q \end{pmatrix}$ is $6-6q$, the payoff from playing B is $4q+2(1-q)=2+2q$ and the payoff from playing C is $3q+4(1-q)=4-q$. These functions are shown in Figure 6.16, where the downward-sloping line plots the function where $A(q)=6-6q$, the upward-sloping line plots the function $B(q)=2+2q$ and the dotted line the function $C(q)=4-q$.

It can be seen from Figure 6.16 that

$$\text{Player1's best reply} = \begin{cases} A & \text{if } 0\le q<\frac{2}{5} \\ \begin{pmatrix} A & C \\ p & 1-p \end{pmatrix} \text{ for any } p\in[0,1] & \text{if } q=\frac{2}{5} \\ C & \text{if } \frac{2}{5}<q<\frac{2}{3} \\ \begin{pmatrix} B & C \\ p & 1-p \end{pmatrix} \text{ for any } p\in[0,1] & \text{if } q=\frac{2}{3} \\ B & \text{if } \frac{2}{3}<q\le 1 \end{cases}$$

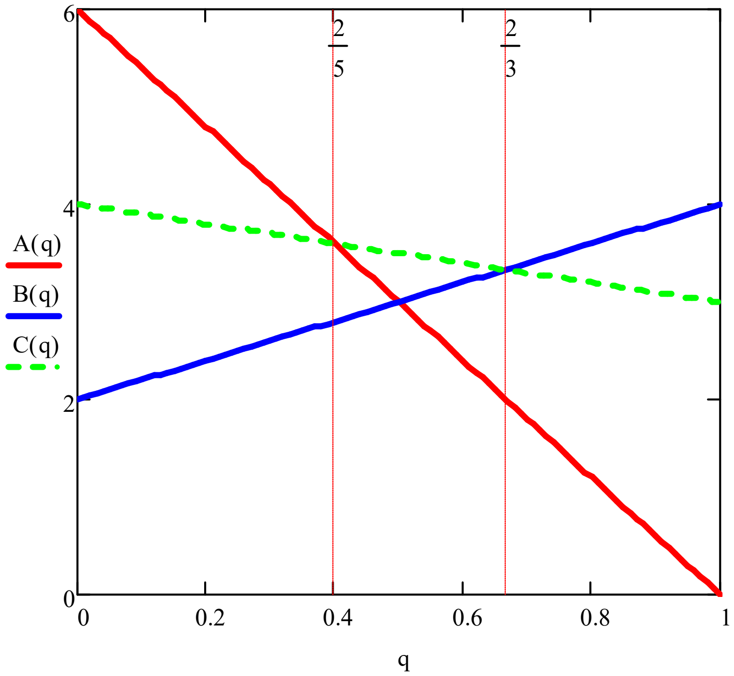

Figure 6.16: The best-reply diagram for Exercise 6.11

Thus if there is a mixed-strategy equilibrium it is either of the form

$$\left[\begin{pmatrix} A & C \\ p & 1-p \end{pmatrix}, \begin{pmatrix} D & E \\ \frac{2}{5} & \frac{3}{5} \end{pmatrix}\right] \quad \text{or of the form} \quad \left[\begin{pmatrix} B & C \\ p & 1-p \end{pmatrix}, \begin{pmatrix} D & E \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}\right].$$

In the first case, where Player 1 chooses B with probability zero, E strictly dominates D for Player 2 and thus $\begin{pmatrix} D & E \\ \frac{2}{5} & \frac{3}{5} \end{pmatrix}$ is not a best reply for Player 2, so that

$$\left[\begin{pmatrix} A & C \\ p & 1-p \end{pmatrix}, \begin{pmatrix} D & E \\ \frac{2}{5} & \frac{3}{5} \end{pmatrix}\right] \qquad \text{is not a Nash equilibrium for any} p.$$

In the second case we need $D(p) = E(p)$, that is, $4p = 2(1-p)$, which yields $p = \frac{1}{3}$. Thus the mixed-strategy Nash equilibrium is $\left[\begin{pmatrix} B & C \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}, \begin{pmatrix} D & E \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}\right]$ with payoffs of $\frac{10}{3}$ for Player 1 and $\frac{4}{3}$ for Player 2. $\square$

Solution to Exercise 6.12. For Player 1, B is strictly dominated by $\begin{pmatrix} A & C \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$; for Player 2, R is strictly dominated by $\begin{pmatrix} L & M \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Eliminating B and R we are left with

		Player 2 L	M
Player 1	A	3 , 5	2 , 0
	C	9 , 0	1 , 5

In this game no player has a strictly dominated strategy. Thus for Player 1 both A and C are rationalizable and for Player 2 both L and M are rationalizable. □

Solution to Exercise 6.13. The statement is false. Consider, for example, the following game:

		Player 2 L	R
	A	3 , 1	0 , 0
Player 1	B	0 , 0	3 , 1
	C	2 , 1	2 , 1

Here both A and B are rationalizable (indeed, they are both part of a Nash equilibrium; note that the cardinal IDSDS procedure leaves the game unchanged). However, the mixture $\begin{pmatrix} A & B \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ (which gives Player 1 a payoff of 1.5, no matter what Player 2 does) *cannot* be a best reply to any mixed strategy of Player 2, since it is strictly dominated by C. □

Solution to Exercise 6.14.

(a) If Player 1 assigns probability $\frac{1}{2}$ to the event "Player 2 will play E and Player 3 will play G" and probability $\frac{1}{2}$ to the event "Player 2 will play F and Player will play H", then A gives Player 1 an expected payoff of 1.5, B an expected payoff of 0, C an expected payoff of 1.5 and D an expected payoff of 2.
Thus D is a best reply to those beliefs.
The functions are as follows: $A(p,q) = 3pq$, $B(p,q) = 3(1-p)q + 3p(1-q)$, $C(p,q) = 3(1-p)(1-q)$, $D(p,q) = 2pq + 2(1-p)(1-q)$.

(b) $A(p,q) = D(p,q)$ at those points (p,q) such that $q = \frac{2-2p}{2-p}$. The set of such points is the continuous curve in the Figure 6.17. The region where $A(p,q) > D(p,q)$ is the region **above** the continuous curve.

(c) $C(p,q) = D(p,q)$ at those points (p,q) such that $q = \frac{1-p}{1+p}$. The set of such points is the dotted curve in the diagram shown in Figure 6.17. The region where $C(p,q) > D(p,q)$ is the region **below** the dotted curve.

(d) $B(p,q) = D(p,q)$ at those points (p,q) such that $q = \frac{2-5p}{5-10p}$ (for $p \neq \frac{1}{2}$). The set of such points is given by the two dashed curves in the diagram below. The region where $B(p,q) > D(p,q)$ is the region between the two dashed curves.

Thus

- in the region strictly above the continuous curve, A is better than D,
- in the region strictly below the dotted curve, C is better than D and
- in the region on and between the continuous curve and the dotted curve, B is better that D.

Hence, at every point in the (p,q) square there is a pure strategy of Player 1 which is strictly better than D. It follows that there is no mixed-strategy σ_{-1} against which D is a best reply. □

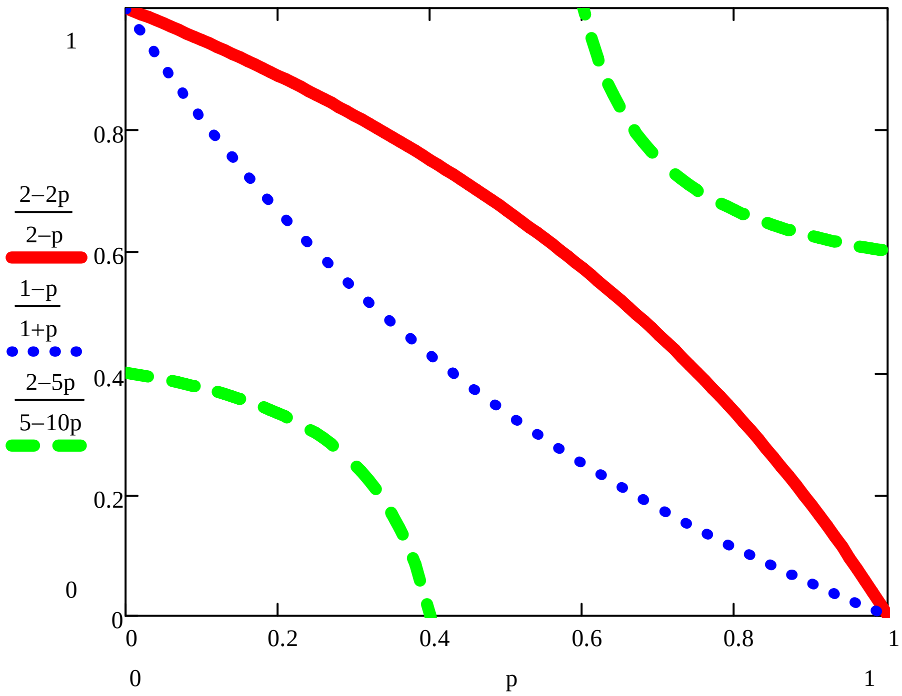

Figure 6.17: The diagram for Exercise 6.14

Solution to Exercise 6.15.

(a) There are n pure-strategy Nash equilibria: at each equilibrium exactly one player jumps in.

(b) Let p be the probability with which each player jumps into the water. Consider a Player i. The probability that none of the other players jump in is $(1-p)^{n-1}$ and thus the probability that somebody else jumps in is $\left[1-(1-p)^{n-1}\right]$.
Player i's payoff if he jumps in is $v-c$, while his expected payoff if he does not jump in is $v\left[1-(1-p)^{n-1}\right]+0(1-p)^{n-1}=v\left[1-(1-p)^{n-1}\right]$.

Thus we need $v-c=v\left[1-(1-p)^{n-1}\right]$, that is, $\boxed{p=1-\left(\frac{c}{v}\right)^{\frac{1}{n-1}}}$, which is strictly between 0 and 1 because $c<v$.

(c) At the Nash equilibrium the probability that nobody jumps in is $(1-p)^n = \left(\frac{c}{v}\right)^{\frac{n}{n-1}}$; thus this is the probability that the dog dies.
Hence, the dog is rescued with the remaining probability $1-\left(\frac{c}{v}\right)^{\frac{n}{n-1}}$.
This is a decreasing function of n. The larger the number of swimmers who are present, the more likely it is that the dog dies.
The plot of this function when $c = 10$ and $v = 12$ is shown in Figure 6.18. □

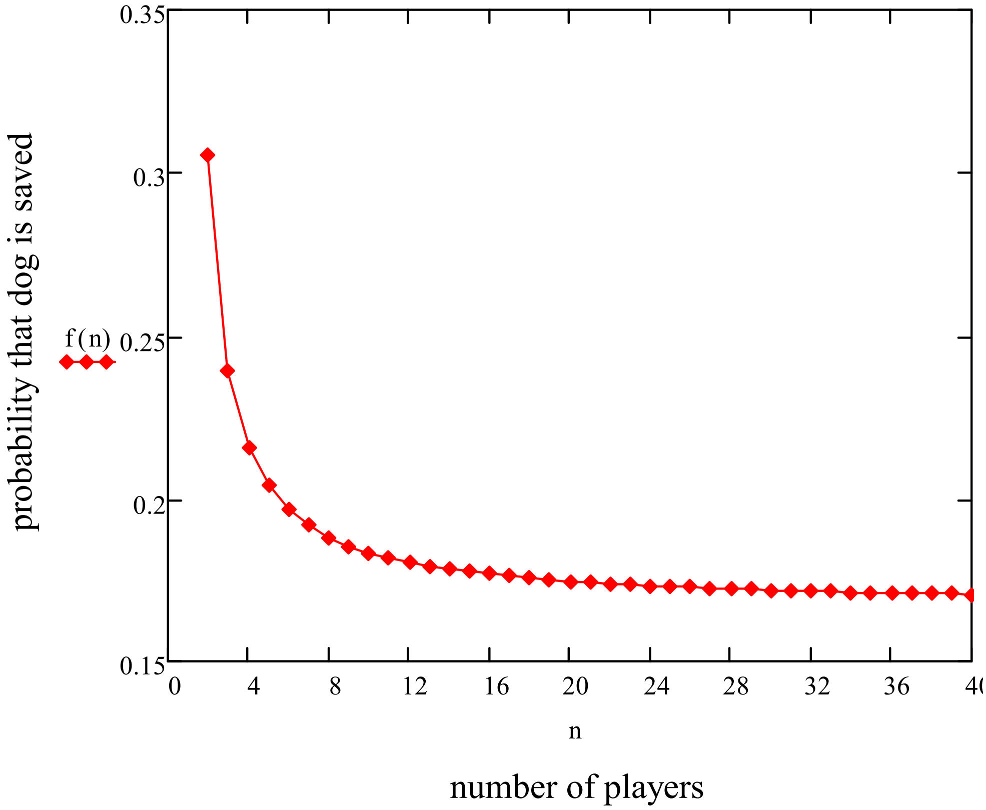

Figure 6.18: The probability that the dog is saved as a function of the number of potential rescuers