3. Perfect-information Games

3.1 Trees, frames and games

Often interactions are not simultaneous but sequential. For example, in the game of Chess the two players, White and Black, take turns moving pieces on the board, having full knowledge of the opponent's (and their own) past moves. Games with sequential interaction are called *dynamic games* or *games in extensive form*. This chapter is devoted to the subclass of dynamic games characterized by *perfect information*, namely the property that, whenever it is her turn to move, a player knows all the preceding moves.

Perfect-information games are represented by means of rooted directed trees.

Definition 3.1.1 A *rooted directed tree* consists of a set of nodes and directed edges joining them.

- The root of the tree has no directed edges leading to it (has indegree 0), while every other node has exactly one directed edge leading to it (has indegree 1).
- There is a unique path (that is, a unique sequence of directed edges) leading from the root to any other node. A node that has no directed edges out of it (has outdegree 0) is called a *terminal node*, while every other node is called a *decision node*.
- We shall denote the set of nodes by X, the set of decision nodes by D and the set of terminal nodes by Z. Thus $X = D \cup Z$.

Definition 3.1.2 A *finite extensive form (or frame) with perfect information* consists of the following items.

- A finite rooted directed tree.
- A set of players $I = \{1,\ldots,n\}$ and a function that assigns one player to every decision node.
- A set of actions A and a function that assigns one action to every directed edge, satisfying the restriction that no two edges out of the same node are assigned the same action.
- A set of outcomes O and a function that assigns an outcome to every terminal node.

■ **Example 3.1** Amy (Player 1) and Beth (Player 2) have decided to dissolve a business partnership whose assets have been valued at \$100,000. The charter of the partnership prescribes that the senior partner, Amy, make an offer concerning the division of the assets to the junior partner, Beth. The junior partner can *Accept*, in which case the proposed division is implemented, or *Reject*, in which case the case goes to litigation.

- Litigating involves a cost of \$20,000 in legal fees for each partner and the typical verdict assigns 60% of the assets to the senior partner and the remaining 40% to the junior partner.
- Suppose, for simplicity, that there is no uncertainty about the verdict (how to model uncertainty will be discussed in a later chapter). Suppose also that there are only two possible offers that Amy can make: a 50-50 split or a 70-30 split.

This situation can be represented as a finite extensive form with perfect information as shown in Figure 3.1. Each outcome is represented as two sums of money: the top one is what Player 1 gets and the bottom one what Player 2 gets. ■

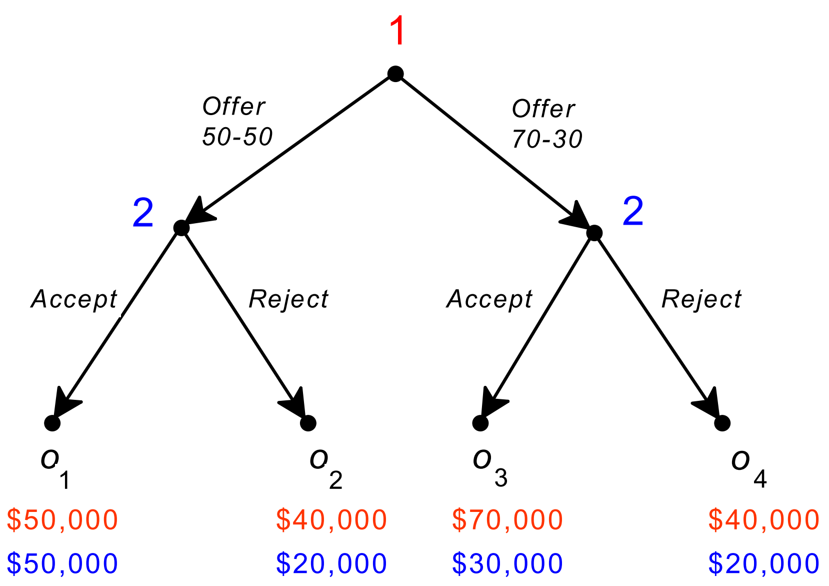


Figure 3.1: A perfect-information extensive form representing the situation described in Example 3.1

What should we expect the players to do in the game of Figure 3.1? Consider the following reasoning, which is called *backward induction* reasoning, because it starts from the end of the game and proceeds backwards towards the root:

- If Player 2 (the junior partner) is offered a 50-50 split then, if she accepts, she will get \$50,000, while, if she rejects, she will get \$20,000 (the court-assigned 40% minus legal fees in the amount of \$20,000); thus, if rational, she will accept.
- Similarly, if Player 2 is offered a 70-30 split then, if she accepts, she will get \$30,000, while, if she rejects, she will get \$20,000 (the court-assigned 40% minus legal fees in the amount of \$20,000); thus, if rational, she will accept.
- Anticipating all of this, Player 1 realizes that, if she offers a 50-50 split then she will end up with \$50,000, while if she offers a 70-30 split then she will end up with \$70,000; thus, if Player 1 is rational and believes that Player 2 is rational, she will offer a 70-30 split and Player 2, being rational, will accept.

The above reasoning suffers from the same flaw as the reasoning described in Chapter 2: it is not a valid argument because it is based on an implicit assumption about how Player 2 ranks the outcomes, which may or may not be correct. For example, Player 2 may feel that she worked as hard as her senior partner and the only fair division is a 50-50 split; indeed she may feel so strongly about this that – if offered an unfair 70-30 split – she would be willing to sacrifice \$10,000 in order to "teach a lesson to Player 1"; in other words, she ranks outcome o_4 above outcome o_3.

Using the terminology introduced in Chapter 2, we say that the situation represented in Figure 3.1 is not a game but a *game-frame*. In order to convert that frame into a game we need to add a ranking of the outcomes for each player.

Definition 3.1.3 A *finite extensive game with perfect information* is a finite extensive form with perfect information together with a ranking $\succsim_i$ of the set of outcomes O, for every player $i \in I$.

As usual, it is convenient to represent the ranking of Player i by means of an ordinal utility function $U_i : O \to \mathbb{R}$. For example, take the extensive form of Figure 3.1 and assume that Player 1 is selfish and greedy, that is, her ranking is:

best	o_3	
	o_1	(or, in the alternative notation, $o_3 \succ_1 o_1 \succ_1 o_2 \sim_1 o_4$).
worst	o_2, o_4	

while Player 2 is concerned with fairness and her ranking is:

best	o_1	
	o_2, o_4	(or, in the alternative notation, $o_1 \succ_2 o_2 \sim_2 o_4 \succ_2 o_3$)
worst	o_3	

Then we can represent the players' preferences using the following utility functions:

outcome $\to$ utility function $\downarrow$	o_1	o_2	o_3	o_4
U_1 (Player 1)	2	1	3	1
U_2 (Player 2)	3	2	1	2

and replace each outcome in Figure 3.1 with a pair of utilities or payoffs, as shown in Figure 3.2, thereby obtaining one of the many possible games based on the frame of Figure 3.1.

Now that we have a game (rather than just a game-frame), we can apply the backward-induction reasoning and conclude that Player 1 will offer a 50-50 split, anticipating that Player 2 would reject the offer of a 70-30 split, and Player 2 will accept Player 1's 50-50 offer. The choices selected by the backward-induction reasoning have been highlighted in Figure 3.2 by doubling the corresponding edges.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 3.6.1 at the end of this chapter.

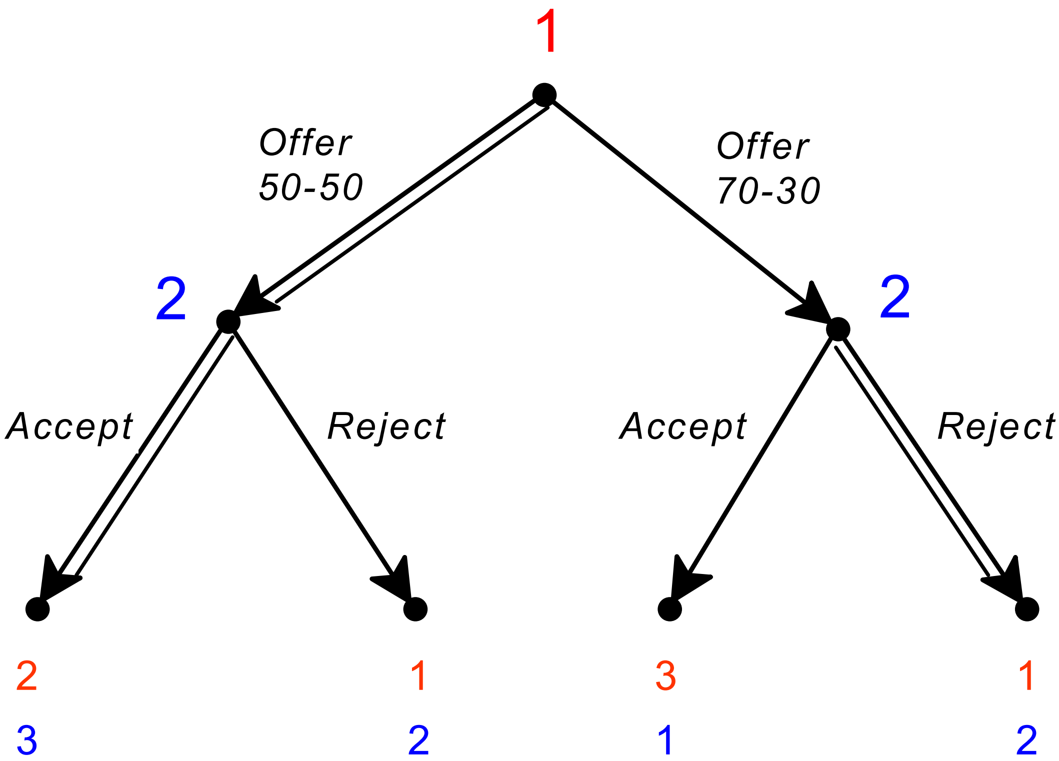


Figure 3.2: A perfect-information game based on the frame of Figure 3.1

3.2 Backward induction

The backward-induction reasoning mentioned above can be formalized as an algorithm for solving any finite perfect-information game. We say that a node is *marked* if a utility vector is associated with it. Initially all and only the terminal nodes are marked; the following procedure provides a way of marking all the nodes.

Definition 3.2.1 The *backward-induction algorithm* is the following procedure for solving a finite perfect-information game:

1. Select a decision node x whose immediate successors are all marked. Let i be the player who moves at x. Select a choice that leads to an immediate successor of x with the highest payoff (or utility) for Player i (highest among the utilities associated with the immediate successors of x). Mark x with the payoff vector associated with the node that follows the selected choice.

2. Repeat the above step until all the nodes have been marked.

Note that, since the game is finite, the above procedure is well defined. In the initial steps one starts at those decision nodes that are followed only by terminal nodes, call them penultimate nodes. After all the penultimate nodes have been marked, there will be unmarked nodes whose immediate successors are all marked and thus the step can be repeated. Note also that, in general, at a decision node there may be several choices that maximize the payoff of the player who moves at that node. If that is the case, then the procedure requires that *one* such choice be selected. This arbitrary selection may lead to the existence of several backward-induction solutions.

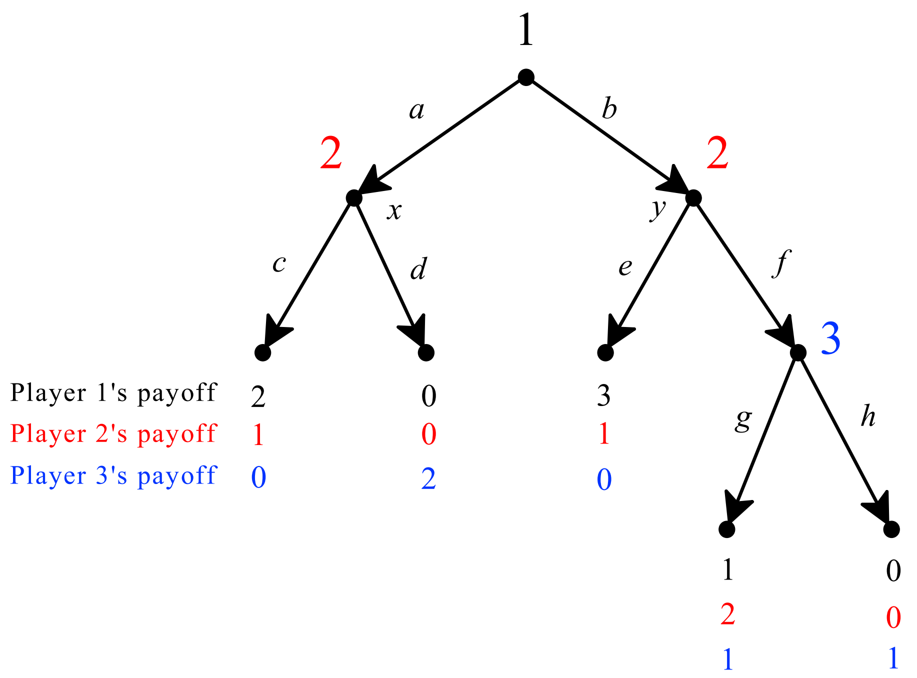


Figure 3.3: A perfect-information game with multiple backward-induction solutions.

For example, in the game of 3.3 starting at node x of Player 2 we select choice c (since it gives Player 2 a higher payoff than d). Then we move on to Player 3's node and we find that both choices there are payoff maximizing for Player 3; thus there are two ways to proceed, as shown in the next two figures.

In Figure 3.4 we show the steps of the backward-induction algorithm with the selection of choice g, while Figure 3.5 shows the steps of the algorithm with the selection of choice h. As before, the selected choices are shown by double edges. In Figures 3.4 and 3.5 the marking of nodes is shown explicitly, but later on we will represent the backward-induction solution more succinctly by merely highlighting the selected choices.

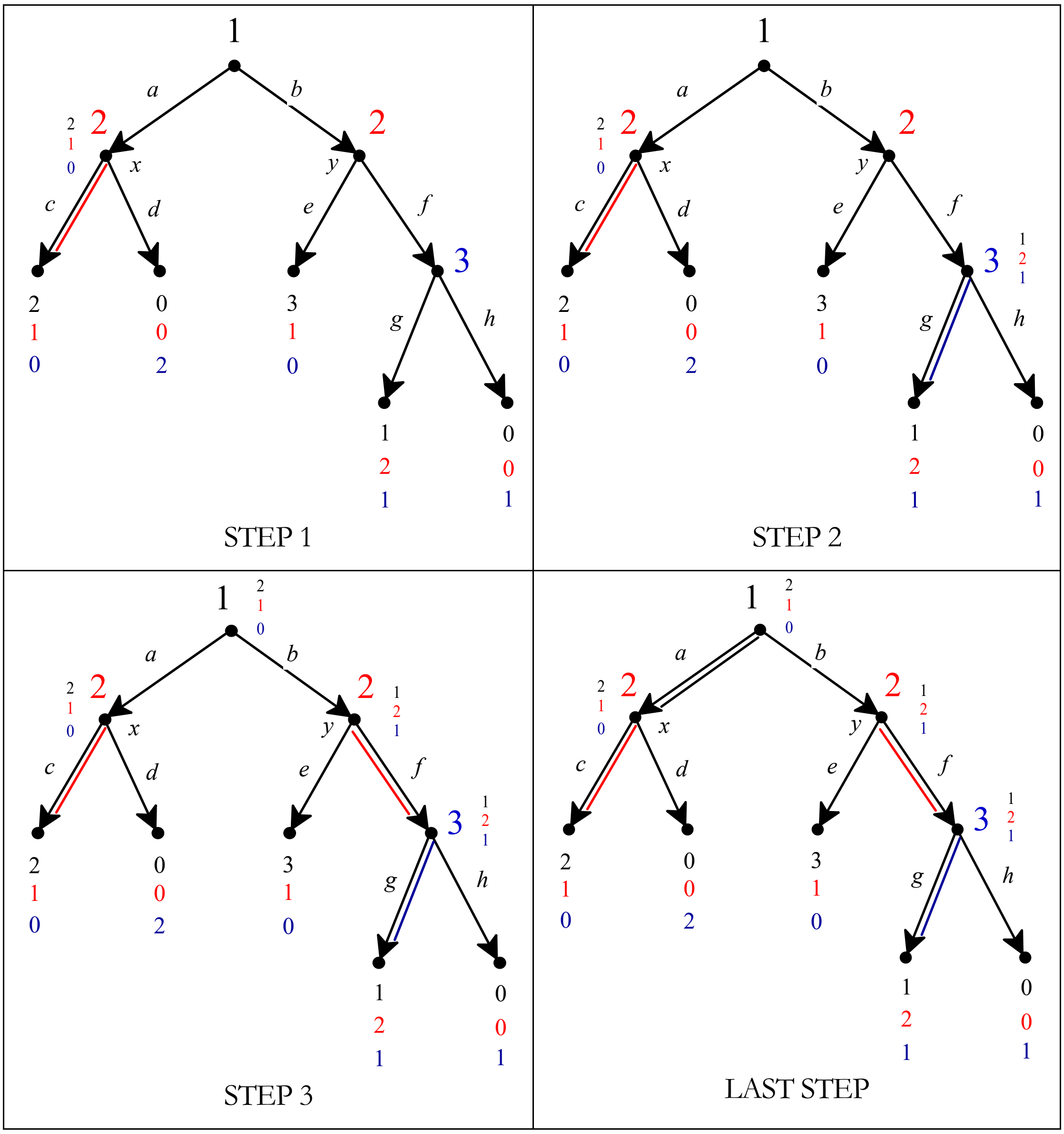


Figure 3.4: One possible output of the backward-induction algorithm applied to the game of Figure 3.3

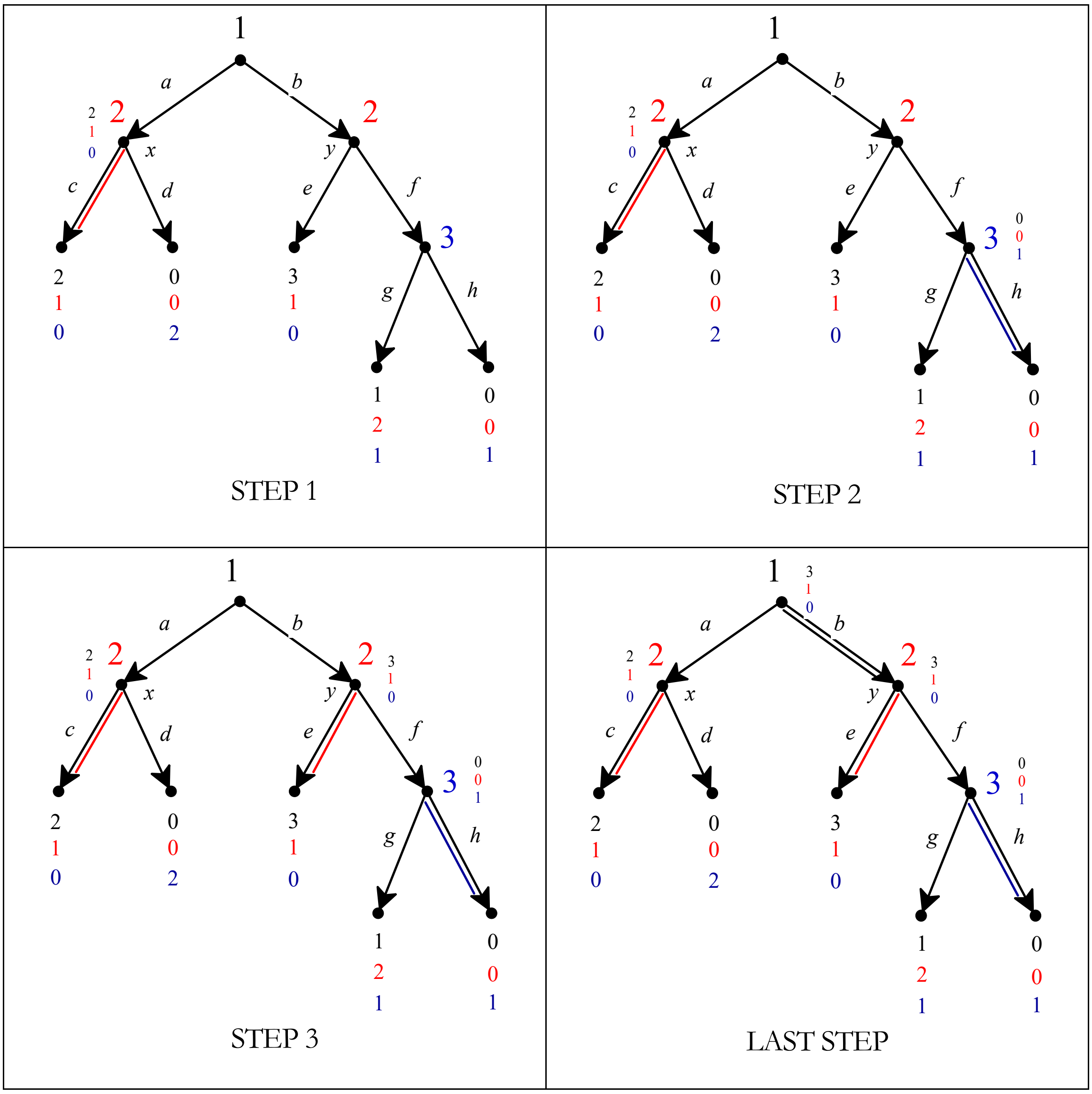


Figure 3.5: Another possible output of the backward-induction algorithm applied to the game of Figure 3.3

How should one define the output of the backward-induction algorithm and the notion of backward-induction solution? What kind of objects are they? Before we answer this question we need to introduce the notion of strategy in a perfect-information game.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 3.6.2 at the end of this chapter.

3.3 Strategies in perfect-information games

A strategy for a player in a perfect-information game is a complete, contingent plan on how to play the game. Consider, for example, the game shown in Figure 3.6 (which reproduces 3.3) and let us focus on Player 2.

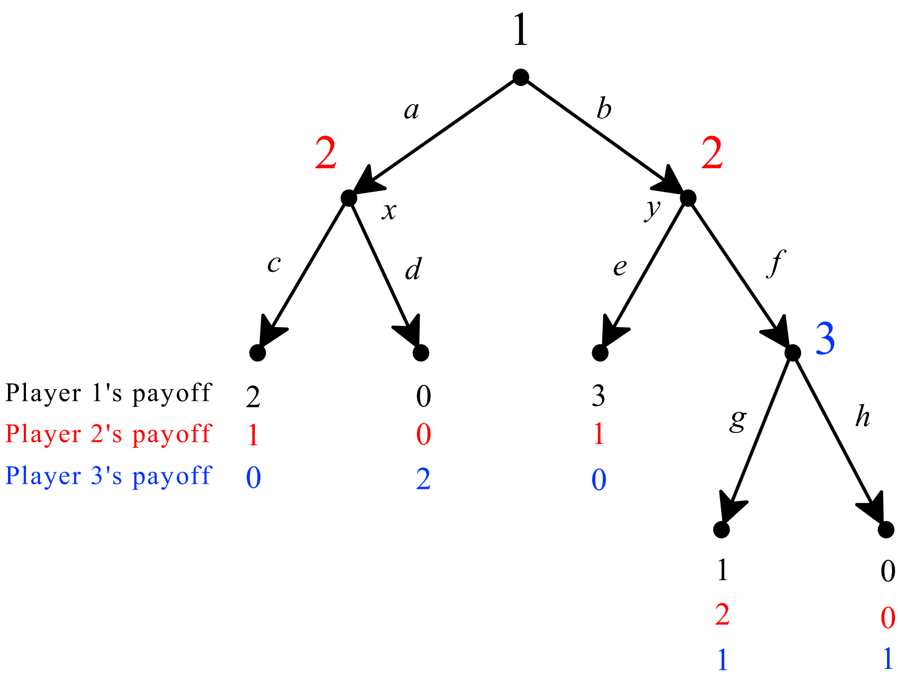


Figure 3.6: Copy of the game of Figure 3.3

Before the game is played, Player 2 does not know what Player 1 will do and thus a complete plan needs to specify what she will do if Player 1 decides to play a and what she will do if Player 1 decides to play b. A possible plan, or strategy, is "if Player 1 chooses a then I will choose c and if Player 1 chooses b then I will choose e", which we can denote more succinctly as (c,e). The other possible plans, or strategies, for Player 2 are $(c,f),(d,e)$ and (d,f). The formal definition of strategy is as follows.

Definition 3.3.1 A *strategy* for a player in a perfect-information game is a list of choices, one for each decision node of that player.

For example, suppose that Player 1 has three decision nodes in a given game: at one node she has three possible choices, a_1, a_2 and a_3, at another node she has two possible choices, b_1 and b_2, and at the third node she has four possible choices, c_1, c_2, c_3 and c_4. Then a strategy for Player 1 in that game can be thought of as a way of filling in three blanks:

$$\left(\underbrace{\qquad\qquad}_{\text{one of } a_1, a_2, a_3}, \underbrace{\qquad\qquad}_{\text{one of } b_1, b_2}, \underbrace{\qquad\qquad}_{\text{one of } c_1, c_2, c_3, c_4} \right).$$

Since there are 3 choices for the first blank, 2 for the second and 4 for the third, the total number of possible strategies for Player 1 in this case would be $3 \times 2 \times 4 = 24$. One strategy is (a_2, b_1, c_1), another strategy is (a_1, b_2, c_4), etc.

It should be noted that the notion of strategy involves redundancies. To see this, consider the game of Figure 3.7. In this game a possible strategy for Player 1 is (a, g), which means that Player 1 is planning to choose a at the root of the tree and would choose g at her other node. But if Player 1 indeed chooses a, then her other node will *not* be reached and thus why should Player 1 make a plan on what to do there? One could justify this redundancy in the notion of strategy in a number of ways:

1. Player 1 is so cautious that she wants her plan to cover also the possibility that she might make mistakes in the implementation of parts of her plan (in this case, she allows for the possibility that – despite her intention to play a – she might end up playing b), or
2. we can think of a strategy as a set of instructions given to a third party on how to play the game on Player 1's behalf, in which case Player 1 might indeed worry about the possibility of mistakes in the implementation and thus want to cover all contingencies.

An alternative justification relies on a different interpretation of the notion of strategy: not as a plan of Player 1 but as a belief in the mind of Player 2 concerning what Player 1 would do. For the moment we will set this issue aside and simply use the notion of strategy as given in Definition 4.2.1.

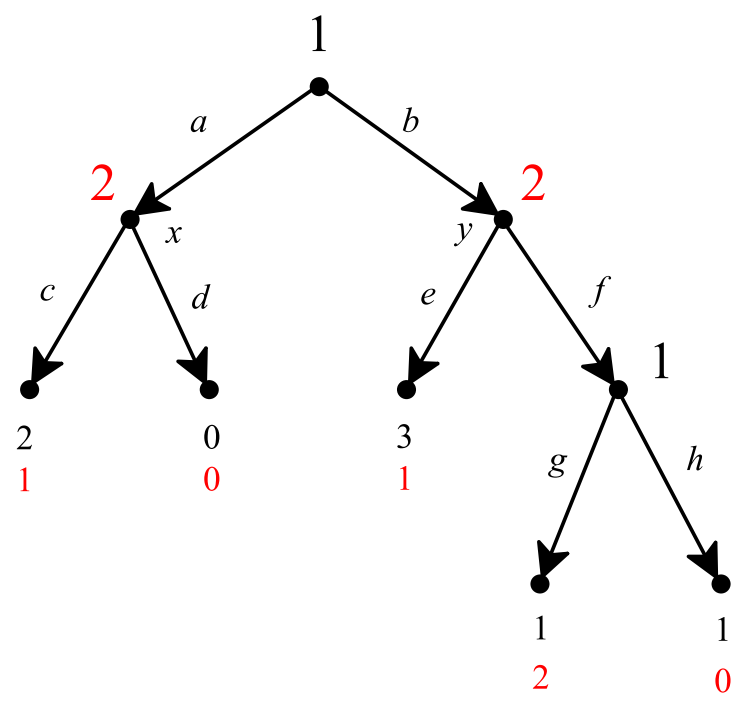


Figure 3.7: A perfect-information game

Using Definition 4.2.1, one can associate with every perfect-information game a strategic-form (or normal-form) game: a strategy profile determines a unique terminal node that is reached if the players act according to that strategy profile and thus a unique vector of payoffs. Figure 3.8 shows the strategic-form associated with the perfect-information game of Figure 3.7, with the Nash equilibria highlighted.

		Player 2			
		ce	*cf*	*de*	*df*
Player 1	*ag*	2 , 1	2 , 1	0 , 0	0 , 0
	ah	2 , 1	2 , 1	0 , 0	0 , 0
	bg	3 , 1	1 , 2	3 , 1	1 , 2
	bh	3 , 1	1 , 0	3 , 1	1 , 0

Figure 3.8: The strategic form of the perfect-information game of Figure 3.7 with the Nash equilibria highlighted

Because of the redundancy discussed above, the strategic form also displays redundancies: in this case the top two rows are identical.

Armed with the notion of strategy, we can now revisit the notion of backward-induction solution. Figure 3.9 shows the two backward-induction solutions of the game of Figure 3.7.

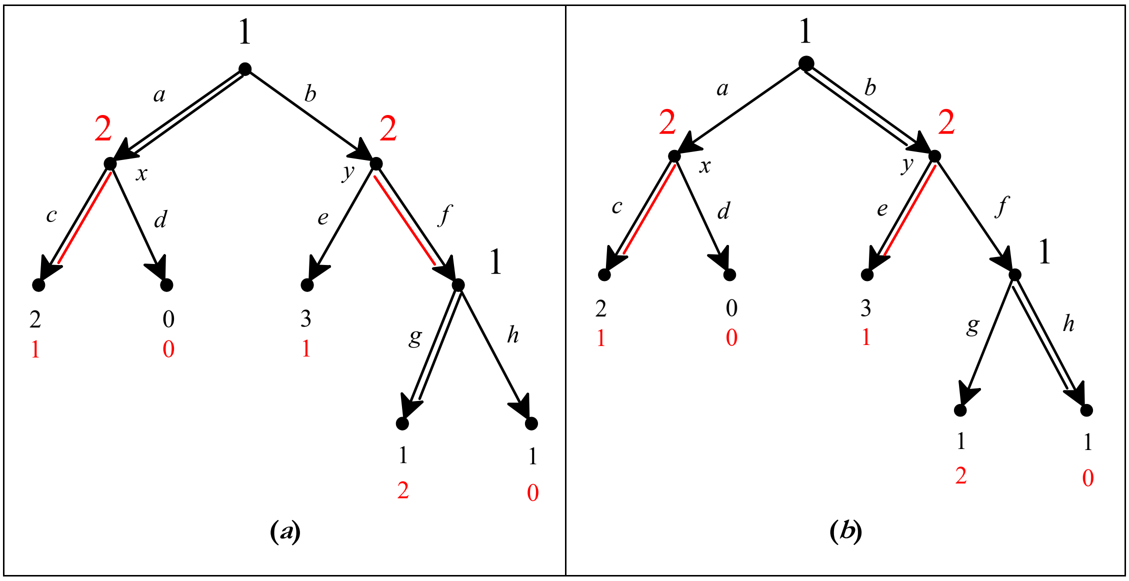


Figure 3.9: The backward-induction solutions of the game of Figure 3.7

It is clear from the definition of backward-induction algorithm (Definition 3.2.1) that the procedure selects a choice at every decision node and thus yields a strategy profile for the entire game: the backward-induction solution shown in Panel (*a*) of Figure 3.9 is the strategy profile $((a,g),(c,f))$, while the backward-induction solution shown in Panel (*b*) is the strategy profile $((b,h),(c,e))$. Both of them are Nash equilibria of the corresponding strategic-form game, but not all the Nash equilibria correspond to backward-induction solutions. The relationship between the two concepts is explained in the next section.

R A backward-induction solution is a strategy profile. Since strategies contain a description of what a player actually does and also of what the player would do in circumstances that do not arise, one often draws a distinction between the backward-induction *solution* and the backward-induction *outcome* which is defined as the sequence of actual moves. For example, the backward-induction outcome associated with the solution $((a,g),(c,f))$ is the play *ac* with corresponding payoff $(2,1)$, while the backward-induction outcome associated with the solution $((b,h),(c,e))$ is the play *be* with corresponding payoff $(3,1)$.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 3.6.3 at the end of this chapter.

3.4 Relationship between backward induction and other solutions

If you have gone through the exercises for the previous three sections, you will have seen that in all those games the backward-induction solutions are also Nash equilibria. This is always true, as stated in the following theorem.

Theorem 3.4.1 Every backward-induction solution of a perfect-information game is a Nash equilibrium of the associated strategic form.

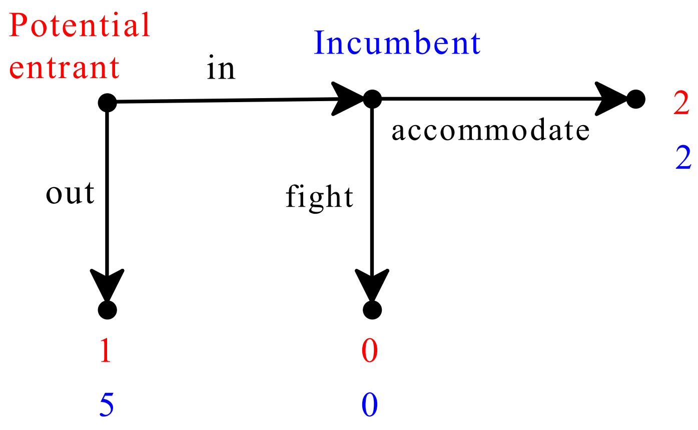
In some games the set of backward-induction solutions coincides with the set of Nash equilibria (see, for example, Exercise 3.9), but typically the set of Nash equilibria is larger than (is a proper superset of) the set of backward-induction solutions (for example the game of Figure 3.7 has two backward-induction solutions – shown in Figure 3.9 – but five Nash equilibria, shown in Figure 3.8).

Nash equilibria that are not backward-induction solutions often involve *incredible threats*. To see this, consider the following game.

An industry is currently a monopoly and the incumbent monopolist is making a profit of \$5 million. A potential entrant is considering whether or not to enter this industry.
- If she does not enter, she will make \$1 million in an alternative investment.
- If she does enter, then the incumbent can either fight entry with a price war whose outcome is that both firms make zero profits, or it can accommodate entry, by sharing the market with the entrant, in which case both firms make a profit of \$2 million.

This situation is illustrated in Figure 3.10 with the associated strategic form. Note that we are assuming that each player is selfish and greedy, that is, cares only about its own profit and prefers more money to less.

The backward-induction solution is *(in, accommodate)* and it is also a Nash equilibrium. However, there is another Nash equilibrium, namely *(out, fight)*. The latter should be discarded as a "rational solution" because it involves an incredible threat on the part of the incumbent, namely that it will fight entry if the potential entrant enters.
- It is true that, if the potential entrant believes the incumbent's threat, then she is better off staying out; however, she should ignore the incumbent's threat because she should realize that – when faced with the *fait accompli* of entry – the incumbent would not want to carry out the threat.



		Chain store	
		fight	accommodate
Potential entrant	In	0 , 0	2 , 2
	Out	1 , 5	1 , 5

Figure 3.10: The entry game

Reinhard Selten (who shared the 1994 Nobel Memorial prize in economics with two other game theorists, John Harsanyi and John Nash) discussed a repeated version of the above entry game, which has become known as *Selten's Chain Store Game*. The story is as follows:

- A chain store is a monopolist in an industry. It owns stores in m different towns ($m \geq 2$).

- In each town the chain store makes \$5 million if left to enjoy its privileged position undisturbed.

- In each town there is a businesswoman who could enter the industry in that town, but earns \$1 million if she chooses not to enter; if she decides to enter, then the monopolist can either fight the entrant, leading to zero profits for both the chain store and the entrant in that town, or it can accommodate entry and share the market with the entrant, in which case both players make \$2 million in that town.

Thus, in each town the interaction between the incumbent monopolist and the potential entrant is as illustrated in Figure 3.10.

However, decisions are made sequentially, as follows:

At date $t(t = 1,\dots,m)$ the businesswoman in town t decides whether or not to enter and if she enters then the chain store decides whether or not to fight in that town.

What happens in town t at date t becomes known to everybody. Thus, for example, the businesswoman in town 2 at date 2 knows what happened in town 1 at date 1 (either that there was no entry or that entry was met with a fight or that entry was accommodated).

Intuition suggests that in this game the threat by the incumbent to fight early entrants might be credible, for the following reason. The incumbent could tell Businesswoman 1 the following:

> "It is true that, if you enter and I fight, I will make zero profits, while by accommodating your entry I would make \$2 million and thus it would seem that it cannot be in my interest to fight you. However, somebody else is watching us, namely Businesswoman 2. If she sees that I have fought your entry then she might fear that I would do the same with her and decide to stay out, in which case in town 2, I would make \$5 million, so that my total profits in towns 1 and 2 would be $(0+5) =$ \$5 million. On the other hand, if I accommodate your entry, then she will be encouraged to entry herself and I will make \$2 million in each town, for a total profit of \$4 million. Hence, as you can see, it is indeed in my interest to fight you and thus you should stay out."

Does the notion of backward induction capture this intuition? To check this, let us consider the case where $m = 2$, so that the extensive game is not too large to draw. It is shown in Figure 3.11, where at each terminal node the top number is the profit of the incumbent monopolist (it is the sum of the profits in the two towns), the middle number is the profit of Businesswoman 1 and the bottom number is the profit of Businesswoman 2. All profits are expressed in millions of dollars. We assume that all the players are selfish and greedy, so that we can take the profit of each player to be that player's payoff. The backward-induction solution is unique and is shown by the thick directed edges in Figure 3.11. The corresponding outcome is that both businesswomen will enter and the incumbent monopolist accommodates entry in both towns.

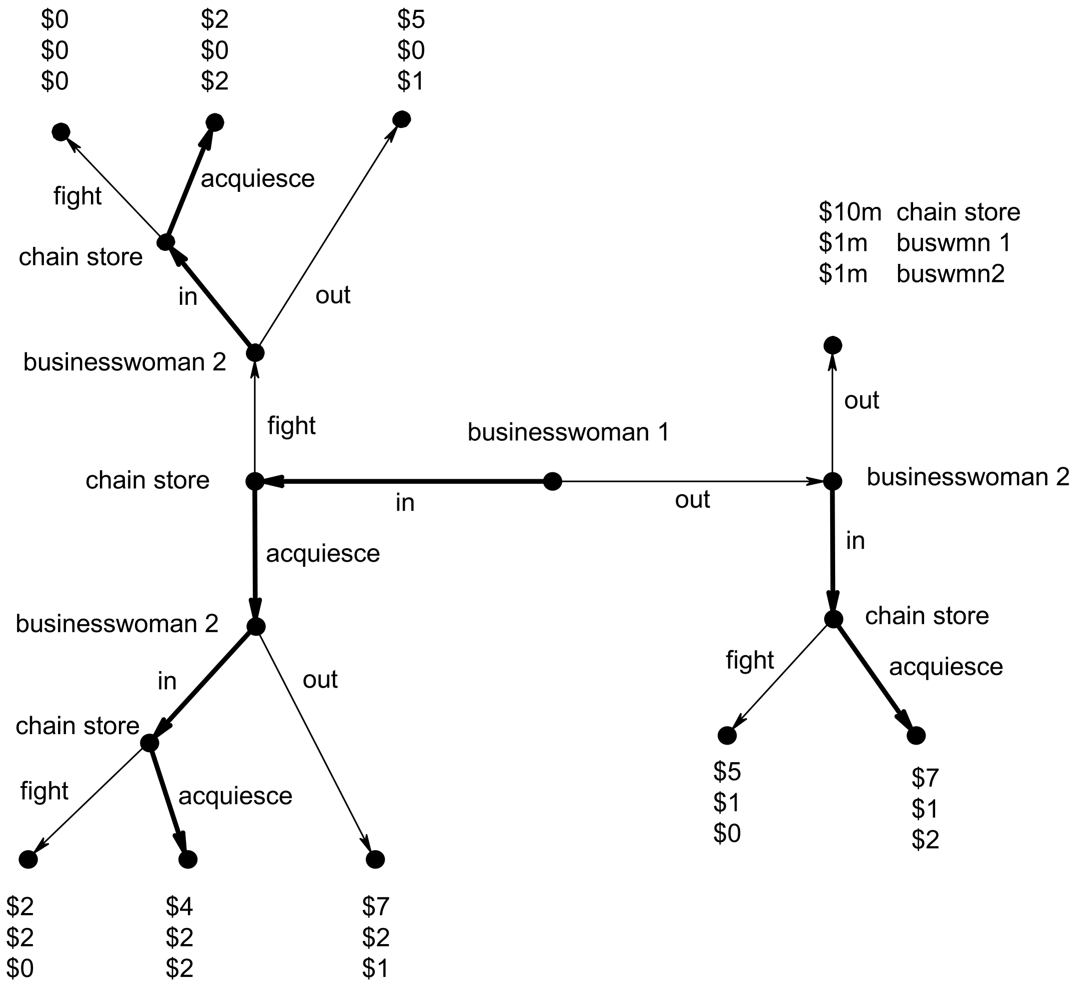


Figure 3.11: Selten's Chain-Store game

Thus the backward-induction solution does not capture the "reputation" argument outlined above. However, the backward-induction solution does seem to capture the notion of rational behavior in this game. Indeed, Businesswoman 1 should reply to the incumbent with the following counter-argument:

> "Your reasoning is not valid. Whatever happens in town 1, it will be common knowledge between you and Businesswoman 2 that your interaction in town 2 will be the last; in particular, nobody else will be watching and thus there won't be an issue of establishing a reputation in the eyes of another player. Hence in town 2 it will be in your interest to accommodate entry, since in essence you will be playing the one-shot entry game of Figure 3.10. Hence a rational Businesswoman 2 will decide to enter in town 2 *whatever happened in town 1*: what you do against me will have no influence on her decision. Thus your "reputation" argument does not apply and it will in fact be in your interest not to fight my entry: your choice will be between a profit of $\$(0+2) = \$2$ million, if you fight me, and a profit of $\$(2+2) = \$4$ million, if you don't fight me. Hence I will enter and you will not fight me."

In order to capture the reputation argument described above we need to allow for some uncertainty in the mind of some of the players, as we will show in a later chapter. In a perfect-information game uncertainty is ruled out by definition.

By Theorem 3.4.1 the notion of backward induction can be seen as a refinement of the notion of Nash equilibrium. Another solution concept that is related to backward induction is the iterated elimination of weakly dominated strategies. Indeed the backward-induction algorithm could be viewed as a step-wise procedure that eliminates dominated choices at decision nodes, and thus strategies that contain those choices. What is the relationship between the two notions? In general this is all that can be said: applying the iterated deletion of weakly dominated strategies to the strategic form associated with a perfect-information game leads to a set of strategy profiles that contains at least one backward-induction solution; however,

(1) it may also contain strategy profiles that are not backward-induction solutions, and

(2) it may fail to contain all the backward-induction solutions, as shown in Exercise 3.8.

3.5 Perfect-information games with two players

We conclude this chapter with a discussion of finite two-player extensive games with perfect information.

We will start with games that have only two outcomes, namely "Player 1 wins" (denoted by W_1) and "Player 2 wins" (denoted by W_2). We assume that Player 1 strictly prefers W_1 to W_2 and Player 2 strictly prefers W_2 to W_1. Thus we can use utility functions with values 0 and 1 and associate with each terminal node either the payoff vector $(1,0)$ (if the outcome is W_1) or the payoff vector $(0,1)$ (if the outcome is W_2). We call these games *win-lose games*. An example of such a game is the following.

■ **Example 3.2** Two players take turns choosing a number from the set $\{1,2,\ldots,10\}$, with Player 1 moving first. The first player who brings the sum of all the chosen numbers to 100 or more wins. ■

The following is one possible play of the game (the bold-face numbers are the ones chosen by Player 1 and the underlined numbers the ones chosen by Player 2):

10, <u>9</u>, **9**, <u>10</u>, **8**, <u>7</u>, **4**, <u>10</u>, **1**, <u>8</u>, **3**, <u>3</u>, **8**, <u>10</u>.

In this play Player 2 wins: before her last move the sum is 90 and with her final choice of 10 she brings the total to 100. However, in this game Player 1 has a *winning strategy*, that is, a strategy that guarantees that he will win, no matter what numbers Player 2 chooses. To see this, we can use backward-induction reasoning. Drawing the tree is not a practical option, since the number of nodes is very large: one needs 10,000 nodes just to represent the first 4 moves! But we can imagine drawing the tree, placing ourselves towards the end of the tree and ask what partial sum represents a "losing position", in the sense that whoever is choosing in that position cannot win, while the other player can then win with his subsequent choice. With some thought one can see that 89 is the largest losing position: whoever moves there can take the sum to any number in the set $\{90,91,\ldots,99\}$, thus coming short of 100, while the other player can then take the sum to 100 with an appropriate choice. What is the largest losing position that precedes 89? The answer is 78: whoever moves at 78 must take the sum to a number in the set $\{79,80,\ldots,88\}$ and then from there the other player can make sure to take the sum to 89 and then we know what happens from there! Repeating this reasoning we see that the losing positions are: 89, 78, 67, 56, 45, 34, 23, 12, 1. Since Player 1 moves first he can choose 1 and put Player 2 in the first losing position; then, whatever Player 2 chooses, Player 1 can put her in the next losing position, namely 12, etc. Recall that a strategy for Player 1 must specify what to do in every possible situation in which he might find himself. In his game Player 1's winning strategy is as follows:

> Start with the number 1. Then, at every turn, choose the number $(11-n)$, where n is the number that was chosen by Player 2 in the immediately preceding turn.

Here is an example of a possible play of the game where Player 1 employs the winning strategy and does in fact win:

1, <u>9</u>, **2**, <u>6</u>, **5**, <u>7</u>, **4**, <u>10</u>, **1**, <u>8</u>, **3**, <u>3</u>, **8**, <u>9</u>, **2**, <u>5</u>, **6**, <u>1</u>, **10**

We can now state a general result about this class of games.

Theorem 3.5.1 In every finite two-player, win-lose game with perfect information one of the two players has a winning strategy.

Although we will not give a detailed proof, the argument of the proof is rather simple. By applying the backward-induction algorithm we assign to every decision node either the payoff vector $(1,0)$ or the payoff vector $(0,1)$. Imagine applying the algorithm up to the point where the immediate successors of the root have been assigned a payoff vector. Two cases are possible.

Case 1: at least one of the immediate successors of the root has been assigned the payoff vector $(1,0)$. In this case Player 1 is the one who has a winning strategy and his initial choice should be such that a node with payoff vector $(1,0)$ is reached and then his future choices should also be such that only nodes with payoff vector $(1,0)$ are reached.

Case 2: all the immediate successors of the root have been assigned the payoff vector $(0,1)$. In this case it is Player 2 who has a winning strategy. An example of a game where it is Player 2 who has a winning strategy is given in Exercise 3.11.

We now turn to finite two-player games where there are three possible outcomes: "Player 1 wins" (W_1), "Player 2 wins" (W_2) and "Draw" (D). We assume that the rankings of the outcomes are as follows: $W_1 \succ_1 D \succ_1 W_2$ and $W_2 \succ_2 D \succ_2 W_1$.
Examples of such games are Tic-Tac-Toe (`http://en.wikipedia.org/wiki/Tic-tac-toe`), Draughts or Checkers (`http://en.wikipedia.org/wiki/Draughts`) and Chess (although there does not seem to be agreement as to whether the rules of Chess guarantee that every possible play of the game is finite). What can we say about such games? The answer is provided by the following theorem.

Theorem 3.5.2 Every finite two-player, perfect-information game with three outcomes: Player 1 wins (W_1), Player 2 wins (W_2) and Draw (D), and preferences $W_1 \succ_1 D \succ_1 W_2$ and $W_2 \succ_2 D \succ_2 W_1$, falls within one of the following three categories:

1. Player 1 has a strategy that guarantees outcome W_1.
2. Player 2 has a strategy that guarantees outcome W_2.
3. Player 1 has a strategy that guarantees that the outcome will be W_1 or D and Player 2 has a strategy that guarantees that the outcome will be W_2 or D, so that, if both players employ these strategies, the outcome will be D.

The logic of the proof is as follows. By applying the backward-induction algorithm we assign to every decision node either the payoff vector $(2,0)$ (corresponding to outcome W_1) or the payoff vector $(0,2)$ (corresponding to outcome W_2) or the payoff vector $(1,1)$ (corresponding to outcome D). Imagine applying the algorithm up to the point where the immediate successors of the root have been assigned a payoff vector. Three cases are possible.

Case 1: at least one of the immediate successors of the root has been assigned the payoff vector $(2,0)$; in this case Player 1 is the one who has a winning strategy.

Case 2: all the immediate successors of the root have been assigned the payoff vector $(0,2)$; in this case it is Player 2 who has a winning strategy.

Case 3: there is at least one immediate successor of the root to which the payoff vector $(1,1)$ has been assigned and all the other immediate successors of the root have been assigned either $(1,1)$ or $(0,2)$. In this case we fall within the third category of Theorem 3.5.2.

Both Tic-Tac-Toe and Checkers fall within the third category (`http://en.wikipedia.org/wiki/Solved_game#Solved_games`). As of the time of writing this book, it is not known to which category the game of Chess belongs.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 3.6.4 at the end of this chapter.

3.6 Exercises

3.6.1 Exercises for Section 3.1: Trees, frames and games

The answers to the following exercises are in Section 4.7 at the end of this chapter.

Exercise 3.1 How could they do that! They abducted Speedy, your favorite tortoise! They asked for \$1,000 in unmarked bills and threatened to kill Speedy if you don't pay. Call the tortoise-napper Mr. T. Let the possible outcomes be as follows:

o_1 : you don't pay and speedy is released
o_2 : you pay \$ 1,000 and speedy is released
o_3 : you don't pay and speedy is killed
o_4 : you pay \$ 1,000 and speedy is killed

You are attached to Speedy and would be willing to pay \$1,000 to get it back. However, you also like your money and you prefer not to pay, conditional on each of the two separate events "Speedy is released" and "Speedy is killed". Thus your ranking of the outcomes is $o_1 \succ_{you} o_2 \succ_{you} o_3 \succ_{you} o_4$. On the other hand, you are not quite sure of what Mr. T's ranking is.

(a) Suppose first that Mr T has communicated that he wants you to go to Central Park tomorrow at 10:00 a.m. and leave the money in a garbage can; he also said that, two miles to the East and at the exact same time, he will decide whether or not to free Speedy in front of the police station and then go and collect his money in Central Park. What should you do?

(b) Suppose that Mr T is not as dumb as in part **(a)** and instead gives you the following instructions: first you leave the money in a garbage can in Central Park and then he will go there to collect the money. He also told you that if you left the money there then he will free Speedy, otherwise he will kill it. Draw an extensive form or frame to represent this situation.

(c) Now we want to construct a game based on the extensive form of part **(b)**. For this we need Mr T's preferences. There are two types of criminals in Mr T's line of work: the professionals and the one-timers. Professionals are in the business for the long term and thus, besides being greedy, worry about reputation; they want it to be known that (1) every time they were paid they honored their promise to free the hostage and (2) their threats are to be taken seriously: every time they were *not* paid, the hostage was killed. The one-timers hit once and then they disappear; they don't try to establish a reputation and the only thing they worry about, besides money, is not to be caught: whether or not they get paid, they prefer to kill the hostage in order to eliminate any kind of evidence (DNA traces, fingerprints, etc.). Construct two games based on the extensive form of part **(b)** representing the two possible types of Mr T.

■

Exercise 3.2 A three-man board, composed of A, B, and C, has held hearings on a personnel case involving an officer of the company. This officer was scheduled for promotion but, prior to final action on his promotion, he made a decision that cost the company a good deal of money. The question is whether he should be (1) promoted anyway, (2) denied the promotion, or (3) fired. The board has discussed the matter at length and is unable to reach unanimous agreement. In the course of the discussion it has become clear to all three of them that their separate opinions are as follows:

- A considers the officer to have been a victim of bad luck, not bad judgment, and wants to go ahead and promote him but, failing that, would keep him rather than fire him.
- B considers the mistake serious enough to bar promotion altogether; he'd prefer to keep the officer, denying promotion, but would rather fire than promote him.
- C thinks the man ought to be fired but, in terms of personnel policy and morale, believes the man ought not to be kept unless he is promoted, i.e., that keeping an officer who has been declared unfit for promotion is even worse than promoting him.

	PROMOTE	KEEP	FIRE
A :	best	middle	worst
B :	worst	best	middle
C :	middle	worst	best

Assume that everyone's preferences among the three outcomes are fully evident as a result of the discussion. The three must proceed to a vote.
Consider the following voting procedure. First A proposes an action (either promote or keep or fire). Then it is B's turn. If B accepts A's proposal, then this becomes the final decision. If B disagrees with A'a proposal, then C makes the final decision (which may be *any of the three*: promote, keep or fire). Represent this situation as an extensive game with perfect information. (Use utility numbers from the set $\{1,2,3\}$.) ■

3.6.2 Exercises for Section 3.2: Backward induction

The answers to the following exercises are in Section 4.7 at the end of this chapter.

Exercise 3.3 Apply the backward-induction algorithm to the two games of Exercise 3.1 Part **(c)**. ■

Exercise 3.4 Apply the backward-induction algorithm to the game of Exercise 3.2. ■

3.6.3 Exercises for Section 3.3: Strategies in perfect-information games

The answers to the following exercises are in Section 4.7 at the end of this chapter.

Exercise 3.5 Write the strategic form of the game of Figure 3.2, find all the Nash equilibria and verify that the backward-induction solution is a Nash equilibrium. ■

Exercise 3.6 Write the strategic form of the game of Figure 3.3, find all the Nash equilibria and verify that the backward-induction solutions are Nash equilibria. ■

Exercise 3.7 Consider the game of Exercise 3.2.

(a) Write down all the strategies of Player B.

(b) How many strategies does Player C have?

■

Exercise 3.8 Consider the perfect-information game shown in Figure 3.12.

(a) Find the backward-induction solutions.

(b) Write down all the strategies of Player 1.

(c) Write down all the strategies of Player 2.

(d) Write the strategic form associated with this game.

(e) Does Player 1 have a dominant strategy?

(f) Does Player 2 have a dominant strategy?

(g) Is there a dominant-strategy equilibrium?

(h) Does Player 1 have any dominated strategies?

(i) Does Player 2 have any dominated strategies?

(j) What do you get when you apply the iterative elimination of weakly dominated strategies?

(k) What are the Nash equilibria?

■

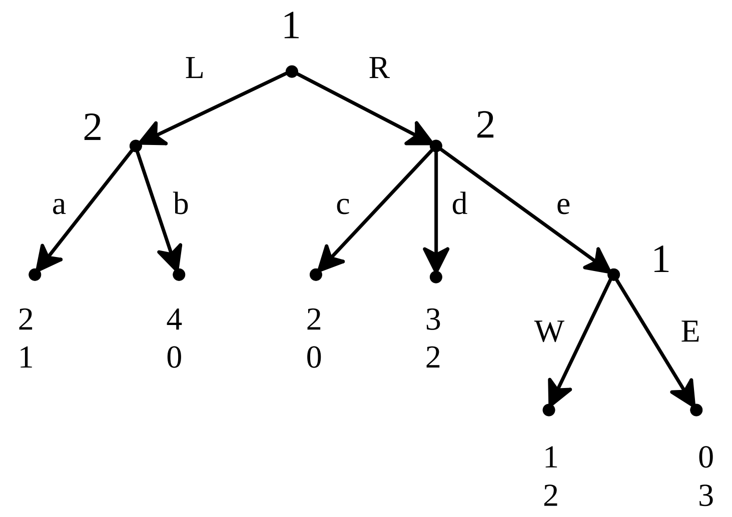


Figure 3.12: The perfect-information game for Exercise 3.8

Exercise 3.9 Consider an industry where there are two firms, a large firm, Firm 1, and a small firm, Firm 2. The two firms produce identical products.
- Let x be the output of Firm 1 and y the output of Firm 2. Industry output is $Q = x + y$.
- The price P at which each unit of output can be sold is determined by the inverse demand function $P = 130 - 10Q$. For example, if Firm 1 produces 4 units and Firm 2 produces 2 units, then industry output is 6 and each unit is sold for $P = 130 - 60 = \$70$.
- For each firm the cost of producing q units of output is $C(q) = 10q + 62.5$.
- Each firm is only interested in its own profits.
- The profit of Firm 1 depends on both x and y and is given by

$$\Pi_1(x, y) = \underbrace{x\,[130 - 10(x+y)]}_{\text{revenue}} - \underbrace{(10x + 62.5)}_{\text{cost}}$$

and similarly the profit function of Firm 2 is given by

$$\Pi_2(x, y) = \underbrace{y\,[130 - 10(x+y)]}_{\text{revenue}} - \underbrace{(10y + 62.5)}_{\text{cost}}.$$

- The two firms play the following sequential game. First Firm 1 chooses its own output x and commits to it; then Firm 2, after having observed Firm 1's output, chooses its own output y; then the price is determined according to the demand function and the two firms collect their own profits. In what follows assume, for simplicity, that x can only be 6 or 6.5 units and y can only be 2.5 or 3 units.

(a) Represent this situation as an extensive game with perfect information.

(b) Solve the game using backward induction.

(c) Write the strategic form associated with the perfect-information game.

(d) Find the Nash equilibria of this game and verify that the backward-induction solutions are Nash equilibria.

Exercise 3.10 Consider the perfect-information game shown in Figure 3.13 where x is an integer.

(a) For every value of x find the backward induction solution(s).

(b) Write the corresponding strategic-form and find all the Nash equilibria. ■

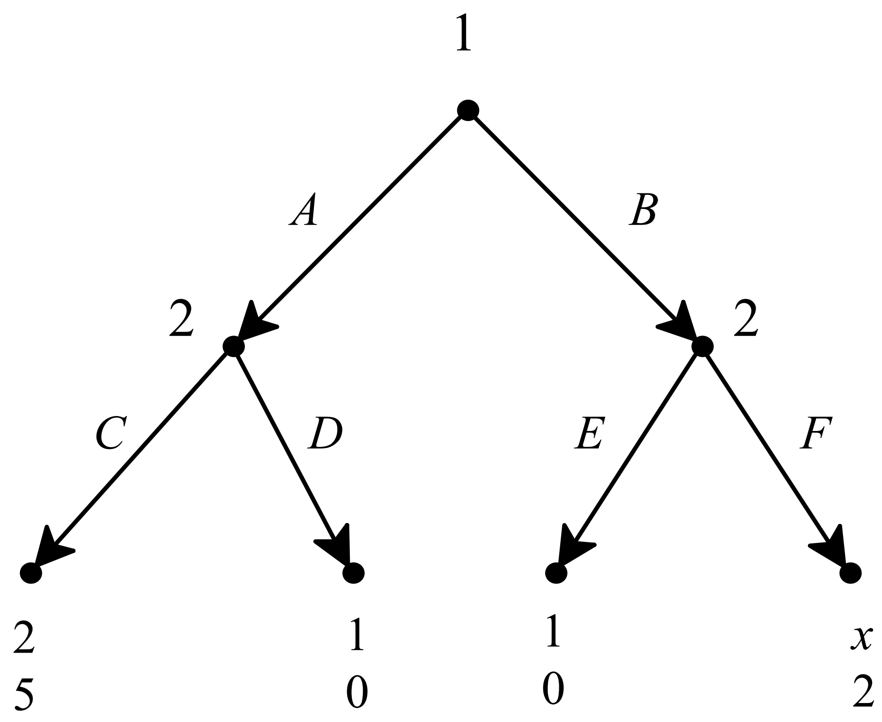


Figure 3.13: A perfect-information game

3.6.4 Exercises for Section 3.5: Two-player games

The answers to the following exercises are in Section 4.7 at the end of this chapter.

Exercise 3.11 Consider the following perfect-information game. Player 1 starts by choosing a number from the set $\{1,2,3,4,5,6,7\}$, then Player 2 chooses a number from this set, then Player 1 again, followed by Player 2, etc. The first player who brings the cumulative sum of all the numbers chosen (up to and including the last one) to 48 or more wins. By Theorem 3.5.1 one of the two players has a winning strategy. Find out who that player is and fully describe the winning strategy. ■

Exercise 3.12 Consider Figure 3.14 and the following two-player, perfect-information game. A coin is placed in the cell marked 'START' (cell A1). Player 1 moves first and can move the coin one cell up (to A2) or one cell to the left (to B1) or one cell diagonally in the left-up direction (to B2). Then Player 2 moves, according to the same rules (e.g. if the coin is in cell B2 then the admissible moves are shown by the directed edges). The players alternate moving the coin. Black cells are not accessible (so that, for example, from A3 the coin can only be moved to A4 or B3 and from F3 it can only be moved to G4, as shown by the directed edge). The player who manages to place the coin in the cell marked 'END' wins.

(a) Represent this game by means of an extensive form with perfect information by drawing the initial part of the tree that covers the first two moves (the first move of Player 1 and the first move of Player 2).

(b) Suppose that the coin is currently in cell G4 and it is Player 1's turn to move. Show that Player 1 has a strategy that allows her to win the game starting from cell G4. Describe the strategy in detail.

(c) Describe a play of the game (from cell A1) where Player 1 wins (describe it by means of the sequence of cells visited by the coin).

(d) Describe a play of the game (from cell A1) where Player 2 wins (describe it by means of the sequence of cells visited by the coin).

(e) Now go back to the beginning of the game. The coin is in cell A1 and player 1 has the first move. By Theorem 3.5.1 one of the two players has a winning strategy. Find out who that player is and fully describe the winning strategy.

■

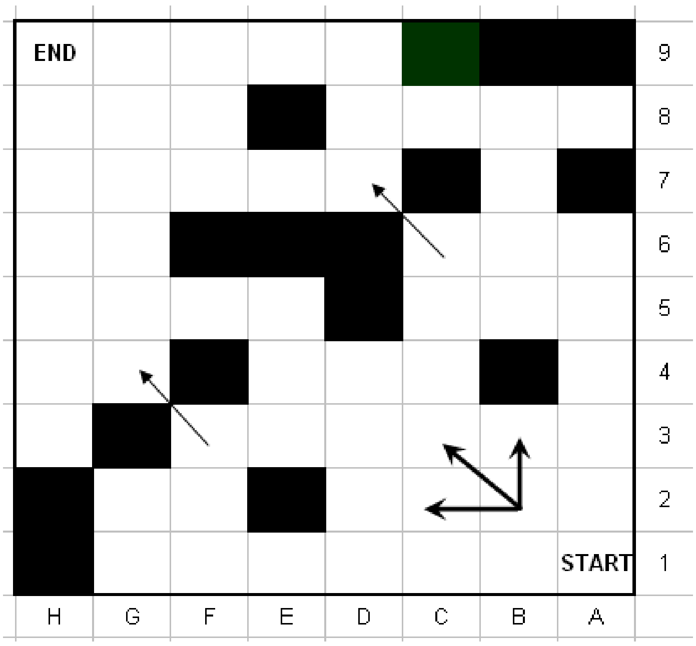


Figure 3.14: The coin game

Exercise 3.13 — ⋆⋆⋆ Challenging Question ⋆⋆⋆.
Two women, Anna and Bess, claim to be the legal owners of a diamond ring that - each claims - has great sentimental value. Neither of them can produce evidence of ownership and nobody else is staking a claim on the ring. Judge Sabio wants the ring to go to the legal owner, but he does not know which of the two women is in fact the legal owner. He decides to proceed as follows. First he announces a fine of $\$F > 0$ and then asks Anna and Bess to play the following game.

Move 1: Anna moves first. Either she gives up her claim to the ring (in which case Bess gets the ring, the game ends and nobody pays the fine) or she asserts her claim, in which case the game proceeds to Move 2.

Move 2: Bess either accepts Anna's claim (in which case Anna gets the ring, the game ends and nobody pays the fine) or challenges her claim. In the latter case, Bess must put in a bid, call it B, and Anna must pay the fine of $\$F$ to Sabio. The game goes on to Move 3.

Move 3: Anna now either matches Bess's bid (in which case Anna gets the ring, Anna pays $\$B$ to Sabio in addition to the fine that she already paid and Bess pays the fine of $\$F$ to Sabio) or chooses not to match (in which case Bess gets the ring and pays her bid of $\$B$ to Sabio and, furthermore, Sabio keeps the fine that Anna already paid).

Denote by C_A the monetary equivalent of getting the ring for Anna (that is, getting the ring is as good, in Anna's mind, as getting $\$C_A$) and C_B the monetary equivalent of getting the ring for Bess. Not getting the ring is considered by both to be as good as getting zero dollars.

(a) Draw an extensive game with perfect information to represent the above situation, assuming that there are only two possible bids: B_1 and B_2. Write the payoffs to Anna and Bess next to each terminal node.

(b) Find the backward-induction solution of the game you drew in part **(a)** for the case where $B_1 > C_A > C_B > B_2 > F > 0$.

Now consider the general case where the bid B can be any non-negative number and assume that both Anna and Bess are very wealthy. Assume also that C_A, C_B and F are positive numbers and that C_A and C_B are common knowledge between Anna and Bess . We want to show that, at the backward-induction solution of the game, the ring always goes to the legal owner. Since we (like Sabio) don't know who the legal owner is, we must consider two cases.

Case 1: the legal owner is Anna. Let us assume that this implies that $C_A > C_B$.

Case 2: the legal owner is Bess. Let us assume that this implies that $C_B > C_A$.

(c) Find the backward-induction solution for Case 1 and show that it implies that the ring goes to Anna.

(d) Find the backward-induction solution for Case 2 and show that it implies that the ring goes to Bess.

(e) How much money does Sabio make at the backward-induction solution? How much money do Ann and Bess end up paying at the backward-induction solution?

■

3.7 Solutions to exercises

Solution to Exercise 3.1.

(a) For you it is a strictly dominant strategy to not pay and thus you should not pay.

(b) The extensive form is shown in Figure 3.15.

(c) For the professional, concern with reputation implies that $o_2 \succ_{MrT} o_4$ and $o_3 \succ_{MrT} o_1$. If we add the reasonable assumption that after all money is what they are after, then we can take the full ranking to be $o_2 \succ_{MrT} o_4 \succ_{MrT} o_3 \succ_{MrT} o_1$.
Representing preferences with ordinal utility functions with values in the set $\{1,2,3,4\}$, we have

outcome → utility function ↓	o_1	o_2	o_3	o_4
U_{you}	4	3	2	1
U_{MrT}	1	4	2	3

The corresponding game is obtained by replacing in Figure 3.15 o_1 with the payoff vector (4,1), o_3 with the payoff vector (2,2), etc.

For the one-timer, the ranking can be taken to be (although this is not the only possibility) $o_4 \succ_{MrT} o_2 \succ_{MrT} o_3 \succ_{MrT} o_1$, with corresponding utility representation:

outcome → utility function ↓	o_1	o_2	o_3	o_4
U_{you}	4	3	2	1
U_{MrT}	1	3	2	4

The corresponding extensive-form game is shown in Figure 3.16. □

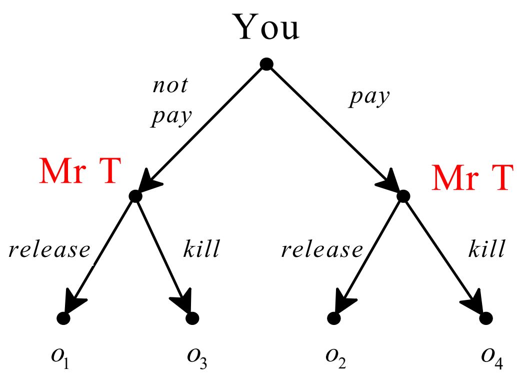


Figure 3.15: The game-frame for Part **(b)** of Exercise 3.1

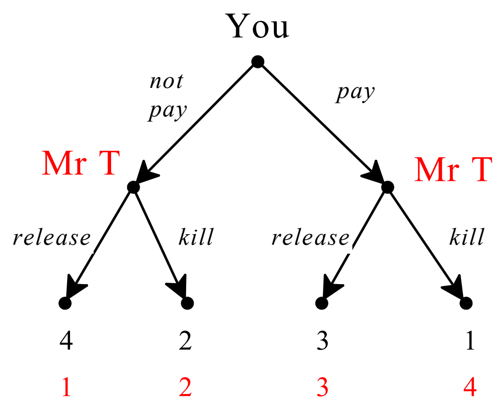


Figure 3.16: The game for Part (**(c)** of Exercise 3.1 when Mr T is a one-timer

Solution to Exercise 3.2. The game is shown in Figure 3.17 ('P' stands for promote, 'K' for keep (without promoting), 'F' for fire). □

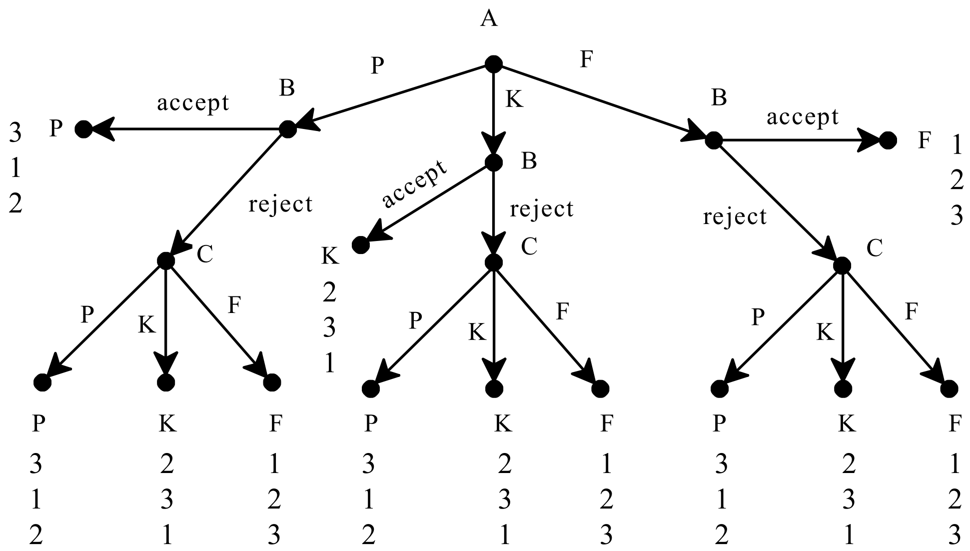


Figure 3.17: The game for Exercise 3.2

Solution to Exercise 3.3. The application of the backward-induction algorithm is shown by double edges in Figure 3.18 for the case of a professional Mr. T and in Figure 3.19 for the case of a one-timer Mr. T. Thus, against a professional you will pay and against a one-timer you would not pay. With the professional you would get Speedy back, with the one-timer you will hold a memorial service for Speedy. □

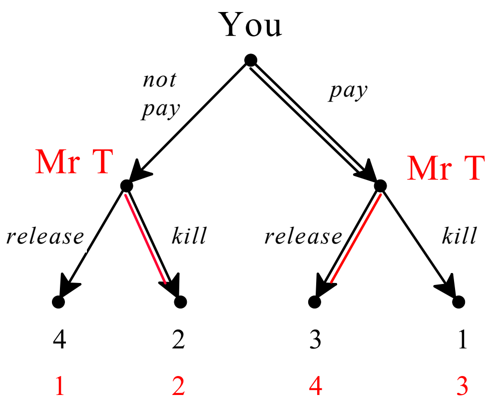


Figure 3.18: The game for Part **(b)** of Exercise 3.3

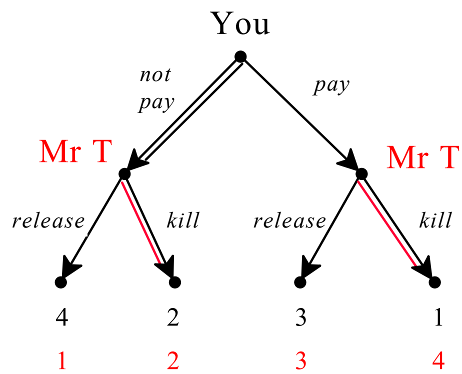


Figure 3.19: The game for Part (**(c)** of Exercise 3.3

Solution to Exercise 3.4. The backward-induction algorithm yields two solutions, shown in Figures 3.20 and 3.21. The difference between the two solutions lies in what Player *B* would do if Player *A* proposed *F*. In both solutions the officer is kept without promotion.□

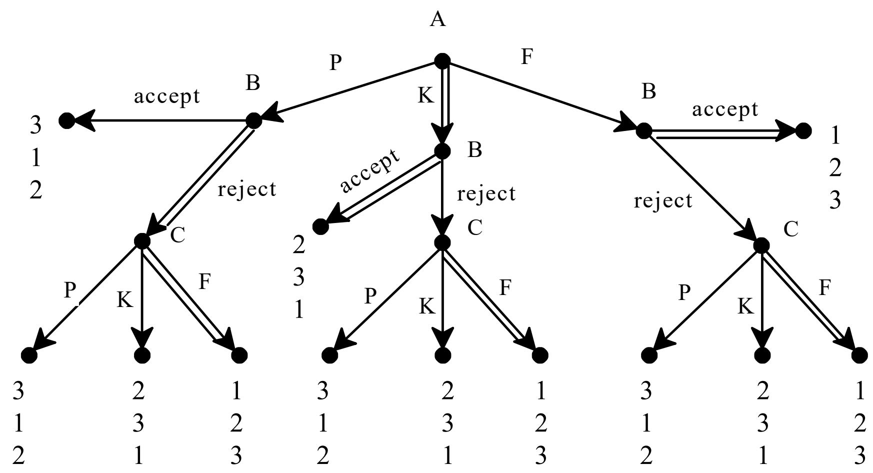


Figure 3.20: The first game for Exercise 3.4

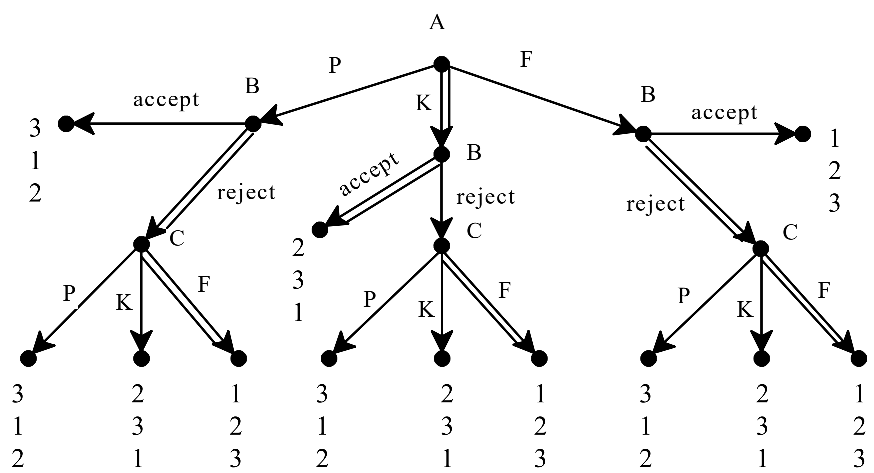


Figure 3.21: The second game for Exercise 3.4

Solution to Exercise 3.5. The game of Figure 3.2 is reproduced in Figure 3.22, with the unique backward-induction solution marked by double edges. The corresponding strategic form is shown In Figure 3.23 (for each of Player 2's strategies, the first element in the pair is what Player 2 would do at her left node and the second element what she would do at her right node). The Nash equilibria are highlighted. One Nash equilibrium, namely *(Offer 50-50,(Accept,Reject))*, corresponds to the backward induction solution, while the other Nash equilibrium, namely *(Offer 70-30,(Reject,Reject))* does not correspond to a backward-induction solution. □

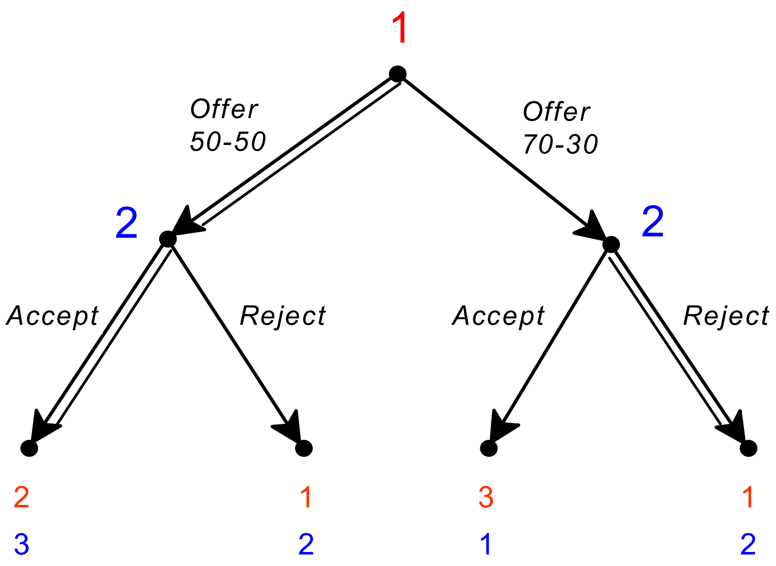


Figure 3.22: The extensive-form game for Exercise 3.5

		Player 2			
		(Accept,Accept)	(Accept,Reject)	(Reject,Accept)	(Reject,Reject)
Player 1	offer 50-50	2 3	2 3	1 2	1 2
	offer 70-30	3 1	1 2	3 1	1 2

Figure 3.23: The strategic-form game for Exercise 3.5

Solution to Exercise 3.6. The game of Figure 3.3 is reproduced in Figure 3.24 with the two backward-induction solutions marked by double edges. The corresponding strategic form is shown in Figure 3.25. The Nash equilibria are highlighted. The backward-induction solutions are $(a,(c,f),g)$ and $(b,(c,e),h)$ and both of them are Nash equilibria. There are three more Nash equilibria which are not backward-induction solutions, namely $(b,(d,f),g)$, $(a,(c,f),h)$ and $(b,(d,e),h)$. □

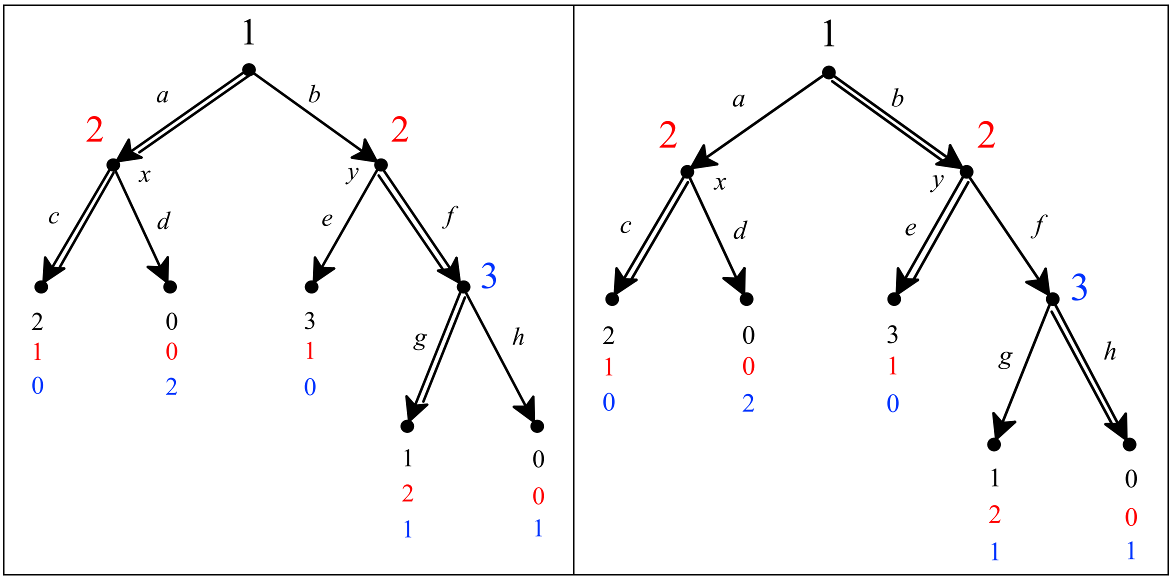


Figure 3.24: The extensive-form game for Exercise 3.6

Player 2

		ce			cf			de			df		
Player 1	a	2	1	0	2	1	0	0	0	2	0	0	2
	b	3	1	0	1	2	1	3	1	0	1	2	1

Player 3: *g*

Player 2

		ce			cf			de			df		
Player 1	a	2	1	0	2	1	0	0	0	2	0	0	2
	b	3	1	0	0	0	1	3	1	0	0	0	1

Player 3: *h*

Figure 3.25: The strategic-form game for Exercise 3.6

Solution to Exercise 3.7. The game of Exercise 3.2 is reproduced in Figure 3.26.

(a) All the possible strategies of Player B are shown in Figure 3.27.

(b) Player C has three decision nodes and three choices at each of her nodes. Thus she has $3 \times 3 \times 3 = 27$ strategies. □

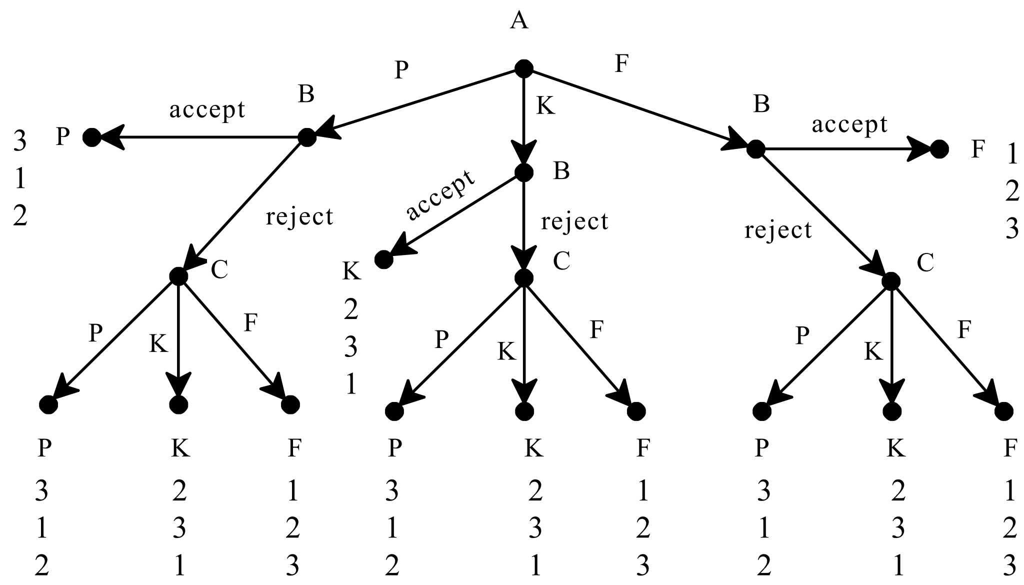


Figure 3.26: The extensive-form game for Exercise 3.7

	If A chooses P	If A chooses K	If A chooses F
1	accept	accept	accept
2	accept	accept	reject
3	accept	reject	accept
4	accept	reject	reject
5	reject	accept	accept
6	reject	accept	reject
7	reject	reject	accept
8	reject	reject	reject

Figure 3.27: The eight strategies of Player B

Solution to Exercise 3.8.

(a) One backward-induction solution is the strategy profile $((L,W),(a,e))$ shown by double edges in Figure 3.28. The corresponding backward-induction outcome is the play *La* with associated payoff vector $(2,1)$. The other backward-induction solution is the strategy profile $((R,W),(a,d))$ shown in Figure 3.29. The corresponding backward-induction outcome is the play *Rd* with associated payoff vector $(3,2)$.

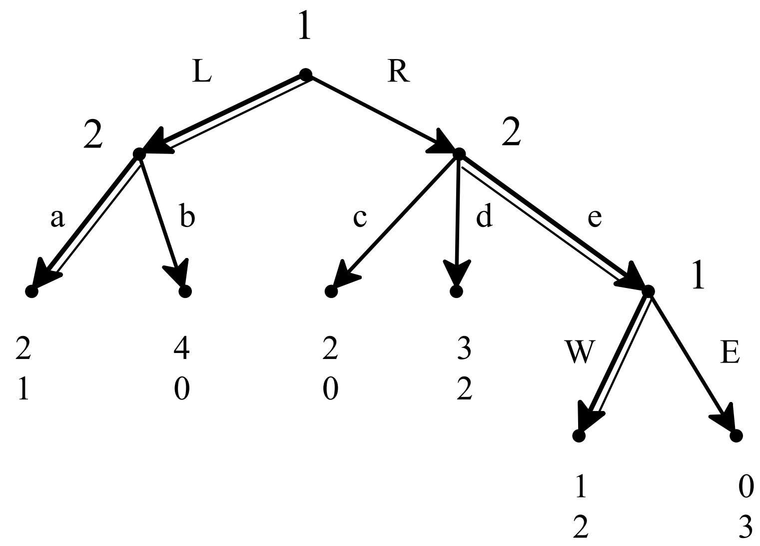


Figure 3.28: One backward-induction solution of the game of Part **(a)** of Exercise 3.8

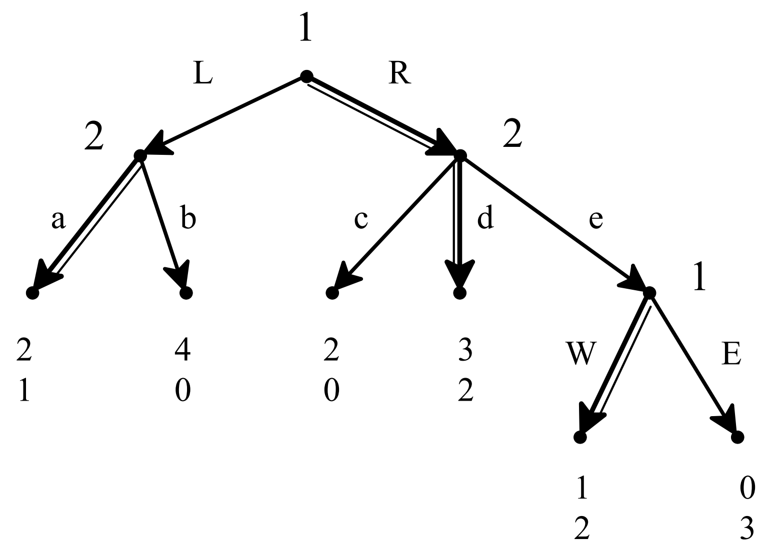


Figure 3.29: A second backward-induction solution of the game of Part **(a)** of Exercise 3.8

(b) Player 1 has four strategies: LW, LE, RW and RE.

(c) Player 2 has six strategies: ac, ad, ae, bc, bd and be.

(d) The strategic form is shown in Figure 3.30.

1 \ 2	ac	ad	ae	bc	bd	be
LW	2 , 1	2 , 1	2 , 1	4 , 0	4 , 0	4 , 0
LE	2 , 1	2 , 1	2 , 1	4 , 0	4 , 0	4 , 0
RW	2 , 0	3 , 2	1 , 2	2 , 0	3 , 2	1 , 2
RE	2 , 0	3 , 2	0 , 3	2 , 0	3 , 2	0 , 3

Figure 3.30: The strategic-form game for Part (d) of Exercise 3.8

(e) Player 1 does not have a dominant strategy.

(f) For Player 2 *ae* is a weakly dominant strategy.

(g) There is no dominant strategy equilibrium.

(h) For Player 1 *RE* is weakly dominated by *RW* (and *LW* and *LE* are equivalent).

(i) For Player 2 *ac* is weakly dominated by *ad* (and *ae*), *ad* is weakly dominated by *ae*, *bc* is (strictly or weakly) dominated by every other strategy, *bd* is weakly dominated by *be* (and by *ae* and *ad*), *be* is weakly dominated by *ae*.
Thus the dominated strategies are: ac, ad, bc, bd and *b*e.

(j) The iterative elimination of weakly dominated strategies yields the following reduced game (in Step 1 eliminate *RE* for Player 1 and ac, ad, bc, bd and *be* for Player 2; in Step 2 eliminate *RW* for Player 1):

Player 1 \ Player 2	ae
LW	2 , 1
LE	2 , 1

Thus we are left with one of the two backward-induction solutions, namely $((L,W),(a,e))$ but also with $((L,E),(a,e))$ which is not a backward-induction solution.

(k) The Nash equilibria are highlighted in Figure 3.31.
There are five Nash equilibria: $(LW, ac), (LE, ac), (RW, ad), (LW, ae)$ and (LE, ae).

□

1 \ 2	ac	ad	ae	bc	bd	be
LW	2 , 1	2 , 1	2 , 1	4 , 0	4 , 0	4 , 0
LE	2 , 1	2 , 1	2 , 1	4 , 0	4 , 0	4 , 0
RW	2 , 0	3 , 2	1 , 2	2 , 0	3 , 2	1 , 2
RE	2 , 0	3 , 2	0 , 3	2 , 0	3 , 2	0 , 3

Figure 3.31: The highlighted cells are the Nash equilibria (for Part **(k)** of Exercise 3.8)

Solution to Exercise 3.9.

(a) The extensive game is shown in Figure 3.32.

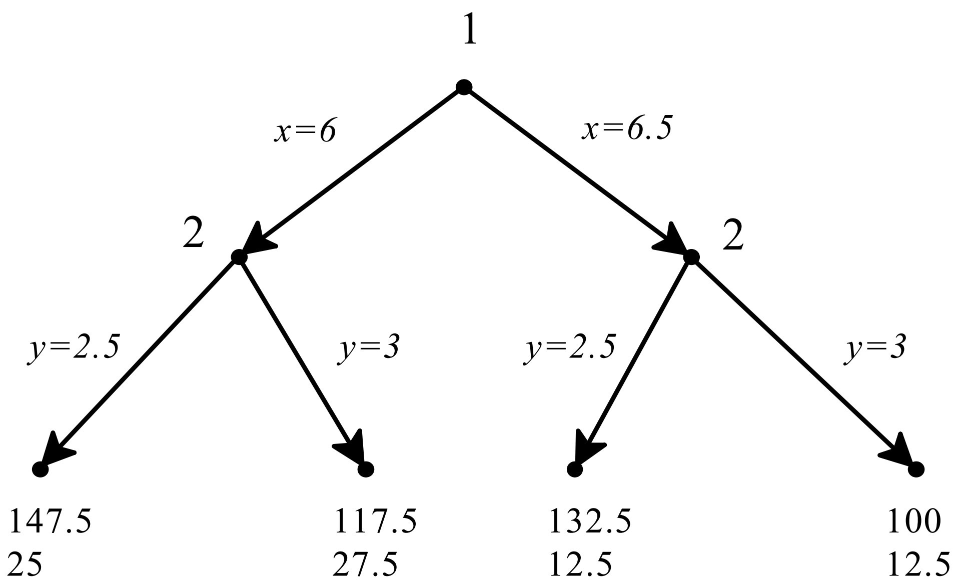


Figure 3.32: The extensive-form game for Exercise 3.9

(b) There are two backward-induction solutions.
The first is the strategy profile shown in Figure 3.33. The corresponding backward-induction outcome is given by Firm 1 producing 6 units and Firm 2 producing 3 units with profits 117.5 for Firm 1 and 27.5 for Firm 2.
The other backward-induction solution is the strategy profile shown in Figure 3.34. The corresponding backward-induction outcome is given by Firm 1 producing 6.5 units and Firm 2 producing 2.5 units with profits 132.5 for Firm 1 and 12.5 for Firm 2.

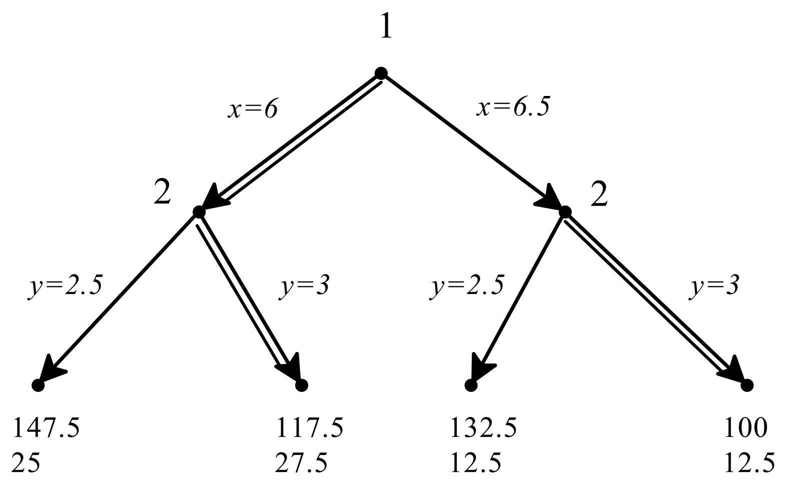


Figure 3.33: One backward-induction solution of the game of Figure 3.32

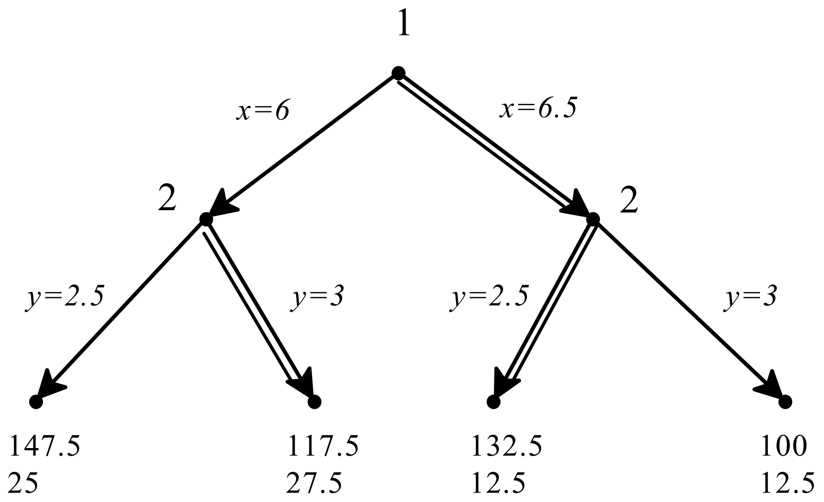


Figure 3.34: A second backward-induction solution of the game of Figure 3.32

(c) The strategic form is shown in Figure 3.35.

		Firm 2			
		(2.5,2.5)	(2.5,3)	(3,2.5)	(3,3)
Firm 1	6	147.5 25	147.5 25	117.5 27.5	117.5 27.5
	6.5	132.5 12.5	100 12.5	132.5 12.5	100 12.5

Figure 3.35: The strategic-form game for Part (d) of Exercise 3.9

(d) The Nash equilibria are highlighted in Figure 3.35. In this game the set of Nash equilibria coincides with the set of backward-induction solutions. □

Solution to Exercise 3.10. The game under consideration is shown in Figure 3.36, where x is an integer.

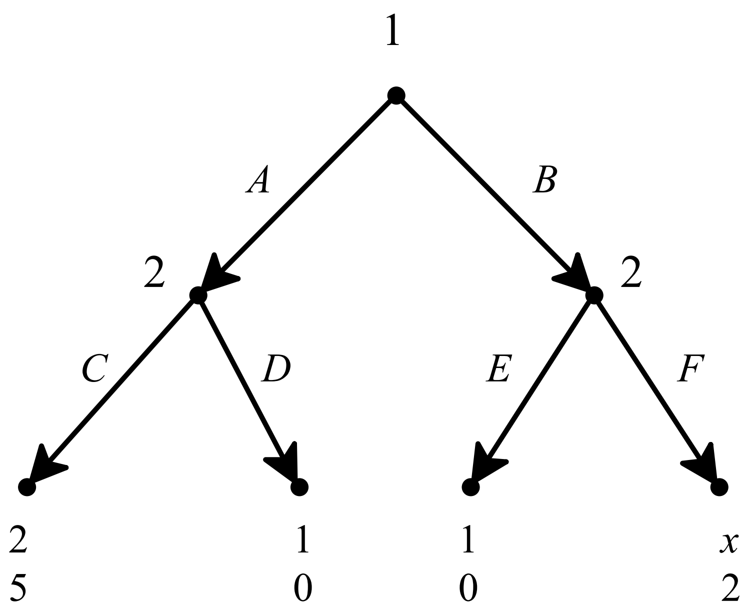


Figure 3.36: The extensive-form game for Part **(a)** of Exercise 3.10

(a) The backward-induction strategy of Player 2 is the same, no matter what x is, namely (C,F). Thus the backward induction solutions are as follows.

- If $x < 2$, there is only one: $(A,(C,F))$.
- If $x = 2$ there are two: $(A,(C,F))$ and $(B,(C,F))$.
- $If x > 2$, there is only one: $(B,(C,F))$.

(b) The strategic form is shown in Figure 3.37. First note that $(\mathbf{A}, (\mathbf{C}, \mathbf{E}))$ **is a Nash equilibrium for every value of** x. Now, depending on the value of x the other Nash equilibria are as follows:

- If $x < 1$, $(A, (C, F))$.
- If $1 \leq x < 2$, $(A, (C, F))$ and $(B, (D, F))$.
- If $x = 2$, $(A, (C, F)), (B, (C, F))$ and $(B, (D, F))$.
- If $x > 2$, $(B, (C, F))$ and $(B, (D, F))$. □

		Player 2			
		CE	CF	DE	DF
Player 1	A	2 **5**	2 **5**	1 **0**	1 **0**
	B	1 **0**	x **2**	1 **0**	x **2**

Figure 3.37: The strategic-form game for Part **(b)** of Exercise 3.10

Solution to Exercise 3.11. Let us find the losing positions. If Player i, with his choice, can bring the sum to **40** then he can win (the other player with her next choice will take the sum to a number between 41 and 47 and then Player i can win with his next choice). Working backwards, the previous losing position is **32** (from here the player who has to move will take the sum to a number between 33 and 39 and after this the opponent can take it to 40). Reasoning backwards, the earlier losing positions are **24**, **16**, **8** and **0**. Thus Player 1 starts from a losing position and therefore it is Player 2 who has a winning strategy. The winning strategy is: at every turn, if Player 1's last choice was n then Player 2 should choose $(\mathbf{8} - \mathbf{n})$. □

Solution to Exercise 3.12.

(a) The initial part of the game is shown in Figure 3.38.

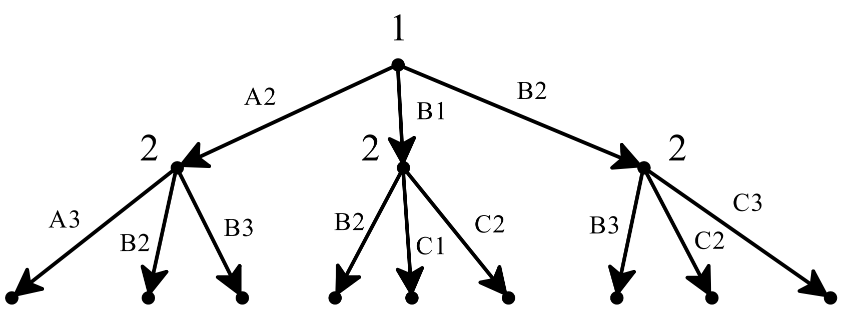


Figure 3.38: The initial part of the game of Part **(a)** of Exercise 3.12

(b) From G4 Player 1 should move the coin to H5. From there Player 2 has to move it to H6 and Player 1 to H7 and Player 2 to H8 and from there Player 1 wins by moving it to H9.

(c) $A1 \xrightarrow{1} B2 \xrightarrow{2} C3 \xrightarrow{1} D4 \xrightarrow{2} E5 \xrightarrow{1} F5 \xrightarrow{2} G6 \xrightarrow{1} H7 \xrightarrow{2} H8 \xrightarrow{1} H9.$

(d) $A1 \xrightarrow{1} B2 \xrightarrow{2} C3 \xrightarrow{1} D4 \xrightarrow{2} E5 \xrightarrow{1} F5 \xrightarrow{2} G6 \xrightarrow{1} G7 \xrightarrow{2} H7 \xrightarrow{1} H8 \xrightarrow{2} H9.$

(e) Using backward induction we can label each cell with a W (meaning that the player who has to move when the coin is there has a winning continuation strategy) or with an L (meaning that the player who has to move when the coin is there can be made to lose).
If all the cells that are accessible from a given cell are marked with a W then that cell must be marked with an L.
If from a cell there is an accessible cell marked with an L then that cell should be marked with a W. See Figure 3.39.
From the picture it is clear that it is Player 1 who has a winning strategy. The winning strategy of Player 1 is: move the coin to cell B1 and from then on, after every move of Player 2, move the coin to a cell marked L. □

END	W	L	W	L				9
W	W	W		W	W	L		8
L	W	L	W	L		W		7
W	W				W	L	W	6
L	W	L	W		L	W	W	5
W	W		W	L	W		L	4
L		L	W	W	W	L	W	3
	W	W		L	W	W	W	2
	L	W	L	W	W	L	**START**	1
H	G	F	E	D	C	B	A	

Figure 3.39: Solution for the coin game

Solution to Exercise 3.13.

(a) The game is shown in Figure 3.40.

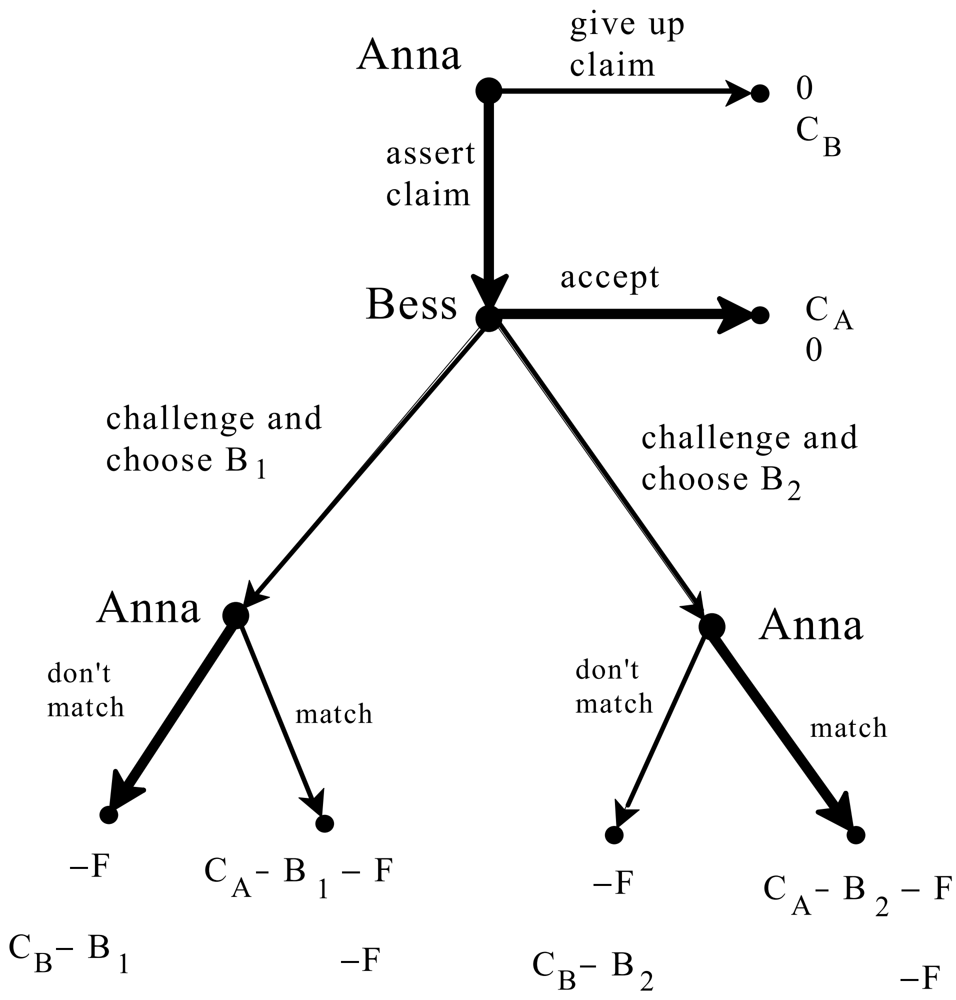


Figure 3.40: The extensive-form game for Part **(a)** of Exercise 3.13

(b) The backward-induction solution is marked by thick arrows in Figure 3.40.

(c) The sequence of moves is shown in Figure 3.41.
Suppose that Anna is the legal owner and values the ring more than Bess does: $C_A > C_B$. At the last node Anna will choose "match" if $C_A > B$ and "don't match" if $B > C_A$. In the first case Bess's payoff will be $-F$, while in the second case it will be $C_B - B$, which is negative since $B > C_A$ and $C_A > C_B$. Thus in either case Bess's payoff would be negative. Hence at her decision node Bess will choose "accept" (Bess can get the ring at this stage only if she bids more than the ring is worth to her). Anticipating this, Anna will assert her claim at the first decision node. Thus at the backward-induction solution the ring goes to Anna, the legal owner. The payoffs are C_A for Anna and 0 for Bess. **Note that no money changes hands**.

(d) Suppose that Bess is the legal owner and values the ring more than Anna does: $C_B > C_A$. At the last node Anna will choose "match" if $C_A > B$ and "don't match" if $B > C_A$. In the first case Bess's payoff will be $-F$, while in the second case it will be $C_B - B$, which will be positive as long as $C_B > B$. Hence at her decision node Bess will choose to challenge and bid any amount B such that $C_B > B > C_A$. Anticipating this, at her first decision node Anna will give up (and get a payoff of 0), because if she asserted her claim then her final payoff would be negative. Thus at the backward-induction solution the ring goes to Bess, the legal owner. The payoffs are 0 for Anna and C_B for Bess. **Note that no money changes hands**.

(e) As pointed out above, in both cases no money changes hands at the backward-induction solution. Thus Judge Sabio collects no money at all and both Ann and Bess pay nothing. □

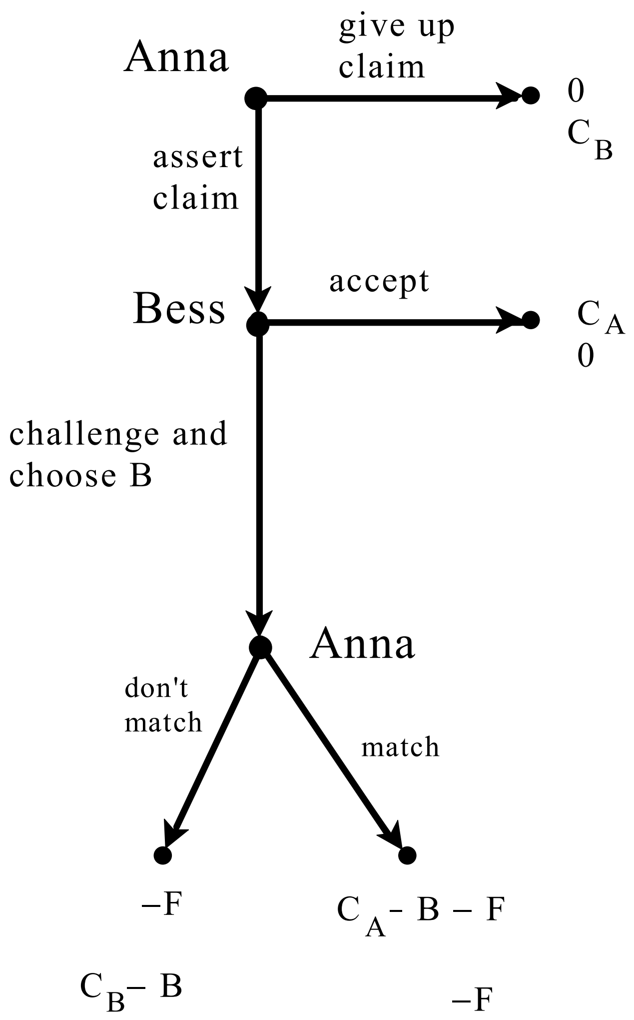


Figure 3.41: The extensive-form game for Part ((**c**) of Exercise 3.13