Chapter 3

Rigid-Body Motions

In the previous chapter, we saw that a minimum of six numbers is needed to specify the position and orientation of a rigid body in three-dimensional physical space. In this chapter we develop a systematic way to describe a rigid body's position and orientation which relies on attaching a reference frame to the body. The configuration of this frame with respect to a fixed reference frame is then represented as a 4×4 matrix. This matrix is an example of an implicit representation of the C-space, as discussed in the previous chapter: the actual six-dimensional space of rigid-body configurations is obtained by applying ten constraints to the 16-dimensional space of 4×4 real matrices.

Such a matrix not only represents the configuration of a frame, but can also be used to (1) translate and rotate a vector or a frame, and (2) change the representation of a vector or a frame from coordinates in one frame to coordinates in another frame. These operations can be performed by simple linear algebra, which is a major reason why we choose to represent a configuration as a 4×4 matrix.

The non-Euclidean (i.e., non-"flat") nature of the C-space of positions and orientations leads us to use a matrix representation. A rigid body's velocity, however, can be represented simply as a point in $\mathbb{R}^6$, defined by three angular velocities and three linear velocities, which together we call a **spatial velocity** or **twist**. More generally, even though a robot's C-space may not be a vector space, the set of feasible velocities at any point in the C-space always forms a vector space. For example, consider a robot whose C-space is the sphere S^2: although the C-space is not flat, at any point on the sphere the space of velocities can be thought of as the plane (a vector space) tangent to that point on the sphere.

Any rigid-body configuration can be achieved by starting from the fixed (home) reference frame and integrating a constant twist for a specified time. Such a motion resembles the motion of a screw, rotating about and translating along the same fixed axis. The observation that all configurations can be achieved by a screw motion motivates a six-parameter representation of the configuration called the **exponential coordinates**. The six parameters can be divided into the parameters describing the direction of the screw axis and a scalar to indicate how far the screw motion must be followed to achieve the desired configuration.

This chapter concludes with a discussion of forces. Just as angular and linear velocities are packaged together into a single vector in $\mathbb{R}^6$, moments (torques) and forces are packaged together into a six-vector called a **spatial force** or **wrench**.

To illustrate the concepts and to provide a synopsis of the chapter, we begin with a motivating planar example. Before doing so, we make some remarks about vector notation.

A Word about Vectors and Reference Frames

A **free vector** is a geometric quantity with a length and a direction. Think of it as an arrow in $\mathbb{R}^n$. It is called "free" because it is not necessarily rooted anywhere; only its length and direction matter. A linear velocity can be viewed as a free vector: the length of the arrow is the speed and the direction of the arrow is the direction of the velocity. A free vector is denoted by an upright text symbol, e.g., v.

If a reference frame and length scale have been chosen for the underlying space in which v lies then this free vector can be moved to a position such that the base of the arrow is at the origin without changing the orientation. The free vector v can then be represented by its coordinates in the reference frame. We write the vector in italics, $v \in \mathbb{R}^n$, where v is at the "head" of the arrow in the frame's coordinates. If a different reference frame and length scale are chosen then the representation v will change but the underlying free vector v is unchanged.

In other words, we say that v is **coordinate free**; it refers to a physical quantity in the underlying space, and it does not care how we represent it. However, v is a representation of v that depends on the choice of coordinate frame.

A point p in physical space can also be represented as a vector. Given a choice of reference frame and length scale for physical space, the point p can be represented as a vector from the reference frame origin to p; its vector

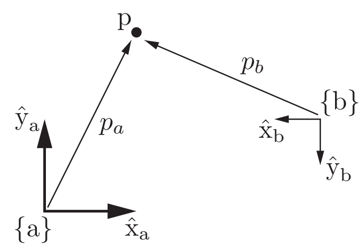


Figure 3.1: The point p exists in physical space, and it does not care how we represent it. If we fix a reference frame {a}, with unit coordinate axes $\hat{\mathrm{x}}_\mathrm{a}$ and $\hat{\mathrm{y}}_\mathrm{a}$, we can represent p as $p_a = (1, 2)$. If we fix a reference frame {b} at a different location, a different orientation, and a different length scale, we can represent p as $p_b = (4, -2)$.

representation is denoted in italics by $p \in \mathbb{R}^n$. Here, as before, a different choice of reference frame and length scale for physical space leads to a different representation $p \in \mathbb{R}^n$ for the same point p in physical space. See Figure 3.1.

In the rest of this book, a choice of length scale will always be assumed, but we will be dealing with reference frames at different positions and orientations. A reference frame can be placed anywhere in space, and any reference frame leads to an equally valid representation of the underlying space and the objects in it. We always assume that exactly one stationary **fixed frame**, or **space frame**, denoted {s}, has been defined. This might be attached to a corner of a room, for example. Similarly, we often assume that at least one frame has been attached to some moving rigid body, such as the body of a quadrotor flying in the room. This **body frame**, denoted {b}, is the stationary frame that is coincident with the body-attached frame at any instant.

While it is common to attach the origin of the {b} frame to some important point on the body, such as its center of mass, this is not necessary. The origin of the {b} frame does not even need to be on the physical body itself, as long as its configuration relative to the body, viewed from an observer stationary relative to the body, is constant.

Important! All frames in this book are stationary, inertial, frames. When we refer to a body frame {b}, we mean a motionless frame that is instantaneously coincident with a frame that is fixed to a (possibly moving) body. This is important to keep in mind, since you may have had a dynamics course that used non-inertial moving frames attached to rotating bodies. Do not confuse these with the stationary, inertial, body frames of this book.

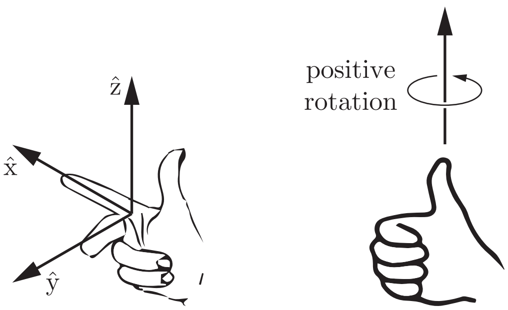


Figure 3.2: (Left) The $\hat{\mathrm{x}}$, $\hat{\mathrm{y}}$, and $\hat{\mathrm{z}}$ axes of a right-handed reference frame are aligned with the index finger, middle finger, and thumb of the right hand, respectively. (Right) A positive rotation about an axis is in the direction in which the fingers of the right hand curl when the thumb is pointed along the axis.

For simplicity, we will usually refer to a body frame as a frame attached to a moving rigid body. Despite this, at any instant, by “body frame” we actually mean the stationary frame that is instantaneously coincident with the frame moving along with the body.

It is worth repeating one more time: **all frames are stationary**.

All reference frames are **right-handed**, as illustrated in Figure 3.2. A positive rotation about an axis is defined as the direction in which the fingers of the right hand curl when the thumb is pointed along the axis (Figure 3.2).

3.1 Rigid-Body Motions in the Plane

Consider the planar body (the gray shape) in Figure 3.3; its motion is confined to the plane. Suppose that a length scale and a fixed reference frame {s} have been chosen as shown, with unit axes $\hat{\mathrm{x}}_{\mathrm{s}}$ and $\hat{\mathrm{y}}_{\mathrm{s}}$. (Throughout this book, the hat notation indicates a unit vector.) Similarly, we attach a reference frame with unit axes $\hat{\mathrm{x}}_{\mathrm{b}}$ and $\hat{\mathrm{y}}_{\mathrm{b}}$ to the planar body. Because this frame moves with the body, it is called the body frame and is denoted {b}.

To describe the configuration of the planar body, only the position and orientation of the body frame with respect to the fixed frame need to be specified. The body-frame origin p can be expressed in terms of the coordinate axes of

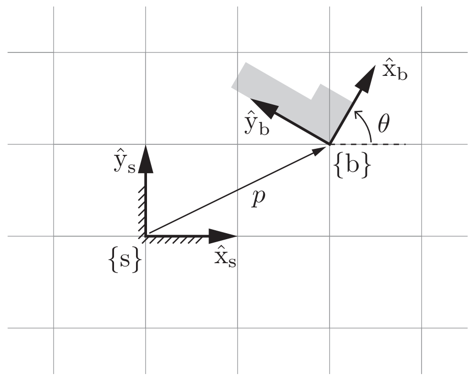


Figure 3.3: The body frame $\{b\}$ is expressed in the fixed-frame coordinates $\{s\}$ by the vector p and the directions of the unit axes $\hat{x}_b$ and $\hat{y}_b$. In this example, $p = (2, 1)$ and $\theta = 60^\circ$, so $\hat{x}_b = (\cos\theta, \sin\theta) = (0.5, 1/\sqrt{2})$ and $\hat{y}_b = (-\sin\theta, \cos\theta) = (-1/\sqrt{2}, 0.5)$.

$\{s\}$ as

$$p = p_x\hat{x}_s + p_y\hat{y}_s. \tag{3.1}$$

You are probably more accustomed to writing this vector as simply $p = (p_x, p_y)$; this is fine when there is no possibility of ambiguity about reference frames, but writing p as in Equation (3.1) clearly indicates the reference frame with respect to which (p_x, p_y) is defined.

The simplest way to describe the orientation of the body frame $\{b\}$ relative to the fixed frame $\{s\}$ is by specifying the angle θ, as shown in Figure 3.3. Another (admittedly less simple) way is to specify the directions of the unit axes $\hat{x}_b$ and $\hat{y}_b$ of $\{b\}$ relative to $\{s\}$, in the form

$$\hat{x}_b = \cos\theta\,\hat{x}_s + \sin\theta\,\hat{y}_s, \tag{3.2}$$

$$\hat{y}_b = -\sin\theta\,\hat{x}_s + \cos\theta\,\hat{y}_s. \tag{3.3}$$

At first sight this seems to be a rather inefficient way of representing the body-frame orientation. However, imagine if the body were to move arbitrarily in three-dimensional space; a single angle θ would not suffice to describe the orientation of the displaced reference frame. We would actually need three angles, but it is not yet clear how to define an appropriate set of three angles. However, expressing the directions of the coordinate axes of $\{b\}$ in terms of coefficients of the coordinate axes of $\{s\}$, as we have done above for the planar case, is straightforward.

Assuming we agree to express everything in terms of $\{s\}$ then, just as the

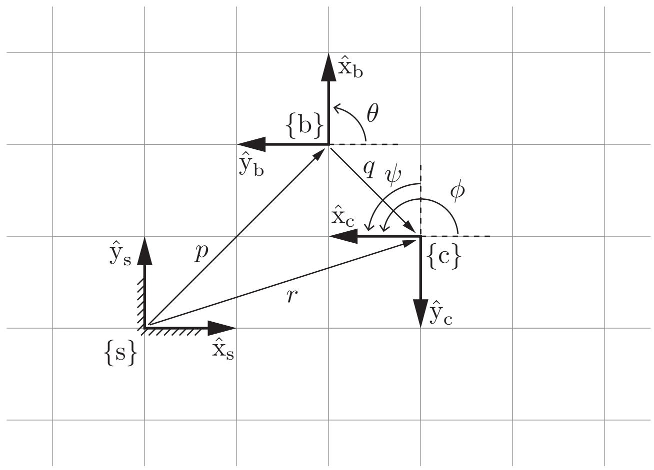


Figure 3.4: The frame $\{b\}$ in $\{s\}$ is given by (P, p), and the frame $\{c\}$ in $\{b\}$ is given by (Q, q). From these we can derive the frame $\{c\}$ in $\{s\}$, described by (R, r). The numerical values of the vectors p, q, and r and the coordinate-axis directions of the three frames are evident from the grid of unit squares.

point p can be represented as a column vector $p \in \mathbb{R}^2$ of the form

$$p = \begin{bmatrix} p_x \\ p_y \end{bmatrix}, \tag{3.4}$$

the two vectors $\hat{x}_b$ and $\hat{y}_b$ can also be written as column vectors and packaged into the following 2×2 matrix P:

$$P = [\hat{x}_b \ \ \hat{y}_b] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{3.5}$$

The matrix P is an example of a **rotation matrix**. Although P consists of four numbers, they are subject to three constraints (each column of P must be a unit vector, and the two columns must be orthogonal to each other), and the one remaining degree of freedom is parametrized by θ. Together, the pair (P, p) provides a description of the orientation and position of $\{b\}$ relative to $\{s\}$.

Now refer to the three frames in Figure 3.4. Repeating the approach above, and expressing $\{c\}$ in $\{s\}$ as the pair (R, r), we can write

$$r = \begin{bmatrix} r_x \\ r_y \end{bmatrix}, \qquad R = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}. \tag{3.6}$$

We could also describe the frame {c} relative to {b}. Letting q denote the vector from the origin of {b} to the origin of {c} expressed in {b} coordinates, and letting Q denote the orientation of {c} relative to {b}, we can write {c} relative to {b} as the pair (Q, q), where

$$q = \begin{bmatrix} q_x \\ q_y \end{bmatrix}, \qquad Q = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}. \tag{3.7}$$

If we know (Q, q) (the configuration of {c} relative to {b}) and (P, p) (the configuration of {b} relative to {s}), we can compute the configuration of {c} relative to {s} as follows:

$$R = PQ \quad \text{(convert } Q \text{ to the \{s\} frame)} \tag{3.8}$$

$$r = Pq + p \quad \text{(convert } q \text{ to the \{s\} frame and vector-sum with } p\text{)}. \tag{3.9}$$

Thus (P, p) not only represents a configuration of {b} in {s}; it can also be used to convert the representation of a point or frame from {b} coordinates to {s} coordinates.

Now consider a rigid body with two frames attached to it, {d} and {c}. The frame {d} is initially coincident with {s}, and {c} is initially described by (R, r) in {s} (Figure 3.5(a)). Then the body is moved in such a way that {d} moves to {d′}, becoming coincident with a frame {b} described by (P, p) in {s}. Where does {c} end up after this motion? Denoting the configuration of the new frame {c′} as (R', r'), you can verify that

$$R' = PR, \tag{3.10}$$

$$r' = Pr + p, \tag{3.11}$$

which is similar to Equations (3.8) and (3.9). The difference is that (P, p) is expressed in the same frame as (R, r), so the equations are not viewed as a change of coordinates, but instead as a **rigid-body displacement** (also known as a **rigid-body motion**): in Figure 3.5(a) transformation 1 rotates {c} according to P and transformation 2 translates it by p in {s}.

Thus we see that a rotation matrix–vector pair such as (P, p) can be used for three purposes:

(a) to represent a configuration of a rigid body in {s} (Figure 3.3);

(b) to change the reference frame in which a vector or frame is represented (Figure 3.4);

(c) to displace a vector or a frame (Figure 3.5(a)).

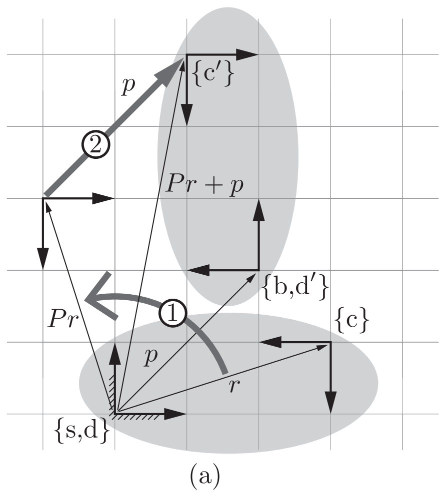


Figure 3.5: (a) The frame {d}, fixed to an elliptical rigid body and initially coincident with {s}, is displaced to {d′} (which is coincident with the stationary frame {b}), by first rotating according to P then translating according to p, where (P, p) is the representation of {b} in {s}. The same transformation takes the frame {c}, also attached to the rigid body, to {c′}. The transformation marked ① rigidly rotates {c} about the origin of {s}, and then transformation ② translates the frame by p expressed in {s}. (b) Instead of viewing this displacement as a rotation followed by a translation, both rotation and translation can be performed simultaneously. The displacement can be viewed as a rotation of $\beta = 90°$ about a fixed point s.

Referring to Figure 3.5(b), note that the rigid-body motion illustrated in Figure 3.5(a), expressed as a rotation followed by a translation, can be obtained by simply rotating the body about a fixed point s by an angle β. This is a planar example of a **screw motion**.[1] The displacement can therefore be parametrized by the three screw coordinates (β, s_x, s_y), where $(s_x, s_y) = (0, 2)$ denotes the coordinates for the point s (i.e., the screw axis out of the page) in the fixed frame {s}.

Another way to represent the screw motion is to consider it as the displacement obtained by following simultaneous angular and linear velocities for a given distance. Inspecting Figure 3.5(b), we see that rotating about s with a unit angular velocity ($\omega = 1$ rad/s) means that a point at the origin of the {s} frame moves at two units per second initially in the $+\hat{x}$-direction of the {s} frame, i.e., $v = (v_x, v_y) = (2, 0)$. We can package these together in the three-

[1]If the displacement is a pure translation without rotation, then s lies at infinity.

vector $\mathcal{S} = (\omega, v_x, v_y) = (1, 2, 0)$, a representation of the **screw axis**. Following this screw axis for an angle $\theta = \pi/2$ yields the final displacement. Thus we can represent the displacement using the three coordinates $\mathcal{S}\theta = (\pi/2, \pi, 0)$. These coordinates have some advantages, and we call these the **exponential coordinates** for the planar rigid-body displacement.

To represent the combination of an angular and a linear velocity, called a **twist**, we take a screw axis $\mathcal{S} = (\omega, v_x, v_y)$, where $\omega = 1$, and scale it by multiplying by some rotation speed, $\dot{\theta}$. The twist is $\mathcal{V} = \mathcal{S}\dot{\theta}$. The net displacement obtained by rotating about the screw axis $\mathcal{S}$ by an angle θ is equivalent to the displacement obtained by rotating about $\mathcal{S}$ at a speed $\dot{\theta} = \theta$ for unit time, so $\mathcal{V} = \mathcal{S}\dot{\theta}$ can also be considered a set of exponential coordinates.

Preview of the remainder of this chapter. In the rest of this chapter we generalize the concepts above to three-dimensional rigid-body motions. For this purpose consider a rigid body occupying three-dimensional physical space, as shown in Figure 3.6. Assume that a length scale for physical space has been chosen, and that both the fixed frame {s} and body frame {b} have been chosen as shown. Throughout this book all reference frames are right-handed – the unit axes $\{\hat{\mathrm{x}}, \hat{\mathrm{y}}, \hat{\mathrm{z}}\}$ always satisfy $\hat{\mathrm{x}} \times \hat{\mathrm{y}} = \hat{\mathrm{z}}$. Denote the unit axes of the fixed frame by $\{\hat{\mathrm{x}}_\mathrm{s}, \hat{\mathrm{y}}_\mathrm{s}, \hat{\mathrm{z}}_\mathrm{s}\}$ and the unit axes of the body frame by $\{\hat{\mathrm{x}}_\mathrm{b}, \hat{\mathrm{y}}_\mathrm{b}, \hat{\mathrm{z}}_\mathrm{b}\}$. Let p denote the vector from the fixed-frame origin to the body-frame origin. In terms of the fixed-frame coordinates, p can be expressed as

$$p = p_1\hat{\mathrm{x}}_\mathrm{s} + p_2\hat{\mathrm{y}}_\mathrm{s} + p_3\hat{\mathrm{z}}_\mathrm{s}. \tag{3.12}$$

The axes of the body frame can also be expressed as

$$\begin{aligned}
\hat{\mathrm{x}}_\mathrm{b} &= r_{11}\hat{\mathrm{x}}_\mathrm{s} + r_{21}\hat{\mathrm{y}}_\mathrm{s} + r_{31}\hat{\mathrm{z}}_\mathrm{s}, && (3.13)\\
\hat{\mathrm{y}}_\mathrm{b} &= r_{12}\hat{\mathrm{x}}_\mathrm{s} + r_{22}\hat{\mathrm{y}}_\mathrm{s} + r_{32}\hat{\mathrm{z}}_\mathrm{s}, && (3.14)\\
\hat{\mathrm{z}}_\mathrm{b} &= r_{13}\hat{\mathrm{x}}_\mathrm{s} + r_{23}\hat{\mathrm{y}}_\mathrm{s} + r_{33}\hat{\mathrm{z}}_\mathrm{s}. && (3.15)
\end{aligned}$$

Defining $p \in \mathbb{R}^3$ and $R \in \mathbb{R}^{3\times 3}$ as

$$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}, \qquad R = [\hat{\mathrm{x}}_\mathrm{b} \;\; \hat{\mathrm{y}}_\mathrm{b} \;\; \hat{\mathrm{z}}_\mathrm{b}] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}, \tag{3.16}$$

the 12 parameters given by (R, p) then provide a description of the position and orientation of the rigid body relative to the fixed frame.

Since the orientation of a rigid body has three degrees of freedom, only three of the nine entries in R can be chosen independently. One three-parameter

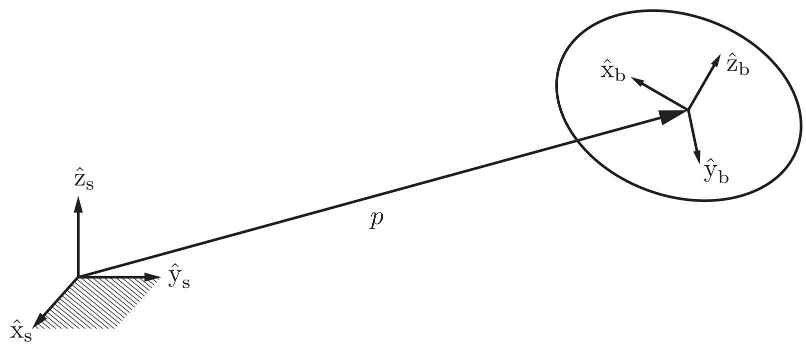


Figure 3.6: Mathematical description of position and orientation.

representation of rotations is provided by the exponential coordinates, which define an axis of rotation and the angle rotated about that axis. We leave other popular representations of orientations (the three-parameter **Euler angles** and the **roll–pitch–yaw angles**, the **Cayley–Rodrigues parameters**, and the **unit quaternions**, which use four variables subject to one constraint) to Appendix B.

We then examine the six-parameter exponential coordinates for the configuration of a rigid body that arise from integrating a six-dimensional twist consisting of the body's angular and linear velocities. This representation follows from the Chasles–Mozzi theorem which states that every rigid-body displacement can be obtained by a finite rotation and translation about a fixed screw axis.

We conclude with a discussion of forces and moments. Rather than treat these as separate three-dimensional quantities, we merge the moment and force vectors into a six-dimensional **wrench**. The twist and wrench, and rules for manipulating them, form the basis for the kinematic and dynamic analyses in subsequent chapters.

3.2 Rotations and Angular Velocities

3.2.1 Rotation Matrices

We argued earlier that, of the nine entries in the rotation matrix R, only three can be chosen independently. We begin by expressing a set of six explicit constraints on the entries of R. Recall that the three columns of R correspond to the body-frame unit axes $\{\hat{x}_b, \hat{y}_b, \hat{z}_b\}$. The following conditions must therefore be satisfied.

(a) The unit norm condition: $\hat{x}_b$, $\hat{y}_b$, and $\hat{z}_b$ are all unit vectors, i.e.,

$$\begin{aligned} r_{11}^2 + r_{21}^2 + r_{31}^2 &= 1, \\ r_{12}^2 + r_{22}^2 + r_{32}^2 &= 1, \\ r_{13}^2 + r_{23}^2 + r_{33}^2 &= 1. \end{aligned} \tag{3.17}$$

(b) The orthogonality condition: $\hat{x}_b \cdot \hat{y}_b = \hat{x}_b \cdot \hat{z}_b = \hat{y}_b \cdot \hat{z}_b = 0$ (here $\cdot$ denotes the inner product), or

$$\begin{aligned} r_{11}r_{12} + r_{21}r_{22} + r_{31}r_{32} &= 0, \\ r_{12}r_{13} + r_{22}r_{23} + r_{32}r_{33} &= 0, \\ r_{11}r_{13} + r_{21}r_{23} + r_{31}r_{33} &= 0. \end{aligned} \tag{3.18}$$

These six constraints can be expressed more compactly as a single set of constraints on the matrix R,

$$R^{\mathrm{T}}R = I, \tag{3.19}$$

where R^{T} denotes the transpose of R and I denotes the identity matrix.

There is still the matter of accounting for the fact that the frame is right-handed (i.e., $\hat{x}_b \times \hat{y}_b = \hat{z}_b$, where $\times$ denotes the cross product) rather than left-handed (i.e., $\hat{x}_b \times \hat{y}_b = -\hat{z}_b$); our six equality constraints above do not distinguish between right- and left-handed frames. We recall the following formula for evaluating the determinant of a 3×3 matrix M: denoting the three columns of M by a, b, and c, respectively, its determinant is given by

$$\det M = a^{\mathrm{T}}(b \times c) = c^{\mathrm{T}}(a \times b) = b^{\mathrm{T}}(c \times a). \tag{3.20}$$

Substituting the columns for R into this formula then leads to the constraint

$$\det R = 1. \tag{3.21}$$

Note that, had the frame been left-handed, we would have $\det R = -1$. In summary, the six equality constraints represented by Equation (3.19) imply that $\det R = \pm 1$; imposing the additional constraint $\det R = 1$ means that only right-handed frames are allowed. The constraint $\det R = 1$ does not change the number of independent continuous variables needed to parametrize R.

The set of 3×3 rotation matrices forms the **special orthogonal group** $SO(3)$, which we now formally define.

Definition 3.1. The **special orthogonal group** $SO(3)$, also known as the group of rotation matrices, is the set of all 3×3 real matrices R that satisfy (i) $R^{\mathrm{T}} R = I$ and (ii) $\det R = 1$.

The set of 2×2 rotation matrices is a subgroup of $SO(3)$ and is denoted $SO(2)$.

Definition 3.2. The **special orthogonal group** $SO(2)$ is the set of all 2×2 real matrices R that satisfy (i) $R^{\mathrm{T}} R = I$ and (ii) $\det R = 1$.

From the definition it follows that every $R \in SO(2)$ can be written

$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

where $\theta \in [0, 2\pi)$. The elements of $SO(2)$ represent planar orientations and the elements of $SO(3)$ represent spatial orientations.

3.2.1.1 Properties of Rotation Matrices

The sets of rotation matrices $SO(2)$ and $SO(3)$ are called groups because they satisfy the properties required of a mathematical group.[2] Specifically, a group consists of a set of elements and an operation on two elements (matrix multiplication for $SO(n)$) such that, for all A, B in the group, the following properties are satisfied:

- **closure:** AB is also in the group.
- **associativity:** $(AB)C = A(BC)$.
- **identity element existence:** There exists an element I in the group (the identity matrix for $SO(n)$) such that $AI = IA = A$.
- **inverse element existence:** There exists an element A^{-1} in the group such that $AA^{-1} = A^{-1}A = I$.

Proofs of these properties are given below, using the fact that the identity matrix I is a trivial example of a rotation matrix.

Proposition 3.3. *The inverse of a rotation matrix $R \in SO(3)$ is also a rotation matrix, and it is equal to the transpose of R, i.e., $R^{-1} = R^{\mathrm{T}}$.*

[2] More specifically, the $SO(n)$ groups are also called *matrix Lie groups* (where "Lie" is pronounced "Lee") because the elements of the group form a differentiable manifold.

Proof. The condition $R^{\mathrm{T}}R = I$ implies that $R^{-1} = R^{\mathrm{T}}$ and $RR^{\mathrm{T}} = I$. Since $\det R^{\mathrm{T}} = \det R = 1$, R^{T} is also a rotation matrix. □

Proposition 3.4. *The product of two rotation matrices is a rotation matrix.*

Proof. Given $R_1, R_2 \in SO(3)$, their product R_1R_2 satisfies $(R_1R_2)^{\mathrm{T}}(R_1R_2) = R_2^{\mathrm{T}}R_1^{\mathrm{T}}R_1R_2 = R_2^{\mathrm{T}}R_2 = I$. Further, $\det R_1R_2 = \det R_1 \cdot \det R_2 = 1$. Thus R_1R_2 satisfies the conditions for a rotation matrix. □

Proposition 3.5. *Multiplication of rotation matrices is associative,* $(R_1R_2)R_3 = R_1(R_2R_3)$*, but generally not commutative,* $R_1R_2 \neq R_2R_1$*. For the special case of rotation matrices in* $SO(2)$*, rotations commute.*

Proof. Associativity and nocommutativity follows from the properties of matrix multiplication in linear algebra. Commutativity for planar rotations follows from a direct calculation. □

Another important property is that the action of a rotation matrix on a vector (e.g., rotating the vector) does not change the length of the vector.

Proposition 3.6. *For any vector* $x \in \mathbb{R}^3$ *and* $R \in SO(3)$*, the vector* $y = Rx$ *has the same length as* x*.*

Proof. This follows from $\|y\|^2 = y^{\mathrm{T}}y = (Rx)^{\mathrm{T}}Rx = x^{\mathrm{T}}R^{\mathrm{T}}Rx = x^{\mathrm{T}}x = \|x\|^2$. □

3.2.1.2 Uses of Rotation Matrices

Analogously to the discussion after Equations 3.10 and (3.11) in Section 3.1, there are three major uses for a rotation matrix R:

(a) to represent an orientation;

(b) to change the reference frame in which a vector or a frame is represented;

(c) to rotate a vector or a frame.

In the first use, R is thought of as representing a frame; in the second and third uses, R is thought of as an operator that acts on a vector or frame (changing its reference frame or rotating it).

To illustrate these uses, refer to Figure 3.7, which shows three different coordinate frames – {a}, {b}, and {c} – representing the same space. These frames are chosen to have the same origin, since we are only representing orientations, but, to make the axes clear, the figure shows the same space drawn three times.

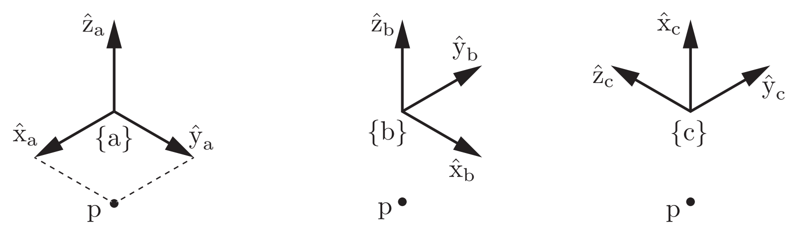


Figure 3.7: The same space and the same point p represented in three different frames with different orientations.

A point p in the space is also shown. Not shown is a fixed space frame {s}, which is aligned with {a}. The orientations of the three frames relative to {s} can be written

$$R_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_b = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_c = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix},$$

and the location of the point p in these frames can be written

$$p_a = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad p_b = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad p_c = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}.$$

Note that {b} is obtained by rotating {a} about $\hat{\mathrm{z}}_\mathrm{a}$ by 90°, and {c} is obtained by rotating {b} about $\hat{\mathrm{y}}_\mathrm{b}$ by −90°.

Representing an orientation When we write R_c, we are implicitly referring to the orientation of frame {c} relative to the fixed frame {s}. We can be more explicit about this by writing it as R_{sc}: we are representing the frame {c} of the second subscript relative to the frame {s} of the first subscript. This notation allows us to express one frame relative to another that is not {s}; for example, R_{bc} is the orientation of {c} relative to {b}.

If there is no possibility of confusion regarding the frames involved, we may simply write R.

Inspecting Figure 3.7, we see that

$$R_{ac} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad R_{ca} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

A simple calculation shows that $R_{ac}R_{ca} = I$; that is, $R_{ac} = R_{ca}^{-1}$ or, equivalently, from Proposition 3.3, $R_{ac} = R_{ca}^{\mathrm{T}}$. In fact, for any two frames {d} and {e},

$$R_{de} = R_{ed}^{-1} = R_{ed}^{\mathrm{T}}.$$

You can verify this fact using any two frames in Figure 3.7.

Changing the reference frame The rotation matrix R_{ab} represents the orientation of {b} in {a}, and R_{bc} represents the orientation of {c} in {b}. A straightforward calculation shows that the orientation of {c} in {a} can be computed as

$$R_{ac} = R_{ab}R_{bc}. \tag{3.22}$$

In the previous equation, R_{bc} can be viewed as a representation of the orientation of {c}, while R_{ab} can be viewed as a mathematical operator that changes the reference frame from {b} to {a}, i.e.,

$$R_{ac} = R_{ab}R_{bc} = \text{change\_reference\_frame\_from\_\{b\}\_to\_\{a\}}\,(R_{bc}).$$

A subscript cancellation rule helps us to remember this property. When multiplying two rotation matrices, if the second subscript of the first matrix matches the first subscript of the second matrix, the two subscripts "cancel" and a change of reference frame is achieved:

$$R_{ab}R_{bc} = R_{a\cancel{b}}R_{\cancel{b}c} = R_{ac}.$$

A rotation matrix is just a collection of three unit vectors, so the reference frame of a vector can also be changed by a rotation matrix using a modified version of the subscript cancellation rule:

$$R_{ab}p_b = R_{a\cancel{b}}p_{\cancel{b}} = p_a.$$

You can verify these properties using the frames and points in Figure 3.7.

Rotating a vector or a frame The final use of a rotation matrix is to rotate a vector or a frame. Figure 3.8 shows a frame {c} initially aligned with {s} with axes $\{\hat{\mathrm{x}}, \hat{\mathrm{y}}, \hat{\mathrm{z}}\}$. If we rotate the frame {c} about a unit axis $\hat{\omega}$ by an amount θ, the new frame, {c′} (light gray), has coordinate axes $\{\hat{\mathrm{x}}', \hat{\mathrm{y}}', \hat{\mathrm{z}}'\}$. The rotation matrix $R = R_{sc'}$ represents the orientation of {c′} relative to {s}, but instead we can think of it as representing the rotation operation that takes {s} to {c′}. Emphasizing our view of R as a rotation operator, instead of as an orientation, we can write

$$R = \mathrm{Rot}(\hat{\omega}, \theta),$$

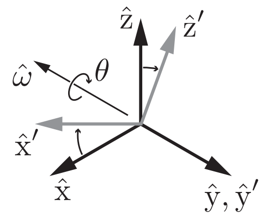


Figure 3.8: A coordinate frame with axes $\{\hat{x}, \hat{y}, \hat{z}\}$ is rotated by θ about a unit axis $\hat{\omega}$ (which is aligned with $-\hat{y}$ in this figure). The orientation of the final frame, with axes $\{\hat{x}', \hat{y}', \hat{z}'\}$, is written as R relative to the original frame.

meaning the operation that rotates the orientation represented by the identity matrix to the orientation represented by R. Examples of rotation operations about coordinate frame axes are

$$\mathrm{Rot}(\hat{x}, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad \mathrm{Rot}(\hat{y}, \theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix},$$

$$\mathrm{Rot}(\hat{z}, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

More generally, as we will see in Section 3.2.3.3, for $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$,

$$\mathrm{Rot}(\hat{\omega}, \theta) =$$
$$\begin{bmatrix} c_\theta + \hat{\omega}_1^2(1 - c_\theta) & \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1 - c_\theta) & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1 - c_\theta) \end{bmatrix},$$

where $s_\theta = \sin\theta$ and $c_\theta = \cos\theta$. Any $R \in SO(3)$ can be obtained by rotating from the identity matrix by some θ about some $\hat{\omega}$. Note also that $\mathrm{Rot}(\hat{\omega}, \theta) = \mathrm{Rot}(-\hat{\omega}, -\theta)$.

Now, say that R_{sb} represents some $\{b\}$ relative to $\{s\}$ and that we want to rotate $\{b\}$ by θ about a unit axis $\hat{\omega}$, i.e., by a rotation $R = \mathrm{Rot}(\hat{\omega}, \theta)$. To be clear about what we mean, we have to specify whether the axis of rotation $\hat{\omega}$ is expressed in $\{s\}$ coordinates or $\{b\}$ coordinates. Depending on our choice, the same numerical $\hat{\omega}$ (and therefore the same numerical R) corresponds to different rotation axes in the underlying space, unless the $\{s\}$ and $\{b\}$ frames are aligned. Letting $\{b'\}$ be the new frame after a rotation by θ about $\hat{\omega}_s = \hat{\omega}$ (the rotation axis $\hat{\omega}$ is considered to be in the fixed frame, $\{s\}$), and letting $\{b''\}$ be the new

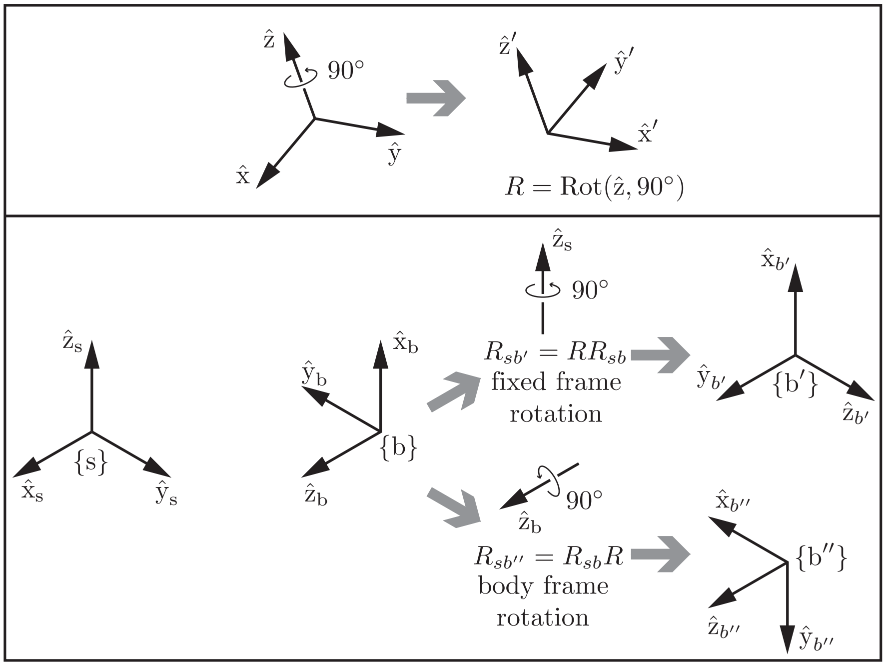


Figure 3.9: (Top) The rotation operator $R = \mathrm{Rot}(\hat{z}, 90^\circ)$ gives the orientation of the right-hand frame in the left-hand frame. (Bottom) On the left are shown a fixed frame {s} and a body frame {b}, which can be expressed as R_{sb}. The quantity RR_{sb} rotates {b} by 90° about the fixed-frame axis $\hat{\mathrm{z}}_\mathrm{s}$ to {b′}. The quantity $R_{sb}R$ rotates {b} by 90° about the body-frame axis $\hat{\mathrm{z}}_\mathrm{b}$ to {b″}.

frame after a rotation by θ about $\hat{\omega}_b = \hat{\omega}$ (the rotation axis $\hat{\omega}$ is considered to be in the body frame {b}), representations of these new frames can be calculated as

$$R_{sb'} = \text{rotate\_by\_}R\text{\_in\_\{s\}\_frame}\ (R_{sb}) = RR_{sb} \tag{3.23}$$

$$R_{sb''} = \text{rotate\_by\_}R\text{\_in\_\{b\}\_frame}\ (R_{sb}) = R_{sb}R. \tag{3.24}$$

In other words, premultiplying by $R = \mathrm{Rot}(\hat{\omega}, \theta)$ yields a rotation about an axis $\hat{\omega}$ considered to be in the fixed frame, and postmultiplying by R yields a rotation about $\hat{\omega}$ considered as being in the body frame.

Rotation by R in the {s} frame and in the {b} frame is illustrated in Figure 3.9.

To rotate a vector v, note that there is only one frame involved, the frame

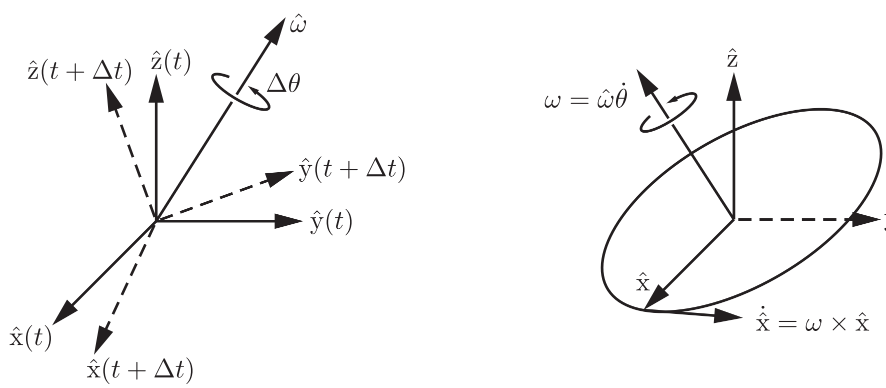


Figure 3.10: (Left) The instantaneous angular velocity vector. (Right) Calculating $\dot{\hat{x}}$.

in which v is represented, and therefore $\hat{\omega}$ must be interpreted as being in this frame. The rotated vector v', in that same frame, is

$$v' = Rv.$$

3.2.2 Angular Velocities

Referring to Figure 3.10(a), suppose that a frame with unit axes $\{\hat{x}, \hat{y}, \hat{z}\}$ is attached to a rotating body. Let us determine the time derivatives of these unit axes. Beginning with $\dot{\hat{x}}$, first note that $\hat{x}$ is of unit length; only the direction of $\hat{x}$ can vary with time (the same goes for $\hat{y}$ and $\hat{z}$). If we examine the body frame at times t and $t+\Delta t$, the change in frame orientation can be described as a rotation of angle $\Delta\theta$ about some unit axis $\hat{w}$ passing through the origin. The axis $\hat{w}$ is coordinate-free; it is not yet represented in any particular reference frame.

In the limit as Δt approaches zero, the ratio $\Delta\theta/\Delta t$ becomes the rate of rotation $\dot{\theta}$, and $\hat{w}$ can similarly be regarded as the instantaneous axis of rotation. In fact, $\hat{w}$ and $\dot{\theta}$ can be combined to define the **angular velocity** w as follows:

$$\mathrm{w} = \hat{\mathrm{w}}\dot{\theta}. \tag{3.25}$$

Referring to Figure 3.10(b), it should be evident that

$$\dot{\hat{x}} = \mathrm{w} \times \hat{x}, \tag{3.26}$$

$$\dot{\hat{y}} = \mathrm{w} \times \hat{y}, \tag{3.27}$$

$$\dot{\hat{z}} = \mathrm{w} \times \hat{z}. \tag{3.28}$$

To express these equations in coordinates, we have to choose a reference frame in which to represent w. We can choose any reference frame, but two natural choices are the fixed frame {s} and the body frame {b}. Let us start with fixed-frame {s} coordinates. Let $R(t)$ be the rotation matrix describing the orientation of the body frame with respect to the fixed frame at time t; $\dot{R}(t)$ is its time rate of change. The first column of $R(t)$, denoted $r_1(t)$, describes x̂ in fixed-frame coordinates; similarly, $r_2(t)$ and $r_3(t)$ respectively describe ŷ and ẑ in fixed-frame coordinates. At a specific time t, let $\omega_s \in \mathbb{R}^3$ be the angular velocity w expressed in fixed-frame coordinates. Then Equations (3.26)–(3.28) can be expressed in fixed-frame coordinates as

$$\dot{r}_i = \omega_s \times r_i, \qquad i = 1, 2, 3.$$

These three equations can be rearranged into the following single 3×3 matrix equation:

$$\dot{R} = [\omega_s \times r_1 \;\; \omega_s \times r_2 \;\; \omega_s \times r_3] = \omega_s \times R. \tag{3.29}$$

To eliminate the cross product on the right in Equation (3.29), we introduce some new notation, rewriting $\omega_s \times R$ as $[\omega_s]R$, where $[\omega_s]$ is a 3×3 **skew-symmetric** matrix representation of $\omega_s \in \mathbb{R}^3$:

Definition 3.7. Given a vector $x = [x_1 \; x_2 \; x_3]^{\mathrm{T}} \in \mathbb{R}^3$, define

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}. \tag{3.30}$$

The matrix $[x]$ is a 3×3 **skew-symmetric** matrix representation of x; that is,

$$[x] = -[x]^{\mathrm{T}}.$$

The set of all 3×3 real skew-symmetric matrices is called $so(3)$.[3]

A useful property involving rotations and skew-symmetric matrices is the following.

Proposition 3.8. *Given any $\omega \in \mathbb{R}^3$ and $R \in SO(3)$, the following always holds:*

$$R[\omega]R^{\mathrm{T}} = [R\omega]. \tag{3.31}$$

[3]The set of skew-symmetric matrices $so(3)$ is called the *Lie algebra* of the Lie group $SO(3)$. It consists of all possible $\dot{R}$ when $R = I$.

Proof. Letting r_i^{T} be the ith row of R, we have

$$\begin{aligned}
R[\omega]R^{\mathrm{T}} &= \begin{bmatrix} r_1^{\mathrm{T}}(\omega \times r_1) & r_1^{\mathrm{T}}(\omega \times r_2) & r_1^{\mathrm{T}}(\omega \times r_3) \\ r_2^{\mathrm{T}}(\omega \times r_1) & r_2^{\mathrm{T}}(\omega \times r_2) & r_2^{\mathrm{T}}(\omega \times r_3) \\ r_3^{\mathrm{T}}(\omega \times r_1) & r_3^{\mathrm{T}}(\omega \times r_2) & r_3^{\mathrm{T}}(\omega \times r_3) \end{bmatrix} \\
&= \begin{bmatrix} 0 & -r_3^{\mathrm{T}}\omega & r_2^{\mathrm{T}}\omega \\ r_3^{\mathrm{T}}\omega & 0 & -r_1^{\mathrm{T}}\omega \\ -r_2^{\mathrm{T}}\omega & r_1^{\mathrm{T}}\omega & 0 \end{bmatrix} \\
&= [R\omega], \qquad (3.32)
\end{aligned}$$

where the second line makes use of the determinant formula for 3×3 matrices, i.e., if M is a 3×3 matrix with columns $\{a, b, c\}$, then $\det M = a^{\mathrm{T}}(b \times c) = c^{\mathrm{T}}(a \times b) = b^{\mathrm{T}}(c \times a)$. □

With the skew-symmetric notation, we can rewrite Equation (3.29) as

$$[\omega_s]R = \dot{R}. \qquad (3.33)$$

We can post-multiply both sides of Equation (3.33) by R^{-1} to get

$$[\omega_s] = \dot{R}R^{-1}. \qquad (3.34)$$

Now let ω_b be w expressed in body-frame coordinates. To see how to obtain ω_b from ω_s and vice versa, we write R explicitly as R_{sb}. Then ω_s and ω_b are two different vector representations of the same angular velocity w and, by our subscript cancellation rule, $\omega_s = R_{sb}\omega_b$. Therefore

$$\omega_b = R_{sb}^{-1}\omega_s = R^{-1}\omega_s = R^{\mathrm{T}}\omega_s. \qquad (3.35)$$

Let us now express this relation in skew-symmetric matrix form:

$$\begin{aligned}
[\omega_b] &= [R^{\mathrm{T}}\omega_s] \\
&= R^{\mathrm{T}}[\omega_s]R \quad \text{(by Proposition 3.8)} \\
&= R^{\mathrm{T}}(\dot{R}R^{\mathrm{T}})R \\
&= R^{\mathrm{T}}\dot{R} = R^{-1}\dot{R}. \qquad (3.36)
\end{aligned}$$

In summary, two equations relate R and $\dot{R}$ to the angular velocity w:

Proposition 3.9. *Let $R(t)$ denote the orientation of the rotating frame as seen from the fixed frame. Denote by* w *the angular velocity of the rotating frame. Then*

$$\dot{R}R^{-1} = [\omega_s], \tag{3.37}$$

$$R^{-1}\dot{R} = [\omega_b], \tag{3.38}$$

where $\omega_s \in \mathbb{R}^3$ is the fixed-frame vector representation of w *and $[\omega_s] \in so(3)$ is its 3×3 matrix representation, and where $\omega_b \in \mathbb{R}^3$ is the body-frame vector representation of* w *and $[\omega_b] \in so(3)$ is its 3×3 matrix representation.*

It is important to note that ω_b is *not* the angular velocity relative to a moving frame. Rather, ω_b is the angular velocity relative to the *stationary* frame {b} that is instantaneously coincident with a frame attached to the moving body.

It is also important to note that the fixed-frame angular velocity ω_s *does not depend on the choice of body frame*. Similarly, the body-frame angular velocity ω_b *does not depend on the choice of fixed frame*. While Equations (3.37) and (3.38) may appear to depend on both frames (since R and $\dot{R}$ individually depend on both {s} and {b}), the product $\dot{R}R^{-1}$ is independent of {b} and the product $R^{-1}\dot{R}$ is independent of {s}.

Finally, an angular velocity expressed in an arbitrary frame {d} can be represented in another frame {c} if we know the rotation that takes {c} to {d}, using our now-familiar subscript cancellation rule:

$$\omega_c = R_{cd}\omega_d.$$

3.2.3 Exponential Coordinate Representation of Rotation

We now introduce a three-parameter representation for rotations, the **exponential coordinates for rotation**. The exponential coordinates parametrize a rotation matrix in terms of a rotation axis (represented by a unit vector $\hat{\omega}$) and an angle of rotation θ about that axis; the vector $\hat{\omega}\theta \in \mathbb{R}^3$ then serves as the three-parameter exponential coordinate representation of the rotation. Writing $\hat{\omega}$ and θ individually is the **axis-angle** representation of a rotation.

The exponential coordinate representation $\hat{\omega}\theta$ for a rotation matrix R can be interpreted equivalently as:

- the axis $\hat{\omega}$ and rotation angle θ such that, if a frame initially coincident with {s} were rotated by θ about $\hat{\omega}$, its final orientation relative to {s} would be expressed by R; or

- the angular velocity $\hat{\omega}\theta$ expressed in {s} such that, if a frame initially coincident with {s} followed $\hat{\omega}\theta$ for one unit of time (i.e., $\hat{\omega}\theta$ is integrated over this time interval), its final orientation would be expressed by R; or
- the angular velocity $\hat{\omega}$ expressed in {s} such that, if a frame initially coincident with {s} followed $\hat{\omega}$ for θ units of time (i.e., $\hat{\omega}$ is integrated over this time interval) its final orientation would be expressed by R.

The latter two views suggest that we consider exponential coordinates in the setting of linear differential equations. Below we briefly review some key results from linear differential equations theory.

3.2.3.1 Essential Results from Linear Differential Equations Theory

Let us begin with the simple scalar linear differential equation

$$\dot{x}(t) = ax(t), \tag{3.39}$$

where $x(t) \in \mathbb{R}$, $a \in \mathbb{R}$ is constant, and the initial condition $x(0) = x_0$ is given. Equation (3.39) has solution

$$x(t) = e^{at}x_0.$$

It is also useful to remember the series expansion of the exponential function:

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots .$$

Now consider the vector linear differential equation

$$\dot{x}(t) = Ax(t), \tag{3.40}$$

where $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ is constant, and the initial condition $x(0) = x_0$ is given. From the above scalar result one can conjecture a solution of the form

$$x(t) = e^{At}x_0 \tag{3.41}$$

where the **matrix exponential** e^{At} now needs to be defined in a meaningful way. Again mimicking the scalar case, we define the matrix exponential to be

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots \tag{3.42}$$

The first question to be addressed is under what conditions this series converges, so that the matrix exponential is well defined. It can be shown that if A is constant and finite then this series is always guaranteed to converge to a finite limit;

the proof can be found in most texts on ordinary linear differential equations and is not covered here.

The second question is whether Equation (3.41), using Equation (3.42), is indeed a solution to Equation (3.40). Taking the time derivative of $x(t) = e^{At}x_0$,

$$\begin{aligned} \dot{x}(t) &= \left(\frac{d}{dt}e^{At}\right)x_0 \\ &= \frac{d}{dt}\left(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots\right)x_0 \\ &= \left(A + A^2t + \frac{A^3t^2}{2!} + \cdots\right)x_0 \\ &= Ae^{At}x_0 \\ &= Ax(t), \end{aligned} \tag{3.43}$$

which proves that $x(t) = e^{At}x_0$ is indeed a solution. That this is a unique solution follows from the basic existence and uniqueness result for linear ordinary differential equations, which we invoke here without proof.

While $AB \neq BA$ for arbitrary square matrices A and B, it is always true that

$$Ae^{At} = e^{At}A \tag{3.44}$$

for any square A and scalar t. You can verify this directly using the series expansion for the matrix exponential. Therefore, in line four of Equation (3.43), A could also have been factored to the right, i.e.,

$$\dot{x}(t) = e^{At}Ax_0.$$

While the matrix exponential e^{At} is defined as an infinite series, closed-form expressions are often available. For example, if A can be expressed as $A = PDP^{-1}$ for some $D \in \mathbb{R}^{n\times n}$ and invertible $P \in \mathbb{R}^{n\times n}$ then

$$\begin{aligned} e^{At} &= I + At + \frac{(At)^2}{2!} + \cdots \\ &= I + (PDP^{-1})t + (PDP^{-1})(PDP^{-1})\frac{t^2}{2!} + \cdots \\ &= P(I + Dt + \frac{(Dt)^2}{2!} + \cdots)P^{-1} \\ &= Pe^{Dt}P^{-1}. \end{aligned} \tag{3.45}$$

If moreover D is diagonal, i.e., $D = \mathrm{diag}\{d_1, d_2, \ldots, d_n\}$, then its matrix exponential is particularly simple to evaluate:

$$e^{Dt} = \begin{bmatrix} e^{d_1 t} & 0 & \cdots & 0 \\ 0 & e^{d_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{d_n t} \end{bmatrix}. \tag{3.46}$$

We summarize the results above in the following proposition.

Proposition 3.10. *The linear differential equation $\dot{x}(t) = Ax(t)$ with initial condition $x(0) = x_0$, where $A \in \mathbb{R}^{n\times n}$ is constant and $x(t) \in \mathbb{R}^n$, has solution*

$$x(t) = e^{At}x_0 \tag{3.47}$$

where

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots. \tag{3.48}$$

The matrix exponential e^{At} further satisifies the following properties:

(a) $d(e^{At})/dt = Ae^{At} = e^{At}A$.

(b) *If $A = PDP^{-1}$ for some $D \in \mathbb{R}^{n\times n}$ and invertible $P \in \mathbb{R}^{n\times n}$ then $e^{At} = Pe^{Dt}P^{-1}$.*

(c) *If $AB = BA$ then $e^A e^B = e^{A+B}$.*

(d) $(e^A)^{-1} = e^{-A}$.

The third property can be established by expanding the exponentials and comparing terms. The fourth property follows by setting $B = -A$ in the third property.

3.2.3.2 Exponential Coordinates of Rotations

The exponential coordinates of a rotation can be viewed equivalently as (1) a unit axis of rotation $\hat{\omega}$ ($\hat{\omega} \in \mathbb{R}^3, \|\hat{\omega}\| = 1$) together with a rotation angle about the axis $\theta \in \mathbb{R}$, or (2) as the 3-vector obtained by multiplying the two together, $\hat{\omega}\theta \in \mathbb{R}^3$. When we represent the motion of a robot joint in the next chapter, the first view has the advantage of separating the description of the joint axis from the motion θ about the axis.

Referring to Figure 3.11, suppose that a three-dimensional vector $p(0)$ is rotated by θ about $\hat{\omega}$ to $p(\theta)$; here we assume that all quantities are expressed

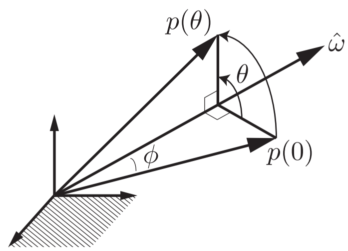


Figure 3.11: The vector $p(0)$ is rotated by an angle θ about the axis $\hat{\omega}$, to $p(\theta)$.

in fixed-frame coordinates. This rotation can be achieved by imagining that $p(0)$ rotates at a constant rate of 1 rad/s (since $\hat{\omega}$ has unit magnitude) from time $t = 0$ to $t = \theta$. Let $p(t)$ denote the path traced by the tip of the vector. The velocity of $p(t)$, denoted $\dot{p}$, is then given by

$$\dot{p} = \hat{\omega} \times p. \tag{3.49}$$

To see why this is true, let ϕ be the constant angle between $p(t)$ and $\hat{\omega}$. Observe that p traces a circle of radius $\|p\| \sin\phi$ about the $\hat{\omega}$-axis. Then $\dot{p}$ is tangent to the path with magnitude $\|p\| \sin\phi$, which is equivalent to Equation (3.49).

The differential equation (3.49) can be expressed as (see Equation (3.30))

$$\dot{p} = [\hat{\omega}]p \tag{3.50}$$

with initial condition $p(0)$. This is a linear differential equation of the form $\dot{x} = Ax$, which we studied earlier; its solution is given by

$$p(t) = e^{[\hat{\omega}]t} p(0).$$

Since t and θ are interchangeable, the equation above can also be written

$$p(\theta) = e^{[\hat{\omega}]\theta} p(0).$$

Let us now expand the matrix exponential $e^{[\hat{\omega}]\theta}$ in series form. A straightforward calculation shows that $[\hat{\omega}]^3 = -[\hat{\omega}]$, and therefore we can replace $[\hat{\omega}]^3$ by $-[\hat{\omega}]$, $[\hat{\omega}]^4$ by $-[\hat{\omega}]^2$, $[\hat{\omega}]^5$ by $-[\hat{\omega}]^3 = [\hat{\omega}]$, and so on, obtaining

$$\begin{aligned} e^{[\hat{\omega}]\theta} &= I + [\hat{\omega}]\theta + [\hat{\omega}]^2 \frac{\theta^2}{2!} + [\hat{\omega}]^3 \frac{\theta^3}{3!} + \cdots \\ &= I + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)[\hat{\omega}] + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)[\hat{\omega}]^2. \end{aligned}$$

Now recall the series expansions for $\sin\theta$ and $\cos\theta$:

$$\begin{aligned}\sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots \\ \cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\end{aligned}$$

The exponential $e^{[\hat{\omega}]\theta}$ therefore simplifies to the following:

Proposition 3.11. *Given a vector $\hat{\omega}\theta \in \mathbb{R}^3$, such that θ is any scalar and $\hat{\omega} \in \mathbb{R}^3$ is a unit vector, the matrix exponential of $[\hat{\omega}]\theta = [\hat{\omega}\theta] \in so(3)$ is*

$$\mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I + \sin\theta\,[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2 \in SO(3). \tag{3.51}$$

Equation (3.51) is also known as **Rodrigues' formula** for rotations.

We have shown how to use the matrix exponential to construct a rotation matrix from a rotation axis $\hat{\omega}$ and an angle θ. Further, the quantity $e^{[\hat{\omega}]\theta}p$ has the effect of rotating $p \in \mathbb{R}^3$ about the fixed-frame axis $\hat{\omega}$ by an angle θ. Similarly, considering that a rotation matrix R consists of three column vectors, the rotation matrix $R' = e^{[\hat{\omega}]\theta}R = \mathrm{Rot}(\hat{\omega}, \theta)R$ is the orientation achieved by rotating R by θ about the axis $\hat{\omega}$ in the fixed frame. Reversing the order of matrix multiplication, $R'' = Re^{[\hat{\omega}]\theta} = R\,\mathrm{Rot}(\hat{\omega}, \theta)$ is the orientation achieved by rotating R by θ about $\hat{\omega}$ in the body frame.

Example 3.12. The frame {b} in Figure 3.12 is obtained by rotation from an initial orientation aligned with the fixed frame {s} about a unit axis $\hat{\omega}_1 = (0, 0.866, 0.5)$ by an angle $\theta_1 = 30° = 0.524$ rad. The rotation matrix representation of {b} can be calculated as

$$\begin{aligned} R &= e^{[\hat{\omega}_1]\theta_1} \\ &= I + \sin\theta_1[\hat{\omega}_1] + (1 - \cos\theta_1)[\hat{\omega}_1]^2 \\ &= I + 0.5\begin{bmatrix} 0 & -0.5 & 0.866 \\ 0.5 & 0 & 0 \\ -0.866 & 0 & 0 \end{bmatrix} + 0.134\begin{bmatrix} 0 & -0.5 & 0.866 \\ 0.5 & 0 & 0 \\ -0.866 & 0 & 0 \end{bmatrix}^2 \\ &= \begin{bmatrix} 0.866 & -0.250 & 0.433 \\ 0.250 & 0.967 & 0.058 \\ -0.433 & 0.058 & 0.899 \end{bmatrix}. \end{aligned}$$

The orientation of the frame {b} can be represented by R or by the unit axis $\hat{\omega}_1 = (0, 0.866, 0.5)$ and the angle $\theta_1 = 0.524$ rad, i.e., the exponential coordinates $\hat{\omega}_1\theta_1 = (0, 0.453, 0.262)$.

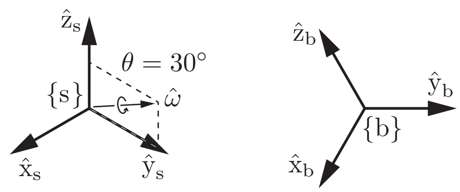


Figure 3.12: The frame {b} is obtained by a rotation from {s} by $\theta_1 = 30^\circ$ about $\hat{\omega}_1 = (0, 0.866, 0.5)$.

If {b} is then rotated by θ_2 about a fixed-frame axis $\hat{\omega}_2 \neq \hat{\omega}_1$, i.e.,

$$R' = e^{[\hat{\omega}_2]\theta_2} R,$$

then the frame ends up at a different location than that reached were {b} to be rotated by θ_2 about an axis expressed as $\hat{\omega}_2$ in the body frame, i.e.,

$$R'' = Re^{[\hat{\omega}_2]\theta_2} \neq R' = e^{[\hat{\omega}_2]\theta_2} R.$$

Our next task is to show that for any rotation matrix $R \in SO(3)$, one can always find a unit vector $\hat{\omega}$ and scalar θ such that $R = e^{[\hat{\omega}]\theta}$.

3.2.3.3 Matrix Logarithm of Rotations

If $\hat{\omega}\theta \in \mathbb{R}^3$ represents the exponential coordinates of a rotation matrix R, then the skew-symmetric matrix $[\hat{\omega}\theta] = [\hat{\omega}]\theta$ is the **matrix logarithm** of the rotation R.[4] The matrix logarithm is the inverse of the matrix exponential. Just as the matrix exponential "integrates" the matrix representation of an angular velocity $[\hat{\omega}]\theta \in so(3)$ for one second to give an orientation $R \in SO(3)$, the matrix logarithm "differentiates" an $R \in SO(3)$ to find the matrix representation of a constant angular velocity $[\hat{\omega}]\theta \in so(3)$ which, if integrated for one second, rotates a frame from I to R. In other words,

$$\begin{array}{rlcl} \exp: & [\hat{\omega}]\theta \in so(3) & \rightarrow & R \in SO(3), \\ \log: & R \in SO(3) & \rightarrow & [\hat{\omega}]\theta \in so(3). \end{array}$$

[4] We use the term "*the* matrix logarithm" to refer both to a specific matrix which is a logarithm of R as well as to the algorithm that calculates this specific matrix. Also, while a matrix R can have more than one matrix logarithm (just as $\sin^{-1}(0)$ has solutions $0, \pi, 2\pi$, etc.), we commonly refer to "the" matrix logarithm, i.e., the unique solution returned by the matrix logarithm algorithm.

To derive the matrix logarithm, let us expand each entry for $e^{[\hat{\omega}]\theta}$ in Equation (3.51),

$$\begin{bmatrix} c_\theta + \hat{\omega}_1^2(1-c_\theta) & \hat{\omega}_1\hat{\omega}_2(1-c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1-c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1-c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1-c_\theta) & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1-c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1-c_\theta) \end{bmatrix}, \tag{3.52}$$

where $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$, and we use again the shorthand notation $s_\theta = \sin\theta$ and $c_\theta = \cos\theta$. Setting the above matrix equal to the given $R \in SO(3)$ and subtracting the transpose from both sides leads to the following:

$$\begin{aligned} r_{32} - r_{23} &= 2\hat{\omega}_1 \sin\theta, \\ r_{13} - r_{31} &= 2\hat{\omega}_2 \sin\theta, \\ r_{21} - r_{12} &= 2\hat{\omega}_3 \sin\theta. \end{aligned}$$

Therefore, as long as $\sin\theta \neq 0$ (or, equivalently, θ is not an integer multiple of π), we can write

$$\begin{aligned} \hat{\omega}_1 &= \frac{1}{2\sin\theta}(r_{32} - r_{23}), \\ \hat{\omega}_2 &= \frac{1}{2\sin\theta}(r_{13} - r_{31}), \\ \hat{\omega}_3 &= \frac{1}{2\sin\theta}(r_{21} - r_{12}). \end{aligned}$$

The above equations can also be expressed in skew-symmetric matrix form as

$$[\hat{\omega}] = \begin{bmatrix} 0 & -\hat{\omega}_3 & \hat{\omega}_2 \\ \hat{\omega}_3 & 0 & -\hat{\omega}_1 \\ -\hat{\omega}_2 & \hat{\omega}_1 & 0 \end{bmatrix} = \frac{1}{2\sin\theta}\left(R - R^{\mathrm{T}}\right). \tag{3.53}$$

Recall that $\hat{\omega}$ represents the axis of rotation for the given R. Because of the $\sin\theta$ term in the denominator, $[\hat{\omega}]$ is not well defined if θ is an integer multiple of π.[5] We address this situation next, but for now let us assume that $\sin\theta \neq 0$ and find an expression for θ. Setting R equal to (3.52) and taking the trace of both sides (recall that the trace of a matrix is the sum of its diagonal entries),

$$\operatorname{tr} R = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta. \tag{3.54}$$

The above follows since $\hat{\omega}_1^2 + \hat{\omega}_2^2 + \hat{\omega}_3^2 = 1$. For any θ satisfying $1 + 2\cos\theta = \operatorname{tr} R$ such that θ is not an integer multiple of π, R can be expressed as the exponential $e^{[\hat{\omega}]\theta}$ with $[\hat{\omega}]$ as given in Equation (3.53).

[5]Singularities such as this are unavoidable for any three-parameter representation of rotation. Euler angles and roll–pitch–yaw angles suffer from similar singularities.

Let us now return to the case $\theta = k\pi$, where k is some integer. When k is an even integer, regardless of $\hat{\omega}$ we have rotated back to $R = I$ so the vector $\hat{\omega}$ is undefined. When k is an odd integer (corresponding to $\theta = \pm\pi, \pm 3\pi, \ldots$, which in turn implies $\operatorname{tr} R = -1$), the exponential formula (3.51) simplifies to

$$R = e^{[\hat{\omega}]\pi} = I + 2[\hat{\omega}]^2. \tag{3.55}$$

The three diagonal terms of Equation (3.55) can be manipulated to give

$$\hat{\omega}_i = \pm\sqrt{\frac{r_{ii} + 1}{2}}, \qquad i = 1, 2, 3. \tag{3.56}$$

The off-diagonal terms lead to the following three equations:

$$\begin{aligned} 2\hat{\omega}_1\hat{\omega}_2 &= r_{12}, \\ 2\hat{\omega}_2\hat{\omega}_3 &= r_{23}, \\ 2\hat{\omega}_1\hat{\omega}_3 &= r_{13}, \end{aligned} \tag{3.57}$$

From Equation (3.55) we also know that R must be symmetric: $r_{12} = r_{21}$, $r_{23} = r_{32}$, $r_{13} = r_{31}$. Equations (3.56) and (3.57) may both be necessary to obtain a solution for $\hat{\omega}$. Once such a solution has been found then $R = e^{[\hat{\omega}]\theta}$, where $\theta = \pm\pi, \pm 3\pi, \ldots$

From the above it can be seen that solutions for θ exist at 2π intervals. If we restrict θ to the interval $[0, \pi]$ then the following algorithm can be used to compute the matrix logarithm of the rotation matrix $R \in SO(3)$.

Algorithm: Given $R \in SO(3)$, find a $\theta \in [0, \pi]$ and a unit rotation axis $\hat{\omega} \in \mathbb{R}^3, \|\hat{\omega}\| = 1$, such that $e^{[\hat{\omega}]\theta} = R$. The vector $\hat{\omega}\theta \in \mathbb{R}^3$ comprises the exponential coordinates for R and the skew-symmetric matrix $[\hat{\omega}]\theta \in so(3)$ is the matrix logarithm of R.

(a) If $R = I$ then $\theta = 0$ and $\hat{\omega}$ is undefined.

(b) If $\operatorname{tr} R = -1$ then $\theta = \pi$. Set $\hat{\omega}$ equal to any of the following three vectors that is a feasible solution:

$$\hat{\omega} = \frac{1}{\sqrt{2(1 + r_{33})}} \begin{bmatrix} r_{13} \\ r_{23} \\ 1 + r_{33} \end{bmatrix} \tag{3.58}$$

or

$$\hat{\omega} = \frac{1}{\sqrt{2(1 + r_{22})}} \begin{bmatrix} r_{12} \\ 1 + r_{22} \\ r_{32} \end{bmatrix} \tag{3.59}$$

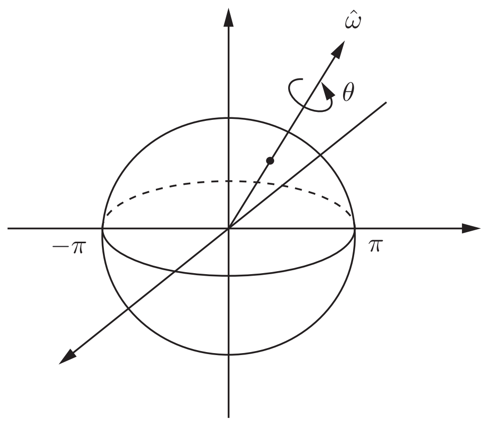


Figure 3.13: $SO(3)$ as a solid ball of radius π. The exponential coordinates $r = \hat{\omega}\theta$ may lie anywhere within the solid ball.

or

$$\hat{\omega} = \frac{1}{\sqrt{2(1 + r_{11})}} \begin{bmatrix} 1 + r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}. \tag{3.60}$$

(Note that if $\hat{\omega}$ is a solution, then so is $-\hat{\omega}$.)

(c) Otherwise $\theta = \cos^{-1}\left(\frac{1}{2}(\operatorname{tr} R - 1)\right) \in [0, \pi)$ and

$$[\hat{\omega}] = \frac{1}{2 \sin \theta}(R - R^{\mathrm{T}}). \tag{3.61}$$

Since every $R \in SO(3)$ satisfies one of the three cases in the algorithm, for every R there exists a matrix logarithm $[\hat{\omega}]\theta$ and therefore a set of exponential coordinates $\hat{\omega}\theta$.

Because the matrix logarithm calculates exponential coordinates $\hat{\omega}\theta$ satisfying $||\hat{\omega}\theta|| \leq \pi$, we can picture the rotation group $SO(3)$ as a solid ball of radius π (see Figure 3.13): given a point $r \in \mathbb{R}^3$ in this solid ball, let $\hat{\omega} = r/\|r\|$ be the unit axis in the direction from the origin to the point r and let $\theta = \|r\|$ be the distance from the origin to r, so that $r = \hat{\omega}\theta$. The rotation matrix corresponding to r can then be regarded as a rotation about the axis $\hat{\omega}$ by an angle θ. For any $R \in SO(3)$ such that $\operatorname{tr} R \neq -1$, there exists a unique r in the interior of the solid ball such that $e^{[r]} = R$. In the event that $\operatorname{tr} R = -1$, $\log R$ is given by two antipodal points on the surface of this solid ball. That is, if there exists some r such that $R = e^{[r]}$ with $\|r\| = \pi$ then $R = e^{[-r]}$ also holds; both r and $-r$ correspond to the same rotation R.

3.3 Rigid-Body Motions and Twists

In this section we derive representations for rigid-body configurations and velocities that extend, but otherwise are analogous to, those in Section 3.2 for rotations and angular velocities. In particular, the homogeneous transformation matrix T is analogous to the rotation matrix R; a screw axis $\mathcal{S}$ is analogous to a rotation axis $\hat{\omega}$; a twist $\mathcal{V}$ can be expressed as $\mathcal{S}\dot{\theta}$ and is analogous to an angular velocity $\omega = \hat{\omega}\dot{\theta}$; and exponential coordinates $\mathcal{S}\theta \in \mathbb{R}^6$ for rigid-body motions are analogous to exponential coordinates $\hat{\omega}\theta \in \mathbb{R}^3$ for rotations.

3.3.1 Homogeneous Transformation Matrices

We now consider representations for the combined orientation and position of a rigid body. A natural choice would be to use a rotation matrix $R \in SO(3)$ to represent the orientation of the body frame {b} in the fixed frame {s} and a vector $p \in \mathbb{R}^3$ to represent the origin of {b} in {s}. Rather than identifying R and p separately, we package them into a single matrix as follows.

Definition 3.13. The **special Euclidean group** $SE(3)$, also known as the group of **rigid-body motions** or **homogeneous transformation matrices** in $\mathbb{R}^3$, is the set of all 4×4 real matrices T of the form

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.62}$$

where $R \in SO(3)$ and $p \in \mathbb{R}^3$ is a column vector.

An element $T \in SE(3)$ will sometimes be denoted (R, p). In this section we will establish some basic properties of $SE(3)$ and explain why we package R and p into this matrix form.

Many robotic mechanisms we have encountered thus far are planar. With planar rigid-body motions in mind, we make the following definition:

Definition 3.14. The special Euclidean group $SE(2)$ is the set of all 3×3 real matrices T of the form

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}, \tag{3.63}$$

where $R \in SO(2)$, $p \in \mathbb{R}^2$, and 0 denotes a row vector of two zeros.

A matrix $T \in SE(2)$ is always of the form

$$T = \begin{bmatrix} r_{11} & r_{12} & p_1 \\ r_{21} & r_{22} & p_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & p_1 \\ \sin\theta & \cos\theta & p_2 \\ 0 & 0 & 1 \end{bmatrix},$$

where $\theta \in [0, 2\pi)$.

3.3.1.1 Properties of Transformation Matrices

We now list some basic properties of transformation matrices, which can be proven by calculation. First, the identity I is a trivial example of a transformation matrix. The first three properties confirm that $SE(3)$ is a group.

Proposition 3.15. *The inverse of a transformation matrix $T \in SE(3)$ is also a transformation matrix, and it has the following form:*

$$T^{-1} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix}. \tag{3.64}$$

Proposition 3.16. *The product of two transformation matrices is also a transformation matrix.*

Proposition 3.17. *The multiplication of transformation matrices is associative, so that $(T_1T_2)T_3 = T_1(T_2T_3)$, but generally not commutative: $T_1T_2 \neq T_2T_1$.*

Before stating the next proposition, we note that, just as in Section 3.1, it is often useful to calculate the quantity $Rx + p$, where $x \in \mathbb{R}^3$ and (R, p) represents T. If we append a '1' to x, making it a four-dimensional vector, this computation can be performed as a single matrix multiplication:

$$T \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Rx + p \\ 1 \end{bmatrix}. \tag{3.65}$$

The vector $[x^{\mathrm{T}}\ 1]^{\mathrm{T}}$ is the representation of x in **homogeneous coordinates**, and accordingly $T \in SE(3)$ is called a homogenous transformation. When, by an abuse of notation, we write Tx, we mean $Rx + p$.

Proposition 3.18. *Given $T = (R, p) \in SE(3)$ and $x, y \in \mathbb{R}^3$, the following hold:*

(a) $\|Tx - Ty\| = \|x - y\|$, *where* $\|\cdot\|$ *denotes the standard Euclidean norm in* $\mathbb{R}^3$, *i.e.,* $\|x\| = \sqrt{x^{\mathrm{T}}x}$.

(b) $\langle Tx - Tz, Ty - Tz\rangle = \langle x - z, y - z\rangle$ *for all* $z \in \mathbb{R}^3$, *where* $\langle\cdot,\cdot\rangle$ *denotes the standard Euclidean inner product in* $\mathbb{R}^3$, $\langle x, y\rangle = x^{\mathrm{T}}y$.

In Proposition 3.18, T is regarded as a transformation on points in $\mathbb{R}^3$; T transforms a point x to Tx. Property (a) then asserts that T preserves distances, while property (b) asserts that T preserves angles. Specifically, if $x, y, z \in \mathbb{R}^3$ represent the three vertices of a triangle then the triangle formed by the transformed vertices $\{Tx, Ty, Tz\}$ has the same set of lengths and angles as those of the triangle $\{x, y, z\}$ (the two triangles are said to be *isometric*). One can easily imagine taking $\{x, y, z\}$ to be the points on a rigid body, in which case $\{Tx, Ty, Tz\}$ represents a displaced version of the rigid body. It is in this sense that $SE(3)$ can be identified with rigid-body motions.

3.3.1.2 Uses of Transformation Matrices

As was the case for rotation matrices, there are three major uses for a transformation matrix T:

(a) to represent the configuration (position and orientation) of a rigid body;

(b) to change the reference frame in which a vector or frame is represented;

(c) to displace a vector or frame.

In the first use, T is thought of as representing the configuration of a frame; in the second and third uses, T is thought of as an operator that acts to change the reference frame or to move a vector or a frame.

To illustrate these uses, we refer to the three reference frames {a}, {b}, and {c}, and the point v, in Figure 3.14. The frames were chosen in such a way that the alignment of their axes is clear, allowing the visual confirmation of calculations.

Representing a configuration. The fixed frame {s} is coincident with {a} and the frames {a}, {b}, and {c}, represented by $T_{sa} = (R_{sa}, p_{sa})$, $T_{sb} =$

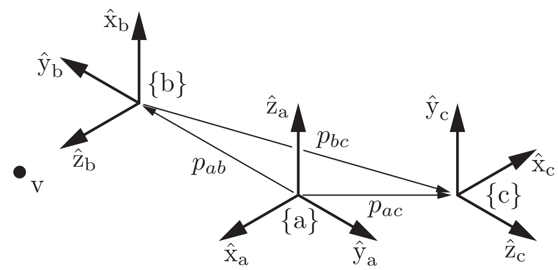


Figure 3.14: Three reference frames in space, and a point v that can be represented in {b} as $v_b = (0, 0, 1.5)$.

(R_{sb}, p_{sb}), and $T_{sc} = (R_{sc}, p_{sc})$, respectively, can be expressed relative to {s} by the rotations

$$R_{sa} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad R_{sb} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad R_{sc} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The location of the origin of each frame relative to {s} can be written

$$p_{sa} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad p_{sb} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \qquad p_{sc} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Since {a} is collocated with {s}, the transformation matrix T_{sa} constructed from (R_{sa}, p_{sa}) is the identity matrix.

Any frame can be expressed relative to any other frame, not just to {s}; for example, $T_{bc} = (R_{bc}, p_{bc})$ represents {b} relative to {c}:

$$R_{bc} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \qquad p_{bc} = \begin{bmatrix} 0 \\ -3 \\ -1 \end{bmatrix}.$$

It can also be shown, using Proposition 3.15, that

$$T_{de} = T_{ed}^{-1}$$

for any two frames {d} and {e}.

Changing the reference frame of a vector or a frame. By a subscript cancellation rule analogous to that for rotations, for any three reference frames {a}, {b}, and {c}, and any vector v expressed in {b} as v_b,

$$T_{ab}T_{bc} = T_{a\cancel{b}}T_{\cancel{b}c} = T_{ac}$$
$$T_{ab}v_b = T_{a\cancel{b}}v_{\cancel{b}} = v_a,$$

where v_a is the vector v expressed in {a}.

Displacing (rotating and translating) a vector or a frame. A transformation matrix T, viewed as the pair $(R,p) = (\text{Rot}(\hat{\omega},\theta),p)$, can act on a frame T_{sb} by rotating it by θ about an axis $\hat{\omega}$ and translating it by p. By a minor abuse of notation, we can extend the 3×3 rotation operator $R = \text{Rot}(\hat{\omega},\theta)$ to a 4×4 transformation matrix that rotates without translating,

$$\text{Rot}(\hat{\omega},\theta) = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix},$$

and we can similarly define a translation operator that translates without rotating,

$$\text{Trans}(p) = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(To parallel the rotation operator more directly, we could write $\text{Trans}(\hat{p},\|p\|)$, a translation along the unit direction $\hat{p}$ by a distance $\|p\|$, but we will use the simpler notation with $p = \hat{p}\|p\|$.)

Whether we pre-multiply or post-multiply T_{sb} by $T = (R,p)$ determines whether the $\hat{\omega}$-axis and p are interpreted as in the fixed frame {s} or in the body frame {b}:

$$\begin{aligned} T_{sb'} &= TT_{sb} = \text{Trans}(p)\,\text{Rot}(\hat{\omega},\theta)T_{sb} \qquad \text{(fixed frame)} \\ &= \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}\begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} RR_{sb} & Rp_{sb}+p \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{3.66}$$

$$\begin{aligned} T_{sb''} &= T_{sb}T = T_{sb}\,\text{Trans}(p)\,\text{Rot}(\hat{\omega},\theta) \qquad \text{(body frame)} \\ &= \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{sb}R & R_{sb}p+p_{sb} \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{3.67}$$

The fixed-frame transformation (corresponding to pre-multiplication by T) can be interpreted as first rotating the {b} frame by θ about an axis $\hat{\omega}$ in the {s}

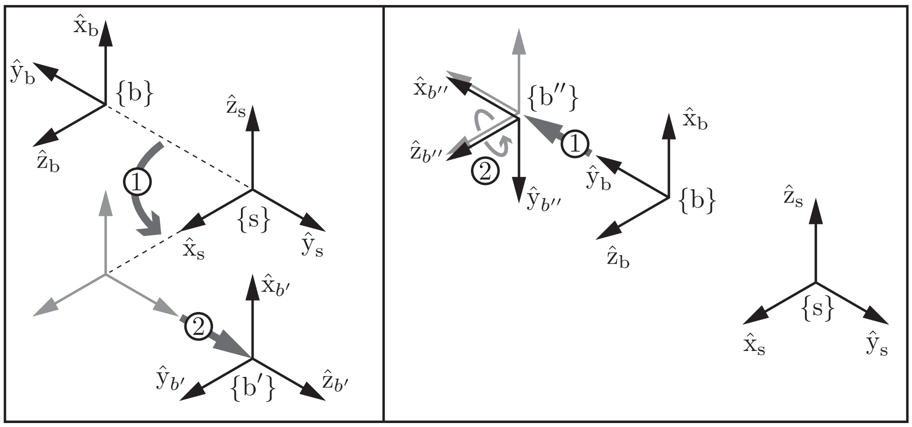


Figure 3.15: Fixed-frame and body-frame transformations corresponding to $\hat{\omega} = (0, 0, 1)$, $\theta = 90°$, and $p = (0, 2, 0)$. (Left) The frame {b} is rotated by 90° about $\hat{\mathrm{z}}_\mathrm{s}$ and then translated by two units in $\hat{\mathrm{y}}_\mathrm{s}$, resulting in the new frame {b′}. (Right) The frame {b} is translated by two units in $\hat{\mathrm{y}}_\mathrm{b}$ and then rotated by 90° about its $\hat{\mathrm{z}}$ axis, resulting in the new frame {b″}.

frame (this rotation will cause the origin of {b} to move if it is not coincident with the origin of {s}), then translating it by p in the {s} frame to get a frame {b′}. The body-frame transformation (corresponding to post-multiplication by T) can be interpreted as first translating {b} by p considered to be in the {b} frame, then rotating about $\hat{\omega}$ in this new body frame (this does not move the origin of the frame) to get {b″}.

Fixed-frame and body-frame transformations are illustrated in Figure 3.15 for a transformation T with $\hat{\omega} = (0, 0, 1)$, $\theta = 90°$, and $p = (0, 2, 0)$, yielding

$$T = (\mathrm{Rot}(\hat{\omega}, \theta), p) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Beginning with the frame {b} represented by

$$T_{sb} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

the new frame {b′} achieved by a fixed-frame transformation TT_{sb} and the new

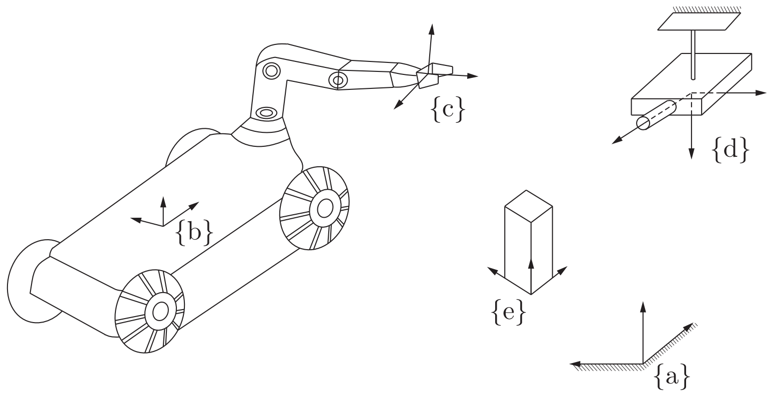


Figure 3.16: Assignment of reference frames.

frame {b″} achieved by a body-frame transformation $T_{sb}T$ are given by

$$TT_{sb} = T_{sb'} = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad T_{sb}T = T_{sb''} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -4 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Example 3.19. Figure 3.16 shows a robot arm mounted on a wheeled mobile platform moving in a room, and a camera fixed to the ceiling. Frames {b} and {c} are respectively attached to the wheeled platform and the end-effector of the robot arm, and frame {d} is attached to the camera. A fixed frame {a} has been established, and the robot must pick up an object with body frame {e}. Suppose that the transformations T_{db} and T_{de} can be calculated from measurements obtained with the camera. The transformation T_{bc} can be calculated using the arm's joint-angle measurements. The transformation T_{ad} is assumed to be known in advance. Suppose these calculated and known transformations are given as follows:

$$T_{db} = \begin{bmatrix} 0 & 0 & -1 & 250 \\ 0 & -1 & 0 & -150 \\ -1 & 0 & 0 & 200 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{aligned}
T_{de} &= \begin{bmatrix} 0 & 0 & -1 & 300 \\ 0 & -1 & 0 & 100 \\ -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
T_{ad} &= \begin{bmatrix} 0 & 0 & -1 & 400 \\ 0 & -1 & 0 & 50 \\ -1 & 0 & 0 & 300 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
T_{bc} &= \begin{bmatrix} 0 & -1/\sqrt{2} & -1/\sqrt{2} & 30 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & -40 \\ 1 & 0 & 0 & 25 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

In order to calculate how to move the robot arm so as to pick up the object, the configuration of the object relative to the robot hand, T_{ce}, must be determined. We know that

$$T_{ab}T_{bc}T_{ce} = T_{ad}T_{de},$$

where the only quantity besides T_{ce} not given to us directly is T_{ab}. However, since $T_{ab} = T_{ad}T_{db}$, we can determine T_{ce} as follows:

$$T_{ce} = (T_{ad}T_{db}T_{bc})^{-1}\, T_{ad}T_{de}.$$

From the given transformations we obtain

$$\begin{aligned}
T_{ad}T_{de} &= \begin{bmatrix} 1 & 0 & 0 & 280 \\ 0 & 1 & 0 & -50 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
T_{ad}T_{db}T_{bc} &= \begin{bmatrix} 0 & -1/\sqrt{2} & -1/\sqrt{2} & 230 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 160 \\ 1 & 0 & 0 & 75 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
(T_{ad}T_{db}T_{bc})^{-1} &= \begin{bmatrix} 0 & 0 & 1 & -75 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 70/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 & 390/\sqrt{2} \\ 0 & 0 & 0 & 1 \end{bmatrix},
\end{aligned}$$

from which T_{ce} is evaluated to be

$$T_{ce} = \begin{bmatrix} 0 & 0 & 1 & -75 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & -260/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 & 130/\sqrt{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3.3.2 Twists

We now consider both the linear and angular velocities of a moving frame. As before, {s} and {b} denote the fixed (space) and moving (body) frames, respectively. Let

$$T_{sb}(t) = T(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix} \tag{3.68}$$

denote the configuration of {b} as seen from {s}. To keep the notation uncluttered, for the time being we write T instead of the usual T_{sb}.

In Section 3.2.2 we discovered that pre- or post-multiplying $\dot{R}$ by R^{-1} results in a skew-symmetric representation of the angular velocity vector, either in fixed- or body-frame coordinates. One might reasonably ask whether a similar property carries over to $\dot{T}$, i.e., whether $T^{-1}\dot{T}$ and $\dot{T}T^{-1}$ carry similar physical interpretations.

Let us first see what happens when we pre-multiply $\dot{T}$ by T^{-1}:

$$\begin{aligned} T^{-1}\dot{T} &= \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} R^{\mathrm{T}}\dot{R} & R^{\mathrm{T}}\dot{p} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{3.69}$$

Recall that $R^{\mathrm{T}}\dot{R} = [\omega_b]$ is just the skew-symmetric matrix representation of the angular velocity expressed in {b} coordinates. Also, $\dot{p}$ is the linear velocity of the origin of {b} expressed in the fixed frame {s}, and $R^{\mathrm{T}}\dot{p} = v_b$ is this linear velocity expressed in the frame {b}. Putting these two observations together, we can conclude that $T^{-1}\dot{T}$ represents the linear and angular velocities of the moving frame relative to the stationary frame {b} currently aligned with the moving frame.

The above calculation of $T^{-1}\dot{T}$ suggests that it is reasonable to merge ω_b and v_b into a single six-dimensional velocity vector. We define the **spatial velocity**

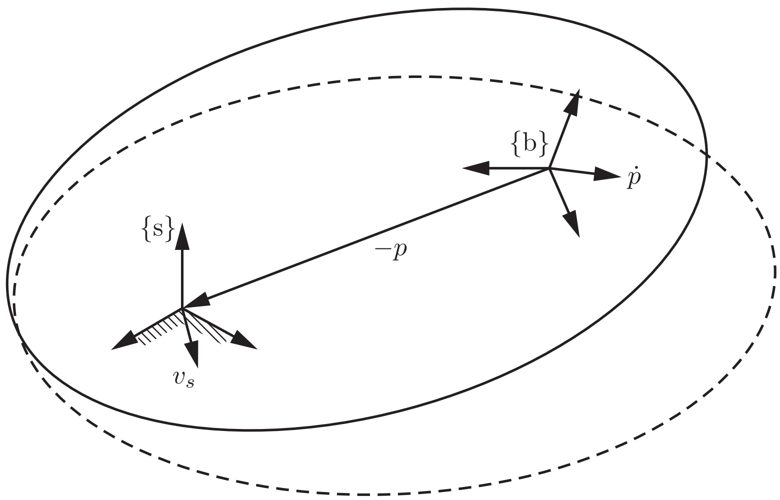


Figure 3.17: Physical interpretation of v_s. The initial (solid line) and displaced (dashed line) configurations of a rigid body.

in the body frame, or simply the **body twist**,[6] to be

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \in \mathbb{R}^6. \tag{3.70}$$

Just as it is convenient to have a skew-symmetric matrix representation of an angular velocity vector, it is convenient to have a matrix representation of a twist, as shown in Equation (3.69). We will stretch the $[\cdot]$ notation, writing

$$T^{-1}\dot{T} = [\mathcal{V}_b] = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} \in se(3), \tag{3.71}$$

where $[\omega_b] \in so(3)$ and $v_b \in \mathbb{R}^3$. The set of all 4×4 matrices of this form is called $se(3)$ and comprises the matrix representations of the twists associated with the rigid-body configurations $SE(3)$.[7]

[6]The term "twist" has been used in different ways in the mechanisms and screw theory literature. In robotics, however, it is common to use the term to refer to a spatial velocity. We mostly use the term "twist" instead of "spatial velocity" to minimize verbiage, e.g., "body twist" versus "spatial velocity in the body frame."

[7]$se(3)$ is called the Lie algebra of the Lie group $SE(3)$. It consists of all possible $\dot{T}$ when $T = I$.

Now that we have a physical interpretation for $T^{-1}\dot{T}$, let us evaluate $\dot{T}T^{-1}$:

$$\begin{aligned}
\dot{T}T^{-1} &= \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \dot{R}R^{\mathrm{T}} & \dot{p} - \dot{R}R^{\mathrm{T}}p \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix}. \qquad (3.72)
\end{aligned}$$

Observe that the skew-symmetric matrix $[\omega_s] = \dot{R}R^{\mathrm{T}}$ is the angular velocity expressed in fixed-frame coordinates but that $v_s = \dot{p} - \dot{R}R^{\mathrm{T}}p$ is ***not*** the linear velocity of the body-frame origin expressed in the fixed frame (that quantity would simply be $\dot{p}$). If we write v_s as

$$v_s = \dot{p} - \omega_s \times p = \dot{p} + \omega_s \times (-p), \qquad (3.73)$$

the physical meaning of v_s can now be inferred: imagining the moving body to be infinitely large, v_s is the instantaneous velocity of the point on this body currently at the fixed-frame origin, expressed in the fixed frame (see Figure 3.17).

As we did for ω_b and v_b, we assemble ω_s and v_s into a six-dimensional twist,

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} \in \mathbb{R}^6, \qquad [\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} = \dot{T}T^{-1} \in se(3), \qquad (3.74)$$

where $[\mathcal{V}_s]$ is the 4×4 matrix representation of $\mathcal{V}_s$. We call $\mathcal{V}_s$ the **spatial velocity in the space frame**, or simply the **spatial twist**.

If we regard the moving body as being infinitely large, there is an appealing and natural symmetry between $\mathcal{V}_s = (\omega_s, v_s)$ and $\mathcal{V}_b = (\omega_b, v_b)$:

(a) ω_b is the angular velocity expressed in {b}, and ω_s is the angular velocity expressed in {s}.

(b) v_b is the linear velocity of a point at the origin of {b} expressed in {b}, and v_s is the linear velocity of a point at the origin of {s} expressed in {s}.

We can obtain $\mathcal{V}_b$ from $\mathcal{V}_s$ as follows:

$$\begin{aligned}
[\mathcal{V}_b] &= T^{-1}\dot{T} \\
&= T^{-1}[\mathcal{V}_s]T. \qquad (3.75)
\end{aligned}$$

Going the other way,

$$[\mathcal{V}_s] = T[\mathcal{V}_b]T^{-1}. \qquad (3.76)$$

Writing out the products in Equation (3.76), we get

$$\mathcal{V}_s = \begin{bmatrix} R[\omega_b]R^{\mathrm{T}} & -R[\omega_b]R^{\mathrm{T}}p + Rv_b \\ 0 & 0 \end{bmatrix}$$

which, using $R[\omega]R^{\mathrm{T}} = [R\omega]$ (Proposition 3.8) and $[\omega]p = -[p]\omega$ for $p, \omega \in \mathbb{R}^3$, can be manipulated into the following relation between $\mathcal{V}_b$ and $\mathcal{V}_s$:

$$\begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}.$$

Because the 6×6 matrix pre-multiplying $\mathcal{V}_b$ is useful for changing the frame of reference for twists and wrenches, as we will see shortly, we give it its own name.

Definition 3.20. Given $T = (R, p) \in SE(3)$, its **adjoint representation** $[\mathrm{Ad}_T]$ is

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}.$$

For any $\mathcal{V} \in \mathbb{R}^6$, the **adjoint map** associated with T is

$$\mathcal{V}' = [\mathrm{Ad}_T]\mathcal{V},$$

which is sometimes also written as

$$\mathcal{V}' = \mathrm{Ad}_T(\mathcal{V}).$$

In terms of the matrix form $[\mathcal{V}] \in se(3)$ of $\mathcal{V} \in \mathbb{R}^6$,

$$[\mathcal{V}'] = T[\mathcal{V}]T^{-1}.$$

The adjoint map satisfies the following properties, verifiable by direct calculation:

Proposition 3.21. *Let $T_1, T_2 \in SE(3)$ and $\mathcal{V} = (\omega, v)$. Then*

$$\mathrm{Ad}_{T_1}\left(\mathrm{Ad}_{T_2}(\mathcal{V})\right) = \mathrm{Ad}_{T_1T_2}(\mathcal{V}) \quad \textit{or} \quad [\mathrm{Ad}_{T_1}][\mathrm{Ad}_{T_2}]\mathcal{V} = [\mathrm{Ad}_{T_1T_2}]\mathcal{V}. \tag{3.77}$$

Also, for any $T \in SE(3)$ the following holds:

$$[\mathrm{Ad}_T]^{-1} = [\mathrm{Ad}_{T^{-1}}], \tag{3.78}$$

The second property follows from the first on choosing $T_1 = T^{-1}$ and $T_2 = T$, so that

$$\mathrm{Ad}_{T^{-1}}\left(\mathrm{Ad}_T(\mathcal{V})\right) = \mathrm{Ad}_{T^{-1}T}(\mathcal{V}) = \mathrm{Ad}_I(\mathcal{V}) = \mathcal{V}. \tag{3.79}$$

3.3.2.1 Summary of Results on Twists

The main results on twists derived thus far are summarized in the following proposition:

Proposition 3.22. *Given a fixed (space) frame* {s}*, a body frame* {b}*, and a differentiable* $T_{sb}(t) \in SE(3)$*, where*

$$T_{sb}(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix}, \tag{3.80}$$

then

$$T_{sb}^{-1}\dot{T}_{sb} = [\mathcal{V}_b] = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} \in se(3) \tag{3.81}$$

is the matrix representation of the **body twist***, and*

$$\dot{T}_{sb}T_{sb}^{-1} = [\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} \in se(3) \tag{3.82}$$

is the matrix representation of the **spatial twist***. The twists $\mathcal{V}_s$ and $\mathcal{V}_b$ are related by*

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b, \tag{3.83}$$

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \begin{bmatrix} R^{\mathrm{T}} & 0 \\ -R^{\mathrm{T}}[p] & R^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = [\mathrm{Ad}_{T_{bs}}]\mathcal{V}_s. \tag{3.84}$$

More generally, for any two frames {c} *and* {d}*, a twist represented as $\mathcal{V}_c$ in* {c} *is related to its representation $\mathcal{V}_d$ in* {d} *by*

$$\mathcal{V}_c = [\mathrm{Ad}_{T_{cd}}]\mathcal{V}_d, \qquad \mathcal{V}_d = [\mathrm{Ad}_{T_{dc}}]\mathcal{V}_c.$$

Again analogously to the case of angular velocities, it is important to realize that, for a given twist, its fixed-frame representation $\mathcal{V}_s$ *does not depend on the choice of the body frame* {b}, and its body-frame representation $\mathcal{V}_b$ *does not depend on the choice of the fixed frame* {s}.

Example 3.23. Figure 3.18 shows a top view of a car, with a single steerable front wheel, driving on a plane. The $\hat{\mathrm{z}}_\mathrm{b}$-axis of the body frame {b} is into the page and the $\hat{\mathrm{z}}_\mathrm{s}$-axis of the fixed frame {s} is out of the page. The angle of the front wheel of the car causes the car's motion to be a pure angular velocity

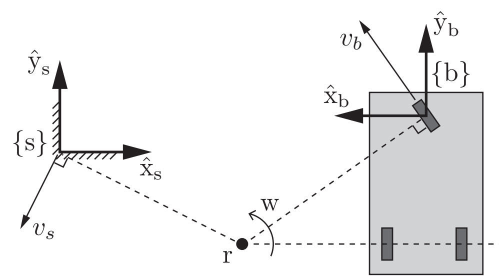


Figure 3.18: The twist corresponding to the instantaneous motion of the chassis of a three-wheeled vehicle can be visualized as an angular velocity w about the point r.

w = 2 rad/s about an axis out of the page at the point r in the plane. Inspecting the figure, we can write r as $r_s = (2, -1, 0)$ or $r_b = (2, -1.4, 0)$, w as $\omega_s = (0, 0, 2)$ or $\omega_b = (0, 0, -2)$, and T_{sb} as

$$T_{sb} = \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0.4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

From the figure and simple geometry, we get

$$v_s = \omega_s \times (-r_s) = r_s \times \omega_s = (-2, -4, 0),$$
$$v_b = \omega_b \times (-r_b) = r_b \times \omega_b = (2.8, 4, 0),$$

and thus obtain the twists $\mathcal{V}_s$ and $\mathcal{V}_b$:

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \\ -4 \\ 0 \end{bmatrix}, \qquad \mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 2.8 \\ 4 \\ 0 \end{bmatrix}.$$

To confirm these results, try calculating $\mathcal{V}_s = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b$.

3.3.2.2 The Screw Interpretation of a Twist

Just as an angular velocity ω can be viewed as $\hat{\omega}\dot{\theta}$, where $\hat{\omega}$ is the unit rotation axis and $\dot{\theta}$ is the rate of rotation about that axis, a twist $\mathcal{V}$ can be interpreted in terms of a **screw axis** $\mathcal{S}$ and a velocity $\dot{\theta}$ about the screw axis.

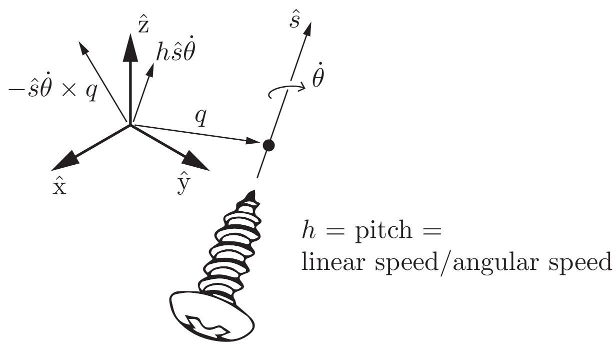


Figure 3.19: A screw axis $\mathcal{S}$ represented by a point q, a unit direction $\hat{s}$, and a pitch h.

A screw axis represents the familiar motion of a screw: rotating about the axis while also translating along the axis. One representation of a screw axis $\mathcal{S}$ is the collection $\{q, \hat{s}, h\}$, where $q \in \mathbb{R}^3$ is any point on the axis, $\hat{s}$ is a unit vector in the direction of the axis, and h is the **screw pitch**, which defines the ratio of the linear velocity along the screw axis to the angular velocity $\dot{\theta}$ about the screw axis (Figure 3.19).

Using Figure 3.19 and geometry, we can write the twist $\mathcal{V} = (\omega, v)$ corresponding to an angular velocity $\dot{\theta}$ about $\mathcal{S}$ (represented by $\{q, \hat{s}, h\}$) as

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s}\dot{\theta} \\ -\hat{s}\dot{\theta} \times q + h\hat{s}\dot{\theta} \end{bmatrix}.$$

Note that the linear velocity v is the sum of two terms: one due to translation along the screw axis, $h\hat{s}\dot{\theta}$, and the other due to the linear motion at the origin induced by rotation about the axis, $-\hat{s}\dot{\theta} \times q$. The first term is in the direction of $\hat{s}$, while the second term is in the plane orthogonal to $\hat{s}$. It is not hard to show that, for any $\mathcal{V} = (\omega, v)$ where $\omega \neq 0$, there exists an equivalent screw axis $\{q, \hat{s}, h\}$ and velocity $\dot{\theta}$, where $\hat{s} = \omega/\|\omega\|$, $\dot{\theta} = \|\omega\|$, $h = \hat{\omega}^{\mathrm{T}} v/\dot{\theta}$, and q is chosen so that the term $-\hat{s}\dot{\theta} \times q$ provides the portion of v orthogonal to the screw axis.

If $\omega = 0$, then the pitch h of the screw is infinite. In this case $\hat{s}$ is chosen as $v/\|v\|$, and $\dot{\theta}$ is interpreted as the linear velocity $\|v\|$ along $\hat{s}$.

Instead of representing the screw axis $\mathcal{S}$ using the cumbersome collection $\{q, \hat{s}, h\}$, with the possibility that h may be infinite and with the nonuniqueness of q (any q along the screw axis may be used), we instead define the screw axis $\mathcal{S}$ using a normalized version of any twist $\mathcal{V} = (\omega, v)$ corresponding to motion along the screw:

(a) If $\omega \neq 0$ then $\mathcal{S} = \mathcal{V}/\|\omega\| = (\omega/\|\omega\|, v/\|\omega\|)$. The screw axis $\mathcal{S}$ is simply $\mathcal{V}$ normalized by the length of the angular velocity vector. The angular velocity about the screw axis is $\dot{\theta} = \|\omega\|$, such that $\mathcal{S}\dot{\theta} = \mathcal{V}$.

(b) If $\omega = 0$ then $\mathcal{S} = \mathcal{V}/\|v\| = (0, v/\|v\|)$. The screw axis $\mathcal{S}$ is simply $\mathcal{V}$ normalized by the length of the linear velocity vector. The linear velocity along the screw axis is $\dot{\theta} = \|v\|$, such that $\mathcal{S}\dot{\theta} = \mathcal{V}$.

This leads to the following definition of a "unit" (normalized) screw axis:

Definition 3.24. For a given reference frame, a **screw axis** $\mathcal{S}$ is written as

$$\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6,$$

where either (i) $\|\omega\| = 1$ or (ii) $\omega = 0$ and $\|v\| = 1$. If (i) holds then $v = -\omega \times q + h\omega$, where q is a point on the axis of the screw and h is the pitch of the screw ($h = 0$ for a pure rotation about the screw axis). If (ii) holds then the pitch of the screw is infinite and the twist is a translation along the axis defined by v.

Important: Although we use the pair (ω, v) for both a normalized screw axis $\mathcal{S}$ (where one of $\|\omega\|$ or $\|v\|$ must be unity) and a general twist $\mathcal{V}$ (where there are no constraints on ω and v), the meaning should be clear from the context.

Since a screw axis $\mathcal{S}$ is just a normalized twist, the 4×4 matrix representation $[\mathcal{S}]$ of $\mathcal{S} = (\omega, v)$ is

$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3), \qquad [\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3), \tag{3.85}$$

where the bottom row of $[\mathcal{S}]$ consists of all zeros. Also, a screw axis represented as $\mathcal{S}_a$ in a frame {a} is related to the representation $\mathcal{S}_b$ in a frame {b} by

$$\mathcal{S}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{S}_b, \qquad \mathcal{S}_b = [\mathrm{Ad}_{T_{ba}}]\mathcal{S}_a.$$

3.3.3 Exponential Coordinate Representation of Rigid-Body Motions

3.3.3.1 Exponential Coordinates of Rigid-Body Motions

In the planar example in Section 3.1, we saw that any planar rigid-body displacement can be achieved by rotating the rigid body about some fixed point in the plane (for a pure translation, this point lies at infinity). A similar result also exists for spatial rigid-body displacements: the **Chasles–Mozzi theorem** states that every rigid-body displacement can be expressed as a displacement along a fixed screw axis $\mathcal{S}$ in space.

By analogy to the exponential coordinates $\hat{\omega}\theta$ for rotations, we define the six-dimensional **exponential coordinates of a homogeneous transformation** T as $\mathcal{S}\theta \in \mathbb{R}^6$, where $\mathcal{S}$ is the screw axis and θ is the distance that must be traveled along the screw axis to take a frame from the origin I to T. If the pitch of the screw axis $\mathcal{S} = (\omega, v)$ is finite then $\|\omega\| = 1$ and θ corresponds to the angle of rotation about the screw axis. If the pitch of the screw is infinite then $\omega = 0$ and $\|v\| = 1$ and θ corresponds to the linear distance traveled along the screw axis.

Also by analogy to the rotation case, we define a matrix exponential (exp) and matrix logarithm (log):

$$\begin{array}{rlcl} \exp: & [\mathcal{S}]\theta \in se(3) & \rightarrow & T \in SE(3), \\ \log: & T \in SE(3) & \rightarrow & [\mathcal{S}]\theta \in se(3). \end{array}$$

We begin by deriving a closed-form expression for the matrix exponential $e^{[\mathcal{S}]\theta}$. Expanding the matrix exponential in series form leads to

$$\begin{aligned} e^{[\mathcal{S}]\theta} &= I + [\mathcal{S}]\theta + [\mathcal{S}]^2\frac{\theta^2}{2!} + [\mathcal{S}]^3\frac{\theta^3}{3!} + \cdots \\ &= \begin{bmatrix} e^{[\omega]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix}, \qquad G(\theta) = I\theta + [\omega]\frac{\theta^2}{2!} + [\omega]^2\frac{\theta^3}{3!} + \cdots \end{aligned} \tag{3.86}$$

Using the identity $[\omega]^3 = -[\omega]$, $G(\theta)$ can be simplified to

$$\begin{aligned} G(\theta) &= I\theta + [\omega]\frac{\theta^2}{2!} + [\omega]^2\frac{\theta^3}{3!} + \cdots \\ &= I\theta + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)[\omega] + \left(\frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \frac{\theta^7}{7!} - \cdots\right)[\omega]^2 \\ &= I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2. \end{aligned} \tag{3.87}$$

Putting everything together leads to the following proposition:

Proposition 3.25. *Let $\mathcal{S} = (\omega, v)$ be a screw axis. If $\|\omega\| = 1$ then, for any distance $\theta \in \mathbb{R}$ traveled along the axis,*

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & \left(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right)v \\ 0 & 1 \end{bmatrix}. \tag{3.88}$$

If $\omega = 0$ *and* $\|v\| = 1$*, then*

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}. \tag{3.89}$$

3.3.3.2 Matrix Logarithm of Rigid-Body Motions

The above derivation essentially provides a constructive proof of the Chasles–Mozzi theorem. That is, given an arbitrary $(R, p) \in SE(3)$, one can always find a screw axis $\mathcal{S} = (\omega, v)$ and a scalar θ such that

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}, \tag{3.90}$$

i.e., the matrix

$$[\mathcal{S}]\theta = \begin{bmatrix} [\omega]\theta & v\theta \\ 0 & 0 \end{bmatrix} \in se(3)$$

is the matrix logarithm of $T = (R, p)$.

Algorithm: Given (R, p) written as $T \in SE(3)$, find a $\theta \in [0, \pi]$ and a screw axis $\mathcal{S} = (\omega, v) \in \mathbb{R}^6$ (where at least one of $\|\omega\|$ and $\|v\|$ is unity) such that $e^{[\mathcal{S}]\theta} = T$. The vector $\mathcal{S}\theta \in \mathbb{R}^6$ comprises the exponential coordinates for T and the matrix $[\mathcal{S}]\theta \in se(3)$ is the matrix logarithm of T.

(a) If $R = I$ then set $\omega = 0$, $v = p/\|p\|$, and $\theta = \|p\|$.

(b) Otherwise, use the matrix logarithm on $SO(3)$ to determine ω (written as $\hat{\omega}$ in the $SO(3)$ algorithm) and θ for R. Then v is calculated as

$$v = G^{-1}(\theta)p \tag{3.91}$$

where

$$G^{-1}(\theta) = \frac{1}{\theta}I - \frac{1}{2}[\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2}\right)[\omega]^2. \tag{3.92}$$

The verification of Equation (3.92) is left as an exercise.

Example 3.26. In this example, the rigid-body motion is confined to the $\hat{\mathrm{x}}_\mathrm{s}$–$\hat{\mathrm{y}}_\mathrm{s}$-plane. The initial frame {b} and final frame {c} in Figure 3.20 can be repre-

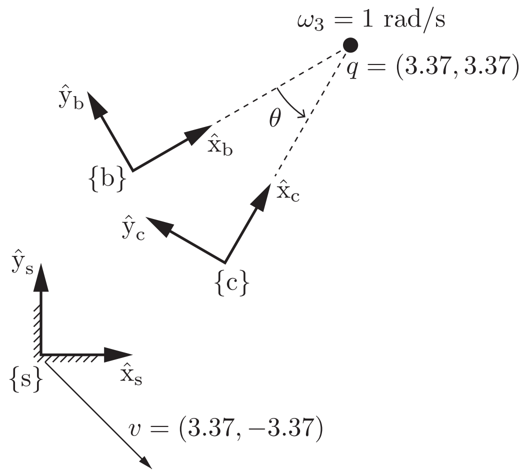


Figure 3.20: Two frames in a plane.

sented by the $SE(3)$ matrices

$$
\begin{aligned}
T_{sb} &= \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 & 1 \\ \sin 30^\circ & \cos 30^\circ & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
T_{sc} &= \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ & 0 & 2 \\ \sin 60^\circ & \cos 60^\circ & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Because the motion occurs in the $\hat{x}_s$–$\hat{y}_s$-plane, the corresponding screw has an axis in the direction of the $\hat{z}_s$-axis and has zero pitch. The screw axis $\mathcal{S} = (\omega, v)$, expressed in $\{s\}$, therefore has the form

$$
\begin{aligned}
\omega &= (0, 0, \omega_3), \\
v &= (v_1, v_2, 0).
\end{aligned}
$$

We seek the screw motion that displaces the frame at T_{sb} to T_{sc}; i.e., $T_{sc} = e^{[\mathcal{S}]\theta} T_{sb}$ or

$$
T_{sc} T_{sb}^{-1} = e^{[\mathcal{S}]\theta},
$$

where

$$[\mathcal{S}] = \begin{bmatrix} 0 & -\omega_3 & 0 & v_1 \\ \omega_3 & 0 & 0 & v_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can apply the matrix logarithm algorithm directly to $T_{sc}T_{sb}^{-1}$ to obtain $[\mathcal{S}]$ (and therefore $\mathcal{S}$) and θ as follows:

$$[\mathcal{S}] = \begin{bmatrix} 0 & -1 & 0 & 3.37 \\ 1 & 0 & 0 & -3.37 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{S} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3.37 \\ -3.37 \\ 0 \end{bmatrix}, \quad \theta = \frac{\pi}{6} \text{ rad (or } 30^\circ\text{)}.$$

The value of $\mathcal{S}$ means that the constant screw axis, expressed in the fixed frame {s}, is represented by an angular velocity of 1 rad/s about $\hat{\mathrm{z}}_\mathrm{s}$ and a linear velocity (of a point currently at the origin of {s}) of $(3.37, -3.37, 0)$ expressed in the {s} frame.

Alternatively, we can observe that the displacement is not a pure translation – T_{sb} and T_{sc} have rotation components that differ by an angle of 30° – and we quickly determine that $\theta = 30^\circ$ and $\omega_3 = 1$. We can also graphically determine the point $q = (q_x, q_y)$ in the $\hat{\mathrm{x}}_\mathrm{s}$–$\hat{\mathrm{y}}_\mathrm{s}$-plane through which the screw axis passes; for our example this point is given by $q = (3.37, 3.37)$.

For planar rigid-body motions such as this one, we could derive a planar matrix logarithm algorithm that maps elements of $SE(2)$ to elements of $se(2)$, which have the form

$$\begin{bmatrix} 0 & -\omega & v_1 \\ \omega & 0 & v_2 \\ 0 & 0 & 0 \end{bmatrix}.$$

3.4 Wrenches

Consider a linear force f acting on a rigid body at a point r. Defining a reference frame {a}, the point r can be represented as $r_a \in \mathbb{R}^3$ and the force f can be represented as $f_a \in \mathbb{R}^3$. This force creates a **torque** or **moment** $m_a \in \mathbb{R}^3$ in the {a} frame:

$$m_a = r_a \times f_a.$$

Note that the point of application of the force along its line of action is immaterial.

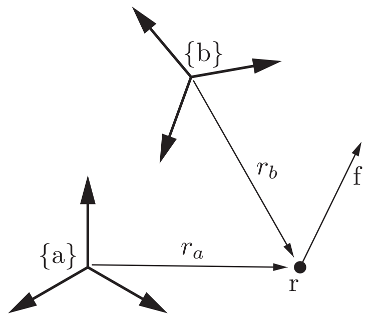


Figure 3.21: Relation between wrench representations $\mathcal{F}_a$ and $\mathcal{F}_b$.

Just as with twists, we can merge the moment and force into a single six-dimensional **spatial force**, or **wrench**, expressed in the {a} frame, $\mathcal{F}_a$:

$$\mathcal{F}_a = \begin{bmatrix} m_a \\ f_a \end{bmatrix} \in \mathbb{R}^6. \tag{3.93}$$

If more than one wrench acts on a rigid body, the total wrench on the body is simply the vector sum of the individual wrenches, provided that the wrenches are expressed in the same frame. A wrench with a zero linear component is called a **pure moment**.

A wrench in the {a} frame can be represented in another frame {b} (Figure 3.21) if T_{ba} is known. One way to derive the relationship between $\mathcal{F}_a$ and $\mathcal{F}_b$ is to derive the appropriate transformations between the individual force and moment vectors on the basis of techniques we have already used.

A simpler and more insightful way to derive the relationship between $\mathcal{F}_a$ and $\mathcal{F}_b$, however, is to (1) use the results we have already derived relating representations $\mathcal{V}_a$ and $\mathcal{V}_b$ of the same twist, and (2) use the fact that the power generated (or dissipated) by an $(\mathcal{F}, \mathcal{V})$ pair must be the same regardless of the frame in which it is represented. (Imagine if we could create power simply by changing our choice of reference frame!) Recall that the dot product of a force and a velocity is a power, and power is a coordinate-independent quantity. Because of this, we know that

$$\mathcal{V}_b^{\mathrm{T}} \mathcal{F}_b = \mathcal{V}_a^{\mathrm{T}} \mathcal{F}_a. \tag{3.94}$$

From Proposition 3.22 we know that $\mathcal{V}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{V}_b$, and therefore Equation (3.94) can be rewritten as

$$\begin{aligned} \mathcal{V}_b^{\mathrm{T}} \mathcal{F}_b &= ([\mathrm{Ad}_{T_{ab}}]\mathcal{V}_b)^{\mathrm{T}} \mathcal{F}_a \\ &= \mathcal{V}_b^{\mathrm{T}} [\mathrm{Ad}_{T_{ab}}]^{\mathrm{T}} \mathcal{F}_a. \end{aligned}$$

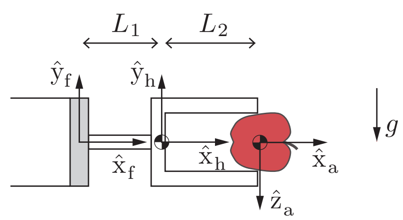


Figure 3.22: A robot hand holding an apple subject to gravity.

Since this must hold for all $\mathcal{V}_b$, this simplifies to

$$\mathcal{F}_b = [\mathrm{Ad}_{T_{ab}}]^{\mathrm{T}} \mathcal{F}_a. \tag{3.95}$$

Similarly,

$$\mathcal{F}_a = [\mathrm{Ad}_{T_{ba}}]^{\mathrm{T}} \mathcal{F}_b. \tag{3.96}$$

Proposition 3.27. *Given a wrench* F*, represented in* {a} *as* $\mathcal{F}_a$ *and in* {b} *as* $\mathcal{F}_b$*, the two representations are related by*

$$\mathcal{F}_b = \mathrm{Ad}^{\mathrm{T}}_{T_{ab}}(\mathcal{F}_a) = [\mathrm{Ad}_{T_{ab}}]^{\mathrm{T}} \mathcal{F}_a, \tag{3.97}$$

$$\mathcal{F}_a = \mathrm{Ad}^{\mathrm{T}}_{T_{ba}}(\mathcal{F}_b) = [\mathrm{Ad}_{T_{ba}}]^{\mathrm{T}} \mathcal{F}_b. \tag{3.98}$$

Since we usually have a fixed space frame {s} and a body frame {b}, we can define a **spatial wrench** $\mathcal{F}_s$ and a **body wrench** $\mathcal{F}_b$.

Example 3.28. The robot hand in Figure 3.22 is holding an apple with a mass of 0.1 kg in a gravitational field $g = 10$ m/s^2 (rounded to keep the numbers simple) acting downward on the page. The mass of the hand is 0.5 kg. What is the force and torque measured by the six-axis force–torque sensor between the hand and the robot arm?

We define frames {f} at the force–torque sensor, {h} at the center of mass of the hand, and {a} at the center of mass of the apple. According to the coordinate axes in Figure 3.22, the gravitational wrench on the hand in {h} is given by the column vector

$$\mathcal{F}_h = (0, 0, 0, 0, -5 \text{ N}, 0)$$

and the gravitational wrench on the apple in {a} is

$$\mathcal{F}_a = (0, 0, 0, 0, 0, 1 \text{ N}).$$

Given $L_1 = 10$ cm and $L_2 = 15$ cm, the transformation matrices T_{hf} and T_{af} are

$$T_{hf} = \begin{bmatrix} 1 & 0 & 0 & -0.1 \text{ m} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad T_{af} = \begin{bmatrix} 1 & 0 & 0 & -0.25 \text{ m} \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The wrench measured by the six-axis force–torque sensor is

$$\begin{aligned} \mathcal{F}_f &= [\mathrm{Ad}_{T_{hf}}]^{\mathrm{T}} \mathcal{F}_h + [\mathrm{Ad}_{T_{af}}]^{\mathrm{T}} \mathcal{F}_a \\ &= [0\ 0\ -0.5 \text{ Nm}\ 0\ -5 \text{ N}\ 0]^{\mathrm{T}} + [0\ 0\ -0.25 \text{ Nm}\ 0\ -1 \text{ N}\ 0]^{\mathrm{T}} \\ &= [0\ 0\ -0.75 \text{ Nm}\ 0\ -6 \text{ N}\ 0]^{\mathrm{T}}. \end{aligned}$$

3.5 Summary

The following table succinctly summarizes some of the key concepts from the chapter, as well as the parallelism between rotations and rigid-body motions. For more details, consult the appropriate section of the chapter.

Rotations	**Rigid-Body Motions**
$R \in SO(3) : 3 \times 3$ matrices $R^{\mathrm{T}}R = I, \det R = 1$	$T \in SE(3) : 4 \times 4$ matrices $T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix},$ where $R \in SO(3), p \in \mathbb{R}^3$
$R^{-1} = R^{\mathrm{T}}$	$T^{-1} = \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix}$
change of coordinate frame: $R_{ab}R_{bc} = R_{ac}, \;\; R_{ab}p_b = p_a$	change of coordinate frame: $T_{ab}T_{bc} = T_{ac}, \;\; T_{ab}p_b = p_a$
rotating a frame {b}: $R = \mathrm{Rot}(\hat{\omega}, \theta)$ $R_{sb'} = RR_{sb}$: rotate θ about $\hat{\omega}_s = \hat{\omega}$ $R_{sb''} = R_{sb}R$: rotate θ about $\hat{\omega}_b = \hat{\omega}$	displacing a frame {b}: $T = \begin{bmatrix} \mathrm{Rot}(\hat{\omega}, \theta) & p \\ 0 & 1 \end{bmatrix}$ $T_{sb'} = TT_{sb}$: rotate θ about $\hat{\omega}_s = \hat{\omega}$ (moves {b} origin), translate p in {s} $T_{sb''} = T_{sb}T$: translate p in {b}, rotate θ about $\hat{\omega}$ in new body frame
unit rotation axis is $\hat{\omega} \in \mathbb{R}^3$, where $\\|\hat{\omega}\\| = 1$	"unit" screw axis is $\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$, where either (i) $\\|\omega\\| = 1$ or (ii) $\omega = 0$ and $\\|v\\| = 1$
	for a screw axis $\{q, \hat{s}, h\}$ with finite h, $\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s} \\ -\hat{s} \times q + h\hat{s} \end{bmatrix}$
angular velocity is $\omega = \hat{\omega}\dot{\theta}$	twist is $\mathcal{V} = \mathcal{S}\dot{\theta}$

continued...

Rotations (cont.)	**Rigid-Body Motions (cont.)**
for any 3-vector, e.g., $\omega \in \mathbb{R}^3$, $[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3)$ identities, $\omega, x \in \mathbb{R}^3, R \in SO(3)$: $[\omega] = -[\omega]^{\mathrm{T}}, [\omega]x = -[x]\omega$, $[\omega][x] = ([x][\omega])^{\mathrm{T}}, R[\omega]R^{\mathrm{T}} = [R\omega]$	for $\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$, $[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3)$ (the pair (ω, v) can be a twist $\mathcal{V}$ or a "unit" screw axis $\mathcal{S}$, depending on the context)
$\dot{R}R^{-1} = [\omega_s], \;\; R^{-1}\dot{R} = [\omega_b]$	$\dot{T}T^{-1} = [\mathcal{V}_s], \;\; T^{-1}\dot{T} = [\mathcal{V}_b]$
	$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$ identities: $[\mathrm{Ad}_T]^{-1} = [\mathrm{Ad}_{T^{-1}}]$, $[\mathrm{Ad}_{T_1}][\mathrm{Ad}_{T_2}] = [\mathrm{Ad}_{T_1 T_2}]$
change of coordinate frame: $\hat{\omega}_a = R_{ab}\hat{\omega}_b, \;\; \omega_a = R_{ab}\omega_b$	change of coordinate frame: $\mathcal{S}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{S}_b, \;\; \mathcal{V}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{V}_b$
exp coords for $R \in SO(3)$: $\hat{\omega}\theta \in \mathbb{R}^3$	exp coords for $T \in SE(3)$: $\mathcal{S}\theta \in \mathbb{R}^6$
$\exp : [\hat{\omega}]\theta \in so(3) \to R \in SO(3)$ $R = \mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} =$ $I + \sin\theta[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2$	$\exp : [\mathcal{S}]\theta \in se(3) \to T \in SE(3)$ $T = e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & * \\ 0 & 1 \end{bmatrix}$ where $* =$ $(I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2)v$
$\log : R \in SO(3) \to [\hat{\omega}]\theta \in so(3)$ algorithm in Section 3.2.3.3	$\log : T \in SE(3) \to [\mathcal{S}]\theta \in se(3)$ algorithm in Section 3.3.3.2
moment change of coord frame: $m_a = R_{ab}m_b$	wrench change of coord frame: $\mathcal{F}_a = (m_a, f_a) = [\mathrm{Ad}_{T_{ba}}]^{\mathrm{T}}\mathcal{F}_b$

3.6 Software

The following functions are included in the software distribution accompanying the book. The code below is in MATLAB format, but it is available in other languages. For more details on the software, consult the code and its documentation.

```
invR = RotInv(R)
```

Computes the inverse of the rotation matrix `R`.

`so3mat = VecToso3(omg)`
Returns the 3×3 skew-symmetric matrix corresponding to `omg`.

`omg = so3ToVec(so3mat)`
Returns the 3-vector corresponding to the 3×3 skew-symmetric matrix `so3mat`.

`[omghat,theta] = AxisAng3(expc3)`
Extracts the rotation axis $\hat{\omega}$ and the rotation amount θ from the 3-vector $\hat{\omega}\theta$ of exponential coordinates for rotation, `expc3`.

`R = MatrixExp3(so3mat)`
Computes the rotation matrix `R` $\in SO(3)$ corresponding to the matrix exponential of `so3mat` $\in so(3)$.

`so3mat = MatrixLog3(R)`
Computes the matrix logarithm `so3mat` $\in so(3)$ of the rotation matrix `R` $\in SO(3)$.

`T = RpToTrans(R,p)`
Builds the homogeneous transformation matrix `T` corresponding to a rotation matrix `R` $\in SO(3)$ and a position vector `p` $\in \mathbb{R}^3$.

`[R,p] = TransToRp(T)`
Extracts the rotation matrix and position vector from a homogeneous transformation matrix `T`.

`invT = TransInv(T)`
Computes the inverse of a homogeneous transformation matrix `T`.

`se3mat = VecTose3(V)`
Returns the $se(3)$ matrix corresponding to a 6-vector twist `V`.

`V = se3ToVec(se3mat)`
Returns the 6-vector twist corresponding to an $se(3)$ matrix `se3mat`.

`AdT = Adjoint(T)`
Computes the 6×6 adjoint representation $[\mathrm{Ad}_T]$ of the homogeneous transformation matrix `T`.

`S = ScrewToAxis(q,s,h)`
Returns a normalized screw axis representation $\mathcal{S}$ of a screw described by a unit

vector `s` in the direction of the screw axis, located at the point `q`, with pitch `h`.

`[S,theta] = AxisAng(expc6)`
Extracts the normalized screw axis $\mathcal{S}$ and the distance traveled along the screw θ from the 6-vector of exponential coordinates $\mathcal{S}\theta$.

`T = MatrixExp6(se3mat)`
Computes the homogeneous transformation matrix `T` $\in SE(3)$ corresponding to the matrix exponential of `se3mat` $\in se(3)$.

`se3mat = MatrixLog6(T)`
Computes the matrix logarithm `se3mat` $\in se(3)$ of the homogeneous transformation matrix `T` $\in SE(3)$.

3.7 Notes and References

The exponential coordinates for rotations introduced in this chapter are also referred to in the kinematics literature as the Euler–Rodrigues parameters. Other representations for rotations such as Euler angles, Cayley–Rodrigues parameters, and unit quaternions are described in Appendix B; further details on these and related parametrizations of the rotation group $SO(3)$ can be found in, e.g., [169, 113, 186, 122, 135].

Classical screw theory has its origins in the works of Mozzi and Chasles, who independently discovered that the motion of a rigid body can be obtained as a rotation about some axis followed by a translation about the same axis [25]. Ball's treatise [6] is often regarded as the classical reference on screw theory, while more modern treatments can be found in Bottema and Roth [18], Angeles [2], and McCarthy [113].

The identification of elements of classical screw theory with the Lie group structure of the rigid body motions $SE(3)$ was first made by Brockett in [20], who went considerably further and showed that the forward kinematics of open chains can be expressed as the product of matrix exponentials (this is the subject of the next chapter). Derivations of the formulas for the matrix exponentials, logarithms, their derivatives, and other related formulas can be found in [92, 129, 131, 122].

3.8 Exercises

Exercise 3.1 In terms of the $\hat{\mathrm{x}}_{\mathrm{s}}$, $\hat{\mathrm{y}}_{\mathrm{s}}$, $\hat{\mathrm{z}}_{\mathrm{s}}$ coordinates of a fixed space frame {s}, the frame {a} has its $\hat{\mathrm{x}}_{\mathrm{a}}$-axis pointing in the direction $(0,0,1)$ and its $\hat{\mathrm{y}}_{\mathrm{a}}$-axis pointing in the direction $(-1,0,0)$, and the frame {b} has its $\hat{\mathrm{x}}_{\mathrm{b}}$-axis pointing in the direction $(1,0,0)$ and its $\hat{\mathrm{y}}_{\mathrm{b}}$-axis pointing in the direction $(0,0,-1)$.

(a) Draw by hand the three frames, at different locations so that they are easy to see.

(b) Write down the rotation matrices R_{sa} and R_{sb}.

(c) Given R_{sb}, how do you calculate R_{sb}^{-1} without using a matrix inverse? Write down R_{sb}^{-1} and verify its correctness using your drawing.

(d) Given R_{sa} and R_{sb}, how do you calculate R_{ab} (again without using matrix inverses)? Compute the answer and verify its correctness using your drawing.

(e) Let $R = R_{sb}$ be considered as a transformation operator consisting of a rotation about $\hat{\mathrm{x}}$ by $-90°$. Calculate $R_1 = R_{sa}R$, and think of R_{sa} as a representation of an orientation, R as a rotation of R_{sa}, and R_1 as the new orientation after the rotation has been performed. Does the new orientation R_1 correspond to a rotation of R_{sa} by $-90°$ about the world-fixed $\hat{\mathrm{x}}_{\mathrm{s}}$-axis or about the body-fixed $\hat{\mathrm{x}}_{\mathrm{a}}$-axis? Now calculate $R_2 = RR_{sa}$. Does the new orientation R_2 correspond to a rotation of R_{sa} by $-90°$ about the world-fixed $\hat{\mathrm{x}}_{\mathrm{s}}$-axis or about the body-fixed $\hat{\mathrm{x}}_{\mathrm{a}}$-axis?

(f) Use R_{sb} to change the representation of the point $p_b = (1,2,3)$ (which is in {b} coordinates) to {s} coordinates.

(g) Choose a point p represented by $p_s = (1,2,3)$ in {s} coordinates. Calculate $p' = R_{sb}p_s$ and $p'' = R_{sb}^{\mathrm{T}}p_s$. For each operation, should the result be interpreted as changing coordinates (from the {s} frame to {b}) without moving the point p or as moving the location of the point without changing the reference frame of the representation?

(h) An angular velocity w is represented in {s} as $\omega_s = (3,2,1)$. What is its representation ω_a in {a}?

(i) By hand, calculate the matrix logarithm $[\hat{\omega}]\theta$ of R_{sa}. (You may verify your answer with software.) Extract the unit angular velocity $\hat{\omega}$ and rotation amount θ. Redraw the fixed frame {s} and in it draw $\hat{\omega}$.

(j) Calculate the matrix exponential corresponding to the exponential coordinates of rotation $\hat{\omega}\theta = (1,2,0)$. Draw the corresponding frame relative to {s}, as well as the rotation axis $\hat{\omega}$.

Exercise 3.2 Let p be a point whose coordinates are $p = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right)$ with respect to the fixed frame $\hat{\mathrm{x}}$–$\hat{\mathrm{y}}$–$\hat{\mathrm{z}}$. Suppose that p is rotated about the fixed-frame $\hat{\mathrm{x}}$-axis by 30 degrees, then about the fixed-frame $\hat{\mathrm{y}}$-axis by 135 degrees, and finally about the fixed-frame $\hat{\mathrm{z}}$-axis by -120 degrees. Denote the coordinates of this newly rotated point by p'.

(a) What are the coordinates p'?

(b) Find the rotation matrix R such that $p' = Rp$ for the p' you obtained in (a).

Exercise 3.3 Suppose that $p_i \in \mathbb{R}^3$ and $p_i' \in \mathbb{R}^3$ are related by $p_i' = Rp_i$, $i = 1, 2, 3$, for some unknown rotation matrix R. Find, if it exists, the rotation R for the three input–output pairs $p_i \mapsto p_i'$, where

$$\begin{aligned} p_1 = (\sqrt{2}, 0, 2) \quad &\mapsto \quad p_1' = (0, 2, \sqrt{2}), \\ p_2 = (1, 1, -1) \quad &\mapsto \quad p_2' = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{2}\right), \\ p_3 = (0, 2\sqrt{2}, 0) \quad &\mapsto \quad p_3' = (-\sqrt{2}, \sqrt{2}, -2). \end{aligned}$$

Exercise 3.4 In this exercise you are asked to prove the property $R_{ab}R_{bc} = R_{ac}$ of Equation (3.22). Define the unit axes of frames {a}, {b}, and {c} by the triplets of orthogonal unit vectors $\{\hat{\mathrm{x}}_\mathrm{a}, \hat{\mathrm{y}}_\mathrm{a}, \hat{\mathrm{z}}_\mathrm{a}\}$, $\{\hat{\mathrm{x}}_\mathrm{b}, \hat{\mathrm{y}}_\mathrm{b}, \hat{\mathrm{z}}_\mathrm{b}\}$, and $\{\hat{\mathrm{x}}_\mathrm{c}, \hat{\mathrm{y}}_\mathrm{c}, \hat{\mathrm{z}}_\mathrm{c}\}$, respectively. Suppose that the unit axes of frame {b} can be expressed in terms of the unit axes of frame {a} by

$$\begin{aligned} \hat{\mathrm{x}}_\mathrm{b} &= r_{11}\hat{\mathrm{x}}_\mathrm{a} + r_{21}\hat{\mathrm{y}}_\mathrm{a} + r_{31}\hat{\mathrm{z}}_\mathrm{a}, \\ \hat{\mathrm{y}}_\mathrm{b} &= r_{12}\hat{\mathrm{x}}_\mathrm{a} + r_{22}\hat{\mathrm{y}}_\mathrm{a} + r_{32}\hat{\mathrm{z}}_\mathrm{a}, \\ \hat{\mathrm{z}}_\mathrm{b} &= r_{13}\hat{\mathrm{x}}_\mathrm{a} + r_{23}\hat{\mathrm{y}}_\mathrm{a} + r_{33}\hat{\mathrm{z}}_\mathrm{a}. \end{aligned}$$

Similarly, suppose that the unit axes of frame {c} can be expressed in terms of the unit axes of frame {b} by

$$\begin{aligned} \hat{\mathrm{x}}_\mathrm{c} &= s_{11}\hat{\mathrm{x}}_\mathrm{b} + s_{21}\hat{\mathrm{y}}_\mathrm{b} + s_{31}\hat{\mathrm{z}}_\mathrm{b}, \\ \hat{\mathrm{y}}_\mathrm{c} &= s_{12}\hat{\mathrm{x}}_\mathrm{b} + s_{22}\hat{\mathrm{y}}_\mathrm{b} + s_{32}\hat{\mathrm{z}}_\mathrm{b}, \\ \hat{\mathrm{z}}_\mathrm{c} &= s_{13}\hat{\mathrm{x}}_\mathrm{b} + s_{23}\hat{\mathrm{y}}_\mathrm{b} + s_{33}\hat{\mathrm{z}}_\mathrm{b}. \end{aligned}$$

From the above prove that $R_{ab}R_{bc} = R_{ac}$.

Exercise 3.5 Find the exponential coordinates $\hat{\omega}\theta \in \mathbb{R}^3$ for the $SO(3)$ matrix

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Exercise 3.6 Given $R = \mathrm{Rot}(\hat{\mathrm{x}}, \pi/2)\mathrm{Rot}(\hat{\mathrm{z}}, \pi)$, find the unit vector $\hat{\omega}$ and angle θ such that $R = e^{[\hat{\omega}]\theta}$.

Exercise 3.7

(a) Given the rotation matrix

$$R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

find all possible values for $\hat{\omega} \in \mathbb{R}^3, \|\hat{\omega}\| = 1$, and $\theta \in [0, 2\pi)$ such that $e^{[\hat{\omega}]\theta} = R$.

(b) The two vectors $v_1, v_2 \in \mathbb{R}^3$ are related by

$$v_2 = Rv_1 = e^{[\hat{\omega}]\theta} v_1$$

where $\hat{\omega} \in \mathbb{R}^3$ has length 1, and $\theta \in [-\pi, \pi]$. Given $\hat{\omega} = (\frac{2}{3}, \frac{2}{3}, \frac{1}{3}), v_1 = (1, 0, 1), v_2 = (0, 1, 1)$, find all the angles θ that satisfy the above equation.

Exercise 3.8

(a) Suppose that we are seeking the logarithm of a rotation matrix R whose trace is -1. From the exponential formula

$$e^{[\hat{\omega}]\theta} = I + \sin\theta\,[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2, \qquad \|\omega\| = 1,$$

and recalling that $\mathrm{tr}\, R = -1$ implies $\theta = \pi$, the above equation simplifies to

$$R = I + 2[\hat{\omega}]^2 = \begin{bmatrix} 1 - 2(\hat{\omega}_2^2 + \hat{\omega}_3^2) & 2\hat{\omega}_1\hat{\omega}_2 & 2\hat{\omega}_1\hat{\omega}_3 \\ 2\hat{\omega}_1\hat{\omega}_2 & 1 - 2(\hat{\omega}_1^2 + \hat{\omega}_3^2) & 2\hat{\omega}_2\hat{\omega}_3 \\ 2\hat{\omega}_1\hat{\omega}_2 & 2\hat{\omega}_2\hat{\omega}_3 & 1 - 2(\hat{\omega}_1^2 + \hat{\omega}_2^2) \end{bmatrix}.$$

Using the fact that $\hat{\omega}_1^2 + \hat{\omega}_2^2 + \hat{\omega}_3^2 = 1$, is it correct to conclude that

$$\hat{\omega}_1 = \sqrt{\frac{r_{11} + 1}{2}}, \qquad \hat{\omega}_2 = \sqrt{\frac{r_{22} + 1}{2}}, \qquad \hat{\omega}_3 = \sqrt{\frac{r_{33} + 1}{2}},$$

where r_{ij} denotes the (i,j)th entry of R, is also a solution?

(b) Using the fact that $[\hat{\omega}]^3 = -[\hat{\omega}]$, the identity $R = I + 2[\hat{\omega}]^2$ can be written in the alternative forms

$$\begin{aligned} R - I &= 2[\hat{\omega}]^2, \\ [\hat{\omega}]\,(R - I) &= 2\,[\hat{\omega}]^3 = -2\,[\hat{\omega}]\,, \\ [\hat{\omega}]\,(R + I) &= 0. \end{aligned}$$

The resulting equation consists of three linear equations in $(\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$. What is the relation between the solution to this linear system and the logarithm of R?

Exercise 3.9 Exploiting the known properties of rotation matrices, determine the minimum number of arithmetic operations (multiplication and division, addition and subtraction) required to multiply two rotation matrices.

Exercise 3.10 Because arithmetic precision is only finite, the numerically obtained product of two rotation matrices is not necessarily a rotation matrix; that is, the resulting rotation A may not exactly satisfy $A^{\mathrm{T}}A = I$ as desired. Devise an iterative numerical procedure that takes an arbitrary matrix $A \in \mathbb{R}^{3\times 3}$ and produces a matrix $R \in SO(3)$ that minimizes

$$\|A - R\|^2 = \mathrm{tr}\,(A - R)(A - R)^{\mathrm{T}}.$$

(Hint: See Appendix D for the relevant background on optimization.)

Exercise 3.11 Properties of the matrix exponential.

(a) Under what conditions on general $A, B \in \mathbb{R}^{n\times n}$ does $e^A e^B = e^{A+B}$ hold?

(b) If $A = [\mathcal{V}_a]$ and $B = [\mathcal{V}_b]$, where $\mathcal{V}_a = (\omega_a, v_a)$ and $\mathcal{V}_b = (\omega_b, v_b)$ are arbitrary twists, then under what conditions on $\mathcal{V}_a$ and $\mathcal{V}_b$ does $e^A e^B = e^{A+B}$ hold? Try to give a physical description of this condition.

Exercise 3.12

(a) Given a rotation matrix $A = \mathrm{Rot}(\hat{\mathrm{z}}, \alpha)$, where $\mathrm{Rot}(\hat{\mathrm{z}}, \alpha)$ indicates a rotation about the $\hat{\mathrm{z}}$-axis by an angle α, find all rotation matrices $R \in SO(3)$ that satisfy $AR = RA$.

(b) Given rotation matrices $A = \mathrm{Rot}(\hat{\mathrm{z}}, \alpha)$ and $B = \mathrm{Rot}(\hat{\mathrm{z}}, \beta)$, with $\alpha \neq \beta$, find all rotation matrices $R \in SO(3)$ that satisfy $AR = RB$.

(c) Given arbitrary rotation matrices $A, B \in SO(3)$, find all solutions $R \in SO(3)$ to the equation $AR = RB$.

Exercise 3.13

(a) Show that the three eigenvalues of a rotation matrix $R \in SO(3)$ each have unit magnitude, and conclude that they can always be written $\{\mu + i\nu, \mu - i\nu, 1\}$, where $\mu^2 + \nu^2 = 1$.

(b) Show that a rotation matrix $R \in SO(3)$ can always be factored in the form

$$R = A \begin{bmatrix} \mu & \nu & 0 \\ -\nu & \mu & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1},$$

where $A \in SO(3)$ and $\mu^2 + \nu^2 = 1$. (Hint: Denote the eigenvector associated with the eigenvalue $\mu + i\nu$ by $x + iy$, $x, y \in \mathbb{R}^3$, and the eigenvector associated with the eigenvalue 1 by $z \in \mathbb{R}^3$. For the purposes of this problem you may assume that the set of vectors $\{x, y, z\}$ can always be chosen to be linearly independent.)

Exercise 3.14 Given $\omega \in \mathbb{R}^3$, $\|\omega\| = 1$, and θ a nonzero scalar, show that

$$\left(I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right)^{-1} = \frac{1}{\theta}I - \frac{1}{2}[\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2}\right)[\omega]^2.$$

(Hint: From the identity $[\omega]^3 = -[\omega]$, express the inverse as a quadratic matrix polynomial in $[\omega]$.)

Exercise 3.15

(a) Given a fixed frame $\{0\}$ and a moving frame $\{1\}$ initially aligned with $\{0\}$, perform the following sequence of rotations on $\{1\}$:

1. Rotate $\{1\}$ about the $\{0\}$ frame $\hat{\mathrm{x}}$-axis by α; call this new frame $\{2\}$.
2. Rotate $\{2\}$ about the $\{0\}$ frame $\hat{\mathrm{y}}$-axis by β; call this new frame $\{3\}$.
3. Rotate $\{3\}$ about the $\{0\}$ frame $\hat{\mathrm{z}}$-axis by γ; call this new frame $\{4\}$.

What is the final orientation R_{04}?

(b) Suppose that the third step above is replaced by the following: "Rotate $\{3\}$ about the $\hat{\mathrm{z}}$-axis of frame $\{3\}$ by γ; call this new frame $\{4\}$." What is the final orientation R_{04}?

(c) Find T_{ca} for the following transformations:

$$T_{ab} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{cb} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Exercise 3.16 In terms of the $\hat{\mathrm{x}}_\mathrm{s}$, $\hat{\mathrm{y}}_\mathrm{s}$, $\hat{\mathrm{z}}_\mathrm{s}$ coordinates of a fixed space frame {s}, frame {a} has its $\hat{\mathrm{x}}_\mathrm{a}$-axis pointing in the direction $(0, 0, 1)$ and its $\hat{\mathrm{y}}_\mathrm{a}$-axis pointing in the direction $(-1, 0, 0)$, and frame {b} has its $\hat{\mathrm{x}}_\mathrm{b}$-axis pointing in the direction $(1, 0, 0)$ and its $\hat{\mathrm{y}}_\mathrm{b}$-axis pointing in the direction $(0, 0, -1)$. The origin of {a} is at $(3, 0, 0)$ in {s} and the origin of {b} is at $(0, 2, 0)$ is {s}.

(a) Draw by hand a diagram showing {a} and {b} relative to {s}.

(b) Write down the rotation matrices R_{sa} and R_{sb} and the transformation matrices T_{sa} and T_{sb}.

(c) Given T_{sb}, how do you calculate T_{sb}^{-1} without using a matrix inverse? Write T_{sb}^{-1} and verify its correctness using your drawing.

(d) Given T_{sa} and T_{sb}, how do you calculate T_{ab} (again without using matrix inverses)? Compute the answer and verify its correctness using your drawing.

(e) Let $T = T_{sb}$ be considered as a transformation operator consisting of a rotation about $\hat{\mathrm{x}}$ by $-90°$ and a translation along $\hat{\mathrm{y}}$ by 2 units. Calculate $T_1 = T_{sa}T$. Does T_1 correspond to a rotation and translation about $\hat{\mathrm{x}}_\mathrm{s}$ and $\hat{\mathrm{y}}_\mathrm{s}$, respectively (a world-fixed transformation of T_{sa}), or a rotation and translation about $\hat{\mathrm{x}}_\mathrm{a}$ and $\hat{\mathrm{y}}_\mathrm{a}$, respectively (a body-fixed transformation of T_{sa})? Now calculate $T_2 = TT_{sa}$. Does T_2 correspond to a body-fixed or world-fixed transformation of T_{sa}?

(f) Use T_{sb} to change the representation of the point $p_b = (1, 2, 3)$ in {b} coordinates to {s} coordinates.

(g) Choose a point p represented by $p_s = (1, 2, 3)$ in {s} coordinates. Calculate $p' = T_{sb}p_s$ and $p'' = T_{sb}^{-1}p_s$. For each operation, should the result be interpreted as changing coordinates (from the {s} frame to {b}) without moving the point p, or as moving the location of the point without changing the reference frame of the representation?

(h) A twist $\mathcal{V}$ is represented in {s} as $\mathcal{V}_s = (3, 2, 1, -1, -2, -3)$. What is its representation $\mathcal{V}_a$ in frame {a}?

(i) By hand, calculate the matrix logarithm $[\mathcal{S}]\theta$ of T_{sa}. (You may verify your answer with software.) Extract the normalized screw axis $\mathcal{S}$ and rotation amount θ. Find the $\{q, \hat{s}, h\}$ representation of the screw axis. Redraw the fixed frame {s} and in it draw $\mathcal{S}$.

(j) Calculate the matrix exponential corresponding to the exponential coordinates of rigid-body motion $\mathcal{S}\theta = (0, 1, 2, 3, 0, 0)$. Draw the corresponding frame relative to {s}, as well as the screw axis $\mathcal{S}$.

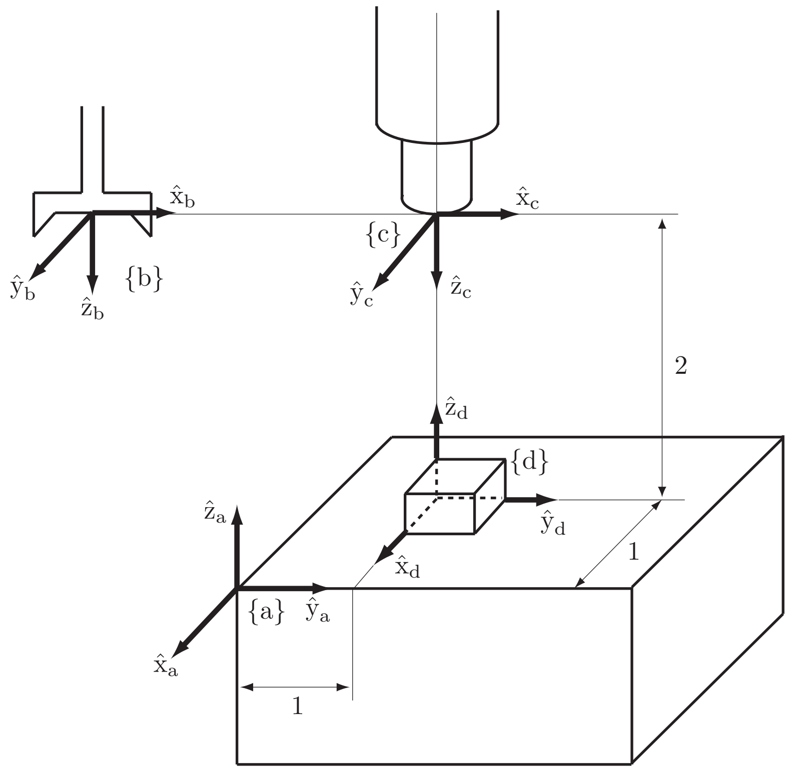


Figure 3.23: Four reference frames defined in a robot's workspace.

Exercise 3.17 Four reference frames are shown in the robot workspace of Figure 3.23: the fixed frame {a}, the end-effector frame effector {b}, the camera frame {c}, and the workpiece frame {d}.

(a) Find T_{ad} and T_{cd} in terms of the dimensions given in the figure.

(b) Find T_{ab} given that

$$T_{bc} = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

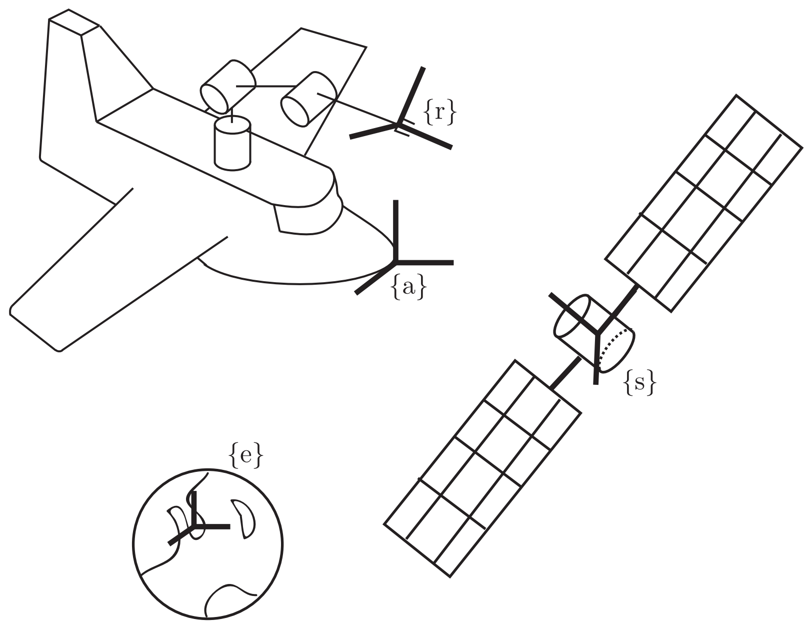


Figure 3.24: A robot arm mounted on a spacecraft.

Exercise 3.18 Consider a robot arm mounted on a spacecraft as shown in Figure 3.24, in which frames are attached to the Earth {e}, a satellite {s}, the spacecraft {a}, and the robot arm {r}, respectively.

(a) Given T_{ea}, T_{ar}, and T_{es}, find T_{rs}.

(b) Suppose that the frame {s} origin as seen from {e} is $(1, 1, 1)$ and that

$$T_{er} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Write down the coordinates of the frame {s} origin as seen from frame {r}.

Exercise 3.19 Two satellites are circling the Earth as shown in Figure 3.25. Frames {1} and {2} are rigidly attached to the satellites in such a way that their $\hat{x}$-axes always point toward the Earth. Satellite 1 moves at a constant speed

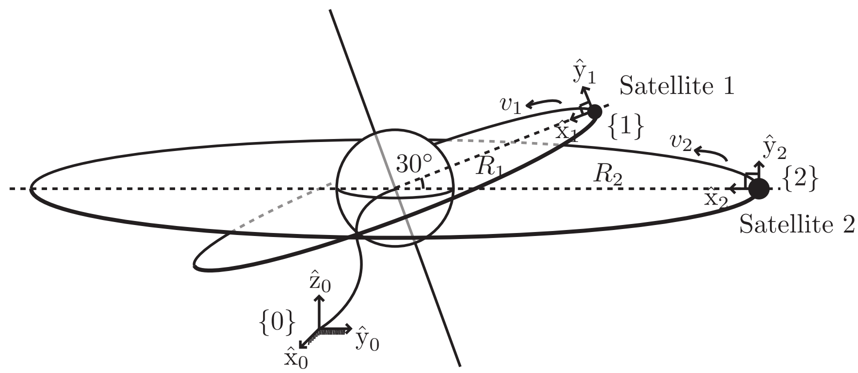


Figure 3.25: Two satellites circling the Earth.

v_1, while satellite 2 moves at a constant speed v_2. To simplify matters, ignore the rotation of the Earth about its own axis. The fixed frame $\{0\}$ is located at the center of the Earth. Figure 3.25 shows the position of the two satellites at $t = 0$.

(a) Derive the frames T_{01}, T_{02} as a function of t.

(b) Using your results from part (a), find T_{21} as a function of t.

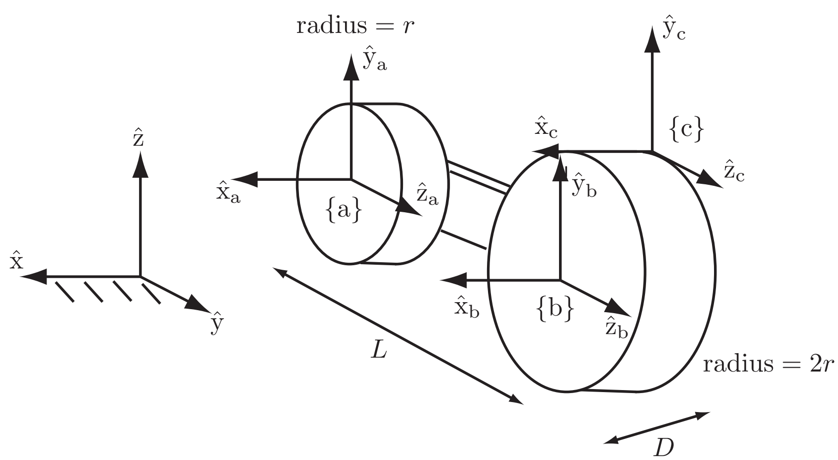


Figure 3.26: A high-wheel bicycle.

Exercise 3.20 Consider the high-wheel bicycle of Figure 3.26, in which the diameter of the front wheel is twice that of the rear wheel. Frames {a} and {b} are attached respectively to the centers of the wheels, and frame {c} is attached to the top of the front wheel. Assuming that the bike moves forward in the $\hat{y}$-direction, find T_{ac} as a function of the front wheel's rotation angle θ (assume $\theta = 0$ at the instant shown in the figure).

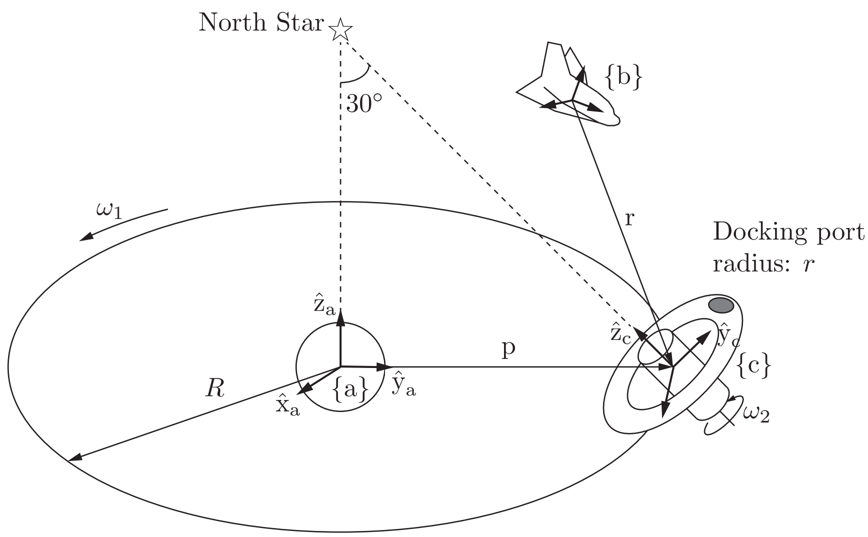


Figure 3.27: A spacecraft and space station.

Exercise 3.21 The space station of Figure 3.27 moves in circular orbit around the Earth, and at the same time rotates about an axis always pointing toward the North Star. Owing to an instrument malfunction, a spacecraft heading toward the space station is unable to locate the docking port. An Earth-based ground station sends the following information to the spacecraft:

$$T_{ab} = \begin{bmatrix} 0 & -1 & 0 & -100 \\ 1 & 0 & 0 & 300 \\ 0 & 0 & 1 & 500 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad p_a = \begin{bmatrix} 0 \\ 800 \\ 0 \end{bmatrix},$$

where p_a is the vector p expressed in {a}-frame coordinates.

(a) From the given information, find r_b, the vector r expressed in {b}-frame coordinates.

(b) Determine T_{bc} at the instant shown in the figure. Assume here that the ŷ- and ẑ-axes of the {a} and {c} frames are coplanar with the docking port.

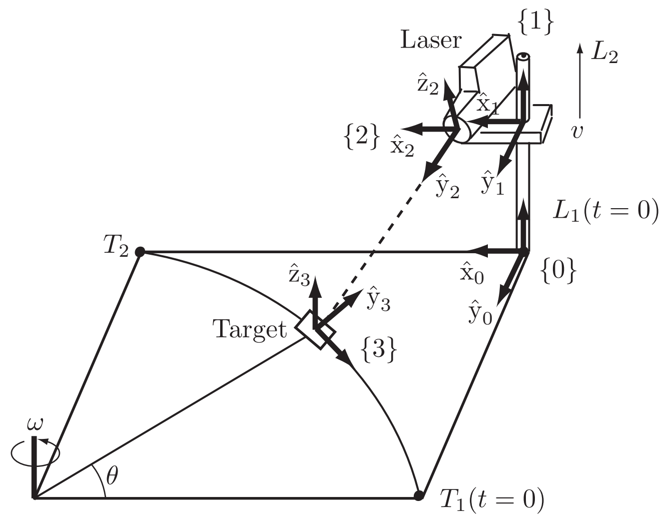


Figure 3.28: A laser tracking a moving target.

Exercise 3.22 A target moves along a circular path at constant angular velocity ω rad/s in the x̂–ŷ-plane, as shown in Figure 3.28. The target is tracked by a laser mounted on a moving platform, rising vertically at constant speed v. Assume that at $t = 0$ the laser and the platform start at L_1, while the target starts at frame T_1.

(a) Derive the frames T_{01}, T_{12}, T_{03} as functions of t.

(b) Using your results from part (a), derive T_{23} as a function of t.

Exercise 3.23 Two toy cars are moving on a round table as shown in Figure 3.29. Car 1 moves at a constant speed v_1 along the circumference of the table, while car 2 moves at a constant speed v_2 along a radius; the positions of the two vehicles at $t = 0$ are shown in the figures.

(a) Find T_{01} and T_{02} as a function of t.

(b) Find T_{12} as a function of t.

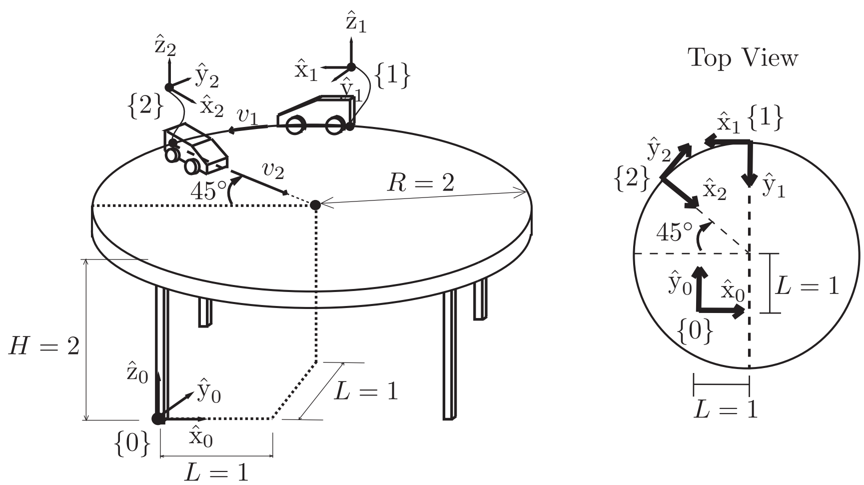


Figure 3.29: Two toy cars on a round table.

Exercise 3.24 Figure 3.30 shows the configuration, at $t = 0$, of a robot arm whose first joint is a screw joint of pitch $h = 2$. The arm's link lengths are $L_1 = 10$, $L_2 = L_3 = 5$, and $L_4 = 3$. Suppose that all joint angular velocities are constant, with values $\omega_1 = \pi/4$, $\omega_2 = \pi/8$, $\omega_3 = -\pi/4$ rad/s. Find $T_{sb}(4) \in SE(3)$, i.e., the configuration of the end-effector frame {b} relative to the fixed frame {s} at time $t = 4$.

Exercise 3.25 A camera is rigidly attached to a robot arm, as shown in Figure 3.31. The transformation $X \in SE(3)$ is constant. The robot arm moves from posture 1 to posture 2. The transformations $A \in SE(3)$ and $B \in SE(3)$ are measured and can be assumed to be known.

(a) Suppose that X and A are given as follows:

$$X = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

What is B?

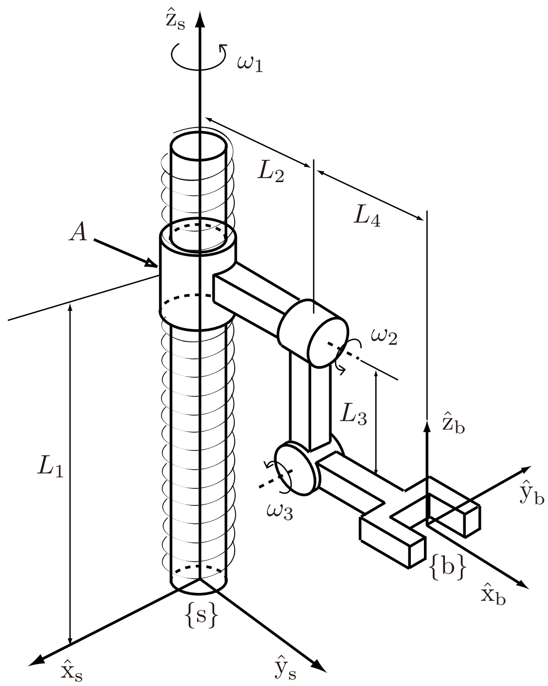


Figure 3.30: A robot arm with a screw joint.

(b) Now suppose that

$$A = \begin{bmatrix} R_A & p_A \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} R_B & p_B \\ 0 & 1 \end{bmatrix}$$

are known and we wish to find

$$X = \begin{bmatrix} R_X & p_X \\ 0 & 1 \end{bmatrix}.$$

Set $R_A = e^{[\alpha]}$ and $R_B = e^{[\beta]}$. What are the conditions on $\alpha \in \mathbb{R}^3$ and $\beta \in \mathbb{R}^3$ for a solution R_X to exist?

(c) Now suppose that we have a set of k equations

$$A_i X = X B_i \quad \text{for } i = 1, \ldots, k.$$

Assume that A_i and B_i are all known. What is the minimum number k for which a unique solution exists?

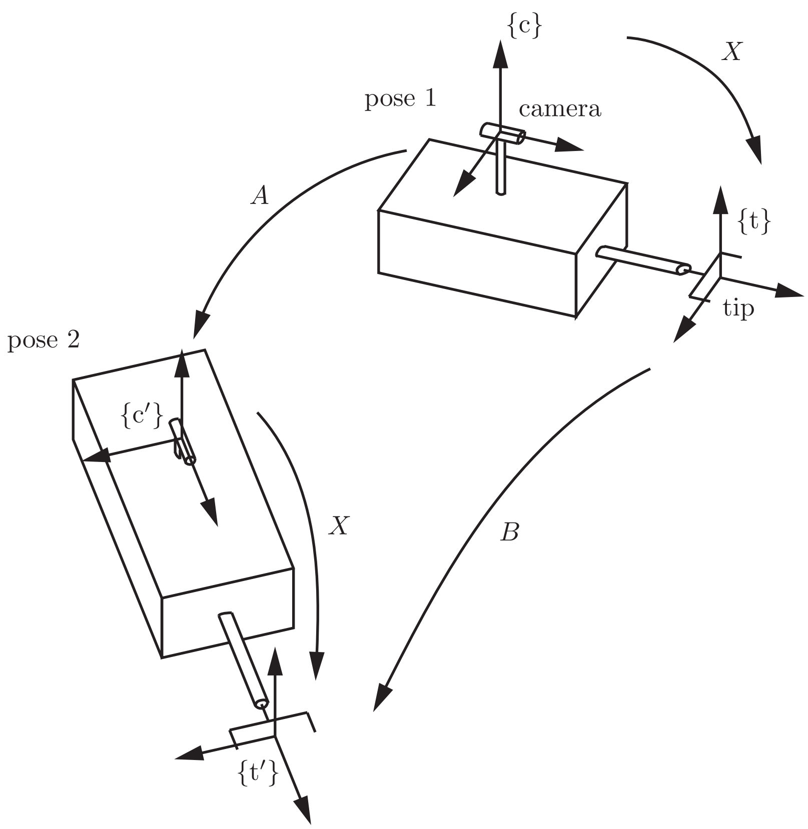


Figure 3.31: A camera rigidly attached to a robot arm.

Exercise 3.26 Draw the screw axis for which $q = (3, 0, 0)$, $\hat{s} = (0, 0, 1)$, and $h = 2$.

Exercise 3.27 Draw the screw axis for the twist $\mathcal{V} = (0, 2, 2, 4, 0, 0)$.

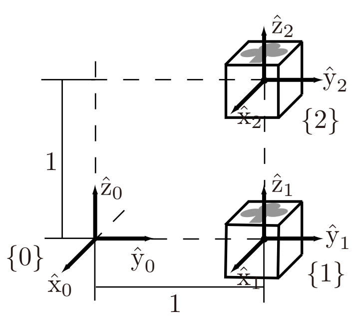
Exercise 3.28 Assume that the space-frame angular velocity is $\omega_s = (1, 2, 3)$

for a moving body with frame {b} at

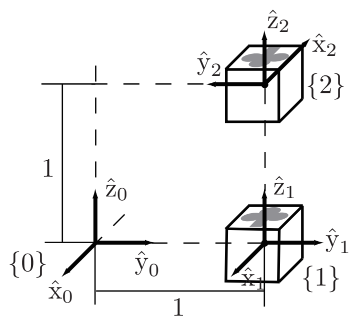
$$R = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

relative to the space frame {s}. Calculate the body's angular velocity ω_b in {b}.

Exercise 3.29 Two frames {a} and {b} are attached to a moving rigid body. Show that the twist of {a} in space-frame coordinates is the same as the twist of {b} in space-frame coordinates.



(a) A first screw motion.



(b) A second screw motion.

Figure 3.32: A cube undergoing two different screw motions.

Exercise 3.30 A cube undergoes two different screw motions from frame {1} to frame {2} as shown in Figure 3.32. In both cases, (a) and (b), the initial configuration of the cube is

$$T_{01} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) For each case, (a) and (b), find the exponential coordinates $\mathcal{S}\theta = (\omega, v)\theta$ such that $T_{02} = e^{[\mathcal{S}]\theta} T_{01}$, where no constraints are placed on ω or v.
(b) Repeat (a), this time with the constraint that $\|\omega\theta\| \in [-\pi, \pi]$.

Exercise 3.31 In Example 3.19 and Figure 3.16, the block that the robot must pick up weighs 1 kg, which means that the robot must provide approximately

10 N of force in the $\hat{z}_e$-direction of the block's frame {e} (which you can assume is at the block's center of mass). Express this force as a wrench $\mathcal{F}_e$ in the {e} frame. Given the transformation matrices in Example 3.19, express this same wrench in the end-effector frame {c} as $\mathcal{F}_c$.

Exercise 3.32 Given two reference frames {a} and {b} in physical space, and a fixed frame {o}, define the distance between frames {a} and {b} as

$$\text{dist}(T_{oa}, T_{ob}) \equiv \sqrt{\theta^2 + ||p_{ab}||^2}$$

where $R_{ab} = e^{[\hat{\omega}]\theta}$. Suppose that the fixed frame is displaced to another frame {o′} and that $T_{o'a} = ST_{oa}$, $T_{o'b} = ST_{o'b}$ for some constant $S = (R_s, p_s) \in SE(3)$.
(a) Evaluate $\text{dist}(T_{o'a}, T_{o'b})$ using the above distance formula.
(b) Under what conditions on S does $\text{dist}(T_{oa}, T_{ob}) = \text{dist}(T_{o'a}, T_{o'b})$?

Exercise 3.33 (a) Find the general solution to the differential equation $\dot{x} = Ax$, where

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}.$$

What happens to the solution $x(t)$ as $t \to \infty$?
(b) Do the same for

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}.$$

What happens to the solution $x(t)$ as $t \to \infty$?

Exercise 3.34 Let $x \in \mathbb{R}^2$, $A \in \mathbb{R}^{2\times 2}$, and consider the linear differential equation $\dot{x}(t) = Ax(t)$. Suppose that

$$x(t) = \begin{bmatrix} e^{-3t} \\ -3e^{-3t} \end{bmatrix}$$

is a solution for the initial condition $x(0) = (1, -3)$, and

$$x(t) = \begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix}$$

is a solution for the initial condition $x(0) = (1, 1)$. Find A and e^{At}.

Exercise 3.35 Given a differential equation of the form $\dot{x} = Ax + f(t)$, where $x \in \mathbb{R}^n$ and $f(t)$ is a given differentiable function of t, show that the general solution can be written

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)} f(s)\, ds.$$

(Hint: Define $z(t) = e^{-At}x(t)$ and evaluate $\dot{z}(t)$.)

Exercise 3.36 Referring to Appendix B, answer the following questions related to ZXZ Euler angles.

(a) Derive a procedure for finding the ZXZ Euler angles of a rotation matrix.

(b) Using the results of (a), find the ZXZ Euler angles for the following rotation matrix:

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Exercise 3.37 Consider a wrist mechanism with two revolute joints θ_1 and θ_2, in which the end-effector frame orientation $R \in SO(3)$ is given by

$$R = e^{[\hat{\omega}_1]\theta_1} e^{[\hat{\omega}_2]\theta_2},$$

with $\hat{\omega}_1 = (0, 0, 1)$ and $\hat{\omega}_2 = (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. Determine whether the following orientation is reachable (that is, find, if it exists, a solution (θ_1, θ_2) for the following R):

$$R = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Exercise 3.38 Show that rotation matrices of the form

$$\begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

can be parametrized using just two parameters θ and ϕ as follows:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \end{bmatrix}.$$

What should the range of values be for θ and ϕ?

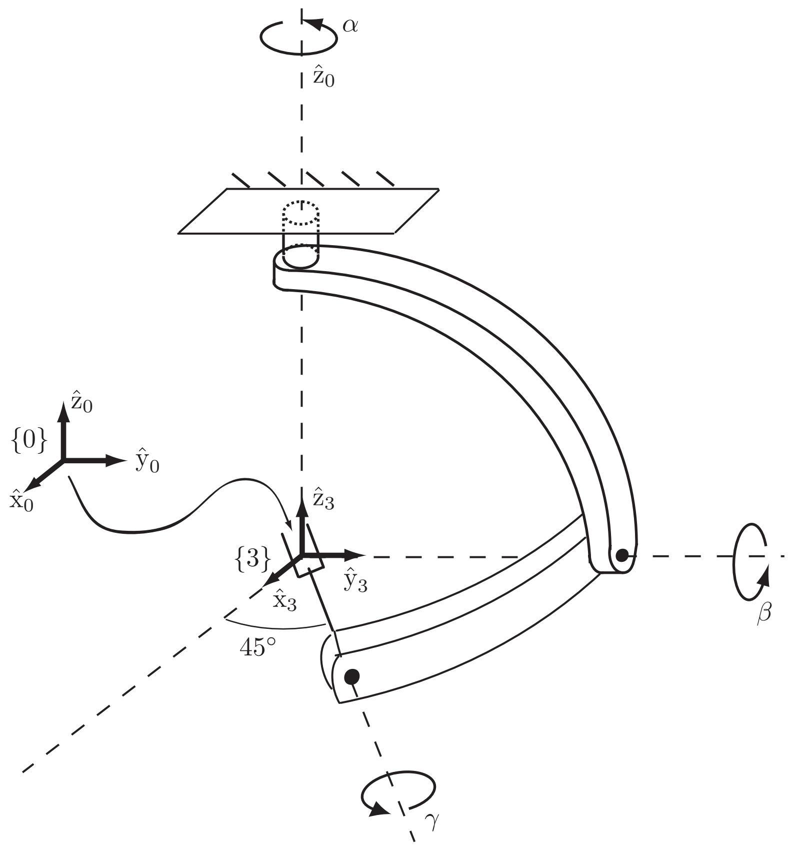


Figure 3.33: A three-degree-of-freedom wrist mechanism.

Exercise 3.39 Figure 3.33 shows a three-dof wrist mechanism in its zero position (i.e., all joint angles are set to zero).

(a) Express the tool-frame orientation $R_{03} = R(\alpha, \beta, \gamma)$ as a product of three rotation matrices.

(b) Find all possible angles (α, β, γ) for the two values of R_{03} given below. If no solution exists, explain why this is so in terms of the analogy between $SO(3)$ and a solid ball of radius π.

(i) $R_{03} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(ii) $R_{03} = e^{[\hat{\omega}]\pi/2}$, where $\hat{\omega} = (0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$.

Exercise 3.40 Refer to Appendix B.

(a) Verify formulas (B.10) and (B.11) for the unit quaternion representation of a rotation $R \in SO(3)$.

(b) Verify formula (B.12) for the rotation matrix R representation of a unit quaternion $q \in S^3$.

(c) Verify the product rule for two unit quaternions. That is, given two unit quaternions $q, p \in S^3$ corresponding respectively to the rotations $R, Q \in SO(3)$, find a formula for the unit quaternion representation of the product $RQ \in SO(3)$.

Exercise 3.41 The Cayley transform of Equation (B.18) in Appendix B can be generalized to higher orders as follows:

$$R = (I - [r])^k (I + [r])^{-k}. \tag{3.99}$$

(a) For the case $k = 2$, show that the rotation R corresponding to r can be computed from the formula

$$R = I - 4\frac{1 - r^{\mathrm{T}}r}{(1 + r^{\mathrm{T}}r)^2}[r] + \frac{8}{(1 + r^{\mathrm{T}}r)^2}[r]^2. \tag{3.100}$$

(b) Conversely, given a rotation matrix R, show that a vector r that satisfies Equation (3.100) can be obtained as

$$r = -\hat{\omega} \tan \frac{\theta}{4}, \tag{3.101}$$

where, as before, $\hat{\omega}$ is the unit vector along the axis of rotation for R, and θ is the corresponding rotation angle. Is this solution unique?

(c) Show that the angular velocity in the body frame obeys the following relation:

$$\dot{r} = \frac{1}{4}\left((1 - r^{\mathrm{T}}r)I + 2[r] + 2rr^{\mathrm{T}}\right)\omega. \tag{3.102}$$

(d) Explain what happens to the singularity at π that exists for the standard Cayley–Rodrigues parameters. Discuss the relative advantages and disadvantages of the modified Cayley–Rodrigues parameters, particularly for order $k = 4$ and higher.

(e) Compare the number of arithmetic operations needed for multiplying two rotation matrices, two unit quaternions, or two Cayley–Rodrigues representations. Which requires the fewest arithmetic operations?

Exercise 3.42 Rewrite the software for Chapter 3 in your favorite programming language.

Exercise 3.43 Write a function that returns "true" if a given 3×3 matrix is within ϵ of being a rotation matrix and "false" otherwise. It is up to you how to define the "distance" between a random 3×3 real matrix and the closest member of $SO(3)$. If the function returns "true," it should also return the "nearest" matrix in $SO(3)$. See, for example, Exercise 3.10.

Exercise 3.44 Write a function that returns "true" if a given 4×4 matrix is within ϵ of an element of $SE(3)$ and "false" otherwise.

Exercise 3.45 Write a function that returns "true" if a given 3×3 matrix is within ϵ of an element of $so(3)$ and "false" otherwise.

Exercise 3.46 Write a function that returns "true" if a given 4×4 matrix is within ϵ of an element of $se(3)$ and "false" otherwise.

Exercise 3.47 The primary purpose of the provided software is to be easy to read and educational, reinforcing the concepts in the book. The code is optimized neither for efficiency nor robustness, nor does it do full error-checking on its inputs.

Familiarize yourself with the whole code in your favorite language by reading the functions and their comments. This should help cement your understanding of the material in this chapter. Then:

(a) Rewrite one function to do full error-checking on its input, and have the function return a recognizable error value if the function is called with an improper input (e.g., an argument to the function is not an element of $SO(3)$, $SE(3)$, $so(3)$, or $se(3)$, as expected).

(b) Rewrite one function to improve its computational efficiency, perhaps by using what you know about properties of rotation or transformation matrices.

(c) Can you reduce the numerical sensitivity of either of the matrix logarithm functions?

Exercise 3.48 Use the provided software to write a program that allows the user to specify an initial configuration of a rigid body by T, a screw axis specified by $\{q, \hat{s}, h\}$ in the fixed frame $\{s\}$, and the total distance traveled along the screw axis, θ. The program should calculate the final configuration $T_1 = e^{[\mathcal{S}]\theta}T$ attained when the rigid body follows the screw $\mathcal{S}$ a distance θ, as well as the intermediate configurations at $\theta/4$, $\theta/2$, and $3\theta/4$. At the initial, intermediate, and final configurations, the program should plot the $\{b\}$ axes of the rigid body. The program should also calculate the screw axis $\mathcal{S}_1$ and the distance θ_1 following $\mathcal{S}_1$ that takes the rigid body from T_1 to the origin and it should plot the screw axis $\mathcal{S}_1$. Test the program with $q = (0, 2, 0)$, $\hat{s} = (0, 0, 1)$, $h = 2$, $\theta = \pi$, and

$$T = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Exercise 3.49 In this chapter, we developed expressions for the matrix exponential for spatial motions mapping elements of $so(3)$ to $SO(3)$ and elements of $se(3)$ to $SE(3)$. Similarly, we developed algorithms for the matrix logarithm going the other direction.

We could also develop matrix exponentials for planar motions, from $so(2)$ to $SO(2)$ and from $se(2)$ to $SE(2)$, as well as the matrix logarithms going from $SO(2)$ to $so(2)$ and $SE(2)$ to $se(2)$. For the $so(2)$ to $SO(2)$ case there is a single exponential coordinate. For the $se(2)$ to $SE(2)$ case there are three exponential coordinates, corresponding to a twist with three elements set to zero, $\mathcal{V} = (0, 0, \omega_z, v_x, v_y, 0)$.

For planar rotations and planar twists we could apply the matrix exponentials and logarithms that we derived for the spatial case by simply expressing the $so(2)$, $SO(2)$, $se(2)$, and $SE(2)$ elements as elements of $so(3)$, $SO(3)$, $se(3)$, and $SE(3)$. Instead, in this problem, write down explicitly the matrix exponential and logarithm for the $so(2)$ to $SO(2)$ case using a single exponential coordinate, and the matrix exponential and logarithm for the $se(2)$ to $SE(2)$ case using three exponential coordinates. Then provide software implementations of each of the four functions in your favorite programming language, and provide execution logs that show that they function as expected.