Chapter

1

Signals, Filters, and Tools

Mixed-signal circuit design requires a fundamental knowledge of signals, signal processing, and circuit design. In this chapter we provide an overview of signals, filtering, and the mathematical tools. The chapter may be a review for the reader; however, we use it to ensure a good foundation to build on in the coming chapters and to provide a quick reference for the mathematical formulas we'll use throughout the book.

1.1 Sinusoidal Signals

Let's take a fundamental look at the sinewave. While there are many ways (equations and formulas) of representing a sinewave, we must remember it is an empirically determined function. Naturally occurring signals, shapes, or constants are determined or described through empirical measurements or observations. For example, π is determined by dividing the circumference of a circle by its diameter

$$\pi = \frac{\text{circumference}}{\text{diameter}} \qquad (1.1)$$

The goal of this section is to provide intuitive discussions that will help create a deeper understanding of what's going on in a circuit or system.

1.1.1 The Pendulum Analogy

Consider the (ideal, that is, lossless) moving pendulum seen in Fig. 1.1a. In this figure the pendulum is moving back and forth between Points 1 and 3 repeatedly over time. As the pendulum leaves Point 1 it starts out slow, gaining maximum speed as it passes Point 2, and finally reaching Point 3. At Point 3 it stops and reverses direction. The time it takes to make this complete journey back to the starting point, Point 1 in this discussion, is the period, T. In Fig. 1.1b we plot the movement of the pendulum along the arched path. We record the position to define a function, $f(t)$, that indicates the pendulum's position at a specific time

$$Position = f(t) = f(t + nT), \text{ where } n \text{ is an integer} \qquad (1.2)$$

This signal, we should all recognize, is a sinusoid or sinewave which repeats its position with a frequency, f_o, of $1/T$.

(a) Ideal (never stops or deviates from
same path) swinging pendulum in motion.
The time it takes to swing from Point 1
to Point 3 and back to Point 1 is the period T.

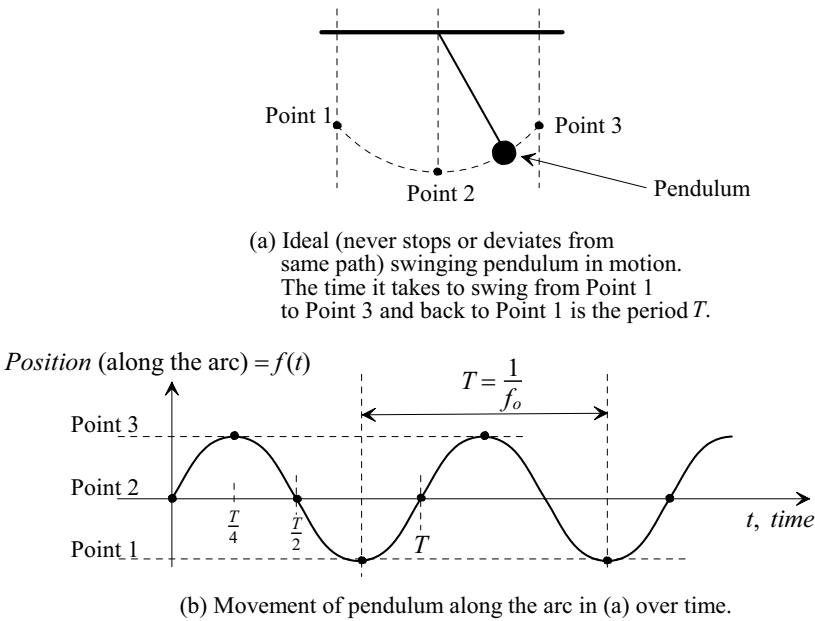(b) Movement of pendulum along the arc in (a) over time.

Figure 1.1 Physical interpretation of a sinewave.

Next, consider the circle seen in Fig. 1.2. One complete rotation around this circle (360 degrees or 2π) is analogous to one complete movement (swing) of our pendulum. We started plotting the pendulum's position at Point 2 in Fig. 1.1b (Point 2, $t = 0$, in Fig. 1.2). After $T/4$ we reach Point 3 in Fig. 1.1b. This corresponds to a 90 degree, or $\pi/2$, movement in our circle. After another $T/4$ seconds we pass back through Point 2. In the circle we've moved 180 degrees. This continues with each swing of the pendulum corresponding to a complete revolution around the circle. Note that we do have some
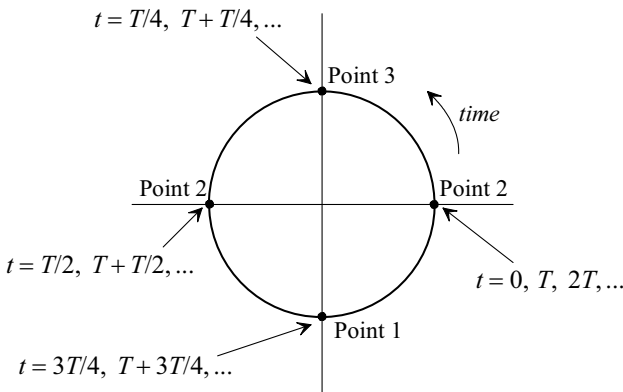

Figure 1.2 Using a circle to describe the movement of the pendulum in Fig. 1.1.

limitations when representing the movement of the pendulum with this circle. For example, what is the amplitude of the sinewave (what is the relative position of the pendulum along the arched path)? We'll address these concerns in a moment. For now let's write, assuming we are using radian angular units,

$$Position = \sin\left(2\pi \cdot \frac{t}{T}\right) = \sin(2\pi f_o \cdot t) \tag{1.3}$$

This function, the sine function, tells us our relative position along the arc (the argument of this function is the angle which relates to the position on the circle in Fig. 1.2). Point 2 corresponds to the function having a value of 0 (and times, $t = 0$, $T/2$, T, $3T/2$, $2T$, ...), Point 3 to a value of $+1$, and Point 1 corresponds to -1. Finally, remember that the values of the sine function in Fig. 1.1.b, and Eq. (1.3), are determined empirically from measured data (e.g., plotting the pendulum's position along the arched path against time).

Describing Amplitude in the x-y Plane

Examine the sinewave in Fig. 1.3a. For the moment we won't concern ourselves with the actual distance the pendulum swings. In Fig. 1.3b we represent the sinewave, at Point i (and Point vi), as a zero length vector along the *x*-axis (the amplitude of the sinewave is 0 at this point in time). As we move towards Point ii in Fig. 1.3a the length of the vector
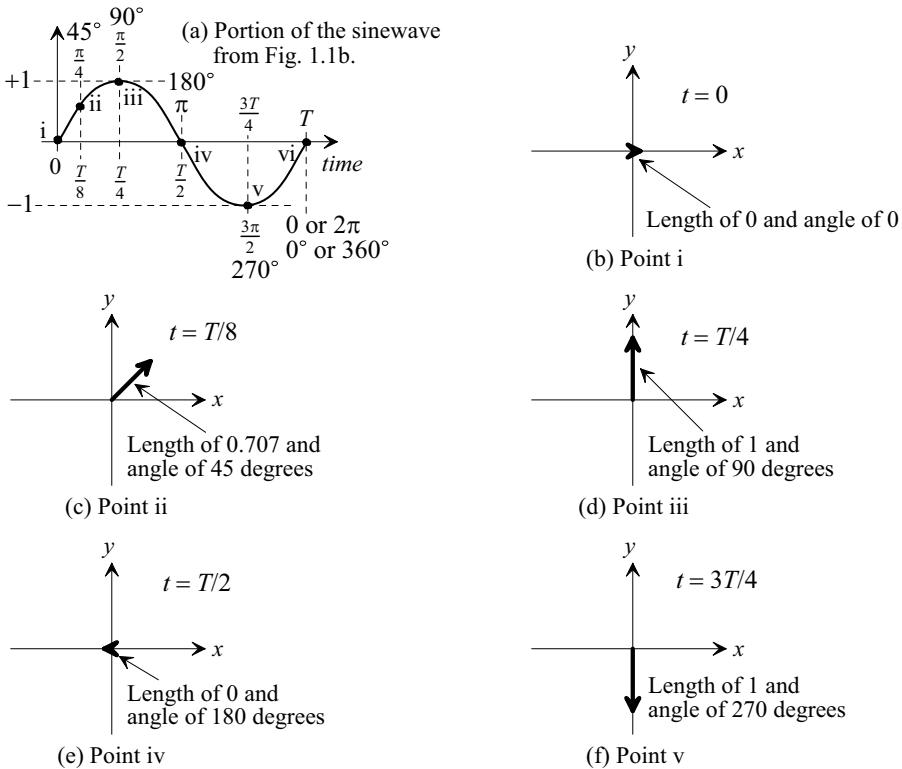

Figure 1.3 A vector swinging around the x-y plane changing both length and angle is used to represent a sinewave.

increases (here indicating an increase in both the x and y directions), Fig. 1.3c. At $T/4$ we are at Point iii in Fig. 1.3a. As seen in Fig. 1.3d the length of the vector is 1 and the angle is 90 degrees. Continuing on towards $T/2$ (Point iv) in Fig. 1.3a the vector is shrinking once again finally reaching a length of 0 and an angle of 180 degrees, Fig. 1.3e. In Fig. 1.3f, Point v, the length is 1 and the angle is 270 degrees. The **key point** here is that a sinewave is represented in the x-y plane by a vector that is changing length and *rotating* around in a circle. Knowing this we can simply represent the sinewave by its peak value and the associated angle between the x-axis and the vector as indicated in Fig. 1.3. For example, as seen in Fig. 1.3d, the peak value of the sinewave occurs when the angle is 90 degrees ($T/4$) so we could write

$$\sin 2\pi f_o \cdot t \rightarrow 1 \angle 90° \rightarrow x = 0, y = 1 \qquad (1.4)$$

knowing the vector representing the sinewave is actually rotating around in the x-y plane with time (and a frequency f_o). Note that we could plot the sinewave more accurately by adding a third axis, time (the z-axis), and showing the corresponding, corkscrew looking, 3-dimensional plot.

In-Phase and Quadrature Signals

Consider moving the sinusoid in Fig. 1.3a a quarter of a cycle (90 degrees, $\pi/2$, or a delay, t_{delay}, of $T/4$) earlier in time, Fig. 1.4a. We write, for this time-shifted signal,

$$\sin \left(\frac{2\pi}{T} \cdot t + \frac{\pi}{2} \right) = \sin \left(\frac{360}{T} \cdot t + 90 \right) = \sin \left(\frac{2\pi}{T} \cdot \left(t + \frac{T}{4} \right) \right) = \sin \left(2\pi f_o \cdot t + \frac{\pi}{2} \right) \qquad (1.5)$$

As the reader probably already knows, this signal function is called cosinusoidal or simply a cosine signal and is described using

$$\sin \left(2\pi f_o \cdot t + \frac{\pi}{2} \right) = \cos \left(2\pi f_o \cdot t \right) \qquad (1.6)$$

We can say that the cosine signal in Fig. 1.4a *leads* the sine signal in Fig. 1.3a by a phase shift, θ, of 90 degrees (= $T/4$ = $\pi/2$). We could also say that the sine signal *lags* the cosine signal by a phase shift of −90 degrees. Note that when we talk about phase shift it's assumed that the frequencies of the two signals are equal. It doesn't make sense to talk about the phase shift between two sinusoids at different frequencies. Finally note that the phase shift is given by

$$\theta = 2\pi \cdot \frac{t_{delay}}{T} = 2\pi f_o \cdot t_{delay} \qquad (1.7)$$

where $t_{delay}/T \times 100\,\%$ is the percentage the delay is of the period.

Figure 1.4b shows the x-y plane plot for a cosine signal. The peak value of the cosine signal occurs when the angle is 0 degrees so we could write

$$\cos 2\pi f_o \cdot t \rightarrow 1 \angle 0° \rightarrow x = 1, y = 0 \qquad (1.8)$$

to represent the signal and not show the rotating vector with time. Note that we can say the cosine signal is *in-phase* (I) since we are representing it above with a zero degree phase shift. We could also say that the sine signal, since it's shifted in time by a quarter cycle, Eq. (1.4), has a *quadrature* (Q) phase shift. Note what happens, looking at the vector representations of the sinusoids in Figs. 1.3b-f and 1.4b, if we create a signal by *adding a sine signal to a cosine signal* or

$$S_{IQ}(t) = \cos 2\pi f_o \cdot t + \sin 2\pi f_o \cdot t \tag{1.9}$$

The resulting signal, when plotted in the *x-y* plane (Fig. 1.5), results in a vector (sinewave) with a length (peak amplitude) of $\sqrt{1^2 + 1^2}$ or $\sqrt{2}$ that simply rotates around in a circle with a frequency of $1/T\,(= f_o)$. In more general terms, Fig. 1.6,

$$S_{IQ}(t) = A_I \cdot \cos 2\pi f_o \cdot t + A_Q \cdot \sin 2\pi f_o \cdot t = \sqrt{A_I^2 + A_Q^2} \cdot \cos\left(2\pi f_o \cdot t + \tan^{-1}\frac{A_Q}{A_I}\right) \tag{1.10}$$

where we define the magnitude and phase as

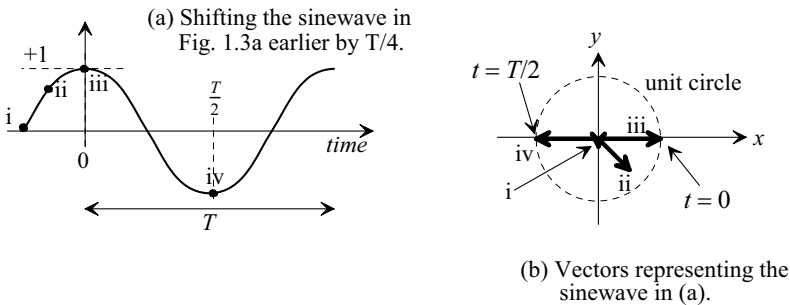$$|S_{IQ}(t)| \angle S_{IQ}(t) = \sqrt{A_I^2 + A_Q^2} \angle \tan^{-1}\frac{A_Q}{A_I} \tag{1.11}$$

(a) Shifting the sinewave in Fig. 1.3a earlier by T/4.

(b) Vectors representing the sinewave in (a).

Figure 1.4 Shifting the sinewave in Fig. 1.3a earlier in time by T/4.

Before going too much further, we should ask why we would want to add *I* and *Q* signals? The answer is that since the *I/Q* sinusoids are at the same frequency, shifted in time by a quarter cycle, we can transmit them with, for example, changes in their respective amplitudes and increase the information sent for a given bandwidth. Note that we can't shift the sinewave seen in Fig. 1.3a by 180 degrees, or *T*/2, and add it to an unshifted sinewave since we would get no signal at all!
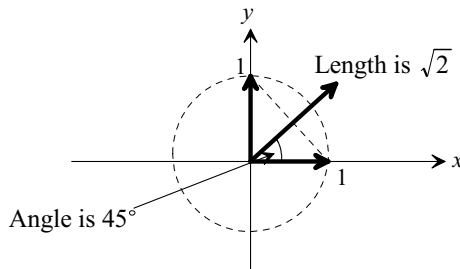

Figure 1.5 Showing how an I/Q signal can be represented in the x-y plane.

Notice that the signals here are very simple. What happens when we start multiplying them together and shifting them in time (changing the phase shift)? We need
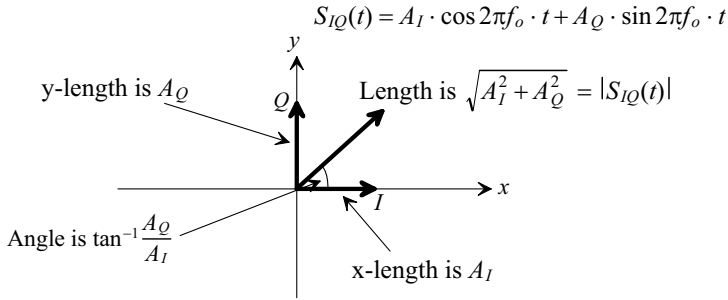
$$S_{IQ}(t) = A_I \cdot \cos 2\pi f_o \cdot t + A_Q \cdot \sin 2\pi f_o \cdot t$$

Figure 1.6 Again, showing how an I/Q signal can be represented in the x-y plane.

to simplify the math! To move towards this goal we'll develop the complex, or *z*-, plane and the frequency-domain representation of signals.

1.1.2 The Complex (*z*-) Plane

Let's attempt (and fail using the *x-y* plane) to simplify our mathematical description of the *IQ* signal given in Eq. (1.10). Recall the following Taylor series expansions

$$e^k = 1 + k + \frac{k^2}{2!} + \frac{k^3}{3!} + \frac{k^4}{4!} + \dots \tag{1.12}$$

$$\cos k = 1 - \frac{k^2}{2!} + \frac{k^4}{4!} - \frac{k^6}{6!} + \frac{k^8}{8!} - \dots \tag{1.13}$$

$$\sin k = k - \frac{k^3}{3!} + \frac{k^5}{5!} - \frac{k^7}{7!} + \frac{k^9}{9!} - \dots \tag{1.14}$$

We can now write

$$\cos k + \sin k = 1 + k - \frac{k^2}{2!} - \frac{k^3}{3!} + \frac{k^4}{4!} + \frac{k^5}{5!} - \frac{k^6}{6!} - \frac{k^7}{7!} + \frac{k^8}{8!} + \frac{k^9}{9!} - \dots \tag{1.15}$$

Comparing Eq. (1.15) to Eq. (1.12) we see that we are close to writing the Taylor's series for e^k. Why is this important? Perhaps the simplest explanation is that if we can represent sinewaves using exponentiation, then multiplying two sinewaves, or shifting a sinewave in time, can be performed using simple addition (of exponents).

The question now is how do we modify things to ensure that all terms are added so that Eq. (1.15) matches Eq. (1.12)? Let's look at the first discrepancy $(-1) \cdot \frac{k^2}{2!}$. The only way to change the polarity of this term is take the square root of -1 and move it inside with k^2. As the reader may know instead of writing $\sqrt{-1}$ for all of these terms we simplify things and write

$$j = \sqrt{-1} \tag{1.16}$$

Numbers using *j* (or *i*) are called *imaginary or complex numbers* (the reason for using the name imaginary will be explained in Ex. 1.1). Imaginary numbers are invaluable for time-shifting and scaling sinusoidal signals. We now rewrite Eq. (1.12) using *j* as

$$e^{jk} = 1 + jk - \frac{k^2}{2!} - j\frac{k^3}{3!} + \frac{k^4}{4!} + j\frac{k^5}{5!} - \frac{k^6}{6!} - j\frac{k^7}{7!} + \dots \tag{1.17}$$

or Euler's formula

$$e^{jk} = \cos k + j \cdot \sin k \qquad (1.18)$$

where the cosine term is a real number and the sine term is the imaginary component of the complex number. Sometimes the following notation is used

$$Re\{e^{jk}\} = \cos k \text{ and } Im\{e^{jk}\} = \sin k \qquad (1.19)$$

Further, with a little algebraic manipulation we can also write

$$\cos k = \frac{e^{jk} + e^{-jk}}{2}, \quad \sin k = \frac{e^{jk} - e^{-jk}}{2j} \qquad (1.20)$$

The next thing we have to discuss is plotting complex numbers. Examine Fig. 1.7. Plotting the real component, x, of a complex number, $x + jy$, follows the same methods we've always used. To plot the imaginary component, we now use the y-axis. Note that the factor j in the complex number simply indicates the imaginary component and shouldn't be included when finding the magnitude of the number, $\sqrt{x^2 + y^2}$ or the phase, θ, of the complex number, $\tan^{-1}\frac{y}{x}$.
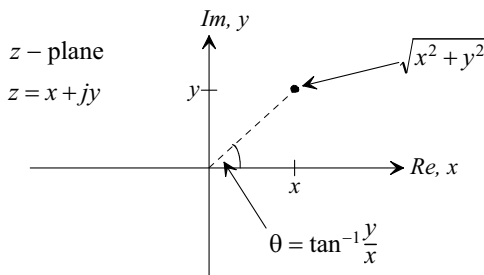

Figure 1.7 The complex plane, plotting the imaginary number

Example 1.1
In the complex plane, plot the signal $e^{j2\pi f_o \cdot t}$. Comment on the resulting plot.

As seen in Fig. 1.8 the magnitude is $|e^{j2\pi f_o \cdot t}| = 1$ and phase shift is $\angle e^{j2\pi f_o \cdot t} = 2\pi f_o \cdot t$. The signal simply swings around and around in the complex
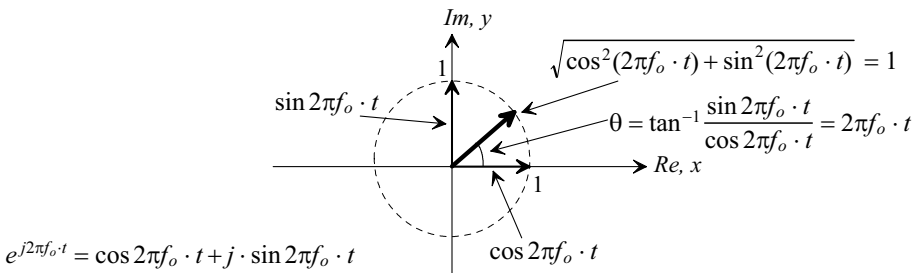

Figure 1.8 Plotting Euler's formula in the z-plane.

plane, on the unit circle, without changing amplitude (both the real and the imaginary components oscillate back and forth between +1 and −1). One complete rotation takes $1/f_o$, or T seconds. We need to pause a moment and ask "If the magnitude of $e^{j2\pi f_o \cdot t}$ is a constant value of 1 isn't it a DC signal?" (answer: no) When a DC signal is represented in the *x-y* plane it doesn't rotate or change amplitude like a sinewave does. (Well, since the frequency of a DC signal, f, is 0, you could say it does rotate around the *x-y* plane just like a sinewave but since $T = \infty$ (=$1/f$) it never leaves the *x*-axis). What this, $e^{j2\pi f_o \cdot t}$, **imaginary signal** can be used for is to introduce or represent delay (phase shift). ∎

To understand this last statement in more detail let's write

$$A \cdot \cos 2\pi f_o \cdot t = Re\{A \cdot e^{j2\pi f_o \cdot t}\} \to A\angle 0 \qquad (1.21)$$

Suppose we want to delay this signal by t_d. We can represent a delay at a particular frequency, f_o, using

$$e^{j2\pi f_o \cdot (-t_d)} \to 1\angle 2\pi f_o \cdot (-t_d) \to 1\angle 2\pi \cdot \frac{-t_d}{T} \qquad (1.22)$$

The delayed cosine signal can be written as

$$A \cdot \cos 2\pi f \cdot (t - t_d) = Re\{A \cdot e^{j2\pi f \cdot t} \cdot e^{j2\pi f \cdot (-t_d)}\} = Re\{A \cdot e^{j2\pi f \cdot (t-t_d)}\} \qquad (1.23)$$

To *simplify the notation* we can drop the real indication, $Re\{\}$. We could also describe the delayed cosine signal in terms of angle notation as

$$A\angle 0 \cdot 1\angle 2\pi \cdot \frac{-t_d}{T} = A \cdot 1\angle\left[0 + 2\pi \cdot \frac{-t_d}{T}\right] = A\angle(-2\pi f_o \cdot t_d) \qquad (1.24)$$

Again noting that when we use angle notation, the *x-y* plane, or the complex plane, we need to know the frequency, f_o, of the input signal since this information isn't present in these representations.

1.2 Comb Filters

A delay can be used to construct a simple, but very useful, filter called a comb filter, Fig. 1.9. Before deriving the equations that characterize this filter let's discuss notation. If our input signal is a sinusoid we can represent it using

$$v_{in}(t) = A \cos 2\pi f \cdot t \qquad (1.25)$$

If we were to keep the frequency fixed, $f = f_o$, and vary time, which we've done up to this point, the output signal of a circuit will simply repeat itself (at the same frequency as the input signal). *What's more useful* is varying the input signal's frequency, f, and looking at the output of the circuit or system (how the amplitude and phase shift vary). To move
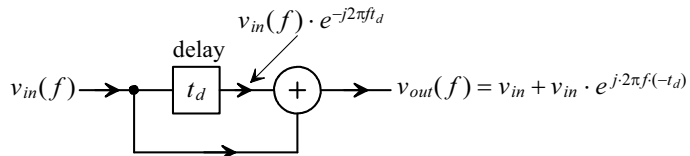

Figure 1.9 A comb filter.

towards this method of characterization let's write the frequency domain description of Eq. (1.25) as $v_{in}(f)$ or simply v_{in}. Representing a signal in the frequency domain will be discussed in the next section.

Returning to our comb filter in Fig. 1.9 we can write

$$v_{out} = v_{in} + v_{in} \cdot e^{j \cdot 2\pi f \cdot (-t_d)} \qquad (1.26)$$

or, using Euler's formula,

$$\frac{v_{out}}{v_{in}} = 1 + e^{j \cdot 2\pi f \cdot (-t_d)} = \overbrace{1 + \cos 2\pi f \cdot (-t_d)}^{\text{Real}} + \overbrace{j \cdot \sin 2\pi f \cdot (-t_d)}^{\text{Imaginary}} \qquad (1.27)$$

The magnitude response of this filter is

$$\left| \frac{v_{out}}{v_{in}} \right| = \sqrt{(1 + \cos 2\pi f \cdot (-t_d))^2 + (\sin 2\pi f \cdot (-t_d))^2} = \sqrt{2(1 + \cos 2\pi f \cdot t_d)} \qquad (1.28)$$

or using

$$1 + \cos x = 2 \cos^2 \frac{x}{2} \qquad (1.29)$$

simply

$$\left| \frac{v_{out}}{v_{in}} \right| = 2 \cdot \left| \cos \pi f \cdot t_d \right| \qquad (1.30)$$

The phase response is given by

$$\angle \frac{v_{out}}{v_{in}} = \tan^{-1} \left[\frac{\sin 2\pi f \cdot (-t_d)}{1 + \cos 2\pi f \cdot (-t_d)} \right] \qquad (1.31)$$

Notice at $f = 1/(2t_d)$ the phase is $\tan^{-1}(0/0)$, which evaluates to ± 90 degrees. Using Eq. (1.29) and

$$\sin x = 2 \sin \frac{x}{2} \cdot \cos \frac{x}{2} \qquad (1.32)$$

the phase response is

$$\angle \frac{v_{out}}{v_{in}} = \pi(-t_d) \cdot f \ \text{ for } \ f < 1/(2t_d) \qquad (1.33)$$

Note that the *phase response is linear* indicating constant delay through the filter, important for distortionless filtering. Figure 1.10 shows the magnitude and frequency response for a comb filter (at this point the reason the filter is called a comb filter should be obvious).

Example 1.2

Figure 1.11 shows one possible implementation of a comb filter. A 50 Ω characteristic impedance co-ax transmission line with an electrical length of 5 ns (delay) is used for the delay element (note how the transmission line is terminated with a 50 ohm resistor and it is assumed that 50 Ω << 5kΩ so that the 5kΩ resistors don't load the output of the transmission line). Determine the comb filter's characteristics (transfer function, v_{out}/v_{in}). Verify your answer with SPICE.
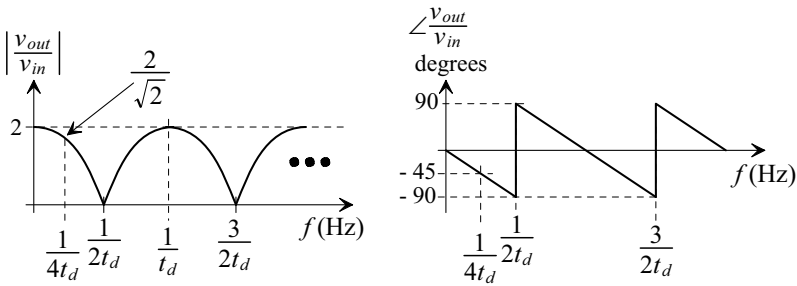
Figure 1.10 Magnitude and phase response of the comb filter in Fig. 1.9.

In order to perform the addition operation seen in Fig. 1.9 we use the two 5k resistors seen in Fig. 1.11. The current through the top resistor must equal the current through the bottom resistor or

$$\frac{v_{in}e^{j2\pi f(-t_d)} - v_{out}}{5k} = \frac{v_{out} - v_{in}}{5k} \qquad (1.34)$$

and thus v_{out} is the average of the signals on the top and bottom of the resistors

$$v_{out} = \frac{v_{in} + v_{in}e^{j2\pi f(-t_d)}}{2} \qquad (1.35)$$

The result is our derivation of the comb filter's magnitude response in Eq. (1.30) is scaled by 2 or

$$\left|\frac{v_{out}}{v_{in}}\right| = |\cos \pi f \cdot t_d| = \left|\cos \pi \frac{f}{200 \text{ MHz}}\right| \qquad (1.36)$$

The phase response is still given by Eq. (1.31). SPICE simulation results showing the filter's frequency response, magnitude and phase, are seen in Fig. 1.12. Note how the filter eliminates from its output, input signals at frequencies that are multiples of $1/2t_d$. This filter can be very useful in communication systems where it's used to isolate, and prevent crosstalk between, transmission channels.

In order to ensure that we can sketch time-domain signals from frequency domain plots consider the case when the input has 1 V peak amplitude and a frequency of 50 MHz. We can see from these plots, and the equations, that the
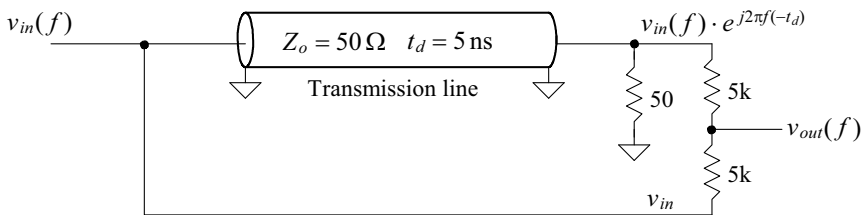

Figure 1.11 Implementation of an analog comb filter.
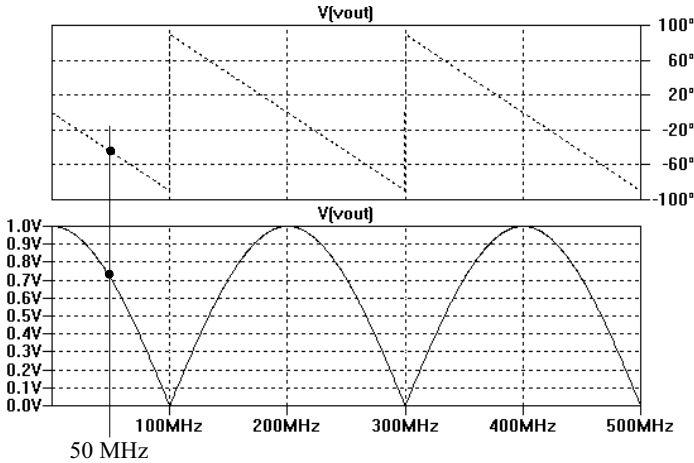
50 MHz

Figure 1.12   Simulating the operation of the comb filter in Fig. 1.11.

output will have an amplitude of 0.707 V and a phase shift of −45 degrees (or the output is lagging the input by 2.5 ns). Time domain simulation results are seen in Fig. 1.13. Note, in this figure, that there is a start-up time, the time it takes the signal to propagate through the transmission line (note the kink at 5 ns), before the filter's behavior follows the equations we derived. ∎
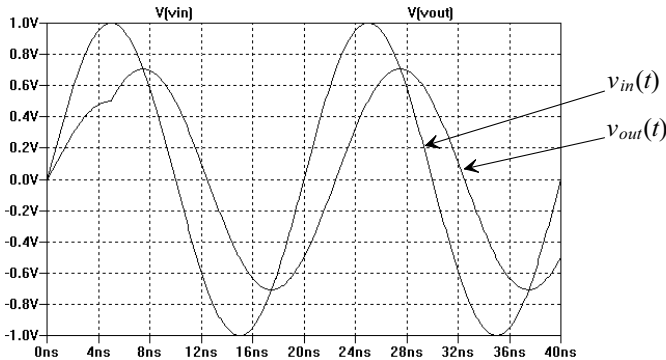

Figure 1.13 Time domain input and output (50 MHz) for the comb filter seen in Fig. 1.11.

1.2.1 The Digital Comb Filter

Notice how, when discussing the comb filter, it was more useful to vary the input signal frequency, f, and look at the output (Fig. 1.12) of the circuit rather than attempting to change the input with time and plot the output. In our comb filter the input signal was delayed by t_d. We represented this time delay in the complex plane using $e^{j2\pi f \cdot (-t_d)}$.

In a digital system we can implement the delay line seen in Figs. 1.9 and 1.11 with a register, Fig. 1.14a. The contents of the register change, in Fig. 1.14a, every T_s seconds
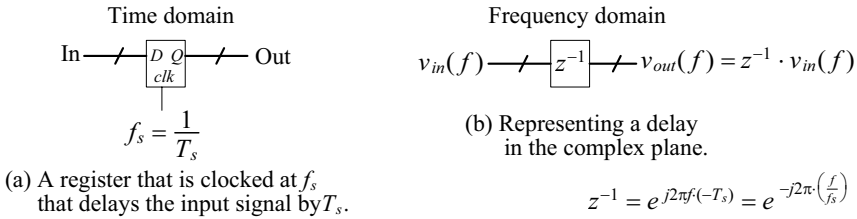
Time domain

In ─/─[D Q / clk]─/─ Out

$$f_s = \frac{1}{T_s}$$

(a) A register that is clocked at f_s that delays the input signal by T_s.

Frequency domain

$$v_{in}(f) \longrightarrow \boxed{z^{-1}} \longrightarrow v_{out}(f) = z^{-1} \cdot v_{in}(f)$$

(b) Representing a delay in the complex plane.

$$z^{-1} = e^{j2\pi f \cdot (-T_s)} = e^{-j2\pi \cdot \left(\frac{f}{f_s}\right)}$$

Figure 1.14 A digital delay.

(so this would correspond to t_d in Fig. 1.11). To simplify the notation, since delays are so common in a digital system that processes signals, we can use

$$z = e^{j2\pi f \cdot T_s} = e^{j2\pi \cdot \frac{f}{f_s}} \text{ or, for a delay, } z^{-1} = e^{j2\pi f \cdot (-T_s)} = e^{-j2\pi \cdot \left(\frac{f}{f_s}\right)} \tag{1.37}$$

This representation for a delay, in the frequency domain, is seen in Fig. 1.14b (**some authors** use Z^{-1} to indicate a delay, $e^{j2\pi f \cdot (-T_s)}$, and to differentiate between the general situation where $z = x + jy$). Figure 1.15 shows how Eq. (1.37) is plotted in the z-plane (the dashed circle). The magnitude of $e^{j2\pi f \cdot T_s}$ is one and its phase shift is $2\pi f \cdot T_s$ or $2\pi \cdot \frac{f}{f_s}$ (plotting $e^{j2\pi f \cdot T_s}$ in the z-plane simply plots the unit circle). Note that plotting a delay, z^{-1} or $e^{j2\pi f \cdot (-T_s)}$, also results in a magnitude of one but the phase shift is now $-2\pi f \cdot T_s$. *As we move through the book we will regularly use Eq.* (1.37) *to evaluate the frequency response of a discrete-time system, see Fig.* 1.21 *and the associated discussion, and to relate the complex number, z to the frequency, f.*

$f = f_s/4,\ 5f_s/4,\ 9f_s/4,\ ...$

$z = x + jy$

Im, y

Plotting $e^{j2\pi\frac{f}{f_s}} = \cos 2\pi\frac{f}{f_s} + j \cdot \sin 2\pi\frac{f}{f_s}$

$f = f_s/2,\ 3f_s/2,\ 5f_s/2,\ ...$

Re, x

$f = 0,\ f_s,\ 2f_s,\ 3f_s,\ ...$

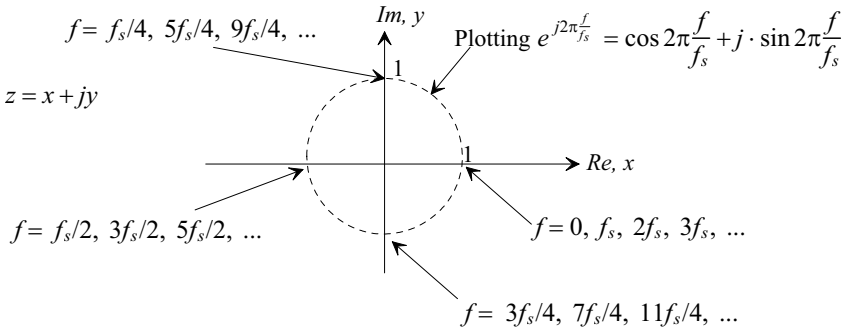$f = 3f_s/4,\ 7f_s/4,\ 11f_s/4,\ ...$

Figure 1.15 Plotting a delay in the complex plane.

Figure 1.16 shows the digital implementation of the comb filter. We can write

$$v_{out} = v_{in}(1 + z^{-1}) \tag{1.38}$$

and thus the transfer function is

$$\frac{v_{out}}{v_{in}} = 1 + z^{-1} = \frac{z+1}{z} \tag{1.39}$$

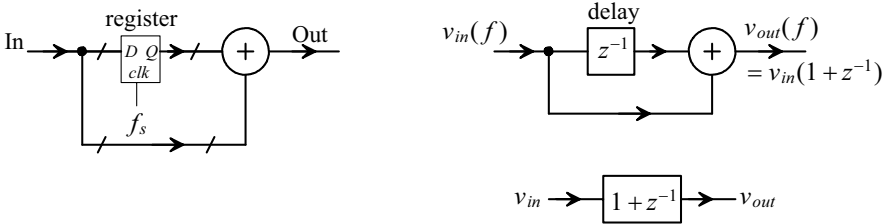or, after reviewing Eqs. (1.27) to (1.33), we get

Figure 1.16 A digital comb filter using one delay. Note how the
output is the average (sum) of consecutive inputs.

$$\left|\frac{v_{out}}{v_{in}}\right| = 2 \cdot \left|\cos \pi \frac{f}{f_s}\right| \text{ and } \angle\frac{v_{out}}{v_{in}} = -\pi \cdot \frac{f}{f_s} \text{ for } f < \frac{f_s}{2} \qquad (1.40)$$

The magnitude and phase response for this digital comb filter is seen in Fig. 1.17. Note that if we apply a constant (DC, $f = 0$) value of 1 to the input of our digital filter, the output goes to 2 verifying that what we get with Eq. (1.40), at least at DC, matches what we see in Fig. 1.17.
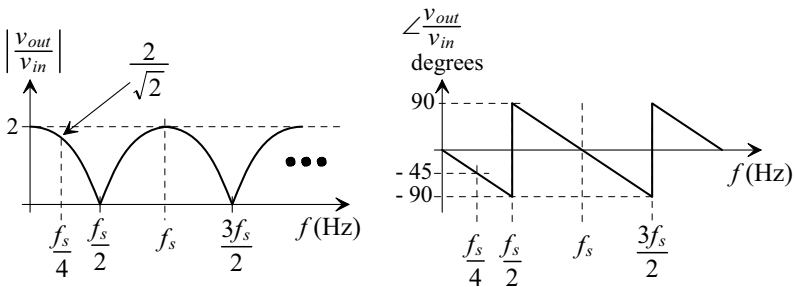

Figure 1.17 Magnitude and phase response of the comb filter in Fig. 1.16.

Note that if we divide the output of the digital comb filter in Fig. 1.16 by two we can think of it as an *averaging filter* (taking two inputs, summing them, and then dividing by two). Let's pause and think about this for a moment. If the input is 8-bits then the output, which is the sum of two 8-bit words, is a 9-bit word. To avoid lowering the resolution of the filter's output (the number of bits in the filter's output word) we can keep all 9-bits and not perform the divide by two (but still call the filter an "averaging filter"...or comb filter). This is **important** because we will regularly perform averaging (which is simply lowpass filtering a signal) to increase the signal's resolution (so the word size should increase!) Note that our definition of averaging here is simply adding a number of input words and outputting the result.

Finally, to show how it's possible for the output of the comb filter in Fig. 1.16 to go to zero, consider averaging two points of an input signal, at a frequency of $f_s/2$, spaced apart by T_s, Fig. 1.18. It may be a good exercise, at this point, to vary the input signal frequency and plot the corresponding outputs to verify the results seen in Fig. 1.17.
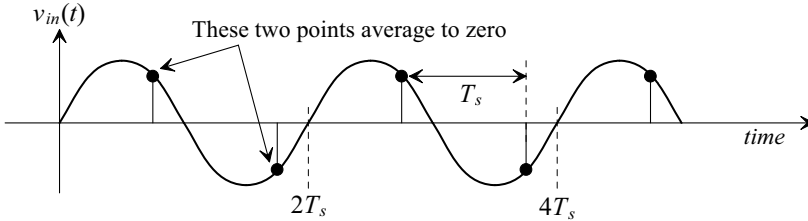
Figure 1.18 Showing how the averaging filter in Fig. 1.16 can have zero output at $f_s/2$.

1.2.2 The Digital Differentiator

The digital comb filter seen in Fig. 1.16 could also be called an averaging filter since its output is the sum of adjacent inputs. The gain of this filter at DC was, as seen in Fig. 1.17, 2. We can also construct a comb filter using a digital *differentiator* or *differencer*, Fig. 1.19. This filter outputs the difference between adjacent input signals.
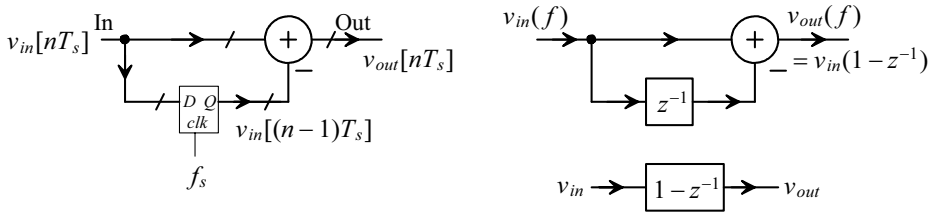

Figure 1.19 A digital comb filter using a digital differentiator.

In the time domain, with an input signal of $v_{in}(t)$ sampled at discrete times nT_s, we can write the output of the digital differentiator filter as

$$v_{out}[nT_s] = v_{in}[nT_s] - v_{in}[(n-1)T_s] \tag{1.41}$$

or, in the frequency domain,

$$v_{out}(f) = v_{in}(f) \cdot (1 - z^{-1}) \tag{1.42}$$

and so the transfer function of the filter is

$$\frac{v_{out}}{v_{in}} = 1 - z^{-1} = \frac{z-1}{z} \tag{1.43}$$

The magnitude response of the digital differentiator is

$$\left| \frac{v_{out}}{v_{in}} \right| = \sqrt{2\left(1 - \cos 2\pi \frac{f}{f_s}\right)} \tag{1.44}$$

or using

$$1 - \cos x = 2\sin^2 \frac{x}{2} \tag{1.45}$$

we get

$$\left| \frac{v_{out}}{v_{in}} \right| = 2\left| \sin \pi \frac{f}{f_s} \right| \tag{1.46}$$
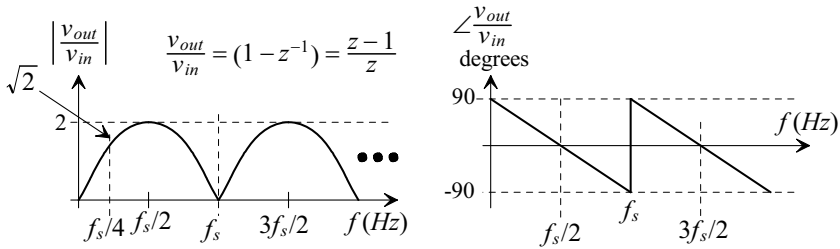
Figure 1.20 Magnitude and phase response of the digital differentiator (also a comb filter).

The phase response is

$$\angle \frac{v_{out}}{v_{in}} = \frac{\pi}{2} - \pi \frac{f}{f_s} \text{ for } 0 < f < f_s \tag{1.47}$$

Figure 1.20 shows the magnitude and phase responses for the digital differentiator in Fig. 1.19. Notice how, for this comb filter, the gain at DC is 0.

1.2.3 An Intuitive Discussion of the *z*-Plane

It will be very helpful in our discussion of mixed-signal circuits and systems to gain an intuitive feel for the frequency response of a discrete-time system, like the digital filters we've discussed in this section (or the switched-capacitor filters in Ch. 3 that are discrete-time, continuous-amplitude, circuits), by looking at the *z*-plane representation of the system. To move toward this goal, consider the transfer function of the simple digital averager seen in Fig. 1.16 with a transfer function of

$$H(f) = \frac{v_{out}(f)}{v_{in}(f)} = H(z) = 1 + z^{-1} = \frac{z+1}{z} \tag{1.48}$$

It is very useful, for an intuitive understanding of the frequency response of a discrete-time system, to plot the poles and zeroes of the transfer function in the *z*-plane. Figure
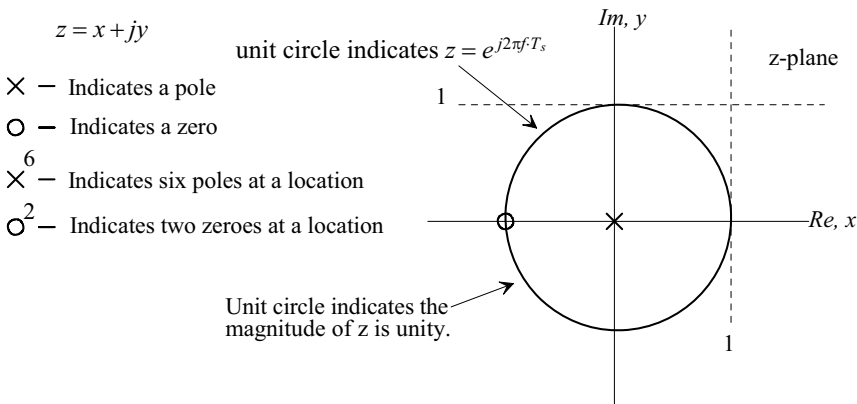

Figure 1.21 The z-plane.

1.21 also shows how Eq. (1.48) can be displayed on the z-plane. A pole is located at $z = 0$ (at the location the denominator goes to zero and the transfer function goes to infinity) and a zero is located at $z = -1$ (at the location where the numerator goes to zero).

The z-plane is usually used to describe the frequency response of a discrete time system, $H(f)$, by assuming the input to the system is a unit magnitude sinusoid with varying frequency, f. This input, $1 \cdot e^{j2\pi\frac{f}{f_s}}$, evaluates the output of the system or

$$H(f) = v_{out}(f) \text{ when } v_{in}(f) = \overbrace{1}^{\text{magnitude}} \cdot \overbrace{e^{j2\pi\frac{f}{f_s}}}^{\text{phase}} \tag{1.49}$$

We should now see that the unit circle, shown in Fig. 1.21, indicates the relationship between z and f when specified by Eq. (1.49) (*this is important*). Therefore, to determine $H(f)$ from a plot of a transfer function's poles and zeroes in the z-plane, we simply evaluate $H(z)$ along the unit circle. To show how this transfer function evaluation is performed, consider Eq. (1.48) and the corresponding plot of its pole and zero shown in Fig. 1.22 along with the magnitude of Eq. (1.48) or Eq. (1.40) plotted against frequency in Fig. 1.17. At DC ($f = 0$ and $z = 1 \cdot e^0 = 1\angle 0$) point A in Fig. 1.22, the gain of the circuit is two and is calculated using

$$|H(f)| = \frac{\text{distance to zero}}{\text{distance to pole}} \tag{1.50}$$

The distance from the zero to point A is 2 while the distance between the pole to point A is 1. Therefore, as shown in Fig. 1.17, the magnitude of $H(f)$ is 2. The phase of the transfer function is calculated along the positive x-axis using

$$\angle H(f) = \angle \text{ of zero} - \angle \text{ of pole} \tag{1.51}$$

which, as seen in Fig. 1.17, results in a phase angle of zero. Next consider evaluating $H(z)$ at $f_s/4$ $\left(f = f_s/4 \text{ and } z = 1 \cdot e^{j\frac{\pi}{2}} = 1\angle 90 \right)$, point B in Fig. 1.22. The distance from the pole to point B is 1 while the distance from the zero is $\sqrt{2}$ resulting in a magnitude $\sqrt{2}$. The
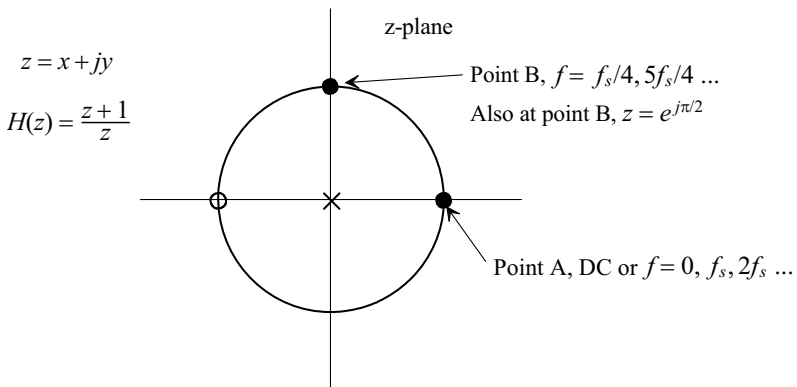

$z = x + jy$

$H(z) = \dfrac{z+1}{z}$

z-plane

Point B, $f = f_s/4, 5f_s/4 \ldots$

Also at point B, $z = e^{j\pi/2}$

Point A, DC or $f = 0, f_s, 2f_s \ldots$

Figure 1.22   The z-plane pole and zero for Eq. (1.48).

angle from the pole along the x-axis to point B is 90°, while the angle from the zero is 45° resulting in an overall phase response of −45° (verify with Fig. 1.17).

Note that (1) any digital filter's or discrete-time system's frequency response is periodic with period f_s (one complete revolution around the unit circle), (2) we normally are only concerned with evaluating $H(z)$ over the top half of the unit circle (from DC to $f_s/2$ [the Nyquist frequency, f_n]), and (3) a pole at the origin has no effect on the magnitude response of $H(z)$ but does affect the phase response (z^{-1}, a pole at the origin, is a delay of one clock cycle). Finally note that the number of poles in $H(z)$ must be greater than or equal to the number of zeroes if the digital filter/system is to be realizable in hardware (the output of the system cannot occur before the system's input).

Example 1.3

Determine, using the graphical approach just discussed, the magnitude and phase of the transfer function indicated by the poles and zeroes plotted in Fig. 1.23 at a frequency of $f_s/4$.
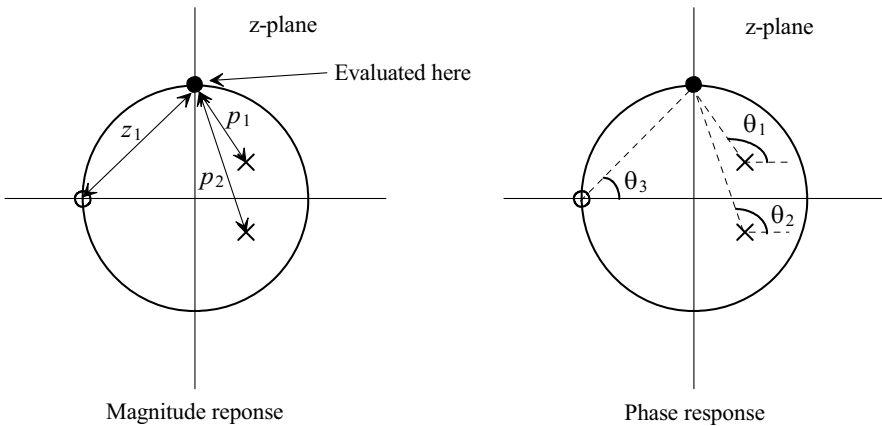

Figure 1.23 The z-plane pole and zero plot for Ex. 1.3.

If we label the length from a pole (zero) to the evaluation point p (z), then the magnitude of the transfer function is given by

$$|H(z)| = \frac{z_1}{p_1 \cdot p_2}$$

Labeling the angles for the poles and zeroes as indicated in the figure, we can write the phase response as

$$\angle H(z) = \theta_3 - \theta_2 - \theta_1 \blacksquare$$

1.2.4 Comb Filters with Multiple Delay Elements

Examine the digital filter seen in Fig. 1.24. The transfer function for this filter is given by

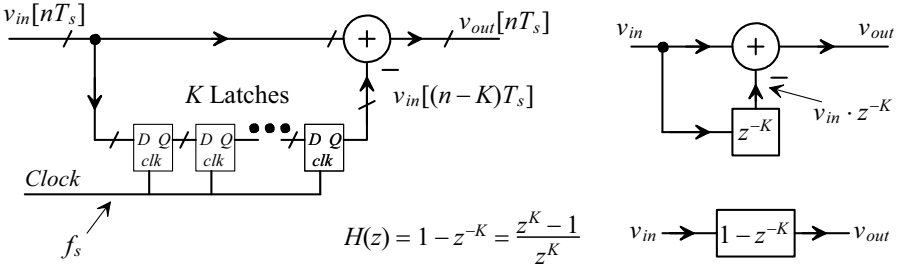$$H(z) = 1 - z^{-K} = \frac{z^K - 1}{z^K} \qquad (1.52)$$

Figure 1.24 A digital comb filter using multiple delay elements.

The magnitude response for this filter is (see Eq. [1.46])

$$\left|\frac{v_{out}}{v_{in}}\right| = 2\left|\sin K\pi \cdot \frac{f}{f_s}\right| \tag{1.53}$$

and the phase response is

$$\angle\frac{v_{out}}{v_{in}} = \frac{\pi}{2} - K\pi \cdot \frac{f}{f_s} \text{ for } 0 < f < f_s/K \tag{1.54}$$

Figure 1.25 shows the magnitude responses and pole-zero plots for comb filters with varying numbers of delay elements, K. For $K = 4$, for example, the zeroes are located at: $1, j, -1,$ and $-j$. In other words the transfer function goes to zero at, DC, $f_s/4, f_s/2, 3f_s/4, f_s,$
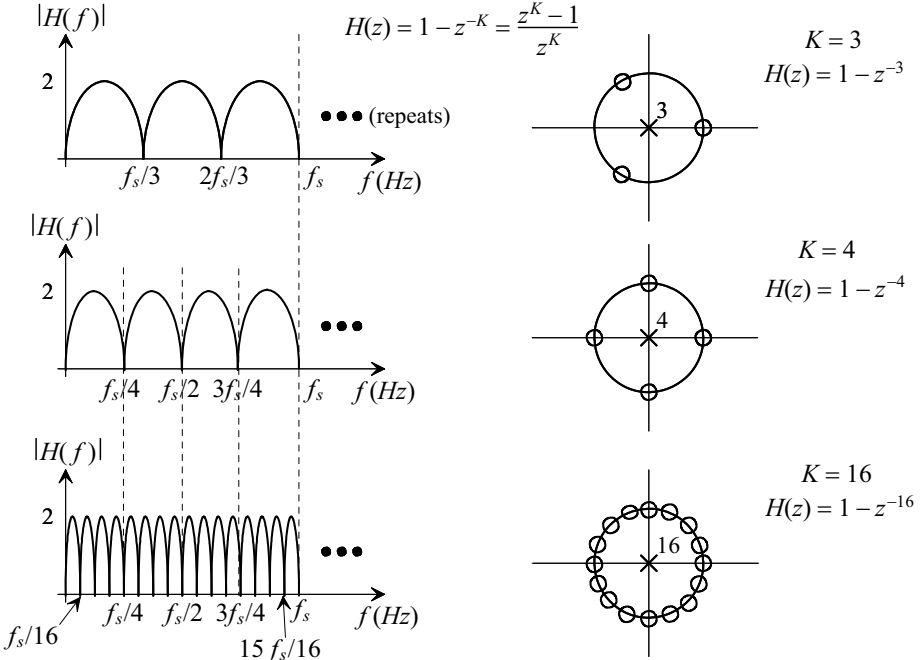

Figure 1.25 Frequency response and z-plane plots for various values of K in a comb filter.

$f_s + f_s/4$, etc.). For $K = 3$ the zeroes are located at: 1, $-\frac{1}{2} + j \cdot \frac{\sqrt{3}}{2}$, and $-\frac{1}{2} - j \cdot \frac{\sqrt{3}}{2}$. Because of their simplicity, comb filters are very useful in mixed-signal design (and used to design lowpass, bandpass, and highpass filters that will be discussed in Ch. 4).

1.2.5 The Digital Integrator

Examine the circuit seen in Fig. 1.26. The output of the circuit, in the time domain, is given by

$$v_{out}[nT_s] = v_{in}[nT_s] + v_{out}[(n-1)T_s] \tag{1.55}$$

or, in the frequency domain,

$$H(z) = H(f) = \frac{v_{out}}{v_{in}} = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1} \tag{1.56}$$

This circuit adds the current input to the sum of the past inputs, performing integration. Note that there isn't a delay in series with the input signal so this circuit is often called a *non-delaying integrator* or *accumulator*. We'll see in a moment that a *delaying integrator* does delay the input signal so it has a transfer function of $z^{-1}/(1 - z^{-1})$.
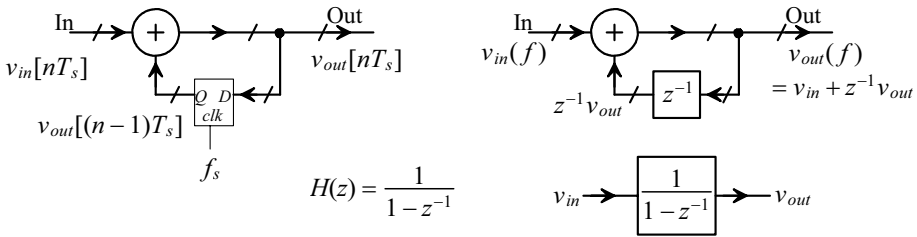

Figure 1.26 The non-delaying digital integrator.

Figure 1.27 shows the *z*-plane representation of the non-delaying integrator along with its magnitude and phase responses. To determine equations for the magnitude and phase responses we can write

$$H(z) = \frac{z}{z - 1} \rightarrow \overbrace{e^{j2\pi\frac{f}{fs}}}^{\text{Phase shift}} \cdot \frac{1}{e^{j2\pi\frac{f}{fs}} - 1} = e^{j2\pi\frac{f}{fs}} \cdot \frac{1}{(-1 + \cos 2\pi\frac{f}{fs}) + j\sin 2\pi\frac{f}{fs}} \tag{1.57}$$

noting $\left| e^{j2\pi\frac{f}{fs}} \right| = 1$ and $\angle e^{j2\pi\frac{f}{fs}} = 2\pi\frac{f}{fs}$. This certainly isn't in the form, $x + jy$, that we are used to plotting, Fig. 1.7, for magnitude and phase responses. Rather it's in the form

$$\frac{1}{a + jb} \tag{1.58}$$

To get Eq. (1.57) into the form $x + jy$ we can multiply it by its complex conjugate or, using Eq. (1.58) to simplify the notation,
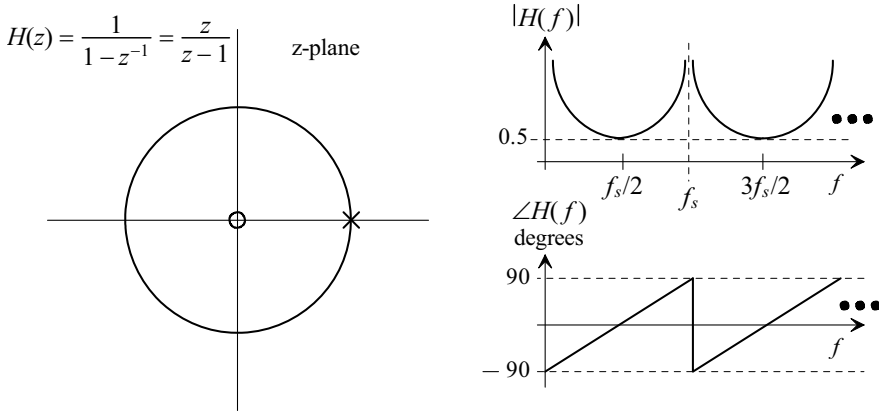
$$H(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}\bigg|$$ z-plane

Figure 1.27 The z-plane representation along with magnitude and phase response for a non-delaying digital integrator.

$$\frac{1}{a+jb} \cdot \underbrace{\frac{a-jb}{a-jb}}_{=1} = \underbrace{\frac{a}{a^2+b^2}}_{\text{Real, }x} + j \underbrace{\frac{-b}{a^2+b^2}}_{\text{Imaginary, }y} \tag{1.59}$$

which results, after reviewing Fig. 1.7, in

$$\left|\frac{1}{a+jb}\right| = \frac{1}{\sqrt{a^2+b^2}} \text{ and } \angle\frac{1}{a+jb} = -\tan^{-1}\frac{b}{a} \tag{1.60}$$

We can now write

$$|H(f)| = \frac{1}{\sqrt{\left(-1+\cos 2\pi\frac{f}{f_s}\right)^2 + \left(\sin 2\pi\frac{f}{f_s}\right)^2}} = \frac{1}{\sqrt{2(1-\cos 2\pi\frac{f}{f_s})}} \tag{1.61}$$

or

$$|H(f)| = \frac{1}{2\left|\sin \pi\frac{f}{f_s}\right|} \tag{1.62}$$

Evaluating the phase response directly from the *z*-plane plot,

$$\angle H(f) = \overbrace{2\pi\frac{f}{f_s}}^{\text{From zero}} - \overbrace{\left(\pi\frac{f}{f_s} + \frac{\pi}{2}\right)}^{\text{From pole}} = 180\frac{f}{f_s} - 90 \text{ (degrees) for } 0 < f < f_s \tag{1.63}$$

At DC the phase contribution from the zero is 0°, while the phase contribution from the pole, at a frequency just above DC, is 90°. The result is an overall phase response of −90°. At $f_s/4$ the phase contribution from the zero is 90°, while the phase contribution from the pole is 135°, resulting in an overall phase response of −45°.

The Delaying Integrator

Figure 1.28 shows the delaying integrator. We get this topology by moving the delay from the feedback path in Fig. 1.26 to the forward path. The output is related to the input using
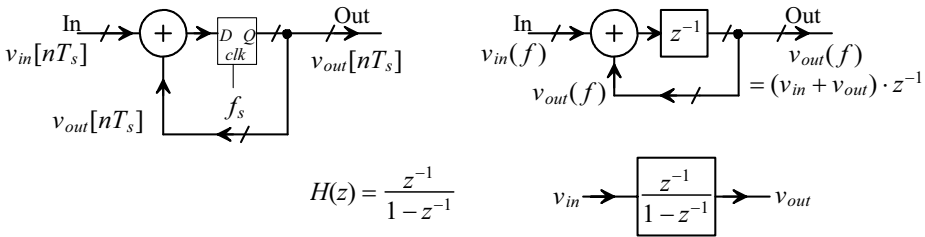
Figure 1.28 The delaying digital integrator.

$$v_{out}[nT_s] = v_{in}[(n-1)T_s] + v_{out}[(n-1)T_s] \qquad (1.64)$$

or

$$H(z) = H(f) = \frac{v_{out}}{v_{in}} = \frac{z^{-1}}{1-z^{-1}} = \frac{1}{z-1} \qquad (1.65)$$

The magnitude response is exactly the same as the non-delaying integrator, Fig. 1.27 and Eq. (1.62). The phase response is given by

$$\angle H(f) = -\pi \frac{f}{f_s} - \frac{\pi}{2} = -180 \frac{f}{f_s} - 90 \text{ for } 0 < f < f_s \qquad (1.66)$$

While the equations describing the integrators in this section were derived using digital implementations, we'll see in the next chapter that they also apply to discrete analog integrators (DAIs).

An Important Note

When a digital signal is generated from an analog signal using an ADC (analog-to-digital converter) signal aliasing can occur. Signal aliasing comes from the sampling process and is discussed in detail in the next chapter. The importance here in this section is that if we use the output of an ADC, clocked at f_s, to provide the input signal to a digital filter we have to restrict ourselves to input frequencies below the Nyquist frequency, $f_s/2$, in order to avoid aliasing. Simulation examples using the filters in this section are available at CMOSedu.com.

1.3 Representing Signals

In the past two sections we've talked about representing sinusoids using the x-y plane with fixed frequency and changing time, Figs. 1.3 and 1.4, and then using the complex, or z-plane, Secs. 1.1.2 and 1.2.3 with variable time and fixed frequency, Fig. 1.8, and then with varying frequency, Fig. 1.22. In this section we discuss representing signals, periodic and non-periodic, that are not single tone (single frequency) sinusoids. Any periodic function can be represented by a sum of sinusoids with varying amplitudes and frequencies (but all sinusoid's frequencies are integer multiples, n, of the original periodic signal we are modeling with the sum of sinewaves). In the first section we discuss the exponential Fourier series representation of a periodic signal. Next we present the Fourier transform which is used to represent non-periodic signals in the frequency domain. It's assumed this material is a review for the reader. Our presentation is focused on making the material useful for reference later in the book.

1.3.1 Exponential Fourier Series

The exponential Fourier series representation of a periodic function with period $T (= 1/f)$ can be expressed using

$$g(t) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{j2\pi nft} \tag{1.67}$$

where, as just mentioned, $n \cdot f$ represents the integer multiples of the original signal's frequency f. The weighting of the sinusoids, c_n, is calculated using

$$c_n = \frac{1}{T} \int_{t}^{t+T} g(t) \cdot e^{-j2\pi nft} dt \tag{1.68}$$

The key thing to note is that we use the Fourier Series to represent signals that are periodic. Examples include the outputs of the sample-and-hold and track-and-hold discussed in the next chapter. Using the Fourier Transform to look at the spectrums of periodic signals can be very challenging for all but the simplest waveforms.

As an example of the use of the Fourier Series consider the squarewave seen in Fig. 1.29. The amplitude and period of this waveform are A and T_s respectively. The time the output pulse is high is T_p. When $T_p = T_s/2$ the squarewave has a 50% duty cycle. Using Eq. (1.68) to calculate the coefficients, with $T_s = 1/f_s$, we get

$$c_n = \frac{1}{T_s} \int_{0}^{T_p} A \cdot e^{-j2\pi nf_s t} dt = \frac{A}{-j2\pi n} \cdot (e^{-j2\pi nf_s T_p} - 1) \tag{1.69}$$

When $n = 0$ we can use l'Hospital's rule to get

$$c_0 = \frac{A}{2} \tag{1.70}$$

Using the results from Sec. 1.2.2 we can write

$$c_n = \frac{-Aj}{2\pi n} \cdot (1 - e^{-j2\pi nf_s T_p}) = \frac{-Aj}{2\pi n} \cdot \sqrt{2(1 - \cos 2\pi nf_s T_p)} \tag{1.71}$$

For the case when $T_p = T_s/2$ this simplifies to

$$c_n = \frac{-Aj}{2\pi n} \cdot \sqrt{2(1 - \cos \pi n)} \tag{1.72}$$

so that the coefficients are zero when n is nonzero and even. When n is odd

$$c_n = \frac{-A}{\pi n} \cdot j \text{ or } |c_n| = \frac{A}{\pi n} \text{ and } \angle c_n = -\frac{\pi}{2} \tag{1.73}$$
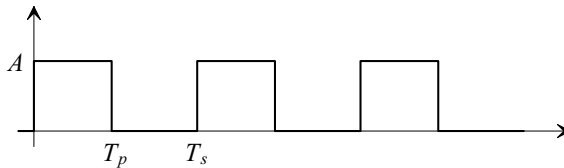

Figure 1.29   Representing a squarewave using exponential Fourier Series.

1.3.2 Fourier Transform

The Fourier Transform of a function $g(t)$ is given by

$$G(f) = \int_{-\infty}^{\infty} g(t) \cdot e^{-j2\pi ft} dt \tag{1.74}$$

while the inverse Fourier Transform is given by

$$g(t) = \int_{-\infty}^{\infty} G(f) \cdot e^{j2\pi ft} df \tag{1.75}$$

Dirac Delta Function (Unit Impulse Response)

The Dirac delta function, sometimes called the unit impulse response, is defined as

$$\delta(t - t_0) = \infty \text{ when } t = t_0 \text{ and } 0 \text{ if } t \neq t_0 \tag{1.76}$$

The discrete version (defined only at discrete time points, that is not continuously) of the Dirac delta function is the Kronecker delta function, or simply delta function, and is given by

$$\delta(t - nT_s) = 1 \text{ when } t = nT_s \text{ and } 0 \text{ if } t \neq nT_s \tag{1.77}$$

Some properties of the Dirac delta function are that

$$\int_{-\infty}^{\infty} \delta(t - t_0) dt = 1 \tag{1.78}$$

As the amplitude tends towards ∞ the width of the function moves to 0 keeping the area equal to 1. Also,

$$\int_{-\infty}^{\infty} f(t) \cdot \delta(t - t_0) dt = f(t_0) \tag{1.79}$$

A few other key properties are that the Fourier Transform of a constant is

$$\int_{-\infty}^{\infty} K \cdot e^{-j2\pi ft} \cdot dt = K \cdot \delta(f) \tag{1.80}$$

The Fourier transform of a delayed signal, $x(t - t_0)$, is

$$Fourier\{x(t - t_0)\} = X(f) \cdot e^{-j2\pi ft_0} \tag{1.81}$$

where the magnitude of the delayed signal is $|X(f)|$ (the delay doesn't affect the magnitude) and the phase shift is $\angle X(f) - 2\pi \cdot \dfrac{t_0}{T} = \angle X(f) - 2\pi f \cdot t_0$. The Fourier Transform of a sinusoid is determined by first writing

$$V_P \cdot \sin(2\pi f_1 t) = V_P \cdot \frac{e^{j2\pi f_1 t} - e^{-j2\pi f_1 t}}{2j} \tag{1.82}$$

then, using Eq. (1.74)

$$Fourier\{V_P \sin(2\pi f_1 t)\} = V_P \int_{-\infty}^{\infty} \frac{e^{j2\pi f_1 t} - e^{-j2\pi f_1 t}}{2j} \cdot e^{-j2\pi ft} dt = V_P \int_{-\infty}^{\infty} \frac{e^{j2\pi(f_1 - f)t} - e^{-j2\pi(f_1 - f)t}}{2j} dt$$

$$\tag{1.83}$$

and thus

$$= \frac{V_P}{2j} \cdot [\delta(f-f_1) - \delta(f+f_1)] = 0 + j \cdot \frac{V_P}{2} \cdot [\delta(f+f_1) - \delta(f-f_1)] \qquad (1.84)$$

The magnitude of this equation is

$$\frac{V_P}{2} \cdot [\delta(f+f_1) - \delta(f-f_1)] \qquad (1.85)$$

and the phase response is

$$\tan^{-1} \frac{\frac{V_P}{2} \cdot [\delta(f+f_1) - \delta(f-f_1)]}{0} = \tan^{-1} \infty = 90° \qquad (1.86)$$

Note that using a single-sided spectrum with positive frequencies only we could rewrite the result as

$$= j \cdot V_P \cdot \delta(f-f_1) \qquad (1.87)$$

Note also that

$$Fourier\{V_P \cos(2\pi f_1 t)\} = \frac{V_P}{2} \cdot [\delta(f+f_1) + \delta(f-f_1)] \qquad (1.88)$$

Lastly note that *negative frequencies* shouldn't confuse us since they simply represent a phase shift. For example,

$$\sin(2\pi(-f)t) = -\sin(2\pi f t) = \sin(2\pi f t + \pi) = \sin(2\pi f t - \pi) \qquad (1.89)$$

ADDITIONAL READING

[1]   M. Weeks, *Digital Signal Processing Using MATLAB and Wavelets*, Infinity Science Press, 2007. ISBN 978-0977858200

[2]   S. Haykin and M. Moher, *An Introduction to Analog and Digital Communications*, Second Edition, John Wiley and Sons, 2006. ISBN 978-0471432227

[3]   R. G. Lyons, *Understanding Digital Signal Processing*, Second Edition, Prentice-Hall, 2004. ISBN 978-0131089891

[4]   P. A. Lynn and W. Fuerst, *Introductory Digital Signal Processing*, Second Edition, John Wiley and Sons, 1998. ISBN 978-0471976318

[5]   L. W. Couch, *Modern Communication Systems: Principles and Applications*, Prentice-Hall, 1995. ISBN 978-0023252860

[6]   E. P. Cunningham, *Digital Filtering: An Introduction*, John Wiley and Sons, 1995. ISBN 978-0471124757

QUESTIONS

1.1   Suggest an alternate physical example, to the swinging pendulum in Fig. 1.1, of sinusoidal motion.

1.2   Add the *z*-axis (time) to the representations of the sinewave seen in Figs. 1.3b-f as discussed following Eq. (1.4).

1.3 Suppose an *IQ* signal is generated using an in-phase component having an amplitude of 0.5 V and a quadrature component having an amplitude of 1 V. Sketch the resulting waveform, the *IQ* signal, in the time domain.

1.4 Figure 1.7 shows how the magnitude and phase are calculated for an imaginary number that resides in the first quadrant of the plane (both real and imaginary components are positive). Show how we calculate the magnitude and phase of an imaginary number in the other quadrants.

1.5 If the output of a system occurs after the corresponding input to the system, is the phase shift positive or negative? Why? What does linear phase indicate?

1.6 Using the SPICE files found at CMOSedu.com, verify, in the time-domain, the frequency response information seen in Fig. 1.10 for input frequencies of $f = 0$ (DC), $1/4t_d$, and $1/2t_d$.

1.7 Repeat question 1.6 for the digital comb filter (averager) seen in Fig. 1.17 for input frequencies of DC, $f_s/4$, and $f_s/2$.

1.8 Plot the magnitude and phase frequency responses of a discrete-time system having the transfer function $(1 + z^{-1})/z^{-2}$. Next, show the location of this system's poles and zeros in the complex plane and verify, using the intuitive method discussed in Sec. 1.2.3, the gain and phase of the response match the frequency response plots when the input signal frequency is 0.

1.9 For the 3 delay element comb filter seen in Fig. 1.25, repeat question 1.6 for input frequencies of 0, $f_s/6$, and $f_s/3$.

1.10 Show how to plot $1/(4 + 3j)$ in the complex plane. What is the magnitude and phase shift of this complex number?

1.11 Determine the *z*-domain representation for the circuit seen in Fig. 1.30. Also, plot the frequency response, both magnitude and phase, and the location of poles and zeroes for this system.
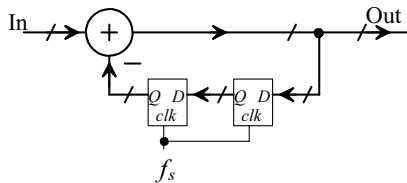

Figure 1.30 Circuit used in question 1.11.

1.12 Repeat question 1.11, and sketch the resulting circuit, if a delay is added to the forward path of the circuit seen in Fig. 1.30.

1.13 Determine the exponential Fourier series representation for the squarewave seen in Fig. 1.29 if it is centered around ground.

1.14 Determine the exponential Fourier series representation for the squarewave seen in Fig. 1.29 for the general case where $T_p \neq T_s/2$.

1.15   What is the Fourier transform of the signal seen in Fig. 1.29?

1.16   What is the area under the Dirac delta function bordered by the x-axis? Why?

1.17   Show how to take the Fourier transform of $\sin(2\pi f_0 t + \theta)$ and $\cos(2\pi f_0 t + \phi)$. Plot the magnitude and phase responses of the transforms.