Appendix A

Summary of Useful Formulas

Chapter 2

- dof = (sum of freedoms of bodies) − (number of independent configuration constraints)
- Grübler's formula is an expression of the previous formula for mechanisms with N links (including ground) and J joints, where joint i has f_i degrees of freedom and $m = 3$ for planar mechanisms or $m = 6$ for spatial mechanisms:
$$\text{dof} = m(N - 1 - J) + \sum_{i=1}^{J} f_i.$$
- Pfaffian velocity constraints take the form $A(\theta)\dot{\theta} = 0$.

Chapter 3

Rotations	Rigid-Body Motions
$R \in SO(3) : 3 \times 3$ matrices $R^{\mathrm{T}} R = I, \det R = 1$	$T \in SE(3) : 4 \times 4$ matrices $T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$, where $R \in SO(3), p \in \mathbb{R}^3$
$R^{-1} = R^{\mathrm{T}}$	$T^{-1} = \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}} p \\ 0 & 1 \end{bmatrix}$
change of coordinate frame: $R_{ab} R_{bc} = R_{ac}, \;\; R_{ab} p_b = p_a$	change of coordinate frame: $T_{ab} T_{bc} = T_{ac}, \;\; T_{ab} p_b = p_a$
rotating a frame {b}: $R = \mathrm{Rot}(\hat{\omega}, \theta)$ $R_{sb'} = R R_{sb}$: rotate θ about $\hat{\omega}_s = \hat{\omega}$ $R_{sb''} = R_{sb} R$: rotate θ about $\hat{\omega}_b = \hat{\omega}$	displacing a frame {b}: $T = \begin{bmatrix} \mathrm{Rot}(\hat{\omega}, \theta) & p \\ 0 & 1 \end{bmatrix}$ $T_{sb'} = T T_{sb}$: rotate θ about $\hat{\omega}_s = \hat{\omega}$ (moves {b} origin), translate p in {s} $T_{sb''} = T_{sb} T$: translate p in {b}, rotate θ about $\hat{\omega}$ in new body frame
unit rotation axis is $\hat{\omega} \in \mathbb{R}^3$, where $\|\hat{\omega}\| = 1$	"unit" screw axis is $\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$, where either (i) $\|\omega\| = 1$ or (ii) $\omega = 0$ and $\|v\| = 1$
	for a screw axis $\{q, \hat{s}, h\}$ with finite h, $\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s} \\ -\hat{s} \times q + h\hat{s} \end{bmatrix}$
angular velocity is $\omega = \hat{\omega}\dot{\theta}$	twist is $\mathcal{V} = \mathcal{S}\dot{\theta}$
for any 3-vector, e.g., $\omega \in \mathbb{R}^3$, $[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3)$ identities, $\omega, x \in \mathbb{R}^3, R \in SO(3)$: $[\omega] = -[\omega]^{\mathrm{T}}, [\omega]x = -[x]\omega$, $[\omega][x] = ([x][\omega])^{\mathrm{T}}, R[\omega]R^{\mathrm{T}} = [R\omega]$	for $\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$, $[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3)$ (the pair (ω, v) can be a twist $\mathcal{V}$ or a "unit" screw axis $\mathcal{S}$, depending on the context)

continued...

Rotations (cont.)	**Rigid-Body Motions (cont.)**
$\dot{R}R^{-1} = [\omega_s], \;\; R^{-1}\dot{R} = [\omega_b]$	$\dot{T}T^{-1} = [\mathcal{V}_s], \;\; T^{-1}\dot{T} = [\mathcal{V}_b]$
	$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$ identities: $[\mathrm{Ad}_T]^{-1} = [\mathrm{Ad}_{T^{-1}}]$, $[\mathrm{Ad}_{T_1}][\mathrm{Ad}_{T_2}] = [\mathrm{Ad}_{T_1 T_2}]$
change of coordinate frame: $\hat{\omega}_a = R_{ab}\hat{\omega}_b, \;\; \omega_a = R_{ab}\omega_b$	change of coordinate frame: $\mathcal{S}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{S}_b, \;\; \mathcal{V}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{V}_b$
exp coords for $R \in SO(3)$: $\hat{\omega}\theta \in \mathbb{R}^3$	exp coords for $T \in SE(3)$: $\mathcal{S}\theta \in \mathbb{R}^6$
$\exp : [\hat{\omega}]\theta \in so(3) \to R \in SO(3)$ $R = \mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} =$ $I + \sin\theta[\hat{\omega}] + (1-\cos\theta)[\hat{\omega}]^2$	$\exp : [\mathcal{S}]\theta \in se(3) \to T \in SE(3)$ $T = e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & * \\ 0 & 1 \end{bmatrix}$ where $* =$ $(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2)v$
$\log : R \in SO(3) \to [\hat{\omega}]\theta \in so(3)$ algorithm in Section 3.2.3.3	$\log : T \in SE(3) \to [\mathcal{S}]\theta \in se(3)$ algorithm in Section 3.3.3.2
moment change of coord frame: $m_a = R_{ab}m_b$	wrench change of coord frame: $\mathcal{F}_a = (m_a, f_a) = [\mathrm{Ad}_{T_{ba}}]^{\mathrm{T}}\mathcal{F}_b$

Chapter 4

- The product of exponentials formula for an open-chain manipulator is

$$\begin{aligned} \text{space frame:} \quad & T = e^{[\mathcal{S}_1]\theta_1} \ldots e^{[\mathcal{S}_n]\theta_n} M, \\ \text{body frame:} \quad & T = M e^{[\mathcal{B}_1]\theta_1} \ldots e^{[\mathcal{B}_n]\theta_n} \end{aligned}$$

  where M is the frame of the end-effector in the space frame when the manipulator is at its home position, $\mathcal{S}_i$ is the spatial twist when joint i rotates (or translates) at unit speed while all other joints are at their zero position, and $\mathcal{B}_i$ is the body twist of the end-effector frame when joint i moves at unit speed and all other joints are at their zero position.

Chapter 5

- For a manipulator end-effector configuration written in coordinates x, the forward kinematics is $x = f(\theta)$, and the differential kinematics is given by $\dot{x} = \frac{\partial f}{\partial \theta}\dot{\theta} = J(\theta)\dot{\theta}$, where $J(\theta)$ is the manipulator Jacobian.

- Written using twists, the relation is $\mathcal{V}_* = J_*(\theta)\dot{\theta}$, where $*$ is either s (space Jacobian) or b (body Jacobian). The columns J_{si}, $i = 2 \ldots n$, of the space Jacobian are
$$J_{si}(\theta) = [\mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}}]\mathcal{S}_i,$$
with $J_{s1} = \mathcal{S}_1$, and the columns J_{bi}, $i = 1 \ldots n-1$, of the body Jacobian are
$$J_{bi}(\theta) = [\mathrm{Ad}_{e^{-[\mathcal{B}_n]\theta_n} \cdots e^{-[\mathcal{B}_{i+1}]\theta_{i+1}}}]\mathcal{B}_i,$$
with $J_{bn} = \mathcal{B}_n$. The spatial twist caused by joint i is only altered by the configurations of joints inboard from joint i (between the joint and the space frame), while the body twist caused by joint i is only altered by the configurations of joints outboard from joint i (between the joint and the body frame).

  The two Jacobians are related by
$$J_b(\theta) = [\mathrm{Ad}_{T_{bs}(\theta)}]J_s(\theta), \quad J_s(\theta) = [\mathrm{Ad}_{T_{sb}(\theta)}]J_b(\theta).$$

- Generalized forces at the joints τ are related to wrenches expressed in the space or end-effector body frame by
$$\tau = J_*^{\mathrm{T}}(\theta)\mathcal{F}_*,$$
where $*$ is s (space frame) or b (body frame).

- The manipulability ellipsoid is defined by
$$\mathcal{V}^{\mathrm{T}}(JJ^{\mathrm{T}})^{-1}\mathcal{V} = 1,$$
where $\mathcal{V}$ may be a set of task-space coordinate velocities $\dot{q}$, a spatial or body twist, or the angular or linear components of a twist, and J is the appropriate Jacobian satisfying $\mathcal{V} = J(\theta)\dot{\theta}$. The principal axes of the manipulability ellipsoid are aligned with the eigenvectors of JJ^{T}, and the semi-axis lengths are the square roots of the corresponding eigenvalues.

- The force ellipsoid is defined by
$$\mathcal{F}^{\mathrm{T}}JJ^{\mathrm{T}}\mathcal{F} = 1,$$
where J is a Jacobian (possibly in terms of a minimum set of task-space coordinates or in terms of the spatial or body wrench) and $\mathcal{F}$ is an end-effector force or wrench satisfying $\tau = J^{\mathrm{T}}\mathcal{F}$. The principal axes of the manipulability ellipsoid are aligned with the eigenvectors of $(JJ^{\mathrm{T}})^{-1}$, and the semi-axis lengths are the square roots of the corresponding eigenvalues.

Chapter 6

- The law of cosines states that $c^2 = a^2 + b^2 - 2ab\cos\gamma$, where a, b, and c are the lengths of the sides of a triangle and γ is the interior angle opposite side c. This formula is often useful to solve inverse kinematics problems.

- Numerical methods are used to solve the inverse kinematics for systems for which closed-form solutions do not exist. A Newton–Raphson method using the Jacobian pseudoinverse $J^\dagger(\theta)$ is outlined below.

  (a) **Initialization**: Given T_{sd} and an initial guess $\theta^0 \in \mathbb{R}^n$. Set $i = 0$.

  (b) Set $[\mathcal{V}_b] = \log\left(T_{sb}^{-1}(\theta^i)T_{sd}\right)$. While $\|\omega_b\| > \epsilon_\omega$ or $\|v_b\| > \epsilon_v$ for small $\epsilon_\omega, \epsilon_v$:
  - Set $\theta^{i+1} = \theta^i + J_b^\dagger(\theta^i)\mathcal{V}_b$.
  - Increment i.

  If J is square and full rank, then $J^\dagger = J^{-1}$. If $J \in \mathbb{R}^{m\times n}$ is full rank (rank m for $n > m$ or rank n for $n < m$), that is, the robot is not at a singularity, the pseudoinverse can be calculated as follows:

  $$\begin{aligned} J^\dagger &= J^{\mathrm{T}}(JJ^{\mathrm{T}})^{-1} && \text{if } n > m \text{ (called a right inverse since } JJ^\dagger = I) \\ J^\dagger &= (J^{\mathrm{T}}J)^{-1}J^{\mathrm{T}} && \text{if } n < m \text{ (called a left inverse since } J^\dagger J = I). \end{aligned}$$

Chapter 8

- The Lagrangian is the kinetic minus the potential energy, $\mathcal{L}(\theta,\dot\theta) = \mathcal{K}(\theta,\dot\theta) - \mathcal{P}(\theta)$.

- The Euler-Lagrange equations are

  $$\tau = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\theta} - \frac{\partial \mathcal{L}}{\partial \theta}.$$

- The equations of motion of a robot can be written in the following equivalent forms:

  $$\begin{aligned} \tau &= M(\theta)\ddot\theta + h(\theta,\dot\theta) \\ &= M(\theta)\ddot\theta + c(\theta,\dot\theta) + g(\theta) \\ &= M(\theta)\ddot\theta + \dot\theta^{\mathrm{T}}\Gamma(\theta)\dot\theta + g(\theta) \\ &= M(\theta)\ddot\theta + C(\theta,\dot\theta)\dot\theta + g(\theta), \end{aligned}$$

where $M(\theta)$ is the $n \times n$ symmetric positive-definite mass matrix, $h(\theta, \dot{\theta})$ is the sum of generalized forces due to gravity and quadratic velocity terms, $c(\theta, \dot{\theta})$ are quadratic velocity forces, $g(\theta)$ are gravitational forces, $\Gamma(\theta)$ is an $n \times n \times n$ matrix of Christoffel symbols of the first kind obtained from partial derivatives of $M(\theta)$ with respect to θ, and $C(\theta, \dot{\theta})$ is the $n \times n$ Coriolis matrix whose (i, j)th entry is given by

$$c_{ij}(\theta, \dot{\theta}) = \sum_{k=1}^{n} \Gamma_{ijk}(\theta)\dot{\theta}_k.$$

If the end-effector of the robot is applying a wrench $\mathcal{F}_{\text{tip}}$ to the environment, the term $J^{\mathrm{T}}(\theta)\mathcal{F}_{\text{tip}}$ should be added to the right-hand side of the robot's dynamic equations.

- The symmetric positive-definite rotational inertia matrix of a rigid body is

$$\mathcal{I}_b = \begin{bmatrix} \mathcal{I}_{xx} & \mathcal{I}_{xy} & \mathcal{I}_{xz} \\ \mathcal{I}_{xy} & \mathcal{I}_{yy} & \mathcal{I}_{yz} \\ \mathcal{I}_{xz} & \mathcal{I}_{yz} & \mathcal{I}_{zz} \end{bmatrix},$$

where

$$\begin{array}{ll} \mathcal{I}_{xx} = \int_{\mathcal{B}}(y^2 + z^2)\rho(x, y, z)dV, & \mathcal{I}_{yy} = \int_{\mathcal{B}}(x^2 + z^2)\rho(x, y, z)dV, \\ \mathcal{I}_{zz} = \int_{\mathcal{B}}(x^2 + y^2)\rho(x, y, z)dV, & \mathcal{I}_{xy} = -\int_{\mathcal{B}} xy\rho(x, y, z)dV, \\ \mathcal{I}_{xz} = -\int_{\mathcal{B}} xz\rho(x, y, z)dV, & \mathcal{I}_{yz} = -\int_{\mathcal{B}} yz\rho(x, y, z)dV, \end{array}$$

$\mathcal{B}$ is the body, dV is a differential volume element, and $\rho(x, y, z)$ is the density function.

- If $\mathcal{I}_b$ is defined in a frame {b} at the center of mass then $\mathcal{I}_q$, the inertia in a frame {q} aligned with {b}, but displaced from the origin of {b} by $q \in \mathbb{R}^3$ in {b} coordinates, is

$$\mathcal{I}_q = \mathcal{I}_b + \mathfrak{m}(q^{\mathrm{T}}qI - qq^{\mathrm{T}})$$

by Steiner's theorem.

- The spatial inertia matrix $\mathcal{G}_b$ expressed in a frame {b} at the center of mass is defined as the 6×6 matrix

$$\mathcal{G}_b = \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m}I \end{bmatrix}.$$

In a frame {a} at a configuration T_{ba} relative to {b}, the spatial inertia matrix is

$$\mathcal{G}_a = [\mathrm{Ad}_{T_{ba}}]^{\mathrm{T}}\mathcal{G}_b[\mathrm{Ad}_{T_{ba}}].$$

- The Lie bracket of two twists $\mathcal{V}_1$ and $\mathcal{V}_2$ is

$$\mathrm{ad}_{\mathcal{V}_1}(\mathcal{V}_2) = [\mathrm{ad}_{\mathcal{V}_1}]\mathcal{V}_2,$$

  where

$$[\mathrm{ad}_{\mathcal{V}}] = \begin{bmatrix} [\omega] & 0 \\ [v] & [\omega] \end{bmatrix} \in \mathbb{R}^{6\times 6}.$$

- The twist-wrench formulation of the rigid-body dynamics of a single rigid body is

$$\mathcal{F}_b = \mathcal{G}_b\dot{\mathcal{V}}_b - [\mathrm{ad}_{\mathcal{V}_b}]^{\mathrm{T}}\mathcal{G}_b\mathcal{V}_b.$$

  The equations have the same form if $\mathcal{F}$, $\mathcal{V}$, and $\mathcal{G}$ are expressed in the same frame, regardless of the frame.

- The kinetic energy of a rigid body is $\frac{1}{2}\mathcal{V}_b^{\mathrm{T}}\mathcal{G}_b\mathcal{V}_b$, and the kinetic energy of an open-chain robot is $\frac{1}{2}\dot{\theta}^{\mathrm{T}}M(\theta)\dot{\theta}$.

- The forward-backward Newton-Euler inverse dynamics algorithm is the following:

  Initialization: Attach a frame $\{0\}$ to the base, frames $\{1\}$ to $\{n\}$ to the centers of mass of links $\{1\}$ to $\{n\}$, and a frame $\{n+1\}$ at the end-effector, fixed in the frame $\{n\}$. Define $M_{i,i-1}$ to be the configuration of $\{i-1\}$ in $\{i\}$ when $\theta_i = 0$. Let $\mathcal{A}_i$ be the screw axis of joint i expressed in $\{i\}$, and $\mathcal{G}_i$ be the 6×6 spatial inertia matrix of link i. Define $\mathcal{V}_0$ to be the twist of the base frame $\{0\}$ expressed in base-frame coordinates. (This quantity is typically zero.) Let $\mathfrak{g} \in \mathbb{R}^3$ be the gravity vector expressed in base-frame-$\{0\}$ coordinates, and define $\dot{\mathcal{V}}_0 = (0, -\mathfrak{g})$. (Gravity is treated as an acceleration of the base in the opposite direction.) Define $\mathcal{F}_{n+1} = \mathcal{F}_{\mathrm{tip}} = (m_{\mathrm{tip}}, f_{\mathrm{tip}})$ to be the wrench applied to the environment by the end-effector expressed in the end-effector frame $\{n+1\}$.

  Forward iterations: Given $\theta, \dot{\theta}, \ddot{\theta}$, for $i = 1$ to n do

$$\begin{aligned} T_{i,i-1} &= e^{-[\mathcal{A}_i]\theta_i}M_{i,i-1}, \\ \mathcal{V}_i &= \mathrm{Ad}_{T_{i,i-1}}(\mathcal{V}_{i-1}) + \mathcal{A}_i\dot{\theta}_i, \\ \dot{\mathcal{V}}_i &= \mathrm{Ad}_{T_{i,i-1}}(\dot{\mathcal{V}}_{i-1}) + \mathrm{ad}_{\mathcal{V}_i}(\mathcal{A}_i)\dot{\theta}_i + \mathcal{A}_i\ddot{\theta}_i. \end{aligned}$$

  Backward iterations: For $i = n$ to 1 do

$$\begin{aligned} \mathcal{F}_i &= \mathrm{Ad}^{\mathrm{T}}_{T_{i+1,i}}(\mathcal{F}_{i+1}) + \mathcal{G}_i\dot{\mathcal{V}}_i - \mathrm{ad}^{\mathrm{T}}_{\mathcal{V}_i}(\mathcal{G}_i\mathcal{V}_i), \\ \tau_i &= \mathcal{F}_i^{\mathrm{T}}\mathcal{A}_i. \end{aligned}$$

- Let $J_{ib}(\theta)$ to be the Jacobian relating $\dot{\theta}$ to the body twist $\mathcal{V}_i$ in link i's center-of-mass frame $\{i\}$. Then the mass matrix $M(\theta)$ of the manipulator can be expressed as

$$M(\theta) = \sum_{i=1}^{n} J_{ib}^{\mathrm{T}}(\theta)\mathcal{G}_i J_{ib}(\theta).$$

- The robot's dynamics $M(\theta)\ddot{\theta} + h(\theta, \dot{\theta})$ can be expressed in the task space as

$$\mathcal{F} = \Lambda(\theta)\dot{\mathcal{V}} + \eta(\theta, \mathcal{V}),$$

  where $\mathcal{F}$ is the wrench applied to the end-effector, $\mathcal{V}$ is the twist of the end-effector, and $\mathcal{F}$, $\mathcal{V}$, and the Jacobian $J(\theta)$ are all defined in the same frame. The task-space mass matrix $\Lambda(\theta)$ and gravity and quadratic velocity forces $\eta(\theta, \mathcal{V})$ are

$$\begin{aligned} \Lambda(\theta) &= J^{-\mathrm{T}} M(\theta) J^{-1}, \\ \eta(\theta, \mathcal{V}) &= J^{-\mathrm{T}} h(\theta, J^{-1}\mathcal{V}) - \Lambda(\theta)\dot{J}J^{-1}\mathcal{V}, \end{aligned}$$

  where $J^{-\mathrm{T}} = (J^{-1})^{\mathrm{T}}$.

- Define two $n \times n$ projection matrices of rank $n - k$

$$\begin{aligned} P(\theta) &= I - A^{\mathrm{T}}(AM^{-1}A^{\mathrm{T}})^{-1}AM^{-1}, \\ P_{\ddot{\theta}}(\theta) &= M^{-1}PM = I - M^{-1}A^{\mathrm{T}}(AM^{-1}A^{\mathrm{T}})^{-1}A \end{aligned}$$

  corresponding to the k Pfaffian constraints acting on the robot, $A(\theta)\dot{\theta} = 0$, $A \in \mathbb{R}^{k \times n}$. Then the $n + k$ constrained equations of motion

$$\begin{aligned} \tau &= M(\theta)\ddot{\theta} + h(\theta, \dot{\theta}) + A^{\mathrm{T}}(\theta)\lambda, \\ A(\theta)\dot{\theta} &= 0 \end{aligned}$$

  can be reduced to the following equivalent forms by eliminating the Lagrange multipliers λ:

$$\begin{aligned} P\tau &= P(M\ddot{\theta} + h), \\ P_{\ddot{\theta}}\ddot{\theta} &= P_{\ddot{\theta}}M^{-1}(\tau - h). \end{aligned}$$

  The matrix P projects away joint force–torque components that act on the constraints without doing work on the robot, and the matrix $P_{\ddot{\theta}}$ projects away acceleration components that do not satisfy the constraints.

Chapter 9

- A straight-line path in joint space is given by $\theta(s) = \theta_{\text{start}} + s(\theta_{\text{end}} - \theta_{\text{start}})$ as s goes from 0 to 1.

- A constant-screw-axis motion of the end-effector from $X_{\text{start}} \in SE(3)$ to X_{end} is $X(s) = X_{\text{start}} \exp(\log(X_{\text{start}}^{-1} X_{\text{end}})s)$ as s goes from 0 to 1.

- The path-constrained dynamics of a robot can be written

$$m(s)\ddot{s} + c(s)\dot{s}^2 + g(s) = \tau \in \mathbb{R}^n$$

  as s goes from 0 to 1.

Chapter 13

- The Lie bracket of two vector fields g_1 and g_2 is the vector field

$$[g_1, g_2] = \left(\frac{\partial g_2}{\partial q} g_1 - \frac{\partial g_1}{\partial q} g_2 \right).$$