Appendix B

Other Representations of Rotations

B.1 Euler Angles

As we established earlier, the orientation of a rigid body can be parametrized by three independent coordinates. For example, consider a rigid body with a body frame {b} attached to it, initially aligned with the space frame {s}. Now rotate the body by α about the body $\hat{\mathrm{z}}_\mathrm{b}$-axis, then by β about the body $\hat{\mathrm{y}}_\mathrm{b}$-axis, and finally by γ about the body $\hat{\mathrm{x}}_\mathrm{b}$-axis. Then (α, β, γ) are the **ZYX Euler angles** representing the final orientation of the body (see Figure B.1). If the successive rotations are made with respect to the body frame, the result corresponds to the final rotation matrix

$$R(\alpha, \beta, \gamma) = I\,\mathrm{Rot}(\hat{\mathrm{z}}, \alpha)\,\mathrm{Rot}(\hat{\mathrm{y}}, \beta)\,\mathrm{Rot}(\hat{\mathrm{x}}, \gamma),$$

where

$$\mathrm{Rot}(\hat{\mathrm{z}}, \alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathrm{Rot}(\hat{\mathrm{y}}, \beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix},$$

$$\mathrm{Rot}(\hat{\mathrm{x}}, \gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}.$$

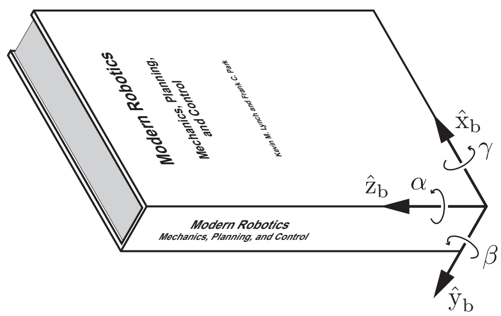

Figure B.1: To understand the ZYX Euler angles, use the corner of a box or a book as the body frame. The ZYX Euler angles correspond to successive rotations of the body about the $\hat{z}_b$-axis by α, the $\hat{y}_b$-axis by β, and the $\hat{x}_b$-axis by γ.

Writing out the entries explicitly, we get

$$R(\alpha,\beta,\gamma) = \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}, \tag{B.1}$$

where s_α is shorthand for $\sin\alpha$, c_α for $\cos\alpha$, etc.

We now ask the following question: given an arbitrary rotation matrix R, does there exist (α,β,γ) satisfying Equation (B.1)? In other words, can the ZYX Euler angles represent all orientations? The answer is yes, and we prove this fact constructively as follows. Let r_{ij} be the (i,j)th element of R. Then, from Equation (B.1), we know that $r_{11}^2 + r_{21}^2 = \cos^2\beta$; as long as $\cos\beta \neq 0$, or equivalently $\beta \neq \pm 90°$, we have two possible solutions for β:

$$\beta = \operatorname{atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)$$

and

$$\beta = \operatorname{atan2}\left(-r_{31}, -\sqrt{r_{11}^2 + r_{21}^2}\right).$$

(The atan2 two-argument arctangent is described at the beginning of Chapter 6.) In the first case β lies in the range $[-90°, 90°]$, while in the second case it lies in the range $[90°, 270°]$. Assuming that the β obtained above is not $\pm 90°$, α and γ can then be determined from the following relations:

$$\begin{aligned} \alpha &= \operatorname{atan2}(r_{21}, r_{11}), \\ \gamma &= \operatorname{atan2}(r_{32}, r_{33}). \end{aligned}$$

In the event that $\beta = \pm 90°$, there exists a one-parameter family of solutions for α and γ. This is most easily seen from Figure B.3. If $\beta = 90°$ then α and γ represent rotations (in the opposite direction) about the same vertical axis. Hence, if $(\alpha, \beta, \gamma) = (\bar{\alpha}, 90°, \bar{\gamma})$ is a solution for a given rotation R then any triple $(\bar{\alpha}', 90°, \bar{\gamma}')$, where $\bar{\alpha}' - \bar{\gamma}' = \bar{\alpha} - \bar{\gamma}$, is also a solution.

B.1.1 Algorithm for Computing the ZYX Euler Angles

Given $R \in SO(3)$, we wish to find angles $\alpha, \gamma \in (-\pi, \pi]$ and $\beta \in [-\pi/2, \pi/2)$ that satisfy

$$R = \begin{bmatrix} \mathrm{c}_\alpha \mathrm{c}_\beta & \mathrm{c}_\alpha \mathrm{s}_\beta \mathrm{s}_\gamma - \mathrm{s}_\alpha \mathrm{c}_\gamma & \mathrm{c}_\alpha \mathrm{s}_\beta \mathrm{c}_\gamma + \mathrm{s}_\alpha \mathrm{s}_\gamma \\ \mathrm{s}_\alpha \mathrm{c}_\beta & \mathrm{s}_\alpha \mathrm{s}_\beta \mathrm{s}_\gamma + \mathrm{c}_\alpha \mathrm{c}_\gamma & \mathrm{s}_\alpha \mathrm{s}_\beta \mathrm{c}_\gamma - \mathrm{c}_\alpha \mathrm{s}_\gamma \\ -\mathrm{s}_\beta & \mathrm{c}_\beta \mathrm{s}_\gamma & \mathrm{c}_\beta \mathrm{c}_\gamma \end{bmatrix}. \tag{B.2}$$

Denote by r_{ij} the (i, j)th entry of R.

(a) If $r_{31} \neq \pm 1$, set

$$\beta = \operatorname{atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right), \tag{B.3}$$

$$\alpha = \operatorname{atan2}(r_{21}, r_{11}), \tag{B.4}$$

$$\gamma = \operatorname{atan2}(r_{32}, r_{33}), \tag{B.5}$$

where the square root is taken to be positive.

(b) If $r_{31} = -1$ then $\beta = \pi/2$, and a one-parameter family of solutions for α and γ exists. One possible solution is $\alpha = 0$ and $\gamma = \operatorname{atan2}(r_{12}, r_{22})$.

(c) If $r_{31} = 1$ then $\beta = -\pi/2$, and a one-parameter family of solutions for α and γ exists. One possible solution is $\alpha = 0$ and $\gamma = -\operatorname{atan2}(r_{12}, r_{22})$.

B.1.2 Other Euler Angle Representations

The ZYX Euler angles can be visualized using the wrist mechanism shown in Figure B.2. The ZYX Euler angles (α, β, γ) refer to the angle of rotation about the three joint axes of this mechanism. In the figure the wrist mechanism is shown in its zero position, i.e., when all three joints are set to zero.

Four reference frames are defined as follows: frame $\{0\}$ is the fixed frame, while frames $\{1\}$, $\{2\}$, and $\{3\}$ are attached to the three links of the wrist mechanism as shown. When the wrist is in the zero position, all four reference frames have the same orientation. At the joint angles (α, β, γ), frame $\{1\}$

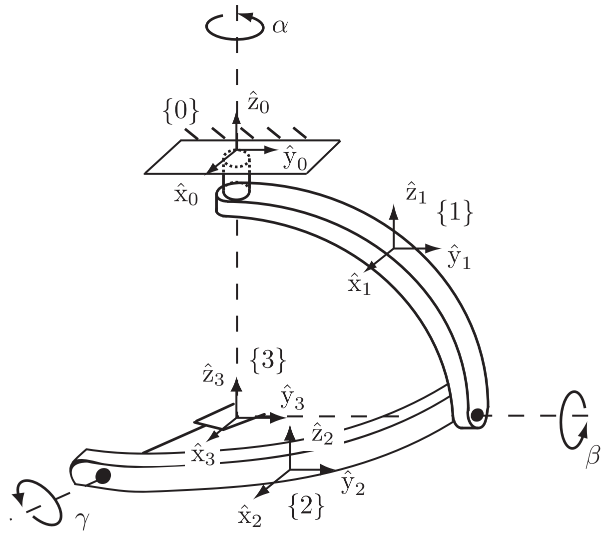

Figure B.2: Wrist mechanism illustrating the ZYX Euler angles.

relative to $\{0\}$ is $R_{01}(\alpha) = \mathrm{Rot}(\hat{z}, \alpha)$, and similarly $R_{12}(\beta) = \mathrm{Rot}(\hat{y}, \beta)$ and $R_{23}(\gamma) = \mathrm{Rot}(\hat{x}, \gamma)$. Therefore $R_{03}(\alpha, \beta, \gamma) = \mathrm{Rot}(\hat{z}, \alpha)\,\mathrm{Rot}(\hat{y}, \beta)\,\mathrm{Rot}(\hat{x}, \gamma)$ as in Equation (B.1).

It should be evident that the choice of zero position for β is, in some sense, arbitrary. That is, we could just as easily have defined the home position of the wrist mechanism to be as in Figure B.3; this would then lead to another three-parameter representation (α, β, γ) for $SO(3)$. In fact, Figure B.3 illustrates the **ZYZ Euler angles**. The resulting rotation matrix can be obtained via the following sequence of rotations, equivalent to rotating the body in Figure B.1 first about the body's $\hat{z}_b$-axis, then about the $\hat{y}_b$-axis, then about the $\hat{z}_b$-axis:

$$\begin{aligned} R(\alpha, \beta, \gamma) &= \mathrm{Rot}(\hat{z}, \alpha)\mathrm{Rot}(\hat{y}, \beta)\mathrm{Rot}(\hat{z}, \gamma) \\ &= \begin{bmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\beta & 0 & s_\beta \\ 0 & 1 & 0 \\ -s_\beta & 0 & c_\beta \end{bmatrix} \begin{bmatrix} c_\gamma & -s_\gamma & 0 \\ s_\gamma & c_\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_\alpha c_\beta c_\gamma - s_\alpha s_\gamma & -c_\alpha c_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta \\ s_\alpha c_\beta c_\gamma + c_\alpha s_\gamma & -s_\alpha c_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta \\ -s_\beta c_\gamma & s_\beta s_\gamma & c_\beta \end{bmatrix}. \end{aligned} \tag{B.6}$$

Just as before, we can show that for every rotation $R \in SO(3)$, there exists a triple (α, β, γ) that satisfies $R = R(\alpha, \beta, \gamma)$ for $R(\alpha, \beta, \gamma)$ as given in Equa-

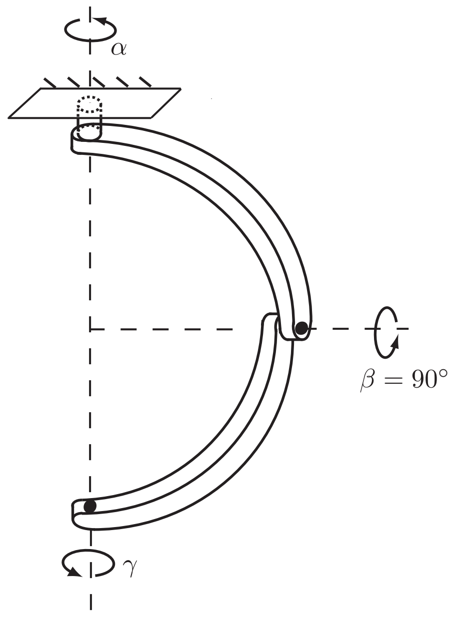

Figure B.3: Configuration corresponding to $\beta = 90°$ for ZYX Euler angles.

tion (B.6). (Of course, the resulting formulas will differ from those for the ZYX Euler angles.)

From the wrist mechanism interpretation of the ZYX and ZYZ Euler angles, it should be evident that, for Euler-angle parametrizations of $SO(3)$, what really matters is that rotation axis 1 is orthogonal to rotation axis 2, and that rotation axis 2 is orthogonal to rotation axis 3 (axes 1 and 3 need not necessarily be orthogonal to each other). Specifically, any sequence of rotations of the form

$$\text{Rot}(\text{axis } 1, \alpha)\text{Rot}(\text{axis } 2, \beta)\text{Rot}(\text{axis } 3, \gamma), \tag{B.7}$$

where axis 1 is orthogonal to axis 2, and axis 2 is orthogonal to axis 3, can serve as a valid three-parameter representation for $SO(3)$. The angle of rotation for the first and third rotations ranges in value over a 2π interval, while that of the second rotation ranges in value over an interval of length π.

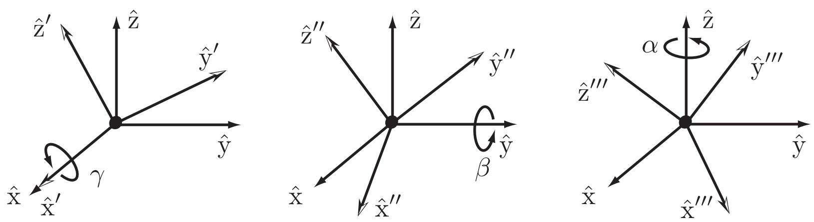

Figure B.4: Illustration of XYZ roll–pitch–yaw angles.

B.2 Roll–Pitch–Yaw Angles

While Euler angles refer to the angles in a sequence of rotations in a body-fixed frame, the **roll–pitch–yaw angles** refer to the angles in a sequence of rotations about axes fixed in the space frame. Referring to Figure B.4, given a frame in the identity configuration (that is, $R = I$), we first rotate this frame by an angle γ about the $\hat{x}$-axis of the fixed frame, then by an angle β about the $\hat{y}$-axis of the fixed frame, and finally by an angle α about the $\hat{z}$-axis of the fixed frame.

Since the three rotations are in the fixed frame, the final orientation is

$$\begin{aligned}
R(\alpha,\beta,\gamma) &= \mathrm{Rot}(\hat{z},\alpha)\mathrm{Rot}(\hat{y},\beta)\mathrm{Rot}(\hat{x},\gamma)I \\
&= \begin{bmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\beta & 0 & s_\beta \\ 0 & 1 & 0 \\ -s_\beta & 0 & c_\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\gamma & -s_\gamma \\ 0 & s_\gamma & c_\gamma \end{bmatrix} I \\
&= \begin{bmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}. \qquad \text{(B.8)}
\end{aligned}$$

This product of three rotations is exactly the same as that for the ZYX Euler angles given in (B.2). We see that the same product of three rotations admits two different physical interpretations: as a sequence of rotations with respect to the body frame (ZYX Euler angles) or, reversing the order of the rotations, as a sequence of rotations with respect to the fixed frame (the XYZ roll–pitch–yaw angles).

The terms roll, pitch, and yaw are often used to describe the rotational motion of a ship or aircraft. In the case of a typical fixed-wing aircraft, for example, suppose a body frame is attached such that the $\hat{x}$-axis is in the direction of forward motion, the $\hat{z}$-axis is the vertical axis pointing downward toward ground (assuming the aircraft is flying level with respect to ground), and the $\hat{y}$-axis extends in the direction of the wing. The roll, pitch, and yaw motions are then defined according to the XYZ roll–pitch–yaw angles (α, β, γ) of Equation (B.8).

B.3 Unit Quaternions

One disadvantage of the exponential coordinates on $SO(3)$ is that, because of the division by $\sin\theta$ in the logarithm formula, the logarithm can be numerically sensitive to small rotation angles θ. The necessary singularity of the three-parameter representation occurs at $R = I$. The **unit quaternions** are an alternative representation of rotations that alleviates this singularity, but at the cost of having a fourth variable in the representation. We now illustrate the definition and use of these coordinates.

Let $R \in SO(3)$ have the exponential coordinate representation $\hat{\omega}\theta$, i.e., $R = e^{[\hat{\omega}]\theta}$, where as usual $\|\hat{\omega}\| = 1$ and $\theta \in [0, \pi]$. The unit quaternion representation of R is constructed as follows. Define $q \in \mathbb{R}^4$ according to

$$q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) \\ \hat{\omega}\sin(\theta/2) \end{bmatrix} \in \mathbb{R}^4. \tag{B.9}$$

As defined, q clearly satisfies $\|q\| = 1$. Geometrically, q is a point lying on the three-dimensional unit sphere in $\mathbb{R}^4$, and for this reason the unit quaternions are also identified with the 3-sphere, denoted S^3. Naturally, among the four coordinates of q, only three can be chosen independently. Recalling that $1 + 2\cos\theta = \mathrm{tr}\ R$, and using the cosine double-angle formula $\cos 2\phi = 2\cos^2\phi - 1$, the elements of q can be obtained directly from the entries of R as follows:

$$q_0 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}, \tag{B.10}$$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{1}{4q_0}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - 2_{12} \end{bmatrix}. \tag{B.11}$$

Going the other way, given a unit quaternion (q_0, q_1, q_2, q_3) the corresponding rotation matrix R is obtained as a rotation about the unit axis, in the direction of (q_1, q_2, q_3), by an angle $2\cos^{-1} q_0$. Explicitly,

$$R = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_0q_2 + q_1q_3) \\ 2(q_0q_3 + q_1q_2) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_0q_1 + q_2q_3) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}. \tag{B.12}$$

From the above explicit formula it should be apparent that both $q \in S^3$ and its antipodal point $-q \in S^3$ produce the same rotation matrix R: for every rotation matrix there exists two unit-quaternion representations that are antipodal to each other.

The final property of the unit quaternions concerns the product of two rotations. Let $R_q, R_p \in SO(3)$ denote two rotation matrices, with unit-quaternion representations $\pm q, \pm p \in S^3$, respectively. The unit-quaternion representation for the product $R_q R_p$ can then be obtained by first arranging the elements of q and p in the form of the following 2×2 complex matrices:

$$Q = \begin{bmatrix} q_0 + iq_1 & q_2 + ip_3 \\ -q_2 + iq_3 & q_0 - iq_1 \end{bmatrix}, \qquad P = \begin{bmatrix} p_0 + ip_1 & p_2 + ip_3 \\ -p_2 + ip_3 & p_0 - ip_1 \end{bmatrix}, \tag{B.13}$$

where i denotes the imaginary unit. Now take the product $N = QP$, where the entries of N are given by

$$N = \begin{bmatrix} n_0 + in_1 & n_2 + in_3 \\ -n_2 + in_3 & n_0 - in_1 \end{bmatrix}. \tag{B.14}$$

The unit quaternion for the product $R_q R_p$ is then $\pm(n_0, n_1, n_2, n_3)$, obtained from the entries of N:

$$\begin{bmatrix} n_0 \\ n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3 \\ q_0p_1 + p_0q_1 + q_2p_3 - q_3p_2 \\ q_0p_2 + p_0q_2 - q_1p_3 + q_3p_1 \\ q_0p_3 + p_0q_3 + q_1p_2 - q_2p_1 \end{bmatrix}. \tag{B.15}$$

B.4 Cayley–Rodrigues Parameters

The Cayley–Rodrigues parameters form another set of widely used local coordinates for $SO(3)$. These parameters can be obtained from the exponential representation on $SO(3)$ as follows: given $R = e^{[\hat{\omega}]\theta}$ for some unit vector $\hat{\omega}$ and angle θ, the Cayley–Rodrigues parameters $r \in \mathbb{R}^3$ are obtained by setting

$$r = \hat{\omega} \tan \frac{\theta}{2}. \tag{B.16}$$

Referring again to the radius-π solid-ball picture of $SO(3)$ (Figure 3.13), the above parametrization has the effect of infinitely "stretching" the radius of this ball via the tangent half-angle function. These parameters can be derived from a general formula attributed to Cayley that is also valid for rotation matrices of arbitrary dimension: if $R \in SO(3)$ such that $\operatorname{tr} R \neq -1$ then $(I - R)(I + R)^{-1}$ is skew symmetric. Denoting this skew-symmetric matrix by $[r]$, it is known that R and $[r]$ are related as follows:

$$R = (I - [r])(I + [r])^{-1}, \tag{B.17}$$

$$[r] = (I - R)(I + R)^{-1}. \tag{B.18}$$

The above two formulas establish a one-to-one correspondence between $so(3)$ and those elements of $SO(3)$ with trace not equal to -1. In the event that $\operatorname{tr} R = -1$, the following alternative formulas can be used to relate $SO(3)$ (this time excluding those with unit trace) and $so(3)$ in a one-to-one fashion:

$$R = -(I - [r])(I + [r])^{-1}, \tag{B.19}$$
$$[r] = (I + R)(I - R)^{-1} \tag{B.20}$$

Furthermore, Equation (B.18) can be explicitly computed as

$$R = \frac{(1 - r^{\mathrm{T}} r)I + 2rr^{\mathrm{T}} + 2[r]}{1 + r^{\mathrm{T}} r} \tag{B.21}$$

with its inverse mapping given by

$$[r] = \frac{R - R^{\mathrm{T}}}{1 + \operatorname{tr} R}. \tag{B.22}$$

(This formula is valid when $\operatorname{tr} R \neq -1$). The vector $r = 0$ therefore corresponds to the identity matrix, and $-r$ represents the inverse of the rotation corresponding to r.

The following two identities also follow from the above formulas:

$$1 + \operatorname{tr} R = \frac{4}{1 + r^{\mathrm{T}} r}, \tag{B.23}$$
$$R - R^{\mathrm{T}} = \frac{4\,[r]}{1 + r^{\mathrm{T}} r}. \tag{B.24}$$

An attractive feature of the Cayley–Rodrigues parameters is the particularly simple form for the composition of two rotation matrices. If r_1 and r_2 denote the Cayley–Rodrigues parameters for two rotations R_1 and R_2, respectively, then the Cayley–Rodrigues parameters for $R_3 = R_1 R_2$, denoted r_3, are given by

$$r_3 = \frac{r_1 + r_2 + (r_1 \times r_2)}{1 - r_1^{\mathrm{T}} r_2} \tag{B.25}$$

In the event that $r_1^{\mathrm{T}} r_2 = 1$, or equivalently $\operatorname{tr}(R_1 R_2) = -1$, the following alternative composition formula can be used. Define

$$s = \frac{r}{\sqrt{1 + r^{\mathrm{T}} r}} \tag{B.26}$$

so that the rotation corresponding to r can be written

$$R = I + 2\sqrt{1 - s^{\mathrm{T}} s}\,[s] + 2[s]^2. \tag{B.27}$$

The direction of s coincides with that of r, and $\|s\| = \sin(\theta/2)$. The composition law now becomes

$$s_3 = s_1\sqrt{1 - s_2^{\mathrm{T}} s_2} + s_2\sqrt{1 - s_1^{\mathrm{T}} s_1} + (s_1 \times s_2) \tag{B.28}$$

Angular velocities and accelerations also admit a simple form in terms of the Cayley–Rodrigues parameters. If $r(t)$ denotes the Cayley–Rodrigues representation of the orientation trajectory $R(t)$ then, in vector form,

$$\omega_s = \frac{2}{1 + \|r\|^2}(r \times \dot{r} + \dot{r}), \tag{B.29}$$

$$\omega_b = \frac{2}{1 + \|r\|^2}(-r \times \dot{r} + \dot{r}). \tag{B.30}$$

The angular acceleration with respect to the space and body frames can now be obtained by time-differentiating the above expressions:

$$\dot{\omega}_s = \frac{2}{1 + \|r\|^2}(r \times \ddot{r} + \ddot{r} - r^{\mathrm{T}}\dot{r}\,\omega_s), \tag{B.31}$$

$$\dot{\omega}_b = \frac{2}{1 + \|r\|^2}(-r \times \ddot{r} + \ddot{r} - r^{\mathrm{T}}\dot{r}\,\omega_b). \tag{B.32}$$