16. The Type-Space Approach

16.1 Types of players

As noted in Chapter 14, the theory of "games of incomplete information"[1] was pioneered by John Harsanyi (1967-68). Harsanyi's approach was developed using a different approach from the one we employed in Chapters 14 and 15, which is based on the interactive knowledge-belief structures introduced in Chapters 8 and 9. The interactive knowledge structures of Chapter 9 are in fact a special case of the more general notion of *interactive Kripke structure*, named after the philosopher and logician Saul Kripke, whose work on this goes back to 1959 and was written while he was still an undergraduate.[2] Although well known among logicians and computer scientists, these structures were not known to game theorists. Perhaps, if Harasnyi had been aware of Kripke's work he might have developed his theory using those structures. We find the interactive Kripke structures more natural and elegant and thus chose to explain the "theory of games of incomplete information" using those structures. In this chapter we will explain the "type-space" approach developed by Harsanyi and show that the two approaches are equivalent. We will limit ourselves to situations of incomplete information involving *static* games. We will begin in the next section with a simple special case, which is often all that one finds in game-theory textbooks, and then explain the general case in Section 16.3.

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[1] We use quotation marks because, strictly speaking, there is no such thing as a game of incomplete information. There are situations of incomplete information involving the playing of a game and what Harsanyi did was to suggest a way of transforming such situations into extensive-form games with imperfect information. Once the so-called "Harsanyi transformation" has been applied, the resulting game is a game (of complete information). Thus the "theory of games of incomplete information" is a theory on how to represent a situation of incomplete information concerning the playing of a game and how to transform it into a dynamic game with imperfect (but complete) information.

[2] Kripke (1959, 1963). For further details the reader is referred to van Ditmarsch et al. (2015).

16.2 Types that know their payoffs

We take as a starting point the case of incomplete information concerning a strategic-form game-frame.[3] In the special case considered in this section, any uncertainty Player i has (if any) concerns either the von Neumann-Morgenstern utility function $U_j : O \to \mathbb{R}$ $(j \neq i)$ of another player (or several other players) or the beliefs of the other player(s) (or both). Within the approach of Chapter 14, we would represent such a situation with an interactive knowledge-belief structure by associating with every state ω a game based on the given game-frame.[4] Let $U_{i,\omega} : O \to \mathbb{R}$ be the utility function of Player i at state ω.

We say that *every player knows her own payoffs* if the following condition is satisfied, for every Player i (recall that $I_i(\omega)$ is the information set of Player i's partition that contains state ω):

$$\text{if} \quad \omega_1, \omega_2 \in \Omega \quad \text{and} \quad \omega_2 \in I_i(\omega_1) \quad \text{then} \quad U_{i,\omega_2} = U_{i,\omega_1}.$$

An example of such a situation of incomplete information is given in Figure 16.1, which reproduces Figure 14.3 of Chapter 14.[5]

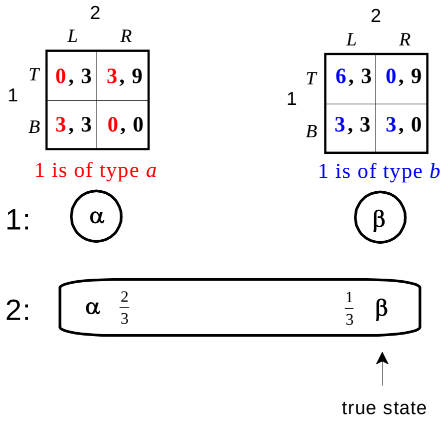

Figure 16.1: A situation of incomplete information represented by means of an interactive knowledge-belief structure.

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[3] Recall (Definition 2.1.1, Chapter 2) that the elements of a strategic-form game-frame $\langle I, (S_1, \ldots, S_n), O, f \rangle$ are as follows: $I = \{1, \ldots, n\}$ is a set of players, S_i is the set of strategies of Player $i \in I$ (and $S = S_1 \times \cdots \times S_n$ is the set of strategy profiles), O is a set of outcomes and $f : S \to O$ is a function that associates with every strategy profile an outcome.

[4] Recall that, given a game-frame, a game based on it is obtained by specifying, for every Player i, a von Neumann-Morgenstern utility function $U_i : O \to \mathbb{R}$ on the set of outcomes.

[5] In this case the game-frame is given by: $I = \{1,2\}, S_1 = \{T,B\}, S_2 = \{L,R\}$ (so that $S = \{(T,L),(T,R),(B,L),(B,R)\}$), $O = \{o_1,o_2,o_3,o_4\}$, $f(T,L) = o_1, f(T,R) = o_2, f(B,L) = o_3$, $f(B,R) = o_4$. The (state-dependent) utility functions are given by $U_{1,\alpha} = \begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ 0 & 3 & 3 & 0 \end{pmatrix}$, $U_{1,\beta} = \begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ 6 & 0 & 3 & 3 \end{pmatrix}$ and $U_{2,\alpha} = U_{2,\beta} = \begin{pmatrix} o_1 & o_2 & o_3 & o_4 \\ 3 & 9 & 3 & 0 \end{pmatrix}$.

In the "type-space" approach the situation illustrated in Figure 16.1 would be represented using "types of players": each type of Player i represents a utility function of Player i as well as Player i's beliefs about the types of the other players. The formal definition is as follows.

Definition 16.2.1 A *static Bayesian game of incomplete information with knowledge of one's own payoffs* consists of the following elements:

- a set $I = \{1,\ldots,n\}$ of players;
- for every Player $i \in I$, a set S_i of strategies (as usual, we denote by S the set of strategy profiles);
- for every Player $i \in I$, a set T_i of possible types; we denote by $T = T_1 \times \ldots \times T_n$ the set of profiles of types and by $T_{-i} = T_1 \times \cdots \times T_{i-1} \times T_{i+1} \times \cdots \times T_n$ the set of profiles of types for the players other than i;
- for every Player i and for every type $t_i \in T_i$ of Player i, a probability distribution $P_{i,t_i} : T_{-i} \to [0,1]$ representing the beliefs of type t_i about the types of the other players.

The beliefs of all the types are said to be *Harsanyi consistent* if there exists a *common prior*, that is, a probability distribution $P : T \to [0,1]$ such that, for every Player i and for every type $t_i \in T_i$ of Player i, P_{i,t_i} coincides with the probability distribution obtained from P by conditioning on the event $\{t_i\}$, that is (denoting a profile $t \in T$ by (t_{-i},t_i)), $P_{i,t_i}(t_{-i}) = \frac{P(t_{-i},t_i)}{\sum\limits_{t'_{-i} \in T_{-i}} P(t'_{-i},t_i)}$.

Let us recast the situation illustrated in Figure 16.1 in the terminology of Definition 16.2.1. First of all, we have that

$I = \{1,2\}, \quad S_1 = \{T,B\}, \quad S_2 = \{L,R\} \quad$ (so that $S = \{(T,L),(T,R),(B,L),(B,R)\}$).

Furthermore, there are two types of Player 1 and only one type of Player 2:

$T_1 = \{t_1^a, t_1^b\}, \quad T_2 = \{t_2\}, \quad$ so that $T = \{(t_1^a,t_2),(t_1^b,t_2)\}$.

The utility functions are given by:

$$U_{1,t_1^a} = \left(\begin{array}{cccc} (T,L) & (T,R) & (B,L) & (B,R) \\ 0 & 3 & 3 & 0 \end{array}\right) \qquad U_{1,t_1^b} = \left(\begin{array}{cccc} (T,L) & (T,R) & (B,L) & (B,R) \\ 6 & 0 & 3 & 3 \end{array}\right)$$

$$\text{and} \qquad U_{2,t_2} = \left(\begin{array}{cccc} (T,L) & (T,R) & (B,L) & (B,R) \\ 3 & 9 & 3 & 0 \end{array}\right).$$

The beliefs are given by $P_{1,t_1^a} = P_{1,t_1^b} = \left(\begin{array}{c} t_2 \\ 1 \end{array}\right)$, $P_{2,t_2} = \left(\begin{array}{cc} t_1^a & t_1^b \\ \frac{2}{3} & \frac{1}{3} \end{array}\right)$

and the common prior is $P = \left(\begin{array}{cc} (t_1^a,t_2) & (t_1^b,t_2) \\ \frac{2}{3} & \frac{1}{3} \end{array}\right)$.

Before we explain in detail how to transform a “state-space” structure into a “type-space” structure and *vice versa*, we give one more example, this time with double-sided incomplete information. Consider the situation illustrated in Figure 16.2, which reproduces Figure 14.8 of Chapter 14.

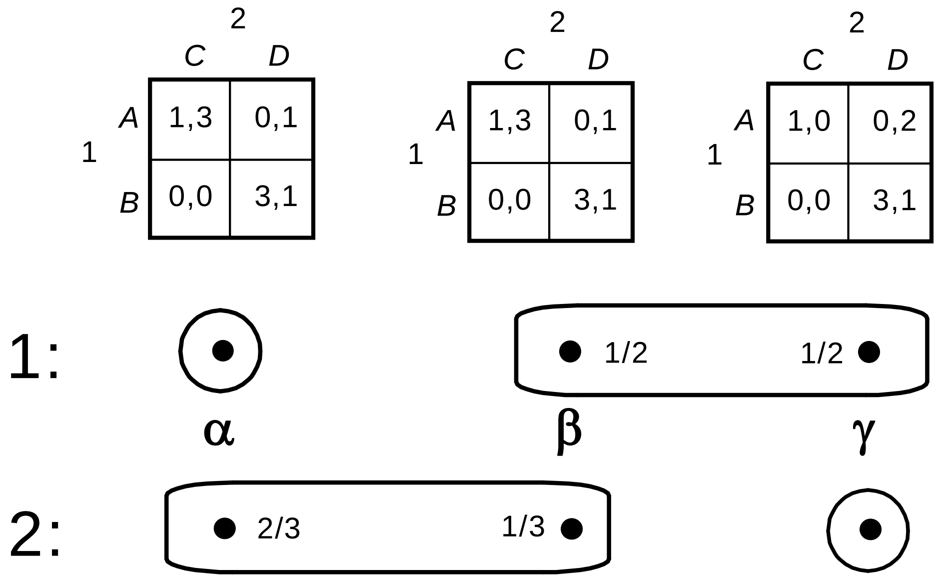


Figure 16.2: A situation with two-sided incomplete information.

In this case we have that each player has two types: we can identify a type of a player with a cell of the player’s information partition.

Thus we have that

$$T_1 = \{t_1^a, t_1^b\}, \quad T_2 = \{t_2^a, t_2^b\}, \quad U_{1,t_1^a} = U_{1,t_1^b} = \begin{pmatrix} (A,C) & (A,D) & (B,C) & (B,D) \\ 1 & 0 & 0 & 3 \end{pmatrix},$$

$$P_{1,t_1^a} = \begin{pmatrix} t_2^a \\ 1 \end{pmatrix}, \quad P_{1,t_1^b} = \begin{pmatrix} t_2^a & t_2^b \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad U_{2,t_2^a} = \begin{pmatrix} (A,C) & (A,D) & (B,C) & (B,D) \\ 3 & 1 & 0 & 1 \end{pmatrix},$$

$$U_{2,t_2^b} = \begin{pmatrix} (A,C) & (A,D) & (B,C) & (B,D) \\ 0 & 2 & 0 & 1 \end{pmatrix}, \quad P_{2,t_2^a} = \begin{pmatrix} t_1^a & t_1^b \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \quad P_{2,t_2^b} = \begin{pmatrix} t_1^b \\ 1 \end{pmatrix}.$$

The common prior is given by $P = \begin{pmatrix} (t_1^a, t_2^a) & (t_1^a, t_2^b) & (t_1^b, t_2^a) & (t_1^b, t_2^b) \\ \frac{2}{4} & 0 & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$.

Thus the two types of Player 1 have the same utility function but different beliefs about the types of Player 2, while the two types of Player 2 differ both in terms of utility function and in terms of beliefs about the types of Player 1.

From these examples it should be clear how to transform a "state-space" model into a "type-space" model.

- First of all, for every Player i, create one type for every cell of Player i's partition, making sure that different cells are associated with different types. In this way we have identified each state with a profile of types.
- Since there is a probability distribution over each information set of Player i, that probability distribution will give us a probability distribution for the associated type of Player i over some set of type-profiles for the other players.
- Finally, since our assumption (to be relaxed in the next section) is that each player knows her own utility function (that is, the utility function of a player does not vary from state to state within the same information set of that player), with each type t_i of Player i is associated a unique utility function U_{i,t_i}.

Conversely, we can convert a "type-space" structure into a "state-space" structure as follows.

- Let the set of states be the set T of profiles of types. For every Player i and for every two states $t, t' \in T$, let t and t' belong to the same information set of Player i (that is, to the same cell of Player i's partition) if and only if Player i's type is the same in t and t': $t' \in I_i(t)$ if and only if $t_i = t_i'$.
- The beliefs of each type of Player i then yield a probability distribution over the information set of Player i corresponding to that type. An example of this transformation is given in Exercise 16.2.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 16.4.1 at the end of this chapter.

16.3 The general case

As pointed out in Chapter 14, it may very well be the case that a rational player does not know her own payoffs (for example, because she is uncertain about what outcomes might occur if she chooses a particular action: see Exercise 14.2, Chapter 14). Uncertainty about a player's own payoffs is compatible with the player knowing her own preferences (that is, how she ranks the outcomes that she considers possible). Definition 16.2.1 is not general enough to encompass such possibilities.

The following, more general, definition allows the utility function of a player to depend not only on the player's own type but also on the types of the other players. Definition 16.3.1 is identical to Definition 16.2.1, except for the starred items and the boldface part in the last item.

Definition 16.3.1 A *static Bayesian game of incomplete information* consists of the following elements:

- a set $I = \{1, \dots n\}$ of players;
- for every Player $i \in I$, a set S_i of strategies (as usual, we denote by S the set of strategy profiles);
- for every Player $i \in I$, a set T_i of possible types; we denote by $T = T_1 \times \dots \times T_n$ the set of profiles of types and by $T_{-i} = T_1 \times \dots \times T_{i-1} \times T_{i+1} \times \cdots \times T_n$ the set of profiles of types for the players other than i;

★ a set $Y \subseteq T$ of relevant profiles of types;

★ for every Player i and for every profile of types $t \in Y$, a utility (or payoff) function $U_{i,t} : S \to \mathbb{R}$;

- for every Player i and for every type $t_i \in T_i$ of Player i, a probability distribution $P_{i,t_i} : T_{-i} \to [0,1]$ representing the beliefs of type t_i about the types of the other players **satisfying the restriction that if $\mathbf{P_{i,t_i}(t_{-i}) > 0}$ then $\mathbf{(t_i, t_{-i}) \in Y}$.**

As in the special case considered in the previous section, also in the general case one can transform a "state-space" structure into a "type-space" structure and vice versa.

Given a "state-space" structure, for every Player i we identify the cells of Player i's partition with the types of Player i (one type for every information set). Since there is a probability distribution over each information set of Player i, that probability distribution will yield the probability distribution for the associated type of Player i over some set of type-profiles for the other players. Finally, having identified each state with a profile of types (since each state belongs to one and only one information set of each player), we can assign to the corresponding type of Player i the utility function of Player i at that state.

We shall illustrate this conversion using the "state-space" structure shown in Figure 16.3 (taken from Exercise 14.2, Chapter 14).

The corresponding "type-space" structure (static Bayesian game of incomplete information) is as follows:

- ○ $I = \{1,2\}$ (letting Bill be Player 1 and Ann Player 2),
- ○ $S_1 = \{g, ng\}$ $\quad S_2 = \{a, r\}$ $\qquad$ (so that $S = \{(g,a),(g,r),(ng,a),(ng,r)\}$).
- ○ There are two types of Player 1 and only one type of Player 2:

  $T_1 = \{t_1^f, t_1^e\}$ (f stands for 'friend' and e for 'enemy'),

  $T_2 = \{t_2\}$ $\qquad$ so that $T = \left\{(t_1^f, t_2), (t_1^e, t_2)\right\}$.
- ○ Thus in this case we have that $Y = T$.

- The utility functions are given by:

$$U_{1,(t_1^f,t_2)} = U_{1,(t_1^e,t_2)} = \begin{pmatrix} (g,a) & (g,r) & (ng,a) & (ng,r) \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

$$U_{2,(t_1^f,t_2)} = \begin{pmatrix} (g,a) & (g,r) & (ng,a) & (ng,r) \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{2,(t_1^e,t_2)} = \begin{pmatrix} (g,a) & (g,r) & (ng,a) & (ng,r) \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

- The beliefs are given by $P_{1,t_1^f} = P_{1,t_1^e} = \begin{pmatrix} t_2 \\ 1 \end{pmatrix}$, $P_{2,t_2} = \begin{pmatrix} t_1^f & t_1^e \\ p & 1-p \end{pmatrix}$

  and the common prior is $P = \begin{pmatrix} (t_1^f,t_2) & (t_1^e,t_2) \\ p & 1-p \end{pmatrix}$.

Conversely, we can convert a "type-space" structure into a "state-space" structure as follows:

- Let the set of states be the set Y of relevant profiles of types.
- For every Player i and for every two states $t,t' \in Y$, let t and t' belong to the same information set of Player i (that is, to the same cell of Player i's partition) if and only if Player i's type is the same in t and t': $t' \in I_i(t)$ if and only if $t_i = t'_i$.
- The beliefs of each type of Player i then yield a probability distribution over the information set of Player i corresponding to that type.

An example of this transformation is given in Exercise 16.3.

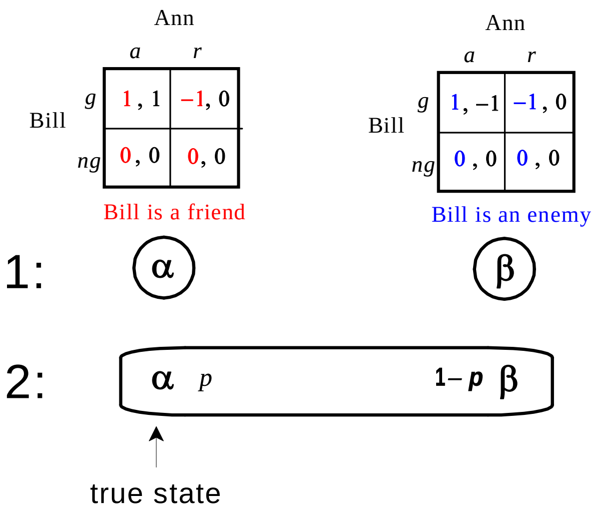


Figure 16.3: The situation of incomplete information described in Exercise 14.2 (Chapter 14) where it is not true that each player knows his/her own payoffs.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 16.4.2 at the end of this chapter.

16.4 Exercises

The answers to the following exercises are in Section 16.5 at the end of this chapter.

16.4.1 Exercises for Section 16.2: Types that know their own payoffs

Exercise 16.1 Transform the situation of incomplete information shown in Figure 16.4, where G_1 and G_2 are the games shown in Figure 16.5, into a “type-space” structure. ■

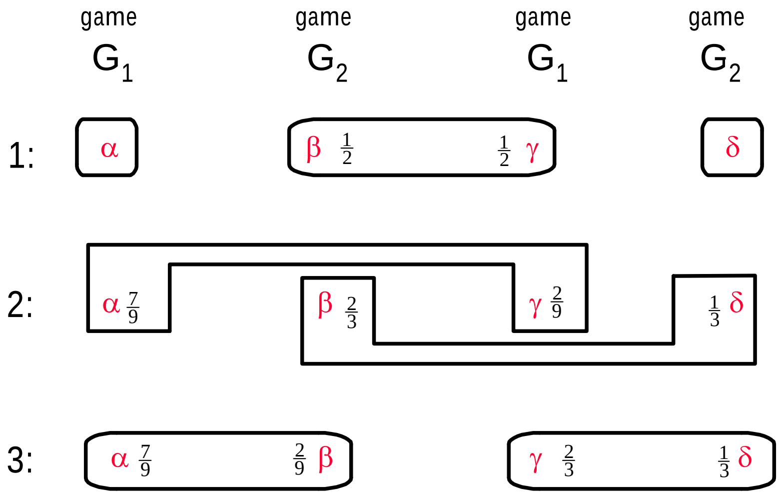


Figure 16.4: A situation of incomplete information. The games G_1 and G_2 are shown in Figure 16.5.

Exercise 16.2 Consider the following Bayesian game of incomplete information:

$I = \{1,2\} \qquad S_1 = \{A,B\} \qquad S_2 = \{C,D\} \qquad T_1 = \{t_1^a, t_1^b\} \qquad T_2 = \{t_2^a, t_2^b, t_2^c\}$

$$U_{1,t_1^a} = \begin{pmatrix} AC & AD & BC & BD \\ 4 & 1 & 0 & 2 \end{pmatrix} \qquad U_{1,t_1^b} = \begin{pmatrix} AC & AD & BC & BD \\ 0 & 0 & 2 & 1 \end{pmatrix}$$

$$U_{2,t_2^a} = U_{2,t_2^b} = \begin{pmatrix} AC & AD & BC & BD \\ 2 & 1 & 2 & 3 \end{pmatrix} \qquad U_{2,t_2^c} = \begin{pmatrix} AC & AD & BC & BD \\ 0 & 2 & 2 & 0 \end{pmatrix}$$

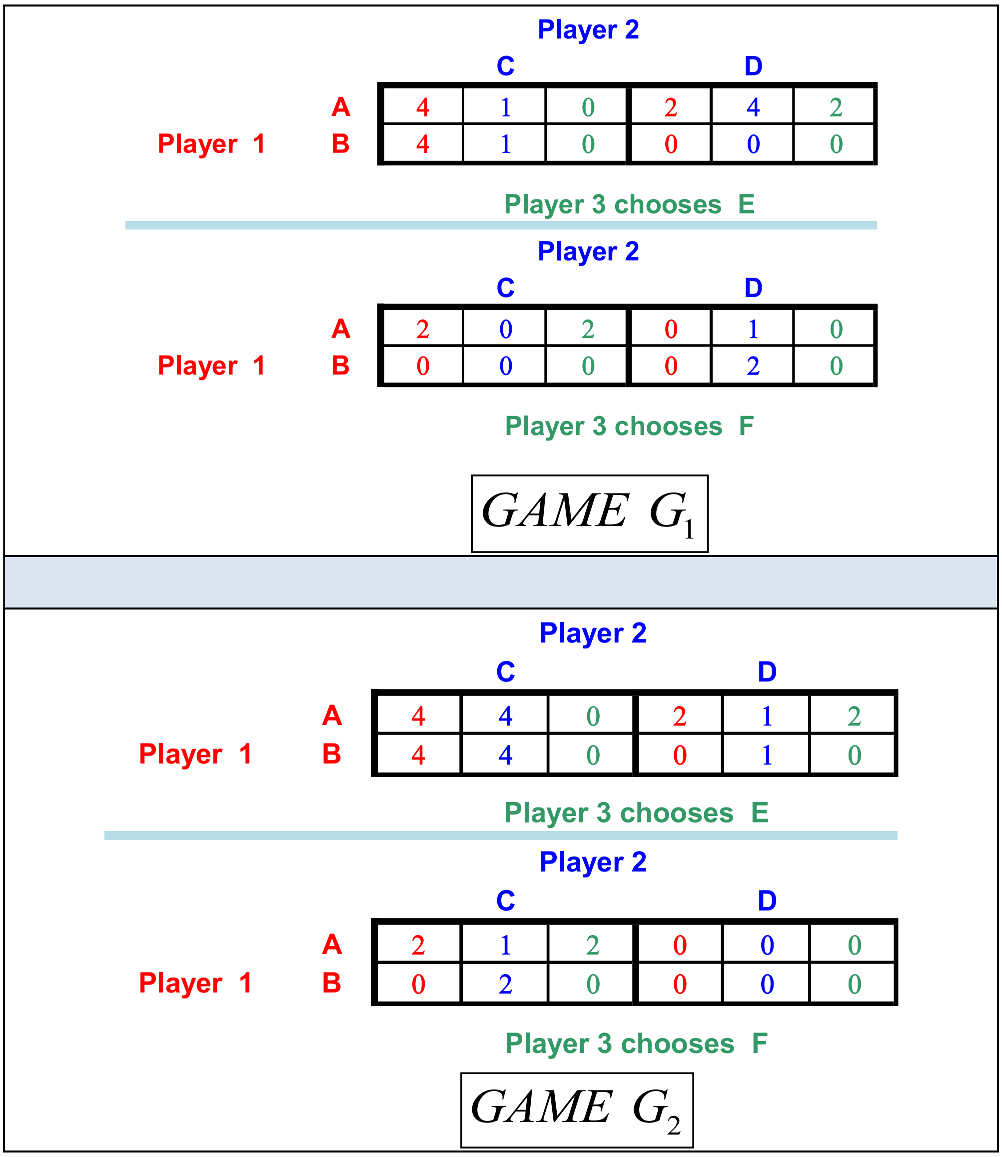
$$P_{1,t_1^a} = \begin{pmatrix} t_2^a & t_2^b & t_2^c \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix}, \quad P_{1,t_1^b} = \begin{pmatrix} t_2^a & t_2^b & t_2^c \\ \frac{4}{13} & \frac{3}{13} & \frac{6}{13} \end{pmatrix}, \quad P_{2,t_2^a} = \begin{pmatrix} t_1^a & t_1^b \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix},$$

$$P_{2,t_2^b} = \begin{pmatrix} t_1^a & t_1^b \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \quad P_{2,t_2^c} = \begin{pmatrix} t_1^a & t_1^b \\ 0 & 1 \end{pmatrix}.$$

(a) Are the beliefs of the types consistent (that is, is there a common prior)?

(b) Transform this type-space structure into an interactive knowledge-belief structure.

■



Player 2

		C			D		
Player 1	A	4	1	0	2	4	2
	B	4	1	0	0	0	0

Player 3 chooses E

Player 2

		C			D		
Player 1	A	2	0	2	0	1	0
	B	0	0	0	0	2	0

Player 3 chooses F

GAME G_1

Player 2

		C			D		
Player 1	A	4	4	0	2	1	2
	B	4	4	0	0	1	0

Player 3 chooses E

Player 2

		C			D		
Player 1	A	2	1	2	0	0	0
	B	0	2	0	0	0	0

Player 3 chooses F

GAME G_2

Figure 16.5: The games for the situation of incomplete information of Figure 16.4.

16.4.2 Exercises for Section 16.3: The general case

Exercise 16.3 Consider the following Bayesian game of incomplete information:

$I = \{1,2,3\} \qquad S_1 = \{A,B\} \qquad S_2 = \{C,D\} \qquad S_3 = \{E,F\}$

$T_1 = \{t_1^a, t_1^b\} \qquad T_2 = \{t_2^a, t_2^b\} \qquad T_3 = \{t_3^a, t_3^b\}$

$Y = \left\{ (t_1^a,t_2^a,t_3^a), (t_1^b,t_2^b,t_3^a), (t_1^b,t_2^a,t_3^b), (t_1^b,t_2^b,t_3^b) \right\}$

$$U_{1,(t_1^a,t_2^a,t_3^a)} = \begin{pmatrix} ACE & ADE & BCE & BDE & ACF & ADF & BCF & BDF \\ 2 & 2 & 3 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$U_{1,(t_1^b,t_2^b,t_3^a)} = U_{1,(t_1^b,t_2^b,t_3^b)} = \begin{pmatrix} ACE & ADE & BCE & BDE & ACF & ADF & BCF & BDF \\ 1 & 2 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$U_{1,(t_1^b,t_2^a,t_3^b)} = \begin{pmatrix} ACE & ADE & BCE & BDE & ACF & ADF & BCF & BDF \\ 0 & 0 & 2 & 1 & 2 & 0 & 0 & 1 \end{pmatrix}$$

$$U_{2,(t_1^a,t_2^a,t_3^a)} = \begin{pmatrix} ACE & ADE & BCE & BDE & ACF & ADF & BCF & BDF \\ 2 & 4 & 2 & 0 & 0 & 1 & 0 & 2 \end{pmatrix}$$

$$U_{2,(t_1^b,t_2^b,t_3^a)} = U_{2,(t_1^b,t_2^b,t_3^b)} = \begin{pmatrix} ACE & ADE & BCE & BDE & ACF & ADF & BCF & BDF \\ 4 & 1 & 4 & 1 & 1 & 0 & 2 & 0 \end{pmatrix}$$

$$U_{2,(t_1^b,t_2^a,t_3^b)} = \begin{pmatrix} ACE & ADE & BCE & BDE & ACF & ADF & BCF & BDF \\ 4 & 3 & 1 & 2 & 0 & 1 & 0 & 2 \end{pmatrix}$$

$$U_{3,(t_1^a,t_2^a,t_3^a)} = \begin{pmatrix} ACE & ADE & BCE & BDE & ACF & ADF & BCF & BDF \\ 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{3,(t_1^b,t_2^b,t_3^a)} = U_{3,(t_1^b,t_2^b,t_3^b)} = \begin{pmatrix} ACE & ADE & BCE & BDE & ACF & ADF & BCF & BDF \\ 3 & 2 & 0 & 1 & 1 & 2 & 2 & 0 \end{pmatrix}$$

$$U_{3,(t_1^b,t_2^a,t_3^b)} = \begin{pmatrix} ACE & ADE & BCE & BDE & ACF & ADF & BCF & BDF \\ 1 & 2 & 0 & 1 & 2 & 0 & 0 & 1 \end{pmatrix}$$

$$P_{1,t_1^a} = \begin{pmatrix} (t_2^a,t_3^a) \\ 1 \end{pmatrix} \qquad P_{1,t_1^b} = \begin{pmatrix} (t_2^b,t_3^a) & (t_2^a,t_3^b) & (t_2^b,t_3^b) \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

$$P_{2,t_2^a} = \begin{pmatrix} (t_1^a,t_3^a) & (t_1^b,t_3^b) \\ \frac{5}{7} & \frac{2}{7} \end{pmatrix} \qquad P_{2,t_2^b} = \begin{pmatrix} (t_1^b,t_3^a) & (t_1^b,t_3^b) \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$P_{3,t_3^a} = \begin{pmatrix} (t_1^a,t_2^a) & (t_1^b,t_2^b) \\ \frac{5}{7} & \frac{2}{7} \end{pmatrix} \qquad P_{3,t_3^b} = \begin{pmatrix} (t_1^b,t_2^a) & (t_1^b,t_2^b) \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

(a) Are the beliefs of the types consistent (that is, is there a common prior)?

(b) Transform this type-space structure into a knowledge-belief structure.

■

Exercise 16.4 — ⋆⋆⋆ Challenging Question ⋆⋆⋆.

Consider the following two-player Bayesian game of incomplete information:

$$I=\{1,2\} \qquad S_1=\{T,B\} \qquad S_2=\{L,R\} \qquad T_1=\{t_1^A,t_1^B\} \qquad T_2=\{t_2^a,t_2^b\},$$

$$U_{1,(t_1^A,t_2^a)}=U_{1,(t_1^A,t_2^b)}=\left(\begin{array}{cccc}(T,L) & (T,R) & (B,L) & (B,R)\\ 6 & 0 & 3 & 3\end{array}\right)$$

$$U_{1,(t_1^B,t_2^a)}=U_{1,(t_1^B,t_2^b)}=\left(\begin{array}{cccc}(T,L) & (T,R) & (B,L) & (B,R)\\ 0 & 3 & 3 & 0\end{array}\right)$$

$$U_{2,(t_1^A,t_2^a)}=U_{2,(t_1^A,t_2^b)}=U_{2,(t_1^B,t_2^a)}=U_{2,(t_1^B,t_2^b)}=\left(\begin{array}{cccc}(T,L) & (T,R) & (B,L) & (B,R)\\ 3 & 9 & 3 & 0\end{array}\right)$$

$$P_{1,t_1^A}=\left(\begin{array}{cc}t_2^a & t_2^b\\ \frac{1}{2} & \frac{1}{2}\end{array}\right) \qquad P_{1,t_1^B}=\left(\begin{array}{cc}t_2^a & t_2^b\\ \frac{3}{5} & \frac{2}{5}\end{array}\right)$$

$$P_{2,t_2^a}=\left(\begin{array}{cc}t_1^A & t_1^B\\ \frac{1}{4} & \frac{3}{4}\end{array}\right) \qquad P_{2,t_2^b}=\left(\begin{array}{cc}t_1^A & t_1^B\\ \frac{1}{3} & \frac{2}{3}\end{array}\right).$$

(a) Transform the situation described above into an interactive knowledge-belief structure. Assume that the true state is where Player 1 is of type t_1^A and Player 2 is of type t_2^a.

(b) Apply the Harsanyi transformation to obtain an extensive-form game.

(c) Write the strategic-form game corresponding to the game of part **(b)**.

(d) Find all the pure-strategy Bayesian Nash equilibria. Does any of these yield a Nash equilibrium in the true game being played?

(e) Of the pure-strategy Bayesian Nash equilibria select one where Player 1 is uncertain about Player 2's choice of action.

For this equilibrium complete the structure of part **(a)** by turning it into a model (that is, associate with each state an action – not a strategy – for each player: see Chapter 10) and verify that at the true state there is common knowledge of rationality.

■

16.5 Solutions to Exercises

Solutions to Exercise 16.1 The structure is as follows (the elements are given as listed in Definition 16.2.1).

$$I=\{1,2,3\} \qquad S_1=\{A,B\} \qquad S_2=\{C,D\} \qquad S_3=\{E,F\}$$

$$T_1=\left\{t_1^a,t_1^b,t_1^c\right\} \qquad T_2=\left\{t_2^a,t_2^b\right\} \qquad T_3=\left\{t_3^a,t_3^b\right\}$$

$$U_{1,t_1^a}=U_{1,t_1^b}=U_{1,t_1^c}=\begin{pmatrix} ACE & ADE & BCE & BDE & ACF & ADF & BCF & BDF \\ 4 & 2 & 4 & 0 & 2 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{2,t_2^a}=\begin{pmatrix} ACE & ADE & BCE & BDE & ACF & ADF & BCF & BDF \\ 1 & 4 & 1 & 0 & 0 & 1 & 0 & 2 \end{pmatrix}$$

$$U_{2,t_2^b}=\begin{pmatrix} ACE & ADE & BCE & BDE & ACF & ADF & BCF & BDF \\ 4 & 1 & 4 & 1 & 1 & 0 & 2 & 0 \end{pmatrix}$$

$$U_{3,t_3^a}=U_{3,t_3^b}=\begin{pmatrix} ACE & ADE & BCE & BDE & ACF & ADF & BCF & BDF \\ 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix}$$

$$P_{1,t_1^a}=\begin{pmatrix} (t_2^a,t_3^a) \\ 1 \end{pmatrix} \qquad P_{1,t_1^b}=\begin{pmatrix} (t_2^b,t_3^a) & (t_2^a,t_3^b) \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad P_{1,t_1^c}=\begin{pmatrix} (t_2^b,t_3^b) \\ 1 \end{pmatrix}$$

$$P_{2,t_2^a}=\begin{pmatrix} (t_1^a,t_3^a) & (t_1^b,t_3^b) \\ \frac{7}{9} & \frac{2}{9} \end{pmatrix} \qquad P_{2,t_2^b}=\begin{pmatrix} (t_1^b,t_3^a) & (t_1^c,t_3^b) \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$P_{3,t_3^a}=\begin{pmatrix} (t_1^a,t_2^a) & (t_1^b,t_2^b) \\ \frac{7}{9} & \frac{2}{9} \end{pmatrix} \qquad P_{3,t_3^b}=\begin{pmatrix} (t_1^b,t_2^a) & (t_1^c,t_2^b) \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

The common prior is given by:

$$P=\begin{pmatrix} t_1^at_2^at_3^a & t_1^at_2^at_3^b & t_1^at_2^bt_3^a & t_1^at_2^bt_3^b & t_1^bt_2^at_3^a & t_1^bt_2^at_3^b & \\ \frac{7}{12} & 0 & 0 & 0 & 0 & \frac{2}{12} & \\ & & & & & & \\ & t_1^bt_2^bt_3^a & t_1^bt_2^bt_3^b & t_1^ct_2^at_3^a & t_1^ct_2^at_3^b & t_1^ct_2^bt_3^a & t_1^ct_2^bt_3^b \\ & \frac{2}{12} & 0 & 0 & 0 & 0 & \frac{1}{12} \end{pmatrix}$$

□

Solutions to Exercise 16.2

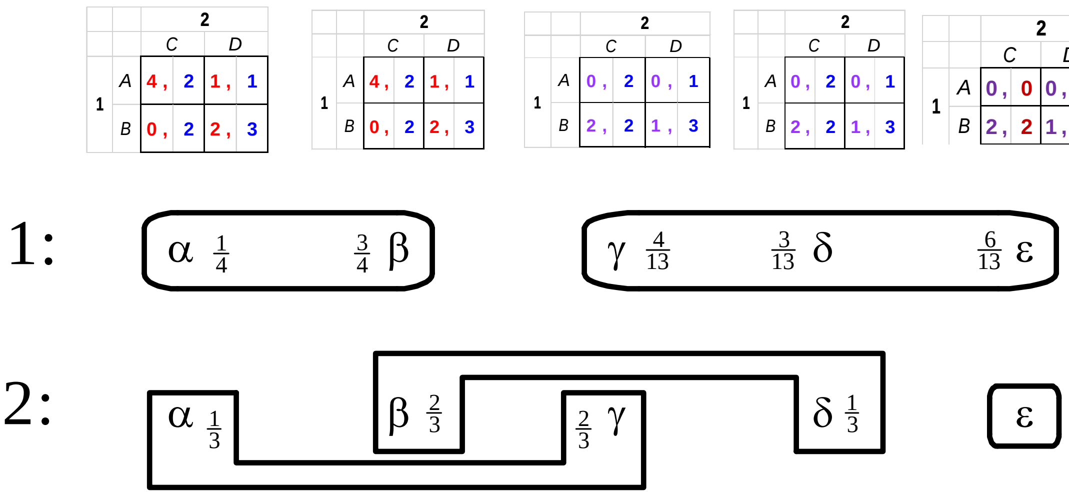
(a) Yes, the following is a common prior:

$$P = \left(\begin{array}{cccccc} (t_1^a, t_2^a) & (t_1^a, t_2^b) & (t_1^a, t_2^c) & (t_1^b, t_2^a) & (t_1^b, t_2^b) & (t_1^b, t_2^c) \\ \frac{2}{21} & \frac{6}{21} & 0 & \frac{4}{21} & \frac{3}{21} & \frac{6}{21} \end{array} \right)$$

(b) The knowledge-belief structure is shown in Figure 16.6. The set of states is

$\Omega = \{\alpha, \beta, \gamma, \delta, \varepsilon\}$, where

$\alpha = (t_1^a, t_2^a)$ $\beta = (t_1^a, t_2^b)$ $\gamma = (t_1^b, t_2^a)$ $\delta = (t_1^b, t_2^b)$, $\varepsilon = (t_1^b, t_2^c)$.



		2	
		C	D
1	A	4 , 2	1 , 1
	B	0 , 2	2 , 3

		2	
		C	D
1	A	4 , 2	1 , 1
	B	0 , 2	2 , 3

		2	
		C	D
1	A	0 , 2	0 , 1
	B	2 , 2	1 , 3

		2	
		C	D
1	A	0 , 2	0 , 1
	B	2 , 2	1 , 3

		2	
		C	D
1	A	0 , 0	0 , 2
	B	2 , 2	1 , 0

Figure 16.6: The interactive knowledge-belief structure representing the Bayesian game of incomplete information of Exercise 16.2.

□

Solutions to Exercise 16.3

(a) Yes: the following is a common prior:

$$P = \begin{pmatrix} t_1^a t_2^a t_3^a & t_1^b t_2^a t_3^b & t_1^b t_2^b t_3^a & t_1^b t_2^b t_3^b \\ \frac{1}{2} & \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \end{pmatrix} \text{ and } P(t) = 0 \text{ for every other } t \in T.$$

(b) The interactive knowledge-belief structure is shown in Figure 16.7, where

$$\alpha = (t_1^a, t_2^a, t_3^a) \qquad \beta = (t_1^b, t_2^b, t_3^a) \qquad \gamma = (t_1^b, t_2^a, t_3^b) \qquad \delta = (t_1^b, t_2^b, t_3^b)$$

and the games G_1, G_2 and G_3 are as shown in Figure 16.8. □

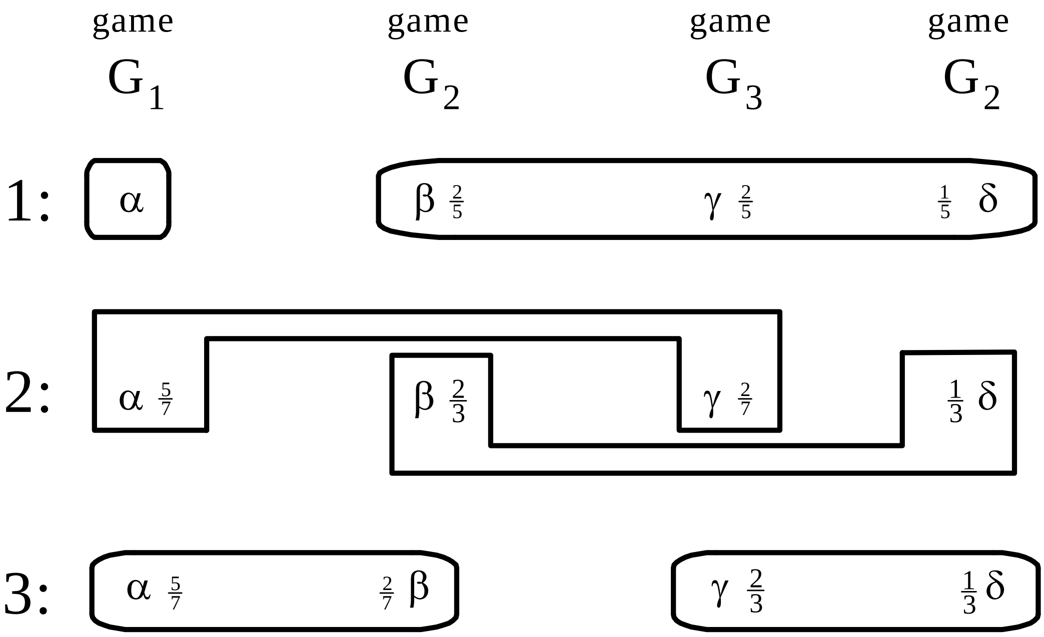
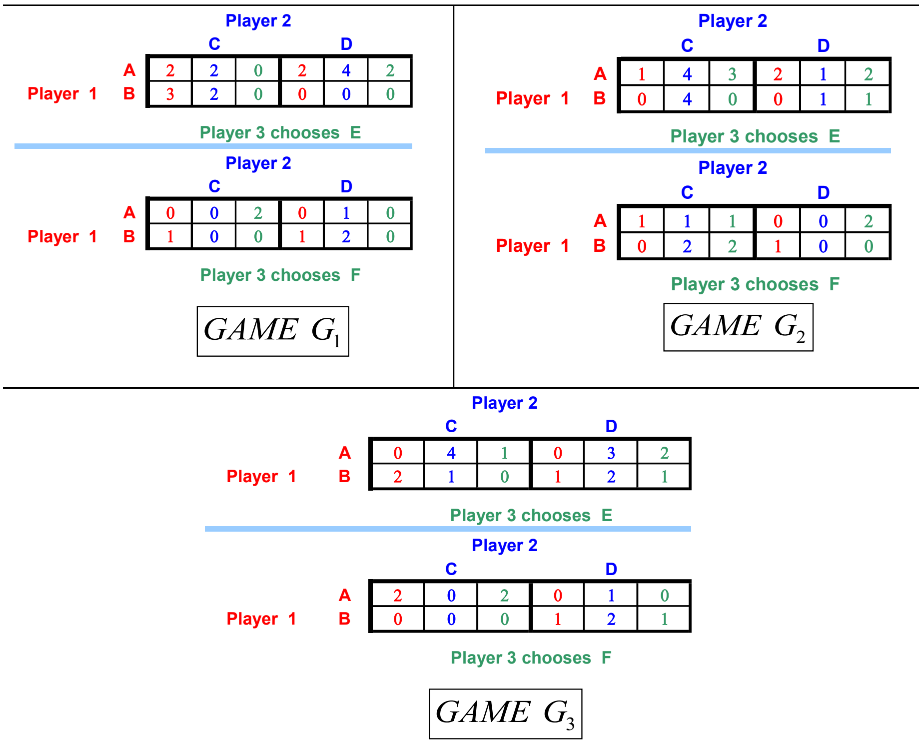


Figure 16.7: The knowledge-belief structure for Exercise 16.3; the games G_1, G_2 and G_3 are as shown in Figure 16.8.



GAME G_1

Player 3 chooses E

Player 1 \ Player 2	C			D		
A	2	2	0	2	4	2
B	3	2	0	0	0	0

Player 3 chooses F

Player 1 \ Player 2	C			D		
A	0	0	2	0	1	0
B	1	0	0	1	2	0

GAME G_2

Player 3 chooses E

Player 1 \ Player 2	C			D		
A	1	4	3	2	1	2
B	0	4	0	0	1	1

Player 3 chooses F

Player 1 \ Player 2	C			D		
A	1	1	1	0	0	2
B	0	2	2	1	0	0

GAME G_3

Player 3 chooses E

Player 1 \ Player 2	C			D		
A	0	4	1	0	3	2
B	2	1	0	1	2	1

Player 3 chooses F

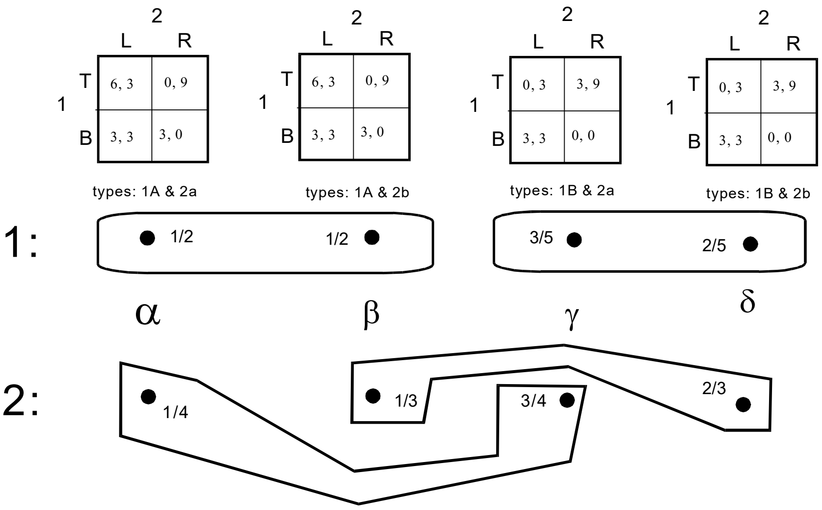
Player 1 \ Player 2	C			D		
A	2	0	2	0	1	0
B	0	0	0	1	2	1

Figure 16.8: The games in the situation of incomplete information shown in Figure 16.7.

Solutions to Exercise 16.4

(a) The interactive knowledge-belief structure is represented in Figure 16.9 where

$$\alpha = \left(t_1^A, t_2^a\right) \qquad \beta = \left(t_1^A, t_2^b\right) \qquad \gamma = \left(t_1^B, t_2^a\right) \qquad \delta = \left(t_1^B, t_2^b\right).$$



types: 1A & 2a

1 \ 2	L	R
T	6, 3	0, 9
B	3, 3	3, 0

types: 1A & 2b

1 \ 2	L	R
T	6, 3	0, 9
B	3, 3	3, 0

types: 1B & 2a

1 \ 2	L	R
T	0, 3	3, 9
B	3, 3	0, 0

types: 1B & 2b

1 \ 2	L	R
T	0, 3	3, 9
B	3, 3	0, 0

Figure 16.9: The interactive knowledge-belief structure for Exercise 16.4.

(b) First of all, there is a common prior: $\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \frac{1}{7} & \frac{1}{7} & \frac{3}{7} & \frac{2}{7} \end{pmatrix}$.

The extensive-form game is shown in Figure 16.10.

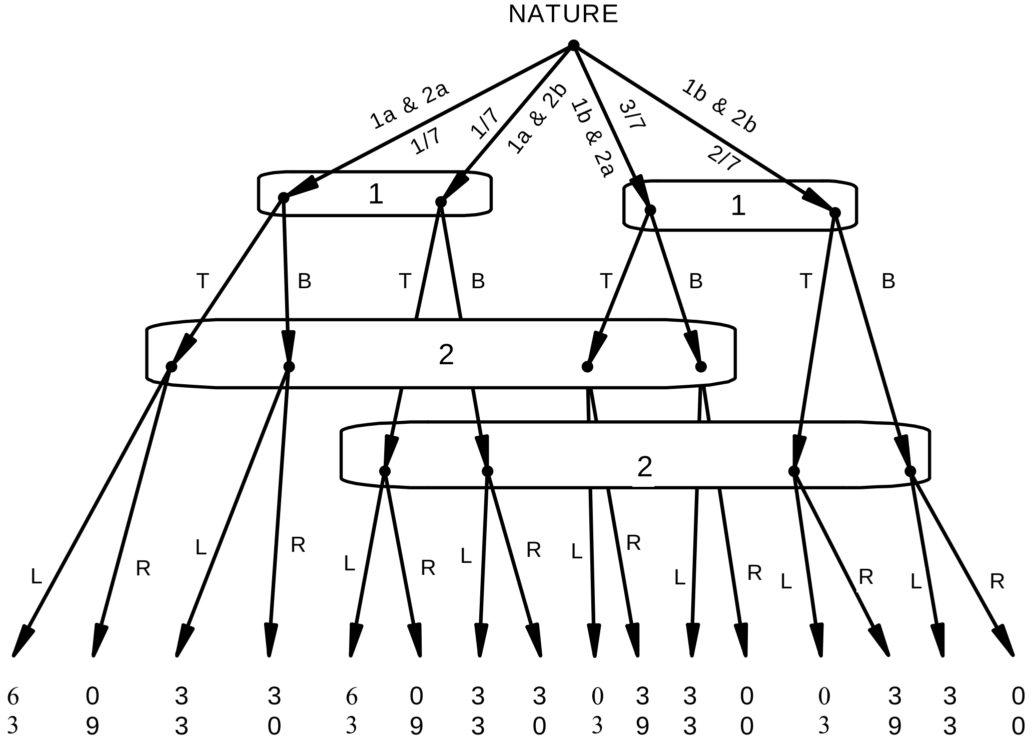


Figure 16.10: The game obtained by applying the Harsanyi transformation to the situation of incomplete information of Figure 16.9.

(c) The strategy profiles give rise to the lotteries shown in Figure 16.11. Thus the strategic form is as shown in Figure 16.12.

(d) The pure-strategy Nash equilibria are: (TB,LL), (TB,LR) and (BT,RR). None of them yields a Nash equilibrium of the game associated with state α, since neither (T,L) nor (B,R) are Nash equilibria of that game.

Player 2

		L if type 2a L if type 2b	L if type 2a R if type 2b	R if type 2a L if type 2b	R if type 2a R if type 2b
Player 1	T if type 1A T if type 1B	1/7 1/7 3/7 2/7 *TL TL TL TL* 6 6 0 0 3 3 3 3	1/7 1/7 3/7 2/7 *TL TR TL TR* 6 0 0 3 3 9 3 9	1/7 1/7 3/7 2/7 *TR TL TR TL* 0 6 3 0 9 3 9 3	1/7 1/7 3/7 2/7 *TR TR TR TR* 0 0 3 3 9 9 9 9
	T if type 1A B if type 1B	1/7 1/7 3/7 2/7 *TL TL BL BL* 6 6 3 3 3 3 3 3	1/7 1/7 3/7 2/7 *TL TR BL BR* 6 0 3 0 3 9 3 0	1/7 1/7 3/7 2/7 *TR TL BR BL* 0 6 0 3 9 3 0 3	1/7 1/7 3/7 2/7 *TR TR BR BR* 0 0 0 0 9 9 0 0
	B if type 1A T if type 1B	1/7 1/7 3/7 2/7 *BL BL TL TL* 3 3 0 0 3 3 3 3	1/7 1/7 3/7 2/7 *BL BR TL TR* 3 3 0 3 3 0 3 9	1/7 1/7 3/7 2/7 *BR BL TR TL* 3 3 3 0 0 3 9 3	1/7 1/7 3/7 2/7 *BR BR TR TR* 3 3 3 3 0 0 9 9
	B if type 1A B if type 1B	1/7 1/7 3/7 2/7 *BL BL BL BL* 3 3 3 3 3 3 3 3	1/7 1/7 3/7 2/7 *BL BR BL BR* 3 3 3 0 3 0 3 0	1/7 1/7 3/7 2/7 *BR BL BR BL* 3 3 0 3 0 3 0 3	1/7 1/7 3/7 2/7 *BR BR BR BR* 3 3 0 0 0 0 0 0

Figure 16.11: The lotteries associated with the strategy profiles for the game of Figure 16.10.

Player 2

		L if type 2a L if type 2b	L if type 2a R if type 2b	R if type 2a L if type 2b	R if type 2a R if type 2b
Player 1	T if type 1A T if type 1B	$\frac{12}{7}, 3$	$\frac{12}{7}, \frac{39}{7}$	$\frac{15}{7}, \frac{45}{7}$	$\frac{15}{7}, 9$
	T if type 1A B if type 1B	$\boxed{\frac{27}{7}, 3}$	$\boxed{\frac{15}{7}, 3}$	$\frac{12}{7}, \frac{18}{7}$	$0, \frac{18}{7}$
	B if type 1A T if type 1B	$\frac{6}{7}, 3$	$\frac{12}{7}, \frac{30}{7}$	$\frac{15}{7}, \frac{36}{7}$	$\boxed{3, \frac{45}{7}}$
	B if type 1A B if type 1B	$3, 3$	$\frac{15}{7}, \frac{12}{7}$	$\frac{12}{7}, \frac{9}{7}$	$\frac{6}{7}, 0$

Figure 16.12: The strategic form of the game of Figure 16.10.

(e) The only pure-strategy equilibrium where Player 1 is uncertain of Player 2's choice is (TB, LR). The corresponding model is shown in Figure 16.13.

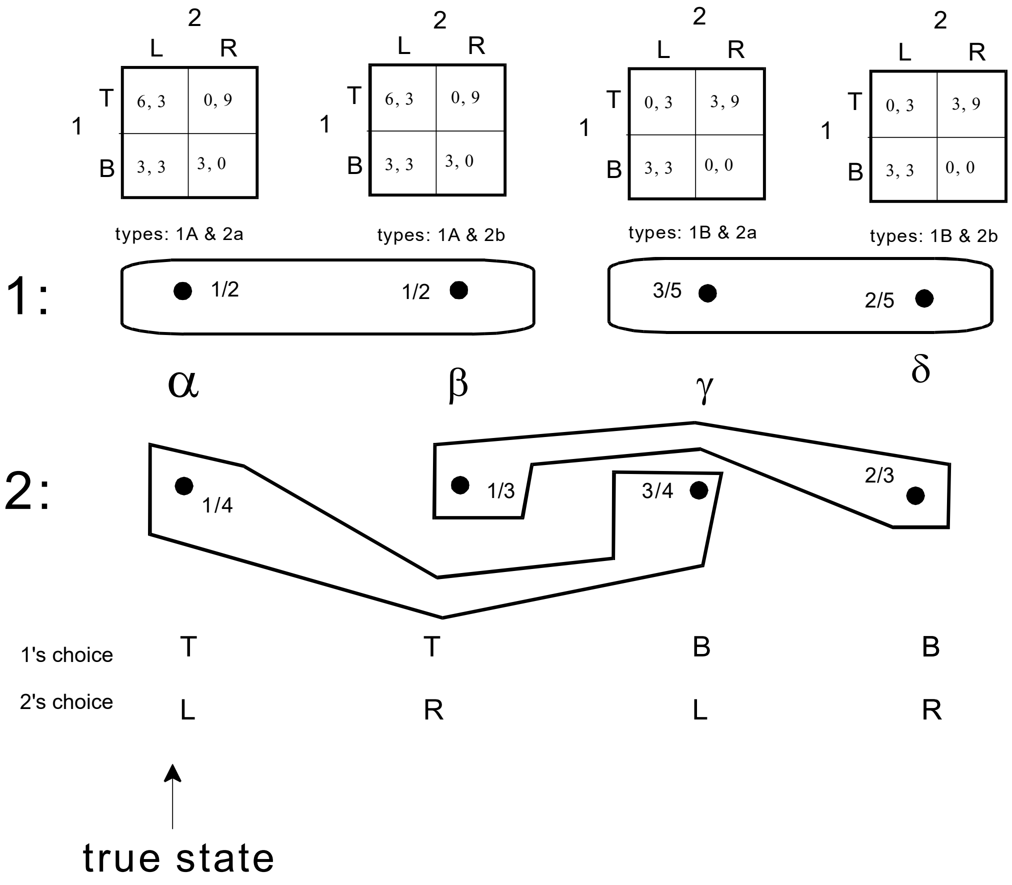


Figure 16.13: A model of the game of Figure 16.10.

There is common knowledge of rationality because at every state both players are rational:

- At states α and β Player 1 has an expected payoff of 3 from both T and B (thus T is a best reply) and at states γ and δ Player 1 has an expected payoff of $\frac{9}{5}$ from B and $\frac{6}{5}$ from T (thus B is a best reply).

- At states α and γ Player 2 has an expected payoff of 3 from L and $\frac{9}{4}$ from R (thus L is a best reply) and at states β and δ Player 2 has an expected payoff of 3 from both L and R (thus R is a best reply). □