Chapter 5

Velocity Kinematics and Statics

In the previous chapter we saw how to calculate the robot end-effector frame's position and orientation for a given set of joint positions. In this chapter we examine the related problem of calculating the twist of the end-effector of an open chain from a given set of joint positions and velocities.

Before we reach the representation of the end-effector twist as $\mathcal{V} \in \mathbb{R}^6$, starting in Section 5.1, let us consider the case where the end-effector configuration is represented by a minimal set of coordinates $x \in \mathbb{R}^m$ and the velocity is given by $\dot{x} = dx/dt \in \mathbb{R}^m$. In this case, the forward kinematics can be written as

$$x(t) = f(\theta(t)),$$

where $\theta \in \mathbb{R}^n$ is a set of joint variables. By the chain rule, the time derivative at time t is

$$\begin{aligned} \dot{x} &= \frac{\partial f(\theta)}{\partial \theta}\frac{d\theta(t)}{dt} = \frac{\partial f(\theta)}{\partial \theta}\dot{\theta} \\ &= J(\theta)\dot{\theta}, \end{aligned}$$

where $J(\theta) \in \mathbb{R}^{m \times n}$ is called the **Jacobian**. The Jacobian matrix represents the linear sensitivity of the end-effector velocity $\dot{x}$ to the joint velocity $\dot{\theta}$, and it is a function of the joint variables θ.

To provide a concrete example, consider a 2R planar open chain (left-hand

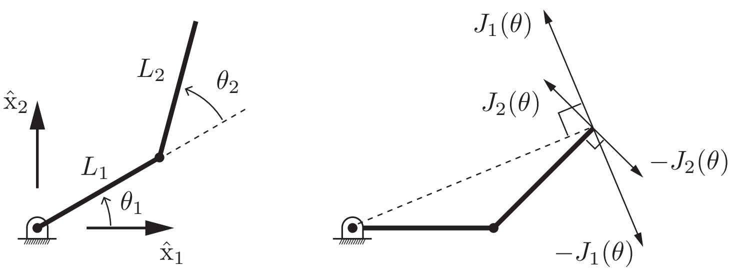


Figure 5.1: (Left) A 2R robot arm. (Right) Columns 1 and 2 of the Jacobian correspond to the endpoint velocity when $\dot{\theta}_1 = 1$ (and $\dot{\theta}_2 = 0$) and when $\dot{\theta}_2 = 1$ (and $\dot{\theta}_1 = 0$), respectively.

side of Figure 5.1) with forward kinematics given by

$$\begin{aligned} x_1 &= L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) \\ x_2 &= L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2). \end{aligned}$$

Differentiating both sides with respect to time yields

$$\begin{aligned} \dot{x}_1 &= -L_1\dot{\theta}_1 \sin\theta_1 - L_2(\dot{\theta}_1 + \dot{\theta}_2)\sin(\theta_1 + \theta_2) \\ \dot{x}_2 &= L_1\dot{\theta}_1 \cos\theta_1 + L_2(\dot{\theta}_1 + \dot{\theta}_2)\cos(\theta_1 + \theta_2), \end{aligned}$$

which can be rearranged into an equation of the form $\dot{x} = J(\theta)\dot{\theta}$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -L_1\sin\theta_1 - L_2\sin(\theta_1+\theta_2) & -L_2\sin(\theta_1+\theta_2) \\ L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) & L_2\cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}. \tag{5.1}$$

Writing the two columns of $J(\theta)$ as $J_1(\theta)$ and $J_2(\theta)$, and the tip velocity $\dot{x}$ as v_{tip}, Equation (5.1) becomes

$$v_{\text{tip}} = J_1(\theta)\dot{\theta}_1 + J_2(\theta)\dot{\theta}_2. \tag{5.2}$$

As long as $J_1(\theta)$ and $J_2(\theta)$ are not collinear, it is possible to generate a tip velocity v_{tip} in any arbitrary direction in the x_1–x_2-plane by choosing appropriate joint velocities $\dot{\theta}_1$ and $\dot{\theta}_2$. Since $J_1(\theta)$ and $J_2(\theta)$ depend on the joint values θ_1 and θ_2, one may ask whether there are any configurations at which $J_1(\theta)$ and $J_2(\theta)$ become collinear. For our example the answer is yes: if θ_2 is $0°$ or $180°$ then, regardless of the value of θ_1, $J_1(\theta)$ and $J_2(\theta)$ will be collinear and the Jacobian $J(\theta)$ becomes a singular matrix. Such configurations are therefore called **singularities**; they are characterized by a situation where the robot tip is unable to generate velocities in certain directions.

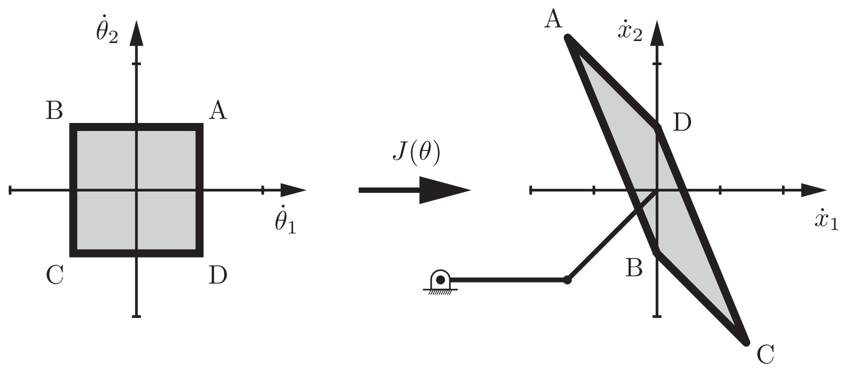


Figure 5.2: Mapping the set of possible joint velocities, represented as a square in the $\dot\theta_1$–$\dot\theta_2$ space, through the Jacobian to find the parallelogram of possible end-effector velocities. The extreme points A, B, C, and D in the joint velocity space map to the extreme points A, B, C, and D in the end-effector velocity space.

Now let's substitute $L_1 = L_2 = 1$ and consider the robot at two different nonsingular postures: $\theta = (0, \pi/4)$ and $\theta = (0, 3\pi/4)$. The Jacobians $J(\theta)$ at these two configurations are

$$J\left(\begin{bmatrix} 0 \\ \pi/4 \end{bmatrix}\right) = \begin{bmatrix} -0.71 & -0.71 \\ 1.71 & 0.71 \end{bmatrix} \quad \text{and} \quad J\left(\begin{bmatrix} 0 \\ 3\pi/4 \end{bmatrix}\right) = \begin{bmatrix} -0.71 & -0.71 \\ 0.29 & -0.71 \end{bmatrix}.$$

The right-hand side of Figure 5.1 illustrates the robot at the $\theta_2 = \pi/4$ configuration. Column i of the Jacobian matrix, $J_i(\theta)$, corresponds to the tip velocity when $\dot\theta_i = 1$ and the other joint velocity is zero. These tip velocities (and therefore columns of the Jacobian) are indicated in Figure 5.1.

The Jacobian can be used to map bounds on the rotational speed of the joints to bounds on v_{tip}, as illustrated in Figure 5.2. Rather than mapping a polygon of joint velocities through the Jacobian as in Figure 5.2, we could instead map a unit circle of joint velocities in the θ_1–θ_2-plane. This circle represents an "iso-effort" contour in the joint velocity space, where total actuator effort is considered to be the sum of squares of the joint velocities. This circle maps through the Jacobian to an ellipse in the space of tip velocities, and this ellipse is referred to as the **manipulability ellipsoid**.[1] Figure 5.3 shows examples of this mapping for the two different postures of the 2R arm. As the manipulator configuration approaches a singularity, the ellipse collapses to a line segment, since the ability of the tip to move in one direction is lost.

[1] A two-dimensional ellipsoid, as in our example, is commonly referred to as an ellipse.

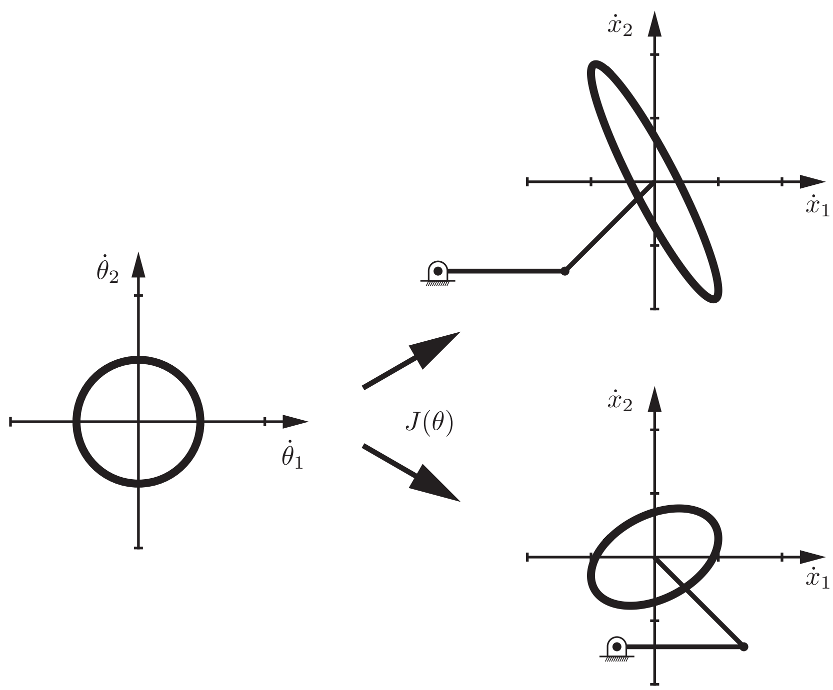


Figure 5.3: Manipulability ellipsoids for two different postures of the 2R planar open chain.

Using the manipulability ellipsoid one can quantify how close a given posture is to a singularity. For example, we can compare the lengths of the major and minor principal semi-axes of the manipulability ellipsoid, respectively denoted $\ell_{\max}$ and $\ell_{\min}$. The closer the ellipsoid is to a circle, i.e., the closer the ratio $\ell_{\max}/\ell_{\min}$ is to 1, the more easily can the tip move in arbitrary directions and thus the more removed it is from a singularity.

The Jacobian also plays a central role in static analysis. Suppose that an external force is applied to the robot tip. What are the joint torques required to resist this external force?

This question can be answered via a conservation of power argument. Assuming that negligible power is used to move the robot, the power measured at the robot's tip must equal the power generated at the joints. Denoting the tip force vector generated by the robot as f_{tip} and the joint torque vector by τ, the

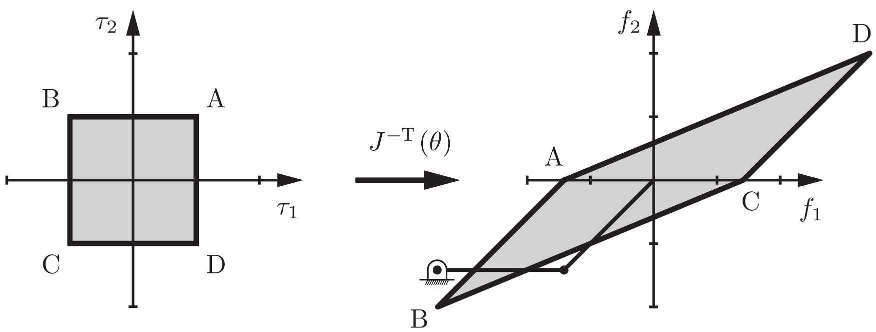


Figure 5.4: Mapping joint torque bounds to tip force bounds.

conservation of power then requires that

$$f_{\mathrm{tip}}^{\mathrm{T}} v_{\mathrm{tip}} = \tau^{\mathrm{T}} \dot{\theta},$$

for all arbitrary joint velocities $\dot{\theta}$. Since $v_{\mathrm{tip}} = J(\theta)\dot{\theta}$, the equality

$$f_{\mathrm{tip}}^{\mathrm{T}} J(\theta)\dot{\theta} = \tau^{\mathrm{T}} \dot{\theta}$$

must hold for all possible $\dot{\theta}$.[2] This can only be true if

$$\tau = J^{\mathrm{T}}(\theta) f_{\mathrm{tip}}. \tag{5.3}$$

The joint torque τ needed to create the tip force f_{tip} is calculated from the equation above.

For our two-link planar chain example, $J(\theta)$ is a square matrix dependent on θ. If the configuration θ is not a singularity then both $J(\theta)$ and its transpose are invertible, and Equation (5.3) can be written

$$f_{\mathrm{tip}} = ((J(\theta))^{\mathrm{T}})^{-1}\tau = J^{-\mathrm{T}}(\theta)\tau. \tag{5.4}$$

Using the equation above one can now determine, under the same static equilibrium assumption, what input torques are needed to generate a desired tip force, e.g., the joint torques needed for the robot tip to push against a wall with

[2] Since the robot is at equilibrium, the joint velocity $\dot{\theta}$ is technically zero. This can be considered the limiting case as $\dot{\theta}$ approaches zero. To be more formal, we could invoke the "principle of virtual work," which deals with infinitesimal joint displacements instead of joint velocities.

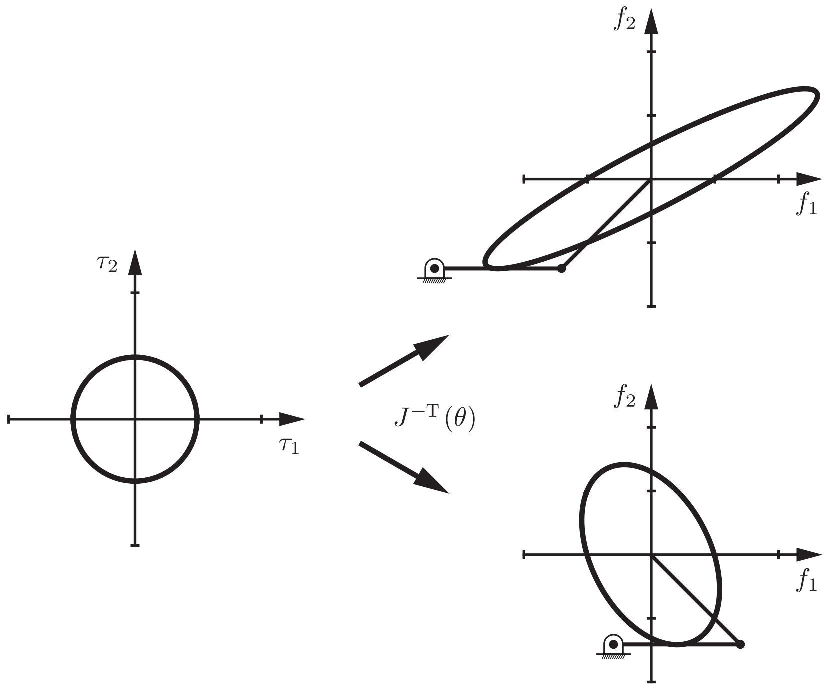


Figure 5.5: Force ellipsoids for two different postures of the 2R planar open chain.

a specified normal force. For a given posture θ of the robot at equilibrium and a set of joint torque limits such as

$$\begin{aligned} -1 \text{ Nm} \leq & \tau_1 \leq 1 \text{ Nm}, \\ -1 \text{ Nm} \leq & \tau_2 \leq 1 \text{ Nm}, \end{aligned}$$

then Equation (5.4) can be used to generate the set of all possible tip forces as indicated in Figure 5.4.

As for the manipulability ellipsoid, a **force ellipsoid** can be drawn by mapping a unit circle "iso-effort" contour in the τ_1–τ_2-plane to an ellipsoid in the f_1–f_2 tip-force plane via the Jacobian transpose inverse $J^{-\mathrm{T}}(\theta)$ (see Figure 5.5). The force ellipsoid illustrates how easily the robot can generate forces in different directions. As is evident from the manipulability and force ellipsoids, if it is easy to generate a tip velocity in a given direction then it is difficult to generate a force in that same direction, and vice versa (Figure 5.6). In fact, for a given robot configuration, the principal axes of the manipulability ellipsoid

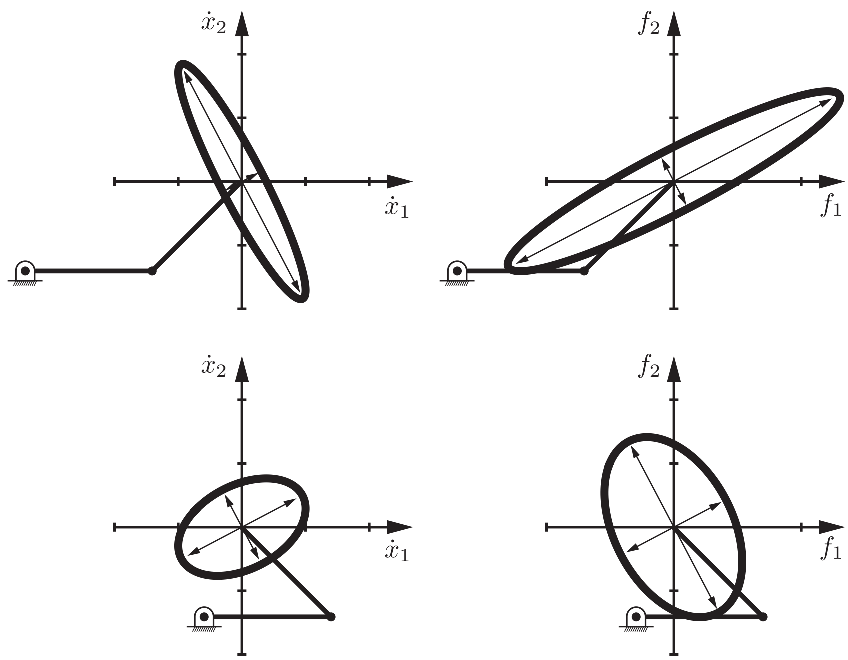


Figure 5.6: Left-hand column: Manipulability ellipsoids at two different arm configurations. Right-hand column: The force ellipsoids for the same two arm configurations.

and force ellipsoid are aligned, and the lengths of the principal semi-axes of the force ellipsoid are the reciprocals of the lengths of the principal semi-axes of the manipulability ellipsoid.

At a singularity, the manipulability ellipsoid collapses to a line segment. The force ellipsoid, on the other hand, becomes infinitely long in a direction orthogonal to the manipulability ellipsoid line segment (i.e., the direction of the aligned links) and skinny in the orthogonal direction. Consider, for example, carrying a heavy suitcase with your arm. It is much easier if your arm hangs straight down under gravity (with your elbow fully straightened at a singularity), because the force you must support passes directly through your joints, therefore requiring no torques about the joints. Only the joint structure bears the load, not the muscles generating torques. The manipulability ellipsoid loses dimension at a singularity and therefore its area drops to zero, but the force ellipsoid's area goes to infinity (assuming that the joints can support the load!).

In this chapter we present methods for deriving the Jacobian for general open chains, where the configuration of the end-effector is expressed as $T \in SE(3)$

and its velocity is expressed as a twist $\mathcal{V}$ in the fixed base frame or the end-effector body frame. We also examine how the Jacobian can be used for velocity and static analysis, including identifying kinematic singularities and determining the manipulability and force ellipsoids. Later chapters on inverse kinematics, motion planning, dynamics, and control make extensive use of the Jacobian and related notions introduced in this chapter.

5.1 Manipulator Jacobian

In the 2R planar open chain example, we saw that, for any joint configuration θ, the tip velocity vector v_{tip} and joint velocity vector $\dot{\theta}$ are linearly related via the Jacobian matrix $J(\theta)$, i.e., $v_{\text{tip}} = J(\theta)\dot{\theta}$. The tip velocity v_{tip} depends on the coordinates of interest for the tip, which in turn determine the specific form of the Jacobian. For example, in the most general case v_{tip} can be taken to be a six-dimensional twist, while, for pure orienting devices such as a wrist, v_{tip} is usually taken to be the angular velocity of the end-effector frame. Other choices for v_{tip} lead to different formulations for the Jacobian. We begin with the general case where v_{tip} is taken to be a six-dimensional twist $\mathcal{V}$.

All the derivations below are mathematical expressions of the same simple idea, embodied in Equation (5.2): given the configuration θ of the robot, the 6-vector $J_i(\theta)$, which is column i of $J(\theta)$, is the twist $\mathcal{V}$ when $\dot{\theta}_i = 1$ and all other joint velocities are zero. This twist is determined in the same way as the joint screw axes were determined in the previous chapter, using a point q_i on joint axis i for revolute joints. The only difference is that the screw axes of the Jacobian depend on the joint variables θ whereas the screw axes for the forward kinematics of Chapter 4 were always for the case $\theta = 0$.

The two standard types of Jacobian that we will consider are: the space Jacobian $J_s(\theta)$ satisfying $\mathcal{V}_s = J_s(\theta)\dot{\theta}$, where each column $J_{si}(\theta)$ corresponds to a screw axis expressed in the fixed space frame {s}; and the body Jacobian $J_b(\theta)$ satisfying $\mathcal{V}_b = J_b(\theta)\dot{\theta}$, where each column $J_{bi}(\theta)$ corresponds to a screw axis expressed in the end-effector frame {b}. We start with the space Jacobian.

5.1.1 Space Jacobian

In this section we derive the relationship between an open chain's joint velocity vector $\dot{\theta}$ and the end-effector's spatial twist $\mathcal{V}_s$. We first recall a few basic properties from linear algebra and linear differential equations: (i) if $A, B \in \mathbb{R}^{n\times n}$ are both invertible then $(AB)^{-1} = B^{-1}A^{-1}$; (ii) if $A \in \mathbb{R}^{n\times n}$ is constant and $\theta(t)$ is a scalar function of t then $d(e^{A\theta})/dt = Ae^{A\theta}\dot{\theta} = e^{A\theta}A\dot{\theta}$; (iii) $(e^{A\theta})^{-1} = e^{-A\theta}$.

Consider an n-link open chain whose forward kinematics is expressed in the following product of exponentials form:

$$T(\theta_1, \ldots, \theta_n) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_n]\theta_n} M. \tag{5.5}$$

The spatial twist $\mathcal{V}_s$ is given by $[\mathcal{V}_s] = \dot{T}T^{-1}$, where

$$\begin{aligned} \dot{T} &= \left(\frac{d}{dt} e^{[\mathcal{S}_1]\theta_1}\right) \cdots e^{[\mathcal{S}_n]\theta_n} M + e^{[\mathcal{S}_1]\theta_1} \left(\frac{d}{dt} e^{[\mathcal{S}_2]\theta_2}\right) \cdots e^{[\mathcal{S}_n]\theta_n} M + \cdots \\ &= [\mathcal{S}_1]\dot{\theta}_1 e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_n]\theta_n} M + e^{[\mathcal{S}_1]\theta_1} [\mathcal{S}_2]\dot{\theta}_2 e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_n]\theta_n} M + \cdots \end{aligned}$$

Also,

$$T^{-1} = M^{-1} e^{-[\mathcal{S}_n]\theta_n} \cdots e^{-[\mathcal{S}_1]\theta_1}.$$

Calculating $\dot{T}T^{-1}$ we obtain

$$[\mathcal{V}_s] = [\mathcal{S}_1]\dot{\theta}_1 + e^{[\mathcal{S}_1]\theta_1}[\mathcal{S}_2]e^{-[\mathcal{S}_1]\theta_1}\dot{\theta}_2 + e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}[\mathcal{S}_3]e^{-[\mathcal{S}_2]\theta_2}e^{-[\mathcal{S}_1]\theta_1}\dot{\theta}_3 + \cdots .$$

The above can also be expressed in vector form by means of the adjoint mapping:

$$\mathcal{V}_s = \underbrace{\mathcal{S}_1}_{J_{s1}} \dot{\theta}_1 + \underbrace{\mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}}(\mathcal{S}_2)}_{J_{s2}} \dot{\theta}_2 + \underbrace{\mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2}}(\mathcal{S}_3)}_{J_{s3}} \dot{\theta}_3 + \cdots \tag{5.6}$$

Observe that $\mathcal{V}_s$ is a sum of n spatial twists of the form

$$\mathcal{V}_s = J_{s1} + J_{s2}(\theta)\dot{\theta}_1 + \cdots + J_{sn}(\theta)\dot{\theta}_n, \tag{5.7}$$

where each $J_{si}(\theta) = (\omega_{si}(\theta), v_{si}(\theta))$ depends explictly on the joint values $\theta \in \mathbb{R}^n$ for $i = 2, \ldots, n$. In matrix form,

$$\begin{aligned} \mathcal{V}_s &= \begin{bmatrix} J_{s1} & J_{s2}(\theta) & \cdots & J_{sn}(\theta) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix} \\ &= J_s(\theta)\dot{\theta}. \end{aligned} \tag{5.8}$$

The matrix $J_s(\theta)$ is said to be the **Jacobian** in fixed (space) frame coordinates, or more simply the **space Jacobian**.

Definition 5.1. Let the forward kinematics of an n-link open chain be expressed in the following product of exponentials form:

$$T = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_n]\theta_n} M. \tag{5.9}$$

The **space Jacobian** $J_s(\theta) \in \mathbb{R}^{6\times n}$ relates the joint rate vector $\dot\theta \in \mathbb{R}^n$ to the spatial twist $\mathcal{V}_s$ via

$$\mathcal{V}_s = J_s(\theta)\dot\theta. \tag{5.10}$$

The ith column of $J_s(\theta)$ is

$$J_{si}(\theta) = \mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}\cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}}(\mathcal{S}_i), \tag{5.11}$$

for $i = 2, \ldots, n$, with the first column $J_{s1} = \mathcal{S}_1$.

To understand the physical meaning behind the columns of $J_s(\theta)$, observe that the ith column is of the form $\mathrm{Ad}_{T_{i-1}}(\mathcal{S}_i)$, where $T_{i-1} = e^{[\mathcal{S}_1]\theta_1}\cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}$; recall that $\mathcal{S}_i$ is the screw axis describing the ith joint axis in terms of the fixed frame with the robot in its zero position. $\mathrm{Ad}_{T_{i-1}}(\mathcal{S}_i)$ is therefore the screw axis describing the ith joint axis after it undergoes the rigid body displacement T_{i-1}. Physically this is the same as moving the first $i-1$ joints from their zero position to the current values $\theta_1, \ldots, \theta_{i-1}$. Therefore, the ith column $J_{si}(\theta)$ of $J_s(\theta)$ is simply the screw vector describing joint axis i, expressed in fixed-frame coordinates, as a function of the joint variables $\theta_1, \ldots, \theta_{i-1}$.

In summary, the procedure for determining the columns J_{si} of $J_s(\theta)$ is similar to the procedure for deriving the joint screws $\mathcal{S}_i$ in the product of exponentials formula $e^{[\mathcal{S}_1]\theta_1}\cdots e^{[\mathcal{S}_n]\theta_n}M$: each column $J_{si}(\theta)$ is the screw vector describing joint axis i, expressed in fixed-frame coordinates, but for arbitrary θ rather than $\theta = 0$.

Example 5.2 (Space Jacobian for a spatial RRRP chain)**.** We now illustrate the procedure for finding the space Jacobian for the spatial RRRP chain of Figure 5.7. Denote the ith column of $J_s(\theta)$ by $J_{si} = (\omega_{si}, v_{si})$. The $[\mathrm{Ad}_{T_{i-1}}]$ matrices are implicit in our calculations of the screw axes of the displaced joint axes.

- Observe that ω_{s1} is constant and in the $\hat{\mathrm{z}}_{\mathrm{s}}$-direction: $\omega_{s1} = (0,0,1)$. Choosing q_1 as the origin, $v_{s1} = (0,0,0)$.

- ω_{s2} is also constant in the $\hat{\mathrm{z}}_{\mathrm{s}}$-direction, so $\omega_{s2} = (0,0,1)$. Choose q_2 as the point $(L_1\mathrm{c}_1, L_1\mathrm{s}_1, 0)$, where $\mathrm{c}_1 = \cos\theta_1$, $\mathrm{s}_1 = \sin\theta_1$. Then $v_{s2} = -\omega_2 \times q_2 = (L_1\mathrm{s}_1, -L_1\mathrm{c}_1, 0)$.

- The direction of ω_{s3} is always fixed in the $\hat{\mathrm{z}}_{\mathrm{s}}$-direction regardless of the values of θ_1 and θ_2, so $\omega_{s3} = (0,0,1)$. Choosing $q_3 = (L_1\mathrm{c}_1 + L_2\mathrm{c}_{12}, L_1\mathrm{s}_1 + L_2\mathrm{s}_{12}, 0)$, where $\mathrm{c}_{12} = \cos(\theta_1+\theta_2)$, $\mathrm{s}_{12} = \sin(\theta_1+\theta_2)$, it follows that $v_{s3} = (L_1\mathrm{s}_1 + L_2\mathrm{s}_{12}, -L_1\mathrm{c}_1 - L_2\mathrm{c}_{12}, 0)$.

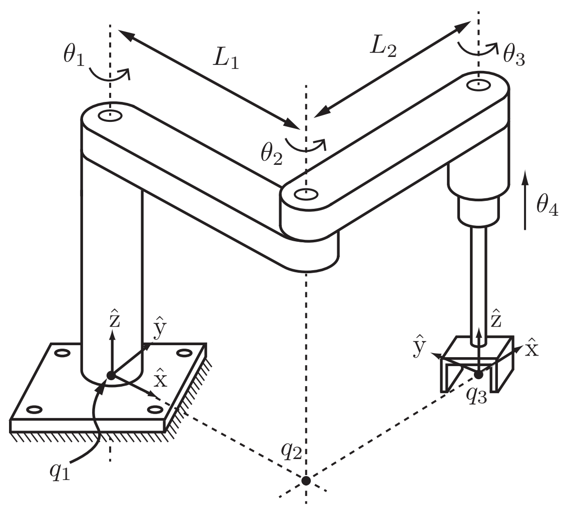


Figure 5.7: Space Jacobian for a spatial RRRP chain.

- Since the final joint is prismatic, $\omega_{s4} = (0, 0, 0)$, and the joint-axis direction is given by $v_{s4} = (0, 0, 1)$.

The space Jacobian is therefore

$$J_s(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & L_1\mathrm{s}_1 & L_1\mathrm{s}_1 + L_2\mathrm{s}_{12} & 0 \\ 0 & -L_1\mathrm{c}_1 & -L_1\mathrm{c}_1 - L_2\mathrm{c}_{12} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Example 5.3 (Space Jacobian for a spatial RRPRRR chain)**.** We now derive the space Jacobian for the spatial RRPRRR chain of Figure 5.8. The base frame is chosen as shown in the figure.

- The first joint axis is in the direction $\omega_{s1} = (0, 0, 1)$. Choosing $q_1 = (0, 0, L_1)$, we get $v_{s1} = -\omega_1 \times q_1 = (0, 0, 0)$.
- The second joint axis is in the direction $\omega_{s2} = (-\mathrm{c}_1, -\mathrm{s}_1, 0)$. Choosing $q_2 = (0, 0, L_1)$, we get $v_{s2} = -\omega_2 \times q_2 = (L_1\mathrm{s}_1, -L_1\mathrm{c}_1, 0)$.

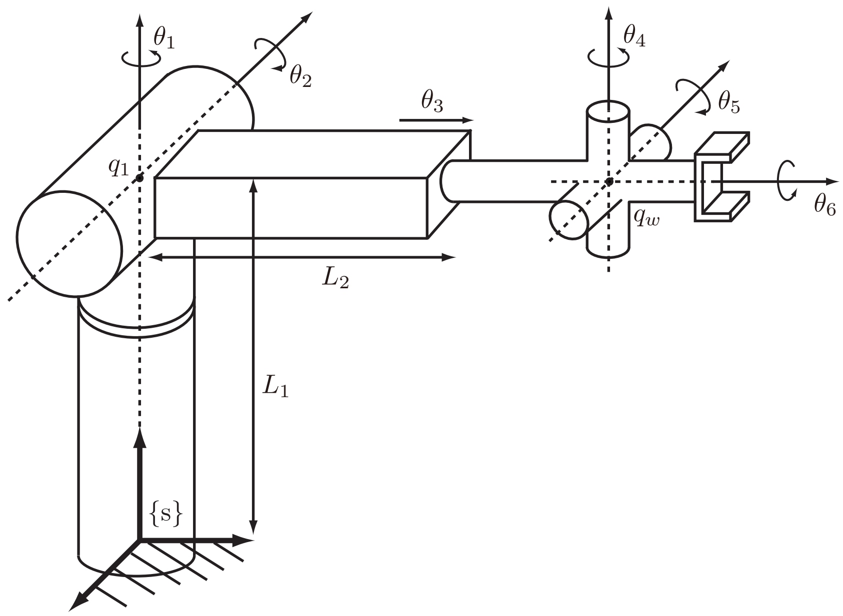


Figure 5.8: Space Jacobian for the spatial RRPRRR chain.

- The third joint is prismatic, so $\omega_{s3} = (0,0,0)$. The direction of the prismatic joint axis is given by

$$v_{s3} = \mathrm{Rot}(\hat{\mathrm{z}}, \theta_1)\mathrm{Rot}(\hat{\mathrm{x}}, -\theta_2) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\mathrm{s}_1\mathrm{c}_2 \\ \mathrm{c}_1\mathrm{c}_2 \\ -\mathrm{s}_2 \end{bmatrix}.$$

- Now consider the wrist portion of the chain. The wrist center is located at the point

$$q_w = \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} + \mathrm{Rot}(\hat{\mathrm{z}}, \theta_1)\mathrm{Rot}(\hat{\mathrm{x}}, -\theta_2) \begin{bmatrix} 0 \\ L_2 + \theta_3 \\ 0 \end{bmatrix} = \begin{bmatrix} -(L_2+\theta_3)\mathrm{s}_1\mathrm{c}_2 \\ (L_2+\theta_3)\mathrm{c}_1\mathrm{c}_2 \\ L_1 - (L_2+\theta_3)\mathrm{s}_2 \end{bmatrix}.$$

  Observe that the directions of the wrist axes depend on θ_1, θ_2, and the preceding wrist axes. These are

$$
\begin{aligned}
\omega_{s4} &= \mathrm{Rot}(\hat{\mathrm{z}},\theta_1)\mathrm{Rot}(\hat{\mathrm{x}},-\theta_2)\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}-\mathrm{s}_1\mathrm{s}_2\\ \mathrm{c}_1\mathrm{s}_2\\ \mathrm{c}_2\end{bmatrix},\\
\omega_{s5} &= \mathrm{Rot}(\hat{\mathrm{z}},\theta_1)\mathrm{Rot}(\hat{\mathrm{x}},-\theta_2)\mathrm{Rot}(\hat{\mathrm{z}},\theta_4)\begin{bmatrix}-1\\0\\0\end{bmatrix} = \begin{bmatrix}-\mathrm{c}_1\mathrm{c}_4+\mathrm{s}_1\mathrm{c}_2\mathrm{s}_4\\ -\mathrm{s}_1\mathrm{c}_4-\mathrm{c}_1\mathrm{c}_2\mathrm{s}_4\\ \mathrm{s}_2\mathrm{s}_4\end{bmatrix},\\
\omega_{s6} &= \mathrm{Rot}(\hat{\mathrm{z}},\theta_1)\mathrm{Rot}(\hat{\mathrm{x}},-\theta_2)\mathrm{Rot}(\hat{\mathrm{z}},\theta_4)\mathrm{Rot}(\hat{\mathrm{x}},-\theta_5)\begin{bmatrix}0\\1\\0\end{bmatrix}\\
&= \begin{bmatrix}-\mathrm{c}_5(\mathrm{s}_1\mathrm{c}_2\mathrm{c}_4+\mathrm{c}_1\mathrm{s}_4)+\mathrm{s}_1\mathrm{s}_2\mathrm{s}_5\\ \mathrm{c}_5(\mathrm{c}_1\mathrm{c}_2\mathrm{c}_4-\mathrm{s}_1\mathrm{s}_4)-\mathrm{c}_1\mathrm{s}_2\mathrm{s}_5\\ -\mathrm{s}_2\mathrm{c}_4\mathrm{c}_5-\mathrm{c}_2\mathrm{s}_5\end{bmatrix}.
\end{aligned}
$$

The space Jacobian can now be computed and written in matrix form as follows:

$$
J_s(\theta) = \begin{bmatrix}\omega_{s1} & \omega_{s2} & 0 & \omega_{s4} & \omega_{s5} & \omega_{s6}\\ 0 & -\omega_{s2}\times q_2 & v_{s3} & -\omega_{s4}\times q_w & -\omega_{s5}\times q_w & -\omega_{s6}\times q_w\end{bmatrix}.
$$

Note that we were able to obtain the entire Jacobian directly, without having to explicitly differentiate the forward kinematic map.

5.1.2 Body Jacobian

In the previous section we derived the relationship between the joint rates and $[\mathcal{V}_s] = \dot{T}T^{-1}$, the end-effector's twist expressed in fixed-frame coordinates. Here we derive the relationship between the joint rates and $[\mathcal{V}_b] = T^{-1}\dot{T}$, the end-effector twist in end-effector-frame coordinates. For this purpose it will be more convenient to express the forward kinematics in the alternative product of exponentials form:

$$
T(\theta) = Me^{[\mathcal{B}_1]\theta_1}e^{[\mathcal{B}_2]\theta_2}\cdots e^{[\mathcal{B}_n]\theta_n}. \tag{5.12}
$$

Computing $\dot{T}$,

$$\begin{aligned}\dot{T} =& Me^{[\mathcal{B}_1]\theta_1}\cdots e^{[\mathcal{B}_{n-1}]\theta_{n-1}}\left(\frac{d}{dt}e^{[\mathcal{B}_n]\theta_n}\right)\\ &+ Me^{[\mathcal{B}_1]\theta_1}\cdots\left(\frac{d}{dt}e^{[\mathcal{B}_{n-1}]\theta_{n-1}}\right)e^{[\mathcal{B}_n]\theta_n}+\cdots\\ =& Me^{[\mathcal{B}_1]\theta_1}\cdots e^{[\mathcal{B}_n]\theta_n}[\mathcal{B}_n]\dot{\theta}_n\\ &+ Me^{[\mathcal{B}_1]\theta_1}\cdots e^{[\mathcal{B}_{n-1}]\theta_{n-1}}[\mathcal{B}_{n-1}]e^{[\mathcal{B}_n]\theta_n}\dot{\theta}_{n-1}+\cdots\\ &+ Me^{[\mathcal{B}_1]\theta_1}[\mathcal{B}_1]e^{[\mathcal{B}_2]\theta_2}\cdots e^{[\mathcal{B}_n]\theta_n}\dot{\theta}_1.\end{aligned}$$

Also,

$$T^{-1} = e^{-[\mathcal{B}_n]\theta_n}\cdots e^{-[\mathcal{B}_1]\theta_1}M^{-1}.$$

Evaluating $T^{-1}\dot{T}$,

$$\begin{aligned}[\mathcal{V}_b] \;=\;& [\mathcal{B}_n]\dot{\theta}_n + e^{-[\mathcal{B}_n]\theta_n}[\mathcal{B}_{n-1}]e^{[\mathcal{B}_n]\theta_n}\dot{\theta}_{n-1}+\cdots\\ &+ e^{-[\mathcal{B}_n]\theta_n}\cdots e^{-[\mathcal{B}_2]\theta_2}[\mathcal{B}_1]e^{[\mathcal{B}_2]\theta_2}\cdots e^{[\mathcal{B}_n]\theta_n}\dot{\theta}_1\end{aligned}$$

or, in vector form,

$$\mathcal{V}_b = \underbrace{\mathcal{B}_n}_{J_{bn}}\dot{\theta}_n + \underbrace{\mathrm{Ad}_{e^{-[\mathcal{B}_n]\theta_n}}(\mathcal{B}_{n-1})}_{J_{b,n-1}}\dot{\theta}_{n-1}+\cdots+\underbrace{\mathrm{Ad}_{e^{-[\mathcal{B}_n]\theta_n}\cdots e^{-[\mathcal{B}_2]\theta_2}}(\mathcal{B}_1)}_{J_{b1}}\dot{\theta}_1. \quad (5.13)$$

The twist $\mathcal{V}_b$ can therefore be expressed as a sum of n body twists:

$$\mathcal{V}_b = J_{b1}(\theta)\dot{\theta}_1+\cdots+J_{bn-1}(\theta)\dot{\theta}_{n-1}+J_{bn}\dot{\theta}_n, \quad (5.14)$$

where each $J_{bi}(\theta) = (\omega_{bi}(\theta), v_{bi}(\theta))$ depends explictly on the joint values θ for $i = 1,\ldots,n-1$. In matrix form,

$$\mathcal{V}_b \;=\; \left[\begin{array}{cccc} J_{b1}(\theta) & \cdots & J_{bn-1}(\theta) & J_{bn}\end{array}\right]\left[\begin{array}{c}\dot{\theta}_1\\ \vdots\\ \dot{\theta}_n\end{array}\right] \;=\; J_b(\theta)\dot{\theta}. \quad (5.15)$$

The matrix $J_b(\theta)$ is the Jacobian in the end-effector- (or body-) frame coordinates and is called, more simply, the **body Jacobian**.

Definition 5.4. Let the forward kinematics of an n-link open chain be expressed in the following product of exponentials form:

$$T = Me^{[\mathcal{B}_1]\theta_1}\cdots e^{[\mathcal{B}_n]\theta_n}. \quad (5.16)$$

The **body Jacobian** $J_b(\theta) \in \mathbb{R}^{6\times n}$ relates the joint rate vector $\dot{\theta} \in \mathbb{R}^n$ to the end-effector twist $\mathcal{V}_b = (\omega_b, v_b)$ via

$$\mathcal{V}_b = J_b(\theta)\dot{\theta}. \tag{5.17}$$

The ith column of $J_b(\theta)$ is

$$J_{bi}(\theta) = \mathrm{Ad}_{e^{-[\mathcal{B}_n]\theta_n}\cdots e^{-[\mathcal{B}_{i+1}]\theta_{i+1}}}(\mathcal{B}_i), \tag{5.18}$$

for $i = n-1,\ldots,1$, with $J_{bn} = \mathcal{B}_n$.

A physical interpretation can be given to the columns of $J_b(\theta)$: each column $J_{bi}(\theta) = (\omega_{bi}(\theta), v_{bi}(\theta))$ of $J_b(\theta)$ is the screw vector for joint axis i, expressed in the coordinates of the end-effector frame rather than those of the fixed frame. The procedure for determining the columns of $J_b(\theta)$ is similar to the procedure for deriving the forward kinematics in the product of exponentials form $Me^{[\mathcal{B}_1]\theta_1}\cdots e^{[\mathcal{B}_n]\theta_n}$, the only difference being that each of the end-effector-frame joint screws $J_{bi}(\theta)$ are expressed for arbitrary θ rather than $\theta = 0$.

5.1.3 Visualizing the Space and Body Jacobian

Another, perhaps simpler, way to derive the formulas for the ith column of the space Jacobian (5.11) and the ith column of the body Jacobian (5.18) comes from inspecting the 5R robot in Figure 5.9. Let's start with the third column, J_{s3}, of the space Jacobian using the left-hand column of Figure 5.9.

The screw corresponding to joint axis 3 is written as $\mathcal{S}_3$ in {s} when the robot is at its zero configuration. Clearly the joint variables θ_3, θ_4, and θ_5 have no impact on the spatial twist resulting from the joint velocity $\dot{\theta}_3$, because they do not displace axis 3 relative to {s}. So we fix those joint variables at zero, making the robot from joint 2 outward a rigid body B. If $\theta_1 = 0$ and θ_2 is arbitrary then the frame {s′} at $T_{ss'} = e^{[\mathcal{S}_2]\theta_2}$ is at the same position and orientation relative to B as frame {s} when $\theta_1 = \theta_2 = 0$. Now, if θ_1 is also arbitrary then the frame {s″} at $T_{ss''} = e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}$ is at the same position and orientation relative to B as frame {s} when $\theta_1 = \theta_2 = 0$. Thus $\mathcal{S}_3$ represents the screw relative to {s″} for arbitrary joint angles θ_1 and θ_2. The column J_{s3}, though, is the screw relative to {s}. The mapping that changes the frame of representation of $\mathcal{S}_3$ from {s″} to {s} is $[\mathrm{Ad}_{T_{ss''}}] = [\mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}}]$, i.e., $J_{s3} = [\mathrm{Ad}_{T_{ss''}}]\mathcal{S}_3$, precisely Equation (5.11) for joint $i = 3$. Equation (5.11) is the generalization of the reasoning above for any joint $i = 2,\ldots,n$.

Now let's derive the third column, J_{b3}, of the body Jacobian by inspecting the right-hand column of Figure 5.9. The screw corresponding to joint 3 is written $\mathcal{B}_3$ in {b} when the robot is at its zero configuration. Clearly the joint

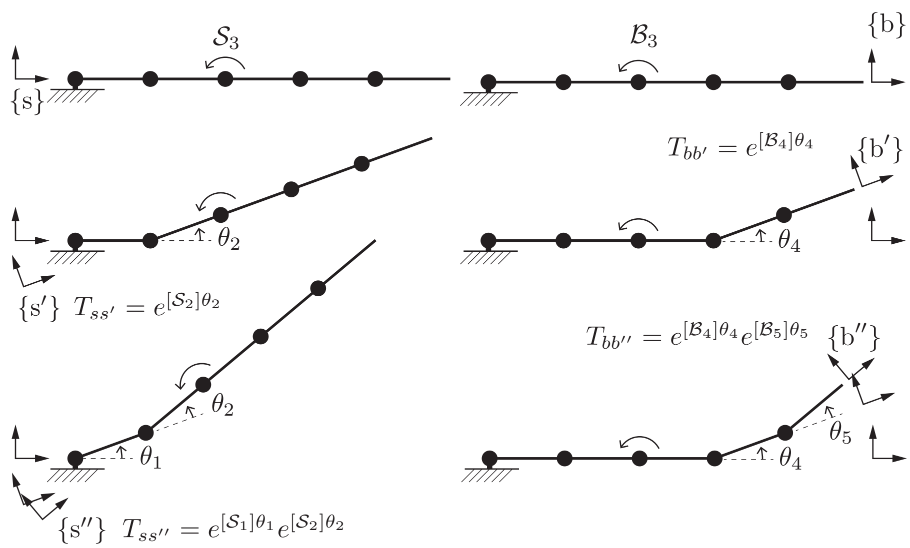


Figure 5.9: A 5R robot. (Left-hand column) Derivation of J_{s3}, the third column of the space Jacobian. (Right-hand column) Derivation of J_{b3}, the third column of the body Jacobian.

variables θ_1, θ_2, and θ_3 have no impact on the body twist resulting from the joint velocity $\dot{\theta}_3$, because they do not displace axis 3 relative to {b}. So we fix those joint variables at zero, making the robot a rigid body B from the base to joint 4. If $\theta_5 = 0$ and θ_4 is arbitrary, then the frame {b′} at $T_{bb'} = e^{[\mathcal{B}_4]\theta_4}$ is the new end-effector frame. Now if θ_5 is also arbitrary, then the frame {b″} at $T_{bb''} = e^{[\mathcal{B}_4]\theta_4} e^{[\mathcal{B}_5]\theta_5}$ is the new end-effector frame. The column J_{b3} is simply the screw axis of joint 3 expressed in {b″}. Since $\mathcal{B}_3$ is expressed in {b}, we have

$$
\begin{aligned}
J_{b3} &= [\mathrm{Ad}_{T_{b''b}}]\mathcal{B}_3 \\
&= [\mathrm{Ad}_{T_{bb''}^{-1}}]\mathcal{B}_3 \\
&= [\mathrm{Ad}_{e^{-[\mathcal{B}_5]\theta_5} e^{-[\mathcal{B}_4]\theta_4}}]\mathcal{B}_3.
\end{aligned}
$$

where we have made use of the fact that $(T_1T_2)^{-1} = T_2^{-1}T_1^{-1}$. This formula for J_{b3} is precisely Equation (5.18) for joint $i = 3$. Equation (5.18) is the generalization of the reasoning above for any joint $i = 1, \ldots, n-1$.

5.1.4 Relationship between the Space and Body Jacobian

Denoting the fixed frame by {s} and the end-effector frame by {b}, the forward kinematics can be written as $T_{sb}(\theta)$. The twist of the end-effector frame can be written in terms of the fixed- and end-effector-frame coordinates as

$$\begin{aligned}[\mathcal{V}_s] &= \dot{T}_{sb}T_{sb}^{-1}, \\ [\mathcal{V}_b] &= T_{sb}^{-1}\dot{T}_{sb},\end{aligned}$$

with $\mathcal{V}_s$ and $\mathcal{V}_b$ related by $\mathcal{V}_s = \mathrm{Ad}_{T_{sb}}(\mathcal{V}_b)$ and $\mathcal{V}_b = \mathrm{Ad}_{T_{bs}}(\mathcal{V}_s)$. The twists $\mathcal{V}_s$ and $\mathcal{V}_b$ are also related to their respective Jacobians via

$$\mathcal{V}_s = J_s(\theta)\dot{\theta}, \tag{5.19}$$

$$\mathcal{V}_b = J_b(\theta)\dot{\theta}. \tag{5.20}$$

Equation (5.19) can therefore be written

$$\mathrm{Ad}_{T_{sb}}(\mathcal{V}_b) = J_s(\theta)\dot{\theta}. \tag{5.21}$$

Applying $[\mathrm{Ad}_{T_{bs}}]$ to both sides of Equation (5.21) and using the general property $[\mathrm{Ad}_X][\mathrm{Ad}_Y] = [\mathrm{Ad}_{XY}]$ of the adjoint map, we obtain

$$\mathrm{Ad}_{T_{bs}}(\mathrm{Ad}_{T_{sb}}(\mathcal{V}_b)) = \mathrm{Ad}_{T_{bs}T_{sb}}(\mathcal{V}_b) = \mathcal{V}_b = \mathrm{Ad}_{T_{bs}}(J_s(q)\dot{\theta}).$$

Since we also have $\mathcal{V}_b = J_b(\theta)\dot{\theta}$ for all $\dot{\theta}$, it follows that $J_s(\theta)$ and $J_b(\theta)$ are related by

$$J_b(\theta) = \mathrm{Ad}_{T_{bs}}\left(J_s(\theta)\right) = [\mathrm{Ad}_{T_{bs}}]J_s(\theta). \tag{5.22}$$

The space Jacobian can in turn be obtained from the body Jacobian via

$$J_s(\theta) = \mathrm{Ad}_{T_{sb}}\left(J_b(\theta)\right) = [\mathrm{Ad}_{T_{sb}}]J_b(\theta). \tag{5.23}$$

The fact that the space and body Jacobians, and the space and body twists, are similarly related by the adjoint map should not be surprising since each column of the space or body Jacobian corresponds to a twist.

An important implication of Equations (5.22) and (5.23) is that $J_b(\theta)$ and $J_s(\theta)$ always have the same rank; this is shown explicitly in Section 5.3 on singularity analysis.

5.1.5 Alternative Notions of the Jacobian

The space and body Jacobians derived above are matrices that relate joint rates to the twist of the end-effector. There exist alternative notions of the

Jacobian that are based on a representation of the end-effector configuration using a minimum set of coordinates q. Such representations are particularly relevant when the task space is considered to be a subspace of $SE(3)$. For example, the configuration of the end-effector of a planar robot could be treated as $q = (x, y, \theta) \in \mathbb{R}^3$ instead of as an element of $SE(2)$.

When using a minimum set of coordinates, the end-effector velocity is not given by a twist $\mathcal{V}$ but by the time derivative of the coordinates $\dot{q}$, and the Jacobian J_a in the velocity kinematics $\dot{q} = J_a(\theta)\dot{\theta}$ is sometimes called the **analytic Jacobian** as opposed to the **geometric Jacobian** in space and body form, described above.[3]

For an $SE(3)$ task space, a typical choice of the minimal coordinates $q \in \mathbb{R}^6$ includes three coordinates for the origin of the end-effector frame in the fixed frame and three coordinates for the orientation of the end-effector frame in the fixed frame. Example coordinates for the orientation include the Euler angles (see Appendix B) and exponential coordinates for rotation.

Example 5.5 (Analytic Jacobian with exponential coordinates for rotation)**.** In this example, we find the relationship between the geometric Jacobian J_b in the body frame and an analytic Jacobian J_a that uses exponential coordinates $r = \hat{\omega}\theta$ to represent the orientation. (Recall that $\|\hat{\omega}\| = 1$ and $\theta \in [0, \pi]$.)

First, consider an open chain with n joints and the body Jacobian

$$\mathcal{V}_b = J_b(\theta)\dot{\theta},$$

where $J_b(\theta) \in \mathbb{R}^{6\times n}$. The angular and linear velocity components of $\mathcal{V}_b = (\omega_b, v_b)$ can be written explicitly as

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = J_b(\theta)\dot{\theta} = \begin{bmatrix} J_\omega(\theta) \\ J_v(\theta) \end{bmatrix} \dot{\theta},$$

where J_ω is the $3 \times n$ matrix corresponding to the top three rows of J_b and J_v is the $3 \times n$ matrix corresponding to the bottom three rows of J_b.

Now suppose that our minimal set of coordinates $q \in \mathbb{R}^6$ is given by $q = (r, x)$, where $x \in \mathbb{R}^3$ is the position of the origin of the end-effector frame and $r = \hat{\omega}\theta \in \mathbb{R}^3$ is the exponential coordinate representation for the rotation. The

[3] The term "geometric Jacobian" has also been used to describe the relationship between joint rates and a representation of the end-effector velocity that combines the rate of change of the position coordinates of the end-effector (which is neither the linear portion of a body twist nor the linear portion of a spatial twist) and a representation of the angular velocity. Unlike a body or spatial twist, which depends only on the body or space frame, respectively, this "hybrid" notion of a spatial velocity depends on the definition of both frames.

coordinate time derivative $\dot{x}$ is related to v_b by a rotation that gives v_b in the fixed coordinates:

$$\dot{x} = R_{sb}(\theta)v_b = R_{sb}(\theta)J_v(\theta)\dot{\theta},$$

where $R_{sb}(\theta) = e^{[r]} = e^{[\hat{\omega}]\theta}$.

The time-derivative $\dot{r}$ is related to the body angular velocity ω_b by

$$\omega_b = A(r)\dot{r},$$

where

$$A(r) = I - \frac{1 - \cos\|r\|}{\|r\|^2}[r] + \frac{\|r\| - \sin\|r\|}{\|r\|^3}[r]^2.$$

(The derivation of this formula is explored in Exercise 5.10.) Provided that the matrix $A(r)$ is invertible, $\dot{r}$ can be obtained from ω_b:

$$\dot{r} = A^{-1}(r)\omega_b = A^{-1}(r)J_\omega(\theta)\dot{\theta}.$$

Putting these together, we obtain

$$\dot{q} = \begin{bmatrix} \dot{r} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} A^{-1}(r) & 0 \\ 0 & R_{sb} \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}, \tag{5.24}$$

i.e., the analytic Jacobian J_a is related to the body Jacobian J_b by

$$J_a(\theta) = \begin{bmatrix} A^{-1}(r) & 0 \\ 0 & R_{sb}(\theta) \end{bmatrix} \begin{bmatrix} J_\omega(\theta) \\ J_v(\theta) \end{bmatrix} = \begin{bmatrix} A^{-1}(r) & 0 \\ 0 & R_{sb}(\theta) \end{bmatrix} J_b(\theta). \tag{5.25}$$

5.1.6 Looking Ahead to Inverse Velocity Kinematics

In the above sections we asked the question "What twist results from a given set of joint velocities?" The answer, written independently of the frame in which the twists are represented, was given by

$$\mathcal{V} = J(\theta)\dot{\theta}.$$

Often we are interested in the inverse question: given a desired twist $\mathcal{V}$, what joint velocities $\dot{\theta}$ are needed? This is a question of inverse velocity kinematics, which is discussed in more detail in Section 6.3. Briefly, if $J(\theta)$ is square (so that the number of joints n is equal to six, the number of elements of a twist) and of full rank then $\dot{\theta} = J^{-1}(\theta)\mathcal{V}$. If $n \neq 6$ or the robot is at a singularity, however, then $J(\theta)$ is not invertible. In the case $n < 6$, arbitrary twists $\mathcal{V}$ cannot be achieved – the robot does not have enough joints. If $n > 6$ then we call the robot **redundant**. In this case, a desired twist $\mathcal{V}$ places six constraints on the joint rates, and the remaining $n-6$ freedoms correspond to internal motions of the robot that are not evident in the motion of the end-effector. For example, if you consider your arm from your shoulder to your palm as a seven-joint open chain, when you place your palm at a fixed configuration in space (e.g., on the surface of a table), you still have one internal degree of freedom corresponding to the position of your elbow.

5.2 Statics of Open Chains

Using our familiar principle of conservation of power, we have

power at the joints = (power to move the robot) + (power at the end-effector)

and, considering the robot to be at static equilibrium (no power is being used to move the robot), we can equate the power at the joints to the power at the end-effector,[4]

$$\tau^{\mathrm{T}}\dot{\theta} = \mathcal{F}_b^{\mathrm{T}}\mathcal{V}_b,$$

where τ is the column vector of the joint torques. Using the identity $\mathcal{V}_b = J_b(\theta)\dot{\theta}$, we get

$$\tau = J_b^{\mathrm{T}}(\theta)\mathcal{F}_b,$$

relating the joint torques to the wrench written in the end-effector frame. Similarly,

$$\tau = J_s^{\mathrm{T}}(\theta)\mathcal{F}_s$$

in the fixed space frame. Independently of the choice of the frame, we can simply write

$$\tau = J^{\mathrm{T}}(\theta)\mathcal{F}. \tag{5.26}$$

If an external wrench $-\mathcal{F}$ is applied to the end-effector when the robot is at equilibrium with joint values θ, Equation (5.26) calculates the joint torques τ needed to generate the opposing wrench $\mathcal{F}$, keeping the robot at equilibrium.[5] This is important in force control of a robot, for example.

One could also ask the opposite question, namely, what is the end-effector wrench generated by a given set of joint torques? If J^{T} is a 6×6 invertible matrix, then clearly $\mathcal{F} = J^{-T}(\theta)\tau$. If the number of joints n is not equal to six then J^{T} is not invertible, and the question is not well posed.

[4] We are considering the limiting case as $\dot{\theta}$ goes to zero, consistent with our assumption that the robot is at equilibrium.

[5] If the robot has to support itself against gravity to maintain static equilibrium, the torques τ must be added to the torques that offset gravity.

If the robot is redundant ($n > 6$) then, even if the end-effector is embedded in concrete, the robot is not immobilized and the joint torques may cause internal motions of the links. The static equilibrium assumption is no longer satisfied, and we need to include dynamics to know what will happen to the robot.

If $n \leq 6$ and $J^{\mathrm{T}} \in \mathbb{R}^{n\times 6}$ has rank n then embedding the end-effector in concrete will immobilize the robot. If $n < 6$, no matter what τ we choose, the robot cannot *actively* generate forces in the $6-n$ wrench directions defined by the null space of J^{T},

$$\mathrm{Null}(J^{\mathrm{T}}(\theta)) = \{\mathcal{F} \mid J^{\mathrm{T}}(\theta)\mathcal{F} = 0\},$$

since no actuators act in these directions. The robot can, however, resist arbitrary externally applied wrenches in the space $\mathrm{Null}(J^{\mathrm{T}}(\theta))$ without moving, owing to the lack of joints that would allow motions due to these forces. For example, consider a motorized rotating door with a single revolute joint ($n = 1$) and an end-effector frame at the door knob. The door can only actively generate a force at the knob that is tangential to the allowed circle of motion of the knob (defining a single direction in the wrench space), but it can resist arbitrary wrenches in the orthogonal five-dimensional wrench space without moving.

5.3 Singularity Analysis

The Jacobian allows us to identify postures at which the robot's end-effector loses the ability to move instantaneously in one or more directions. Such a posture is called a **kinematic singularity**, or simply a **singularity**. Mathematically, a singular posture is one in which the Jacobian $J(\theta)$ fails to be of maximal rank. To understand why, consider the body Jacobian $J_b(\theta)$, whose columns are denoted J_{bi}, $i = 1, \ldots, n$. Then

$$\begin{aligned}
\mathcal{V}_b &= \left[\begin{array}{cccc} J_{b1}(\theta) & \cdots & J_{bn-1}(\theta) & J_{bn} \end{array}\right] \left[\begin{array}{c} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_{n-1} \\ \dot{\theta}_n \end{array}\right] \\
&= J_{b1}(\theta)\dot{\theta}_1 + \cdots + J_{bn-1}(\theta)\dot{\theta}_{n-1} + J_{bn}\dot{\theta}_n.
\end{aligned}$$

Thus, the tip frame can achieve twists that are linear combinations of the J_{bi}. As long as $n \geq 6$, the maximum rank that $J_b(\theta)$ can attain is six. Singular postures correspond to those values of θ at which the rank of $J_b(\theta)$ drops below the maximum possible value; at such postures the tip frame loses the ability to generate instantaneous spatial velocities in in one or more dimensions. This loss of mobility at a singularity is accompanied by the ability to resist arbitrary wrenches in the direction corresponding to the lost mobility.

The mathematical definition of a kinematic singularity is independent of the choice of body or space Jacobian. To see why, recall the relationship between $J_s(\theta)$ and $J_b(\theta)$: $J_s(\theta) = \mathrm{Ad}_{T_{sb}}(J_b(\theta)) = [\mathrm{Ad}_{T_{sb}}]J_b(\theta)$ or, more explicitly,

$$J_s(\theta) = \begin{bmatrix} R_{sb} & 0 \\ [p_{sb}]R_{sb} & R_{sb} \end{bmatrix} J_b(\theta).$$

We now claim that the matrix $[\mathrm{Ad}_{T_{sb}}]$ is always invertible. This can be established by examining the linear equation

$$\begin{bmatrix} R_{sb} & 0 \\ [p_{sb}]R_{sb} & R_{sb} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

Its unique solution is $x = y = 0$, implying that the matrix $[\mathrm{Ad}_{T_{sb}}]$ is invertible. Since multiplying any matrix by an invertible matrix does not change its rank, it follows that

$$\text{rank } J_s(\theta) = \text{rank } J_b(\theta),$$

as claimed; singularities of the space and body Jacobian are one and the same.

Kinematic singularities are also independent of the choice of fixed frame and end-effector frame. Choosing a different fixed frame is equivalent to simply relocating the robot arm, which should have absolutely no effect on whether a particular posture is singular. This obvious fact can be verified by referring to Figure 5.10(a). The forward kinematics with respect to the original fixed frame is denoted $T(\theta)$, while the forward kinematics with respect to the relocated fixed frame is denoted $T'(\theta) = PT(\theta)$, where $P \in SE(3)$ is constant. Then the body Jacobian of $T'(\theta)$, denoted $J_b'(\theta)$, is obtained from $(T')^{-1}\dot{T}'$. A simple calculation reveals that

$$(T')^{-1}\dot{T}' = (T^{-1}P^{-1})(P\dot{T}) = T^{-1}\dot{T},$$

i.e., $J_b'(\theta) = J_b(\theta)$, so that the singularities of the original and relocated robot arms are the same.

To see that singularities are independent of the end-effector frame, refer to Figure 5.10(b) and suppose the forward kinematics for the original end-effector frame is given by $T(\theta)$ while the forward kinematics for the relocated end-effector frame is $T'(\theta) = T(\theta)Q$, where $Q \in SE(3)$ is constant. This time, looking at the space Jacobian – recall that singularities of $J_b(\theta)$ coincide with those of $J_s(\theta)$ – let $J_s'(\theta)$ denote the space Jacobian of $T'(\theta)$. A simple calculation reveals that

$$\dot{T}'(T')^{-1} = (\dot{T}Q)(Q^{-1}T^{-1}) = \dot{T}T^{-1}.$$

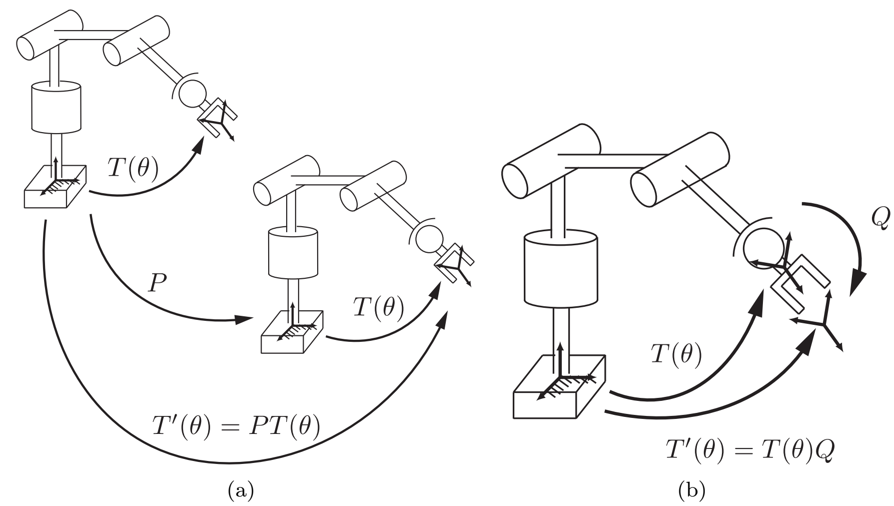


Figure 5.10: Kinematic singularities are invariant with respect to the choice of fixed and end-effector frames. (a) Choosing a different fixed frame, which is equivalent to relocating the base of the robot arm; (b) choosing a different end-effector frame.

That is, $J_s'(\theta) = J_s(\theta)$, so that the kinematic singularities are invariant with respect to the choice of end-effector frame.

In the remainder of this section we consider some common kinematic singularities that occur in six-dof open chains with revolute and prismatic joints. We now know that either the space or body Jacobian can be used for our analysis; we use the space Jacobian in the examples below.

Case I: Two Collinear Revolute Joint Axes

The first case we consider is one in which two revolute joint axes are collinear (see Figure 5.11(a)). Without loss of generality these joint axes can be labeled 1 and 2. The corresponding columns of the Jacobian are

$$J_{s1}(\theta) = \begin{bmatrix} \omega_{s1} \\ -\omega_{s1} \times q_1 \end{bmatrix} \quad \text{and} \quad J_{s2}(\theta) = \begin{bmatrix} \omega_{s2} \\ -\omega_{s2} \times q_2 \end{bmatrix}.$$

Since the two joint axes are collinear, we must have $\omega_{s1} = \pm\omega_{s2}$; let us assume the positive sign. Also, $\omega_{si} \times (q_1 - q_2) = 0$ for $i = 1, 2$. Then $J_{s1} = J_{s2}$, the set $\{J_{s1}, J_{s2}, \ldots, J_{s6}\}$ cannot be linearly independent, and the rank of $J_s(\theta)$ must be less than six.

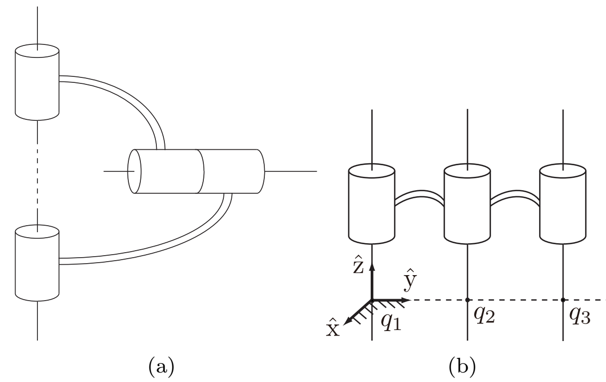


Figure 5.11: (a) A kinematic singularity in which two joint axes are collinear. (b) A kinematic singularity in which three revolute joint axes are parallel and coplanar.

Case II: Three Coplanar and Parallel Revolute Joint Axes

The second case we consider is one in which three revolute joint axes are parallel and also lie on the same plane (three coplanar axes: see Figure 5.11(b)). Without loss of generality we label these as joint axes 1, 2, and 3. In this case we choose the fixed frame as shown in the figure; then

$$J_s(\theta) = \begin{bmatrix} \omega_{s1} & \omega_{s1} & \omega_{s1} & \cdots \\ 0 & -\omega_{s1} \times q_2 & -\omega_{s1} \times q_3 & \cdots \end{bmatrix}.$$

Since q_2 and q_3 are points on the same unit axis, it is not difficult to verify that the first three columns cannot be linearly independent.

Case III: Four Revolute Joint Axes Intersecting at a Common Point

Here we consider the case where four revolute joint axes intersect at a common point (Figure 5.12). Again, without loss of generality, label these axes from 1 to 4. In this case we choose the fixed-frame origin to be the common point of intersection, so that $q_1 = \cdots = q_4 = 0$, and therefore

$$J_s(\theta) = \begin{bmatrix} \omega_{s1} & \omega_{s2} & \omega_{s3} & \omega_{s4} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \end{bmatrix}.$$

The first four columns clearly cannot be linearly independent; one can be written as a linear combination of the other three. Such a singularity occurs, for example, when the wrist center of an elbow-type robot arm is directly above the shoulder.

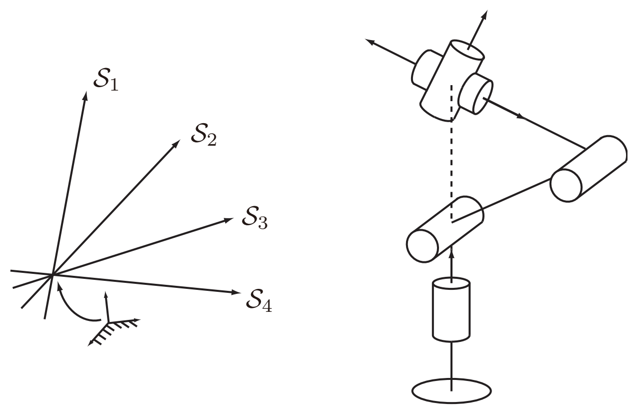


Figure 5.12: A kinematic singularity in which four revolute joint axes intersect at a common point.

Case IV: Four Coplanar Revolute Joints

Here we consider the case in which four revolute joint axes are coplanar. Again, without loss of generality, label these axes from 1 to 4. Choose a fixed frame such that the joint axes all lie on the x–y-plane; in this case the unit vector $\omega_{si} \in \mathbb{R}^3$ in the direction of joint axis i is of the form

$$\omega_{si} = \begin{bmatrix} \omega_{six} \\ \omega_{siy} \\ 0 \end{bmatrix}.$$

Similarly, any reference point $q_i \in \mathbb{R}^3$ lying on joint axis i is of the form

$$q_i = \begin{bmatrix} q_{ix} \\ q_{iy} \\ 0 \end{bmatrix},$$

and subsequently

$$v_{si} = -\omega_{si} \times q_i = \begin{bmatrix} 0 \\ 0 \\ \omega_{siy} q_{ix} - \omega_{six} q_{iy} \end{bmatrix}.$$

The first four columns of the space Jacobian $J_s(\theta)$ are

$$\begin{bmatrix} \omega_{s1x} & \omega_{s2x} & \omega_{s3x} & \omega_{s4x} \\ \omega_{s1y} & \omega_{s2y} & \omega_{s3y} & \omega_{s4y} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \omega_{s1y}q_{1x} - \omega_{s1x}q_{1y} & \omega_{s2y}q_{2x} - \omega_{s2x}q_{2y} & \omega_{s3y}q_{3x} - \omega_{s3x}q_{3y} & \omega_{s4y}q_{4x} - \omega_{s4x}q_{4y} \end{bmatrix}$$

and cannot be linearly independent since they only have three nonzero components.

Case V: Six Revolute Joints Intersecting a Common Line

The final case we consider is six revolute joint axes intersecting a common line. Choose a fixed frame such that the common line lies along the $\hat{z}$-axis, and select the intersection between this common line and joint axis i as the reference point $q_i \in \mathbb{R}^3$ for axis i; each q_i is thus of the form $q_i = (0, 0, q_{iz})$, and

$$v_{si} = -\omega_{si} \times q_i = (\omega_{siy}q_{iz}, -\omega_{six}q_{iz}, 0),$$

for $i = 1, \ldots, 6$. The space Jacobian $J_s(\theta)$ thus becomes

$$\begin{bmatrix} \omega_{s1x} & \omega_{s2x} & \omega_{s3x} & \omega_{s4x} & \omega_{s5x} & \omega_{s6x} \\ \omega_{s1y} & \omega_{s2y} & \omega_{s3y} & \omega_{s4y} & \omega_{s5y} & \omega_{s6y} \\ \omega_{s1z} & \omega_{s2z} & \omega_{s3z} & \omega_{s4z} & \omega_{s5z} & \omega_{s6z} \\ \omega_{s1y}q_{1z} & \omega_{s2y}q_{2z} & \omega_{s3y}q_{3z} & \omega_{s4y}q_{4z} & \omega_{s5y}q_{5z} & \omega_{s6y}q_{6z} \\ -\omega_{s1x}q_{1z} & -\omega_{s2x}q_{2z} & -\omega_{s3x}q_{3z} & -\omega_{s4x}q_{4z} & -\omega_{s5x}q_{5z} & -\omega_{s6x}q_{6z} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which is clearly singular.

5.4 Manipulability

In the previous section we saw that, at a kinematic singularity, a robot's end-effector loses the ability to translate or rotate in one or more directions. A kinematic singularity presents a binary proposition – a particular configuration is either kinematically singular or it is not – and it is reasonable to ask if a nonsingular configuration is "close" to being singular. The answer is yes; in fact, one can even determine the directions in which the end-effector's ability to move is diminished, and to what extent. The manipulability ellipsoid allows one to visualize geometrically the directions in which the end-effector moves with least effort or with greatest effort.

Manipulability ellipsoids are illustrated for a 2R planar arm in Figure 5.3. The Jacobian is given by Equation (5.1).

For a general n-joint open chain and a task space with coordinates $q \in \mathbb{R}^m$, where $m \leq n$, the manipulability ellipsoid corresponds to the end-effector velocities for joint rates $\dot{\theta}$ satisfying $\|\dot{\theta}\| = 1$, a unit sphere in the n-dimensional joint-velocity space.[6] Assuming J is invertible, the unit joint-velocity condition can be written

$$\begin{aligned} 1 &= \dot{\theta}^{\mathrm{T}}\dot{\theta} \\ &= (J^{-1}\dot{q})^{\mathrm{T}}(J^{-1}\dot{q}) \\ &= \dot{q}^{\mathrm{T}}J^{-T}J^{-1}\dot{q} \\ &= \dot{q}^{\mathrm{T}}(JJ^{\mathrm{T}})^{-1}\dot{q} = \dot{q}^{\mathrm{T}}A^{-1}\dot{q}. \end{aligned} \tag{5.27}$$

If J is full rank (i.e., of rank m), the matrix $A = JJ^{\mathrm{T}} \in \mathbb{R}^{m\times m}$ is square, symmetric, and positive definite, as is A^{-1}.

Consulting a textbook on linear algebra, we see that for any symmetric positive-definite $A^{-1} \in \mathbb{R}^{m\times m}$, the set of vectors $\dot{q} \in \mathbb{R}^m$ satisfying

$$\dot{q}^{\mathrm{T}}A^{-1}\dot{q} = 1$$

defines an ellipsoid in the m-dimensional space. Letting v_i and λ_i be the eigenvectors and eigenvalues of A, the directions of the principal axes of the ellipsoid are v_i and the lengths of the principal semi-axes are $\sqrt{\lambda_i}$, as illustrated in Figure 5.13. Furthermore, the volume V of the ellipsoid is proportional to the product of the semi-axis lengths:

$$V \text{ is proportional to } \sqrt{\lambda_1\lambda_2\cdots\lambda_m} = \sqrt{\det(A)} = \sqrt{\det(JJ^{\mathrm{T}})}.$$

For the geometric Jacobian J (either J_b in the end-effector frame or J_s in the fixed frame), we can express the $6 \times n$ Jacobian as

$$J(\theta) = \begin{bmatrix} J_\omega(\theta) \\ J_v(\theta) \end{bmatrix},$$

[6] A two-dimensional ellipsoid is usually referred to as an "ellipse," and an ellipsoid in more than three dimensions is often referred to as a "hyperellipsoid," but here we use the term ellipsoid independently of the dimension. Similarly, we refer to a "sphere" independently of the dimension, instead of using "circle" for two dimensions and "hypersphere" for more than three dimensions.

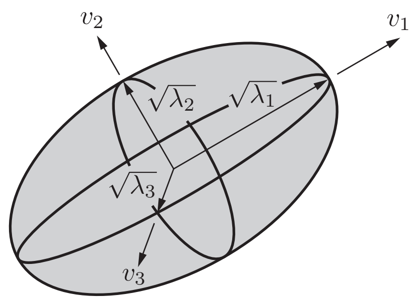


Figure 5.13: An ellipsoid visualization of $\dot{q}^{\mathrm{T}} A^{-1} \dot{q} = 1$ in the $\dot{q}$ space $\mathbb{R}^3$, where the principal semi-axis lengths are the square roots of the eigenvalues λ_i of A and the directions of the principal semi-axes are the eigenvectors v_i.

where J_ω comprises the top three rows of J and J_v the bottom three rows of J. It makes sense to separate the two because the units of angular velocity and linear velocity are different. This leads to two three-dimensional manipulability ellipsoids, one for angular velocities and one for linear velocities. These manipulability ellipsoids have principal semi-axes aligned with the eigenvectors of A, with lengths given by the square roots of the eigenvalues, where $A = J_\omega J_\omega^{\mathrm{T}}$ for the angular velocity manipulability ellipsoid and $A = J_v J_v^{\mathrm{T}}$ for the linear velocity manipulability ellipsoid.

When calculating the linear-velocity manipulability ellipsoid, it generally makes more sense to use the body Jacobian J_b instead of the space Jacobian J_s, since we are usually interested in the linear velocity of a point at the origin of the end-effector frame rather than that of a point at the origin of the fixed-space frame.

Apart from the geometry of the manipulability ellipsoid, it can be useful to assign a single scalar measure defining how easily the robot can move at a given posture. One measure is the ratio of the longest and shortest semi-axes of the manipulability ellipsoid,

$$\mu_1(A) = \frac{\sqrt{\lambda_{\max}(A)}}{\sqrt{\lambda_{\min}(A)}} = \sqrt{\frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}} \geq 1,$$

where $A = JJ^{\mathrm{T}}$. When $\mu_1(A)$ is low (i.e., close to 1) then the manipulability ellipsoid is nearly spherical or **isotropic**, meaning that it is equally easy to move in any direction. This situation is generally desirable. As the robot approaches a singularity, however, $\mu_1(A)$ goes to infinity.

A similar measure $\mu_2(A)$ is just the square of $\mu_1(A)$, which is known as the

condition number of the matrix $A = JJ^{\mathrm{T}}$,

$$\mu_2(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)} \geq 1.$$

Again, smaller values (close to 1) are preferred. The condition number of a matrix is commonly used to characterize the sensitivity of the result of multiplying that matrix by a vector to small errors in the vector.

A final measure is simply proportional to the volume of the manipulability ellipsoid,

$$\mu_3(A) = \sqrt{\lambda_1 \lambda_2 \cdots} = \sqrt{\det(A)}.$$

In this case, unlike the first two measures, a larger value is better.

Just like the manipulability ellipsoid, a force ellipsoid can be drawn for joint torques τ satisfying $\|\tau\| = 1$. Beginning from $\tau = J^{\mathrm{T}}(\theta)\mathcal{F}$, we arrive at similar results to those above, except that now the ellipsoid satisfies

$$1 = f^{\mathrm{T}} JJ^{\mathrm{T}} f = f^{\mathrm{T}} B^{-1} f,$$

where $B = (JJ^{\mathrm{T}})^{-1} = A^{-1}$. For the force ellipsoid, the matrix B plays the same role as A in the manipulability ellipsoid; it is the eigenvectors and the square roots of eigenvalues of B that define the shape of the force ellipsoid.

Since eigenvectors of any invertible matrix A are also eigenvectors of $B = A^{-1}$, the principal axes of the force ellipsoid are aligned with the principal axes of the manipulability ellipsoid. Furthermore, since the eigenvalues of $B = A^{-1}$ associated with each principal axis are the reciprocals of the corresponding eigenvalues of A, the lengths of the principal semi-axes of the force ellipsoid are given by $1/\sqrt{\lambda_i}$, where λ_i are the eigenvalues of A. Thus the force ellipsoid is obtained from the manipulability ellipsoid simply by stretching the manipulability ellipsoid along each principal axis i by a factor $1/\lambda_i$. Furthermore, since the volume V_A of the manipulability ellipsoid is proportional to the product of the semi-axes, $\sqrt{\lambda_1 \lambda_2 \cdots}$, and the volume V_B of the force ellipsoid is proportional to $1/\sqrt{\lambda_1 \lambda_2 \cdots}$, the product of the two volumes $V_A V_B$ is constant *independently* of the joint variables θ. Therefore, positioning the robot to increase the manipulability-ellipsoid volume measure $\mu_3(A)$ simultaneously reduces the force-ellipsoid volume measure $\mu_3(B)$. This also explains the observation made at the start of the chapter that, as the robot approaches a singularity, V_A goes to zero while V_B goes to infinity.

5.5 Summary

- Let the forward kinematics of an n-link open chain be expressed in the following product of exponentials form:

  $$T(\theta) = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_n]\theta_n} M.$$

  The space Jacobian $J_s(\theta) \in \mathbb{R}^{6\times n}$ relates the joint rate vector $\dot{\theta} \in \mathbb{R}^n$ to the spatial twist $\mathcal{V}_s$, via $\mathcal{V}_s = J_s(\theta)\dot{\theta}$. The ith column of $J_s(\theta)$ is given by

  $$J_{si}(\theta) = \mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}}(\mathcal{S}_i),$$

  for $i = 2, \ldots, n$, with the first column $J_{s1} = \mathcal{S}_1$. The screw vector J_{si} for joint i is expressed in space-frame coordinates, with the joint values θ assumed to be arbitrary rather than zero.

- Let the forward kinematics of an n-link open chain be expressed in the following product of exponentials form:

  $$T(\theta) = M e^{[\mathcal{B}_1]\theta_1} \cdots e^{[\mathcal{B}_n]\theta_n}.$$

  The body Jacobian $J_b(\theta) \in \mathbb{R}^{6\times n}$ relates the joint rate vector $\dot{\theta} \in \mathbb{R}^n$ to the end-effector body twist $\mathcal{V}_b = (\omega_b, v_b)$ via $\mathcal{V}_b = J_b(\theta)\dot{\theta}$. The ith column of $J_b(\theta)$ is given by

  $$J_{bi}(\theta) = \mathrm{Ad}_{e^{-[\mathcal{B}_n]\theta_n} \cdots e^{-[\mathcal{B}_{i+1}]\theta_{i+1}}}(\mathcal{B}_i),$$

  for $i = n-1, \ldots, 1$, with $J_{bn} = \mathcal{B}_n$. The screw vector J_{bi} for joint i is expressed in body-frame coordinates, with the joint values θ assumed to be arbitrary rather than zero.

- The body and space Jacobians are related via

  $$\begin{aligned} J_s(\theta) &= [\mathrm{Ad}_{T_{sb}}] J_b(\theta), \\ J_b(\theta) &= [\mathrm{Ad}_{T_{bs}}] J_s(\theta), \end{aligned}$$

  where $T_{sb} = T(\theta)$.

- Consider a spatial open chain with n one-dof joints that is assumed to be in static equilibrium. Let $\tau \in \mathbb{R}^n$ denote the vector of the joint torques and forces and $\mathcal{F} \in \mathbb{R}^6$ be the wrench applied at the end-effector, in either space- or body-frame coordinates. Then τ and $\mathcal{F}$ are related by

  $$\tau = J_b^{\mathrm{T}}(\theta)\mathcal{F}_b = J_s^{\mathrm{T}}(\theta)\mathcal{F}_s.$$

- A kinematically singular configuration for an open chain, or more simply a kinematic singularity, is any configuration $\theta \in \mathbb{R}^n$ at which the rank of the Jacobian is not maximal. For six-dof spatial open chains consisting of revolute and prismatic joints, some common singularities include (i) two collinear revolute joint axes; (ii) three coplanar and parallel revolute joint axes; (iii) four revolute joint axes intersecting at a common point; (iv) four coplanar revolute joints; and (v) six revolute joints intersecting a common line.

- The manipulability ellipsoid describes how easily the robot can move in different directions. For a Jacobian J, the principal axes of the manipulability ellipsoid are defined by the eigenvectors of JJ^{T} and the corresponding lengths of the principal semi-axes are the square roots of the eigenvalues.

- The force ellipsoid describes how easily the robot can generate forces in different directions. For a Jacobian J, the principal axes of the force ellipsoid are defined by the eigenvectors of $(JJ^{\mathrm{T}})^{-1}$ and the corresponding lengths of the principal semi-axes are the square roots of the eigenvalues.

- Measures of the manipulability and force ellipsoids include the ratio of the longest principal semi-axis to the shortest; the square of this measure; and the volume of the ellipsoid. The first two measures indicate that the robot is far from being singular if they are small (close to 1).

5.6 Software

Software functions associated with this chapter are listed below.

`Jb = JacobianBody(Blist,thetalist)`
Computes the body Jacobian $J_b(\theta) \in \mathbb{R}^{6\times n}$ given a list of joint screws $\mathcal{B}_i$ expressed in the body frame and a list of joint angles.

`Js = JacobianSpace(Slist,thetalist)`
Computes the space Jacobian $J_s(\theta) \in \mathbb{R}^{6\times n}$ given a list of joint screws $\mathcal{S}_i$ expressed in the fixed space frame and a list of joint angles.

5.7 Notes and References

One of the key advantages of the PoE formulation is in the derivation of the Jacobian; the columns of the Jacobian are simply the (configuration-dependent)

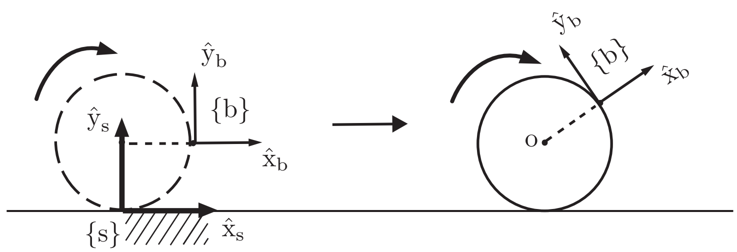


Figure 5.14: A rolling wheel.

screws for the joint axes. Compact closed-form expressions for the columns of the Jacobian are also obtained because taking derivatives of matrix exponentials is particularly straightforward.

There is extensive literature on the singularity analysis of 6R open chains. In addition to the three cases presented in this chapter, other cases are examined in [122] and in exercises at the end of this chapter, including the case when some of the revolute joints are replaced by prismatic joints. Many of the mathematical techniques and analyses used in open chain singularity analysis can also be used to determine the singularities of parallel mechanisms, which are the subject of Chapter 7.

The concept of a robot's manipulability was first formulated in a quantitative way by Yoshikawa [195]. There is now a vast literature on the manipulability analysis of open chains, see, e.g., [75, 134].

5.8 Exercises

Exercise 5.1 A wheel of unit radius is rolling to the right at a rate of 1 rad/s (see Figure 5.14; the dashed circle shows the wheel at $t = 0$).

(a) Find the spatial twist $\mathcal{V}_s(t)$ as a function of t.

(b) Find the linear velocity of the {b}-frame origin expressed in {s}-frame coordinates.

Exercise 5.2 The 3R planar open chain of Figure 5.15(a) is shown in its zero position.

(a) Suppose that the tip must apply a force of 5 N in the $\hat{x}_s$-direction of the {s} frame, with zero component in the $\hat{y}_s$-direction and zero moment about the $\hat{z}_s$ axis. What torques should be applied at each joint?

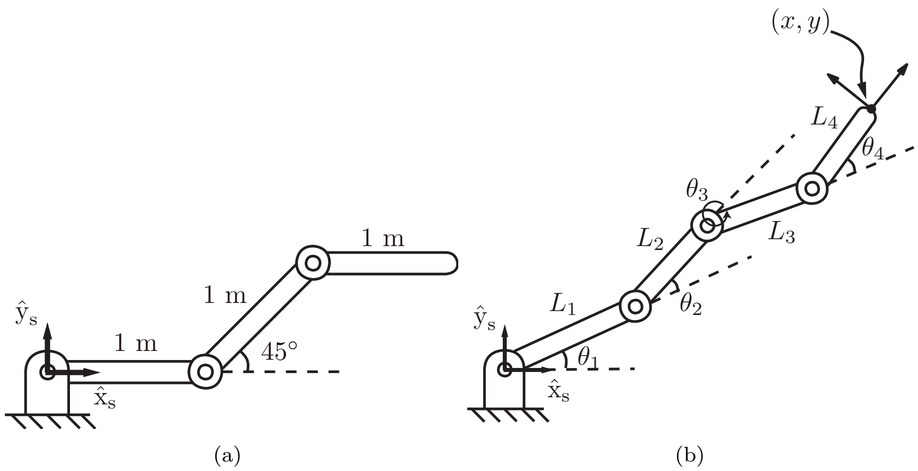


Figure 5.15: (a) A 3R planar open chain. The length of each link is 1 m. (b) A 4R planar open chain.

(b) Suppose that now the tip must apply a force of 5 N in the $\hat{y}_s$-direction. What torques should be applied at each joint?

Exercise 5.3 Answer the following questions for the 4R planar open chain of Figure 5.15(b).

(a) For the forward kinematics of the form

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} e^{[\mathcal{S}_4]\theta_4} M,$$

write down $M \in SE(2)$ and each $\mathcal{S}_i = (\omega_{zi}, v_{xi}, v_{yi}) \in \mathbb{R}^3$.

(b) Write down the body Jacobian.

(c) Suppose that the chain is in static equilibrium at the configuration $\theta_1 = \theta_2 = 0, \theta_3 = \pi/2, \theta_4 = -\pi/2$ and that a force $f = (10, 10, 0)$ and a moment $m = (0, 0, 10)$ are applied to the tip (both f and m are expressed with respect to the fixed frame). What are the torques experienced at each joint?

(d) Under the same conditions as (c), suppose that a force $f = (-10, 10, 0)$ and a moment $m = (0, 0, -10)$, also expressed in the fixed frame, are applied to the tip. What are the torques experienced at each joint?

(e) Find all kinematic singularities for this chain.

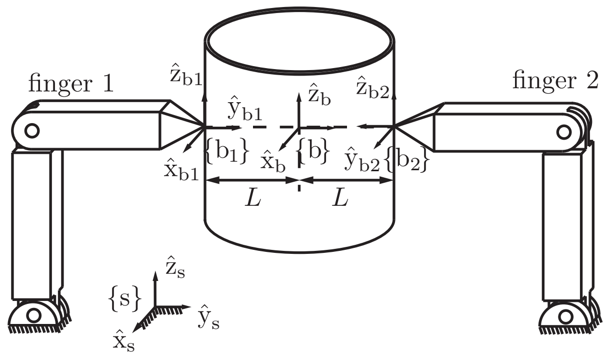


Figure 5.16: Two fingers grasping a can.

Exercise 5.4 Figure 5.16 shows two fingers grasping a can. Frame {b} is attached to the center of the can. Frames $\{b_1\}$ and $\{b_2\}$ are attached to the can at the two contact points as shown. The force $f_1 = (f_{1,x}, f_{1,y}, f_{1,z})$ is the force applied by fingertip 1 to the can, expressed in $\{b_1\}$ coordinates. Similarly, $f_2 = (f_{2,x}, f_{2,y}, f_{2,z})$ is the force applied by fingertip 2 to the can, expressed in $\{b_2\}$ coordinates.

(a) Assume that the system is in static equilibrium, and find the total wrench $\mathcal{F}_b$ applied by the two fingers to the can. Express your result in {b} coordinates.

(b) Suppose that $\mathcal{F}_{\text{ext}}$ is an arbitrary external wrench applied to the can ($\mathcal{F}_{\text{ext}}$ is also expressed in frame-{b} coordinates). Find all $\mathcal{F}_{\text{ext}}$ that cannot be resisted by the fingertip forces.

Exercise 5.5 Referring to Figure 5.17, a rigid body, shown at the top right, rotates about the point (L, L) with angular velocity $\dot{\theta} = 1$.

(a) Find the position of point P on the moving body relative to the fixed reference frame {s} in terms of θ.

(b) Find the velocity of point P in terms of the fixed frame.

(c) What is T_{sb}, the configuration of frame {b}, as seen from the fixed frame {s}?

(d) Find the twist of T_{sb} in body coordinates.

(e) Find the twist of T_{sb} in space coordinates.

(f) What is the relationship between the twists from (d) and (e)?

(g) What is the relationship between the twist from (d) and $\dot{P}$ from (b)?

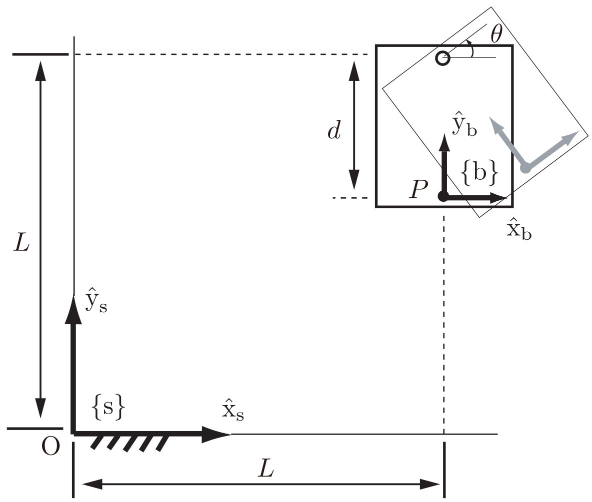


Figure 5.17: A rigid body rotating in the plane.

(h) What is the relationship between the twist from (e) and $\dot{P}$ from (b)?

Exercise 5.6 Figure 5.18 shows a design for a new amusement park ride. A rider sits at the location indicated by the moving frame {b}. The fixed frame {s} is attached to the top shaft as shown. The dimensions indicated in the figure are $R = 10$ m and $L = 20$ m, and the two joints each rotate at a constant angular velocity of 1 rad/s.

(a) Suppose $t = 0$ at the instant shown in the figure. Find the linear velocity v_b and angular velocity ω_b of the rider as functions of time t. Express your answer in frame-{b} coordinates.

(b) Let p be the linear coordinates expressing the position of the rider in {s}. Find the linear velocity $\dot{p}(t)$.

Exercise 5.7 The RRP robot in Figure 5.19 is shown in its zero position.

(a) Write down the screw axes in the space frame. Evaluate the forward kinematics when $\theta = (90°, 90°, 1)$. Hand-draw or use a computer to show the arm and the end-effector frame in this configuration. Obtain the space Jacobian J_s for this configuration.

(b) Write down the screw axes in the end-effector body frame. Evaluate the forward kinematics when $\theta = (90°, 90°, 1)$ and confirm that you get the same result as in part (a). Obtain the body Jacobian J_b for this configu-

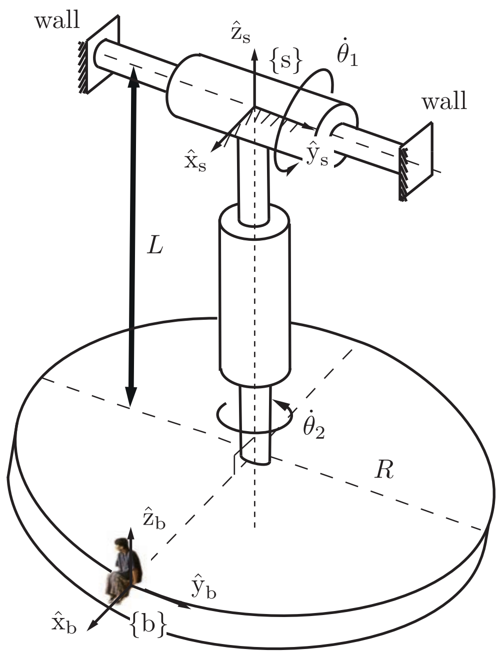


Figure 5.18: A new amusement park ride.

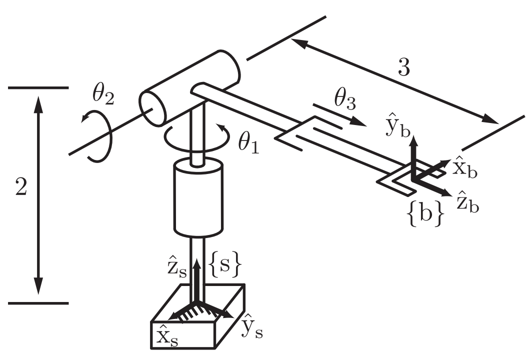


Figure 5.19: RRP robot shown in its zero position.

ration.

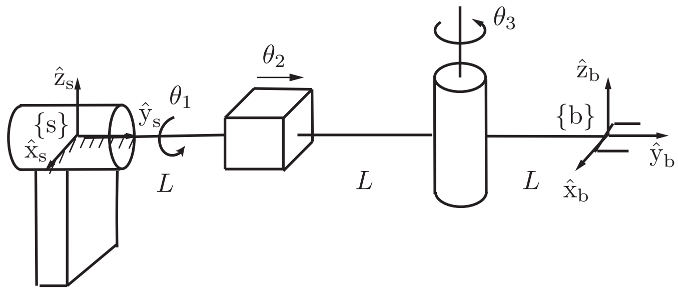


Figure 5.20: RPR robot.

Exercise 5.8 The RPR robot of Figure 5.20 is shown in its zero position. The fixed and end-effector frames are respectively denoted {s} and {b}.

(a) Find the space Jacobian $J_s(\theta)$ for arbitrary configurations $\theta \in \mathbb{R}^3$.

(b) Assume the manipulator is in its zero position. Suppose that an external force $f \in \mathbb{R}^3$ is applied to the {b} frame origin. Find all the directions in which f can be resisted by the manipulator with $\tau = 0$.

Exercise 5.9 Find the kinematic singularities of the 3R wrist given the forward kinematics

$$R = e^{[\hat{\omega}_1]\theta_1} e^{[\hat{\omega}_2]\theta_2} e^{[\hat{\omega}_3]\theta_3},$$

where $\hat{\omega}_1 = (0, 0, 1)$, $\hat{\omega}_2 = (1/\sqrt{2}, 0, 1/\sqrt{2})$, and $\hat{\omega}_3 = (1, 0, 0)$.

Exercise 5.10 In this exercise, for an n-link open chain we derive the analytic Jacobian corresponding to the exponential coordinates on $SO(3)$.

(a) Given an $n \times n$ matrix $A(t)$ parametrized by t that is also differentiable with respect to t, its exponential $X(t) = e^{A(t)}$ is then an $n \times n$ matrix that is always nonsingular. Prove the following:

$$\begin{aligned} X^{-1}\dot{X} &= \int_0^1 e^{-A(t)s}\dot{A}(t)e^{A(t)s}ds, \\ \dot{X}X^{-1} &= \int_0^1 e^{A(t)s}\dot{A}(t)e^{-A(t)s}ds. \end{aligned}$$

(Hint: The formula

$$\frac{d}{d\epsilon}e^{(A+\epsilon B)t}|_{\epsilon=0} = \int_0^t e^{As}Be^{A(t-s)}ds$$

may be useful.)

(b) Use the result above to show that, for $r(t) \in \mathbb{R}^3$ and $R(t) = e^{[r(t)]}$, the angular velocity in the body frame, $[\omega_b] = R^{\mathrm{T}}\dot{R}$, is related to $\dot{r}$ by

$$\begin{aligned} \omega_b &= A(r)\dot{r}, \\ A(r) &= I - \frac{1-\cos\|r\|}{\|r\|^2}[r] + \frac{\|r\|-\sin\|r\|}{\|r\|^3}[r]^2. \end{aligned}$$

(c) Derive the corresponding formula relating the angular velocity in the space frame, $[\omega_s] = \dot{R}R^{\mathrm{T}}$, to $\dot{r}$.

Exercise 5.11 The spatial 3R open chain of Figure 5.21 is shown in its zero position. Let p be the coordinates of the origin of $\{b\}$ expressed in $\{s\}$.

(a) In its zero position, suppose we wish to make the end-effector move with linear velocity $\dot{p} = (10, 0, 0)$. What are the required input joint velocities $\dot{\theta}_1, \dot{\theta}_2$, and $\dot{\theta}_3$?

(b) Suppose that the robot is in the configuration $\theta_1 = 0, \theta_2 = 45°, \theta_3 = -45°$. Assuming static equilibrium, suppose that we wish to generate an end-effector force $f_b = (10, 0, 0)$, where f_b is expressed with respect to the end-effector frame $\{b\}$. What are the required input joint torques τ_1, τ_2, and τ_3?

(c) Under the same conditions as in (b), suppose that we now seek to generate an end-effector moment $m_b = (10, 0, 0)$, where m_b is expressed with respect to the end-effector frame $\{b\}$. What are the required input joint torques τ_1, τ_2, τ_3?

(d) Suppose that the maximum allowable torques for each joint motor are

$$\|\tau_1\| \le 10, \quad \|\tau_2\| \le 20, \qquad \text{and} \qquad \|\tau_3\| \le 5.$$

In the zero position, what is the maximum force that can be applied by the tip in the end-effector-frame $\hat{\mathrm{x}}$-direction?

Exercise 5.12 The RRRP chain of Figure 5.22 is shown in its zero position. Let p be the coordinates of the origin of $\{b\}$ expressed in $\{s\}$.

(a) Determine the body Jacobian $J_b(\theta)$ when $\theta_1 = \theta_2 = 0, \theta_3 = \pi/2, \theta_4 = L$.

(b) Find $\dot{p}$ when $\theta_1 = \theta_2 = 0, \theta_3 = \pi/2, \theta_4 = L$ and $\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \dot{\theta}_4 = 1$.

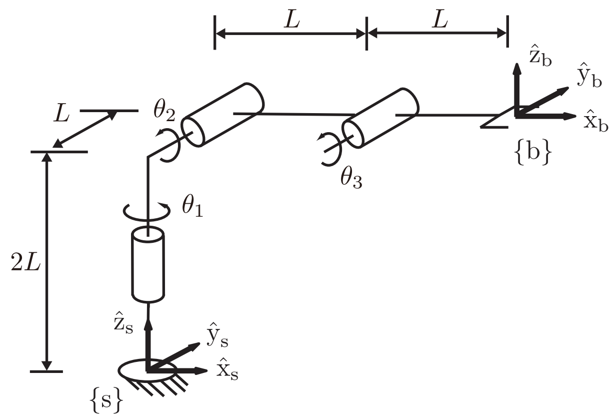


Figure 5.21: A spatial 3R open chain.

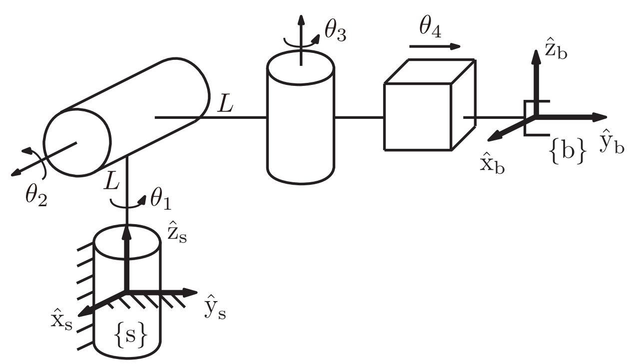


Figure 5.22: An RRRP spatial open chain.

Exercise 5.13 For the 6R spatial open chain of Figure 5.23,

(a) Determine its space Jacobian $J_s(\theta)$.

(b) Find its kinematic singularities. Explain each singularity in terms of the alignment of the joint screws and of the directions in which the end-effector loses one or more degrees of freedom of motion.

Exercise 5.14 Show that a six-dof spatial open chain is at a kinematic sin-

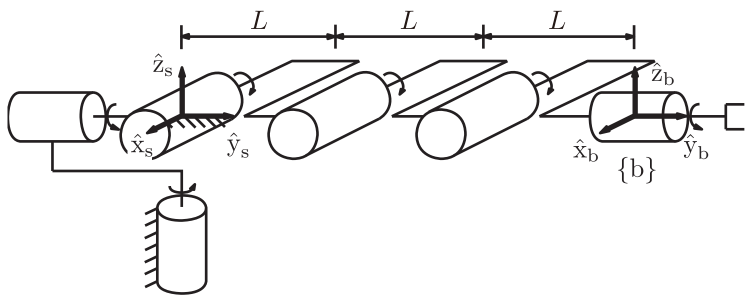


Figure 5.23: Singularities of a 6R open chain.

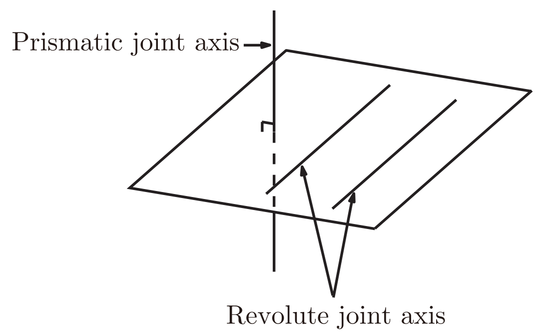


Figure 5.24: A kinematic singularity involving prismatic and revolute joints.

gularity when any two of its revolute joint axes are parallel, and any prismatic joint axis is normal to the plane spanned by the two parallel revolute joint axes (see Figure 5.24).

Exercise 5.15 The spatial PRRRRP open chain of Figure 5.25 is shown in its zero position.

(a) At the zero position, find the first three columns of the space Jacobian.
(b) Find all configurations for which the first three columns of the space Jacobian become linearly dependent.
(c) Suppose that the chain is in the configuration $\theta_1 = \theta_2 = \theta_3 = \theta_5 = \theta_6 = 0$, $\theta_4 = 90°$. Assuming static equilibrium, suppose that a pure force $f_b = (10, 0, 10)$, where f_b is expressed in terms of the end-effector frame, is applied to the origin of the end-effector frame. Find the torques τ_1, τ_2,

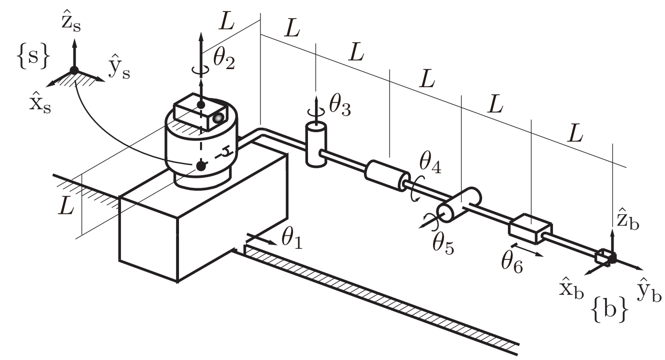


Figure 5.25: A spatial PRRRRP open chain.

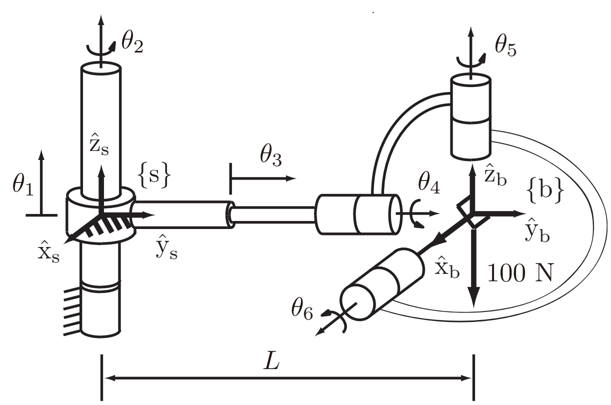


Figure 5.26: A PRRRRR spatial open chain.

and τ_3 experienced at the first three joints.

Exercise 5.16 Consider the PRRRRR spatial open chain of Figure 5.26, shown in its zero position. The distance from the origin of the fixed frame to the origin of the end-effector frame at the home position is L.

(a) Determine the first three columns of the space Jacobian J_s.

(b) Determine the last two columns of the body Jacobian J_b.

(c) For what value of L is the home position a singularity?

(d) In the zero position, what joint forces and torques τ must be applied in order to generate a pure end-effector force of 100 N in the $-\hat{z}_b$-direction?

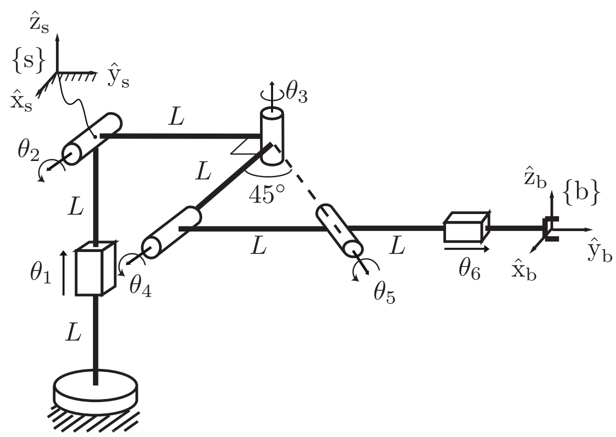


Figure 5.27: A PRRRRP robot.

Exercise 5.17 The PRRRRP robot of Figure 5.27 is shown in its zero position.
(a) Find the first three columns of the space Jacobian $J_s(\theta)$.
(b) Assuming the robot is in its zero position and $\dot{\theta} = (1, 0, 1, -1, 2, 0)$, find the spatial twist $\mathcal{V}_s$.
(c) Is the zero position a kinematic singularity? Explain your answer.

Exercise 5.18 The six-dof RRPRPR open chain of Figure 5.28 has a fixed frame {s} and an end-effector frame {b} attached as shown. At its zero position, joint axes 1, 2, and 6 lie in the ŷ–ẑ-plane of the fixed frame, and joint axis 4 is aligned along the fixed-frame x̂-axis.
(a) Find the first three columns of the space Jacobian $J_s(\theta)$.
(b) At the zero position, let $\dot{\theta} = (1, 0, 1, -1, 2, 0)$. Find the spatial twist $\mathcal{V}_s$.
(c) Is the zero position a kinematic singularity? Explain your answer.

Exercise 5.19 The spatial PRRRRP open chain of Figure 5.29 is shown in its zero position.
(a) Determine the first four columns of the space Jacobian $J_s(\theta)$.
(b) Determine whether the zero position is a kinematic singularity.
(c) Calculate the joint forces and torques required for the tip to apply the following end-effector wrenches:
  (i) $\mathcal{F}_s = (0, 1, -1, 1, 0, 0)$.

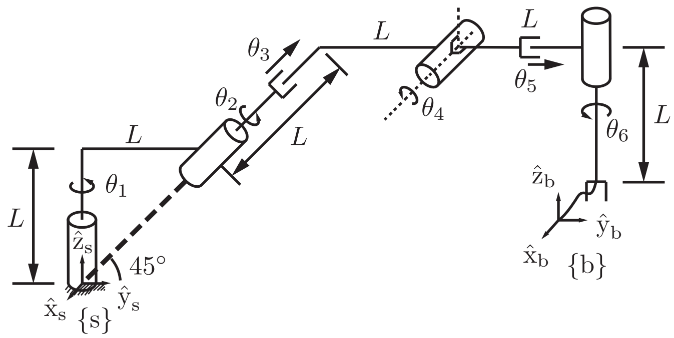


Figure 5.28: An RRPRPR open chain shown at its zero position.

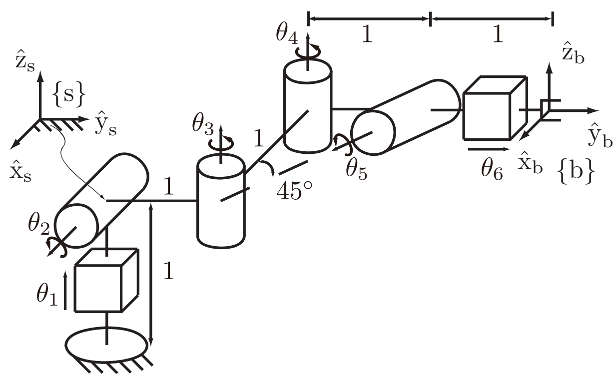


Figure 5.29: A spatial PRRRRP open chain with a skewed joint axis.

(ii) $\mathcal{F}_s = (1, -1, 0, 1, 0, -1)$.

Exercise 5.20 The spatial RRPRRR open chain of Figure 5.30 is shown in its zero position.

(a) For the fixed frame {0} and tool (end-effector) frame {t} as shown, express the forward kinematics in the product of exponentials form

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} e^{[\mathcal{S}_4]\theta_4} e^{[\mathcal{S}_5]\theta_5} e^{[\mathcal{S}_6]\theta_6} M.$$

(b) Find the first three columns of the space Jacobian $J_s(\theta)$.

(c) Suppose that the fixed frame {0} is moved to another location {0′} as shown in the figure. Find the first three columns of the space Jacobian

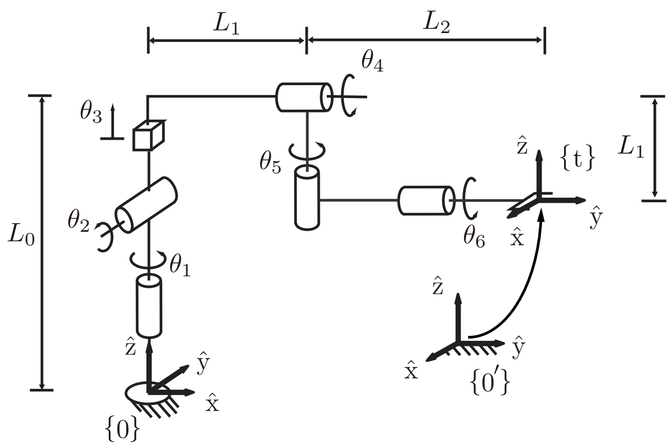


Figure 5.30: A spatial RRPRRR open chain.

$J_s(\theta)$ with respect to this new fixed frame.

(d) Determine whether the zero position is a kinematic singularity and, if so, provide a geometric description in terms of the joint screw axes.

Exercise 5.21 Figure 5.31 shows an RRPRRR exercise robot used for stroke patient rehabilitation.

(a) Assume the manipulator is in its zero position. Suppose that $M_{0c} \in SE(3)$ is the displacement from frame $\{0\}$ to frame $\{c\}$ and $M_{ct} \in SE(3)$ is the displacement from frame $\{c\}$ to frame $\{t\}$. Express the forward kinematics T_{0t} in the form

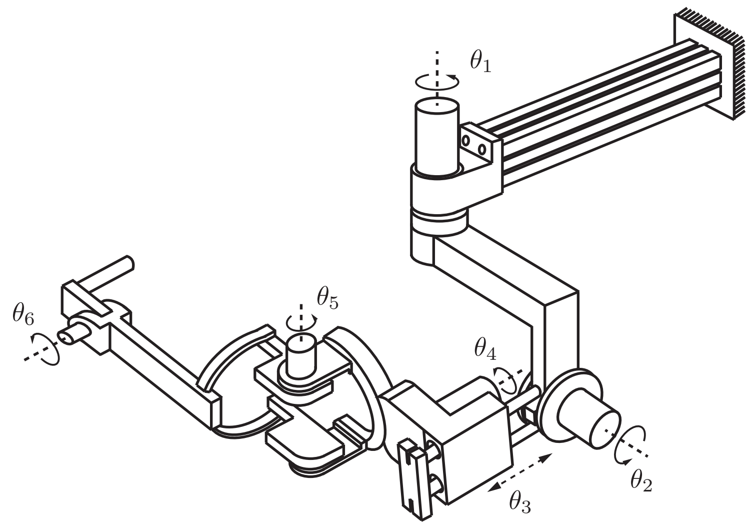
$$T_{0t} = e^{[\mathcal{A}_1]\theta_1} e^{[\mathcal{A}_2]\theta_2} M_{0c} e^{[\mathcal{A}_3]\theta_3} e^{[\mathcal{A}_4]\theta_4} M_{ct} e^{[\mathcal{A}_5]\theta_5} e^{[\mathcal{A}_6]\theta_6}.$$

Find $\mathcal{A}_2, \mathcal{A}_4$, and $\mathcal{A}_5$.

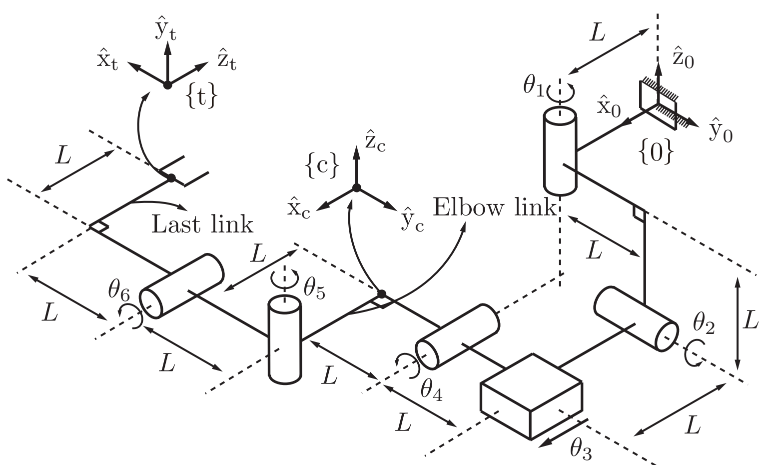
(b) Suppose that $\theta_2 = 90°$ and all the other joint variables are fixed at zero. Set the joint velocities to $(\dot\theta_1, \dot\theta_2, \dot\theta_3, \dot\theta_4, \dot\theta_5, \dot\theta_6) = (1, 0, 1, 0, 0, 1)$, and find the spatial twist $\mathcal{V}_s$ in frame-$\{0\}$ coordinates.

(c) Is the configuration described in part (b) a kinematic singularity? Explain your answer.

(d) Suppose that a person now operates the rehabilitation robot. At the configuration described in part (b), a wrench $\mathcal{F}_{\text{elbow}}$ is applied to the elbow link, and a wrench $\mathcal{F}_{\text{tip}}$ is applied to the last link. Both $\mathcal{F}_{\text{elbow}}$ and $\mathcal{F}_{\text{tip}}$ are expressed in frame $\{0\}$ coordinates and are given by $\mathcal{F}_{\text{elbow}} =$



(a) Rehabilitation robot ARMin III [123]. Figure courtesy of ETH Zürich.



(b) Kinematic model of the ARMin III.

Figure 5.31: The ARMin III rehabilitation robot.

$(1, 0, 0, 0, 0, 1)$ and $\mathcal{F}_{\text{tip}} = (0, 1, 0, 1, 1, 0)$. Find the joint forces and torques

τ that must be applied for the robot to maintain static equilibrium.

Exercise 5.22 Consider an n-link open chain, with reference frames attached to each link. Let

$$T_{0k} = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_k]\theta_k} M_k, \qquad k = 1, \ldots, n$$

be the forward kinematics up to link frame $\{k\}$. Let $J_s(\theta)$ be the space Jacobian for T_{0n}; $J_s(\theta)$ has columns J_{si} as shown below:

$$J_s(\theta) = \left[\begin{array}{ccc} J_{s1}(\theta) & \cdots & J_{sn}(\theta) \end{array} \right].$$

Let $[\mathcal{V}_k] = \dot{T}_{0k} T_{0k}^{-1}$ be the twist of link frame $\{k\}$ in fixed frame $\{0\}$ coordinates.

(a) Derive explicit expressions for $\mathcal{V}_2$ and $\mathcal{V}_3$.

(b) On the basis of your results from (a), derive a recursive formula for $\mathcal{V}_{k+1}$ in terms of $\mathcal{V}_k$, $J_{s1}, \ldots, J_{s,k+1}$, and $\dot{\theta}$.

Exercise 5.23 Write a program that allows the user to enter the link lengths L_1 and L_2 of a 2R planar robot (Figure 5.32) and a list of robot configurations (each defined by the joint angles (θ_1, θ_2)) and plots the manipulability ellipse at each of those configurations. The program should plot the arm (as two line segments) at each configuration and the manipulability ellipse centered at the endpoint of the arm. Choose the same scaling for all the ellipses so that they can be easily visualized (e.g., the ellipse should usually be shorter than the arm but not so small that you cannot easily see it). The program should also print the three manipulability measures μ_1, μ_2, and μ_3 for each configuration.

(a) Choose $L_1 = L_2 = 1$ and plot the arm and its manipulability ellipse at the four configurations $(-10°, 20°), (60°, 60°), (135°, 90°)$, and $(190°, 160°)$. At which of these configurations does the manipulability ellipse appear most isotropic? Does this agree with the manipulability measures calculated by the program?

(b) Does the ratio of the length of the major axis of the manipulability ellipse and the length of the minor axis depend on θ_1? On θ_2? Explain your answers.

(c) Choose $L_1 = L_2 = 1$. Hand-draw the following: the arm at $(-45°, 90°)$; the endpoint linear velocity vector arising from $\dot{\theta}_1 = 1$ rad/s and $\dot{\theta}_2 = 0$; the endpoint linear velocity vector arising from $\dot{\theta}_1 = 0$ and $\dot{\theta}_2 = 1$ rad/s; and the vector sum of these two vectors to get the endpoint linear velocity when $\dot{\theta}_1 = 1$ rad/s and $\dot{\theta}_2 = 1$ rad/s.

Exercise 5.24 Modify the program in the previous exercise to plot the force

Figure 5.32: Left: The 2R robot arm. Right: The arm at four different configurations.

ellipse. Demonstrate it at the same four configurations as in the first part of the previous exercise.

Exercise 5.25 The kinematics of the 6R UR5 robot are given in Section 4.1.2.

(a) Give the numerical space Jacobian J_s when all joint angles are $\pi/2$. Separate the Jacobian matrix into an angular velocity portion J_ω (the joint rates act on the angular velocity) and a linear velocity portion J_v (the joint rates act on the linear velocity).

(b) For this configuration, calculate the directions and lengths of the principal semi-axes of the three-dimensional angular-velocity manipulability ellipsoid (based on J_ω) and the directions and lengths of the principal semi-axes of the three-dimensional linear-velocity manipulability ellipsoid (based on J_v).

(c) For this configuration, calculate the directions and lengths of the principal semi-axes of the three-dimensional moment (torque) force ellipsoid (based on J_ω) and the directions and lengths of the principal semi-axes of the three-dimensional linear force ellipsoid (based on J_v).

Exercise 5.26 The kinematics of the 7R WAM robot are given in Section 4.1.3.

(a) Give the numerical body Jacobian J_b when all joint angles are $\pi/2$. Separate the Jacobian matrix into an angular-velocity portion J_ω (the joint rates act on the angular velocity) and a linear-velocity portion J_v (the joint rates act on the linear velocity).

(b) For this configuration, calculate the directions and lengths of the principal semi-axes of the three-dimensional angular-velocity manipulability ellipsoid (based on J_ω) and the directions and lengths of the principal semi-axes of the three-dimensional linear-velocity manipulability ellipsoid (based on J_v).

(c) For this configuration, calculate the directions and lengths of the principal semi-axes of the three-dimensional moment (torque) force ellipsoid (based on J_ω) and the directions and lengths of the principal semi-axes of the three-dimensional linear force ellipsoid (based on J_v).

Exercise 5.27 Examine the software functions for this chapter in your favorite programming language. Verify that they work in the way that you expect. Can you make them more computationally efficient?