

Figure 22.10 The articulation points, bridges, and biconnected components of a connected, undirected graph for use in Problem 22-2. The articulation points are the heavily shaded vertices, the bridges are the heavily shaded edges, and the biconnected components are the edges in the shaded regions, with a *bcc* numbering shown.

these definitions. We can determine articulation points, bridges, and biconnected components using depth-first search. Let $G_\pi = (V, E_\pi)$ be a depth-first tree of G .

- a. Prove that the root of G_π is an articulation point of G if and only if it has at least two children in G_π .
- b. Let v be a nonroot vertex of G_π . Prove that v is an articulation point of G if and only if v has a child s such that there is no back edge from s or any descendant of s to a proper ancestor of v .
- c. Let

$$v.\text{low} = \min \begin{cases} v.d, \\ w.d : (u, w) \text{ is a back edge for some descendant } u \text{ of } v. \end{cases}$$

Show how to compute $v.\text{low}$ for all vertices $v \in V$ in $O(E)$ time.

- d. Show how to compute all articulation points in $O(E)$ time.
- e. Prove that an edge of G is a bridge if and only if it does not lie on any simple cycle of G .
- f. Show how to compute all the bridges of G in $O(E)$ time.
- g. Prove that the biconnected components of G partition the nonbridge edges of G .
- h. Give an $O(E)$ -time algorithm to label each edge e of G with a positive integer $e.\text{bcc}$ such that $e.\text{bcc} = e'.\text{bcc}$ if and only if e and e' are in the same biconnected component.

22-3 Euler tour

An **Euler tour** of a strongly connected, directed graph $G = (V, E)$ is a cycle that traverses each edge of G exactly once, although it may visit a vertex more than once.

- a. Show that G has an Euler tour if and only if $\text{in-degree}(v) = \text{out-degree}(v)$ for each vertex $v \in V$.
- b. Describe an $O(E)$ -time algorithm to find an Euler tour of G if one exists. (*Hint: Merge edge-disjoint cycles.*)

22-4 Reachability

Let $G = (V, E)$ be a directed graph in which each vertex $u \in V$ is labeled with a unique integer $L(u)$ from the set $\{1, 2, \dots, |V|\}$. For each vertex $u \in V$, let $R(u) = \{v \in V : u \rightsquigarrow v\}$ be the set of vertices that are reachable from u . Define $\min(u)$ to be the vertex in $R(u)$ whose label is minimum, i.e., $\min(u)$ is the vertex v such that $L(v) = \min \{L(w) : w \in R(u)\}$. Give an $O(V + E)$ -time algorithm that computes $\min(u)$ for all vertices $u \in V$.

Chapter notes

Even [103] and Tarjan [330] are excellent references for graph algorithms.

Breadth-first search was discovered by Moore [260] in the context of finding paths through mazes. Lee [226] independently discovered the same algorithm in the context of routing wires on circuit boards.

Hopcroft and Tarjan [178] advocated the use of the adjacency-list representation over the adjacency-matrix representation for sparse graphs and were the first to recognize the algorithmic importance of depth-first search. Depth-first search has been widely used since the late 1950s, especially in artificial intelligence programs.

Tarjan [327] gave a linear-time algorithm for finding strongly connected components. The algorithm for strongly connected components in Section 22.5 is adapted from Aho, Hopcroft, and Ullman [6], who credit it to S. R. Kosaraju (unpublished) and M. Sharir [314]. Gabow [119] also developed an algorithm for strongly connected components that is based on contracting cycles and uses two stacks to make it run in linear time. Knuth [209] was the first to give a linear-time algorithm for topological sorting.

Electronic circuit designs often need to make the pins of several components electrically equivalent by wiring them together. To interconnect a set of n pins, we can use an arrangement of $n - 1$ wires, each connecting two pins. Of all such arrangements, the one that uses the least amount of wire is usually the most desirable.

We can model this wiring problem with a connected, undirected graph $G = (V, E)$, where V is the set of pins, E is the set of possible interconnections between pairs of pins, and for each edge $(u, v) \in E$, we have a weight $w(u, v)$ specifying the cost (amount of wire needed) to connect u and v . We then wish to find an acyclic subset $T \subseteq E$ that connects all of the vertices and whose total weight

$$w(T) = \sum_{(u,v) \in T} w(u, v)$$

is minimized. Since T is acyclic and connects all of the vertices, it must form a tree, which we call a *spanning tree* since it “spans” the graph G . We call the problem of determining the tree T the *minimum-spanning-tree problem*.¹ Figure 23.1 shows an example of a connected graph and a minimum spanning tree.

In this chapter, we shall examine two algorithms for solving the minimum-spanning-tree problem: Kruskal’s algorithm and Prim’s algorithm. We can easily make each of them run in time $O(E \lg V)$ using ordinary binary heaps. By using Fibonacci heaps, Prim’s algorithm runs in time $O(E + V \lg V)$, which improves over the binary-heap implementation if $|V|$ is much smaller than $|E|$.

The two algorithms are greedy algorithms, as described in Chapter 16. Each step of a greedy algorithm must make one of several possible choices. The greedy strategy advocates making the choice that is the best at the moment. Such a strategy does not generally guarantee that it will always find globally optimal solutions

¹The phrase “minimum spanning tree” is a shortened form of the phrase “minimum-weight spanning tree.” We are not, for example, minimizing the number of edges in T , since all spanning trees have exactly $|V| - 1$ edges by Theorem B.2.

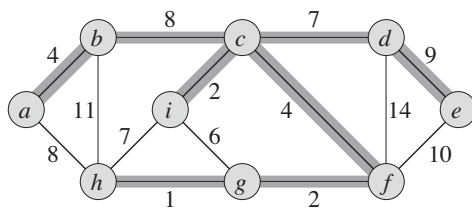


Figure 23.1 A minimum spanning tree for a connected graph. The weights on edges are shown, and the edges in a minimum spanning tree are shaded. The total weight of the tree shown is 37. This minimum spanning tree is not unique: removing the edge (b, c) and replacing it with the edge (a, h) yields another spanning tree with weight 37.

to problems. For the minimum-spanning-tree problem, however, we can prove that certain greedy strategies do yield a spanning tree with minimum weight. Although you can read this chapter independently of Chapter 16, the greedy methods presented here are a classic application of the theoretical notions introduced there.

Section 23.1 introduces a “generic” minimum-spanning-tree method that grows a spanning tree by adding one edge at a time. Section 23.2 gives two algorithms that implement the generic method. The first algorithm, due to Kruskal, is similar to the connected-components algorithm from Section 21.1. The second, due to Prim, resembles Dijkstra’s shortest-paths algorithm (Section 24.3).

Because a tree is a type of graph, in order to be precise we must define a tree in terms of not just its edges, but its vertices as well. Although this chapter focuses on trees in terms of their edges, we shall operate with the understanding that the vertices of a tree T are those that some edge of T is incident on.

23.1 Growing a minimum spanning tree

Assume that we have a connected, undirected graph $G = (V, E)$ with a weight function $w : E \rightarrow \mathbb{R}$, and we wish to find a minimum spanning tree for G . The two algorithms we consider in this chapter use a greedy approach to the problem, although they differ in how they apply this approach.

This greedy strategy is captured by the following generic method, which grows the minimum spanning tree one edge at a time. The generic method manages a set of edges A , maintaining the following loop invariant:

Prior to each iteration, A is a subset of some minimum spanning tree.

At each step, we determine an edge (u, v) that we can add to A without violating this invariant, in the sense that $A \cup \{(u, v)\}$ is also a subset of a minimum spanning

tree. We call such an edge a **safe edge** for A , since we can add it safely to A while maintaining the invariant.

GENERIC-MST(G, w)

```

1   $A = \emptyset$ 
2  while  $A$  does not form a spanning tree
3      find an edge  $(u, v)$  that is safe for  $A$ 
4       $A = A \cup \{(u, v)\}$ 
5  return  $A$ 

```

We use the loop invariant as follows:

Initialization: After line 1, the set A trivially satisfies the loop invariant.

Maintenance: The loop in lines 2–4 maintains the invariant by adding only safe edges.

Termination: All edges added to A are in a minimum spanning tree, and so the set A returned in line 5 must be a minimum spanning tree.

The tricky part is, of course, finding a safe edge in line 3. One must exist, since when line 3 is executed, the invariant dictates that there is a spanning tree T such that $A \subseteq T$. Within the **while** loop body, A must be a proper subset of T , and therefore there must be an edge $(u, v) \in T$ such that $(u, v) \notin A$ and (u, v) is safe for A .

In the remainder of this section, we provide a rule (Theorem 23.1) for recognizing safe edges. The next section describes two algorithms that use this rule to find safe edges efficiently.

We first need some definitions. A **cut** $(S, V - S)$ of an undirected graph $G = (V, E)$ is a partition of V . Figure 23.2 illustrates this notion. We say that an edge $(u, v) \in E$ **crosses** the cut $(S, V - S)$ if one of its endpoints is in S and the other is in $V - S$. We say that a cut **respects** a set A of edges if no edge in A crosses the cut. An edge is a **light edge** crossing a cut if its weight is the minimum of any edge crossing the cut. Note that there can be more than one light edge crossing a cut in the case of ties. More generally, we say that an edge is a **light edge** satisfying a given property if its weight is the minimum of any edge satisfying the property.

Our rule for recognizing safe edges is given by the following theorem.

Theorem 23.1

Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function w defined on E . Let A be a subset of E that is included in some minimum spanning tree for G , let $(S, V - S)$ be any cut of G that respects A , and let (u, v) be a light edge crossing $(S, V - S)$. Then, edge (u, v) is safe for A .

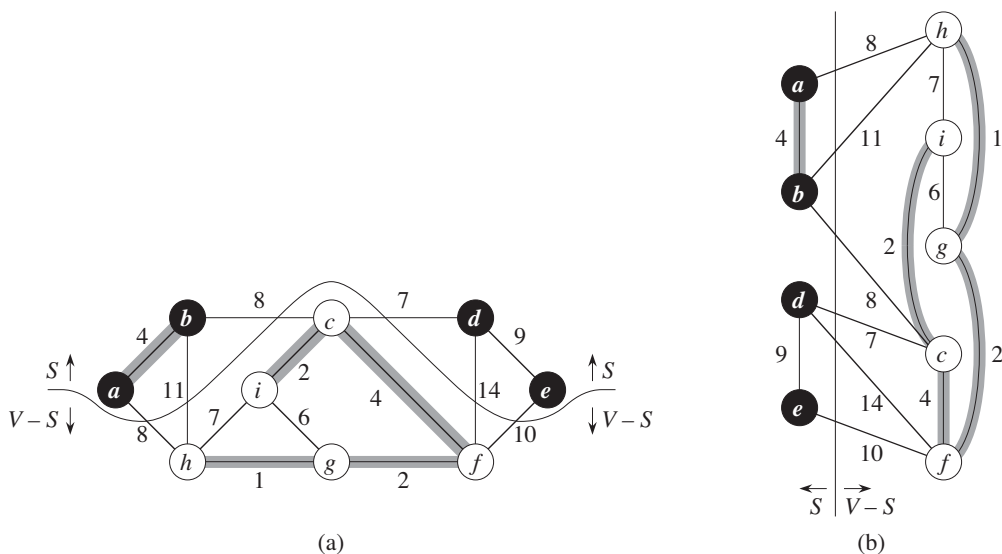


Figure 23.2 Two ways of viewing a cut $(S, V - S)$ of the graph from Figure 23.1. **(a)** Black vertices are in the set S , and white vertices are in $V - S$. The edges crossing the cut are those connecting white vertices with black vertices. The edge (d, c) is the unique light edge crossing the cut. A subset A of the edges is shaded; note that the cut $(S, V - S)$ respects A , since no edge of A crosses the cut. **(b)** The same graph with the vertices in the set S on the left and the vertices in the set $V - S$ on the right. An edge crosses the cut if it connects a vertex on the left with a vertex on the right.

Proof Let T be a minimum spanning tree that includes A , and assume that T does not contain the light edge (u, v) , since if it does, we are done. We shall construct another minimum spanning tree T' that includes $A \cup \{(u, v)\}$ by using a cut-and-paste technique, thereby showing that (u, v) is a safe edge for A .

The edge (u, v) forms a cycle with the edges on the simple path p from u to v in T , as Figure 23.3 illustrates. Since u and v are on opposite sides of the cut $(S, V - S)$, at least one edge in T lies on the simple path p and also crosses the cut. Let (x, y) be any such edge. The edge (x, y) is not in A , because the cut respects A . Since (x, y) is on the unique simple path from u to v in T , removing (x, y) breaks T into two components. Adding (u, v) reconnects them to form a new spanning tree $T' = T - \{(x, y)\} \cup \{(u, v)\}$.

We next show that T' is a minimum spanning tree. Since (u, v) is a light edge crossing $(S, V - S)$ and (x, y) also crosses this cut, $w(u, v) \leq w(x, y)$. Therefore,

$$\begin{aligned} w(T') &= w(T) - w(x, y) + w(u, v) \\ &\leq w(T). \end{aligned}$$

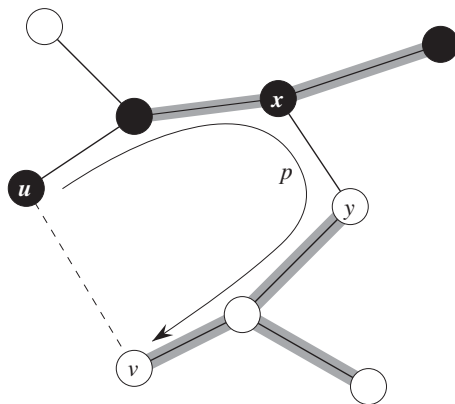


Figure 23.3 The proof of Theorem 23.1. Black vertices are in S , and white vertices are in $V - S$. The edges in the minimum spanning tree T are shown, but the edges in the graph G are not. The edges in A are shaded, and (u, v) is a light edge crossing the cut $(S, V - S)$. The edge (x, y) is an edge on the unique simple path p from u to v in T . To form a minimum spanning tree T' that contains (u, v) , remove the edge (x, y) from T and add the edge (u, v) .

But T is a minimum spanning tree, so that $w(T) \leq w(T')$; thus, T' must be a minimum spanning tree also.

It remains to show that (u, v) is actually a safe edge for A . We have $A \subseteq T'$, since $A \subseteq T$ and $(x, y) \notin A$; thus, $A \cup \{(u, v)\} \subseteq T'$. Consequently, since T' is a minimum spanning tree, (u, v) is safe for A . ■

Theorem 23.1 gives us a better understanding of the workings of the GENERIC-MST method on a connected graph $G = (V, E)$. As the method proceeds, the set A is always acyclic; otherwise, a minimum spanning tree including A would contain a cycle, which is a contradiction. At any point in the execution, the graph $G_A = (V, A)$ is a forest, and each of the connected components of G_A is a tree. (Some of the trees may contain just one vertex, as is the case, for example, when the method begins: A is empty and the forest contains $|V|$ trees, one for each vertex.) Moreover, any safe edge (u, v) for A connects distinct components of G_A , since $A \cup \{(u, v)\}$ must be acyclic.

The **while** loop in lines 2–4 of GENERIC-MST executes $|V| - 1$ times because it finds one of the $|V| - 1$ edges of a minimum spanning tree in each iteration. Initially, when $A = \emptyset$, there are $|V|$ trees in G_A , and each iteration reduces that number by 1. When the forest contains only a single tree, the method terminates.

The two algorithms in Section 23.2 use the following corollary to Theorem 23.1.

Corollary 23.2

Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function w defined on E . Let A be a subset of E that is included in some minimum spanning tree for G , and let $C = (V_C, E_C)$ be a connected component (tree) in the forest $G_A = (V, A)$. If (u, v) is a light edge connecting C to some other component in G_A , then (u, v) is safe for A .

Proof The cut $(V_C, V - V_C)$ respects A , and (u, v) is a light edge for this cut. Therefore, (u, v) is safe for A . ■

Exercises**23.1-1**

Let (u, v) be a minimum-weight edge in a connected graph G . Show that (u, v) belongs to some minimum spanning tree of G .

23.1-2

Professor Sabatier conjectures the following converse of Theorem 23.1. Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function w defined on E . Let A be a subset of E that is included in some minimum spanning tree for G , let $(S, V - S)$ be any cut of G that respects A , and let (u, v) be a safe edge for A crossing $(S, V - S)$. Then, (u, v) is a light edge for the cut. Show that the professor's conjecture is incorrect by giving a counterexample.

23.1-3

Show that if an edge (u, v) is contained in some minimum spanning tree, then it is a light edge crossing some cut of the graph.

23.1-4

Give a simple example of a connected graph such that the set of edges $\{(u, v) : \text{there exists a cut } (S, V - S) \text{ such that } (u, v) \text{ is a light edge crossing } (S, V - S)\}$ does not form a minimum spanning tree.

23.1-5

Let e be a maximum-weight edge on some cycle of connected graph $G = (V, E)$. Prove that there is a minimum spanning tree of $G' = (V, E - \{e\})$ that is also a minimum spanning tree of G . That is, there is a minimum spanning tree of G that does not include e .

23.1-6

Show that a graph has a unique minimum spanning tree if, for every cut of the graph, there is a unique light edge crossing the cut. Show that the converse is not true by giving a counterexample.

23.1-7

Argue that if all edge weights of a graph are positive, then any subset of edges that connects all vertices and has minimum total weight must be a tree. Give an example to show that the same conclusion does not follow if we allow some weights to be nonpositive.

23.1-8

Let T be a minimum spanning tree of a graph G , and let L be the sorted list of the edge weights of T . Show that for any other minimum spanning tree T' of G , the list L is also the sorted list of edge weights of T' .

23.1-9

Let T be a minimum spanning tree of a graph $G = (V, E)$, and let V' be a subset of V . Let T' be the subgraph of T induced by V' , and let G' be the subgraph of G induced by V' . Show that if T' is connected, then T' is a minimum spanning tree of G' .

23.1-10

Given a graph G and a minimum spanning tree T , suppose that we decrease the weight of one of the edges in T . Show that T is still a minimum spanning tree for G . More formally, let T be a minimum spanning tree for G with edge weights given by weight function w . Choose one edge $(x, y) \in T$ and a positive number k , and define the weight function w' by

$$w'(u, v) = \begin{cases} w(u, v) & \text{if } (u, v) \neq (x, y), \\ w(x, y) - k & \text{if } (u, v) = (x, y). \end{cases}$$

Show that T is a minimum spanning tree for G with edge weights given by w' .

23.1-11 ★

Given a graph G and a minimum spanning tree T , suppose that we decrease the weight of one of the edges not in T . Give an algorithm for finding the minimum spanning tree in the modified graph.

23.2 The algorithms of Kruskal and Prim

The two minimum-spanning-tree algorithms described in this section elaborate on the generic method. They each use a specific rule to determine a safe edge in line 3 of GENERIC-MST. In Kruskal's algorithm, the set A is a forest whose vertices are all those of the given graph. The safe edge added to A is always a least-weight edge in the graph that connects two distinct components. In Prim's algorithm, the set A forms a single tree. The safe edge added to A is always a least-weight edge connecting the tree to a vertex not in the tree.

Kruskal's algorithm

Kruskal's algorithm finds a safe edge to add to the growing forest by finding, of all the edges that connect any two trees in the forest, an edge (u, v) of least weight. Let C_1 and C_2 denote the two trees that are connected by (u, v) . Since (u, v) must be a light edge connecting C_1 to some other tree, Corollary 23.2 implies that (u, v) is a safe edge for C_1 . Kruskal's algorithm qualifies as a greedy algorithm because at each step it adds to the forest an edge of least possible weight.

Our implementation of Kruskal's algorithm is like the algorithm to compute connected components from Section 21.1. It uses a disjoint-set data structure to maintain several disjoint sets of elements. Each set contains the vertices in one tree of the current forest. The operation $\text{FIND-SET}(u)$ returns a representative element from the set that contains u . Thus, we can determine whether two vertices u and v belong to the same tree by testing whether $\text{FIND-SET}(u)$ equals $\text{FIND-SET}(v)$. To combine trees, Kruskal's algorithm calls the UNION procedure.

$\text{MST-KRUSKAL}(G, w)$

```

1   $A = \emptyset$ 
2  for each vertex  $v \in G.V$ 
3       $\text{MAKE-SET}(v)$ 
4  sort the edges of  $G.E$  into nondecreasing order by weight  $w$ 
5  for each edge  $(u, v) \in G.E$ , taken in nondecreasing order by weight
6      if  $\text{FIND-SET}(u) \neq \text{FIND-SET}(v)$ 
7           $A = A \cup \{(u, v)\}$ 
8           $\text{UNION}(u, v)$ 
9  return  $A$ 
```

Figure 23.4 shows how Kruskal's algorithm works. Lines 1–3 initialize the set A to the empty set and create $|V|$ trees, one containing each vertex. The **for** loop in lines 5–8 examines edges in order of weight, from lowest to highest. The loop

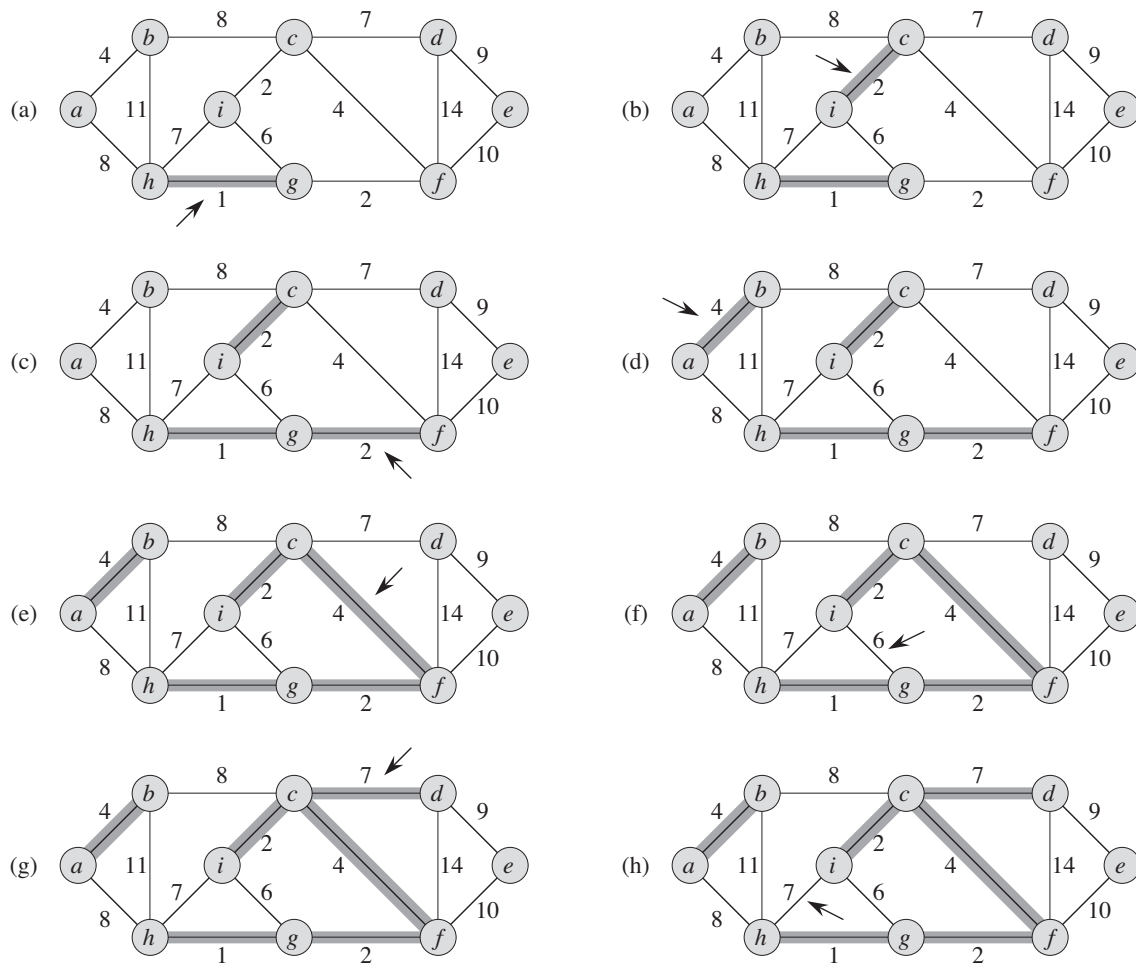


Figure 23.4 The execution of Kruskal's algorithm on the graph from Figure 23.1. Shaded edges belong to the forest A being grown. The algorithm considers each edge in sorted order by weight. An arrow points to the edge under consideration at each step of the algorithm. If the edge joins two distinct trees in the forest, it is added to the forest, thereby merging the two trees.

checks, for each edge (u, v) , whether the endpoints u and v belong to the same tree. If they do, then the edge (u, v) cannot be added to the forest without creating a cycle, and the edge is discarded. Otherwise, the two vertices belong to different trees. In this case, line 7 adds the edge (u, v) to A , and line 8 merges the vertices in the two trees.

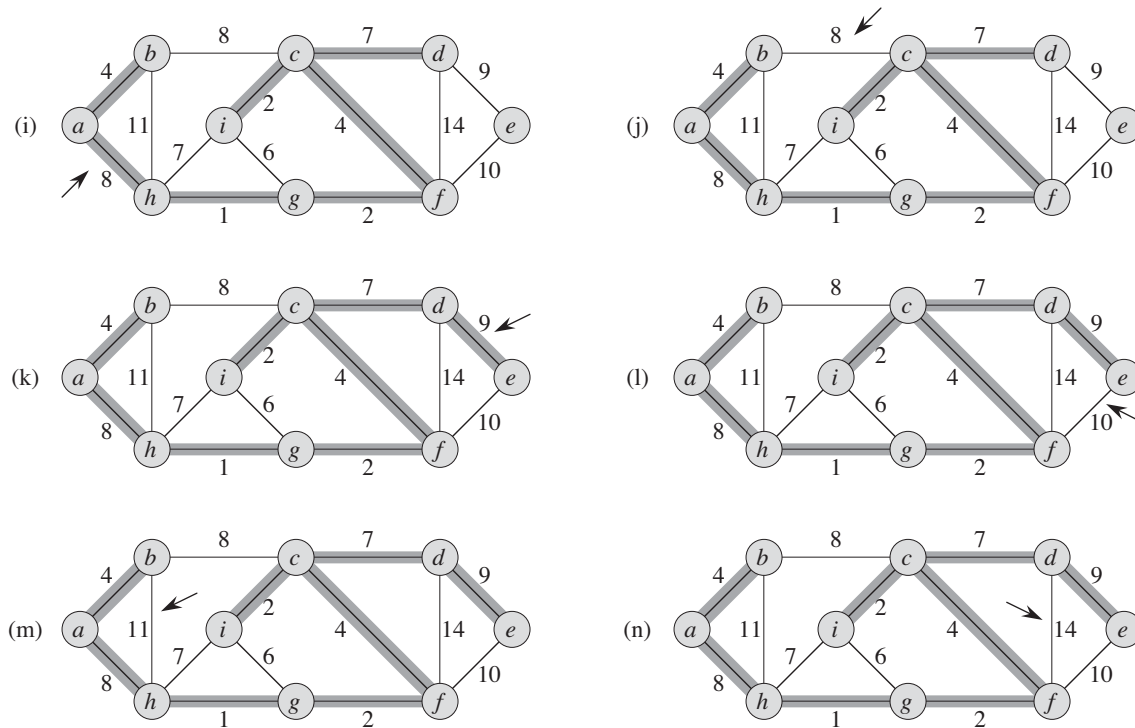


Figure 23.4, continued Further steps in the execution of Kruskal's algorithm.

The running time of Kruskal's algorithm for a graph $G = (V, E)$ depends on how we implement the disjoint-set data structure. We assume that we use the disjoint-set-forest implementation of Section 21.3 with the union-by-rank and path-compression heuristics, since it is the asymptotically fastest implementation known. Initializing the set A in line 1 takes $O(1)$ time, and the time to sort the edges in line 4 is $O(E \lg E)$. (We will account for the cost of the $|V|$ MAKE-SET operations in the **for** loop of lines 2–3 in a moment.) The **for** loop of lines 5–8 performs $O(E)$ FIND-SET and UNION operations on the disjoint-set forest. Along with the $|V|$ MAKE-SET operations, these take a total of $O((V + E) \alpha(V))$ time, where α is the very slowly growing function defined in Section 21.4. Because we assume that G is connected, we have $|E| \geq |V| - 1$, and so the disjoint-set operations take $O(E \alpha(V))$ time. Moreover, since $\alpha(|V|) = O(\lg V) = O(\lg E)$, the total running time of Kruskal's algorithm is $O(E \lg E)$. Observing that $|E| < |V|^2$, we have $\lg |E| = O(\lg V)$, and so we can restate the running time of Kruskal's algorithm as $O(E \lg V)$.

Prim's algorithm

Like Kruskal's algorithm, Prim's algorithm is a special case of the generic minimum-spanning-tree method from Section 23.1. Prim's algorithm operates much like Dijkstra's algorithm for finding shortest paths in a graph, which we shall see in Section 24.3. Prim's algorithm has the property that the edges in the set A always form a single tree. As Figure 23.5 shows, the tree starts from an arbitrary root vertex r and grows until the tree spans all the vertices in V . Each step adds to the tree A a light edge that connects A to an isolated vertex—one on which no edge of A is incident. By Corollary 23.2, this rule adds only edges that are safe for A ; therefore, when the algorithm terminates, the edges in A form a minimum spanning tree. This strategy qualifies as greedy since at each step it adds to the tree an edge that contributes the minimum amount possible to the tree's weight.

In order to implement Prim's algorithm efficiently, we need a fast way to select a new edge to add to the tree formed by the edges in A . In the pseudocode below, the connected graph G and the root r of the minimum spanning tree to be grown are inputs to the algorithm. During execution of the algorithm, all vertices that are *not* in the tree reside in a min-priority queue Q based on a *key* attribute. For each vertex v , the attribute $v.key$ is the minimum weight of any edge connecting v to a vertex in the tree; by convention, $v.key = \infty$ if there is no such edge. The attribute $v.\pi$ names the parent of v in the tree. The algorithm implicitly maintains the set A from GENERIC-MST as

$$A = \{(v, v.\pi) : v \in V - \{r\} - Q\} .$$

When the algorithm terminates, the min-priority queue Q is empty; the minimum spanning tree A for G is thus

$$A = \{(v, v.\pi) : v \in V - \{r\}\} .$$

MST-PRIM(G, w, r)

```

1  for each  $u \in G.V$ 
2       $u.key = \infty$ 
3       $u.\pi = \text{NIL}$ 
4   $r.key = 0$ 
5   $Q = G.V$ 
6  while  $Q \neq \emptyset$ 
7       $u = \text{EXTRACT-MIN}(Q)$ 
8      for each  $v \in G.Adj[u]$ 
9          if  $v \in Q$  and  $w(u, v) < v.key$ 
10              $v.\pi = u$ 
11              $v.key = w(u, v)$ 
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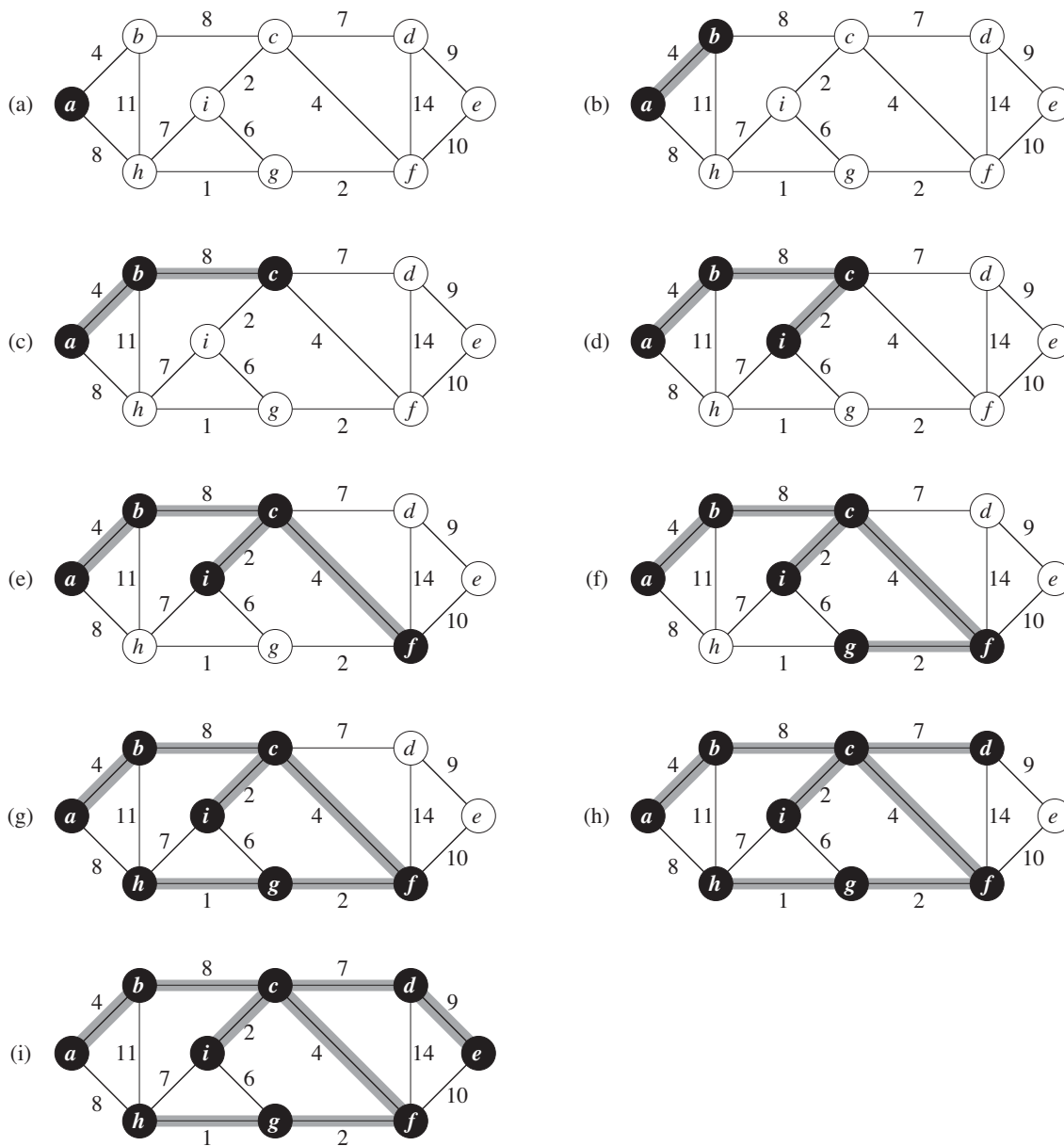


Figure 23.5 The execution of Prim's algorithm on the graph from Figure 23.1. The root vertex is a . Shaded edges are in the tree being grown, and black vertices are in the tree. At each step of the algorithm, the vertices in the tree determine a cut of the graph, and a light edge crossing the cut is added to the tree. In the second step, for example, the algorithm has a choice of adding either edge (b, c) or edge (a, h) to the tree since both are light edges crossing the cut.

Figure 23.5 shows how Prim's algorithm works. Lines 1–5 set the key of each vertex to ∞ (except for the root r , whose key is set to 0 so that it will be the first vertex processed), set the parent of each vertex to NIL, and initialize the min-priority queue Q to contain all the vertices. The algorithm maintains the following three-part loop invariant:

Prior to each iteration of the **while** loop of lines 6–11,

1. $A = \{(v, v.\pi) : v \in V - \{r\} - Q\}$.
2. The vertices already placed into the minimum spanning tree are those in $V - Q$.
3. For all vertices $v \in Q$, if $v.\pi \neq \text{NIL}$, then $v.\text{key} < \infty$ and $v.\text{key}$ is the weight of a light edge $(v, v.\pi)$ connecting v to some vertex already placed into the minimum spanning tree.

Line 7 identifies a vertex $u \in Q$ incident on a light edge that crosses the cut $(V - Q, Q)$ (with the exception of the first iteration, in which $u = r$ due to line 4). Removing u from the set Q adds it to the set $V - Q$ of vertices in the tree, thus adding $(u, u.\pi)$ to A . The **for** loop of lines 8–11 updates the *key* and π attributes of every vertex v adjacent to u but not in the tree, thereby maintaining the third part of the loop invariant.

The running time of Prim's algorithm depends on how we implement the min-priority queue Q . If we implement Q as a binary min-heap (see Chapter 6), we can use the BUILD-MIN-HEAP procedure to perform lines 1–5 in $O(V)$ time. The body of the **while** loop executes $|V|$ times, and since each EXTRACT-MIN operation takes $O(\lg V)$ time, the total time for all calls to EXTRACT-MIN is $O(V \lg V)$. The **for** loop in lines 8–11 executes $O(E)$ times altogether, since the sum of the lengths of all adjacency lists is $2|E|$. Within the **for** loop, we can implement the test for membership in Q in line 9 in constant time by keeping a bit for each vertex that tells whether or not it is in Q , and updating the bit when the vertex is removed from Q . The assignment in line 11 involves an implicit DECREASE-KEY operation on the min-heap, which a binary min-heap supports in $O(\lg V)$ time. Thus, the total time for Prim's algorithm is $O(V \lg V + E \lg V) = O(E \lg V)$, which is asymptotically the same as for our implementation of Kruskal's algorithm.

We can improve the asymptotic running time of Prim's algorithm by using Fibonacci heaps. Chapter 19 shows that if a Fibonacci heap holds $|V|$ elements, an EXTRACT-MIN operation takes $O(\lg V)$ amortized time and a DECREASE-KEY operation (to implement line 11) takes $O(1)$ amortized time. Therefore, if we use a Fibonacci heap to implement the min-priority queue Q , the running time of Prim's algorithm improves to $O(E + V \lg V)$.

Exercises

23.2-1

Kruskal's algorithm can return different spanning trees for the same input graph G , depending on how it breaks ties when the edges are sorted into order. Show that for each minimum spanning tree T of G , there is a way to sort the edges of G in Kruskal's algorithm so that the algorithm returns T .

23.2-2

Suppose that we represent the graph $G = (V, E)$ as an adjacency matrix. Give a simple implementation of Prim's algorithm for this case that runs in $O(V^2)$ time.

23.2-3

For a sparse graph $G = (V, E)$, where $|E| = \Theta(V)$, is the implementation of Prim's algorithm with a Fibonacci heap asymptotically faster than the binary-heap implementation? What about for a dense graph, where $|E| = \Theta(V^2)$? How must the sizes $|E|$ and $|V|$ be related for the Fibonacci-heap implementation to be asymptotically faster than the binary-heap implementation?

23.2-4

Suppose that all edge weights in a graph are integers in the range from 1 to $|V|$. How fast can you make Kruskal's algorithm run? What if the edge weights are integers in the range from 1 to W for some constant W ?

23.2-5

Suppose that all edge weights in a graph are integers in the range from 1 to $|V|$. How fast can you make Prim's algorithm run? What if the edge weights are integers in the range from 1 to W for some constant W ?

23.2-6 ★

Suppose that the edge weights in a graph are uniformly distributed over the half-open interval $[0, 1)$. Which algorithm, Kruskal's or Prim's, can you make run faster?

23.2-7 ★

Suppose that a graph G has a minimum spanning tree already computed. How quickly can we update the minimum spanning tree if we add a new vertex and incident edges to G ?

23.2-8

Professor Borden proposes a new divide-and-conquer algorithm for computing minimum spanning trees, which goes as follows. Given a graph $G = (V, E)$, partition the set V of vertices into two sets V_1 and V_2 such that $|V_1|$ and $|V_2|$ differ

by at most 1. Let E_1 be the set of edges that are incident only on vertices in V_1 , and let E_2 be the set of edges that are incident only on vertices in V_2 . Recursively solve a minimum-spanning-tree problem on each of the two subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Finally, select the minimum-weight edge in E that crosses the cut (V_1, V_2) , and use this edge to unite the resulting two minimum spanning trees into a single spanning tree.

Either argue that the algorithm correctly computes a minimum spanning tree of G , or provide an example for which the algorithm fails.

Problems

23-1 Second-best minimum spanning tree

Let $G = (V, E)$ be an undirected, connected graph whose weight function is $w : E \rightarrow \mathbb{R}$, and suppose that $|E| \geq |V|$ and all edge weights are distinct.

We define a second-best minimum spanning tree as follows. Let \mathcal{T} be the set of all spanning trees of G , and let T' be a minimum spanning tree of G . Then a *second-best minimum spanning tree* is a spanning tree T such that $w(T) = \min_{T'' \in \mathcal{T} - \{T'\}} \{w(T'')\}$.

- a. Show that the minimum spanning tree is unique, but that the second-best minimum spanning tree need not be unique.
- b. Let T be the minimum spanning tree of G . Prove that G contains edges $(u, v) \in T$ and $(x, y) \notin T$ such that $T - \{(u, v)\} \cup \{(x, y)\}$ is a second-best minimum spanning tree of G .
- c. Let T be a spanning tree of G and, for any two vertices $u, v \in V$, let $\max[u, v]$ denote an edge of maximum weight on the unique simple path between u and v in T . Describe an $O(V^2)$ -time algorithm that, given T , computes $\max[u, v]$ for all $u, v \in V$.
- d. Give an efficient algorithm to compute the second-best minimum spanning tree of G .

23-2 Minimum spanning tree in sparse graphs

For a very sparse connected graph $G = (V, E)$, we can further improve upon the $O(E + V \lg V)$ running time of Prim's algorithm with Fibonacci heaps by preprocessing G to decrease the number of vertices before running Prim's algorithm. In particular, we choose, for each vertex u , the minimum-weight edge (u, v) incident on u , and we put (u, v) into the minimum spanning tree under construction. We

then contract all chosen edges (see Section B.4). Rather than contracting these edges one at a time, we first identify sets of vertices that are united into the same new vertex. Then we create the graph that would have resulted from contracting these edges one at a time, but we do so by “renaming” edges according to the sets into which their endpoints were placed. Several edges from the original graph may be renamed the same as each other. In such a case, only one edge results, and its weight is the minimum of the weights of the corresponding original edges.

Initially, we set the minimum spanning tree T being constructed to be empty, and for each edge $(u, v) \in E$, we initialize the attributes $(u, v).orig = (u, v)$ and $(u, v).c = w(u, v)$. We use the *orig* attribute to reference the edge from the initial graph that is associated with an edge in the contracted graph. The *c* attribute holds the weight of an edge, and as edges are contracted, we update it according to the above scheme for choosing edge weights. The procedure MST-REDUCE takes inputs G and T , and it returns a contracted graph G' with updated attributes *orig'* and *c'*. The procedure also accumulates edges of G into the minimum spanning tree T .

MST-REDUCE(G, T)

```

1  for each  $v \in G.V$ 
2       $v.mark = \text{FALSE}$ 
3      MAKE-SET( $v$ )
4  for each  $u \in G.V$ 
5      if  $u.mark == \text{FALSE}$ 
6          choose  $v \in G.Adj[u]$  such that  $(u, v).c$  is minimized
7          UNION( $u, v$ )
8           $T = T \cup \{(u, v).orig\}$ 
9           $u.mark = v.mark = \text{TRUE}$ 
10  $G'.V = \{\text{FIND-SET}(v) : v \in G.V\}$ 
11  $G'.E = \emptyset$ 
12 for each  $(x, y) \in G.E$ 
13      $u = \text{FIND-SET}(x)$ 
14      $v = \text{FIND-SET}(y)$ 
15     if  $(u, v) \notin G'.E$ 
16          $G'.E = G'.E \cup \{(u, v)\}$ 
17          $(u, v).orig' = (x, y).orig$ 
18          $(u, v).c' = (x, y).c$ 
19     else if  $(x, y).c < (u, v).c'$ 
20          $(u, v).orig' = (x, y).orig$ 
21          $(u, v).c' = (x, y).c$ 
22 construct adjacency lists  $G'.Adj$  for  $G'$ 
23 return  $G'$  and  $T$ 
```

- a. Let T be the set of edges returned by MST-REDUCE, and let A be the minimum spanning tree of the graph G' formed by the call MST-PRIM(G', c', r), where c' is the weight attribute on the edges of $G'.E$ and r is any vertex in $G'.V$. Prove that $T \cup \{(x, y).orig' : (x, y) \in A\}$ is a minimum spanning tree of G .
- b. Argue that $|G'.V| \leq |V|/2$.
- c. Show how to implement MST-REDUCE so that it runs in $O(E)$ time. (*Hint*: Use simple data structures.)
- d. Suppose that we run k phases of MST-REDUCE, using the output G' produced by one phase as the input G to the next phase and accumulating edges in T . Argue that the overall running time of the k phases is $O(kE)$.
- e. Suppose that after running k phases of MST-REDUCE, as in part (d), we run Prim's algorithm by calling MST-PRIM(G', c', r), where G' , with weight attribute c' , is returned by the last phase and r is any vertex in $G'.V$. Show how to pick k so that the overall running time is $O(E \lg \lg V)$. Argue that your choice of k minimizes the overall asymptotic running time.
- f. For what values of $|E|$ (in terms of $|V|$) does Prim's algorithm with preprocessing asymptotically beat Prim's algorithm without preprocessing?

23-3 Bottleneck spanning tree

A **bottleneck spanning tree** T of an undirected graph G is a spanning tree of G whose largest edge weight is minimum over all spanning trees of G . We say that the value of the bottleneck spanning tree is the weight of the maximum-weight edge in T .

- a. Argue that a minimum spanning tree is a bottleneck spanning tree.

Part (a) shows that finding a bottleneck spanning tree is no harder than finding a minimum spanning tree. In the remaining parts, we will show how to find a bottleneck spanning tree in linear time.

- b. Give a linear-time algorithm that given a graph G and an integer b , determines whether the value of the bottleneck spanning tree is at most b .
- c. Use your algorithm for part (b) as a subroutine in a linear-time algorithm for the bottleneck-spanning-tree problem. (*Hint*: You may want to use a subroutine that contracts sets of edges, as in the MST-REDUCE procedure described in Problem 23-2.)

23-4 Alternative minimum-spanning-tree algorithms

In this problem, we give pseudocode for three different algorithms. Each one takes a connected graph and a weight function as input and returns a set of edges T . For each algorithm, either prove that T is a minimum spanning tree or prove that T is not a minimum spanning tree. Also describe the most efficient implementation of each algorithm, whether or not it computes a minimum spanning tree.

a. MAYBE-MST-A(G, w)

```

1  sort the edges into nonincreasing order of edge weights  $w$ 
2   $T = E$ 
3  for each edge  $e$ , taken in nonincreasing order by weight
4      if  $T - \{e\}$  is a connected graph
5           $T = T - \{e\}$ 
6  return  $T$ 

```

b. MAYBE-MST-B(G, w)

```

1   $T = \emptyset$ 
2  for each edge  $e$ , taken in arbitrary order
3      if  $T \cup \{e\}$  has no cycles
4           $T = T \cup \{e\}$ 
5  return  $T$ 

```

c. MAYBE-MST-C(G, w)

```

1   $T = \emptyset$ 
2  for each edge  $e$ , taken in arbitrary order
3       $T = T \cup \{e\}$ 
4      if  $T$  has a cycle  $c$ 
5          let  $e'$  be a maximum-weight edge on  $c$ 
6           $T = T - \{e'\}$ 
7  return  $T$ 

```

Chapter notes

Tarjan [330] surveys the minimum-spanning-tree problem and provides excellent advanced material. Graham and Hell [151] compiled a history of the minimum-spanning-tree problem.

Tarjan attributes the first minimum-spanning-tree algorithm to a 1926 paper by O. Borůvka. Borůvka's algorithm consists of running $O(\lg V)$ iterations of the

procedure MST-REDUCE described in Problem 23-2. Kruskal's algorithm was reported by Kruskal [222] in 1956. The algorithm commonly known as Prim's algorithm was indeed invented by Prim [285], but it was also invented earlier by V. Jarník in 1930.

The reason underlying why greedy algorithms are effective at finding minimum spanning trees is that the set of forests of a graph forms a graphic matroid. (See Section 16.4.)

When $|E| = \Omega(V \lg V)$, Prim's algorithm, implemented with Fibonacci heaps, runs in $O(E)$ time. For sparser graphs, using a combination of the ideas from Prim's algorithm, Kruskal's algorithm, and Borůvka's algorithm, together with advanced data structures, Fredman and Tarjan [114] give an algorithm that runs in $O(E \lg^* V)$ time. Gabow, Galil, Spencer, and Tarjan [120] improved this algorithm to run in $O(E \lg \lg^* V)$ time. Chazelle [60] gives an algorithm that runs in $O(E \hat{\alpha}(E, V))$ time, where $\hat{\alpha}(E, V)$ is the functional inverse of Ackermann's function. (See the chapter notes for Chapter 21 for a brief discussion of Ackermann's function and its inverse.) Unlike previous minimum-spanning-tree algorithms, Chazelle's algorithm does not follow the greedy method.

A related problem is **spanning-tree verification**, in which we are given a graph $G = (V, E)$ and a tree $T \subseteq E$, and we wish to determine whether T is a minimum spanning tree of G . King [203] gives a linear-time algorithm to verify a spanning tree, building on earlier work of Komlós [215] and Dixon, Rauch, and Tarjan [90].

The above algorithms are all deterministic and fall into the comparison-based model described in Chapter 8. Karger, Klein, and Tarjan [195] give a randomized minimum-spanning-tree algorithm that runs in $O(V + E)$ expected time. This algorithm uses recursion in a manner similar to the linear-time selection algorithm in Section 9.3: a recursive call on an auxiliary problem identifies a subset of the edges E' that cannot be in any minimum spanning tree. Another recursive call on $E - E'$ then finds the minimum spanning tree. The algorithm also uses ideas from Borůvka's algorithm and King's algorithm for spanning-tree verification.

Fredman and Willard [116] showed how to find a minimum spanning tree in $O(V + E)$ time using a deterministic algorithm that is not comparison based. Their algorithm assumes that the data are b -bit integers and that the computer memory consists of addressable b -bit words.

Professor Patrick wishes to find the shortest possible route from Phoenix to Indianapolis. Given a road map of the United States on which the distance between each pair of adjacent intersections is marked, how can she determine this shortest route?

One possible way would be to enumerate all the routes from Phoenix to Indianapolis, add up the distances on each route, and select the shortest. It is easy to see, however, that even disallowing routes that contain cycles, Professor Patrick would have to examine an enormous number of possibilities, most of which are simply not worth considering. For example, a route from Phoenix to Indianapolis that passes through Seattle is obviously a poor choice, because Seattle is several hundred miles out of the way.

In this chapter and in Chapter 25, we show how to solve such problems efficiently. In a *shortest-paths problem*, we are given a weighted, directed graph $G = (V, E)$, with weight function $w : E \rightarrow \mathbb{R}$ mapping edges to real-valued weights. The *weight* $w(p)$ of path $p = \langle v_0, v_1, \dots, v_k \rangle$ is the sum of the weights of its constituent edges:

$$w(p) = \sum_{i=1}^k w(v_{i-1}, v_i) .$$

We define the *shortest-path weight* $\delta(u, v)$ from u to v by

$$\delta(u, v) = \begin{cases} \min\{w(p) : u \xrightarrow{p} v\} & \text{if there is a path from } u \text{ to } v , \\ \infty & \text{otherwise .} \end{cases}$$

A *shortest path* from vertex u to vertex v is then defined as any path p with weight $w(p) = \delta(u, v)$.

In the Phoenix-to-Indianapolis example, we can model the road map as a graph: vertices represent intersections, edges represent road segments between intersections, and edge weights represent road distances. Our goal is to find a shortest path from a given intersection in Phoenix to a given intersection in Indianapolis.

Edge weights can represent metrics other than distances, such as time, cost, penalties, loss, or any other quantity that accumulates linearly along a path and that we would want to minimize.

The breadth-first-search algorithm from Section 22.2 is a shortest-paths algorithm that works on unweighted graphs, that is, graphs in which each edge has unit weight. Because many of the concepts from breadth-first search arise in the study of shortest paths in weighted graphs, you might want to review Section 22.2 before proceeding.

Variants

In this chapter, we shall focus on the **single-source shortest-paths problem**: given a graph $G = (V, E)$, we want to find a shortest path from a given **source** vertex $s \in V$ to each vertex $v \in V$. The algorithm for the single-source problem can solve many other problems, including the following variants.

Single-destination shortest-paths problem: Find a shortest path to a given **destination** vertex t from each vertex v . By reversing the direction of each edge in the graph, we can reduce this problem to a single-source problem.

Single-pair shortest-path problem: Find a shortest path from u to v for given vertices u and v . If we solve the single-source problem with source vertex u , we solve this problem also. Moreover, all known algorithms for this problem have the same worst-case asymptotic running time as the best single-source algorithms.

All-pairs shortest-paths problem: Find a shortest path from u to v for every pair of vertices u and v . Although we can solve this problem by running a single-source algorithm once from each vertex, we usually can solve it faster. Additionally, its structure is interesting in its own right. Chapter 25 addresses the all-pairs problem in detail.

Optimal substructure of a shortest path

Shortest-paths algorithms typically rely on the property that a shortest path between two vertices contains other shortest paths within it. (The Edmonds-Karp maximum-flow algorithm in Chapter 26 also relies on this property.) Recall that optimal substructure is one of the key indicators that dynamic programming (Chapter 15) and the greedy method (Chapter 16) might apply. Dijkstra's algorithm, which we shall see in Section 24.3, is a greedy algorithm, and the Floyd-Warshall algorithm, which finds shortest paths between all pairs of vertices (see Section 25.2), is a dynamic-programming algorithm. The following lemma states the optimal-substructure property of shortest paths more precisely.

Lemma 24.1 (Subpaths of shortest paths are shortest paths)

Given a weighted, directed graph $G = (V, E)$ with weight function $w : E \rightarrow \mathbb{R}$, let $p = \langle v_0, v_1, \dots, v_k \rangle$ be a shortest path from vertex v_0 to vertex v_k and, for any i and j such that $0 \leq i \leq j \leq k$, let $p_{ij} = \langle v_i, v_{i+1}, \dots, v_j \rangle$ be the subpath of p from vertex v_i to vertex v_j . Then, p_{ij} is a shortest path from v_i to v_j .

Proof If we decompose path p into $v_0 \xrightarrow{p_{0i}} v_i \xrightarrow{p_{ij}} v_j \xrightarrow{p_{jk}} v_k$, then we have that $w(p) = w(p_{0i}) + w(p_{ij}) + w(p_{jk})$. Now, assume that there is a path p'_{ij} from v_i to v_j with weight $w(p'_{ij}) < w(p_{ij})$. Then, $v_0 \xrightarrow{p_{0i}} v_i \xrightarrow{p'_{ij}} v_j \xrightarrow{p_{jk}} v_k$ is a path from v_0 to v_k whose weight $w(p_{0i}) + w(p'_{ij}) + w(p_{jk})$ is less than $w(p)$, which contradicts the assumption that p is a shortest path from v_0 to v_k . ■

Negative-weight edges

Some instances of the single-source shortest-paths problem may include edges whose weights are negative. If the graph $G = (V, E)$ contains no negative-weight cycles reachable from the source s , then for all $v \in V$, the shortest-path weight $\delta(s, v)$ remains well defined, even if it has a negative value. If the graph contains a negative-weight cycle reachable from s , however, shortest-path weights are not well defined. No path from s to a vertex on the cycle can be a shortest path—we can always find a path with lower weight by following the proposed “shortest” path and then traversing the negative-weight cycle. If there is a negative-weight cycle on some path from s to v , we define $\delta(s, v) = -\infty$.

Figure 24.1 illustrates the effect of negative weights and negative-weight cycles on shortest-path weights. Because there is only one path from s to a (the path $\langle s, a \rangle$), we have $\delta(s, a) = w(s, a) = 3$. Similarly, there is only one path from s to b , and so $\delta(s, b) = w(s, a) + w(a, b) = 3 + (-4) = -1$. There are infinitely many paths from s to c : $\langle s, c \rangle$, $\langle s, c, d, c \rangle$, $\langle s, c, d, c, d, c \rangle$, and so on. Because the cycle $\langle c, d, c \rangle$ has weight $6 + (-3) = 3 > 0$, the shortest path from s to c is $\langle s, c \rangle$, with weight $\delta(s, c) = w(s, c) = 5$. Similarly, the shortest path from s to d is $\langle s, c, d \rangle$, with weight $\delta(s, d) = w(s, c) + w(c, d) = 11$. Analogously, there are infinitely many paths from s to e : $\langle s, e \rangle$, $\langle s, e, f, e \rangle$, $\langle s, e, f, e, f, e \rangle$, and so on. Because the cycle $\langle e, f, e \rangle$ has weight $3 + (-6) = -3 < 0$, however, there is no shortest path from s to e . By traversing the negative-weight cycle $\langle e, f, e \rangle$ arbitrarily many times, we can find paths from s to e with arbitrarily large negative weights, and so $\delta(s, e) = -\infty$. Similarly, $\delta(s, f) = -\infty$. Because g is reachable from f , we can also find paths with arbitrarily large negative weights from s to g , and so $\delta(s, g) = -\infty$. Vertices h, i , and j also form a negative-weight cycle. They are not reachable from s , however, and so $\delta(s, h) = \delta(s, i) = \delta(s, j) = \infty$.

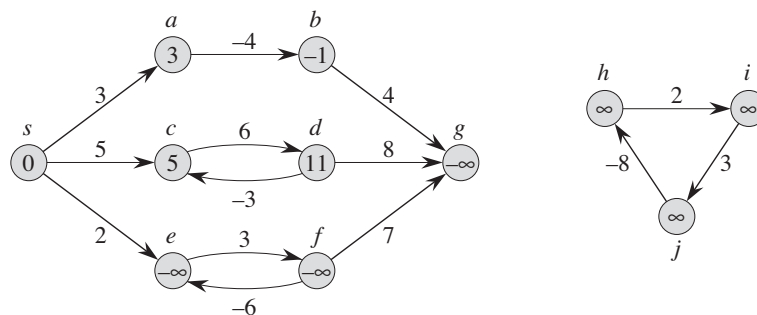


Figure 24.1 Negative edge weights in a directed graph. The shortest-path weight from source s appears within each vertex. Because vertices e and f form a negative-weight cycle reachable from s , they have shortest-path weights of $-\infty$. Because vertex g is reachable from a vertex whose shortest-path weight is $-\infty$, it, too, has a shortest-path weight of $-\infty$. Vertices such as h, i , and j are not reachable from s , and so their shortest-path weights are ∞ , even though they lie on a negative-weight cycle.

Some shortest-paths algorithms, such as Dijkstra's algorithm, assume that all edge weights in the input graph are nonnegative, as in the road-map example. Others, such as the Bellman-Ford algorithm, allow negative-weight edges in the input graph and produce a correct answer as long as no negative-weight cycles are reachable from the source. Typically, if there is such a negative-weight cycle, the algorithm can detect and report its existence.

Cycles

Can a shortest path contain a cycle? As we have just seen, it cannot contain a negative-weight cycle. Nor can it contain a positive-weight cycle, since removing the cycle from the path produces a path with the same source and destination vertices and a lower path weight. That is, if $p = \langle v_0, v_1, \dots, v_k \rangle$ is a path and $c = \langle v_i, v_{i+1}, \dots, v_j \rangle$ is a positive-weight cycle on this path (so that $v_i = v_j$ and $w(c) > 0$), then the path $p' = \langle v_0, v_1, \dots, v_i, v_{j+1}, v_{j+2}, \dots, v_k \rangle$ has weight $w(p') = w(p) - w(c) < w(p)$, and so p cannot be a shortest path from v_0 to v_k .

That leaves only 0-weight cycles. We can remove a 0-weight cycle from any path to produce another path whose weight is the same. Thus, if there is a shortest path from a source vertex s to a destination vertex v that contains a 0-weight cycle, then there is another shortest path from s to v without this cycle. As long as a shortest path has 0-weight cycles, we can repeatedly remove these cycles from the path until we have a shortest path that is cycle-free. Therefore, without loss of generality we can assume that when we are finding shortest paths, they have no cycles, i.e., they are simple paths. Since any acyclic path in a graph $G = (V, E)$

contains at most $|V|$ distinct vertices, it also contains at most $|V| - 1$ edges. Thus, we can restrict our attention to shortest paths of at most $|V| - 1$ edges.

Representing shortest paths

We often wish to compute not only shortest-path weights, but the vertices on shortest paths as well. We represent shortest paths similarly to how we represented breadth-first trees in Section 22.2. Given a graph $G = (V, E)$, we maintain for each vertex $v \in V$ a **predecessor** $v.\pi$ that is either another vertex or NIL. The shortest-paths algorithms in this chapter set the π attributes so that the chain of predecessors originating at a vertex v runs backwards along a shortest path from s to v . Thus, given a vertex v for which $v.\pi \neq \text{NIL}$, the procedure $\text{PRINT-PATH}(G, s, v)$ from Section 22.2 will print a shortest path from s to v .

In the midst of executing a shortest-paths algorithm, however, the π values might not indicate shortest paths. As in breadth-first search, we shall be interested in the **predecessor subgraph** $G_\pi = (V_\pi, E_\pi)$ induced by the π values. Here again, we define the vertex set V_π to be the set of vertices of G with non-NIL predecessors, plus the source s :

$$V_\pi = \{v \in V : v.\pi \neq \text{NIL}\} \cup \{s\} .$$

The directed edge set E_π is the set of edges induced by the π values for vertices in V_π :

$$E_\pi = \{(v.\pi, v) \in E : v \in V_\pi - \{s\}\} .$$

We shall prove that the π values produced by the algorithms in this chapter have the property that at termination G_π is a “shortest-paths tree”—informally, a rooted tree containing a shortest path from the source s to every vertex that is reachable from s . A shortest-paths tree is like the breadth-first tree from Section 22.2, but it contains shortest paths from the source defined in terms of edge weights instead of numbers of edges. To be precise, let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \rightarrow \mathbb{R}$, and assume that G contains no negative-weight cycles reachable from the source vertex $s \in V$, so that shortest paths are well defined. A **shortest-paths tree** rooted at s is a directed subgraph $G' = (V', E')$, where $V' \subseteq V$ and $E' \subseteq E$, such that

1. V' is the set of vertices reachable from s in G ,
2. G' forms a rooted tree with root s , and
3. for all $v \in V'$, the unique simple path from s to v in G' is a shortest path from s to v in G .

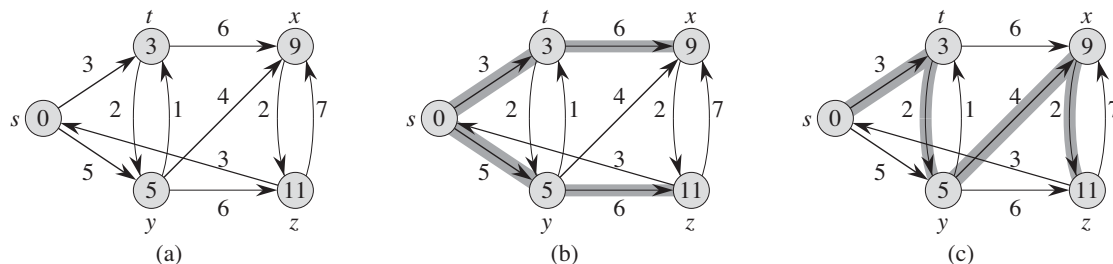


Figure 24.2 (a) A weighted, directed graph with shortest-path weights from source s . (b) The shaded edges form a shortest-paths tree rooted at the source s . (c) Another shortest-paths tree with the same root.

Shortest paths are not necessarily unique, and neither are shortest-paths trees. For example, Figure 24.2 shows a weighted, directed graph and two shortest-paths trees with the same root.

Relaxation

The algorithms in this chapter use the technique of **relaxation**. For each vertex $v \in V$, we maintain an attribute $v.d$, which is an upper bound on the weight of a shortest path from source s to v . We call $v.d$ a **shortest-path estimate**. We initialize the shortest-path estimates and predecessors by the following $\Theta(V)$ -time procedure:

INITIALIZE-SINGLE-SOURCE(G, s)

- 1 **for** each vertex $v \in G.V$
- 2 $v.d = \infty$
- 3 $v.\pi = \text{NIL}$
- 4 $s.d = 0$

After initialization, we have $v.\pi = \text{NIL}$ for all $v \in V$, $s.d = 0$, and $v.d = \infty$ for $v \in V - \{s\}$.

The process of **relaxing** an edge (u, v) consists of testing whether we can improve the shortest path to v found so far by going through u and, if so, updating $v.d$ and $v.\pi$. A relaxation step¹ may decrease the value of the shortest-path

¹It may seem strange that the term “relaxation” is used for an operation that tightens an upper bound. The use of the term is historical. The outcome of a relaxation step can be viewed as a relaxation of the constraint $v.d \leq u.d + w(u, v)$, which, by the triangle inequality (Lemma 24.10), must be satisfied if $u.d = \delta(s, u)$ and $v.d = \delta(s, v)$. That is, if $v.d \leq u.d + w(u, v)$, there is no “pressure” to satisfy this constraint, so the constraint is “relaxed.”

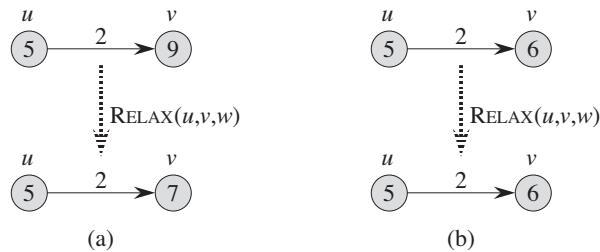


Figure 24.3 Relaxing an edge (u, v) with weight $w(u, v) = 2$. The shortest-path estimate of each vertex appears within the vertex. **(a)** Because $v.d > u.d + w(u, v)$ prior to relaxation, the value of $v.d$ decreases. **(b)** Here, $v.d \leq u.d + w(u, v)$ before relaxing the edge, and so the relaxation step leaves $v.d$ unchanged.

estimate $v.d$ and update v 's predecessor attribute $v.\pi$. The following code performs a relaxation step on edge (u, v) in $O(1)$ time:

```

RELAX( $u, v, w$ )
1  if  $v.d > u.d + w(u, v)$ 
2       $v.d = u.d + w(u, v)$ 
3       $v.\pi = u$ 

```

Figure 24.3 shows two examples of relaxing an edge, one in which a shortest-path estimate decreases and one in which no estimate changes.

Each algorithm in this chapter calls INITIALIZE-SINGLE-SOURCE and then repeatedly relaxes edges. Moreover, relaxation is the only means by which shortest-path estimates and predecessors change. The algorithms in this chapter differ in how many times they relax each edge and the order in which they relax edges. Dijkstra's algorithm and the shortest-paths algorithm for directed acyclic graphs relax each edge exactly once. The Bellman-Ford algorithm relaxes each edge $|V| - 1$ times.

Properties of shortest paths and relaxation

To prove the algorithms in this chapter correct, we shall appeal to several properties of shortest paths and relaxation. We state these properties here, and Section 24.5 proves them formally. For your reference, each property stated here includes the appropriate lemma or corollary number from Section 24.5. The latter five of these properties, which refer to shortest-path estimates or the predecessor subgraph, implicitly assume that the graph is initialized with a call to INITIALIZE-SINGLE-SOURCE(G, s) and that the only way that shortest-path estimates and the predecessor subgraph change are by some sequence of relaxation steps.

Triangle inequality (Lemma 24.10)

For any edge $(u, v) \in E$, we have $\delta(s, v) \leq \delta(s, u) + w(u, v)$.

Upper-bound property (Lemma 24.11)

We always have $v.d \geq \delta(s, v)$ for all vertices $v \in V$, and once $v.d$ achieves the value $\delta(s, v)$, it never changes.

No-path property (Corollary 24.12)

If there is no path from s to v , then we always have $v.d = \delta(s, v) = \infty$.

Convergence property (Lemma 24.14)

If $s \rightsquigarrow u \rightarrow v$ is a shortest path in G for some $u, v \in V$, and if $u.d = \delta(s, u)$ at any time prior to relaxing edge (u, v) , then $v.d = \delta(s, v)$ at all times afterward.

Path-relaxation property (Lemma 24.15)

If $p = \langle v_0, v_1, \dots, v_k \rangle$ is a shortest path from $s = v_0$ to v_k , and we relax the edges of p in the order $(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$. This property holds regardless of any other relaxation steps that occur, even if they are intermixed with relaxations of the edges of p .

Predecessor-subgraph property (Lemma 24.17)

Once $v.d = \delta(s, v)$ for all $v \in V$, the predecessor subgraph is a shortest-paths tree rooted at s .

Chapter outline

Section 24.1 presents the Bellman-Ford algorithm, which solves the single-source shortest-paths problem in the general case in which edges can have negative weight. The Bellman-Ford algorithm is remarkably simple, and it has the further benefit of detecting whether a negative-weight cycle is reachable from the source. Section 24.2 gives a linear-time algorithm for computing shortest paths from a single source in a directed acyclic graph. Section 24.3 covers Dijkstra's algorithm, which has a lower running time than the Bellman-Ford algorithm but requires the edge weights to be nonnegative. Section 24.4 shows how we can use the Bellman-Ford algorithm to solve a special case of linear programming. Finally, Section 24.5 proves the properties of shortest paths and relaxation stated above.

We require some conventions for doing arithmetic with infinities. We shall assume that for any real number $a \neq -\infty$, we have $a + \infty = \infty + a = \infty$. Also, to make our proofs hold in the presence of negative-weight cycles, we shall assume that for any real number $a \neq \infty$, we have $a + (-\infty) = (-\infty) + a = -\infty$.

All algorithms in this chapter assume that the directed graph G is stored in the adjacency-list representation. Additionally, stored with each edge is its weight, so that as we traverse each adjacency list, we can determine the edge weights in $O(1)$ time per edge.

24.1 The Bellman-Ford algorithm

The **Bellman-Ford algorithm** solves the single-source shortest-paths problem in the general case in which edge weights may be negative. Given a weighted, directed graph $G = (V, E)$ with source s and weight function $w : E \rightarrow \mathbb{R}$, the Bellman-Ford algorithm returns a boolean value indicating whether or not there is a negative-weight cycle that is reachable from the source. If there is such a cycle, the algorithm indicates that no solution exists. If there is no such cycle, the algorithm produces the shortest paths and their weights.

The algorithm relaxes edges, progressively decreasing an estimate $v.d$ on the weight of a shortest path from the source s to each vertex $v \in V$ until it achieves the actual shortest-path weight $\delta(s, v)$. The algorithm returns TRUE if and only if the graph contains no negative-weight cycles that are reachable from the source.

```

BELLMAN-FORD( $G, w, s$ )
1  INITIALIZE-SINGLE-SOURCE( $G, s$ )
2  for  $i = 1$  to  $|G.V| - 1$ 
3      for each edge  $(u, v) \in G.E$ 
4          RELAX( $u, v, w$ )
5  for each edge  $(u, v) \in G.E$ 
6      if  $v.d > u.d + w(u, v)$ 
7          return FALSE
8  return TRUE

```

Figure 24.4 shows the execution of the Bellman-Ford algorithm on a graph with 5 vertices. After initializing the d and π values of all vertices in line 1, the algorithm makes $|V| - 1$ passes over the edges of the graph. Each pass is one iteration of the **for** loop of lines 2–4 and consists of relaxing each edge of the graph once. Figures 24.4(b)–(e) show the state of the algorithm after each of the four passes over the edges. After making $|V| - 1$ passes, lines 5–8 check for a negative-weight cycle and return the appropriate boolean value. (We’ll see a little later why this check works.)

The Bellman-Ford algorithm runs in time $O(VE)$, since the initialization in line 1 takes $\Theta(V)$ time, each of the $|V| - 1$ passes over the edges in lines 2–4 takes $\Theta(E)$ time, and the **for** loop of lines 5–7 takes $O(E)$ time.

To prove the correctness of the Bellman-Ford algorithm, we start by showing that if there are no negative-weight cycles, the algorithm computes correct shortest-path weights for all vertices reachable from the source.

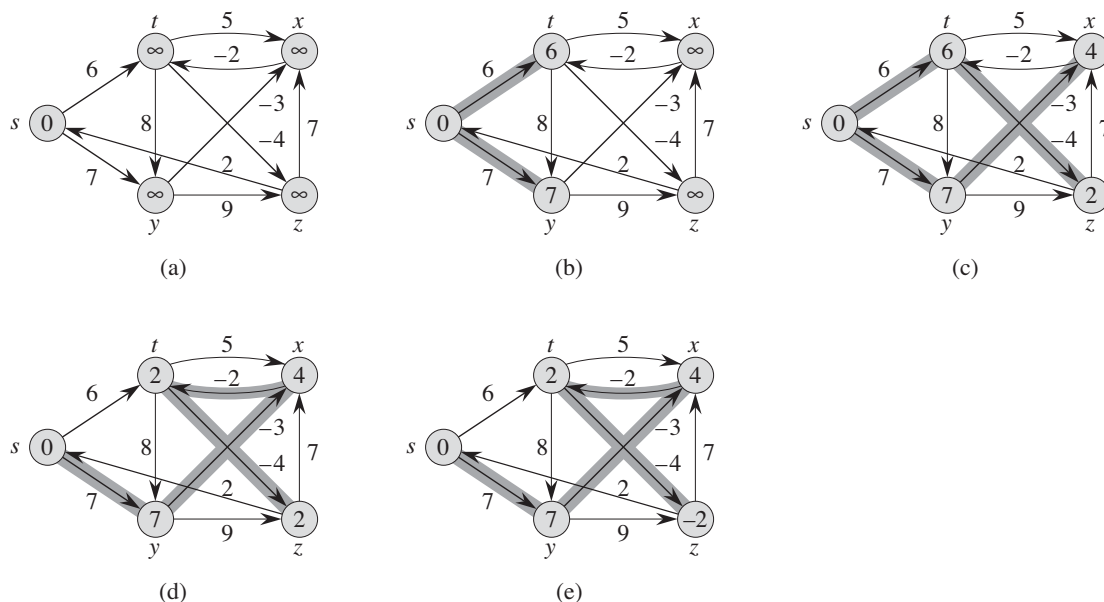


Figure 24.4 The execution of the Bellman-Ford algorithm. The source is vertex s . The d values appear within the vertices, and shaded edges indicate predecessor values: if edge (u, v) is shaded, then $v.\pi = u$. In this particular example, each pass relaxes the edges in the order (t, x) , (t, y) , (t, z) , (x, t) , (y, x) , (y, z) , (z, x) , (z, s) , (s, t) , (s, y) . (a) The situation just before the first pass over the edges. (b)–(e) The situation after each successive pass over the edges. The d and π values in part (e) are the final values. The Bellman-Ford algorithm returns TRUE in this example.

Lemma 24.2

Let $G = (V, E)$ be a weighted, directed graph with source s and weight function $w : E \rightarrow \mathbb{R}$, and assume that G contains no negative-weight cycles that are reachable from s . Then, after the $|V| - 1$ iterations of the **for** loop of lines 2–4 of BELLMAN-FORD, we have $v.d = \delta(s, v)$ for all vertices v that are reachable from s .

Proof We prove the lemma by appealing to the path-relaxation property. Consider any vertex v that is reachable from s , and let $p = \langle v_0, v_1, \dots, v_k \rangle$, where $v_0 = s$ and $v_k = v$, be any shortest path from s to v . Because shortest paths are simple, p has at most $|V| - 1$ edges, and so $k \leq |V| - 1$. Each of the $|V| - 1$ iterations of the **for** loop of lines 2–4 relaxes all $|E|$ edges. Among the edges relaxed in the i th iteration, for $i = 1, 2, \dots, k$, is (v_{i-1}, v_i) . By the path-relaxation property, therefore, $v.d = v_k.d = \delta(s, v_k) = \delta(s, v)$. ■

Corollary 24.3

Let $G = (V, E)$ be a weighted, directed graph with source vertex s and weight function $w : E \rightarrow \mathbb{R}$, and assume that G contains no negative-weight cycles that are reachable from s . Then, for each vertex $v \in V$, there is a path from s to v if and only if BELLMAN-FORD terminates with $v.d < \infty$ when it is run on G .

Proof The proof is left as Exercise 24.1-2. ■

Theorem 24.4 (Correctness of the Bellman-Ford algorithm)

Let BELLMAN-FORD be run on a weighted, directed graph $G = (V, E)$ with source s and weight function $w : E \rightarrow \mathbb{R}$. If G contains no negative-weight cycles that are reachable from s , then the algorithm returns TRUE, we have $v.d = \delta(s, v)$ for all vertices $v \in V$, and the predecessor subgraph G_π is a shortest-paths tree rooted at s . If G does contain a negative-weight cycle reachable from s , then the algorithm returns FALSE.

Proof Suppose that graph G contains no negative-weight cycles that are reachable from the source s . We first prove the claim that at termination, $v.d = \delta(s, v)$ for all vertices $v \in V$. If vertex v is reachable from s , then Lemma 24.2 proves this claim. If v is not reachable from s , then the claim follows from the no-path property. Thus, the claim is proven. The predecessor-subgraph property, along with the claim, implies that G_π is a shortest-paths tree. Now we use the claim to show that BELLMAN-FORD returns TRUE. At termination, we have for all edges $(u, v) \in E$,

$$\begin{aligned} v.d &= \delta(s, v) \\ &\leq \delta(s, u) + w(u, v) \quad (\text{by the triangle inequality}) \\ &= u.d + w(u, v), \end{aligned}$$

and so none of the tests in line 6 causes BELLMAN-FORD to return FALSE. Therefore, it returns TRUE.

Now, suppose that graph G contains a negative-weight cycle that is reachable from the source s ; let this cycle be $c = \langle v_0, v_1, \dots, v_k \rangle$, where $v_0 = v_k$. Then,

$$\sum_{i=1}^k w(v_{i-1}, v_i) < 0. \tag{24.1}$$

Assume for the purpose of contradiction that the Bellman-Ford algorithm returns TRUE. Thus, $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$ for $i = 1, 2, \dots, k$. Summing the inequalities around cycle c gives us

$$\begin{aligned}
\sum_{i=1}^k v_i \cdot d &\leq \sum_{i=1}^k (v_{i-1} \cdot d + w(v_{i-1}, v_i)) \\
&= \sum_{i=1}^k v_{i-1} \cdot d + \sum_{i=1}^k w(v_{i-1}, v_i) .
\end{aligned}$$

Since $v_0 = v_k$, each vertex in c appears exactly once in each of the summations $\sum_{i=1}^k v_i \cdot d$ and $\sum_{i=1}^k v_{i-1} \cdot d$, and so

$$\sum_{i=1}^k v_i \cdot d = \sum_{i=1}^k v_{i-1} \cdot d .$$

Moreover, by Corollary 24.3, $v_i \cdot d$ is finite for $i = 1, 2, \dots, k$. Thus,

$$0 \leq \sum_{i=1}^k w(v_{i-1}, v_i) ,$$

which contradicts inequality (24.1). We conclude that the Bellman-Ford algorithm returns TRUE if graph G contains no negative-weight cycles reachable from the source, and FALSE otherwise. ■

Exercises

24.1-1

Run the Bellman-Ford algorithm on the directed graph of Figure 24.4, using vertex z as the source. In each pass, relax edges in the same order as in the figure, and show the d and π values after each pass. Now, change the weight of edge (z, x) to 4 and run the algorithm again, using s as the source.

24.1-2

Prove Corollary 24.3.

24.1-3

Given a weighted, directed graph $G = (V, E)$ with no negative-weight cycles, let m be the maximum over all vertices $v \in V$ of the minimum number of edges in a shortest path from the source s to v . (Here, the shortest path is by weight, not the number of edges.) Suggest a simple change to the Bellman-Ford algorithm that allows it to terminate in $m + 1$ passes, even if m is not known in advance.

24.1-4

Modify the Bellman-Ford algorithm so that it sets $v.d$ to $-\infty$ for all vertices v for which there is a negative-weight cycle on some path from the source to v .

24.1-5 ★

Let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \rightarrow \mathbb{R}$. Give an $O(VE)$ -time algorithm to find, for each vertex $v \in V$, the value $\delta^*(v) = \min_{u \in V} \{\delta(u, v)\}$.

24.1-6 ★

Suppose that a weighted, directed graph $G = (V, E)$ has a negative-weight cycle. Give an efficient algorithm to list the vertices of one such cycle. Prove that your algorithm is correct.

24.2 Single-source shortest paths in directed acyclic graphs

By relaxing the edges of a weighted dag (directed acyclic graph) $G = (V, E)$ according to a topological sort of its vertices, we can compute shortest paths from a single source in $\Theta(V + E)$ time. Shortest paths are always well defined in a dag, since even if there are negative-weight edges, no negative-weight cycles can exist.

The algorithm starts by topologically sorting the dag (see Section 22.4) to impose a linear ordering on the vertices. If the dag contains a path from vertex u to vertex v , then u precedes v in the topological sort. We make just one pass over the vertices in the topologically sorted order. As we process each vertex, we relax each edge that leaves the vertex.

DAG-SHORTEST-PATHS(G, w, s)

```

1  topologically sort the vertices of  $G$ 
2  INITIALIZE-SINGLE-SOURCE( $G, s$ )
3  for each vertex  $u$ , taken in topologically sorted order
4      for each vertex  $v \in G.Adj[u]$ 
5          RELAX( $u, v, w$ )
```

Figure 24.5 shows the execution of this algorithm.

The running time of this algorithm is easy to analyze. As shown in Section 22.4, the topological sort of line 1 takes $\Theta(V + E)$ time. The call of INITIALIZE-SINGLE-SOURCE in line 2 takes $\Theta(V)$ time. The **for** loop of lines 3–5 makes one iteration per vertex. Altogether, the **for** loop of lines 4–5 relaxes each edge exactly once. (We have used an aggregate analysis here.) Because each iteration of the inner **for** loop takes $\Theta(1)$ time, the total running time is $\Theta(V + E)$, which is linear in the size of an adjacency-list representation of the graph.

The following theorem shows that the DAG-SHORTEST-PATHS procedure correctly computes the shortest paths.

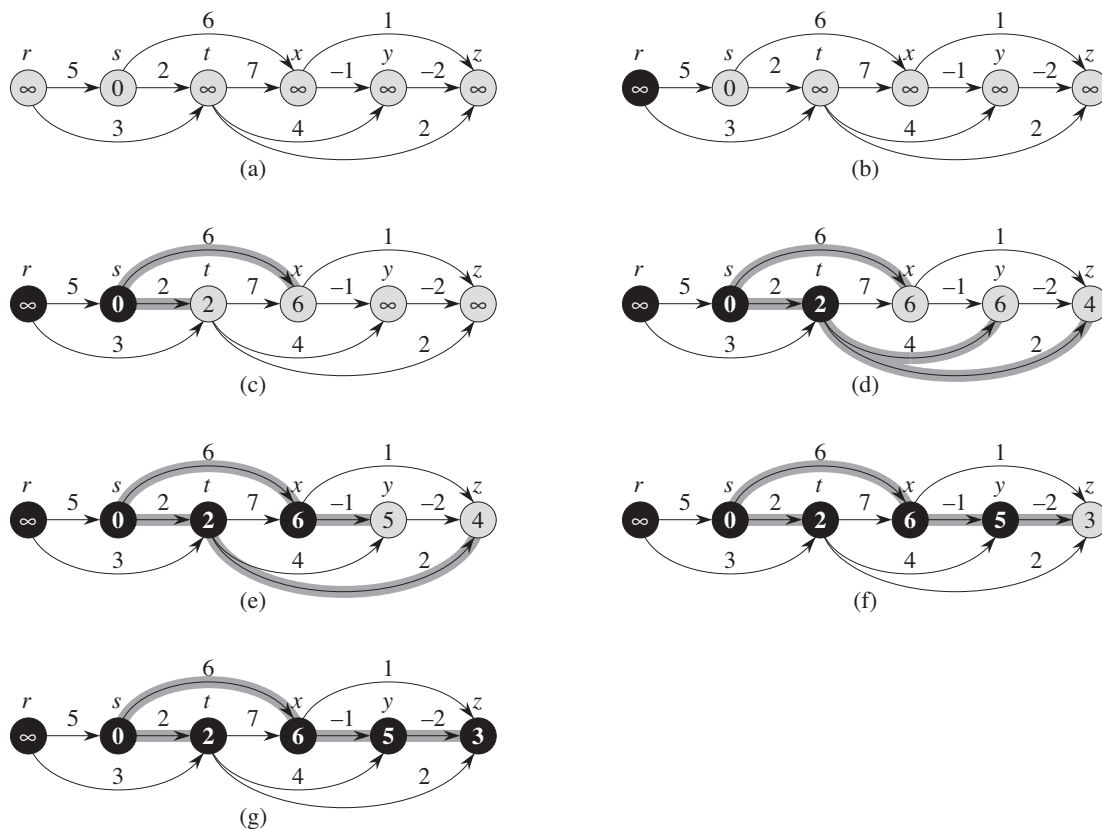


Figure 24.5 The execution of the algorithm for shortest paths in a directed acyclic graph. The vertices are topologically sorted from left to right. The source vertex is s . The d values appear within the vertices, and shaded edges indicate the π values. (a) The situation before the first iteration of the **for** loop of lines 3–5. (b)–(g) The situation after each iteration of the **for** loop of lines 3–5. The newly blackened vertex in each iteration was used as u in that iteration. The values shown in part (g) are the final values.

Theorem 24.5

If a weighted, directed graph $G = (V, E)$ has source vertex s and no cycles, then at the termination of the DAG-SHORTEST-PATHS procedure, $v.d = \delta(s, v)$ for all vertices $v \in V$, and the predecessor subgraph G_π is a shortest-paths tree.

Proof We first show that $v.d = \delta(s, v)$ for all vertices $v \in V$ at termination. If v is not reachable from s , then $v.d = \delta(s, v) = \infty$ by the no-path property. Now, suppose that v is reachable from s , so that there is a shortest path $p = \langle v_0, v_1, \dots, v_k \rangle$, where $v_0 = s$ and $v_k = v$. Because we pro-

cess the vertices in topologically sorted order, we relax the edges on p in the order $(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k)$. The path-relaxation property implies that $v_i.d = \delta(s, v_i)$ at termination for $i = 0, 1, \dots, k$. Finally, by the predecessor-subgraph property, G_π is a shortest-paths tree. ■

An interesting application of this algorithm arises in determining critical paths in **PERT chart**² analysis. Edges represent jobs to be performed, and edge weights represent the times required to perform particular jobs. If edge (u, v) enters vertex v and edge (v, x) leaves v , then job (u, v) must be performed before job (v, x) . A path through this dag represents a sequence of jobs that must be performed in a particular order. A **critical path** is a *longest* path through the dag, corresponding to the longest time to perform any sequence of jobs. Thus, the weight of a critical path provides a lower bound on the total time to perform all the jobs. We can find a critical path by either

- negating the edge weights and running DAG-SHORTEST-PATHS, or
- running DAG-SHORTEST-PATHS, with the modification that we replace “ ∞ ” by “ $-\infty$ ” in line 2 of INITIALIZE-SINGLE-SOURCE and “ $>$ ” by “ $<$ ” in the RELAX procedure.

Exercises

24.2-1

Run DAG-SHORTEST-PATHS on the directed graph of Figure 24.5, using vertex r as the source.

24.2-2

Suppose we change line 3 of DAG-SHORTEST-PATHS to read

3 **for** the first $|V| - 1$ vertices, taken in topologically sorted order

Show that the procedure would remain correct.

24.2-3

The PERT chart formulation given above is somewhat unnatural. In a more natural structure, vertices would represent jobs and edges would represent sequencing constraints; that is, edge (u, v) would indicate that job u must be performed before job v . We would then assign weights to vertices, not edges. Modify the DAG-SHORTEST-PATHS procedure so that it finds a longest path in a directed acyclic graph with weighted vertices in linear time.

²“PERT” is an acronym for “program evaluation and review technique.”

24.2-4

Give an efficient algorithm to count the total number of paths in a directed acyclic graph. Analyze your algorithm.

24.3 Dijkstra's algorithm

Dijkstra's algorithm solves the single-source shortest-paths problem on a weighted, directed graph $G = (V, E)$ for the case in which all edge weights are nonnegative. In this section, therefore, we assume that $w(u, v) \geq 0$ for each edge $(u, v) \in E$. As we shall see, with a good implementation, the running time of Dijkstra's algorithm is lower than that of the Bellman-Ford algorithm.

Dijkstra's algorithm maintains a set S of vertices whose final shortest-path weights from the source s have already been determined. The algorithm repeatedly selects the vertex $u \in V - S$ with the minimum shortest-path estimate, adds u to S , and relaxes all edges leaving u . In the following implementation, we use a min-priority queue Q of vertices, keyed by their d values.

```

DIJKSTRA( $G, w, s$ )
1  INITIALIZE-SINGLE-SOURCE( $G, s$ )
2   $S = \emptyset$ 
3   $Q = G.V$ 
4  while  $Q \neq \emptyset$ 
5       $u = \text{EXTRACT-MIN}(Q)$ 
6       $S = S \cup \{u\}$ 
7      for each vertex  $v \in G.Adj[u]$ 
8          RELAX( $u, v, w$ )

```

Dijkstra's algorithm relaxes edges as shown in Figure 24.6. Line 1 initializes the d and π values in the usual way, and line 2 initializes the set S to the empty set. The algorithm maintains the invariant that $Q = V - S$ at the start of each iteration of the **while** loop of lines 4–8. Line 3 initializes the min-priority queue Q to contain all the vertices in V ; since $S = \emptyset$ at that time, the invariant is true after line 3. Each time through the **while** loop of lines 4–8, line 5 extracts a vertex u from $Q = V - S$ and line 6 adds it to set S , thereby maintaining the invariant. (The first time through this loop, $u = s$.) Vertex u , therefore, has the smallest shortest-path estimate of any vertex in $V - S$. Then, lines 7–8 relax each edge (u, v) leaving u , thus updating the estimate $v.d$ and the predecessor $v.\pi$ if we can improve the shortest path to v found so far by going through u . Observe that the algorithm never inserts vertices into Q after line 3 and that each vertex is extracted from Q

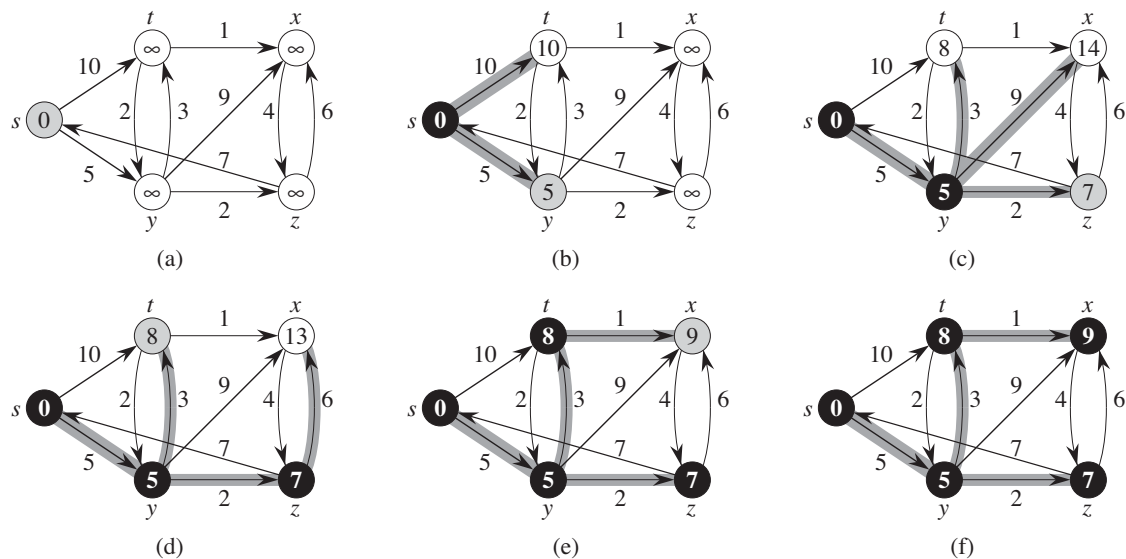


Figure 24.6 The execution of Dijkstra's algorithm. The source s is the leftmost vertex. The shortest-path estimates appear within the vertices, and shaded edges indicate predecessor values. Black vertices are in the set S , and white vertices are in the min-priority queue $Q = V - S$. (a) The situation just before the first iteration of the **while** loop of lines 4–8. The shaded vertex has the minimum d value and is chosen as vertex u in line 5. (b)–(f) The situation after each successive iteration of the **while** loop. The shaded vertex in each part is chosen as vertex u in line 5 of the next iteration. The d values and predecessors shown in part (f) are the final values.

and added to S exactly once, so that the **while** loop of lines 4–8 iterates exactly $|V|$ times.

Because Dijkstra's algorithm always chooses the “lightest” or “closest” vertex in $V - S$ to add to set S , we say that it uses a greedy strategy. Chapter 16 explains greedy strategies in detail, but you need not have read that chapter to understand Dijkstra's algorithm. Greedy strategies do not always yield optimal results in general, but as the following theorem and its corollary show, Dijkstra's algorithm does indeed compute shortest paths. The key is to show that each time it adds a vertex u to set S , we have $u.d = \delta(s, u)$.

Theorem 24.6 (Correctness of Dijkstra's algorithm)

Dijkstra's algorithm, run on a weighted, directed graph $G = (V, E)$ with non-negative weight function w and source s , terminates with $u.d = \delta(s, u)$ for all vertices $u \in V$.

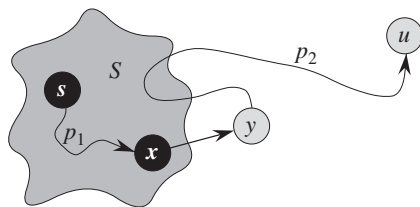


Figure 24.7 The proof of Theorem 24.6. Set S is nonempty just before vertex u is added to it. We decompose a shortest path p from source s to vertex u into $s \xrightarrow{p_1} x \rightarrow y \xrightarrow{p_2} u$, where y is the first vertex on the path that is not in S and $x \in S$ immediately precedes y . Vertices x and y are distinct, but we may have $s = x$ or $y = u$. Path p_2 may or may not reenter set S .

Proof We use the following loop invariant:

At the start of each iteration of the **while** loop of lines 4–8, $v.d = \delta(s, v)$ for each vertex $v \in S$.

It suffices to show for each vertex $u \in V$, we have $u.d = \delta(s, u)$ at the time when u is added to set S . Once we show that $u.d = \delta(s, u)$, we rely on the upper-bound property to show that the equality holds at all times thereafter.

Initialization: Initially, $S = \emptyset$, and so the invariant is trivially true.

Maintenance: We wish to show that in each iteration, $u.d = \delta(s, u)$ for the vertex added to set S . For the purpose of contradiction, let u be the first vertex for which $u.d \neq \delta(s, u)$ when it is added to set S . We shall focus our attention on the situation at the beginning of the iteration of the **while** loop in which u is added to S and derive the contradiction that $u.d = \delta(s, u)$ at that time by examining a shortest path from s to u . We must have $u \neq s$ because s is the first vertex added to set S and $s.d = \delta(s, s) = 0$ at that time. Because $u \neq s$, we also have that $S \neq \emptyset$ just before u is added to S . There must be some path from s to u , for otherwise $u.d = \delta(s, u) = \infty$ by the no-path property, which would violate our assumption that $u.d \neq \delta(s, u)$. Because there is at least one path, there is a shortest path p from s to u . Prior to adding u to S , path p connects a vertex in S , namely s , to a vertex in $V - S$, namely u . Let us consider the first vertex y along p such that $y \in V - S$, and let $x \in S$ be y 's predecessor along p . Thus, as Figure 24.7 illustrates, we can decompose path p into $s \xrightarrow{p_1} x \rightarrow y \xrightarrow{p_2} u$. (Either of paths p_1 or p_2 may have no edges.)

We claim that $y.d = \delta(s, y)$ when u is added to S . To prove this claim, observe that $x \in S$. Then, because we chose u as the first vertex for which $u.d \neq \delta(s, u)$ when it is added to S , we had $x.d = \delta(s, x)$ when x was added

to S . Edge (x, y) was relaxed at that time, and the claim follows from the convergence property.

We can now obtain a contradiction to prove that $u.d = \delta(s, u)$. Because y appears before u on a shortest path from s to u and all edge weights are non-negative (notably those on path p_2), we have $\delta(s, y) \leq \delta(s, u)$, and thus

$$\begin{aligned} y.d &= \delta(s, y) \\ &\leq \delta(s, u) \\ &\leq u.d \quad (\text{by the upper-bound property}) . \end{aligned} \tag{24.2}$$

But because both vertices u and y were in $V - S$ when u was chosen in line 5, we have $u.d \leq y.d$. Thus, the two inequalities in (24.2) are in fact equalities, giving

$$y.d = \delta(s, y) = \delta(s, u) = u.d .$$

Consequently, $u.d = \delta(s, u)$, which contradicts our choice of u . We conclude that $u.d = \delta(s, u)$ when u is added to S , and that this equality is maintained at all times thereafter.

Termination: At termination, $Q = \emptyset$ which, along with our earlier invariant that $Q = V - S$, implies that $S = V$. Thus, $u.d = \delta(s, u)$ for all vertices $u \in V$. ■

Corollary 24.7

If we run Dijkstra's algorithm on a weighted, directed graph $G = (V, E)$ with nonnegative weight function w and source s , then at termination, the predecessor subgraph G_π is a shortest-paths tree rooted at s .

Proof Immediate from Theorem 24.6 and the predecessor-subgraph property. ■

Analysis

How fast is Dijkstra's algorithm? It maintains the min-priority queue Q by calling three priority-queue operations: INSERT (implicit in line 3), EXTRACT-MIN (line 5), and DECREASE-KEY (implicit in RELAX, which is called in line 8). The algorithm calls both INSERT and EXTRACT-MIN once per vertex. Because each vertex $u \in V$ is added to set S exactly once, each edge in the adjacency list $Adj[u]$ is examined in the **for** loop of lines 7–8 exactly once during the course of the algorithm. Since the total number of edges in all the adjacency lists is $|E|$, this **for** loop iterates a total of $|E|$ times, and thus the algorithm calls DECREASE-KEY at most $|E|$ times overall. (Observe once again that we are using aggregate analysis.)

The running time of Dijkstra's algorithm depends on how we implement the min-priority queue. Consider first the case in which we maintain the min-priority

queue by taking advantage of the vertices being numbered 1 to $|V|$. We simply store $v.d$ in the v th entry of an array. Each INSERT and DECREASE-KEY operation takes $O(1)$ time, and each EXTRACT-MIN operation takes $O(V)$ time (since we have to search through the entire array), for a total time of $O(V^2 + E) = O(V^2)$.

If the graph is sufficiently sparse—in particular, $E = o(V^2/\lg V)$ —we can improve the algorithm by implementing the min-priority queue with a binary min-heap. (As discussed in Section 6.5, the implementation should make sure that vertices and corresponding heap elements maintain handles to each other.) Each EXTRACT-MIN operation then takes time $O(\lg V)$. As before, there are $|V|$ such operations. The time to build the binary min-heap is $O(V)$. Each DECREASE-KEY operation takes time $O(\lg V)$, and there are still at most $|E|$ such operations. The total running time is therefore $O((V + E) \lg V)$, which is $O(E \lg V)$ if all vertices are reachable from the source. This running time improves upon the straightforward $O(V^2)$ -time implementation if $E = o(V^2/\lg V)$.

We can in fact achieve a running time of $O(V \lg V + E)$ by implementing the min-priority queue with a Fibonacci heap (see Chapter 19). The amortized cost of each of the $|V|$ EXTRACT-MIN operations is $O(\lg V)$, and each DECREASE-KEY call, of which there are at most $|E|$, takes only $O(1)$ amortized time. Historically, the development of Fibonacci heaps was motivated by the observation that Dijkstra's algorithm typically makes many more DECREASE-KEY calls than EXTRACT-MIN calls, so that any method of reducing the amortized time of each DECREASE-KEY operation to $o(\lg V)$ without increasing the amortized time of EXTRACT-MIN would yield an asymptotically faster implementation than with binary heaps.

Dijkstra's algorithm resembles both breadth-first search (see Section 22.2) and Prim's algorithm for computing minimum spanning trees (see Section 23.2). It is like breadth-first search in that set S corresponds to the set of black vertices in a breadth-first search; just as vertices in S have their final shortest-path weights, so do black vertices in a breadth-first search have their correct breadth-first distances. Dijkstra's algorithm is like Prim's algorithm in that both algorithms use a min-priority queue to find the "lightest" vertex outside a given set (the set S in Dijkstra's algorithm and the tree being grown in Prim's algorithm), add this vertex into the set, and adjust the weights of the remaining vertices outside the set accordingly.

Exercises

24.3-1

Run Dijkstra's algorithm on the directed graph of Figure 24.2, first using vertex s as the source and then using vertex z as the source. In the style of Figure 24.6, show the d and π values and the vertices in set S after each iteration of the **while** loop.