4. General Dynamic Games

4.1 Imperfect Information

There are many situations where players have to make decisions with only partial information about previous moves by other players. Here is an example from my professional experience: in order to discourage copying and cheating in exams, I prepare two versions of the exam, print one version on white paper and the other on pink paper and distribute the exams in such a way that if a student gets, say, the white version then the students on his left and right have the pink version. For simplicity let us assume that there is only one question in the exam. What matters for my purpose is not that the question is indeed different in the two versions, but rather that the students *believe* that they are different and thus refrain from copying from their neighbors. The students, however, are not naïve and realize that I might be bluffing; indeed, introducing differences between the two versions of the exam involves extra effort on my part. Consider a student who finds himself in the embarrassing situation of not having studied for the final exam and is tempted to copy from his neighbor, whom he knows to be a very good student. Let us assume that, if he does not copy, then he turns in a blank exam; in this case, because of his earlier grades in the quarter, he will get a C; on the other hand, if he copies he will get an A if the two versions are identical but will be caught cheating and get an F if the two versions are slightly different. How can we represent such a situation?

Clearly this is a situation in which decisions are made sequentially: first the Professor decides whether to write identical versions (albeit printed on different-color paper) or different versions and then the Student chooses between copying and leaving the exam blank. We can easily represent this situation using a tree as we did with the case of perfect-information games, but the crucial element here is the fact that the Student *does not know* whether the two versions are identical or different. In order to represent this uncertainty (or lack of information) in the mind of the Student, we use the notion of *information set*. An information set for a player is a collection of decision nodes of that player and the interpretation is that the player does not know at which of these nodes he is making his decision. Graphically, we represent an information set by enclosing the corresponding nodes in a rounded rectangle. Figure 4.1 represents the situation described above.

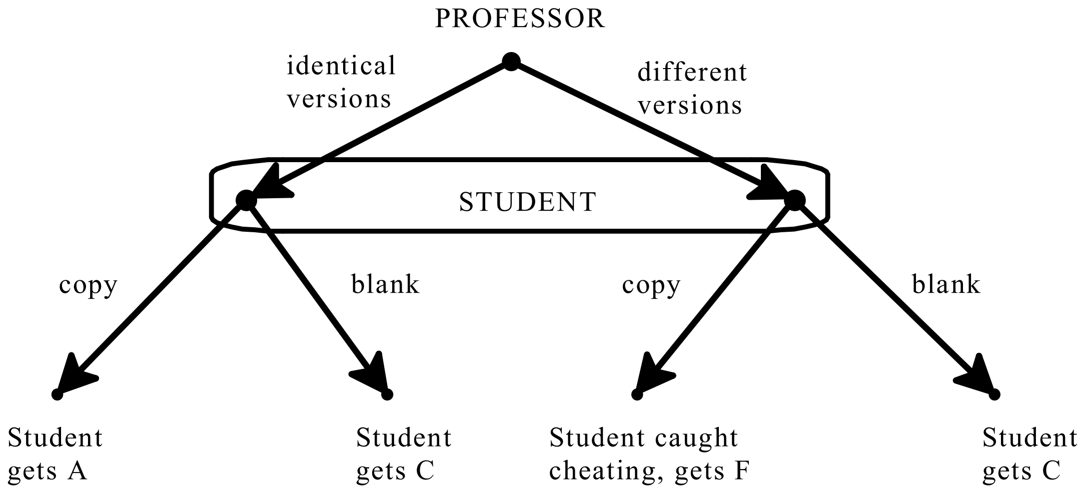


Figure 4.1: An extensive form, or frame, with imperfect information

As usual we need to distinguish between a game-frame and a game. Figure 4.1 depicts a game-frame: in order to obtain a game from it we need to add a ranking of the outcomes for each player. For the moment we shall ignore payoffs and focus on frames. A game-frame such as the one shown in Figure 4.1 is called an *extensive form (or frame) with imperfect information:* in this example it is the Student who has imperfect information, or uncertainty, about the earlier decision of the Professor.

Before we give the definition of extensive form, we need to introduce some additional terminology and notation. Given a directed tree and two nodes x and y we say that *y is a successor of x*, or *x is a predecessor of y*, if there is a sequence of directed edges from x to y.[1]

A *partition* of a set H is a collection $\mathscr{H} = \{H_1, \ldots, H_m\}$ $(m \geq 1)$ of non-empty subsets of H such that:

(1) any two elements of $\mathscr{H}$ are disjoint (if $H_j, H_k \in \mathscr{H}$ with $j \neq k$ then $H_j \cap H_k = \emptyset$) and

(2) the elements of $\mathscr{H}$ cover H: $H_1 \cup \ldots \cup H_m = H$.

[1]If the sequence consists of a single directed edge then we say that y is an *immediate successor* of x or x is the *immediate predecessor* of y.

The definition of extensive form given below allows for perfect information as a special case. The first four items of Definition 4.1.1 (marked by the bullet symbol •), coincide with Definition 3.1.1 in Chapter 3 (which covers the case of perfect information); what is new is the additional item marked by the symbol ★.

Definition 4.1.1 A *finite extensive form (or frame) with perfect recall* consists of the following items.

- A finite rooted directed tree.
- A set of players $I = \{1, \ldots, n\}$ and a function that assigns one player to every decision node.
- A set of actions A and a function that assigns one action to every directed edge, satisfying the restriction that no two edges out of the same node are assigned the same action.
- A set of outcomes O and a function that assigns an outcome to every terminal node.

★ For every player $i \in I$, a partition $\mathscr{D}_i$ of the set D_i of decision nodes assigned to player i (thus $\mathscr{D}_i$ is a collection of mutually disjoint subsets of D_i whose union is equal to D_i). Each element of $\mathscr{D}_i$ is called an *information set of player i*.
The elements of $\mathscr{D}_i$ satisfy the following restrictions:
(1) the actions available at any two nodes in the same information set must be the same (that is, for every $D \in \mathscr{D}_i$, if $x, y \in D$ then the outdegree of x is equal to the outdegree of y and the set of actions assigned to the directed edges out of x is equal to the set of actions assigned to the directed edges out of y),
(2) if x and y are two nodes in the same information set then it is not the case that one node is a predecessor of the other,
(3) each player has *perfect recall* in the sense that if node $x \in D \in \mathscr{D}_i$ is a predecessor of node $y \in D' \in \mathscr{D}_i$ (thus, by (2), $D \neq D'$), and a is the action assigned to the directed edge out of x in the sequence of edges leading from x to y, then for every node $z \in D'$ there is a predecessor $w \in D$ such that the action assigned to the directed edge out of w in the sequence of edges leading from w to z is that same action a.

The perfect-recall restriction says that if a player takes action a at an information set and later on has to move again, then at the later time she remembers that she took action a at that earlier information set (because every node she is uncertain about at the later time comes after taking action a at that information set). Perfect recall can be interpreted as requiring that *a player always remember what she knew in the past and what actions she herself took in the past*.

Figure 4.2 shows two examples of violation of perfect recall. In the frame shown in Panel (*i*) Player 1 first chooses between a and b and then chooses between c and d having forgotten his previous choice: he does not remember *what* he chose previously. In the frame shown in Panel (*ii*) when Player 2 has to choose between e and f she is uncertain whether this is the first time she moves (left node) or the second time (right node): she is uncertain *whether* she moved in the past.

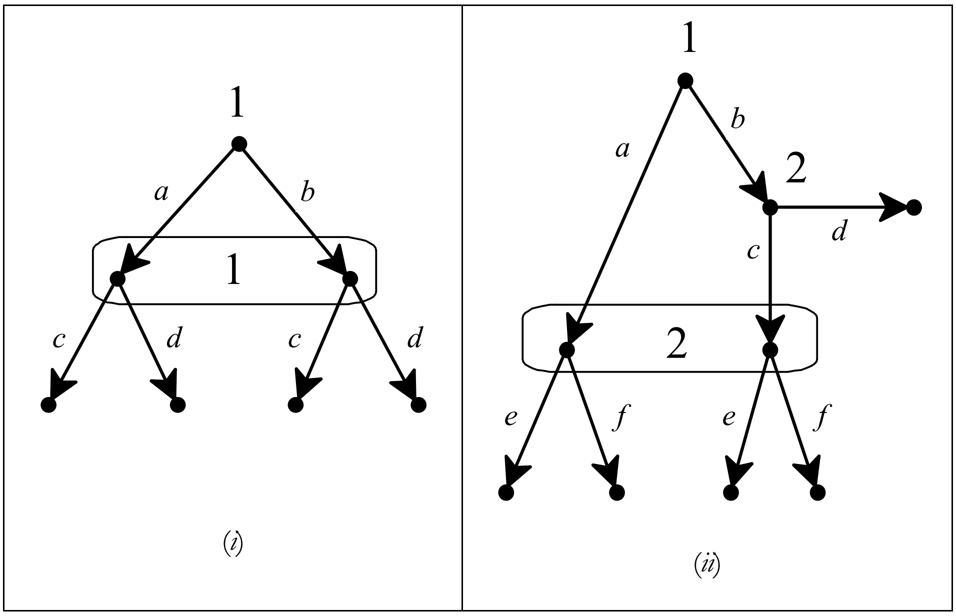


Figure 4.2: Examples of violations of perfect recall

If every information set of every player consists of a single node, then the frame is said to be a *perfect-information frame:* it is easy to verify that, in this case, the last item of Definition 4.1.1 (marked by the symbol ★) is trivially satisfied and thus Definition 4.1.1 coincides with Definition 3.1.1 (Chapter 3). Otherwise (that is, if at least one player has at least one information set that consists of at least two nodes), the frame is said to have *imperfect information.* An example of an extensive frame with imperfect information is the one shown in Figure 4.1. We now give two more examples. In order to simplify the figures, when representing an extensive frame we enclose an information set in a rounded rectangle if and only if that information set contains at least two nodes.

■ **Example 4.1** There are three players, Ann, Bob and Carla. Initially, Ann and Bob are in the same room and Carla is outside the room. Ann moves first, chooses either a red card or a black card from a full deck of cards, shows it to Bob and puts it, face down, on the table. Now Carla enters the room and Bob makes a statement to Carla: he either says "Ann chose a Red card" or he says "Ann chose a Black card"; Bob could be lying or could be telling the truth. After hearing Bob's statement Carla guesses the color of the card that was picked by Ann. The card is then turned and if Carla's guess was correct then Ann and Bob give $1 each to Carla, otherwise Carla gives $1 to each of Ann and Bob. When drawing an extensive frame to represent this situation, it is important to be careful about what Carla knows, when she makes her guess, and what she is uncertain about. The extensive frame is shown in Figure 4.3. ■

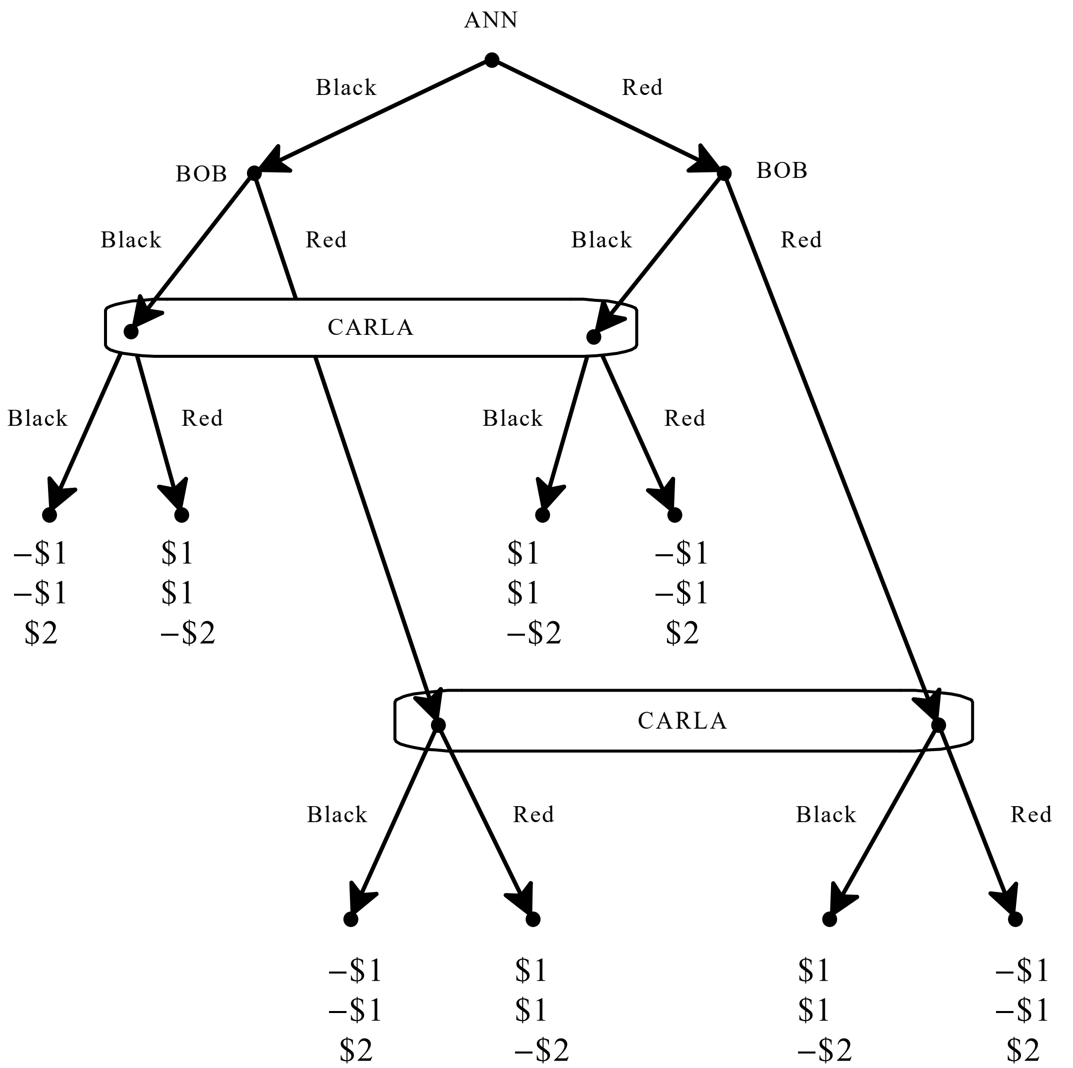


Figure 4.3: The extensive form, or frame, representing Example 4.1

- Carla's top information set captures the situation she is in after hearing Bob say "Ann chose a black card" and not knowing if he is telling the truth (left node) or he is lying (right node).
- Carla's bottom information set captures the alternative situation where she hears Bob say "Ann chose a red card" and does not know if he is lying (left node) or telling the truth (right node).
- In both situations Carla knows something, namely what Bob tells her, but lacks information about something else, namely what Ann chose.
- The fact that Bob knows the color of the card chosen by Ann is captured by giving Bob two information sets, each consisting of a single node: Bob's left node represents the situation he is in when he sees that Ann picked a black card, while his right node represents the situation he is in when he sees that Ann picked a red card.

■ **Example 4.2** Yvonne and Fran were both interviewed for the same job, but only one person can be hired. The employer told each candidate: "don't call me, I will call you if I want to offer you the job". He also told them that he desperately needs to fill the position and thus, if turned down by one candidate, he will automatically make the offer to the other candidate, without revealing whether he is making a first offer or a "recycled" offer. This situation is represented in the extensive frame shown in Figure 4.4. ■

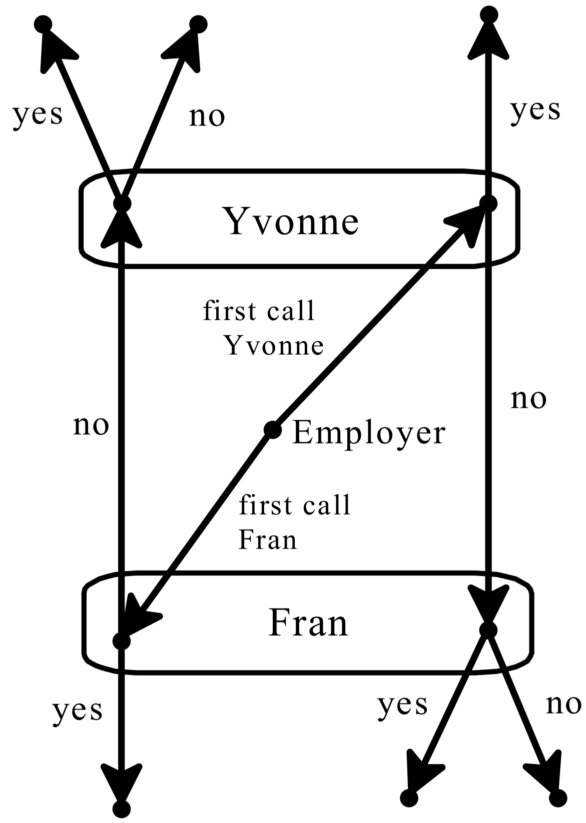


Figure 4.4: The extensive form, or frame, representing Example 4.2

As before, in order to obtain a *game* from an extensive *frame* all we need to do is add a ranking of the outcomes for each player. As usual, the best way to represent such rankings is by means of an ordinal utility function for each player and thus represent an extensive-form game by associating a vector of utilities with each terminal node. For instance, expanding on Example 4.2, suppose that the employer only cares about whether the position is filled or not, prefers filling the position to not filling it, but is indifferent between filling it with Yvonne or with Fran; thus we can assign a utility of 1 for the employer to every outcome where one of the two candidates accepts the offer and a utility of 0 to every other outcome.

Yvonne's favorite outcome is to be hired if she was the recipient of the first call by the employer; her second best outcome is not to be hired and her worst outcome is to accept a recycled offer (in the latter case Fran would take pleasure telling Yvonne "You took that job?! It was offered to me but I turned it down. Who, in her right mind, would want that job? What's wrong with you?!"). Thus for Yvonne we can use utilities of 2 (if she accepts a first offer), 1 (if she is not hired) and 0 (if she accepts a recycled offer). Finally, suppose that Fran has preferences similar (but symmetric) to Yvonne's. Then the extensive frame of Figure 4.4 gives rise to the extensive game shown in Figure 4.5.

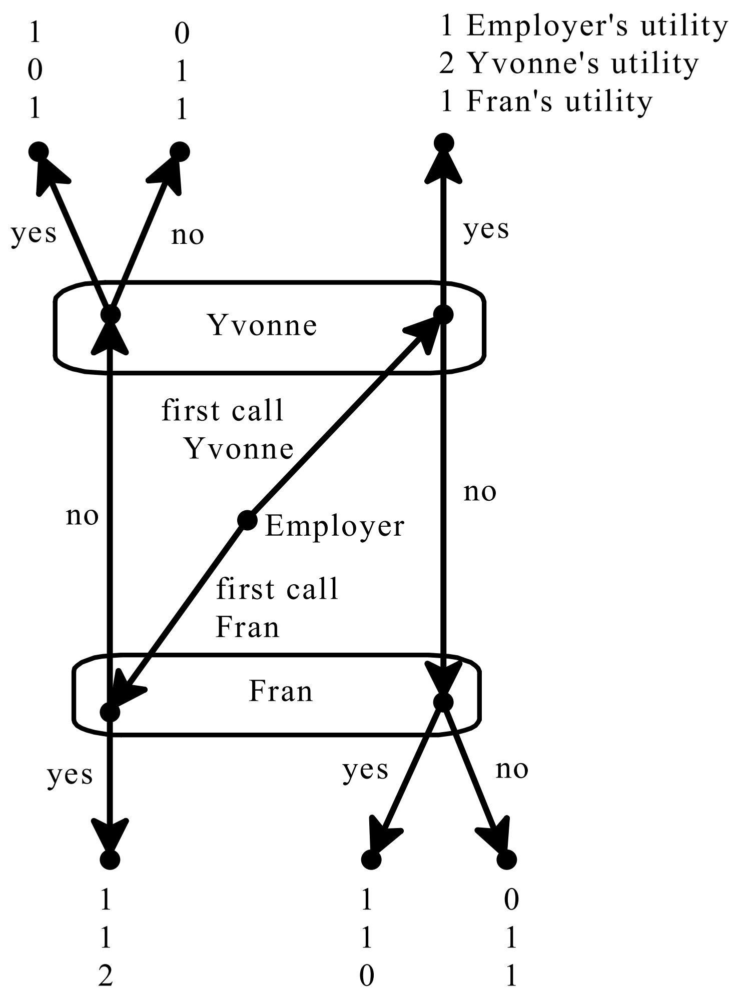


Figure 4.5: A game based on the extensive form of Figure 4.4

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 4.6.1 at the end of this chapter.

4.2 Strategies

The notion of strategy for general extensive games is the same as before: a strategy for Player i is a complete, contingent plan that covers all the possible situations Player i might find herself in. In the case of a perfect-information game a "possible situation" for a player is a decision node of that player; in the general case, where there may be imperfect information, a "possible situation" for a player is an *information set* of that player.

The following definition reduces to Definition 3.3.1 (Chapter 3) if the game is a perfect-information game (where each information set consists of a single node).

Definition 4.2.1 A *strategy* for a player in an extensive-form game is a list of choices, one for every information set of that player.

For example, in the game of Figure 4.5, Yvonne has only one information set and thus a strategy for her is what to do at that information set, namely either say Yes or say No. Yvonne cannot make the plan "if the employer calls me first I will say Yes and if he calls me second I will say No", because when she receives the call she is not told if this is a first call or a recycled call and thus she cannot make her decision dependent on information she does not have.

As in the case of perfect-information games, the notion of strategy allows us to associate with every extensive-form game a strategic-form game. For example, the strategic form associated with the game of Figure 4.5 is shown in Figure 4.6 with the Nash equilibria highlighted.

Fran: Yes

		Yvonne: Yes			Yvonne: No		
Employer	first call Yvonne	1	2	1	1	1	0
	first call Fran	1	1	2	1	1	2

Fran: No

		Yvonne: Yes			Yvonne: No		
Employer	first call Yvonne	1	2	1	0	1	1
	first call Fran	1	0	1	0	1	1

Figure 4.6: The strategic form of the game of Figure 4.5 with the Nash equilibria highlighted

As another example, consider the extensive form of Figure 4.3 and view it as a game by assuming that each player is selfish and greedy (only cares about how much money he/she gets and prefers more money to less). Then the associated strategic form is shown in Figure 4.7, where Bob's strategy (x, y) means "I say x if Ann chose a black card and I say y if Ann chose a red card". Thus (R, B) means "if Ann chose a black card I say Red and if Ann chose a red card I say Black" (that is, Bob plans to lie in both cases). Similarly, Carla's strategy (x, y) means "I guess x if Bob tells me Black and I guess y if Bob tells me Red". Thus (B, R) means "if Bob tells me Black I guess Black and if Bob tells me Red I guess Red" (that is, Carla plans to repeat what Bob says).

		BOB			
		B , B	R , R	B , R	R , B
ANN	B	−1 , −1 , 2	−1 , −1 , 2	−1 , −1 , 2	−1 , −1 , 2
	R	1 , 1 , −2	1 , 1 , −2	1 , 1 , −2	1 , 1 , −2

CARLA: B , B

		BOB			
		B , B	R , R	B , R	R , B
ANN	B	1 , 1 , −2	1 , 1 , −2	1 , 1 , −2	1 , 1 , −2
	R	−1 , −1 , 2	−1 , −1 , 2	−1 , −1 , 2	−1 , −1 , 2

CARLA: R , R

		BOB			
		B , B	R , R	B , R	R , B
ANN	B	−1 , −1 , 2	1 , 1 , −2	−1 , −1 , 2	1 , 1 , −2
	R	1 , 1 , −2	−1 , −1 , 2	−1 , −1 , 2	1 , 1 , −2

CARLA: B , R

		BOB			
		B , B	R , R	B , R	R , B
ANN	B	1 , 1 , −2	−1 , −1 , 2	1 , 1 , −2	−1 , −1 , 2
	R	−1 , −1 , 2	1 , 1 , −2	1 , 1 , −2	−1 , −1 , 2

CARLA: R , B

Figure 4.7: The strategic form of the game of Figure 4.3

In order to "solve" an extensive-form game we could simply construct the associated strategic-form game and look for the Nash equilibria. However, we saw in Chapter 3 that in the case of perfect-information games not all Nash equilibria of the associated strategic form can be considered "rational solutions" and we introduced the notion of backward induction to select the "reasonable" Nash equilibria. What we now need is a generalization of the notion of backward induction that can be applied to general extensive-form games. This generalization is called *subgame-perfect equilibrium*. First we need to define the notion of subgame.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 4.6.2 at the end of this chapter.

4.3 Subgames

Roughly speaking, a subgame of an extensive-form game is a portion of the game that could be a game in itself. What we need to be precise about is the meaning of "portion of the game".

Definition 4.3.1 A *proper subgame* of an extensive-form game is obtained as follows:

1. Start from a decision node x, different from the root, whose information set consists of node x only and enclose in an oval node x itself and all its successors.
2. If the oval does not "cut" any information sets (that is, there is no information set S and two nodes $y, z \in S$ such that y is a successor of x while z is not) then what is included in the oval is a proper subgame, otherwise it is not.

The reason why we use the qualifier 'proper' is that one could start from the root, in which case one would end up taking the entire game and consider this as a (trivial) subgame (just like any set is a subset of itself; a proper subgame is analogous to a proper subset).

Consider, for example, the extensive-form game of Figure 4.8. There are three possible starting points for identifying a proper subgame: nodes x, y and z (the other nodes fail to satisfy condition (1) of Definition 4.3.1).

1. Starting from node x and including all of its successors, we do indeed obtain a proper subgame, which is the portion included in the oval on the left.
2. Starting from node y and including all of its successors we obtain the portion of the game that is included in the oval on the right; in this case, condition **(2)** of Definition 4.3.1 is violated, since we are cutting the top information set of Player 3; hence the portion of the game inside this oval is not a proper subgame.
3. Finally, starting from node z and including all of its successors, we do obtain a proper subgame, which is the portion included in the oval at the bottom.

Thus the game of Figure 4.8 has two proper subgames.

Definition 4.3.2 A proper subgame of an extensive-form game is called *minimal* if it does not strictly contain another proper subgame (that is, if there is no other proper subgame which is strictly contained in it).

For example, the game shown in Figure 4.9 on the following page has three proper subgames, one starting at node x, another at node y and the third at node z. The ones starting at nodes x and z are minimal subgames, while the one that starts at node y is not a minimal subgame, since it strictly contains the one that starts at node z.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 4.6.3 at the end of this chapter.

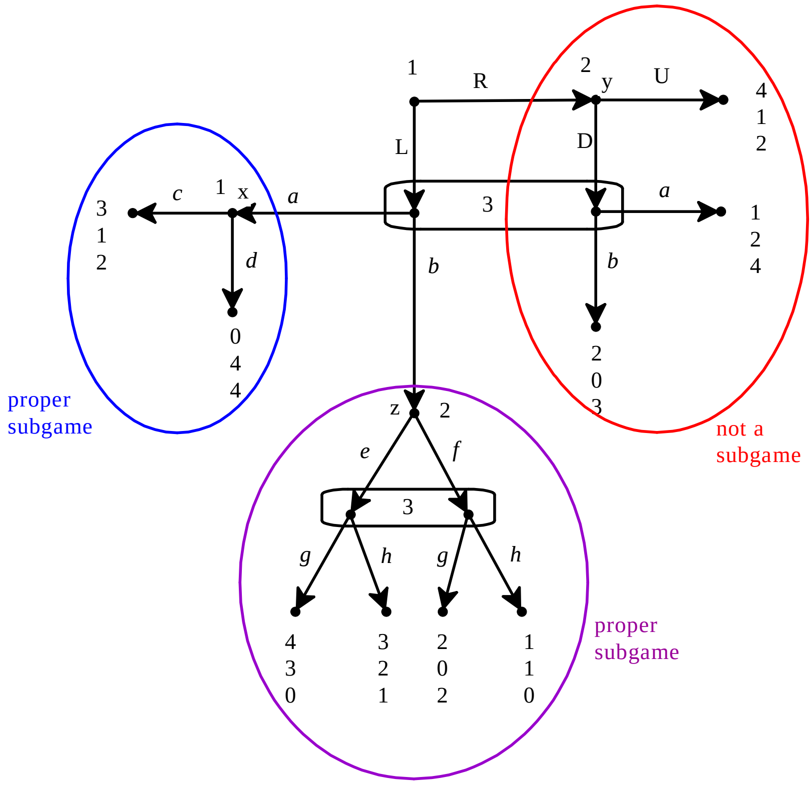


Figure 4.8: An extensive-form game with two proper subgames.

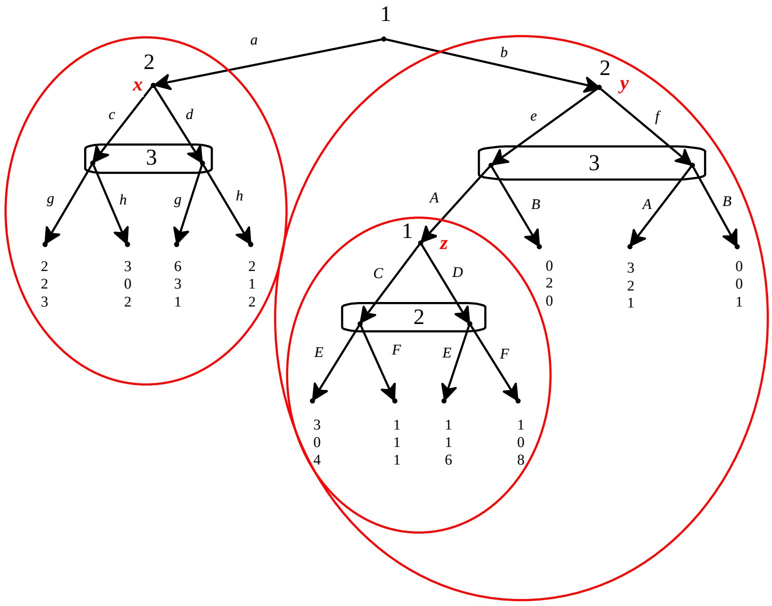


Figure 4.9: An extensive-form game with three proper subgames, two of which are minimal.

4.4 Subgame-perfect equilibrium

A subgame-perfect equilibrium of an extensive-form game is a Nash equilibrium of the entire game which remains an equilibrium in every proper subgame. Consider an extensive-form game and let s be a strategy profile for that game. Let G be a proper subgame. Then the *restriction of s to G*, denoted by $s|_G$, is that part of s which prescribes choices at every information set of G and only at those information sets.
For example, consider the extensive-form game of Figure 4.9 and the strategy profile

$$\left(\underbrace{(a,C)}_{\text{1's strategy}}, \underbrace{(d,f,E)}_{\text{2's strategy}}, \underbrace{(h,B)}_{\text{3's strategy}} \right)$$

Let G be the subgame that starts at node y of Player 2. Then

$$s|_G = \left(\underbrace{C}_{\text{1's strategy in G}}, \underbrace{(f,E)}_{\text{2's strategy in G}}, \underbrace{B}_{\text{3's strategy in G}} \right)$$

Definition 4.4.1 . Given an extensive-form game, let s be a strategy profile for the entire game. Then s is a *subgame-perfect equilibrium* if

1. s is a Nash equilibrium of the entire game and
2. for every proper subgame G, $s|_G$ (the restriction of s to G) is a Nash equilibrium of G.

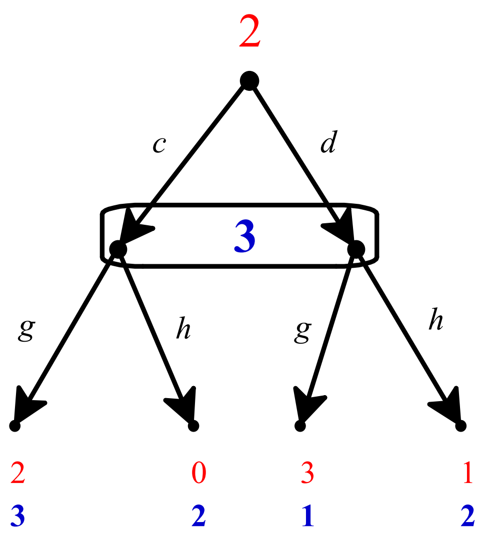
For example, consider again the extensive-form game of Figure 4.9 and the strategy profile $s = ((a,C),(d,f,E),(h,B))$. Then s is a Nash equilibrium of the entire game: Player 1's payoff is 2 and if he were to switch to any strategy where he plays b his payoff would be 0; Player 2's payoff is 1 and if she were to switch to any strategy where she plays c her payoff would be 0; Player 3's payoff is 2 and if he were to switch to any strategy where he plays g his payoff would be 1. However, s is not a subgame-perfect equilibrium, because the restriction of s to the proper subgame that starts at node z of Player 1, namely (C,E), is not a Nash equilibrium of that subgame: in that subgame, for Player 2 the unique best reply to C is F.

One way of finding the subgame-perfect equilibria of a given game is to first find the Nash equilibria and then, for each of them, check if it satisfies condition **(2)** of Definition 4.4.1. However, this is not a practical way to proceed. A quicker and easier way is to apply the following algorithm, which generalizes the backward-induction algorithm for games with perfect information (Definition 3.2.1, Chapter 3).

Definition 4.4.2 Given an extensive-form game, the *subgame-perfect equilibrium algorithm* is the following procedure.

1. Start with a minimal proper subgame and select a Nash equilibrium of it.
2. Delete the selected proper subgame and replace it with the payoff vector associated with the selected Nash equilibrium, making a note of the strategies that constitute the Nash equilibrium. This yields a smaller extensive-form game.
3. Repeat Steps 1 and 2 in the smaller game so obtained.

For example, let us apply the algorithm to the game of Figure 4.9. Begin with the proper subgame that starts at node x of Player 2, shown in Figure 4.10 with its associated strategic form, where the unique Nash equilibrium (d,h) is highlighted. Note that this is a game only between Players 2 and 3 and thus in Figure 4.10 we only show the payoffs of these two players.



		Player 3	
		g	h
Player 2	c	2 3	0 2
	d	3 1	1 2

Figure 4.10: A minimal proper subgame of the game of Figure 4.9 and its strategic form

Now we delete the proper subgame, thereby turning node x into a terminal node to which we attach the full payoff vector associated, in the original game, with the terminal node following history adh, namely $(2,1,2)$.

Hence we obtain the smaller game shown in Figure 4.11.

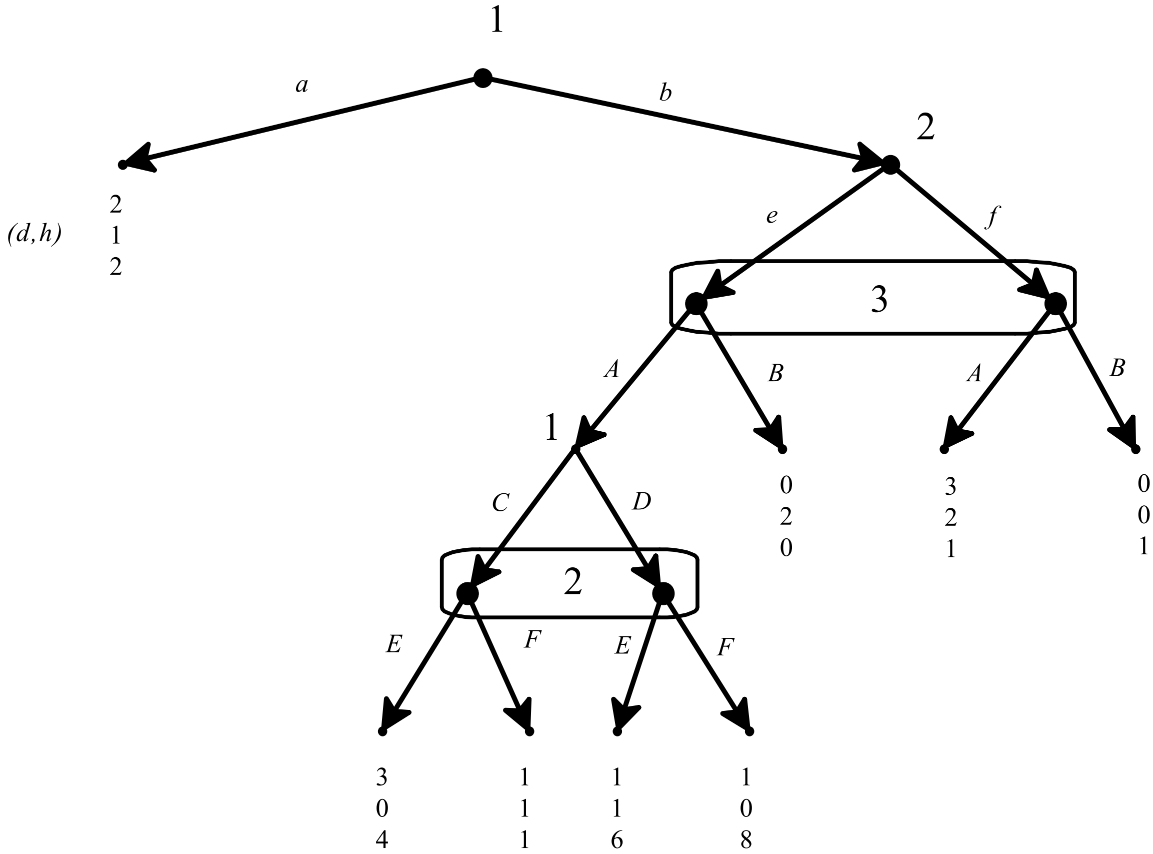
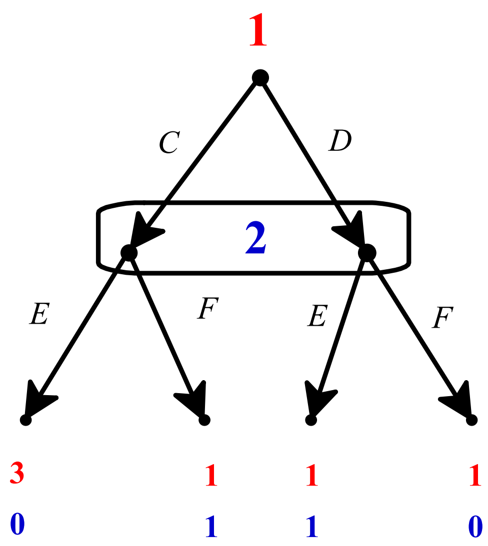


Figure 4.11: The reduced game after replacing a proper minimal subgame in the game of Figure 4.9

Now, in the reduced game of Figure 4.11 we select the only minimal proper subgame, namely the one that starts at the bottom decision node of Player 1. This subgame is shown in Figure 4.12 together with its associated strategic form. The unique Nash equilibrium of this subgame is (C,F).

Then, in the reduced game of Figure 4.11, we replace the selected proper subgame with the payoff vector associated with the history *beACF*, namely $(1,1,1)$, thus obtaining the smaller game shown in Figure 4.13. The game of Figure 4.13 has a unique proper subgame, which has a unique Nash equilibrium, namely (f,A).



Player 2

Player 1		E	F
	C	3 0	1 1
	D	1 1	1 0

Figure 4.12: A minimal proper subgame of the game of Figure 4.11 and its strategic form

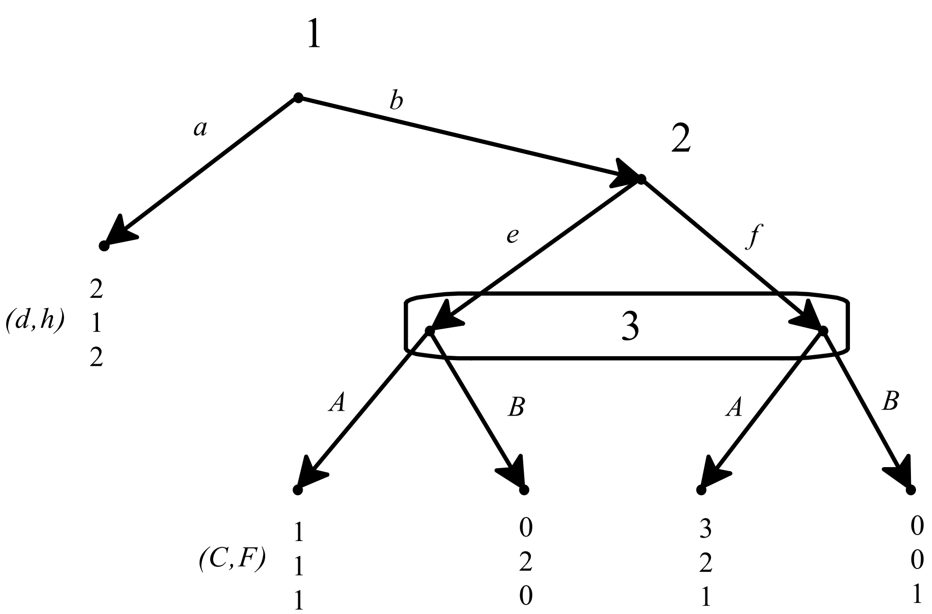


Figure 4.13: The reduced game after replacing a proper minimal subgame in the game of Figure 4.11

Replacing the subgame with the payoff vector associated with the history bfA we get the smaller game shown in Figure 4.14.

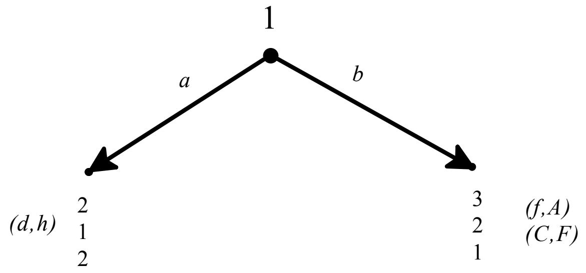


Figure 4.14: The reduced game after replacing the subgame in the game of Figure 4.13

In the reduced game of Figure 4.14 the unique Nash equilibrium is b. Now patching together the choices selected during the application of the algorithm we get the following subgame-perfect equilibrium for the game of Figure 4.9: $((b,C),(d,f,F),(h,A))$.

As a second example, consider the game of Figure 4.15 (which reproduces Figure 4.8).

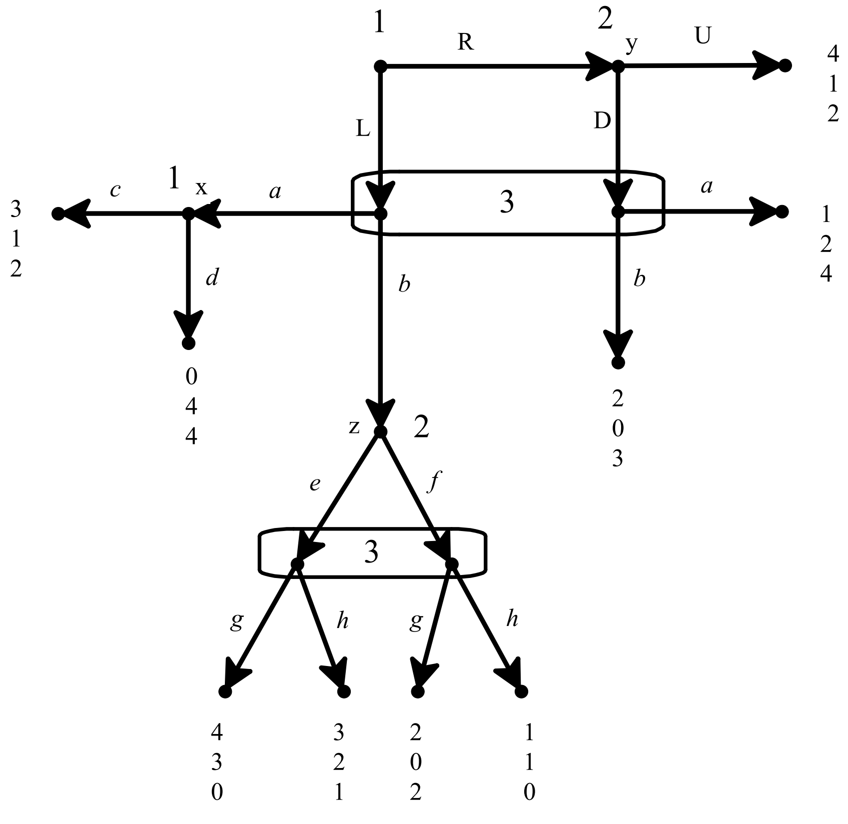
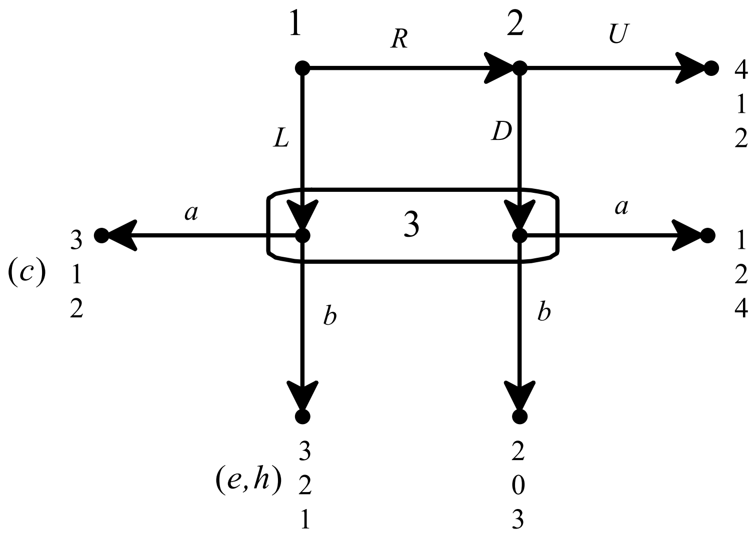


Figure 4.15: Copy of Figure 4.8

Begin with the subgame that starts at node x and replace it with the payoff vector $(3,1,2)$. Next replace the subgame that starts at node z with the payoff vector $(3,2,1)$ which corresponds to the Nash equilibrium (e,h) of that subgame, so that the game is reduced to the one shown in Figure 4.16, together with its strategic form.



Player 2

		U			D		
	L	3	1	2	3	1	2
Player 1	**R**	4	1	2	1	2	4

Player 3: ***a***

Player 2

		U			D		
	L	3	2	1	3	2	1
Player 1	**R**	4	1	2	2	0	3

Player 3: b

Figure 4.16: The game of Figure 4.15 reduced after solving the proper subgames, together with the associated strategic form with the Nash equilibria highlighted

The reduced game of Figure 4.16 has two Nash equilibria: (L,D,a) and (R,U,b). Thus the game of Figure 4.15 has two subgame-perfect equilibria:

$$\Bigg(\underbrace{(L,c)}_{\text{Player 1}}, \underbrace{(D,e)}_{\text{Player 2}}, \underbrace{(a,h)}_{\text{Player 3}} \Bigg) \quad \text{and} \quad \Bigg(\underbrace{(R,c)}_{\text{Player 1}}, \underbrace{(U,e)}_{\text{Player 2}}, \underbrace{(b,h)}_{\text{Player 3}} \Bigg).$$

R

- As shown in the last example, it is possible that – when applying the subgame-perfect equilibrium algorithm – one encounters a proper subgame or a reduced game that has several Nash equilibria. In this case one Nash equilibrium must be selected in order to continue the procedure and in the end one obtains one subgame-perfect equilibrium. One then has to repeat the procedure by selecting a different Nash equilibrium and thus obtain a different subgame-perfect equilibrium, and so on. This is similar to what happens with the backward-induction algorithm in perfect-information games.
- It is also possible that – when applying the subgame-perfect equilibrium algorithm –one encounters a proper subgame or a reduced game that has no Nash equilibria.[2] In such a case the game under consideration does not have any subgame-perfect equilibria.
- When applied to perfect-information games, the notion of subgame-perfect equilibrium coincides with the notion of backward-induction solution. Thus subgame-perfect equilibrium is a generalization of backward induction.
- For extensive-form games that have no proper subgames (for example, the game of Figure 4.3) the set of Nash equilibria coincides with the set of subgame-perfect equilibria. In general, however, the notion of subgame-perfect equilibrium is a refinement of the notion of Nash equilibrium.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 4.6.4 at the end of this chapter.

4.5 Games with chance moves

So far we have only considered games where the outcomes do not involve any uncertainty. As a way of introducing the topic discussed in Part II, in this section we consider games where uncertain, probabilistic events are incorporated in the extensive form.

We begin with an example: There are three cards, one black and two red. They are shuffled well and put face down on the table. Adele picks the top card, looks at it without showing it to Ben and then tells Ben either "the top card is black" or "the top card is red": she could be telling the truth or she could be lying. Ben then has to guess the true color of the top card. If he guesses correctly he gets \$9 from Adele, otherwise he gives her \$9. How can we represent this situation?

Whether the top card is black or red is not the outcome of a player's decision, but the outcome of a random event, namely the shuffling of the cards. In order to capture this random event we introduce a fictitious player called *Nature* or *Chance*. We assign a probability distribution to Nature's "choices". In this case, since one card is black and the other two are red, the probability that the top card is black is $\frac{1}{3}$ and the probability that the top card is red is $\frac{2}{3}$. Note that we don't assign payoffs to Nature and thus the only 'real' players are Adele and Ben. The situation can be represented as shown in Figure 4.17, where the numbers associated with the terminal nodes are dollar amounts.

[2] We will see in Part II that, when payoffs are cardinal and one allows for mixed strategies, then every finite game has at least one Nash equilibrium in mixed strategies.

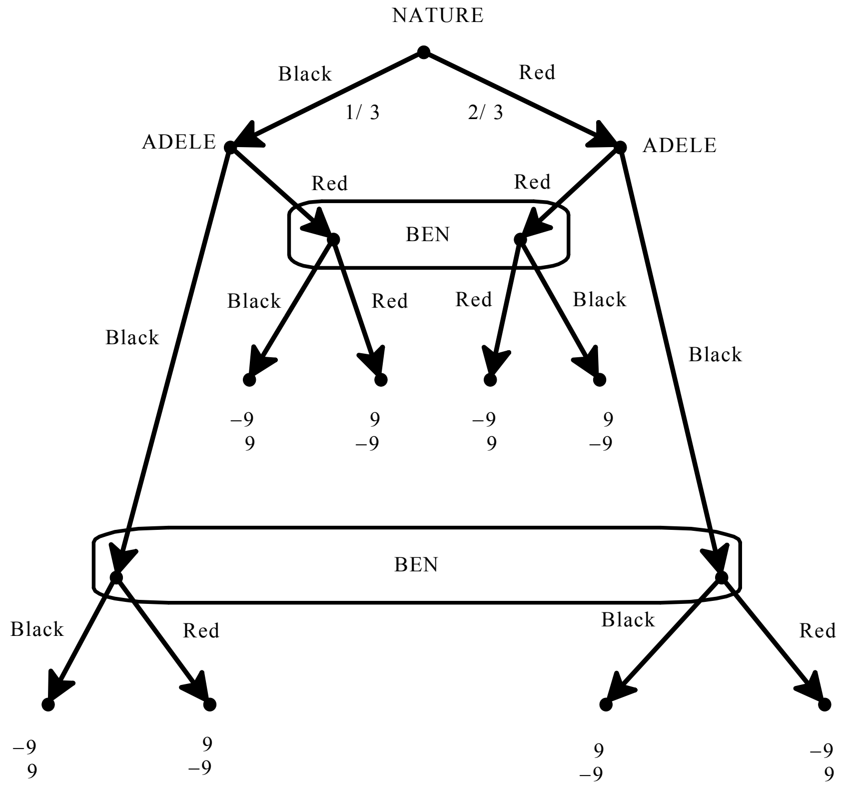


Figure 4.17: An extensive form with a chance move

Clearly, the notion of strategy is not affected by the presence of chance moves. In the game of Figure 4.17 Adele has four strategies and so does Ben. However, we do encounter a difficulty when we try to write the associated strategic form. For example, consider the following strategy profile: $((B,R),(B,B))$ where Adele's strategy is to be truthful (say "Black" if she sees a black card and say "Red" if she sees a red card) and Ben's strategy is to guess Black no matter what Adele says. What is the outcome in this case? It depends on what the true color of the top card is and thus the outcome is a probabilistic one:

$$\begin{pmatrix} \textit{outcome} & \text{Adele gives \$9 to Ben} & \text{Ben gives \$9 to Adele} \\ \textit{probability} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

We call such probabilistic outcomes *lotteries*. In order to convert the game-frame into a game we need to specify how the players rank probabilistic outcomes. Consider the case where Adele is selfish and greedy, in the sense that she only cares about her own wealth and she prefers more money to less. Then, from her point of view, the above probabilistic outcome reduces to the following monetary lottery $\begin{pmatrix} -\$9 & \$9 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$.

If Ben is also selfish and greedy, then he views the same outcome as the lottery $\begin{pmatrix} \$9 & -\$9 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$.

How do we convert a lottery into a payoff or utility? The general answer to this question will be provided in Chapter 5. Here we consider one possibility, which is particularly simple.

Definition 4.5.1 Given a lottery whose outcomes are sums of money

$$\begin{pmatrix} \$x_1 & \$x_2 & \dots & \$x_n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}$$

(with $p_i \geq 0$, for all $i = 1, 2, ..., n$, and $p_1 + p_2 + \cdots + p_n = 1$) the *expected value* of the lottery is the following sum of money: $(x_1 p_1 + x_2 p_2 + \cdots + x_n p_n)$.
We call lotteries whose outcomes are sums of money, *money lotteries*.

For example, the expected value of the lottery

$$\begin{pmatrix} \$5 & \$15 & \$25 \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix}$$

is

$$\$\left[5\left(\tfrac{1}{5}\right) + 15\left(\tfrac{2}{5}\right) + 25\left(\tfrac{2}{5}\right)\right] = \$(1 + 6 + 10) = \$17$$

and the expected value of the lottery

$$\begin{pmatrix} -\$9 & \$9 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

is \$3.

Definition 4.5.2 A player is defined to be *risk neutral* if she considers a money lottery to be just as good as its expected value. Hence a risk neutral person ranks money lotteries according to their expected value.[a]

[a] It is important to stress that our focussing on the case of risk neutrality should *not* be taken to imply that a rational individual ought to be risk neutral nor that risk neutrality is empirically particularly relevant. At this stage we assume risk neutrality only because it yields a very simple type of preference over money lotteries and allows us to introduce the notion of backward induction without the heavy machinery of expected utility theory.

For example, consider the following money lotteries:

$$L_1 = \begin{pmatrix} \$5 & \$15 & \$25 \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix}, \; L_2 = \begin{pmatrix} \$16 \\ 1 \end{pmatrix} \text{ and } L_3 = \begin{pmatrix} \$0 & \$32 & \$48 \\ \frac{5}{8} & \frac{1}{8} & \frac{1}{4} \end{pmatrix}.$$

The expected value of L_1 is \$17 and the expected value of both L_2 and L_3 is \$16. Thus a risk-neutral player would have the following ranking: $L_1 \succ L_2 \sim L_3$, that is, she would prefer L_1 to L_2 and be indifferent between L_2 and L_3.

For a selfish and greedy player who is risk neutral we can take the expected value of a money lottery as the utility of that lottery. For example, if we make the assumption that, in the extensive form of Figure 4.17, Adele and Ben are selfish, greedy and risk neutral then we can associate a strategic-form game to it as shown in Figure 4.18. Note that inside each cell we have two numbers: the first is the utility (= expected value) of the underlying money lottery as perceived by Adele and the second number is the utility (= expected value) of the underlying money lottery as perceived by Ben.

The first element of Adele's strategy is what she says if she sees a black card and the second element is what she says if she sees a red card. The first element of Ben's strategy is what he guesses if Adele says "Red", the second element is what he guesses if Adele says "Black".

		Ben			
		BB	*BR*	*RB*	*RR*
Adele	*BB*	3 −3	−3 3	3 −3	−3 3
	BR	3 −3	9 −9	−9 9	−3 3
	RB	3 −3	−9 9	9 −9	−3 3
	RR	3 −3	3 −3	−3 3	−3 3

Figure 4.18: The strategic form of the game of Figure 4.17 when the two players are selfish, greedy and risk neutral.

We conclude this section with one more example.

■ **Example 4.3** There are three unmarked, opaque envelopes. One contains \$100, one contains \$200 and the third contains \$300. They are shuffled well and then one envelope is given to Player 1 and another is given to Player 2 (the third one remains on the table).

- Player 1 opens her envelope and checks its content without showing it to Player 2. Then she either says "pass" – in which case each player gets to keep his/her envelope – or she asks Player 2 to trade his envelope for hers.
- Player 2 is not allowed to see the content of his envelope and has to say either Yes or No. If he says No, then the two players get to keep their envelopes. If Player 2 says Yes, then they trade envelopes and the game ends. Each player is selfish, greedy and risk neutral.

This situation is represented by the extensive-form game shown in Figure 4.19, where $(100,200)$ means that Player 1 gets the envelope with \$100 and Player 2 gets the envelope with \$200, etc.; P stands for "pass" and T for "suggest a trade"; Y for "Yes" and N for "No". ■

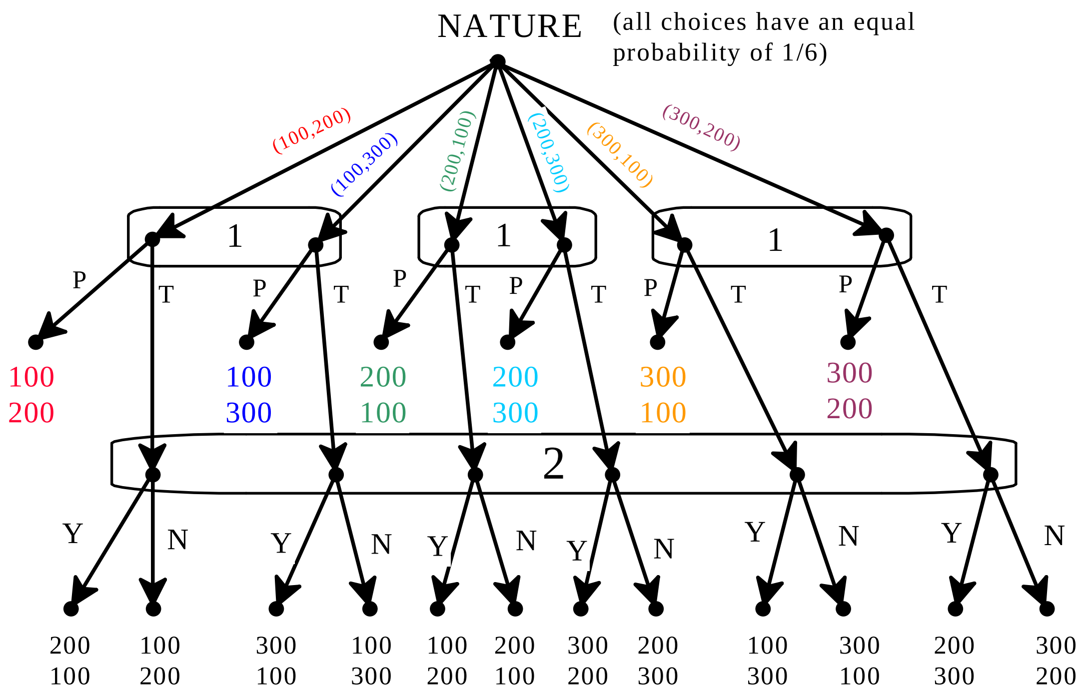


Figure 4.19: The extensive-form game of Example 4.3

In this game Player 1 has eight strategies. One possible strategy is: "if I get \$100 I will pass, if I get \$200 I will propose a trade, if I get \$300 I will pass": we will use the shorthand PTP for this strategy. Similarly for the other strategies. Player 2 has only two strategies: Yes and No.

The strategic form associated with the game of Figure 4.19 is shown in Figure 4.20, where the Nash equilibria are highlighted.

Player 1	Player 2: Y	Player 2: N
PPP	200 , 200	200 , 200
PPT	150 , 250	200 , 200
PTP	200 , 200	200 , 200
PTT	150 , 250	200 , 200
TPP	250 , 150	200 , 200
TPT	200 , 200	200 , 200
TTP	250 , 150	200 , 200
TTT	200 , 200	200 , 200

Figure 4.20: The strategic form of the game of Figure 4.19

How did we get those payoffs? Consider, for example, the first cell. Given the strategies PPP and Y, the outcomes are:
$(\$100, \$200)$ with probability $\frac{1}{6}$, $(\$100, \$300)$ with probability $\frac{1}{6}$,
$(\$200, \$100)$ with probability $\frac{1}{6}$, $(\$200, \$300)$ with probability $\frac{1}{6}$,
$(\$300, \$100)$ with probability $\frac{1}{6}$, $(\$300, \$200)$ with probability $\frac{1}{6}$.

Being risk neutral, Player 1 views his corresponding money lottery as equivalent to getting its expected value $\$(100+100+200+200+300+300)(\frac{1}{6}) = \$200$. Similarly for Player 2 and for the other cells.

Since the game of Figure 4.19 has no proper subgames, all the Nash equilibria are also subgame-perfect equilibria. Are some of the Nash equilibria more plausible than others? For Player 1 all the strategies are weakly dominated, except for TPP and TTP. Elimination of the weakly dominated strategies leads to a game where Y is strictly dominated for Player 2. Thus one could argue that (TPP, N) and (TTP, N) are the most plausible equilibria; in both of them Player 2 refuses to trade.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 4.6.5 at the end of this chapter.

4.6 Exercises

4.6.1 Exercises for Section 4.1: Imperfect information

The answers to the following exercises are in Section 4.7 at the end of this chapter.

Exercise 4.1 Amy and Bill simultaneously write a bid on a piece of paper. The bid can only be either 2 or 3. A referee then looks at the bids, announces the amount of the lowest bid (without revealing who submitted it) and invites Amy to either pass or double her initial bid.

- The outcome is determined by comparing Amy's final bid to Bill's bid: if one is greater than the other then the higher bidder gets the object and pays his/her own bid; if they are equal then Bill gets the object and pays his bid.

Represent this situation by means of two alternative extensive frames.
Note: (1) when there are simultaneous moves we have a choice as to which player we select as moving first: the important thing is that the second player does not know what the first player did;
(2) when representing, by means of information sets, what a player is uncertain about, we typically assume that a player is smart enough to deduce relevant information, even if that information is not explicitly given to him/her. ■

Exercise 4.2 Consider the following situation. An incumbent monopolist decides at date 1 whether to build a small plant or a large plant. At date 2 a potential entrant observes the plant built by the incumbent and decides whether or not to enter.

- If she does not enter then her profit is 0 while the incumbent's profit is \$25 million with a small plant and \$20 million with a large plant.
- If the potential entrant decides to enter, she pays a cost of entry equal to \$K million.
- At date 3 the two firms simultaneously decide whether to produce high output or low output.
- The profits of the firms are as shown in the following table, where 'L' means 'low output' and 'H' means 'high output' (these figure do not include the cost of entry for the entrant; thus you need to subtract that cost for the entrant); in each cell, the first number is the profit of the entrant (in millions of dollars) and the second is the profit of the incumbent.

		Incumbent: L	Incumbent: H
Entrant	L	10 , 10	7 , 7
	H	7 , 6	4 , 3

If Incumbent has small plant

		Incumbent: L	Incumbent: H
Entrant	L	10 , 7	5 , 9
	H	7 , 3	4 , 5

If Incumbent has large plant

Draw an extensive-form game that represents this situation, assuming that each player is selfish and greedy (that is, cares only about its own profits and prefers more money to less). ■

4.6.2 Exercises for Section 4.2: Strategies

The answers to the following exercises are in Section 4.7 at the end of this chapter.

Exercise 4.3 Write the strategic-form game-frame of the extensive form of Exercise 4.1 (that is, instead of writing payoffs in each cell, you write the outcome). Verify that the strategic forms of the two possible versions of the extensive form are identical. ■

Exercise 4.4 Consider the extensive-form game of Exercise 4.2.

(a) Write down in words one of the strategies of the potential entrant.
(b) How many strategies does the potential entrant have?
(c) Write the strategic-form game associated with the extensive-form game.
(d) Find the Nash equilibria for the case where $K = 2$.

■

4.6.3 Exercises for Section 4.3: Subgames

The answers to the following exercises are in Section 4.7 at the end of this chapter.

Exercise 4.5 How many proper subgames does the extensive form of Figure 4.3 have?
namaste ■

Exercise 4.6 How many proper subgames does the extensive form of Figure 4.5 have?
namaste ■

Exercise 4.7 Consider the extensive game Figure 4.21.

(a) How many proper subgames does the game have?
(b) How many of those proper subgames are minimal?

■

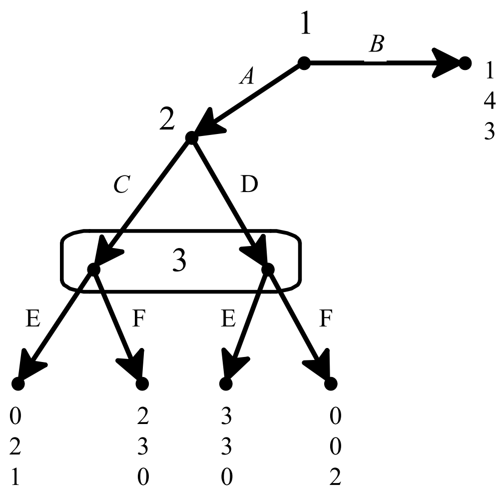


Figure 4.21: The game of Exercise 4.7

Exercise 4.8 Consider the extensive game Figure 4.22.

(a) How many proper subgames does the game have?

(b) How many of those proper subgames are minimal?

■

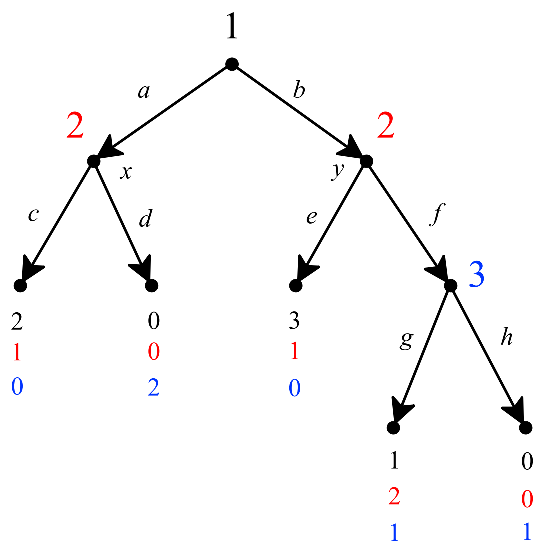


Figure 4.22: The game of Exercise 4.8

4.6.4 Exercises for Section 4.4: Subgame-perfect equilibrium

The answers to the following exercises are in Section 4.7 at the end of this chapter.

Exercise 4.9 Find the Nash equilibria and the subgame-perfect equilibria of the game shown in Figure 4.23. ■

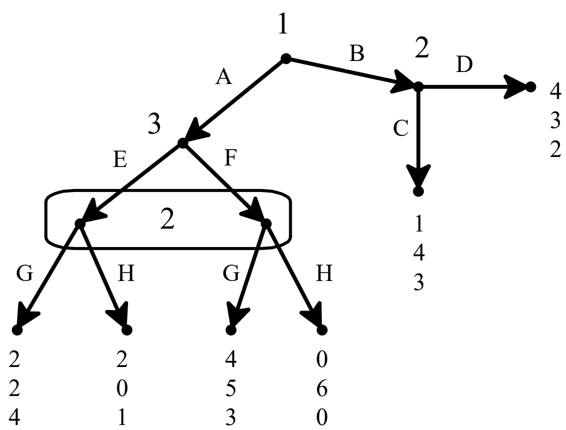


Figure 4.23: The game of Exercise 4.9

Exercise 4.10 Find the Nash equilibria and the subgame-perfect equilibria of the game shown in Figure 4.24. ■

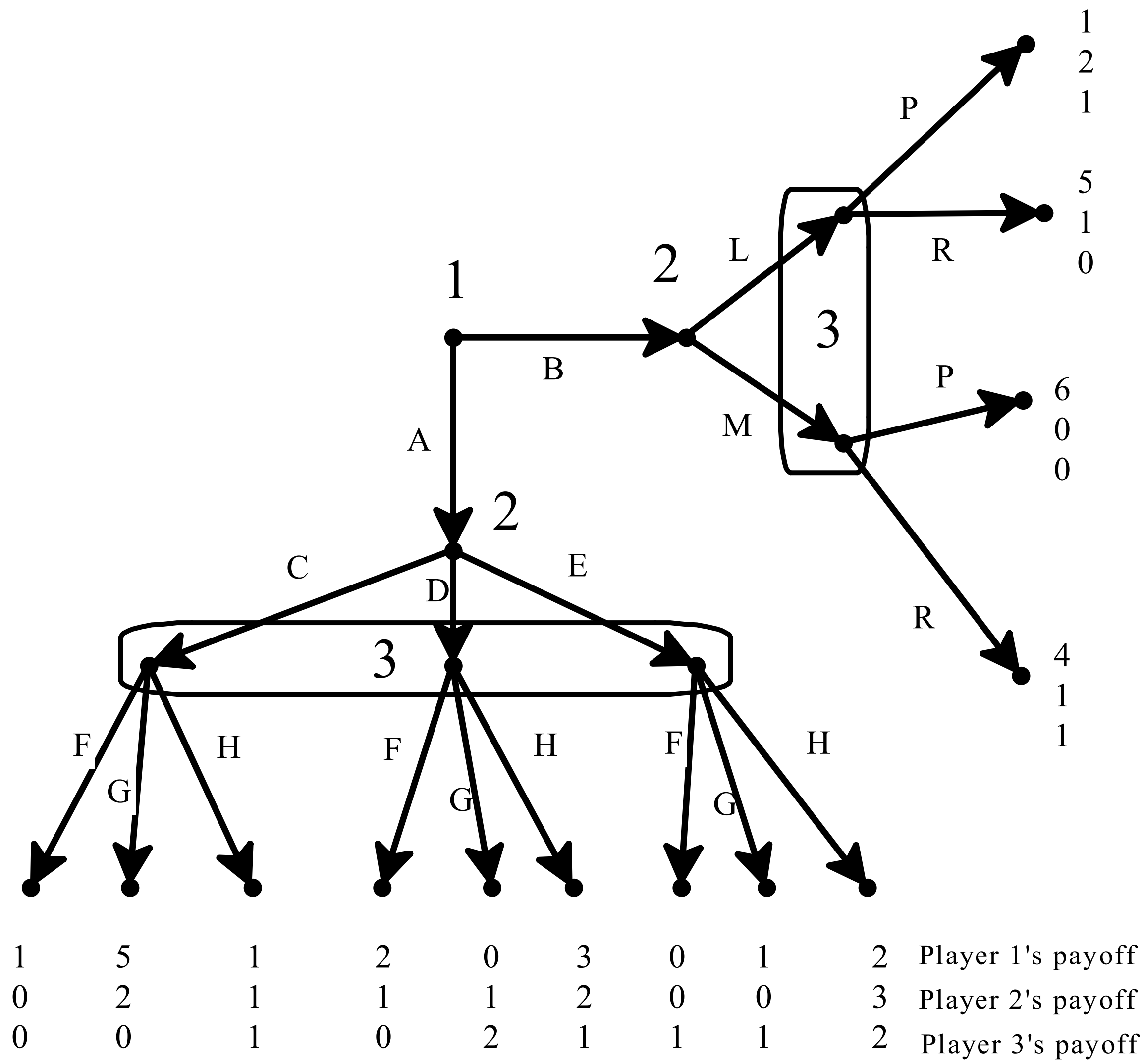


Figure 4.24: The game of Exercise 4.10

Exercise 4.11 Find the subgame-perfect equilibria of the game shown in Figure 4.25, assuming the following about the players' preferences. Both Amy and Bill are selfish and greedy, are interested in their own net gain, Amy values the object at \$5 and Bill at \$4. ■

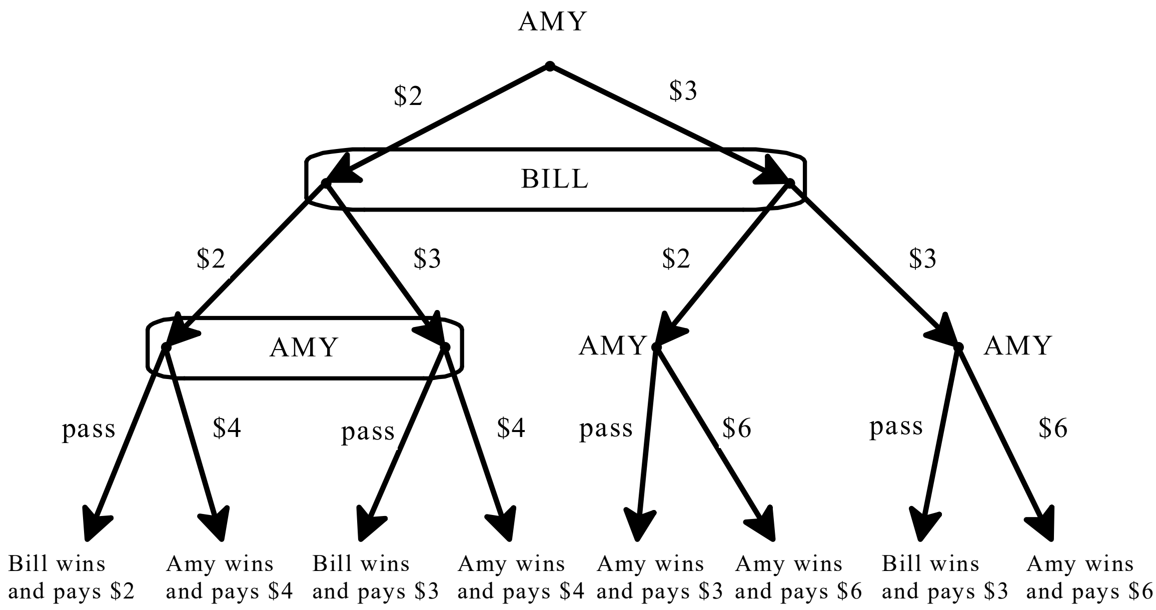


Figure 4.25: The game of Exercise 4.11

4.6.5 Exercises for Section 4.5: Games with chance moves

The answers to the following exercises are in Section 4.7 at the end of this chapter.

Exercise 4.12 Modify the game of Example 4.3 as follows: Player 2 is allowed to privately check the content of his envelope before he decides whether or not to accept Player 1's proposal.

(a) Represent this situation as an extensive-form game game.

(b) List all the strategies of Player 1 and all the strategies of Player 2. ■

Exercise 4.13 Three players, Avinash, Brian and John, play the following game. Two cards, one red and the other black, are shuffled well and put face down on the table. Brian picks the top card, looks at it without showing it to the other players (Avinash and John) and puts it back face down on the table. Then Brian whispers either "Black" or "Red" in Avinash's ear, making sure that John doesn't hear. Avinash then tells John either "Black" or "Red". Note that both players could be lying. Finally John announces either "Black" or "Red" and this exciting game ends.

The payoffs are as follows: if John's final announcement matches the true color of the card Brian looked at, then Brian and Avinash give \$2 each to John. In every other case John gives \$2 each to Brian and Avinash.

(a) Represent this situation as an extensive-form.

(b) Write the corresponding strategic form assuming that the players are selfish, greedy and risk neutral. [At least try to fill in a few cells in at least one table.]

■

Exercise 4.14 Consider the following highly simplified version of Poker.

- There are three cards, marked A, B and C. A beats B and C, B beats C.
- There are two players, Yvonne and Zoe. Each player contributes \$1 to the pot before the game starts. The cards are then shuffled and the top card is given, face down, to Yvonne and the second card (face down) to Zoe. Each player looks at, and only at, her own card: she does not see the card of the other player nor the remaining card.
- Yvonne, the first player, may pass, or bet \$1. If she passes, the game ends, the cards are turned and the pot goes to the high-card holder (recall that A beats B and C, B beats C).
- If Yvonne bets, then Zoe can fold, in which case the game ends and the pot goes to Yvonne, or Zoe can see by betting \$1, in which case the game ends, the cards are turned and the pot goes to the high-card holder. Both players are selfish, greedy and risk neutral.

(a) Draw the extensive-form game.

(b) How many strategies does Yvonne have?

(c) How many strategies does Zoe have?

(d) Consider the following strategies. For Yvonne: If A pass, if B pass, if C bet.
For Zoe: if Yvonne bets, I will fold no matter which card I get.
Calculate the corresponding payoffs.

(e) Redo the same with the following strategies. For Yvonne: If A pass, if B pass, if C bet. For Zoe: see always (that is, no matter what card she gets).

(f) Now that you have understood how to calculate the payoffs, represent the entire game as a normal form game, assigning the rows to Yvonne and the columns to Zoe. [This might take you the entire night, so make sure you have a lot of coffee!]

(g) What strategies of Yvonne are weakly dominated? What strategies of Zoe are weakly dominated?

(h) What do you get when you apply the procedure of iterative elimination of weakly dominated strategies?

■

Exercise 4.15 — ⋆⋆⋆ Challenging Question ⋆⋆⋆.

In an attempt to reduce the deficit, the government of Italy has decided to sell a 14^{th} century palace near Rome. The palace is in disrepair and is not generating any revenue for the government. From now on we will call the government Player G. A Chinese millionaire has offered to purchase the palace for $\$p$. Alternatively, Player G can organize an auction among n interested parties $(n \geq 2)$. The participants to the auction (we will call them players) have been randomly assigned labels $1, 2, \ldots, n$. Player i is willing to pay up to $\$p_i$ for the palace, where $\$p_i$ is a positive integer. For the auction assume the following:

1. it is a simultaneous, sealed-bid second-price auction,
2. bids must be non-negative integers,
3. each player only cares about his own wealth,
4. the tie-breaking rule for the auction is that the palace is given to that player who has the lowest index (e.g. if the highest bid was submitted by Players 3, 7 and 12 then the palace is given to Player 3).

All of the above is commonly known among everybody involved, as is the fact that for every $i, j \in \{1, \ldots, n\}$ with $i \neq j$, $p_i \neq p_j$.

We shall consider four different scenarios. In all scenarios you can assume that the p_i's are common knowledge.

Scenario 1. Player G first decides whether to sell the palace to the Chinese millionaire or make a public and irrevocable decision to auction it.

(a) Draw the extensive form of this game for the case where $n = 2$ and the only possible bids are \$1 and \$2. [List payoffs in the following order: first G then 1 then 2; don't forget that this is a *second-price* auction.]

(b) For the general case where $n \geq 2$ and every positive integer is a possible bid, find a pure-strategy subgame-perfect equilibrium of this game. What are the players' payoffs at the equilibrium?

Scenario 2. Here we assume that $n = 2$, and $p_1 > p_2 + 1 > 2$.

- First Player G decides whether to sell the palace to the Chinese or make a public and irrevocable decision to auction it. In the latter case he first asks Player 2 to publicly announce whether or not he is going to participate in the auction.
- If Player 2 says Yes, then he has to pay \$1 to Player G as a participation fee, which is non-refundable. If he says No, then she is out of the game.
- After Player 2 has made his announcement (and paid his fee if he decided to participate), Player 1 is asked to make the same decision (participate and pay a non-refundable fee of \$1 to Player G or stay out); Player 1 knows Player 2's decision when he makes his own decision.

After both players have made their decisions, player G proceeds as follows:

- if both 1 and 2 said Yes, then he makes them play a simultaneous second-price auction,
- if only one player said Yes, then he is asked to put an amount $\$x$ of his choice in an envelope (where x is a positive integer) and give it to Player G in exchange for the palace,
- if both 1 and 2 said No, then G is no longer bound by his commitment to auction the palace and he sells it to the Chinese.

(c) Draw the extensive form of this game for the case where the only possible bids are \$1 and \$2 and also $x \in \{1,2\}$ [List payoffs in the following order: first G then 1 then 2; again, don't forget that this is a *second-price* auction.]

(d) For the general case where all possible bids are allowed (subject to being positive integers) and x can be any positive integer, find a pure-strategy subgame-perfect equilibrium of this game. What are the players' payoffs at the equilibrium?

Scenario 3. Same as Scenario 2; the only difference is that if both Players 1 and 2 decide to participate in the auction then Player G gives to the loser the fraction a (with $0 < a < 1$) of the amount paid by the winner in the auction (note that player G still keeps 100% of the participation fees). This is publicly announced at the beginning and is an irrevocable commitment.

(e) For the general case where all possible bids are allowed (subject to being positive integers) find a subgame-perfect equilibrium of this game. What are the players' payoff at the equilibrium?

Scenario 4. Player G tells the Chinese millionaire the following:

> "First you (= the Chinese) say Yes or No; if you say No I will sell you the palace at the price that you offered me, namely \$100 (that is, we now assume that $p = 100$); if you say Yes then we play the following perfect information game. I start by choosing a number from the set $\{1,2,3\}$, then you (= the Chinese) choose a number from this set, then I choose again, followed by you, etc. The first player who brings the cumulative sum of all the numbers chosen (up to and including the last one) to 40 wins. If you win I will sell you the palace for \$50, while if I win I will sell you the palace for \$200."

Thus there is no auction in this scenario. Assume that the Chinese would actually be willing to pay up to \$300 for the palace.

(f) Find a pure-strategy subgame-perfect equilibrium of this game.

■

4.7 Solutions to exercises

Solution to Exercise 4.1. One possible extensive frame is shown in Figure 4.26, where Amy moves first. Note that we have only one non-trivial information set for Amy, while each of the other three consists of a single node. The reason is as follows: if Amy initially bids \$3 and Bill bids \$2 then the referee announces "the lowest bid was \$2"; this announcement does not directly reveal to Amy that Bill's bid was \$2, but she can figure it out from her knowledge that her own bid was \$3; similarly, if the initial two bids are both \$3 then the referee announces "the lowest bid was \$3", in which case Amy is able to figure out that Bill's bid was also \$3. If we included those two nodes in the same information set for Amy, we would not show much faith in Amy's reasoning ability!

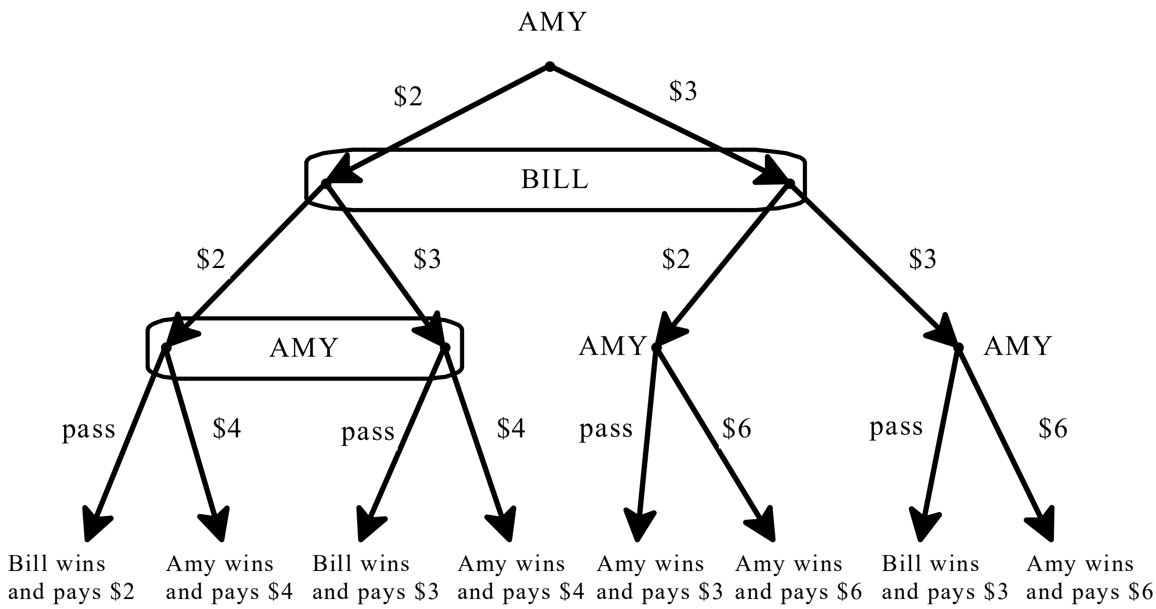


Figure 4.26: One possible game-frame for Exercise 4.1

Another possible extensive frame is shown in Figure 4.27, where Bill moves first. □

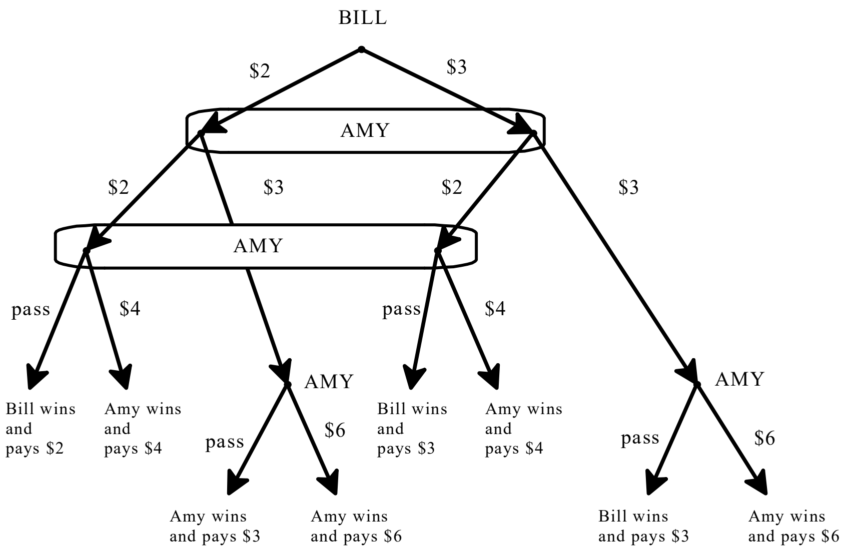


Figure 4.27: Another possible game-frame for Exercise 4.1

Solution to Exercise 4.2. The extensive form is shown in Figure 4.28 (the top number is the Potential Entrant's payoff and the bottom number is the Incumbent's payoff: since the players are selfish and greedy we can take a players' utility of an outcome to be the profit of that player at that outcome). □

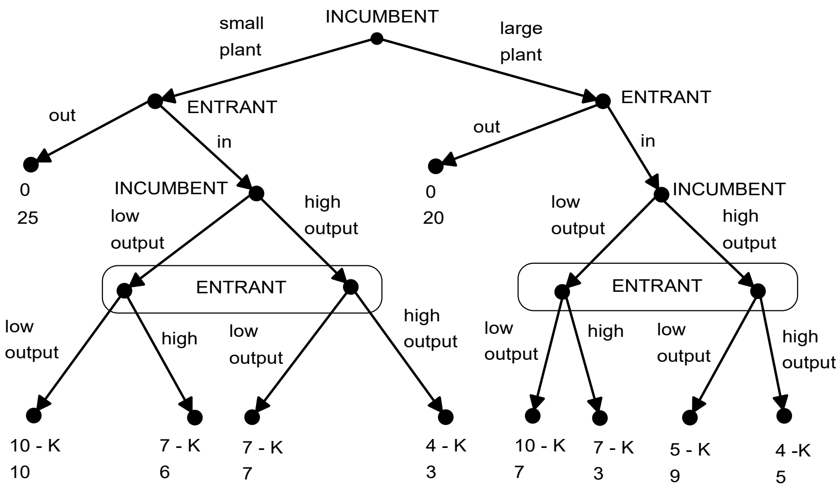


Figure 4.28: The extensive-form game for Exercise 4.2

Solution to Exercise 4.3. The strategic form is shown in Figure 4.29.
Amy's strategy (x,y,w,z) means: at the beginning I bid \$$x$, at the non-trivial information set on the left I choose y, at the singleton node in the middle I choose w and at the singleton node on the right I choose z. The numbers are bid amounts and P stands for "Pass". □

		BILL	
		bid \$2	bid \$3
	\$2, P, P, \$P	Bill wins pays 2	Bill wins pays 3
	2 , P, P, 6	Bill wins pays 2	Bill wins pays 3
	2, P, 6, P	Bill wins pays 2	Bill wins pays 3
	2, P, 6, 6	Bill wins pays 2	Bill wins pays 3
A	2, 4, P, P	Amy wins pays 4	Amy wins pays 4
M	2, 4, P, 6	Amy wins pays 4	Amy wins pays 4
Y	2, 4, 6, P	Amy wins pays 4	Amy wins pays 4
	2, 4, 6, 6	Amy wins pays 4	Amy wins pays 4
	3, P, P, P	Amy wins pays 3	Bill wins pays 3
	3 , P, P, 6	Amy wins pays 3	Amy wins pays 6
	3, P, 6, P	Amy wins pays 6	Bill wins pays 3
	3, P, 6, 6	Amy wins pays 6	Amy wins pays 6
	3, 4, P, P	Amy wins pays 3	Bill wins pays 3
	3, 4, P, 6	Amy wins pays 3	Amy wins pays 6
	3, 4, 6, P	Amy wins pays 6	Bill wins pays 3
	3, 4, 6, 6	Amy wins pays 6	Amy wins pays 6

Figure 4.29: The strategic form for Exercise 4.3

Solution to Exercise 4.4.

(a) The potential entrant has four information sets, hence a strategy has to specify what she would do in each of the four situations. A possible strategy is: "if the incumbent chooses a small plant I stay out, if the incumbent chooses a large plant I enter, if small plant and I entered then I choose low output, if large plant and I entered then I choose high output".

(b) The potential entrant has $2^4 = 16$ strategies.

(c) The strategic form is shown in Figure 4.30.

(d) For the case where $K = 2$ the Nash equilibria are highlighted in Figure 4.30. □

INCUMBENT

		SLL	SLH	SHL	SHH	LLL	LLH	LHL	LHH
	OOLL	0, 25	0, 25	0, 25	0, 25	0, 20	0, 20	0, 20	0, 20
	OOLH	0, 25	0, 25	0, 25	0, 25	0, 20	0, 20	0, 20	0, 20
	OOHL	0, 25	0, 25	0, 25	0, 25	0, 20	0, 20	0, 20	0, 20
	OOHH	0, 25	0, 25	0, 25	0, 25	0, 20	0, 20	0, 20	0, 20
E	OILL	0, 25	0, 25	0, 25	0, 25	10-K, 7	5-K, 9	10-K, 7	5-K, 9
N	OILH	0, 25	0, 25	0, 25	0, 25	7-K, 3	4-K, 5	7-K, 3	4-K, 5
T	OIHL	0, 25	0, 25	0, 25	0, 25	10-K, 7	5-K, 9	10-K, 7	5-K, 9
R	OIHH	0, 25	0, 25	0, 25	0, 25	7-K, 3	4-K, 5	7-K, 3	4-K, 5
A	IOLL	10-K, 10	10-K, 10	7-K, 7	7-K, 7	0, 20	0, 20	0, 20	0, 20
N	IOLH	10-K, 10	10-K, 10	7-K, 7	7-K, 7	0, 20	0, 20	0, 20	0, 20
T	IOHL	7-K, 6	7-K, 6	4-K, 3	4-K, 3	0, 20	0, 20	0, 20	0, 20
	IOHH	7-K, 6	7-K, 6	4-K, 3	4-K, 3	0, 20	0, 20	0, 20	0, 20
	IILL	10-K, 10	10-K, 10	7-K, 7	7-K, 7	10-K, 7	5-K, 9	10-K, 7	5-K, 9
	IILH	10-K, 10	10-K, 10	7-K, 7	7-K, 7	7-K, 3	4-K, 5	7-K, 3	4-K, 5
	IIHL	7-K, 6	7-K, 6	4-K, 3	4-K, 3	10-K, 7	5-K, 9	10-K, 7	5-K, 9
	IIHH	7-K, 6	7-K, 6	4-K, 3	4-K, 3	7-K, 3	4-K, 5	7-K, 3	4-K, 5

Figure 4.30: The strategic form for Exercise 4.4

Solution to Exercise 4.5. There are no proper subgames. □

Solution to Exercise 4.6. There are no proper subgames. □

Solution to Exercise 4.7.
(a) Only one proper subgame: it starts at Player 2's node.
(b) Since it is the only subgame, it is minimal. □

Solution to Exercise 4.8. The game under consideration is shown in Figure 4.31.

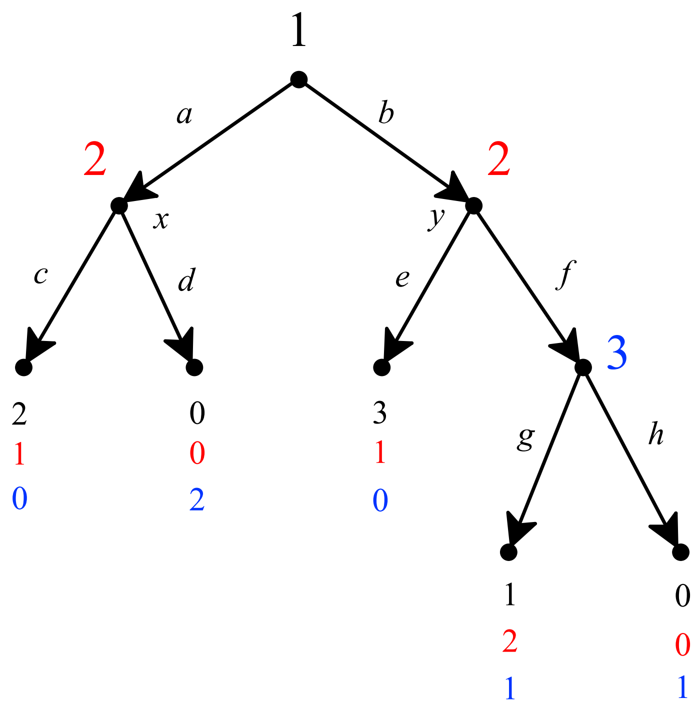


Figure 4.31: The game considered in Exercise 4.8

(a) There are three proper subgames: one starting at node x, one starting at node y and one starting at the node of Player 3.
(b) Two: the one starting at node x and the one starting at the decision node of Player 3. In a perfect-information game a minimal proper subgame is one that starts at a decision node followed only by terminal nodes. □

Solution to Exercise 4.9. The strategic form is shown in Figure 4.32.
The Nash equilibria are: $(A,(G,C),E)$ and $(B,(H,C),F)$.
The extensive-form game has two proper subgames. The one on the left has a unique Nash equilibrium, (G,E), and the one on the right has a unique Nash equilibrium, C.

Hence the game reduces to the game shown in Figure 4.33. In that game A is the unique optimal choice. Hence there is only one subgame-perfect equilibrium, namely $(A,(G,C),E)$. □

Player 2

		GC			GD			HC			HD		
Player 1	A	2	2	4	2	2	4	2	0	1	2	0	1
	B	1	4	3	4	3	2	1	4	3	4	3	2

Player 3: E

Player 2

		GC			GD			HC			HD		
Player 1	A	4	5	3	4	5	3	0	6	0	0	6	0
	B	1	4	3	4	3	2	1	4	3	4	3	2

Player 3: F

Figure 4.32: The strategic form for Exercise 4.9

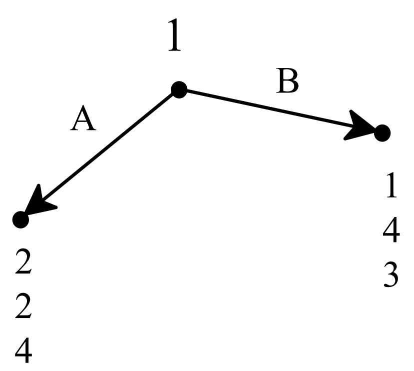


Figure 4.33: The reduced extensive-form game for Exercise 4.9

Solution to Exercise 4.10. Consider first the subgame that starts at Player 2's decision node following choice A of Player 1. The strategic-form of this game is shown in Figure 4.34 (where only the payoff of Players 2 and 3 are shown). The unique Nash equilibrium is (E,H).

		Player 3		
		F	G	H
Player 2	C	0 , 0	2 . 0	1 , 1
	D	1 , 0	1 , 2	2 , 1
	E	0 ,1	0 , 1	3 , 2

Figure 4.34: The strategic-form of subgame that starts at Player 2's decision node following choice A of Player 1

Now consider the subgame that starts at Player 2's decision node following choice B of Player 1. The strategic-form of this game is shown in Figure 4.35. This game has two Nash equilibria: (L,P) and (M,R).

		Player 3	
		P	R
Player 2	L	2 , 1	1 , 0
	M	0 ,0	1 , 1

Figure 4.35: The strategic-form of subgame that starts at Player 2's decision node following choice B of Player 1

Thus there are two subgame-perfect equilibria of the entire game:

1. Player 1's strategy: A; Player 2's strategy: E if A and L if B;
   Player 3's strategy: H if A and P if B.
2. Player 1's strategy: B; Player 2's strategy: E if A and M if B;
   Player 3's strategy: H if A and R if B. □

Solution to Exercise 4.11. Given the players' preferences, we can assign the following utilities to the outcomes:

outcome	Amy's utility	Bill's utility
Amy wins and pays \$3	2	0
Amy wins and pays \$4	1	0
Amy wins and pays \$6	−1	0
Bill wins and pays \$2	0	2
Bill wins and pays \$3	0	1

In the extensive form there are only two proper subgames: they start at Amy's singleton information sets (the two nodes on the right).

- In both subgames Amy will choose to pass (since she values the object at \$5 and is thus not willing to pay \$6).
- Replacing Amy's left singleton node with payoffs of 2 for Amy and 0 for Bill, and Amy's right singleton node with payoffs of 0 for Amy and 1 for Bill, we get a reduced game whose associated strategic form is as follows (in Amy's strategy the first component is the initial bet and the second component is her choice at her information set following Bill's choices after Amy's bet of \$2):

		Bill: \$2	Bill: \$3
Amy	\$2, pass	0 , 2	0 , 1
	\$2, \$4	1 , 0	1 , 0
	\$3, pass	2 , 0	0 , 1
	\$3, \$4	2 , 0	0 , 1

This game has one Nash equilibrium: ((\$2,\$4),\$3)). Thus the initial extensive-form game has one subgame-perfect equilibrium, which is as follows: Amy's strategy is (\$2, \$4, pass, pass), Bill's strategy is \$3. The corresponding outcome is: Amy wins the auction and pays \$4. □

Solution to Exercise 4.12.

(a) The extensive-form game is shown in Figure 4.36.

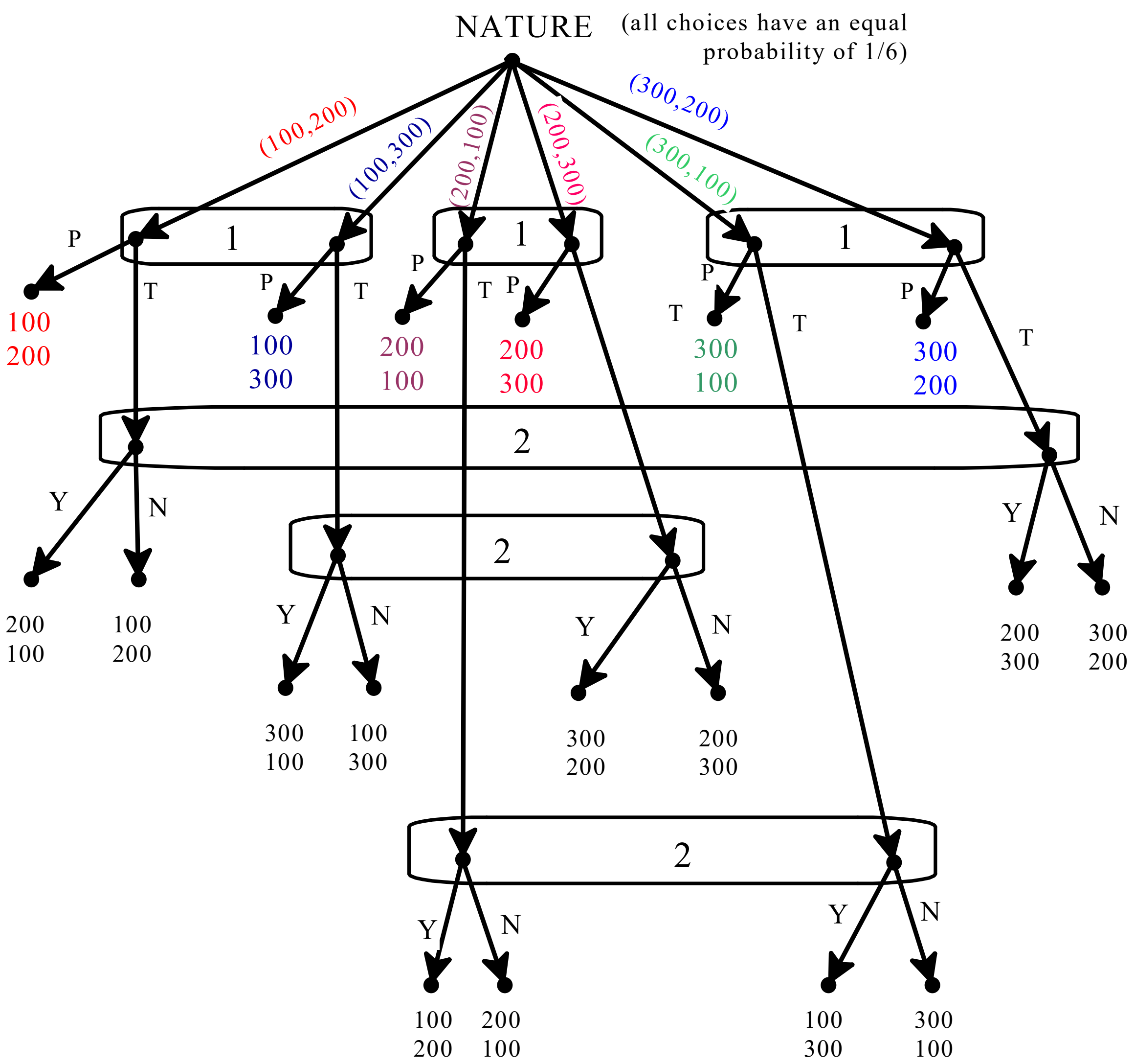


Figure 4.36: The extensive-form game for Exercise 4.12

(b) Player 1's strategies are the same as in Example 4.3. Player 2 now has 8 strategies. Each strategy has to specify how to reply to Player 1's proposal depending on the sum he (Player 2) has. Thus one possible strategy is: if I have \$100 I say No, if I have \$200 I say Yes and if I have \$300 I say No. □

Solution to Exercise 4.13.

(a) The extensive-form game is shown in Figure 4.37

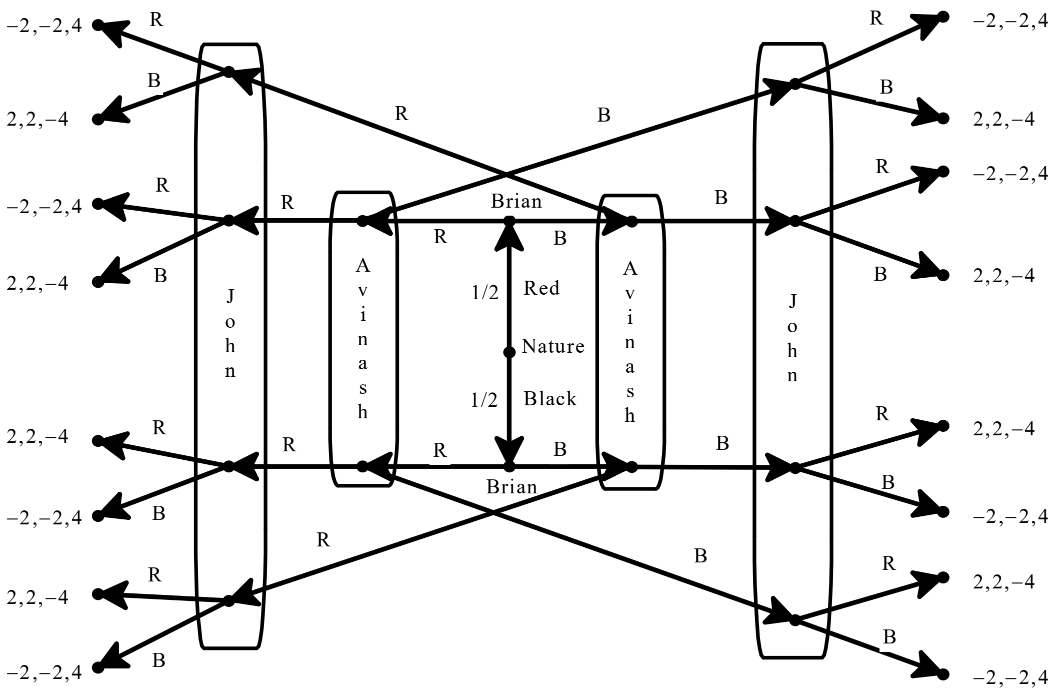


Figure 4.37: The extensive-form game for Part **(a)** of Exercise 4.13

(b) Each player has two information sets, two choices at each information set, hence four strategies.
The strategic form is shown in Figure 4.38
(interpretation: for Avinash "if B, R, if R, B" means "if Brian tells me B then I say R and if Brian tells me R then I say B"; similarly for the other strategies and for the other players).

John: if B, B if R, R

Brian \ Avinash	if B, B if R, R	if B, B if R, B	if B, R if R, R	if B, R if R, B
if B, B if R, R	−2 , −2 , 4	0 , 0 , 0	0 , 0 , 0	2 , 2 , −4
if B, B if R, B	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0
if B, R if R, R	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0
if B, R if R, B	2 , 2 , −4	0 , 0 , 0	0 , 0 , 0	−2 , −2 , 4

John: if B, B if R, B

Brian \ Avinash	if B, B if R, R	if B, B if R, B	if B, R if R, R	if B, R if R, B
if B, B if R, R	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0
if B, B if R, B	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0
if B, R if R, R	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0
if B, R if R, B	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0

John: if B, R if R, R

Brian \ Avinash	if B, B if R, R	if B, B if R, B	if B, R if R, R	if B, R if R, B
if B, B if R, R	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0
if B, B if R, B	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0
if B, R if R, R	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0
if B, R if R, B	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0

John: if B, R if R, B

Brian \ Avinash	if B, B if R, R	if B, B if R, B	if B, R if R, R	if B, R if R, B
if B, B if R, R	2 , 2 , −4	0 , 0 , 0	0 , 0 , 0	−2 , −2 , 4
if B, B if R, B	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0
if B, R if R, R	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0
if B, R if R, B	−2 , −2 , 4	0 , 0 , 0	0 , 0 , 0	2 , 2 , −4

Figure 4.38: The strategic form for Part **(b)** of Exercise 4.13

How can we fill in the payoffs without spending more than 24 hours on this problem? There is a quick way of doing it. First of all, when John's strategy is to guess Black, no matter what Avinash says, he has a 50% chance of being right and a 50% chance of being wrong. Thus his expected payoff is $\frac{1}{2}(4)+\frac{1}{2}(-4)=0$ and the expected payoff of each of the other two players is $\frac{1}{2}(2)+\frac{1}{2}(-2)=0$. This explains why the second table is filled with the same payoff vector, namely $(0,0,0)$. The same reasoning applies to the case where when John's strategy is to guess Red, no matter what Avinash says (leading to the third table, filled with the same payoff vector $(0,0,0)$).

For the remaining strategies of John's, one can proceed as follows:
Start with the two colors, B and R. Under B write T (for true) if Brian's strategy says "if B then B" and write F (for false) if Brian's strategy says "if B then R"; similarly, under R write T (for true) if Brian's strategy says "if R then R" and write F (for false) if Brian's strategy says "if R then B".

In the next row, in the B column rewrite what is in the previous row if Avinash's strategy says "if B then B" and change a T into an F or an F into a T if Avinash's strategy says "if B then R". Similarly for the R column. Now repeat the same for John (in the B column a T remains a T and an F remains an F is John's strategy is "if B then B", while a T is changed into an F and an F is changed into a T if John's strategy is "if B then R").

Now in each column the payoffs are $(-2,-2,4)$ if the last row has a T and $(2,2,-4)$ if the last row has an F. The payoffs are then given by $\frac{1}{2}$ the payoff in the left column plus $\frac{1}{2}$ the payoff in the right column. For example, for the cell in the second row, third column of the third table we have the calculations shown in Figure 4.39. □

	B	R
Brian's strategy: if B, B and if R,B	T	F
Avinash's strategy: if B, R and if R,R	F	T
John's strategy: if B, R and if R,R	F	T
Payoffs	(2 , 2 , –4)	(–2 , –2 , 4)

Expected payoffs:

$$\boxed{\begin{array}{l}\frac{1}{2}(2,2,-4)+\frac{1}{2}(-2,-2,4)\\ =(0,0,0)\end{array}}$$

Figure 4.39: The calculations for the expected payoffs

Solution to Exercise 4.14.

(a) The extensive-form representation of the simplified poker game is shown in Figure 4.40 (the top number is Yvonne's net take in dollars and the bottom number is Zoe's net take).

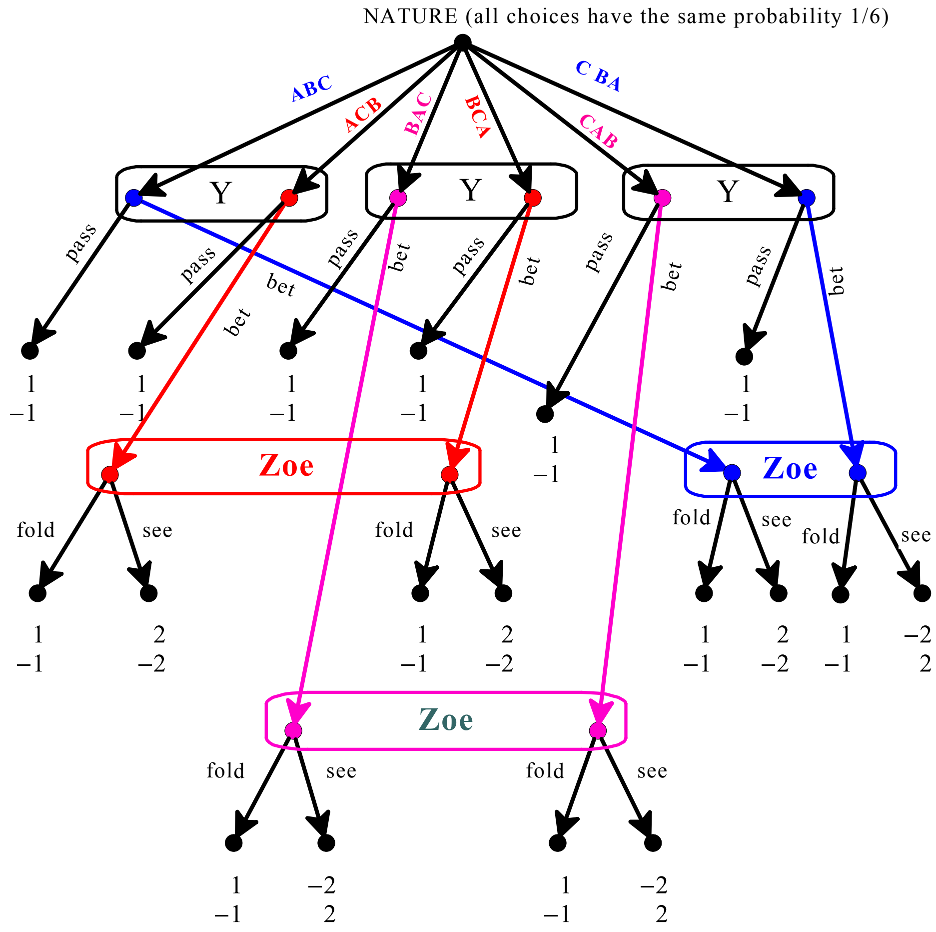


Figure 4.40: The extensive-form game for Exercise 4.14

(b) Yvonne has eight strategies (three information sets, two choices at each information set, thus $2 \times 2 \times 2 = 8$ possible strategies).

(c) Similarly, Zoe has eight strategies.

(d) Yvonne uses the strategy "If A pass, if B pass, if C bet" and Zoe uses the strategy "If A fold, if B fold, if C fold"). The table below shows how to compute the expected net payoff for Yvonne. Zoe's expected net payoff is the negative of that.

Top card is:	A	A	B	B	C	C
Second card is:	B	C	A	C	A	B
Probability:	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
Yvonne's action:	*pass*	*pass*	*pass*	*pass*	*bet*	*bet*
Zoe's action:	$--$	$--$	$--$	$--$	*fold*	*fold*
Yvonne's payoff:	1	1	-1	1	1	1

Yvonne's expected payoff: $\frac{1}{6}(1+1-1+1+1+1) = \frac{4}{6}$.

(e) Yvonne uses the strategy "If A pass, if B pass, if C bet" and Zoe uses the strategy "see with any card". The table below shows how to compute the expected net payoff for Yvonne. Zoe's expected net payoff is the negative of that.

Top card is:	A	A	B	B	C	C
Second card is:	B	C	A	C	A	B
Probability:	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
Yvonne's action:	*pass*	*pass*	*pass*	*pass*	*bet*	*bet*
Zoe's action:	——	——	——	——	*see*	*see*
Yvonne's payoff:	1	1	−1	1	−2	−2

Yvonne's expected payoff: $\frac{1}{6}(1+1-1+1-2-2) = -\frac{2}{6}$.

(f) The strategic form is shown in Figure 4.41.

ZOE

YVONNE	If A fold, If B fold, If C fold	If A see, If B see, If C see	If A see, If B fold, If C fold	If A fold, If B see, If C fold	If A fold, If B fold, If C see	If A see, If B see, If C fold	If A see, If B fold, If C see	If A fold, If B see, If C see
If A pass, if B pass if C pass	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0
If A bet, if B bet if C bet	1, −1	0, 0	0, 0	4/6, −4/6	8/6, −8/6	−2/6, 2/6	2/6, −2/6	1, −1
If A bet, if B pass if C pass	0, 0	2/6, −2/6	0, 0	1/6, −1/6	1/6, −1/6	1/6, −1/6	1/6, −1/6	2/6, −2/6
If A pass, if B bet if C pass	2/6, −2/6	0, 0	−1/6, 1/6	2/6, −2/6	3/6, −3/6	−1/6, 1/6	0, 0	3/6, −3/6
If A pass, if B pass if C bet	4/6, −4/6	−2/6, 2/6	1/6, −1/6	1/6, −1/6	4/6, −4/6	−2/6, 2/6	1/6, −1/6	1/6, −1/6
If A bet, if B bet if C pass	2/6, −2/6	2/6, −2/6	−1/6, 1/6	3/6, −3/6	4/6, −4/6	0, 0	1/6, −1/6	5/6, −5/6
If A bet, if B pass if C bet	4/6, −4/6	0, 0	1/6, −1/6	2/6, −2/6	5/6, −5/6	−1/6, 1/6	2/6, −2/6	3/6, −3/6
if A Pass, If B or C, Bet,	1, −1	−2/6, 2/6	0, 0	3/6, −3/6	7/6, −7/6	−3/6, 3/6	1/6, −1/6	4/6, −4/6

Figure 4.41: The strategic form for Part **(f)** of Exercise 4.14

(g) Let $\triangleright$ denote weak dominance, that is, $a \triangleright b$ means that a weakly dominates b.
FOR YVONNE (row player): 3^{rd} row $\triangleright 1^{st}$ row, $6^{th} \triangleright 4^{th}$, $7^{th} \triangleright 4^{th}$, $7^{th} \triangleright 5^{th}$, $2^{nd} \triangleright 8^{th}$.
FOR ZOE (column player): 1^{st} col $\triangleright$ 5^{th} col, $3^{rd} \triangleright 1^{st}$, $3^{rd} \triangleright 4^{th}$, $3^{rd} \triangleright 5^{th}$, $3^{rd} \triangleright 7^{th}$, $3^{rd} \triangleright 8^{th}$, $2^{nd} \triangleright 8^{th}$, $4^{th} \triangleright 5^{th}$, $4^{th} \triangleright 8^{th}$, $6^{th} \triangleright 2^{nd}$, $6^{th} \triangleright 4^{th}$, $6^{th} \triangleright 5^{th}$, $6^{th} \triangleright 7^{th}$, $6^{th} \triangleright 8^{th}$, $7^{th} \triangleright 4^{th}$, $7^{th} \triangleright 5^{th}$, $7^{th} \triangleright 8^{th}$.

(h) Eliminating rows 1, 4, 5 and 8 and all columns except 3 and 6 we are left with the reduced game shown below:

		Zoe: See only if A	Zoe: See only with A or B
Yvonne	Bet always	0 , 0	$-\frac{2}{6}$, $\frac{2}{6}$
	Bet only if A	0 , 0	$\frac{1}{6}$, $-\frac{1}{6}$
	Bet only if A or B	$-\frac{1}{6}$, $\frac{1}{6}$	0 , 0
	Bet only if A or C	$\frac{1}{6}$, $-\frac{1}{6}$	$-\frac{1}{6}$, $\frac{1}{6}$

In this reduced game, the second row dominates the first and the third. Eliminating them we are led to the reduced game shown below, which is a remarkable simplification of the original strategic form:

		Zoe: See only if A	Zoe: See only with A or B
Yvonne	Bet only if A	0 , 0	$\frac{1}{6}$, $-\frac{1}{6}$
	Bet only if A or C	$\frac{1}{6}$, $-\frac{1}{6}$	$-\frac{1}{6}$, $\frac{1}{6}$

□

Solution to Exercise 4.15.

(a) The extensive-form is shown in Figure 4.42.

(b) In the auction subgame for every player it is a weakly dominant strategy to bid his own value. Thus a natural Nash equilibrium for this subgame is the dominant-strategy equilibrium (although there are other Nash equilibria and one could choose any one of the alternative Nash equilibria).
Let $p_j = \max\{p_1, \dots, p_n\}$ be the highest value and $p_k = \max\{p_1, \dots, p_n\} \setminus \{p_j\}$ be the second highest value. Then the auction, if it takes place, will be won by Player j and he will pay p_k. Hence there are three cases.

Case 1: $p > p_k$. In this case Player G will sell to the Chinese (and the strategy of Player i in the subgame is to bid p_i), G's payoff is p and the payoff of Player i is 0.

Case 2: $p < p_k$. In this case Player G announces the auction, the strategy of Player i in the subgame is to bid p_i, the winner is Player j and he pays p_k, so that the payoff of G is p_k, the payoff of player j is $p_j - p_k$ and the payoff of every other player is 0.

Case 3: $p = p_k$. In this case there are two subgame-perfect equilibria: one as in Case 1 and the other as in Case 2 and G is indifferent between the two.

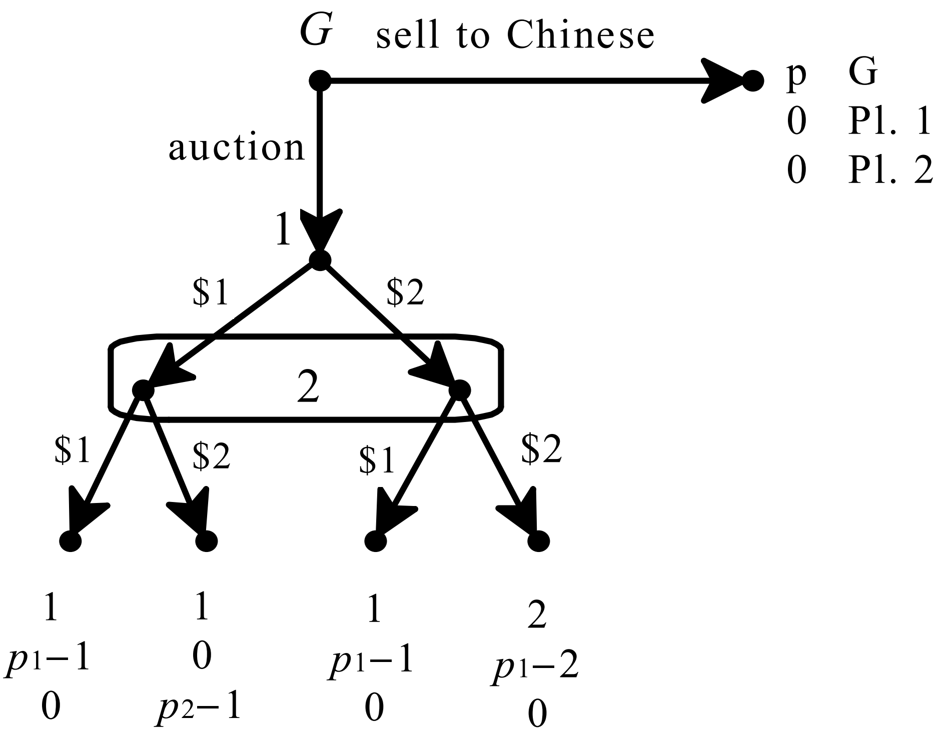


Figure 4.42: The extensive form for Part **(a)** of Exercise 4.15

(c) The extensive-form game is shown in Figure 4.43.

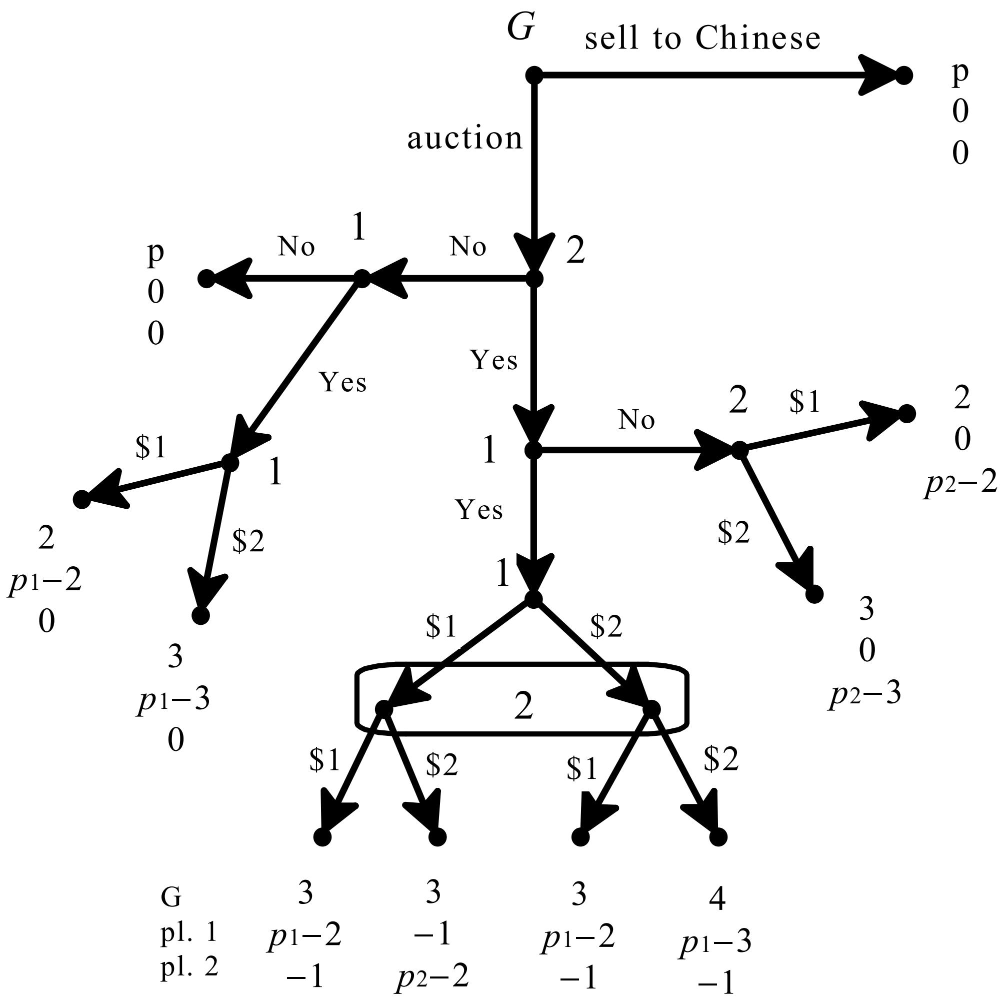


Figure 4.43: The extensive-form game for Part **(c)** of Exercise 4.15

.

(d) In the simultaneous subgame after both players have said Yes, the participation fee paid is a sunk cost and for every player bidding the true value is a weakly dominant strategy. Thus the outcome there is as follows: Player 1 bids p_1, gets the palace by paying p_2, G's payoff is (p_2+2), 1's payoff is (p_1-p_2-1) and Player 2's payoff is -1.
In the subgames where one player said No and the other said Yes the optimal choice is obviously $x=1$, with payoffs of 2 for Player G, 0 for the player who said No and p_i-2 for the player who said Yes.
Thus the game reduces to the one shown in Panel A of Figure 4.44.
By assumption, $p_1 > p_2+1 > 2$, so that $p_1-p_2-1 > 0$ and $p_1-2 > 0$. Thus at the bottom node and at the left node Player 1 prefers Yes to No.
Thus the game reduces to the one shown in Panel B of Figure 4.44.

Hence Player 2 will say No. The subgame-perfect equilibrium is as follows: (1) if $p > 2$ then player G will sell to the Chinese (and the choices off the equilibrium path are as explained above) and the payoffs are $(p,0,0)$; (2) if $p < 2$ then G chooses to auction, 2 says No, 1 says Yes and then offers \$1 and the payoffs are $(2, p_1 - 2, 0)$ (and the choices off the equilibrium path are as explained above); (3) if $p = 2$ then there are two equilibria: one as in (1) and the other as in (2).

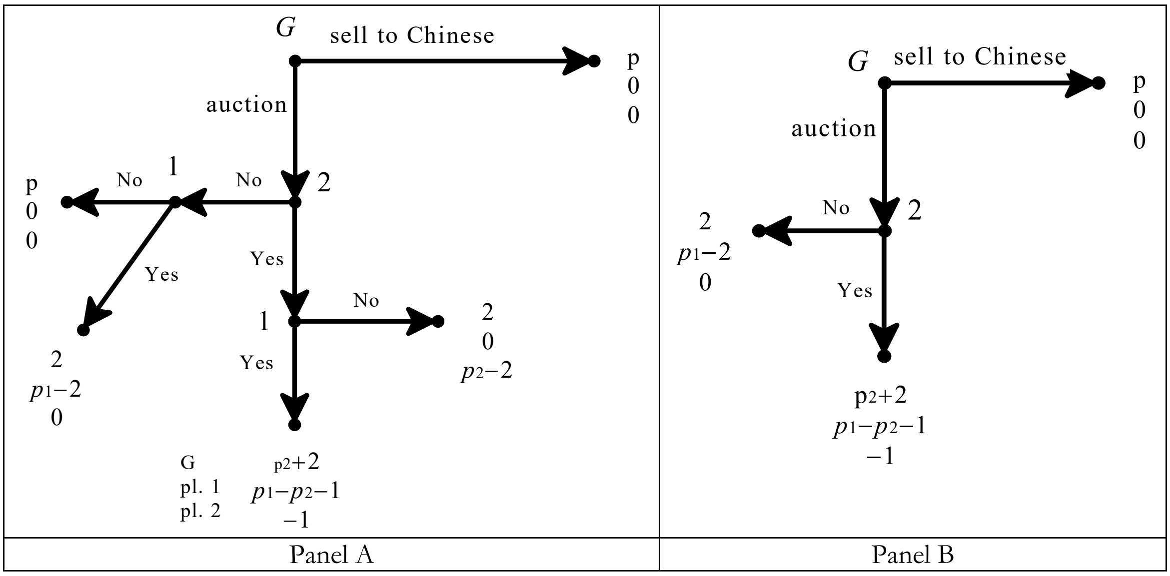


Figure 4.44: The extensive-form game for Part (d) of Exercise 4.15

(e) When the loser is given the fraction a of the amount paid by the winner (that is, the loser is given the fraction a of his own bid), **it is no longer true that bidding one's true value is a dominant strategy**. In fact, (p_1, p_2) is not even a Nash equilibrium any more. To see this, imagine that Player 1's true value is 10 and Player 2's true value is 6 and $a = 50\%$. Then if Player 1 bids 10 and 2 bids 6, Player 2 ends up losing the auction but being given \$3, while if he increased his bid to 8 then he would still lose the auction but receive \$4. This shows that there cannot be a Nash equilibrium where Player 2 bids less than Player 1. Now there are several Nash equilibria of the auction, for example, all pairs (b_1, b_2) with $b_1 = b_2 = b$ and $p_1 \leq b < p_1$ provided that $p_1 - b \geq a(b-1)$, that is, $b \leq \frac{p_1+a}{1+a}$ (but there are more: for example all pairs (b_1, b_2) with $b_1 = b_2 = b$ and $b < p_2$ provided that $p_1 - b \geq a(b-1)$ and $ab \geq p_2 - b$)). Thus to find a subgame-perfect equilibrium of the game one first has to select a Nash equilibrium of the auction game and then apply backward induction to see if the players would want to say Yes or No to the auction, etc.

(f) Let us start by considering the perfect-information game that is played if the Chinese says Yes. This is a game similar to the one discussed in Example 3.2 (Chapter 3, Section 3.5). We first determine the losing positions. Whoever has to move when the sum is 36 cannot win. Thus 36 is a losing position. Working backwards, the losing positions are 32, 28, 24, 20, 16, 12, 8, 4 and 0. Thus the first player (= player G) starts from a losing position: whatever his initial choice, he can be made to choose the second time when the sum is 4, and then 8, etc. Hence **the second player (= the Chinese) has a winning strategy**, which is as follows: if Player G just chose n, then choose $(4-n)$. If the Chinese says Yes and then follows this strategy he can guarantee that he will buy the palace for \$50. Thus the subgame-perfect equilibrium of this game is: the Chinese says Yes and uses the winning strategy in the ensuing game, while for Player G we can pick any arbitrary choices (so that, in fact, there are many subgame-perfect equilibria, but they share the same outcome). □