14. Static Games

14.1 Interactive situations with incomplete information

An implicit assumption in game theory is that the game being played is common knowledge among the players. The expression "incomplete information" refers to situations where some of the elements of the game (e.g. the actions available to the players, the possible outcomes, the players' preferences, etc.) are *not* common knowledge. In such situations the knowledge and beliefs of the players about the game need to be made an integral part of the description of the situation. Pioneering work in this direction was done by John Harsanyi (1967, 1968), who was the recipient of the 1994 Nobel Memorial prize in economics (together with John Nash and Reinhard Selten). Harsanyi suggested a method for converting a situation of incomplete information into an extensive game with imperfect information (this is the so-called *Harsanyi transformation*). The theory of games of incomplete information has been developed for the case of von Neumann-Morgenstern payoffs and the solution concept proposed by Harsanyi is *Bayesian Nash equilibrium*, which is merely Nash equilibrium of the imperfect-information game that is obtained by applying the Harsanyi transformation.

Although the approach put forward by Harsanyi was in terms of "types" of players and of probability distributions over types, we shall develop the theory using the so-called "state-space" approach, which makes use of the interactive knowledge-belief structures developed in Chapters 8 and 9. In Chapter 16 we will explain the "type-space" approach and show how to convert one type of structure into the other.

The distinction between imperfect and incomplete information is not at all the same as that between perfect and imperfect information. To say that a game (in extensive form) has *imperfect* information is to say that there is at least one player who may have to make a choice in a situation where she does not know what choices were made previously by other players. To say that there is *incomplete* information is to say that there is at least one player who does not quite know what game she is playing.

Consider, for example, the players' preferences. In some situations it is not overly restrictive to assume that the players' preferences are common knowledge: for instance, in the game of chess, it seems quite plausible to assume that it is common knowledge that both players prefer winning to either drawing or losing (and prefer drawing to losing).

But think of a contractual dispute. Somebody claims that you owe him a sum of money and threatens to sue you if you don't pay. We can view this situation as a two-player game: you have two strategies, "pay" and "not pay", and he has two strategies, "sue (if no payment)" and "not sue (if no payment)". In order to determine your best choice, you need to know how he will respond to your refusal to pay. A lawsuit is costly for him as well as for you; if he is the "aggressive" type he will sue you; if he is the "not aggressive" type he will drop the dispute. If you don't know what type he is, you are in a situation of incomplete information.

As explained above, we have a situation of incomplete information whenever at least one of the players does not know some aspects of the game: it could be the preferences of her opponents, the choices available to her opponents or the set of possible outcomes. We shall focus almost exclusively on the case where the uncertainty concerns the preferences of the players, while everything else will be assumed to be common knowledge (for an exception see Exercise 14.2). Harsanyi argued that every situation can be reduced to this case. For example, he argued that if Player 1 does not know whether Player 2 has available only choices a and b or also choice c, we can model this as a situation where there are two possible "states of the world" in both of which Player 2 has three choices available, a, b and c, but in one of the two states choice c gives an extremely low payoff to Player 2, so that she would definitely not choose c.

The interactive knowledge-belief structures developed in Chapters 8 and 9 are sufficiently rich to model situations of incomplete information. The states in these structures can be used to express any type of uncertainty. In Chapter 10 we interpreted the states in terms of the actual choices of the players, thus representing uncertainty in the mind of a player about the behavior of another player. In that case the game was assumed to be common knowledge among the players (that is, it was the same game at every state) and what varied from one state to another was the choice of at least one player. If we want to represent a situation where one player is not sure what game she is playing, all we have to do is interpret the states in terms of games, that is, assign different games to different states.

We shall begin with games in strategic form with cardinal payoffs where only one player is uncertain about what game is being played. We use the expression "one-sided incomplete information" to refer to these situations.

14.2 One-sided complete information

Let us model the following situation: the game being played is the one shown in Figure 14.1; call this the "true" game.

		2	
		L	R
1	T	6 , 3	0 , 9
	B	3 , 3	3 , 0

Figure 14.1: The "true" game.

Player 1 knows that she is playing this game, while Player 2 is uncertain as to whether she is playing this game or a different game, shown in Figure 14.2, where the payoffs of Player 1 are different.

		2	
		L	R
1	T	0 , 3	3 , 9
	B	3 , 3	0 , 0

Figure 14.2: The alternative game in Player 2's mind.

For convenience, let us refer to the "true game" of Figure 14.1 as the game where Player 1 is of type b and the game of Figure 14.2 as the game where Player 1 is of type a. Suppose that Player 2 assigns probability $\frac{2}{3}$ to Player 1 being of type a and probability $\frac{1}{3}$ to Player 1 being of type b.

The description is not complete yet because we need to specify whether Player 1 has any uncertainty concerning the beliefs of Player 2; let us assume that the beliefs of Player 2 are common knowledge. We also need to specify whether Player 2 is uncertain about the state of mind of Player 1; let us assume that it is common knowledge that Player 2 knows that Player 1 knows what game is being played. This is a long verbal description! A picture is worth a thousand words and indeed the simple knowledge-belief structure shown in Figure 14.3 captures all of the above.

There are two states, α and β: α is interpreted as a (counterfactual) state where the game of Figure 14.2 is played, while β is interpreted as the state where the "true" game of Figure 14.1 is played. We capture the fact that latter is the "true" game, that is, the game which is actually played, by designating state β as "the true state".[1]

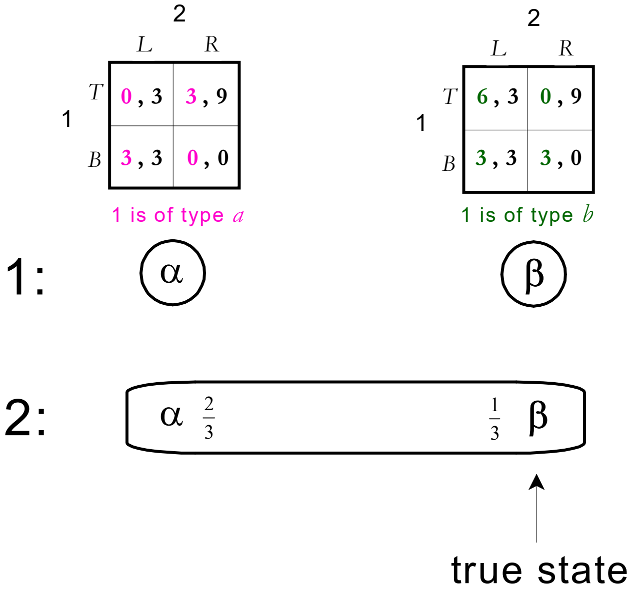

Figure 14.3: A situation of one-sided incomplete information

It is easy to check that, in the knowledge-belief structure of Figure 14.3, at state β all of the elements of the verbal description given above are true. Anything that is constant across states is common knowledge: thus (*i*) the payoffs of Player 2, (*ii*) the beliefs of Player 2, namely $\begin{pmatrix} \alpha & \beta \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$, and (*iii*) the fact that Player 1 knows what game is being played, that is, that

- if the game being played is the one of Figure 14.2 (associated with state α) then Player 1 knows that they are playing that game and
- if the game being played is the one of Figure 14.1 (associated with state β) then Player 1 knows that they are playing that game.

Note that, at the true state β, Player 1 knows more than Player 2, namely which of the two games is actually being played.

R In the situation illustrated in Figure 14.3, at every state each player knows his/her own payoffs. It may seem natural to make this a requirement of rationality: shouldn't a rational player know her own payoffs? The answer is: Yes for preferences and No for payoffs. A rational player should know how she ranks the possible outcomes (her preferences over outcomes); however, it is perfectly rational to be uncertain about what outcome will follow a particular action and thus about one's own payoff. For

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[1] This is something that is almost never done in the literature, but it is an important element of a description of a situation of incomplete information: what is the actual state of affairs?

> example, you know that you prefer accepting a wrapped box containing a thoughtful gift (payoff of 1) to accepting a wrapped box containing an insulting gift (payoff of -1): you know your preferences over these two outcomes. However, you may be uncertain about the intentions of the gift-giver and thus you are uncertain about what outcome will follow if you accept the gift: you don't know if your payoff will be 1 or -1. Thus, although you know your payoff *function*, you may be uncertain about what your *payoff* will be if you accept the gift. This example is developed in Exercise 14.2.

Figure 14.3 illustrates a situation that has to do with games, but is not a game. Harsanyi's insight was that we can transform that situation into an extensive-form game with imperfect information, as follows. We start with a chance move, where Nature chooses the state; then Player 1 is informed of Nature's choice and makes her decision between *T* and *B*; Player 2 then makes his choice between *L* and *R*, without being informed of Nature's choice (to capture her uncertainty about the game) and without being informed of Player 1's choice (to capture the simultaneity of the game). The game is shown in Figure 14.4 .

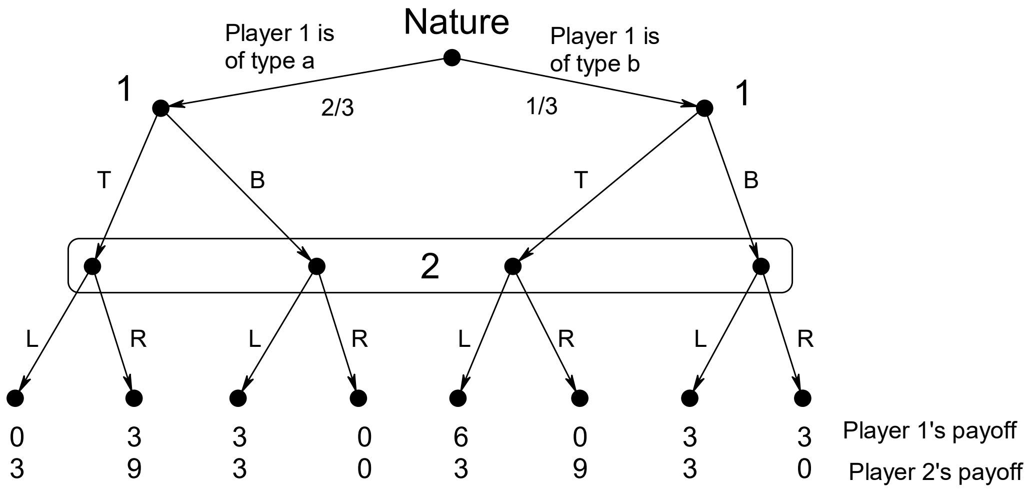

Figure 14.4: The extensive-form game obtained from the incomplete-information situation of Figure 14.3 by applying the Harsanyi transformation.

The reduction of the situation illustrated in Figure 14.3 to the extensive-form game of Figure 14.4 is called the *Harsanyi transformation.* Once a game has been obtained by means of the Harsanyi transformation, it can be solved using, for example, the notion of Nash equilibrium. The Nash equilibria of a game corresponding to a situation of incomplete information are called *Bayesian Nash equilibria.*

Note that these are nothing more than Nash equilibria: the extra term 'Bayesian' is merely a hint that the game being solved is meant to represent a situation of incomplete information.

Note also that the Harsanyi transformation involves some loss of information: in particular, in the resulting game one can no longer tell what the true state is, that is, what the actual game being played is.

To find the Bayesian Nash equilibria of the game of Figure 14.4 we can construct the corresponding strategic-form game, which is shown in Figure 14.5.[2]

		Player 2	
		L	*R*
Player 1	*T* if type *a* *T* if type *b*	2 , 3	2 , 9
	T if type *a* *B* if type *b*	1 , 3	**3 , 6**
	B if type *a* *T* if type *b*	**4 , 3**	0 , 3
	B if type *a* *B* if type *b*	3 , 3	1 , 0

Figure 14.5: The strategic-form game corresponding to the game of Figure 14.4.

There are two pure-strategy Bayesian Nash equilibria: $((T,B),R)$ and $((B,T),L)$ (where (T,B) means "T if type a and B if type b" and (B,T) means "B if type a and T if type b"). How should we interpret them?

Let us consider, for example, the Bayesian Nash equilibrium $((T,B),R)$. This is a Nash equilibrium of the game of Figure 14.4; however, we assumed that the true state was β, where Player 1 is of type b and thus, in the actual game (the one associated with state β), the actual play is (B,R), which is not a Nash equilibrium of that game (because, while B is a best reply to R, R is not a best reply to B).

This is not surprising, since Player 1 knows that she is playing that game, while Player 2 attaches only probability $\frac{1}{3}$ to that game being the actual game. Thus the first observation is that a Bayesian Nash equilibrium of a "game of incomplete information" does not imply that the players play a Nash equilibrium in the actual (or "true") game.

The second observation has to do with Player 1's strategy. By definition, Player 1's strategy in the game of Figure 14.4 consists of a pair of choices, one for the case where she is informed that her type is a and the other for the case where she is informed that her type is b: in the Bayesian Nash equilibrium under consideration, Player 1's strategy (T,B) means "Player 1 plays T if of type a and plays B if of type b". However, Player 1 knows that she is of type b: she knows what game she is playing.

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[2] Recall that, when constructing the strategic-form game associated with an extensive-form game with chance moves, payoffs are *expected* payoffs: see Chapter 7.

Why, then, should she formulate a plan on how she would play in a counterfactual world where her type was a (that is, in a counterfactual game)? The answer is that Player 1's strategy (T,B) is best understood not as a contingent plan of action formulated by Player 1, but as a complex object that incorporates (1) the actual choice (namely to play B) made by Player 1 in the game that she knows she is playing and (2) a belief in the mind of Player 2 about what Player 1 would do in the two games that, as far as Player 2 knows, are actual possibilities.

An alternative (and easier) way to find the Bayesian Nash equilibria of the game of Figure 14.4 is to use the notion of weak sequential equilibrium for that game.
For example, to verify that $((B,T),L)$ is a Nash equilibrium we can reason as follows: If the strategy of Player 1 is (B,T), then Bayesian updating requires Player 2 to have the following beliefs: probability $\frac{2}{3}$ on the second node from the left and probability $\frac{1}{3}$ on the third node from the left.

Given these beliefs, playing L yields him an expected payoff of $\frac{2}{3}(3)+\frac{1}{3}(3)=3$ while playing R yields him an expected payoff of $\frac{2}{3}(0)+\frac{1}{3}(9)=3$; thus any strategy (pure or mixed) is sequentially rational: in particular L is sequentially rational.

If Player 2 plays L then at her left node Player 1 gets 0 with T and 3 with B, so that B is sequentially rational; at her right node Player 1 gets 6 with T and 3 with B, so that T is sequentially rational. Hence $((B,T),L)$ with the stated beliefs is a weak sequential equilibrium, implying that $((B,T),L)$ is a Nash equilibrium (Theorem 11.3.1, Chapter 11).

Definition 14.2.1 In a "game of one-sided incomplete information" a pure-strategy Bayesian Nash equilibrium where the "informed" player makes the same choice at every singleton node is called a *pooling equilibrium*, while a pure-strategy Bayesian Nash equilibrium where the "informed" player makes different choices at different nodes is called a *separating equilibrium*.

Thus in the game of Figure 14.4 there are no pooling equilibria: all the pure-strategy Bayesian Nash equilibria are separating equilibria.

R Although we have only looked at the case of two players, situations of one-sided incomplete information can involve any number of players, as long as only one player is uncertain about the game being played (while all the other players are not). An example of a situation of one-sided incomplete information with three players will be given in the next chapter with reference to Selten's chain-store game (which we studied in Chapter 3, Section 3.4).

We conclude this section with an example of a situation of one-sided incomplete information involving a two-player game where each player has an infinite number of strategies. This is an incomplete-information version of Cournot's game of competition between two firms (Chapter 2, Section 2.7). This example involves the use of calculus and some readers might want to skip it.

Consider a Cournot duopoly (that is, an industry consisting of two firms, which compete in output levels) with inverse demand given by $P(Q) = 34 - Q$, where $Q = q_1 + q_2$ is total industry output (q_1 is the output of Firm 1 and q_2 is the output of Firm 2).

It is common knowledge between the two firms that Firm 1's cost function is given by: $C_1(q_1) = 6q_1$. Firm 2's cost function is $C_2(q_2) = 9q_2$. Firm 2 knows this, while Firm 1 believes that Firm 2's cost function is $C_2(q_2) = 9q_2$ with probability $\frac{1}{3}$ and $C_2(q_2) = 3q_2$ with probability $\frac{2}{3}$ (Firm 2 could be a new entrant to the industry using an old technology, or could have just invented a new technology).

Firm 1 knows that Firm 2 knows its own cost function. Everything that Firm 1 knows is common knowledge between the two firms. Thus we have the situation of one-sided incomplete information illustrated in Figure 14.6.

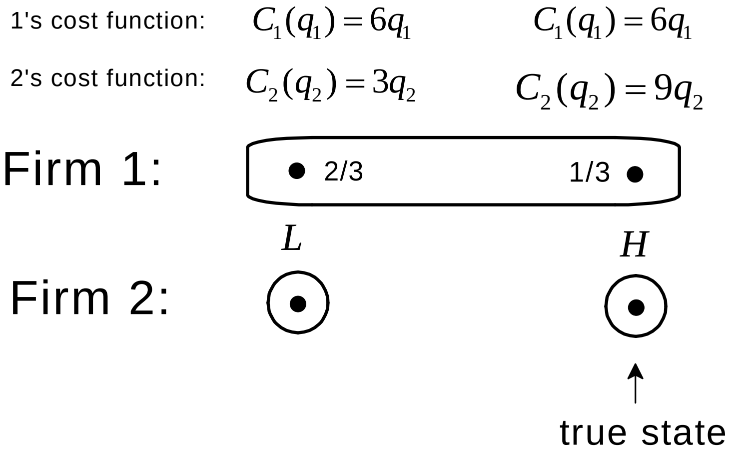

Figure 14.6: A one-sided incomplete information situation involving two firms.

Let us find a Bayesian Nash equilibrium of the game obtained by applying the Harsanyi transformation to this situation of incomplete information and compare it to the Nash equilibrium of the complete information game where Firm 2's cost function is common knowledge.

In the complete information case where it is common knowledge that Firm 2's cost function is $C_2(q_2) = 9q_2$, the profit (= payoff) functions of the two firms are given by:

$$\pi_1(q_1,q_2) = [34 - (q_1 + q_2)]\, q_1 - 6q_1$$
$$\pi_2(q_1,q_2) = [34 - (q_1 + q_2)]\, q_2 - 9q_2$$

The Nash equilibrium is given by the solution to the following pair of equations:

$$\frac{\partial \pi_1(q_1,q_2)}{\partial q_1} = 34 - 2q_1 - q_2 - 6 = 0$$
$$\frac{\partial \pi_2(q_1,q_2)}{\partial q_2} = 34 - q_1 - 2q_2 - 9 = 0$$

The solution is $q_1 = \frac{31}{3} = 10.33$ and $\mathrm{q}_2 = \frac{22}{3} = 7.33$.

Next we go back to the incomplete-information situation illustrated in Figure 14.6. Although it is not possible to draw the extensive-form game that results from applying the Harsanyi transformation (because of the infinite number of possible output levels) we can nevertheless sketch the game as shown in Figure 14.7 (where H means "high cost" and L "low cost"):

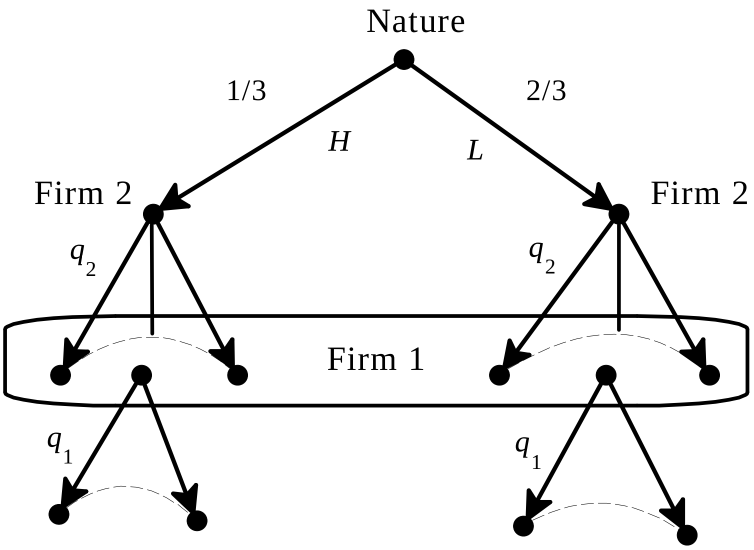

Figure 14.7: Sketch of the infinite extensive-form game obtained by applying the Harsanyi transformation to the incomplete-information situation of Figure 14.6.

To find a Bayesian Nash equilibrium it is easiest to think in terms of weak sequential equilibrium: if the strategy profile $\left(\left(\hat{q}_2^H, \hat{q}_2^L\right), \hat{q}_1\right)$ is part of a weak sequential equilibrium, then – by sequential rationality – $\hat{q}_2^H$ must maximize the expression

$$[(34 - \hat{q}_1 - q_2^H)q_2^H - 9q_2^H]$$

and $\hat{q}_2^L$ must maximize the expression

$$[(34 - \hat{q}_1 - q_2^L)q_2^L - 3q_2^L].$$

Furthermore, by Bayesian updating, Firm 1 must assign probability $\frac{1}{3}$ to the node following choice $\hat{q}_2^H$ and probability $\frac{2}{3}$ to the node following choice $\hat{q}_2^L$ so that $\hat{q}_1$ must maximize the expression

$$\tfrac{1}{3}\left[(34 - q_1 - \hat{q}_2^H)q_1 - 6q_1\right] + \tfrac{2}{3}\left[(34 - q_1 - \hat{q}_2^L)q_1 - 6q_1\right].$$

There is a unique solution given by $\hat{q}_1 = 9,\ \hat{q}_2^H = 8,\ \hat{q}_2^L = 11$. Since the true state is where Firm 2's cost is high, the actual output levels are $\hat{q}_1 = 9$ for Firm 1 and $\hat{q}_2^H = 8$ for Firm 2. Thus in the incomplete information case Firm 2's output is higher than in the complete information case and Firm 1's output is lower than in the complete information case.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 14.5.2 at the end of this chapter.

14.3 Two-sided incomplete information

Let us now consider a situation involving two players, both of whom face some uncertainty. Such a situation is called a *two-sided incomplete-information situation*. Note that for such a situation to arise, it is not necessary that both players are uncertain about some "objective" aspect of the game (such as the preferences of the opponent): one of the two players might simply be uncertain about the beliefs of the other player.

A situation of this sort is illustrated in Figure 14.8.

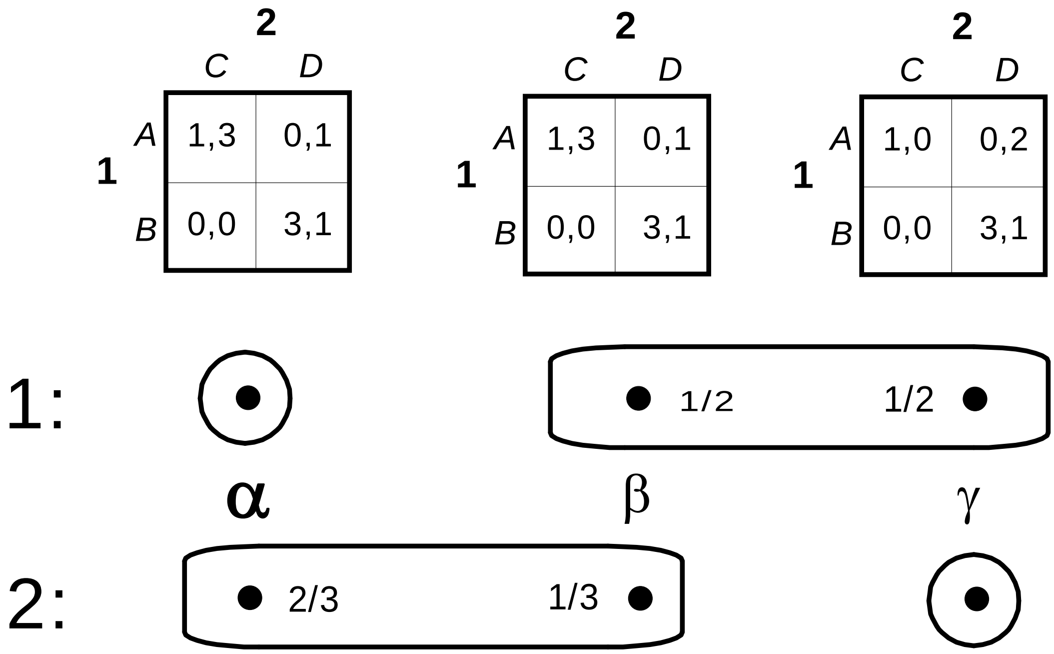

Figure 14.8: A two-sided incomplete-information situation.

Let G be the game associated with states α and β (it is the same game) and G' the game associated with state γ.

Suppose that the true state is α. Then all of the following are true at state α:

(a) both Player 1 and Player 2 know that they are playing game G (that is, neither player has any uncertainty about the objective aspects of the game),

(b) Player 1 knows that Player 2 knows that they are playing game G,

(c) Player 2 is uncertain as to whether Player 1 knows that they are playing G (which is the case if the actual state is α) or whether Player 1 is uncertain (if the actual state is β) between the possibility that they are playing game G and the possibility that they are playing game G' and considers the two possibilities equally likely;

furthermore, Player 2 attaches probability $\frac{2}{3}$ to the first case (where Player 1 knows that they are playing game G) and probability $\frac{1}{3}$ to the second case (where Player 1 is uncertain between game G and game G'),

(d) Player 1 knows the state of uncertainty of Player 2 (concerning Player 1, as described in part **(c)** above),

(e) The payoffs of Player 1 are common knowledge; furthermore, it is common knowledge that Player 2 knows his own payoffs.

In principle, the Harsanyi transformation can be applied also to situations of two-sided incomplete information. However, in such cases there is an issue concerning the probabilities with which Nature chooses the states.

In the case of one-sided incomplete information we can take Nature's probabilities to be the beliefs of the uninformed player, but in the case of two-sided incomplete information we have two uninformed players and thus two sets of beliefs.

For example, if we look at state β in Figure 14.8, we have two different probabilities assigned to that state: $\frac{1}{2}$ by Player 1 and $\frac{1}{3}$ by Player 2. Which of the two should we take as Nature's probability for state β? The answer is: neither of them. What we should take as Nature's probabilities is a probability distribution over the set of states $\{\alpha, \beta, \gamma\}$ that reflects the beliefs of *both* players. We have encountered such a notion before, in Chapter 9: it is the notion of a *common prior* (Definition 9.5.1). A *common prior* is a probability distribution over the set of states that yields the players' beliefs upon conditioning on the information represented by a cell of an information partition.

For example, in the situation illustrated in Figure 14.8 we are seeking a probability distribution

$$\nu : \{\alpha, \beta, \gamma\} \to [0,1]$$

such that

$$\nu(\beta|\{\beta,\gamma\}) = \frac{\nu(\beta)}{\nu(\beta)+\nu(\gamma)} = \tfrac{1}{2},$$

$$\nu(\alpha|\{\alpha,\beta\}) = \frac{\nu(\alpha)}{\nu(\alpha)+\nu(\beta)} = \tfrac{2}{3}.$$

(and, of course, $\nu(\alpha)+\nu(\beta)+\nu(\gamma) = 1$).

In this case a common prior exists and is given by $\nu(\alpha) = \frac{2}{4}$ and $\nu(\beta) = \nu(\gamma) = \frac{1}{4}$.

Using this common prior to assign probabilities to Nature's choices we obtain the imperfect-information game shown in Figure 14.9.

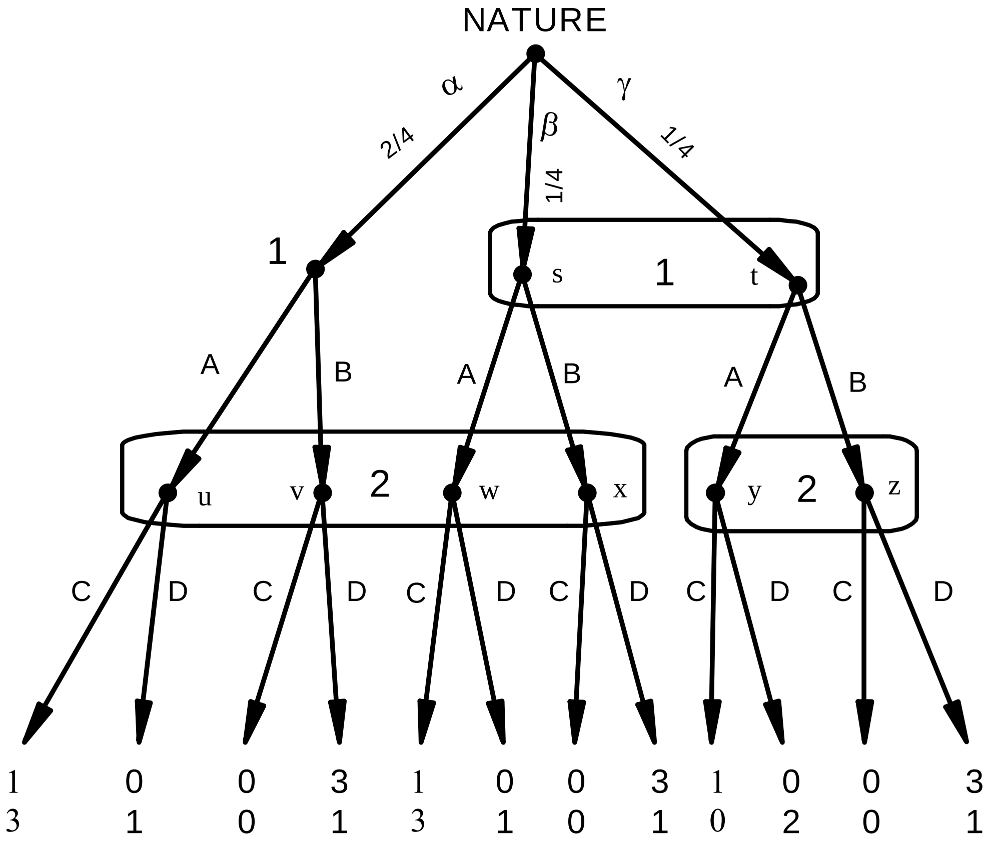

Figure 14.9: The extensive-form game obtained by applying the Harsanyi transformation to the incomplete-information situation of Figure 14.8.

The following pure-strategy profile is Bayesian Nash equilibrium of the game of Figure 14.9:

- Player 1's strategy is *AB* (that is, he plays *A* if informed that the state is α and plays *B* if informed that the state is either β or γ) and
- Player 2's strategy is *CD* (that is, she plays *C* at her information set on the left and *D* at her information set on the right).

To verify that this is a Bayesian Nash equilibrium it is easier to verify that it is a weak sequential equilibrium together with the following system of belief, obtained by using Bayesian updating (note that every information set is reached with positive probability by the strategy profile):

$$\mu = \left(\begin{array}{cc|cccc|cc} s & t & u & v & w & x & y & z \\ \frac{1}{2} & \frac{1}{2} & \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 & 1 \end{array} \right).$$

Let us check sequential rationality.

We begin with Player 1:

- at the singleton node on the left, A gives Player 1 a payoff of 1 (given Player 2's choice of C) while B gives him a payoff of 0,
  hence A is sequentially rational;
- at the information set on the right, given his beliefs and given Player 2's strategy CD, choosing A gives him an expected payoff of $\frac{1}{2}(1)+\frac{1}{2}(0)=\frac{1}{2}$
  while B gives him an expected payoff of $\frac{1}{2}(0)+\frac{1}{2}(3)=\frac{3}{2}$,
  hence B is sequentially rational.

Now consider Player 2:

- at her information set on the left, given her beliefs,
  C gives her an expected payoff of $\frac{2}{3}(3)+\frac{1}{3}(0)=2$,
  while D gives her an expected payoff of $\frac{2}{3}(1)+\frac{1}{3}(1)=1$,
  hence C is sequentially rational;
- at her information set on the right, given her beliefs,
  C gives her a payoff of 0
  while D gives her a payoff of 1,
  hence D is sequentially rational.

The existence of a common prior is essential in order to be able to apply the Harsanyi transformation to a situation of two-sided incomplete information. In some cases a common prior does *not* exist (see Exercise 14.6) and thus the Harsanyi transformation cannot be carried out.

R Besides the conceptual issues that arise in general with respect to the notion of Nash equilibrium, the notion of Bayesian Nash equilibrium for games with incomplete information raises the further issue of how one should understand or justify the notion of a common prior. This issue is not a trivial one and has been the object of debate in the literature.[3]

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 14.5.3 at the end of this chapter.

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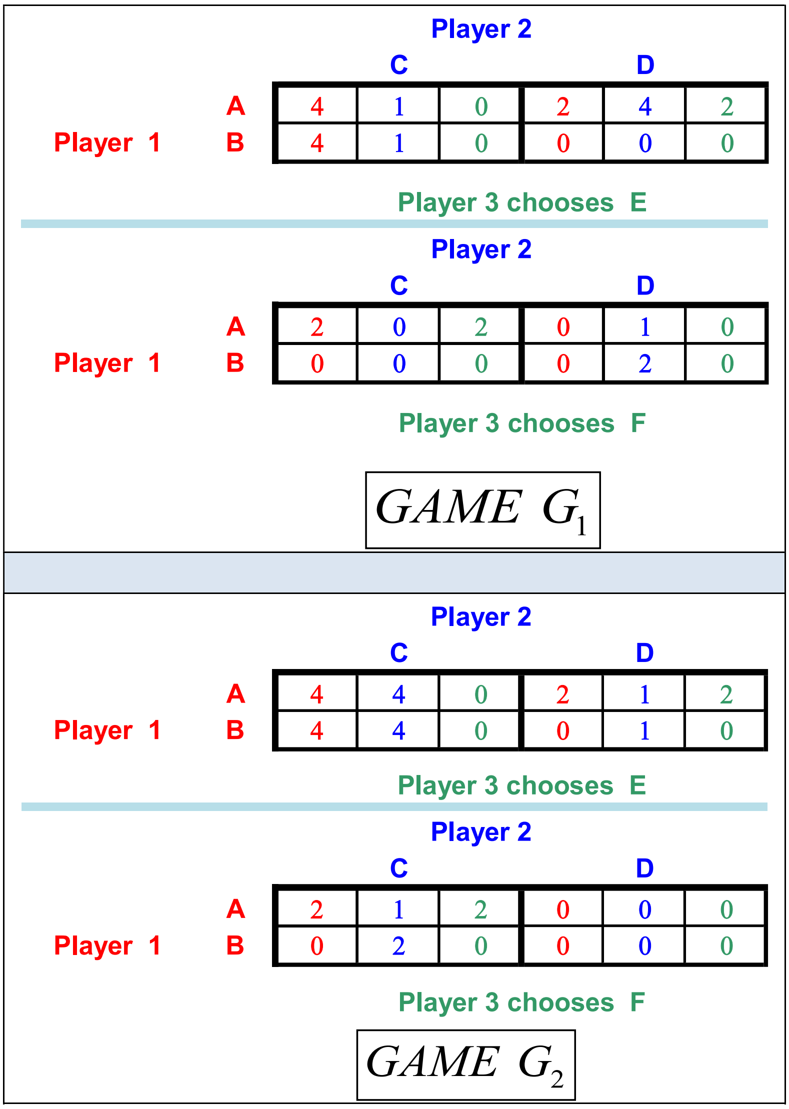
[3] See, for example, Bonanno and Nehring (1999), Gul (1998) and Morris (1995).

14.4 Multi-sided incomplete information

So far we have only considered strategic-form games with two players. However, the analysis extends easily to games involving more than two players. If there are $n \geq 3$ players and only one player has uncertainty about some aspects of the game, while the others have no uncertainty whatsoever, then we have a situation of one-sided incomplete information; if two or more players have uncertainty (not necessarily all about the game but some possibly about the beliefs of other players) then we have a *multi-sided incomplete-information situation* (the two-sided case being a special case).

We will consider an example of this below. Let G_1 and G_2 be the three-player games in strategic form (with cardinal payoffs) shown in Figure 14.10.



Player 2

Player 1

	C			D		
A	4	1	0	2	4	2
B	4	1	0	0	0	0

Player 3 chooses E

Player 2

Player 1

	C			D		
A	2	0	2	0	1	0
B	0	0	0	0	2	0

Player 3 chooses F

GAME G_1

Player 2

Player 1

	C			D		
A	4	4	0	2	1	2
B	4	4	0	0	1	0

Player 3 chooses E

Player 2

Player 1

	C			D		
A	2	1	2	0	0	0
B	0	2	0	0	0	0

Player 3 chooses F

GAME G_2

Figure 14.10: Two three-player games in strategic form with cardinal payoffs.

Consider the multi-sided situation of incomplete information illustrated in Figure 14.11, where with each state is associated one of the two games of Figure 14.10.

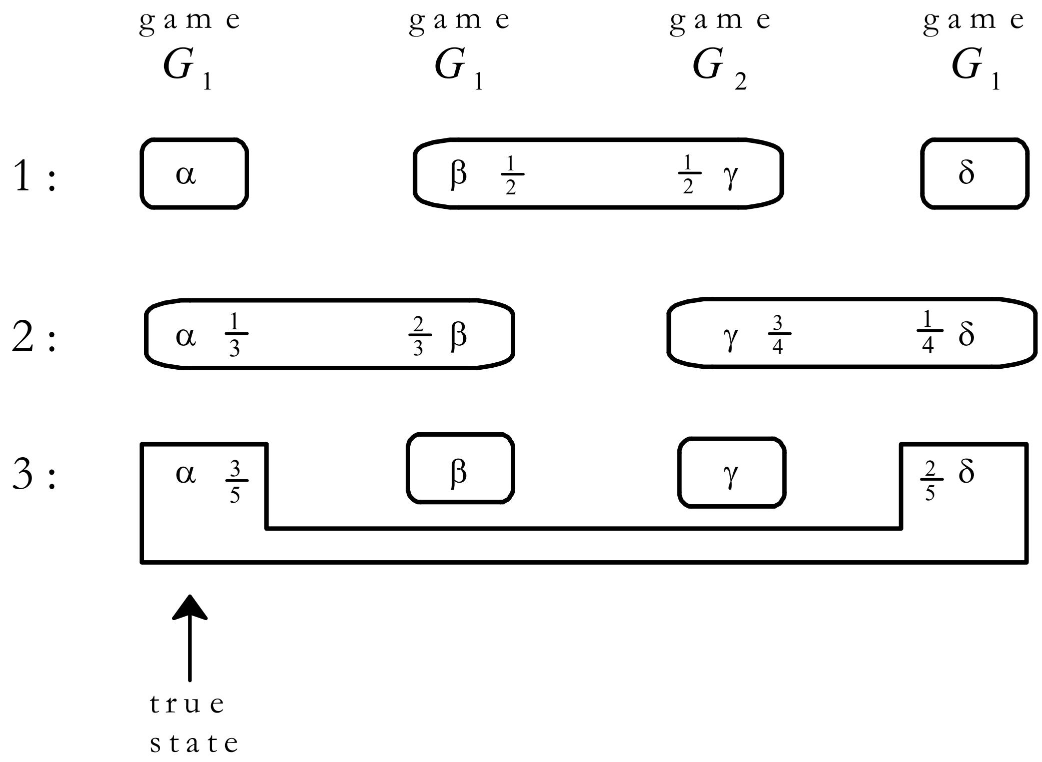


Figure 14.11: A three-sided situation of incomplete information.

At the true state α, all of the following are true:

(a) All three players know that they are playing game G_1,

(b) Player 1 knows that Players 2 and 3 know that the actual game is G_1;

(c) Player 2 knows that Player 3 knows that the actual game is G_1, but is uncertain as to whether Player 1 knows or is uncertain; furthermore, Player 2 assigns probability $\frac{2}{3}$ to Player 1 being uncertain;

(d) Player 3 knows that Player 1 knows that the actual game is G_1, but is uncertain as to whether Player 2 knows or is uncertain; furthermore, Player 3 assigns probability $\frac{2}{5}$ to Player 2 being uncertain;

(e) The payoffs of Players 1 and 3 are common knowledge.

The beliefs of the players in the situation illustrated in Figure 14.11 are compatible with each other, in the sense that there exists a common prior. In fact the following is a common prior: $\nu = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \frac{3}{17} & \frac{6}{17} & \frac{6}{17} & \frac{2}{17} \end{pmatrix}$.

Thus we can apply the Harsanyi transformation and obtain the game shown in Figure 14.12.

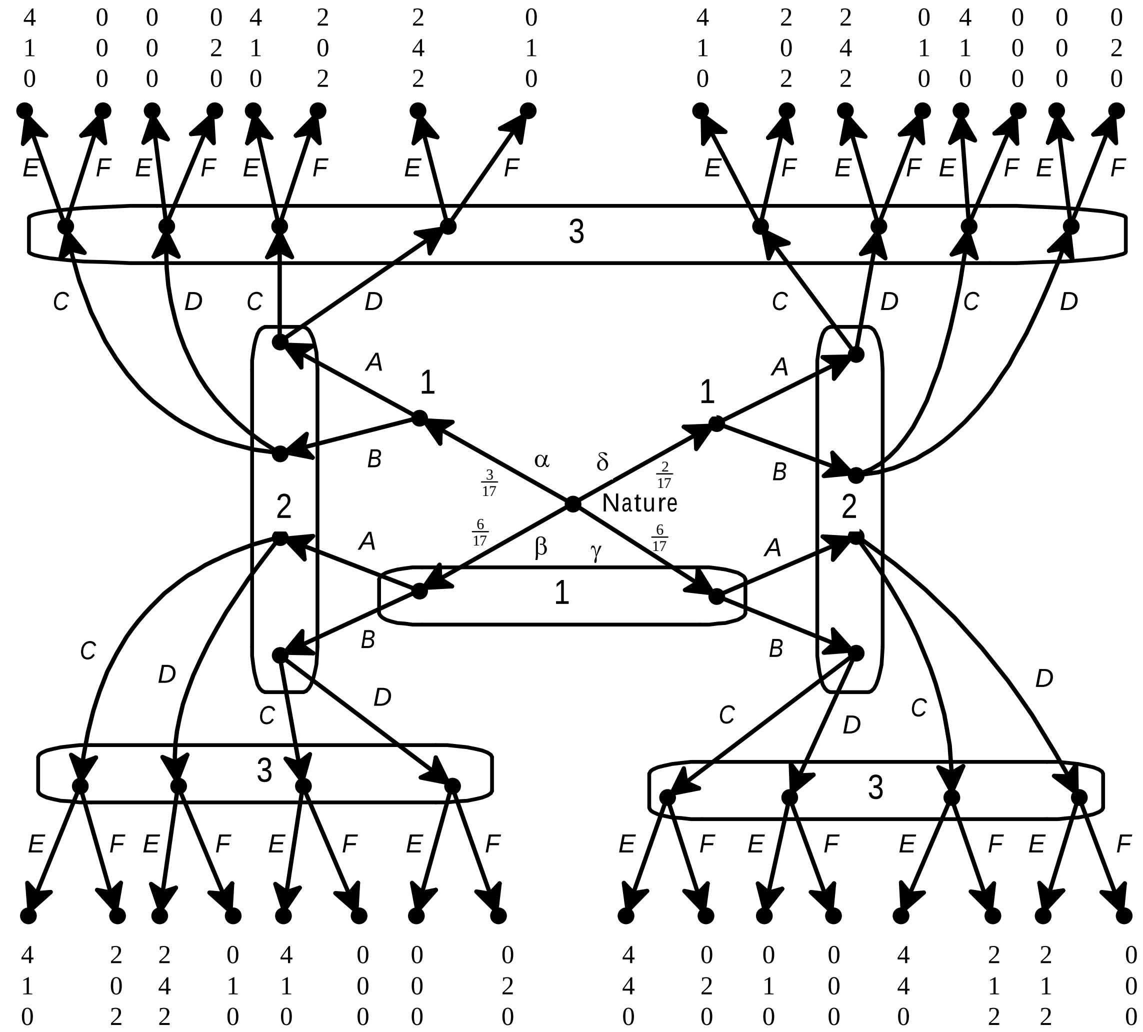


Figure 14.12: The extensive-form game obtained by applying the Harsanyi transformation to the incomplete-information situation of Figure 14.11.

The task of finding Bayesian Nash equilibria for the game of Figure 14.12 is left as an exercise (see Exercise 14.7)

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 14.5.3 at the end of this chapter.

14.5 Exercises

14.5.1 Exercises for Section 14.2: One-sided incomplete information.

The answers to the following exercises are in Section 14.6 at the end of the chapter.

Exercise 14.1

Albert and Bill play a Game of Chicken[a] under incomplete information. The situation is as follows.

- Choices are made simultaneously.
- Each player can choose “swerve” or “don’t swerve.”
  If a player swerves then he is a chicken.
  If he does not swerve, then he is a rooster if the other player swerves,
  but he gets injured if the other player does not swerve.
- Each player can be a “normal” type or a “reckless” type.
  A “normal” type gets a payoff of 100 from being a rooster, 85 from being a chicken, and zero from being injured.
  A “reckless” type gets a payoff of 100 from being a rooster, 50 from being injured, and zero from being a chicken.
- As a matter of fact, both Albert and Bill are normal; it is common knowledge between them that Bill is normal, but Bill thinks that there is a 20% chance that Albert is reckless (and an 80% chance that Albert is normal).
- Bill knows that Albert knows whether he (= Albert) is reckless or normal. Everything that Bill knows is common knowledge between Albert and Bill.

(a) Construct an interactive knowledge-belief structure that represents the situation of incomplete information described above.

(b) Apply the Harsanyi transformation to obtain an extensive-form game.

(c) Construct the strategic-form associated with the extensive-form game of part **(b)**.

(d) Find the pure-strategy Bayesian Nash equilibria and classify them as either pooling or separating (Definition 14.2.1).

■

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[a] In the original interpretation of this game, two drivers drive towards each other on a collision course: at least one must swerve in order to avoid a deadly crash; if one driver swerves and the other does not, the one who swerved will be called a "chicken", meaning a coward. Hence the name “game of chicken”.

Exercise 14.2

Bill used to be Ann's boyfriend. Today it is Ann's birthday. Bill can either not give a gift to Ann or give her a nicely wrapped gift. If offered a gift, Ann can either accept or reject. This would be a pretty simple situation, if it weren't for the fact that Ann does not know if Bill is still a friend or has become an enemy. If he is a friend, she expects a nice present from him. If Bill is an enemy, she expects to find a humiliating thing in the box (he is known to have given dead frogs to his "not so nice friends" when he was in third grade!). The preferences are as follows.

- Bill's favorite outcome (payoff = 1) occurs when he offers a gift and it is accepted (in either case: if he is a friend, he enjoys seeing her unwrap a nice present, and if he is an enemy, he revels in the cruelty of insulting Ann with a humiliating "gift"). Whether he is a friend or an enemy, Bill prefers having not extended a gift (payoff = 0) to enduring the humiliation of a rejected gift (payoff = -1).
- Ann prefers accepting a gift coming from a friend (payoff = 1) to refusing a gift (payoff = 0); the worst outcome for her (payoff = -1) is one where she accepts a gift from an enemy.
- Ann attaches probability p (with $0 < p < 1$) to the event that Bill is a friend (and $1-p$ to Bill being an enemy); however, Ann knows that Bill knows whether he is a friend or an enemy.
- Everything that Ann knows is common knowledge between Ann and Bill.
- As a matter of fact, Bill is a friend.

(a) Construct an interactive knowledge-belief structure that represents the situation of incomplete information described above.

(b) Apply the Harsanyi transformation to obtain an extensive-form game.

(c) Construct the strategic-form associated with the extensive-form game.

(d) Find all the pure-strategy Bayesian Nash equilibria and classify them as either pooling or separating.

(e) Suppose that $p = \frac{1}{4}$. Is the outcome associated with a pure-strategy Bayesian Nash equilibrium Pareto efficient?[a]

■

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[a] An outcome is Pareto efficient if there is no other outcome that is Pareto superior to it (Definition 2.2.4, Chapter 2).

14.5.2 Exercises for Section 14.3: Two-sided incomplete information.

The answers to the following exercises are in Section 14.6 at the end of the chapter.

Exercise 14.3
Consider the situation of two-sided incomplete information illustrated in Figure 14.13 (where the true state is α).

Use the Harsanyi transformation to represent this incomplete-information situation as an extensive-form game. Be explicit about how you calculated the probabilities for Nature's choices. ■

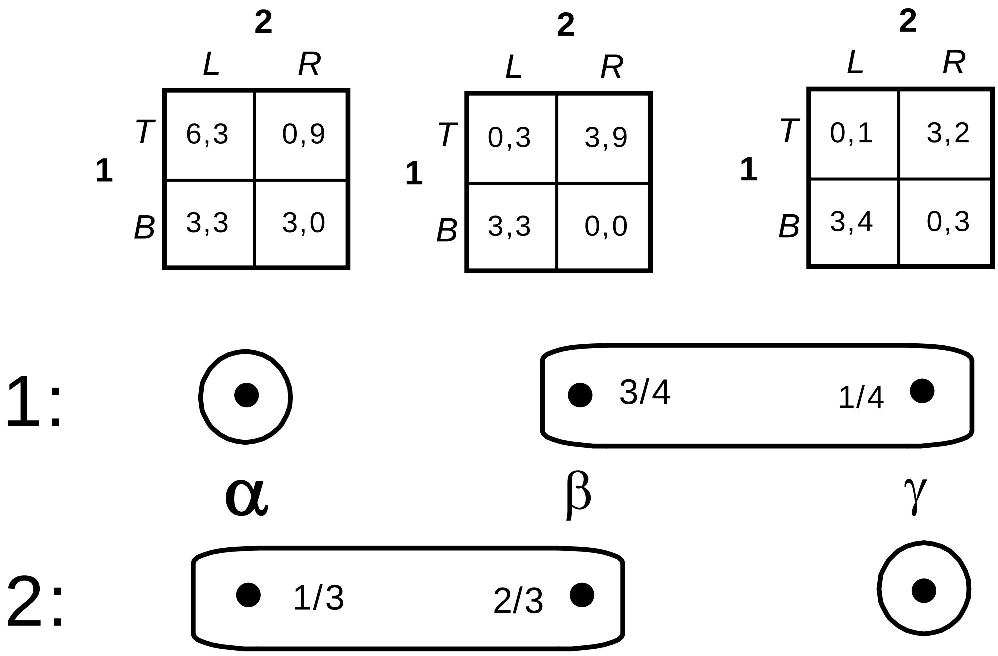


Figure 14.13: A two-sided incomplete-information situation.

Exercise 14.4

Consider the following congestion situation (a variant of the so called El Farol Bar Problem: see `https://en.wikipedia.org/wiki/El_Farol_Bar_problem`).

- ○ Two students at a college are simultaneously deciding between going to a bar or going home. The bar is extremely small and it gets congested when more than one person is there.
- ○ In principle, there are two types of students. One type, call it the b type, prefers going to the bar if he is the only customer there (in which case he gets a utility of 20) but dislikes congestion and gets a utility of -20 if he goes to the bar and is not the only customer there; furthermore, for the b type, the utility of being at home is 0.
- ○ The other type of student, call him the not-b type, prefers being at home (in which case his utility is 20) to being at the bar; being at the bar alone gives him a utility of 0 but being at the bar with other customers is very stressful and gives him a utility of -40. Let $G(b_1, b_2)$ be the game where both players are b types, $G(b_1, \text{not-}b_2)$ the game where Player 1 is a b type and Player 2 a not-b type, etc.
- ○ Assume that all payoffs are von Neumann-Morgenstern payoffs.

(a) Write the four possible strategic-form games.

(b) **(b.1)** For each of the games of part **(a)** find the pure-strategy Nash equilibria.
(b.2) For game $G(b_1, b_2)$ find also a mixed-strategy equilibrium where each choice is made with positive probability.

(c) Assume now that, as a matter of fact, both players are b types. However, it is a situation of incomplete information where it is common knowledge that each player knows his own type but is uncertain about the type of the other player and assigns probability $\frac{1}{5}$ to the other player being the same type as he is and probability $\frac{4}{5}$ to the other player being of the opposite type. Draw an interactive knowledge-belief structure that represents this situation of incomplete information.

(d) Use the Harsanyi transformation to represent the above situation of incomplete information as an extensive-form game.

(e) For the game of part **(d)**, pick one strategy of Player 1 and explain in words what it means.

(f) For the game of part **(d)**, write the corresponding strategic-form game.

(g) Find a pure-strategy Bayesian Nash equilibrium of the game of part **(d)**.

(h) For the Bayesian Nash equilibrium of part **(g)**,
  - **(h.1)** find where the players are actually going,
  - **(h.2)** find the actual payoffs of the players in the game that they are actually playing (that is, at the true state),
  - **(h.3)** do their actual choices yield a Nash equilibrium of the game that they are actually playing?

(i) If you didn't know what the true state was but you knew the game of part **(d)**, what probability would you assign to the event that the players would end-up making actual choices that constitute a Nash equilibrium of the true game that they are playing?

■

Exercise 14.5

Consider the situation of two-sided incomplete information illustrated in Figure 14.14 (where the true state is α).

(a) Use the Harsanyi transformation to represent this situation as an extensive-form game. Be explicit about how you calculated the probabilities for Nature.

(b) Write all the pure strategies of Player 1 and all the pure strategies of Player 2.

(c) Consider the following pure-strategy profile: Player 1 plays T always and Player 2 plays L always. What belief system, paired with this strategy profile, would satisfy Bayesian updating?

■

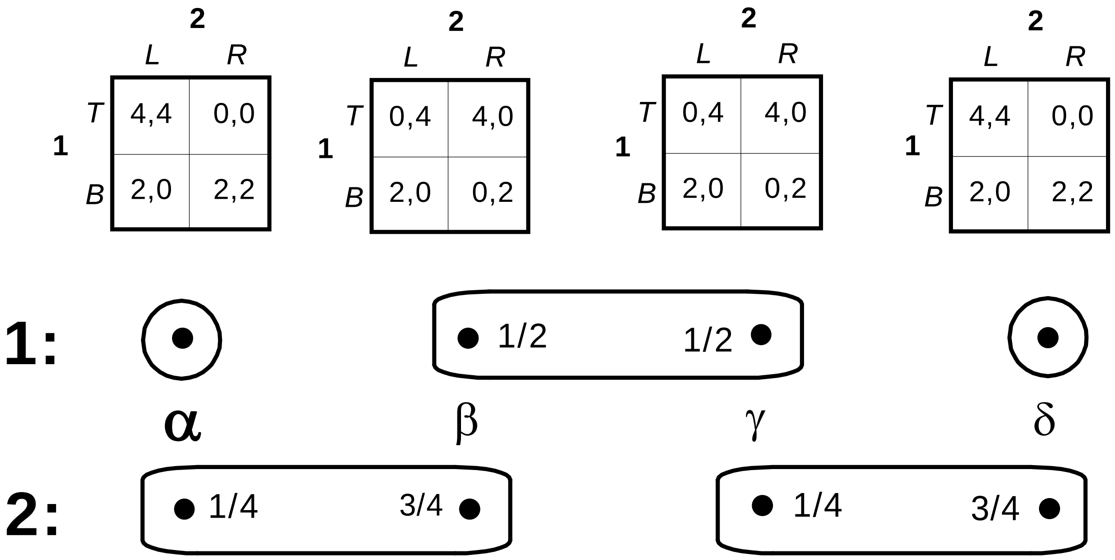


Figure 14.14: A two-sided situation of incomplete information.

14.5.3 Exercises for Section 14.4: Multi-sided incomplete information.

The answers to the following exercises are in Section 14.6 at the end of the chapter.

Exercise 14.6

Consider the three-sided situation of incomplete information shown in Figure 14.15 (what games G_1 and G_2 are is irrelevant to the following question). For what values of p can the Harsanyi transformation be carried out? ■

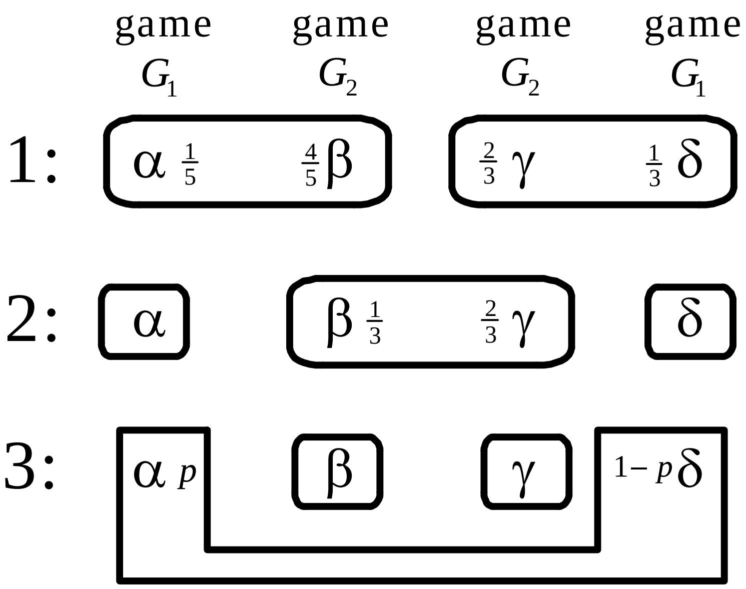
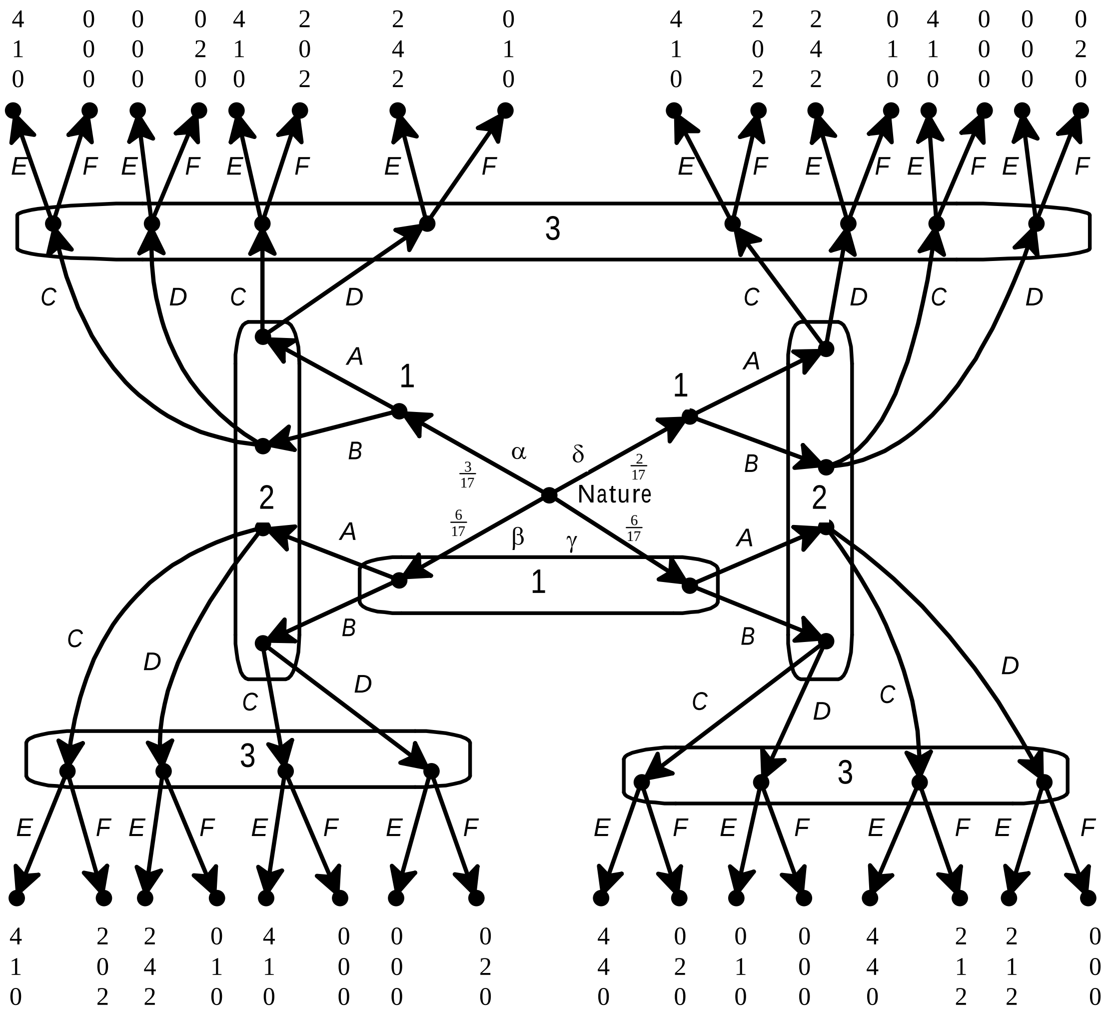


Figure 14.15: A three-sided situation of incomplete information.

Exercise 14.7

Find a pure-strategy Bayesian Nash equilibrium of the game of Figure 14.12 which is reproduced on the next page.

[Hint: it is best to think in terms of weak sequential equilibrium.] ■


Nature
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Exercise 14.8 — ⋆⋆⋆Challenging Question ⋆⋆⋆.

Consider the following situation. It is common knowledge between Players 1 and 2 that tomorrow one of three states will occur: a, b or c.

It is also common knowledge between them that

- if state a materializes, then Player 1 will only know that either a or b occurred and Player 2 will only know that either a or c occurred,
- if state b materializes, then Player 1 will only know that either a or b occurred and Player 2 will know that b occurred,
- if state c materializes, then Player 1 will know that c occurred while Player 2 will only know that either a or c occurred.

Tomorrow they will play the following *simultaneous* game: each will report, confidentially, one of her two possible states of information to a third party (for example, Player 1 can only report either $\{a,b\}$ or $\{c\}$).

Note that lying is a possibility: for example, if the state is a Player 1 can choose to report $\{c\}$.

Let R_1 be the report of Player 1 and R_2 the report of Player 2. The third party, who always knows the true state, then proceeds as follows:

(1) if the reports are compatible, in the sense that

$$R_1 \cap R_2 \neq \emptyset,$$

then he gives the players the following sums of money (which depend on $R_1 \cap R_2$):

If the true state is a :

$R_1 \cap R_2 =$	a	b	c
1 gets	\$5	\$4	\$6
2 gets	\$5	\$6	\$4

If the true state is b :

$R_1 \cap R_2 =$	a	b	c
1 gets	\$5	\$4	\$4
2 gets	\$0	\$1	\$1

If the true state is c :

$R_1 \cap R_2 =$	a	b	c
1 gets	\$0	\$1	\$1
2 gets	\$5	\$4	\$4

(2) if the reports are incompatible, in the sense that

$$R_1 \cap R_2 = \emptyset,$$

then he gives the players the following sums of money:

The true state is	a	b	c
1 gets	\$5	\$4	\$1
2 gets	\$5	\$1	\$4

(a) Represent this situation of incomplete information by means of an interactive knowledge structure (for the moment do not worry about beliefs).

(b) Apply the Harsanyi transformation to the situation represented in part **(a)** to obtain an extensive-form frame (again, at this stage, do not worry about probabilities).

(c) Suppose first that both players have no idea what the probabilities of the states are and are not willing to form subjective probabilities.

It is common knowledge that each player is selfish (i.e. only cares about how much money she herself gets) and greedy (i.e. prefers more money to less) and ranks sets of outcomes according to the worst outcome, in the sense that she is indifferent between sets X and Y if and only if the worst outcome in X is equal to the worst outcome in Y and prefers X to Y if and only if the worst outcome in X is better than the worst outcome in Y.

(c.1) Write the normal-form (or strategic-form) of the game of part **(a)**.

(c.2) Find all the pure-strategy Nash equilibria of this game.

(c.3) Among the Nash equilibria, is there one where each player tells the truth?

(d) Suppose now that it is common knowledge between the players that there are objective probabilities for the states as follows:

$$\begin{pmatrix} a & b & c \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

This time assume that it is common knowledge that both players are selfish, greedy and risk-neutral. (Thus ignore now the preferences described in part **(c)**.)

(d.1) Suppose that Player 2 expects Player 1 to report truthfully. Is it rational for Player 2 to also report truthfully?

(d.2) Is "always lying" for each player part of a pure-strategy weak sequential equilibrium? Prove your claim.

■

14.6 Solutions to Exercises

Solutions to Exercise 14.1

(a) The structure is shown in Figure 14.16 (*s* means "swerve", *ds* means "don't swerve"):

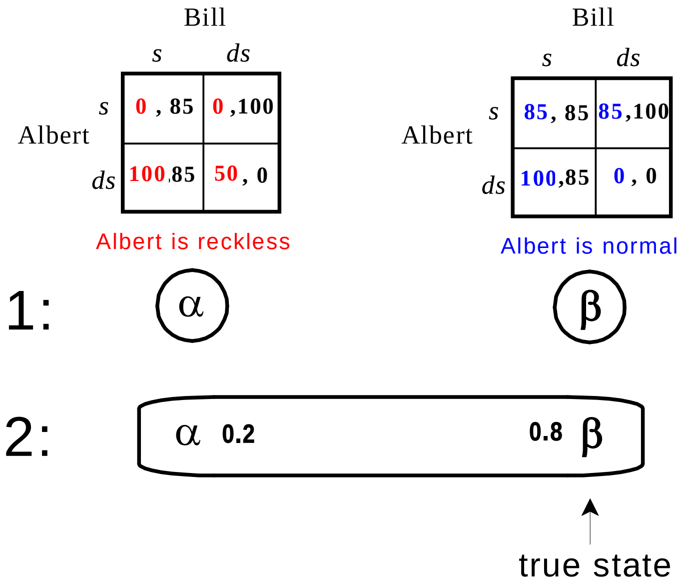

Figure 14.16: The one-sided incomplete information situation for Exercise 14.1.

(b) The Harsanyi transformation yields the game shown in Figure 14.17.

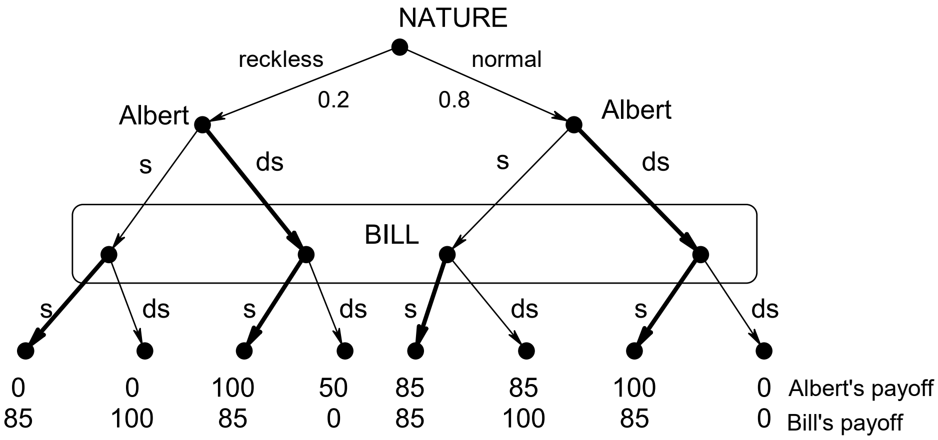

Figure 14.17: The game obtained by applying the Harsanyi transformation to the situation of Figure 14.16.

(c) The corresponding strategic-form game is shown in Figure 14.18.

Bill

		s	*ds*
Albert	*s, s*	(0.8)85 = **68 , 85**	(0.8)85 = **68 , 100**
	s, ds	(0.8)100 = **80 , 85**	**0 ,** (0.2)100 = **20**
	ds, s	(0.8)85 +(0.2)100 = **88 , 85**	(0.8)85 + (0.2)50 = **78 ,** (0.8)100 = **80**
	ds, ds	**100 , 85**	(0.2)50 = **10 , 0**

Figure 14.18: The strategic form of the game of Figure 14.17.

(d) There is only one pure-strategy Bayesian Nash equilibrium, namely $((ds, ds), s)$ and it is a pooling equilibrium.

☐

Solutions to Exercise 14.2

(a) The structure is shown in Figure 14.19
(g means "gift", ng means "no gift", a means "accept' and r means "reject").

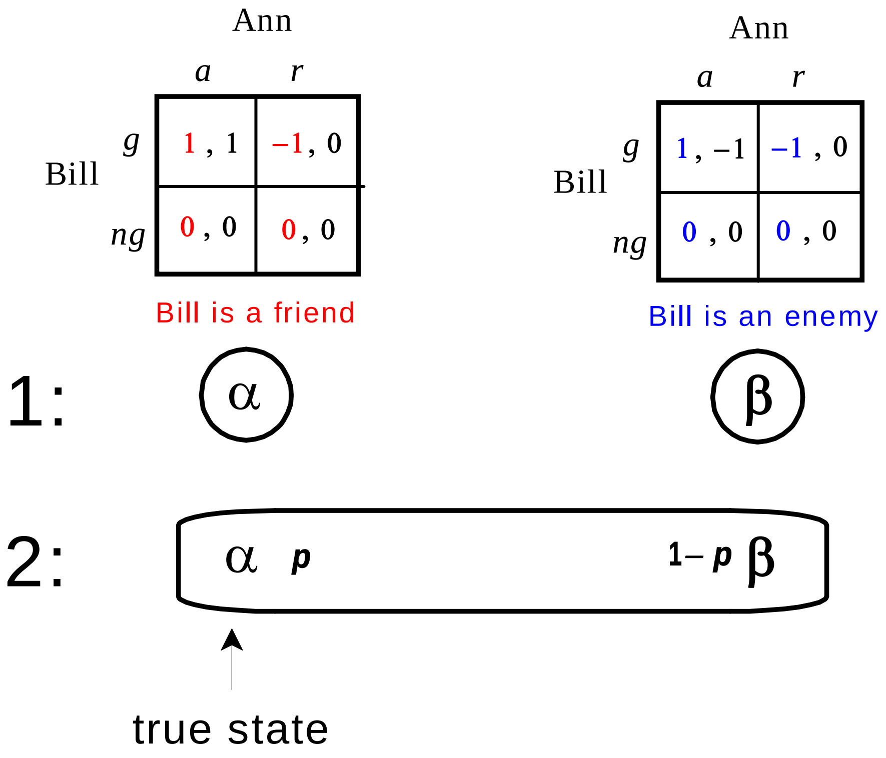

Figure 14.19: The one-sided incomplete-information structure for Exercise 14.2.

(b) The Harsanyi transformation yields the game shown in Figure 14.20.

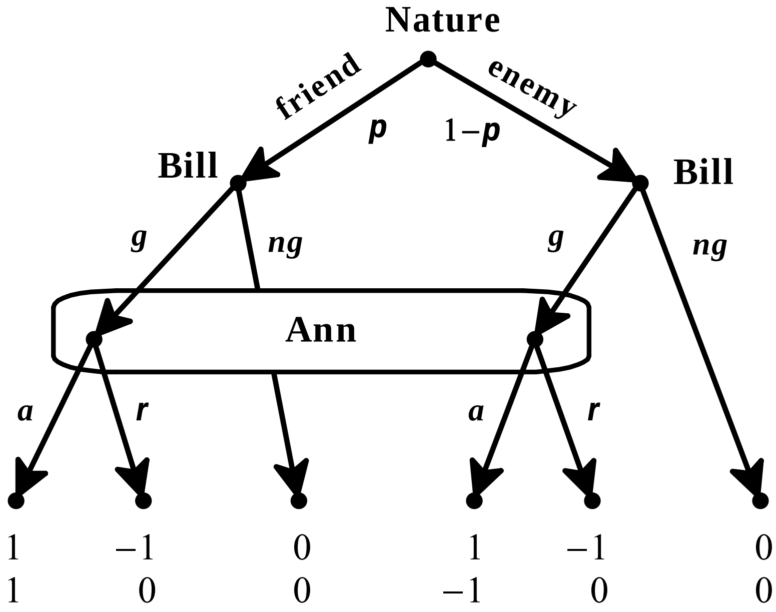

Figure 14.20: The game obtained by applying the Harsanyi transformation to the incomplete-information situation of Figure 14.19.

(c) The corresponding strategic-form game is shown in Figure 14.21.

Ann

Bill	*a*	*r*
g, g	$1\ ,\ 2p-1$	$-1\ ,\ 0$
g, ng	$p\ ,\ p$	$-p\ ,\ 0$
ng, g	$1-p\ ,\ p-1$	$p-1\ ,\ 0$
ng, ng	$0\ ,\ 0$	$0\ ,\ 0$

Figure 14.21: The strategic form of the game of Figure 14.20.

(d) If $p \geq \frac{1}{2}$ then there are two pure-strategy Bayesian Nash equilibria: $((g,g),a)$ and $((ng,ng)$, $r)$. Both of them are pooling equilibria. If $p < \frac{1}{2}$ then there is only one Bayesian -Nash equilibrium: $((ng,ng)$, $r)$ (a pooling equilibrium).

(e) If $p = \frac{1}{4}$ the only Bayesian-Nash equilibrium is $((ng,ng),r)$ and the outcome is that Bill does not offer a gift to Ann. This outcome is Pareto inefficient in the true game because, given that the true state of affairs is one where Bill is a friend, a Pareto superior outcome would be one where Bill offers a gift and Ann accepts.

□

Solutions to Exercise 14.3 The game is shown in Figure 14.22.

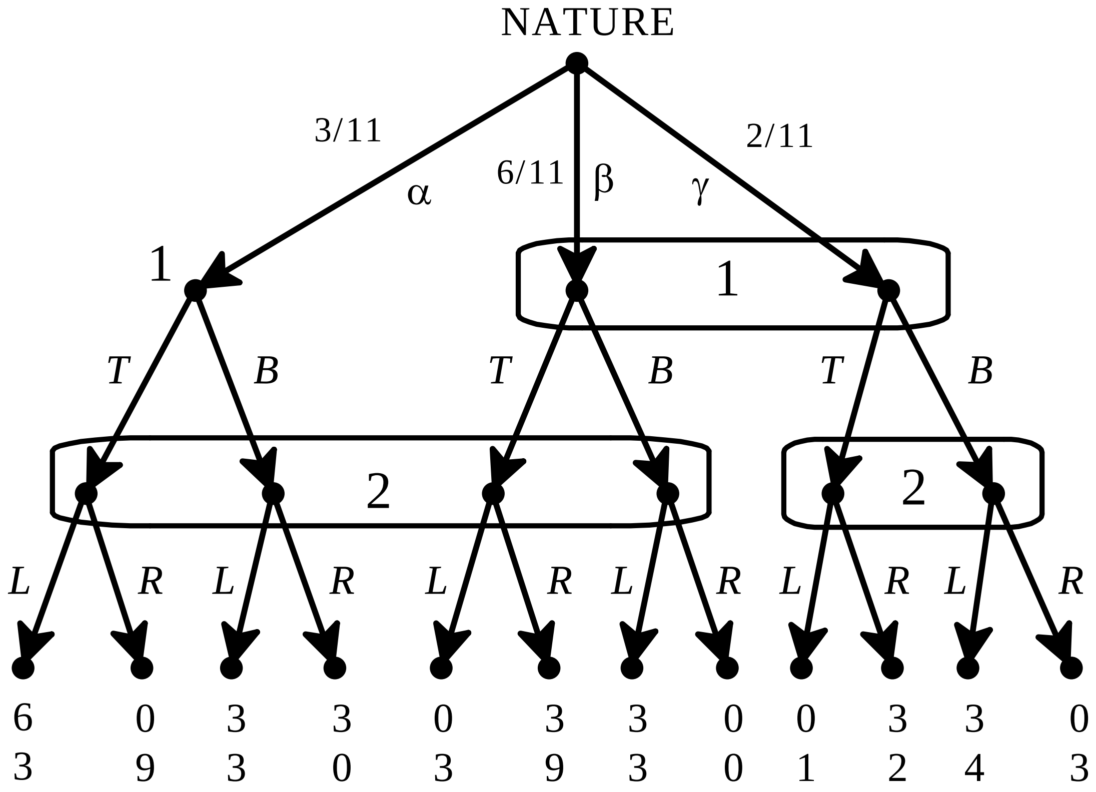

Figure 14.22: The game for exercise 14.3.

Nature's probabilities are obtained by solving the following system of equations:

$$\frac{p_\alpha}{p_\alpha + p_\beta} = \frac{1}{3}$$

$$\frac{p_\beta}{p_\beta + p_\gamma} = \frac{3}{4}$$

$$p_\alpha + p_\beta + p_\gamma = 1.$$

□

Solutions to Exercise 14.4

(a) The four games are shown in Figure 14.23.

		Player 2	
		B	H
Player 1	B	−20 −20	20 0
	H	0 20	0 0

The game $G(b_1,b_2)$.
Nash equilibria: (H,B) and (B,H)

		Player 2	
		B	H
Player 1	B	−20 −40	20 20
	H	0 0	0 20

The game $G(b_1,\text{not-}b_2)$.
Nash equilibrium: (B,H)

		Player 2	
		B	H
Player 1	B	−40 −20	0 0
	H	20 20	20 0

The game $G(\text{not-}b_1,b_2)$.
Nash equilibrium: (H,B)

		Player 2	
		B	H
Player 1	B	−40 −40	0 20
	H	20 0	20 20

The game $G(\text{not-}b_1,\text{not-}b_2)$.
Nash equilibrium: (H,H)

Figure 14.23: The four games of part **(a)** of Exercise 14.4.

(b) (b.1) The Nash equilibria are written under each figure in Figure 14.23.

(b.2) Let p be the probability that Player 1 plays B and q the probability that Player 2 plays B. Then the mixed-strategy equilibrium is given by the solution to $-20q+20(1-q)=0$ and $-20p+20(1-p)=0$, which is $p=q=\frac{1}{2}$.

(c) The structure is shown in Figure 14.24.

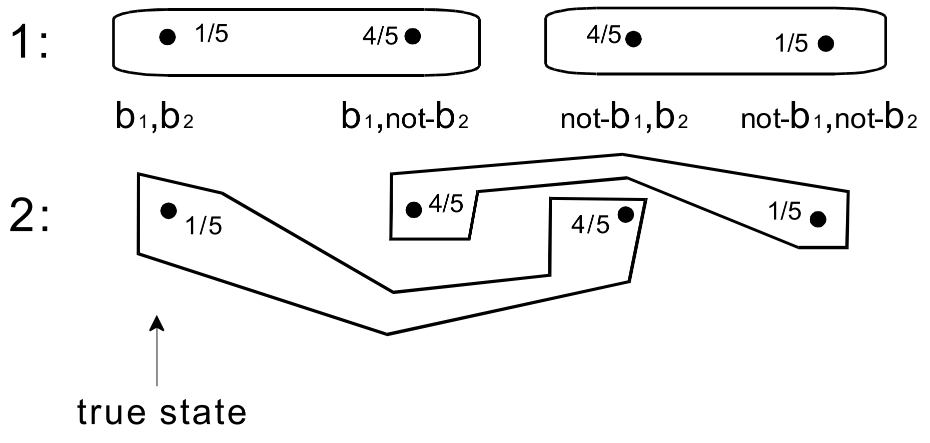

Figure 14.24: The two-sided incomplete-information structure of part **(c)** of Exercise 14.4

(d) First of all, note that the common prior is

$$\begin{pmatrix} (b_1,b_2) & (b_1,\text{not -}b_2) & (\text{not -}b_1,b_2) & (\text{not -}b_1,\text{not -}b_2) \\ \frac{1}{10} & \frac{4}{10} & \frac{4}{10} & \frac{1}{10} \end{pmatrix}.$$

The extensive-form game is shown in Figure 14.25.

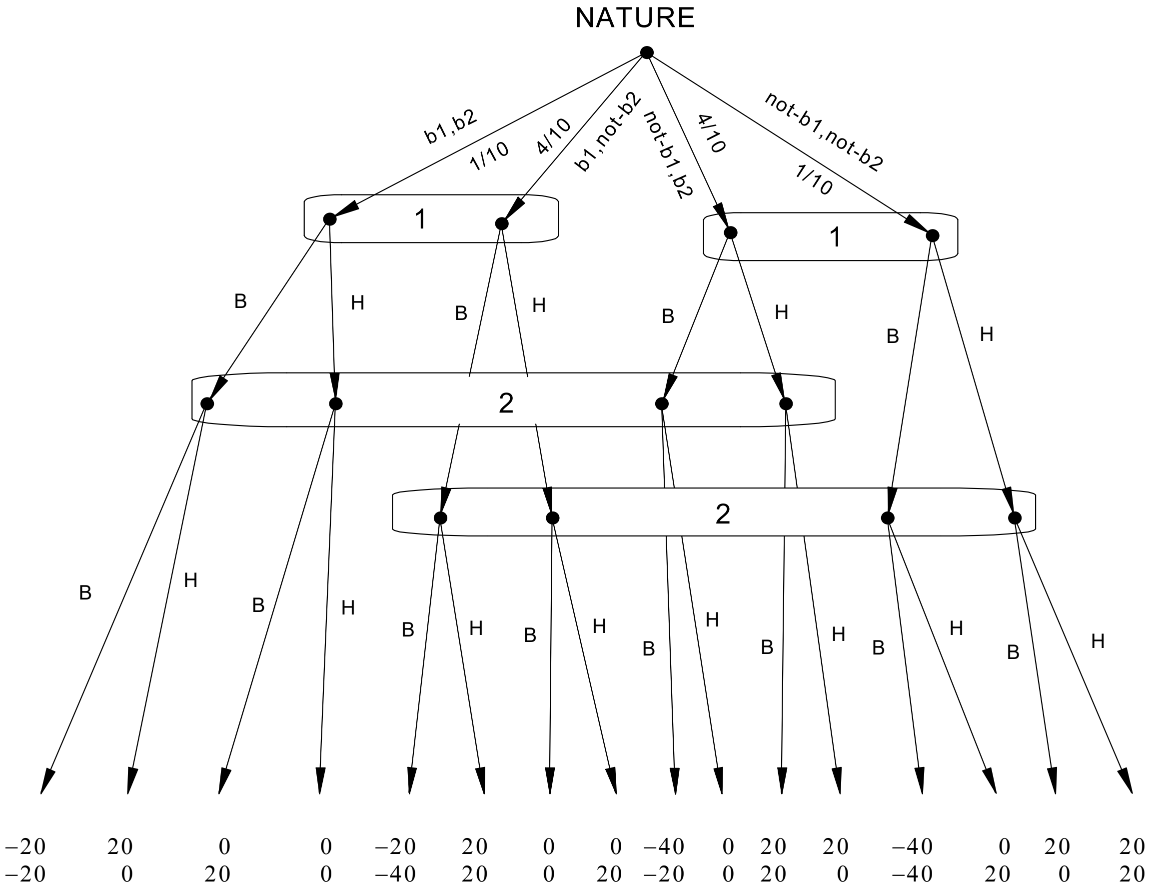

Figure 14.25: The game obtained by applying the Harsanyi transformation to the incomplete-information situation of Figure 14.24

(e) One possible strategy for Player 1 is (B,*H*) which means "if I am the *b* type then I go to the bar and if I am the not-*b* type then I go home".

(f) The strategic form is shown in Figure 14.26.

		Player 2			
		B,B	***B,H***	***H,B***	***H,H***
Player 1	***B,B***	–30, –30	–10 , 0	–10 , –20	10, 10
	B,H	0, –10	16, 16	4 , –16	20, 10
	H,B	–20 , –10	–16 , 4	–4 , –4	0, 10
	H,H	10 , 10	10 , 20	10 , 0	10 , 10

Figure 14.26: The strategic form of the game of Figure 14.25.

(g) There is a unique pure-strategy Nash equilibrium, namely $((B,H),(B,H))$ where each player goes to the bar if he is a b type and goes home if he is a not-b type. This can be found either the long way, by filling in all the payoffs in the above matrix, or by reasoning as follows: For each player, going home is strictly better than going to the bar if the player is of type not-b, no matter what she anticipates the other player doing (in other words, H strictly dominates B at the information set where not-b_i holds). Thus the question is what to do if you are of type b. You know that the other player is going home if he is of type not-b, thus you only need to consider his choice if he is of type b; if his plan is to go home also in that case, then B gives you 20 for sure and H gives you 0 for sure, hence B is better; if his plan is to go to the bar, then H gives you 0 for sure while B gives you the lottery

$$\begin{pmatrix} -20 & 20 \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix},$$

that is, an expected payoff of 12; hence B is better in that case too.

(h) At the true state, both players prefer going to the bar, thus

- **(h.1)** they both end up going to the bar and
- **(h.2)** they both get a payoff of -20.
- **(h.3)** (B,B) is not a Nash equilibrium of the game that they are actually playing (game $G(b_1,b_2)$).

(i) If the true game is $G(b_1,b_2)$ they end up playing (B,B) which is not a Nash equilibrium of that game,
if the true game is $G(b_1,\text{not - }b_2)$ they end up playing (B,H) which is a Nash equilibrium of that game,
if the true game is $G(\text{not - }b_1,b_2)$ they end up playing (H,B) which is a Nash equilibrium of that game
and if the true game is $G(\text{not - }b_1,\text{not - }b_2)$ they end up playing (H,H) which is a Nash equilibrium of that game.
Since the probability of $G(b_1,b_2)$ is $\frac{1}{10}$, the probability that they end up playing a Nash equilibrium of the actual game is $\frac{9}{10}$. □

Solutions to Exercise 14.5

(a) The game is shown in Figure 14.27.

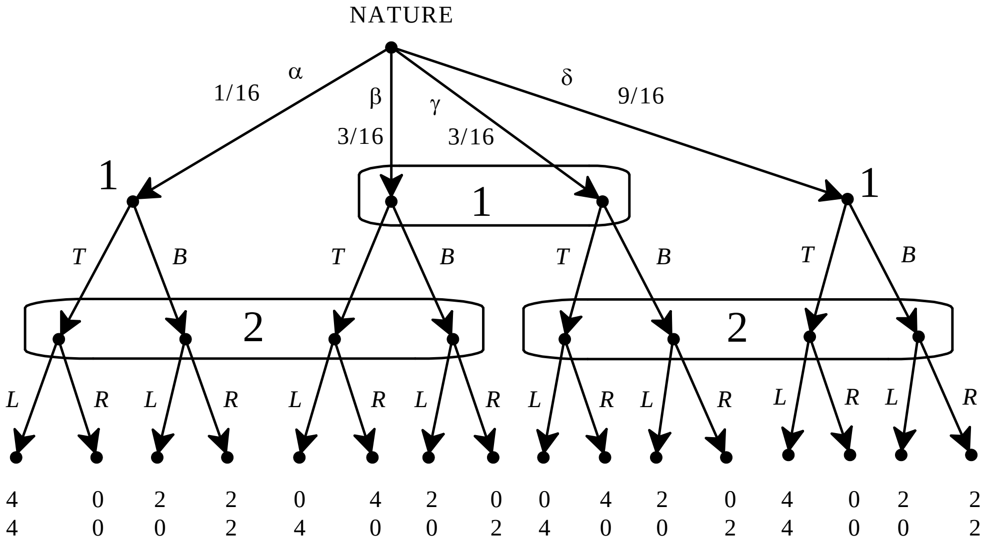

Figure 14.27: The extensive-form game for Exercise 14.5.

Nature's probabilities are obtained solving the following system of equations:

$$\frac{p_\alpha}{p_\alpha+p_\beta}=\frac{1}{4},\quad \frac{p_\beta}{p_\beta+p_\gamma}=\frac{1}{2},\quad \frac{p_\gamma}{p_\gamma+p_\delta}=\frac{3}{4},\quad p_\alpha+p_\beta+p_\gamma+p_\delta=1.$$

(b) Player 1 has 8 strategies: *TTT, TTB, TBT, TBB, BTT, BTB, BBT, BBB*. Player 2 has four strategies: *LL, LR, RL, RR*.

(c) Player 1's beliefs must be $\frac{1}{2}$ at the left node and $\frac{1}{2}$ at the right node of his middle information set.
Player 2's beliefs at her information set on the left must be: $\frac{1}{4}$ at the left-most node and $\frac{3}{4}$ at the third node from the left. The same is true for the other information set of Player 2. □

Solutions to Exercise 14.6

The Harsanyi transformation requires that there be a common prior. Thus we need a probability distribution $\nu : \{\alpha,\beta,\gamma,\delta\} \to (0,1)$ such that:

$$\textbf{(1)}\ \frac{\nu(\alpha)}{\nu(\alpha)+\nu(\beta)}=\frac{1}{5},\quad \textbf{(2)}\ \frac{\nu(\gamma)}{\nu(\gamma)+\nu(\delta)}=\frac{2}{3},\quad \textbf{(3)}\ \frac{\nu(\beta)}{\nu(\beta)+\nu(\gamma)}=\frac{1}{3}\quad \text{and}\quad \textbf{(4)}\ \frac{\nu(\alpha)}{\nu(\alpha)+\nu(\delta)}=p.$$

From **(1)** we get $\nu(\beta)=4\nu(\alpha)$, from **(2)** we get $\nu(\gamma)=2\nu(\delta)$ from **(3)** we get $\nu(\gamma)=2\nu(\delta)$; these three equalities, together with $\alpha+\beta+\gamma+\delta=1$ yield a unique solution, namely $\nu=\begin{pmatrix}\alpha & \beta & \gamma & \delta\\ \frac{1}{17} & \frac{4}{17} & \frac{8}{17} & \frac{4}{17}\end{pmatrix}$. This is a common prior if and only if

$$p=\frac{\nu(\alpha)}{\nu(\alpha)+\nu(\delta)}=\frac{\frac{1}{17}}{\frac{1}{17}+\frac{4}{17}}=\frac{1}{5}.$$

Thus $p=\frac{1}{5}$ is the only value that makes it possible to apply the Harsanyi transformation. □

Solutions to Exercise 14.7 The game under consideration is reproduced in Figure 14.28.

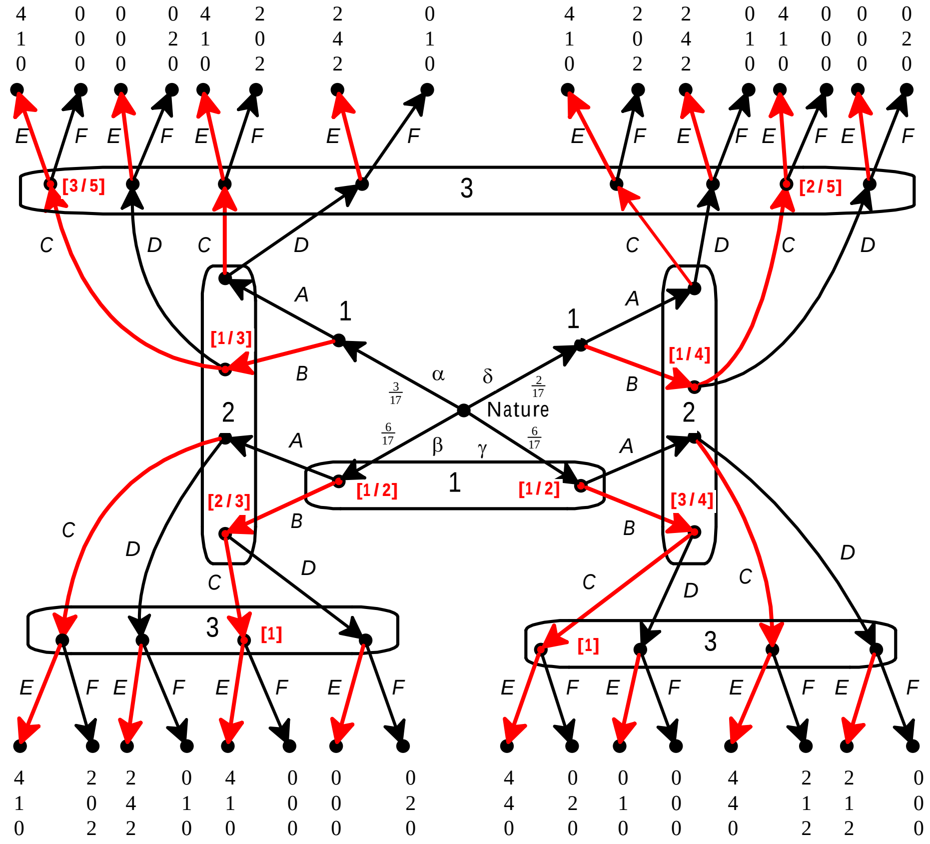

Figure 14.28: The game for Exercise 14.7.

Given the complexity of the game, it is definitely not a good idea to construct the corresponding strategic form. It is best to think in terms of weak sequential equilibrium. Consider the assessment (σ, μ), highlighted in the Figure 14.28 by thick arrows, where $\sigma = (BBB, CC, EEE)$ (that is, σ is the pure-strategy profile where Player 1 chooses B at each of his three information sets, Player 2 chooses C at each of her two information sets and Player 3 chooses E at each of her three information sets) and μ is the following system of beliefs (shown in square brackets in Figure 14.28):

- Player 1 attaches probability $\frac{1}{2}$ to each node in his bottom information set;
- Player 2, at her information set on the left, attaches probability $\frac{1}{3}$ to the second node from the top and probability $\frac{2}{3}$ to the bottom node and,
  at her information set on the right, attaches probability $\frac{1}{4}$ to the second node from the top and probability $\frac{3}{4}$ to the bottom node;
- Player 3, at her top information set, attaches probability $\frac{3}{5}$ to the left-most node and probability $\frac{2}{5}$ to the second node from the right and, at her bottom-left information set, attaches probability 1 to the third node from the left and, at her bottom-right information set, attaches probability 1 to the left-most node.

Let us verify that (σ, μ) is a weak sequential equilibrium. The beliefs described above are obtained from σ using Bayesian updating. Thus we only need to check sequential rationality.

For Player 1:

1. at the top-left node, both A and B give the same payoff (namely, 4), thus B is sequentially rational;
2. the same is true at the top-right node;
3. at the bottom information set both A and B give an expected payoff of $\frac{1}{2}(4)+\frac{1}{2}(4)=4$, thus B is sequentially rational.

For Player 2:

1. at the left information set C gives an expected payoff of $\frac{1}{3}(1)+\frac{2}{3}(1)=1$ while D gives $\frac{1}{3}(0)+\frac{2}{3}(0)=0$, thus C is sequentially rational;
2. at the information on the right C gives an expected payoff of $\frac{1}{4}(1)+\frac{3}{4}(4)=\frac{13}{4}$ while D gives an expected payoff of $\frac{1}{4}(0)+\frac{3}{4}(1)=\frac{3}{4}$, thus C is sequentially rational.

For Player 3:

1. at the top information set both E and F give an expected payoff of $\frac{3}{5}(0)+\frac{2}{5}(0)=0$, thus E is sequentially rational;
2. at the bottom-left information set both E and F give a payoff of 0, thus E is sequentially rational;
3. the same is true at the bottom-right information set. □

Solutions to Exercise 14.8

(a) The structure is shown in Figure 14.29.

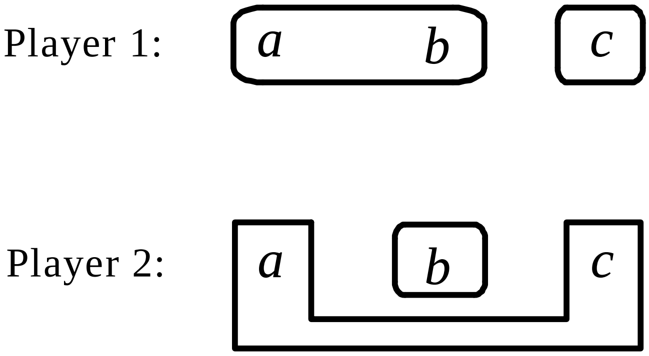

Figure 14.29: The two-sided incomplete-information structure for Exercise 14.8.

(b) The game is shown in Figure 14.30.

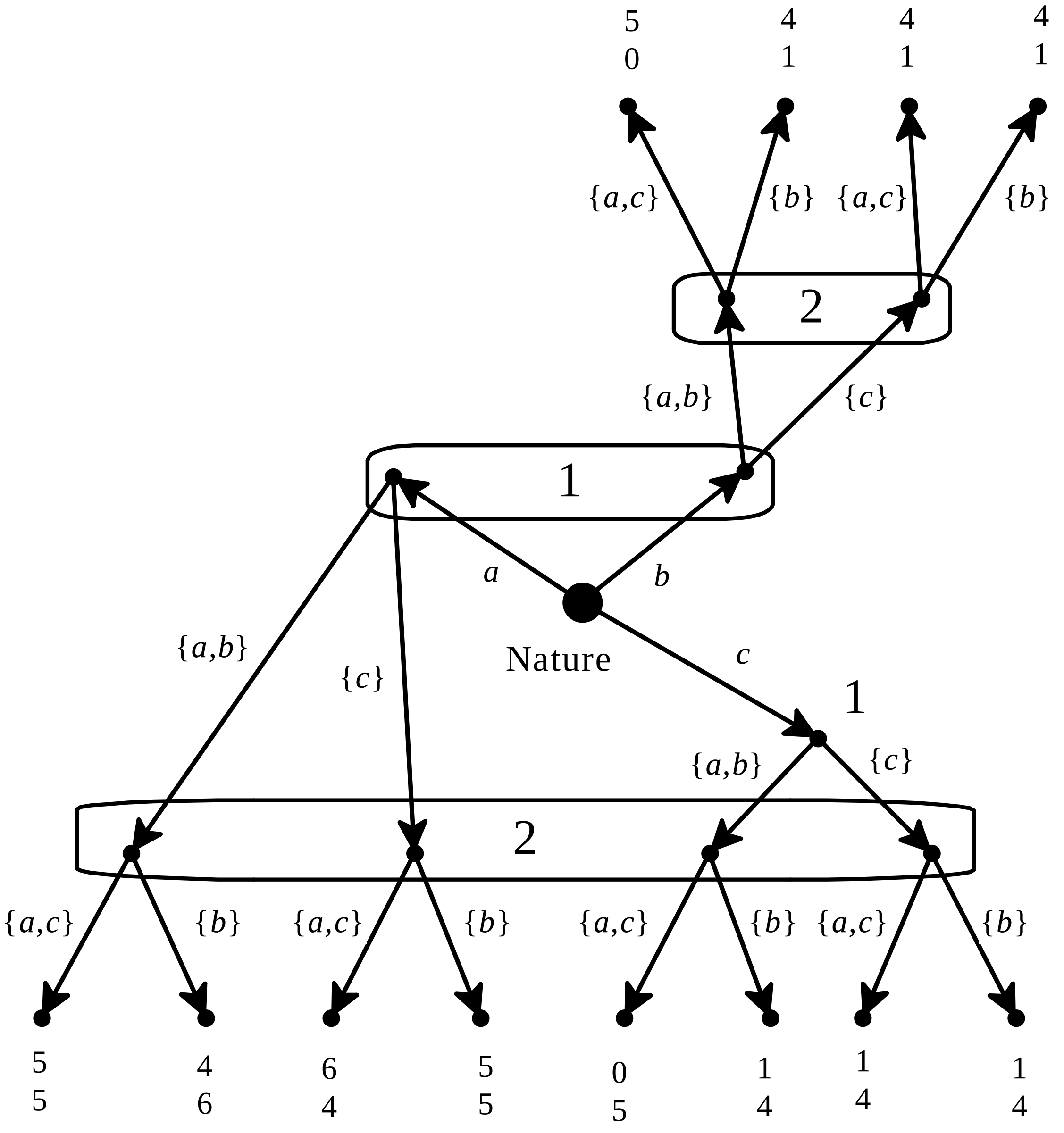

Figure 14.30: The extensive-form game for the incomplete-information situation of Figure 14.29

(c) (c.1) The strategic form is as follows (where the strategy (x,y) for Player 1 means x if $\{a,b\}$ and y if $\{c\}$ and the strategy (z,w) for Player 2 means z if $\{a,c\}$ and w if $\{b\}$). Given inside each cell are the sets of outcomes:

		Player 2			
		$(\{a,c\},\{a,c\})$	$(\{a,c\},\{b\})$	$(\{b\},\{a,c\})$	$(\{b\},\{b\})$
	$(\{a,b\},\{a,b\})$	$\{(5,5),(5,0),(0,5)\}$	$\{(5,5),(4,1),(0,5)\}$	$\{(4,6),(5,0),(1,4)\}$	$\{(4,6),(4,1),(1,4)\}$
Pl	$(\{a,b\},\{c\})$	$\{(5,5),(5,0),(1,4)\}$	$\{(5,5),(4,1),(1,4)\}$	$\{(4,6),(5,0),(1,4)\}$	$\{(4,6),(4,1),(1,4)\}$
1	$(\{c\},\{a,b\})$	$\{(6,4),(4,1),(0,5)\}$	$\{(6,4),(4,1),(0,5)\}$	$\{(5,5),(4,1),(1,4)\}$	$\{(5,5),(4,1),(1,4)\}$
	$(\{c\},\{c\})$	$\{(6,4),(4,1),(1,4)\}$	$\{(6,4),(4,1),(1,4)\}$	$\{(5,5),(4,1),(1,4)\}$	$\{(5,5),(4,1),(1,4)\}$

Taking as payoffs the smallest sum of money in each cell (for the corresponding player) the game can be written as follows:

		Player 2			
		$(\{a,c\},\{a,c\})$	$(\{a,c\},\{b\})$	$(\{b\},\{a,c\})$	$(\{b\},\{b\})$
	$(\{a,b\},\{a,b\})$	0 , 0	0 , 1	1 , 0	1 , 1
Pl	$(\{a,b\},\{c\})$	1 , 0	1 , 1	1 , 0	1 , 1
1	$(\{c\},\{a,b\})$	0 , 1	0 , 1	1 , 1	1 , 1
	$(\{c\},\{c\})$	1 , 1	1 , 1	1 , 1	1 , 1

(c.2) There are 9 Nash equilibria, namely all the strategy profiles with payoffs $(1,1)$.

(c.3) Truth telling is represented by the following strategy profile and it is one of the Nash equilibria: $\big((\{a,b\},\{c\})\ (\{a,c\},\{b\})\big)$.

(d) (d.1) No. If the state is b then it is a good idea for Player 2 to report truthfully because $\{a,c\}$ yields her 0 while $\{b\}$ yields her 1.
But if the state is either a or c then, by Bayesian updating, Player 2 must assign probability $\frac{1}{2}$ to the left-most node and probability $\frac{1}{2}$ to the right-most node of her bottom information set;
thus her expected payoff from reporting $\{a,c\}$ is $\frac{1}{2}(5)+\frac{1}{2}(4)=4.5$
while the expected payoff from reporting $\{b\}$ is $\frac{1}{2}(6)+\frac{1}{2}(4)=5$.

(d.2) Yes. "Always lie" corresponds to the strategy profile $\big((\{c\},\{a,b\})\ (\{b\},\{a,c\})\big)$.
By Bayesian updating the corresponding beliefs must be: for Player 1 $\left(\frac{2}{3},\frac{1}{3}\right)$ and for Player 2 $\left(0,\frac{1}{2},\frac{1}{2},0\right)$ at the bottom information set and (0,1) at the top information set.

Sequential rationality is then satisfied at every information set: for Player 1 at the top information set $\{c\}$ gives an expected payoff of $\frac{2}{3}(5)+\frac{1}{3}(4)=\frac{14}{3}$
while $\{a,b\}$ gives $\frac{2}{3}(4)+\frac{1}{3}(5)=\frac{13}{3}$

and at the singleton node on the right $\{a,b\}$ gives 1 and so does $\{c\}$.

For Player 2 at the bottom information set $\{b\}$ gives an expected payoff of $\frac{1}{2}(5)+\frac{1}{2}(4)=4.5$ and $\{a,c\}$ gives $\frac{1}{2}(4)+\frac{1}{2}(5)=4.5$,

and at the top information set both $\{a,c\}$ and $\{b\}$ give 1. □