7

Simultaneous-Move Games: Mixed Strategies

IN OUR STUDY of simultaneous-move games in Chapter 4, we came across a class of games that the solution methods described there could not solve; in fact, games in that class have no Nash equilibria in pure strategies. To predict outcomes for such games, we need an extension of our concepts of strategies and equilibria. This is to be found in the randomization of moves, which is the focus of this chapter.

Consider the tennis-point game from the end of Chapter 4. This game is zero sum; the interests of the two tennis players are exactly opposite. Evert wants to hit her passing shot to whichever side—down the line (DL) or crosscourt (CC)—is not covered by Navratilova, whereas Navratilova wants to cover the side to which Evert hits her shot. In Chapter 4, we pointed out that in such a situation, any systematic choice by Evert will be exploited by Navratilova to her own advantage and therefore to Evert's disadvantage. Conversely, Evert can exploit any systematic choice by Navratilova. To avoid being thus exploited, each player wants to keep the other guessing, which can be done by acting unsystematically or randomly.

However, randomness doesn't mean choosing each shot half the time or alternating between the two. The latter would itself be a systematic action open to exploitation, and a 60–40 or 75–25 random mix may be better than 50–50 depending on the situation. In this chapter, we develop methods for calculating the best mix and discuss how well this theory helps us understand actual play in such games.

Our method for calculating the best mix can also be applied to non-zero-sum games. However, in such games the players' interests can partially coincide, so when player B exploits A's systematic choice to her own advantage, it is not necessarily to A's disadvantage. Therefore, the logic of keeping the other player guessing is weaker or even absent altogether in non-zero-sum games. We will discuss whether and when mixed-strategy equilibria make sense in such games.

We start this chapter with a discussion of mixing in two-by-two games and with the most direct method for calculating best responses and finding a mixed-strategy equilibrium. Many of the concepts and methods we develop in Section 2 continue to be valid in more general games, and Sections 6 and 7 extend these methods to games where players may have more than two pure strategies. We conclude with some general observations about how to mix strategies in practice and with some evidence on whether mixing is observed in reality.

1 WHAT IS A MIXED STRATEGY?

When players choose to act unsystematically, they pick from among their pure strategies in some random way. In the tennis-point game, Navratilova and Evert each choose from two initially given pure strategies, DL and CC. We call a random mixture of these two pure strategies a mixed strategy.

Such mixed strategies cover a whole continuous range. At one extreme, DL could be chosen with probability 1 (for sure), meaning that CC is never chosen (probability 0); this "mixture" is just the pure strategy DL. At the other extreme, DL could be chosen with probability 0 and CC with probability 1; this "mixture" is the same as pure CC. In between is the whole set of possibilities: DL chosen with probability 75% (0.75) and CC with probability 25% (0.25); or both chosen with probabilities 50% (0.5) each; or DL with probability 1/3 (33.33 . . . %) and CC with probability 2/3 (66.66 . . . %); and so on.[1]

The payoffs from a mixed strategy are defined as the corresponding probability-weighted averages of the payoffs from its constituent pure strategies. For example, in the tennis game of Section 7 of Chapter 4, against Navratilova's DL, Evert's payoff from DL is 50 and from CC is 90. Therefore, the payoff of Evert's

[1] When a chance event has just two possible outcomes, people often speak of the odds in favor of or against one of the outcomes. If the two possible outcomes are labeled A and B, and the probability of A is p so that the probability of B is $(1 - p)$, then the ratio $p/(1 - p)$ gives the odds in favor of A, and the reverse ratio $(1 - p)/p$ gives the odds against A. Thus, when Evert chooses CC with probability 0.25 (25%), the odds against her choosing CC are 3 to 1, and the odds in favor of it are 1 to 3. This terminology is often used in betting contexts, so those of you who misspent your youth in that way will be more familiar with it. However, this usage does not readily extend to situations in which three or more outcomes are possible, so we avoid its use here.

mixture (0.75 DL, 0.25 CC) against Navratilova's DL is $0.75 \times 50 + 0.25 \times 90 = 37.5 + 22.5 = 60$. This is Evert's **expected payoff** from this particular mixed strategy.[2]

The probability of choosing one or the other pure strategy is a continuous variable that ranges from 0 to 1. Therefore, mixed strategies are just special kinds of continuously variable strategies like those we studied in Chapter 5. Each pure strategy is an extreme special case where the probability of choosing that pure strategy equals 1.

The notion of Nash equilibrium also extends easily to include mixed strategies. Nash equilibrium is defined as a list of mixed strategies, one for each player, such that the choice of each is her best choice, in the sense of yielding the highest expected payoff for her, given the mixed strategies of the others. Allowing for mixed strategies in a game solves the problem of possible nonexistence of Nash equilibrium, which we encountered for pure strategies, automatically and almost entirely. Nash's celebrated theorem shows that, under very general circumstances (which are broad enough to cover all the games that we meet in this book and many more besides), a Nash equilibrium in mixed strategies exists.

At this broadest level, therefore, incorporating mixed strategies into our analysis does not entail anything different from the general theory of continuous strategies developed in Chapter 5. However, the special case of mixed strategies does bring with it several special conceptual as well as methodological matters and therefore deserves separate study.

2 MIXING MOVES

We begin with the tennis example of Section 7 of Chapter 4, which did not have a Nash equilibrium in pure strategies. We show how the extension to mixed strategies remedies this deficiency, and we interpret the resulting equilibrium as one in which each player keeps the other guessing.

A. The Benefit of Mixing

We reproduce in Figure 7.1 the payoff matrix of Figure 4.14. In this game, if Evert always chooses DL, Navratilova will then cover DL and hold Evert's payoff down

[2] Game theory assumes that players will calculate and try to maximize their expected payoffs when probabilistic mixtures of strategies or outcomes are included. We consider this further in the appendix to this chapter, but for now we proceed to use it, with just one important note. The word *expected* in "expected payoff" is a technical term from probability and statistics. It merely denotes a probability-weighted average. It does not mean this is the payoff that the player should expect in the sense of regarding it as her right or entitlement.

		NAVRATILOVA	
		DL	CC
EVERT	DL	50, 50	80, 20
	CC	90, 10	20, 80

FIGURE 7.1 No Equilibrium in Pure Strategies

to 50. Similarly, if Evert always chooses CC, Navratilova will choose to cover CC and hold Evert down to 20. If Evert can only choose one of her two basic (pure) strategies and Navratilova can predict that choice, Evert's better (or less bad) pure strategy will be DL, yielding her a payoff of 50.

But suppose Evert is not restricted to using only pure strategies and can choose a mixed strategy, perhaps one in which the probability of playing DL on any one occasion is 75%, or 0.75; this makes her probability of playing CC 25%, or 0.25. Using the method outlined in Section 1, we can calculate *Navratilova's* expected payoff against this mixture as

$$0.75 \times 50 + 0.25 \times 10 = 37.5 + 2.5 = 40 \text{ if she covers DL, and}$$
$$0.75 \times 20 + 0.25 \times 80 = 15 + 20 = 35 \text{ if she covers CC.}$$

If Evert chooses this 75–25 mixture, the expected payoffs show that Navratilova can best exploit it by covering DL.

When Navratilova chooses DL to best exploit Evert's 75–25 mix, her choice works to Evert's disadvantage because this is a zero-sum game. Evert's expected payoffs are

$$0.75 \times 50 + 0.25 \times 90 = 37.5 + 22.5 = 60 \text{ if Navratilova covers DL, and}$$
$$0.75 \times 80 + 0.25 \times 20 = 60 + 5 = 65 \text{ if Navratilova covers CC.}$$

By choosing DL, Navratilova holds Evert down to 60 rather than 65. But notice that Evert's payoff with the mixture is still better than the 50 she would get by playing purely DL or the 20 she would get by playing purely CC.[3]

The 75–25 mix, while improving Evert's expected payoff relative to her pure strategies, does leave Evert's strategy open to some exploitation by Navratilova. By choosing to cover DL she can hold Evert down to a lower expected payoff than when she chooses CC. Ideally, Evert would like to find a mix that would

[3] Not every mixed strategy will perform better than the pure strategies. For example, if Evert mixes 50–50 between DL and CC, Navratilova can hold Evert's expected payoff down to 50, exactly the same as from pure DL. And a mixture that attaches a probability of less than 30% to DL will be worse for Evert than pure DL. We ask you to verify these statements as a useful exercise to acquire the skill of calculating expected payoffs and comparing strategies.

be exploitation proof—a mix that would leave Navratilova no obvious choice of pure strategy to use against it. Evert's exploitation-proof mixture must have the property that Navratilova gets the same expected payoff against it by covering DL or CC; it must keep Navratilova indifferent between her two pure strategies. We call this the **opponent's indifference property**; it is the key to mixed-strategy equilibria in non-zero-sum games, as we see later in this chapter.

To find the exploitation-proof mix requires taking a more general approach to describing Evert's mixed strategy so that we can solve algebraically for the appropriate mixture probabilities. For this approach, we denote the probability of Evert choosing DL by the algebraic symbol p, so the probability of choosing CC is $1 - p$. We refer to this mixture as Evert's p-mix for short.

Against the p-mix, Navratilova's expected payoffs are

$$50p + 10(1 - p) \text{ if she covers DL, and}$$
$$20p + 80(1 - p) \text{ if she covers CC.}$$

For Evert's strategy, her p-mix, to be exploitation proof, these two expected payoffs for Navratilova should be equal. That implies $50p + 10(1 - p) = 20p + 80(1 - p)$; or $30p = 70(1 - p)$; or $100p = 70$; or $p = 0.7$. Thus, Evert's exploitation-proof mix uses DL with probability 70% and CC with probability 30%. With these mixture probabilities, Navratilova gets the same expected payoff from each of her pure strategies and therefore cannot exploit any one of them to her advantage (or Evert's disadvantage in this zero-sum game). And Evert's expected payoff from this mixed strategy is

$$50 \times 0.7 + 90 \times 0.3 = 35 + 27 = 62 \text{ if Navratilova covers DL, and also}$$
$$80 \times 0.7 + 20 \times 0.3 = 56 + 6 = 62 \text{ if Navratilova covers CC.}$$

This expected payoff is better than the 50 that Evert would get if she used the pure strategy DL and better than the 60 from the 75–25 mixture. We now know this mixture is exploitation proof, but is it Evert's optimal or equilibrium mixture?

B. Best Responses and Equilibrium

To find the equilibrium mixtures in this game, we return to the method of best-response analysis originally described in Chapter 4 and extended to games with continuous strategies in Chapter 5. Our first task is to identify Evert's best response to—her best choice of p for—each of Navratilova's possible strategies. Since those strategies can also be mixed, they are similarly described by the probability with which she covers DL. Label this q, so $1 - q$ is the probability that Navratilova covers CC. We refer to Navratilova's mixed strategy as her q-mix and now look for Evert's best choice of p at each of Navratilova's possible choices of q.

Using Figure 7.1, we see that Evert's p-mix gets her the expected payoff

$50p + 90(1 - p)$ if Navratilova chooses DL, and
$80p + 20(1 - p)$ if Navratilova chooses CC.

Therefore against Navratilova's q-mix, Evert's expected payoff is

$$[50p + 90(1 - p)]q + [80p + 20(1 - p)]\,(1 - q)\,.$$

Rearranging the terms, Evert's expected payoff becomes

$$\begin{aligned}&[50q + 80(1 - q)]p + [90q + 20(1 - q)]\,(1 - p)\\ &= [90q + 20(1 - q)] + [50q + 80(1 - q) - 90q - 20(1 - q)]p\\ &= [20 + 70q] + [60 - 100q]p\end{aligned}$$

and we use this expected payoff to help us find Evert's best response values of p.

We are trying to identify the p that maximizes Evert's payoff at each value of q, so the key question is how her expected payoff expression varies with p. What matters is the coefficient on p: $[60 - 100q]$. Specifically, it matters whether that coefficient is positive (in which case Evert's expected payoff increases as p increases) or negative (in which case Evert's expected payoff decreases as p increases). Clearly, the sign of the coefficient depends on q, the critical value of q being the one that makes $60 - 100q = 0$. That q value is 0.6.

When Navratilova's $q < 0.6$, $[60 - 100q]$ is positive, Evert's expected payoff increases as p increases, and her best choice is $p = 1$, or the pure strategy DL. Similarly, when Navratilova's $q > 0.6$, Evert's best choice is $p = 0$, or the pure-strategy CC. If Navratilova's $q = 0.6$, Evert gets the same expected payoff regardless of p, and any mixture between DL and CC is just as good as any other; any p from 0 to 1 can be a best response. We summarize this for future reference:

If $q < 0.6$, best response is $p = 1$ (pure DL).
If $q = 0.6$, any p-mix is a best response.
If $q > 0.6$, best response is $p = 0$ (pure CC).

As a quick confirmation of intuition, observe that when q is low (Navratilova is sufficiently unlikely to cover DL), Evert should choose DL, and when q is high (Navratilova is sufficiently likely to cover DL), Evert should choose CC. The exact sense of "sufficiently," and therefore the switching point $q = 0.6$, of course depends on the specific payoffs in the example.[4]

We said earlier that mixed strategies are just a special kind of continuous strategy, with the probability being the continuous variable. Now we have found Evert's best p corresponding to each of Navratilova's choices of q. In other words,

[4] If, in some numerical problem you are trying to solve, the expected payoff lines for the pure strategies do not intersect, that would indicate that one pure strategy was best for all of the opponent's mixtures. Then this player's best response would always be that pure strategy.

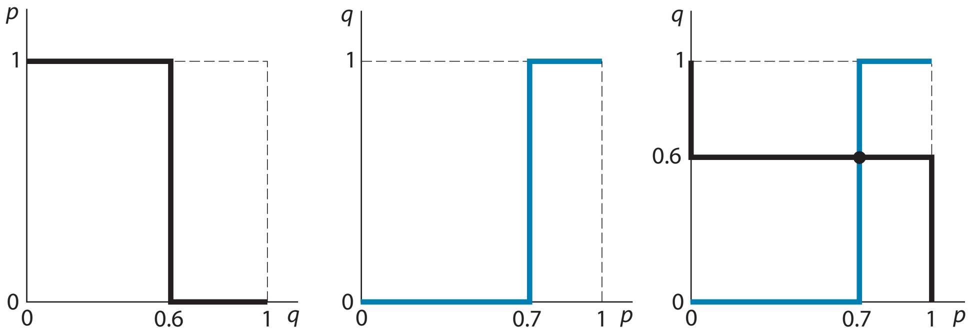

FIGURE 7.2 Best Responses and Equilibrium in the Tennis Point

we have found Evert's best-response rule, and we can graph it exactly as we did in Chapter 5.

We show this graph in the left-hand panel of Figure 7.2, with q on the horizontal axis and p on the vertical axis. Both are probabilities, limited to the range from 0 to 1. For q less than 0.6, p is at its upper limit of 1; for q greater than 0.6, p is at its lower limit of 0. At $q = 0.6$, all values of p between 0 and 1 are equally "best" for Evert; therefore the best response is the vertical line between 0 and 1. This is a new flavor of best-response graph; unlike the steadily rising or falling lines or curves of Chapter 5, it is flat over two intervals of q and jumps down in a step at the point where the two intervals meet. But conceptually it is just like any other best-response graph.

Similarly, Navratilova's best-response rule—her best q-mix corresponding to each of Evert's p-mixes—can be calculated; we leave this for you to do so you can consolidate your understanding of the idea and the algebra. You should also check the intuition of Navratilova's choices as we did for Evert. We just state the result:

If $p < 0.7$, best response is $q = 0$ (pure CC).
If $p = 0.7$, any q-mix is a best response.
If $p > 0.7$, best response is $q = 1$ (pure DL).

This best-response rule for Navratilova is graphed in the middle panel of Figure 7.2.

The right-hand panel in Figure 7.2 combines the other two panels by reflecting the left graph across the diagonal ($p = q$ line) so that p is on the horizontal axis and q on the vertical axis and then superimposing this graph on the middle graph. Now the blue and black curves meet at exactly one point, where $p = 0.7$ and $q = 0.6$. Here each player's mixture choice is a best response to the other's choice, so the pair constitutes a Nash equilibrium in mixed strategies.

This representation of best-response rules includes pure strategies as special cases corresponding to the extreme values of p and q. So we can see that the best-response curves do not have any points in common at any of the sides of the square where each value of p and q equals either 0 or 1; this shows us that the game does not have any pure-strategy equilibria, as we checked directly in Section 7 of Chapter 4. The mixed-strategy equilibrium in this example is the unique Nash equilibrium in the game.

You can also calculate Navratilova's exploitation-proof choice of q using the same method as we used in Section 2.A for finding Evert's exploitation-proof p. You will get the answer $q = 0.6$. Thus, the two exploitation-proof choices are indeed best responses to each other, and they are the Nash equilibrium mixtures for the two players.

In fact, if all you want to do is to find a mixed-strategy equilibrium of a zero-sum game where each player has just two pure strategies, you don't have to go through the detailed construction of best-response curves, graph them, and look for their intersection. You can write down the exploitation-proofness equations from Section 2.A for each player's mixture and solve them. If the solution has both probabilities between 0 and 1, you have found what you want. If the solution includes a probability that is negative, or greater than 1, then the game does not have a mixed-strategy equilibrium; you should go back and look for a pure-strategy equilibrium. For games where a player has more than two pure strategies, we examine solution techniques in Sections 6 and 7.

3 NASH EQUILIBRIUM AS A SYSTEM OF BELIEFS AND RESPONSES

When the moves in a game are simultaneous, neither player can respond to the other's actual choice. Instead, each takes her best action in light of what she thinks the other might be choosing at that instant. In Chapter 4, we called such thinking a player's belief about the other's strategy choice. We then interpreted Nash equilibrium as a configuration where such beliefs are correct, so each chooses her best response to the actual actions of the other. This concept proved useful for understanding the structures and outcomes of many important types of games, most notably the prisoners' dilemma, coordination games, and chicken.

However, in Chapter 4 we considered only pure-strategy Nash equilibria. Therefore, a hidden assumption went almost unremarked—namely, that each player was sure or confident in her belief that the other would choose a particular pure strategy. Now that we are considering more general mixed strategies, the concept of belief requires a corresponding reinterpretation.

Players may be unsure about what others might be doing. In the coordination game in Chapter 4, in which Harry wanted to meet Sally, he might be unsure whether she would go to Starbucks or Local Latte, and his belief might be that there was a 50–50 chance that she would go to either one. And in the tennis example, Evert might recognize that Navratilova was trying to keep her (Evert) guessing and would therefore be unsure of which of the available actions Navratilova would play. In Chapter 2, Section 4, we labeled this as strategic uncertainty, and in Chapter 4 we mentioned that such uncertainty can give rise to mixed-strategy equilibria. Now we develop this idea more fully.

It is important, however, to distinguish between being unsure and having incorrect beliefs. For example, in the tennis example, Navratilova cannot be sure of what Evert is choosing on any one occasion. But she can still have correct beliefs about Evert's mixture—namely, about the probabilities with which Evert chooses between her two pure strategies. Having correct beliefs about mixed actions means knowing or calculating or guessing the correct probabilities with which the other player chooses from among her underlying basic or pure actions. In the equilibrium of our example, it turned out that Evert's equilibrium mixture was 70% DL and 30% CC. If Navratilova believes that Evert will play DL with 70% probability and CC with 30% probability, then her belief, although uncertain, will be correct in equilibrium.

Thus, we have an alternative and mathematically equivalent way to define Nash equilibrium in terms of beliefs: each player forms beliefs about the probabilities of the mixture that the other is choosing and chooses her own best response to this. A Nash equilibrium in mixed strategies occurs when the beliefs are correct, in the sense just explained.

In the next section, we consider mixed strategies and their Nash equilibria in non-zero-sum games. In such games, there is no general reason that the other player's pursuit of her own interests should work against your interests. Therefore, it is not in general the case that you would want to conceal your intentions from the other player, and there is no general argument in favor of keeping the other player guessing. However, because moves are simultaneous, each player may still be subjectively unsure of what action the other is taking and therefore may have uncertain beliefs that in turn lead her to be unsure about how she should act. This can lead to mixed-strategy equilibria, and their interpretation in terms of subjectively uncertain but correct beliefs proves particularly important.

4 MIXING IN NON-ZERO-SUM GAMES

The same mathematical method used to find mixed-strategy equilibria in zero-sum games—namely, exploitation-proofness or the opponent's indifference property—can be applied to non-zero-sum games as well, and it can reveal mixed-strategy equilibria in some of them. However, in such games the players' interests may coincide to some extent. Therefore, the fact that the other player will exploit your systematic choice of strategy to her advantage need not work out to your disadvantage, as was the case with zero-sum interactions. In a coordination game of the kind we studied in Chapter 4, for example, the players are better able to coordinate if each can rely on the other's acting systematically; random actions only increase the risk of coordination failure. As a result, mixed-strategy equilibria have a weaker rationale, and sometimes no rationale at all, in non-zero-sum games. Here we examine mixed-strategy equilibria in some prominent non-zero-sum games and discuss their relevance or lack thereof.

A. Will Harry Meet Sally? Assurance, Pure Coordination, and Battle of the Sexes

We illustrate mixing in non-zero-sum games by using the assurance version of the meeting game. For your convenience, we reproduce its table (Figure 4.11) as Figure 7.3. We consider the game from Sally's perspective first. If she is confident that Harry will go to Starbucks, she also should go to Starbucks. If she is confident that Harry will go to Local Latte, so should she. But if she is unsure about Harry's choice, what is her own best choice?

To answer this question, we must give a more precise meaning to the uncertainty in Sally's mind. (The technical term for this uncertainty, in the theory of probability and statistics, is her subjective uncertainty. In the context where the uncertainty is about another player's action in a game, it is also strategic uncertainty; recall the distinctions we discussed in Chapter 2, Section 2.D.) We gain precision by stipulating the probability with which Sally thinks Harry will choose one café or the other. The probability of Harry's choosing Local Latte can be any real number between 0 and 1 (that is, between 0% and 100%). We cover all possible cases by using algebra, letting the symbol p denote the probability (in Sally's mind) that Harry chooses Starbucks; the variable p can take on any real value between 0 and 1. Then $(1 - p)$ is the probability (again in Sally's mind) that Harry chooses Local Latte. In other words, we describe Sally's strategic uncertainty as follows: she thinks that Harry is using a mixed strategy, mixing the two pure strategies, Starbucks and Local Latte, in proportions or probabilities p and $(1 - p)$, respectively. We call this mixed strategy Harry's p-mix, even though for the moment it is purely an idea in Sally's mind.

		SALLY	
		Starbucks	Local Latte
HARRY	Starbucks	1, 1	0, 0
	Local Latte	0, 0	2, 2

FIGURE 7.3 Assurance

Given her uncertainty, Sally can calculate the expected payoffs from her actions when they are played against her belief about Harry's p-mix. If she chooses Starbucks, it will yield her $1 \times p + 0 \times (1 - p) = p$. If she chooses Local Latte, it will yield her $0 \times p + 2 \times (1 - p) = 2 \times (1 - p)$. When p is high, $p > 2(1 - p)$; so if Sally is fairly sure that Harry is going to Starbucks, then she does better by also going to Starbucks. Similarly, when p is low, $p < 2(1 - p)$; if Sally is fairly sure that Harry is going to Local Latte, then she does better by going to Local Latte. If $p = 2(1 - p)$, or $3p = 2$, or $p = 2/3$, the two choices give Sally the same expected payoff. Therefore, if she believes that $p = 2/3$, she might be unsure about her own choice, so she might dither between the two.

Harry can figure this out, and that makes him unsure about Sally's choice. Thus, Harry also faces subjective strategic uncertainty. Suppose in his mind Sally will choose Starbucks with probability q and Local Latte with probability $(1 - q)$. Similar reasoning shows that Harry should choose Starbucks if $q > 2/3$ and Local Latte if $q < 2/3$. If $q = 2/3$, he will be indifferent between the two actions and unsure about his own choice.

Now we have the basis for a mixed-strategy equilibrium with $p = 2/3$ and $q = 2/3$. In such an equilibrium, these p and q values are simultaneously the actual mixture probabilities and the subjective beliefs of each player about the other's mixture probabilities. The correct beliefs sustain each player's own indifference between the two pure strategies and therefore each player's willingness to mix between the two. This matches exactly the concept of a Nash equilibrium as a system of self-fulfilling beliefs and responses described in Section 3.

The key to finding the mixed-strategy equilibrium is that Sally is willing to mix between her two pure strategies only if her subjective uncertainty about Harry's choice is just right—that is, if the value of p in Harry's p-mix is just right. Algebraically, this idea is borne out by solving for the equilibrium value of p by using the equation $p = 2(1 - p)$, which ensures that Sally gets the same expected payoff from her two pure strategies when each is matched against Harry's p-mix. When the equation holds in equilibrium, it is as if Harry's mixture probabilities are doing the job of keeping Sally indifferent. We emphasize the "as if" because in this game, Harry has no reason to keep Sally indifferent; the outcome is merely a property of the equilibrium. Still, the general idea is worth remembering: in a mixed-strategy Nash equilibrium, each person's mixture probabilities keep the other player indifferent between her pure strategies. We derived this opponent's indifference property in the zero-sum discussion above, and now we see that it remains valid even in non-zero-sum games.

However, the mixed-strategy equilibrium has some very undesirable properties in the assurance game. First, it yields both players rather low expected payoffs. The formulas for Sally's expected payoffs from her two actions, p and $2(1 - p)$, both equal $2/3$ when $p = 2/3$. Similarly, Harry's expected payoffs against Sally's equilibrium q-mix for $q = 2/3$ are also both $2/3$. Thus, each player gets 2/3 in the mixed-strategy equilibrium. In Chapter 4, we found two pure-strategy equilibria for this game; even the worse of them (both choosing Starbucks) yields the players 1 each, and the better one (both choosing Local Latte) yields them 2 each.

The reason the two players fare so badly in the mixed-strategy equilibrium is that when they choose their actions independently and randomly, they create a significant probability of going to different places; when that happens, they do not meet, and each gets a payoff of 0. Harry and Sally fail to meet if one goes to Starbucks and the other goes to Local Latte or vice versa. The probability of this happening when both are using their equilibrium mixtures is $2 \times (2/3) \times (1/3) = 4/9$.[5] Similar problems exist in the mixed-strategy equilibria of most non-zero-sum games.

A second undesirable property of the mixed-strategy equilibrium here is that it is very fragile. If either player departs ever so slightly from the exact values $p = 2/3$ or $q = 2/3$, the best choice of the other tips to one pure strategy. Once one player chooses a pure strategy, then the other also does better by choosing the same pure strategy, and play moves to one of the two pure-strategy equilibria. This instability of mixed-strategy equilibria is also common to many non-zero-sum games. However, some important non-zero-sum games do have mixed-strategy equilibria that are not so fragile. One example considered later in this chapter and in Chapter 12 is the mixed-strategy equilibrium in the game chicken, which has an interesting evolutionary interpretation.

Given the analysis of the mixed-strategy equilibrium in the assurance version of the meeting game, you can now probably guess the mixed-strategy equilibria for the related non-zero-sum meeting games. In the pure-coordination version (see Figure 4.10), the payoffs from meeting in the two cafés are the same, so the mixed-strategy equilibrium will have $p = 1/2$ and $q = 1/2$. In the battle-of-the-sexes variant (see Figure 4.12), Sally prefers to meet at Local Latte because her payoff is 2 rather than the 1 that she gets from meeting at Starbucks. Her decision hinges on whether her subjective probability of Harry's going to Starbucks is greater than or less than 2/3. (Sally's payoffs here are similar to those in the assurance version, so the critical p is the same.) Harry prefers to meet at Starbucks, so his decision hinges on whether his subjective probability of Sally's going to Starbucks is greater than or less than 1/3. Therefore, the mixed-strategy Nash equilibrium has $p = 2/3$ and $q = 1/3$.

[5] The probability that each chooses Starbucks in equilibrium is 2/3. The probability that each chooses Local Latte is 1/3. The probability that one chooses Starbucks while the other chooses Local Latte is $(2/3) \times (1/3)$. But that can happen two different ways (once when Harry chooses Starbucks and Sally chooses Local Latte, and again when the choices are reversed) so the total probability of not meeting is $2 \times (2/3) \times (1/3)$. See the appendix to this chapter for more details on the algebra of probabilities.

B. Will James Meet Dean? Chicken

The non-zero-sum game of chicken also has a mixed-strategy equilibrium that can be found using the same method developed above, although its interpretations are slightly different. Recall that this is a game between James and Dean, who are trying to *avoid* a meeting; the game table, originally introduced in Figure 4.13, is reproduced here as Figure 7.4.

If we introduce mixed strategies, James's p-mix will entail a probability p of swerving and a probability $1 - p$ of going straight. Against that p-mix, Dean gets $0 \times p - 1 \times (1 - p) = p - 1$ if he chooses Swerve and $1 \times p - 2 \times (1 - p) = 3p - 2$ if he chooses Straight. Comparing the two, we see that Dean does better by choosing swerve when $p - 1 > 3p - 2$, or when $2p < 1$, or when $p < 1/2$, that is, when p is low and James's is more likely to choose Straight. Conversely, when p is high and James is more likely to choose Swerve, then Dean does better by choosing Straight. If James' p-mix has p exactly equal to $1/2$, then Dean is indifferent between his two pure actions; he is therefore equally willing to mix between the two. Similar analysis of the game from James's perspective when considering his options against Dean's q-mix yields the same results. Therefore, $p = 1/2$ and $q = 1/2$ is a mixed-strategy equilibrium of this game.

The properties of this equilibrium have some similarities but also some differences when compared with the mixed-strategy equilibria of the meeting game. Here, each player's expected payoff in the mixed-strategy equilibrium is low $(-1/2)$. This is bad, as was the case in the meeting game, but unlike in that game, the mixed-strategy equilibrium payoff is not worse for both players than either of the two pure-strategy equilibria. In fact, because player interests are somewhat opposed here, each player will do strictly better in the mixed-strategy equilibrium than in the pure-strategy equilibrium that entails his choosing Swerve.

This mixed-strategy equilibrium is again unstable, however. If James increases his probability of choosing Straight to just slightly above 1/2, this change tips Dean's choice to pure Swerve. Then (Straight, Swerve) becomes the pure-strategy equilibrium. If James instead lowers his probability of choosing

		DEAN	
		Swerve (Chicken)	Straight (Tough)
JAMES	Swerve (Chicken)	0, 0	−1, 1
	Straight (Tough)	1, −1	−2, −2

FIGURE 7.4 Chicken

Straight slightly below 1/2, Dean chooses Straight, and the game goes to the other pure-strategy equilibrium.[6]

In this section, we found mixed-strategy equilibria in several non-zero-sum games by solving the equations that come from the opponent's indifference property. We already know from Chapter 4 that these games also have other equilibria in pure strategies. Best-response curves can give a comprehensive picture, displaying all Nash equilibria at once. As you already know all of the equilibria from the two separate analyses, we do not spend time and space graphing the best-response curves here. We merely note that when there are two pure-strategy equilibria and one mixed-strategy equilibrium, as in the examples above, you will find that the best-response curves cross in three different places, one for each of the Nash equilibria. We also invite you to graph best-response curves for similar games at the end of this chapter, with full analyses presented (as usual) in the solutions to the solved exercises.

5 GENERAL DISCUSSION OF MIXED-STRATEGY EQUILIBRIA

Now that we have seen how to find mixed-strategy equilibria in both zero-sum and non-zero-sum games, it is worthwhile to consider some additional features of these equilibria. In particular, we highlight in this section some general properties of mixed-strategy equilibria. We also introduce you to some results that seem counterintuitive at first, until you fully analyze the game in question.

A. Weak Sense of Equilibrium

The opponent's indifference property described in Section 2 implies that in a mixed-strategy equilibrium, each player gets the same expected payoff from each of her two pure strategies, and therefore also gets the same expected payoff from any mixture between them. Thus, mixed-strategy equilibria are Nash equilibria only in a weak sense. When one player is choosing her equilibrium mix, the other has no positive reason to deviate from her own equilibrium mix. But she would not do any worse if she chose another mix or even one of her pure strategies. Each player is indifferent between her pure strategies, or indeed between any mixture of them, so long as the other player is playing her correct (equilibrium) mix.

[6] In Chapter 12, we consider a different kind of stability, namely evolutionary stability. The question in the evolutionary context is whether a stable mix of Straight and Swerve choosers can arise and persist in a population of chicken players. The answer is yes, and the proportions of the two types are exactly equal to the probabilities of playing each action in the mixed-strategy equilibrium. Thus, we derive a new and different motivation for that equilibrium in this game.

This seems to undermine the basis for mixed-strategy Nash equilibria as the solution concept for games. Why should a player choose her appropriate mixture when the other player is choosing her own? Why not just do the simpler thing by choosing one of her pure strategies? After all, the expected payoff is the same. The answer is that to do so would not be a Nash equilibrium; it would not be a stable outcome, because then the other player would deviate from her mixture. If Evert says to herself "When Navratilova is choosing her best mix ($q = 0.6$), I get the same payoff from DL, CC, or any mixture. So why bother to mix; why don't I just play DL?" then Navratilova can do better by switching to her pure strategy of covering DL. Similarly, if Harry chooses pure Starbucks in the assurance meeting game, then Sally can get a higher payoff in equilibrium (1 instead of 2/3) by switching from her 50–50 mix to her pure Starbucks as well.

B. Counterintuitive Changes in Mixture Probabilities in Zero-Sum Games

Games with mixed-strategy equilibria may exhibit some features that seem counterintuitive at first glance. The most interesting of them is the change in the equilibrium mixes that follow a change in the structure of a game's payoffs. To illustrate, we return to Evert and Navratilova and their tennis point.

Suppose that Navratilova works on improving her skills covering down the line to the point where Evert's success using her DL strategy against Navratilova's covering DL drops to 30% from 50%. This improvement in Navratilova's skill alters the payoff table, including the mixed strategies for each player, from that illustrated in Figure 7.1. We present the new table in Figure 7.5.

The only change from the table in Figure 7.1 has occurred in the upper-left-hand cell, where our earlier 50 for Evert is now a 30 and the 50 for Navratilova is now a 70. This change in the payoff table does not lead to a game with a pure-strategy equilibrium because the players still have opposing interests; Navratilova still wants their choices to coincide, and Evert still wants their choices to differ. We still have a game in which mixing will occur.

But how will the equilibrium mixes in this new game differ from those calculated in Section 2? At first glance, many people would argue that Navratilova should cover DL more often now that she has gotten so much better at doing so. Thus, the assumption is that her equilibrium q-mix should be more heavily

		NAVRATILOVA	
		DL	CC
EVERT	DL	30, 70	80, 20
	CC	90, 10	20, 80

FIGURE 7.5 Changed Payoffs in the Tennis Point

weighted toward DL, and her equilibrium q should be higher than the 0.6 calculated before.

But when we calculate Navratilova's q-mix by using the condition of Evert's indifference between her two pure strategies, we get $30q + 80(1 - q) = 90q + 20(1 - q)$, or $q = 0.5$. The actual equilibrium value for q, 50%, has exactly the opposite relation to the original q of 60% than what many people's intuition predicts.

Although the intuition seems reasonable, it misses an important aspect of the theory of strategy: the interaction between the two players. Evert will also be reassessing her equilibrium mix after the change in payoffs, and Navratilova must take the new payoff structure *and* Evert's behavior into account when determining her new mix. Specifically, because Navratilova is now so much better at covering DL, Evert uses CC more often in her mix. To counter that, Navratilova covers CC more often, too.

We can see this more explicitly by calculating Evert's new mixture. Her equilibrium p must equate Navratilova's expected payoff from covering DL, $30p + 90(1 - p)$, with her expected payoff from covering CC, $80p + 20(1 - p)$. So we have $30p + 90(1 - p) = 80p + 20(1 - p)$, or $90 - 60p = 20 + 60p$, or $120p = 70$. Thus, Evert's p must be $7/12$, which is 0.583, or 58.3%. Comparing this new equilibrium p with the original 70% calculated in Section 2 shows that Evert has significantly decreased the number of times she sends her shot DL in response to Navratilova's improved skills. Evert has taken into account the fact that she is now facing an opponent with better DL coverage, and so she does better to play DL less frequently in her mixture. By virtue of this behavior, Evert makes it better for Navratilova also to decrease the frequency of her DL play. Evert would now exploit any other choice of mix by Navratilova, in particular a mix heavily favoring DL.

So is Navratilova's skill improvement wasted? No, but we must judge it properly—not by how often one strategy or the other gets used but by the resulting payoffs. When Navratilova uses her new equilibrium mix with $q = 0.5$, Evert's success percentage from either of her pure strategies is $(30 \times 0.5) + (80 \times 0.5) = (90 \times 0.5) + (20 \times 0.5) = 55$. This is less than Evert's success percentage of 62 in the original example. Thus, Navratilova's average payoff also rises from 38 to 45, and she does benefit by improving her DL coverage.

Unlike the counterintuitive result that we saw when we considered Navratilova's strategic response to the change in payoffs, we see here that her response is absolutely intuitive when considered in light of her expected payoff. In fact, players' expected payoff responses to changed payoffs can never be counterintuitive, although strategic responses, as we have seen, can be.[7] The most interesting

[7] For a general theory of the effect that changing the payoff in a particular cell has on the equilibrium mixture and the expected payoffs in equilibrium, see Vincent Crawford and Dennis Smallwood, "Comparative Statics of Mixed-Strategy Equilibria in Noncooperative Games," *Theory and Decision*, vol. 16 (May 1984), pp. 225–32.

aspect of this counterintuitive outcome in players' strategic responses is the message that it sends to tennis players and to strategic game players more generally. The result here is equivalent to saying that Navratilova should improve her down-the-line coverage so that she does not have to use it so often.

Next, we present an even more general and more surprising result about changes in mixture probabilities. The opponent's indifference condition means that each player's equilibrium mixture probabilities depend only on the other player's payoffs, not on her own. Consider the assurance game of Figure 7.3. Suppose Sally's payoff from meeting in Local Latte increases from 2 to 3, while all other payoffs remain unchanged. Now, against Harry's p-mix, Sally gets $1 \times p + 0 \times (1 - p) = p$ if she chooses Starbucks, and $0 \times p + 3 \times (1 - p) = 3 - 3p$ if she chooses Local Latte. Her indifference condition is $p = 3 - 3p$, or $4p = 3$, or $p = 3/4$, compared with the value of $2/3$ we found earlier for Harry's p-mix in the original game. The calculation of Harry's indifference condition is unchanged and yields $q = 2/3$ for Sally's equilibrium strategy. The change in Sally's payoffs changes Harry's mixture probabilities, not Sally's! In Exercise S13, you will have the opportunity to prove that this is true quite generally: my equilibrium mixing proportions do not change with my own payoffs, only with my opponent's payoffs.

C. Risky and Safe Choices in Zero-Sum Games

In sports, some strategies are relatively safe; they do not fail disastrously even if anticipated by the opponent but do not do very much better even if unanticipated. Other strategies are risky; they do brilliantly if the other side is not prepared for them but fail miserably if the other side is ready. In American football, on third down with a yard to go, a run up the middle is safe and a long pass is risky. An interesting question arises because some third-and-one situations have more at stake than others. For example, making the play from your opponent's 10-yard line has a much greater impact on a possible score than making the play from your own 20-yard line. The question is, when the stakes are higher, should you play the risky strategy more or less often than when the stakes are lower?

To make this concrete, consider the success probabilities shown in Figure 7.6. (Note that, while in the tennis game we used percentages between 0 and 100, here we use probabilities between 0 and 1.) The offense's safe play is the run; the probability of a successful first down is 60% if the defense anticipates a run versus 70% if the defense anticipates a pass. The offense's risky play is the pass because the success probability depends much more on what the defense does; the probability of success is 80% if the defense anticipates a run and only 30% if it anticipates a pass.

		DEFENSE EXPECTS	
		Run	Pass
OFFENSE PLAYS	Run	0.6	0.7
	Pass	0.8	0.3

FIGURE 7.6 Probability of Offense's Success on Third Down with One Yard to Go

Suppose that when the offense succeeds with its play, it earns a payoff equal to V, and if the play fails the payoff is 0. The payoff V could be some number of points, such as three for a field-goal situation or seven for a touchdown situation. Alternatively, it could represent some amount of status or money that the team earns, perhaps $V = 100$ for succeeding in a game-winning play in an ordinary game or $V = 1{,}000{,}000$ for clinching victory in the Super Bowl.[8]

The actual game table between Offense and Defense, illustrated in Figure 7.7, contains expected payoffs to each player. Those expected payoffs average between the success payoff of V and the failure payoff of 0. For example, the expected payoff to the Offense of playing Run when the Defense expects Run is: $0.6 \times V + 0.4 \times 0 = 0.6V$. The zero-sum nature of the game means the Defense's payoff in that cell is $-0.6V$. You can similarly compute the expected payoffs for each other cell of the table to verify that the payoffs shown below are correct.

In the mixed-strategy equilibrium, Offense's probability p of choosing Run is determined by the opponent's indifference property. The correct p therefore satisfies:

$$p[-0.6V] + (1 - p)[-0.8V] = p[-0.7V] + (1 - p)[-0.3V].$$

Notice that we can divide both sides of this equation by V to eliminate V entirely from the calculation for p.[9] Then the simplified equation becomes $-0.6p - 0.8(1 - p) = -0.7p - 0.3(1 - p)$, or $0.1p = 0.5(1 - p)$. Solving this reduced equation yields $p = 5/6$, so Offense will play Run with high probability in its optimal mixture. This safer play is often called the "percentage play" because it is the normal play in such situations. The risky play (Pass) is played only occasionally to keep the opponent guessing or, in football commentators' terminology, "to keep the defense honest."

[8] Note that V is not necessarily a monetary amount; it can be an amount of utility that captures aversion to risk. We investigate issues pertaining to risk in great detail in Chapter 8 and attitudes toward risk and expected utility in the appendix to that chapter.

[9] This result comes from the fact that we can eliminate V entirely from the opponent's indifference equation, so it does not depend on the particular success probabilities specified in Figure 7.6. The result is therefore quite general for mixed-strategy games where each payoff equals a success probability times a success value.

		DEFENSE	
		Run	Pass
OFFENSE	Run	$0.6V$, $-0.6V$	$0.7V$, $-0.7V$
	Pass	$0.8V$, $-0.8V$	$0.3V$, $-0.3V$

FIGURE 7.7 The Third-and-One Game

The interesting part of this result is that the expression for p is completely independent of V. That is, the theory says that you should mix the percentage play and the risky play in exactly the same proportions on a big occasion as you would on a minor occasion. This result runs against the intuition of many people. They think that the risky play should be engaged in less often when the occasion is more important. Throwing a long pass on third down with a yard to go may be fine on an ordinary Sunday afternoon in October, but doing so in the Super Bowl is too risky.

So which is right: theory or intuition? We suspect that readers will be divided on this issue. Some will think that the sports commentators are wrong and will be glad to have found a theoretical argument to refute their claims. Others will side with the commentators and argue that bigger occasions call for safer play. Still others may think that bigger risks should be taken when the prizes are bigger, but even they will find no support in the theory, which says that the size of the prize or the loss should make no difference to the mixture probabilities.

On many previous occasions when discrepancies between theory and intuition arose, we argued that the discrepancies were only apparent, that they were the result of failing to make the theory sufficiently general or rich enough to capture all the features of the situation that created the intuition, and that improving the theory removed the discrepancy. This one is different: the problem is fundamental to the calculation of payoffs from mixed strategies as probability-weighted averages or expected payoffs. And almost all of existing game theory has this starting point.[10]

[10] Vincent P. Crawford, "Equilibrium Without Independence," *Journal of Economic Theory*, vol. 50, no. 1 (February 1990), pp. 127–54; and James Dow and Sergio Werlang, "Nash Equilibrium Under Knightian Uncertainty," *Journal of Economic Theory*, vol. 64, no. 2 (December 1994), pp. 305–24, are among the few research papers that suggest alternative foundations for game theory. And our exposition of this problem in the first edition of this book inspired an article that uses such new methods on it: Simon Grant, Atsushi Kaji, and Ben Polak, "Third Down and a Yard to Go: Recursive Expected Utility and the Dixit-Skeath Conundrum," *Economic Letters*, vol. 73, no. 3 (December 2001), pp. 275–86. Unfortunately, it uses more advanced concepts than those available at the introductory level of this book.

6 MIXING WHEN ONE PLAYER HAS THREE OR MORE PURE STRATEGIES

Our discussion of mixed strategies to this point has been confined to games in which each player has only two pure strategies, as well as mixes between them. In many strategic situations, each player has available a larger number of pure strategies, and we should be ready to calculate equilibrium mixes for those cases as well. However, these calculations get complicated quite quickly. For truly complex games, we would turn to a computer to find the mixed-strategy equilibrium. But for some small games, it is possible to calculate equilibria by hand quite easily. The calculation process gives us a better understanding of how the equilibrium works than can be obtained just from looking at a computer-generated solution. Therefore, in this section and the next one, we solve some larger games.

Here we consider zero-sum games in which one of the players has only two pure strategies, whereas the other has more. In such games, we find that the player who has three (or more) pure strategies typically uses only two of them in equilibrium. The others do not figure in his mix; they get zero probabilities. We must determine which ones are used and which ones are not.[11]

Our example is that of the tennis-point game augmented by giving Evert a third type of return. In addition to going down the line or crosscourt, she now can consider using a lob (a slower but higher and longer return). The equilibrium depends on the payoffs of the lob against each of Navratilova's two defensive stances. We begin with the case that is most likely to arise and then consider a coincidental or exceptional case.

A. A General Case

Evert now has three pure strategies in her repertoire: DL, CC, and Lob. We leave Navratilova with just two pure strategies, Cover DL or Cover CC. The payoff table for this new game can be obtained by adding a Lob row to the table in Figure 7.1. The result is shown in Figure 7.8. We have assumed that Evert's payoffs from the Lob are between the best and the worst she can get with DL and CC, and not too different against Navratilova's covering DL or CC. We have shown not only the payoffs from the pure strategies, but also those for Evert's three pure strategies against

[11] Even when a player has only two pure strategies, he may not use one of them in equilibrium. The other player then generally finds one of his strategies to be better against the one that the first player does use. In other words, the equilibrium "mixtures" collapse to the special case of pure strategies. But when one or both players have three or more strategies, we can have a genuinely mixed-strategy equilibrium where some of the pure strategies go unused.

		NAVRATILOVA		
		DL	CC	q-mix
EVERT	DL	50, 50	80, 20	$50q + 80(1 - q)$, $50q + 20(1 - q)$
	CC	90, 10	20, 80	$90q + 20(1 - q)$, $10q + 80(1 - q)$
	Lob	70, 30	60, 40	$70q + 60(1 - q)$, $30q + 40(1 - q)$

FIGURE 7.8 Payoff Table for Tennis Point with Lob

Navratilova's q-mix. [We do not show a row for Evert's p-mix because we don't need it. It would require two probabilities, say p_1 for DL and p_2 for CC, and then that for the Lob would be $(1 - p_1 - p_2)$. We show you how to solve for equilibrium mixtures of this type in the following section.]

Technically, before we begin looking for a mixed-strategy equilibrium, we should verify that there is no pure-strategy equilibrium. This is easy to do, however, so we leave it to you and turn to mixed strategies.

We will use the logic of best responses to consider Navratilova's optimal choice of q. In Figure 7.9 we show Evert's expected payoffs (success percentages) from playing each of her pure strategies DL, CC, and Lob as the q in Navratilova's q-mix varies over its full range from 0 to 1. These graphs are just those of Evert's payoff expressions in the right-hand column of Figure 7.8. For each q, if Navratilova were to choose that q-mix in equilibrium, Evert's best response would be to choose the strategy that gives her (Evert) the highest payoff. We show this set of best-response outcomes for Evert with the thicker lines in Figure 7.9; in mathematical jargon this is the *upper envelope* of the three payoff lines. Navratilova wants to choose her own best possible q—the q that makes her own payoff as large as possible (thereby making Evert's payoff as low as possible)—from this set of Evert's best responses.

To be more precise about Navratilova's optimal choice of q, we must calculate the coordinates of the kink points in the line showing her worst-case (Evert's best-case) outcomes. The value of q at the leftmost kink in this line makes Evert indifferent between DL and Lob. That q must equate the two payoffs from DL and Lob when used against the q-mix. Setting those two expressions equal gives us $50q + 80(1 - q) = 70q + 60(1 - q)$, or $q = 20/40 = 1/2 = 50\%$. Evert's expected payoff at this point is $50 \times 0.5 + 80 \times 0.5 = 70 \times 0.5 + 60 \times 0.5 = 65$. At the second (rightmost) kink, Evert is indifferent between CC and Lob. Thus, the q value at this kink is the one that equates the CC and Lob payoff expressions. Setting $90q + 20(1 - q) = 70q + 60(1 - q)$, we find $q = 40/60 = 2/3 = 66.7\%$.

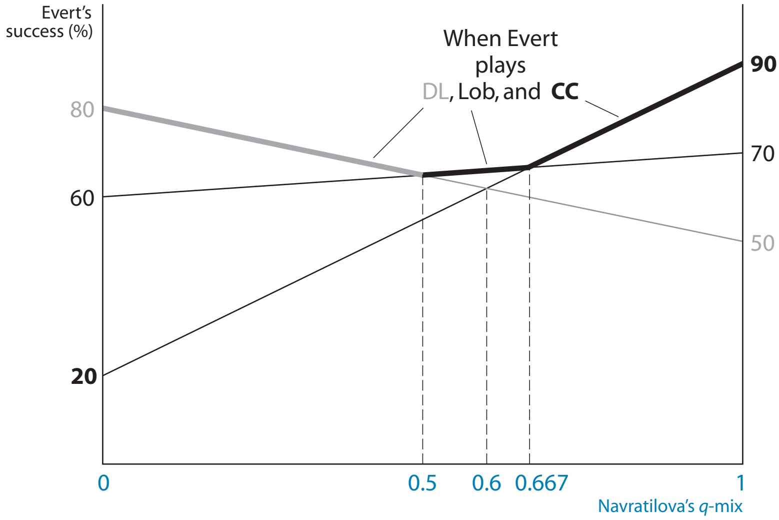

FIGURE 7.9 Diagrammatic Solution for Navratilova's q-Mix

Here, Evert's expected payoff is $90 \times 0.667 + 20 \times 0.333 = 70 \times 0.667 + 60 \times 0.333 = 66.67$. Therefore, Navratilova's best (or least bad) choice of q is at the left kink, namely $q = 0.5$. Evert's expected payoff is 65, so Navratilova's is 35.

When Navratilova chooses $q = 0.5$, Evert is indifferent between DL and Lob, and either of these choices gives her a better payoff than does CC. Therefore, Evert will not use CC at all in equilibrium. CC will be an unused strategy in her equilibrium mix.

Now we can proceed with the equilibrium analysis as if this were a game with just two pure strategies for each player: DL and CC for Navratilova, and DL and Lob for Evert. We are back in familiar territory. Therefore, we leave the calculation to you and just tell you the result. Evert's optimal mixture in this game entails her using DL with probability 0.25 and Lob with probability 0.75. Evert's expected payoff from this mixture, taken against Navratilova's DL and CC, respectively, is $50 \times 0.25 + 70 \times 0.75 = 80 \times 0.25 + 60 \times 0.75 = 65$, as of course it should be.

We could not have started our analysis with this two-by-two game because we did not know in advance which of her three strategies Evert would not use. But we can be confident that in the general case, there will be one such strategy. When the three expected payoff lines take the most general positions, they intersect pair by pair rather than all crossing at a single point. Then the upper envelope has the shape that we see in Figure 7.9. Its lowest point is defined by the intersection of the payoff lines associated with two of the three strategies. The payoff from the third strategy lies below the intersection at this point, so the player choosing among the three strategies does not use that third one.

B. Exceptional Cases

The positions and intersections of the three lines of Figure 7.9 depend on the payoffs specified for the pure strategies. We chose the payoffs for that particular game to show a general configuration of the lines. But if the payoffs stand in very specific relationships to each other, we can get some exceptional configurations with different results. We describe the possibilities here but leave it to you to redraw the diagrams for these cases.

First, if Evert's payoffs from Lob against Navratilova's DL and CC are equal, then the line for Lob is horizontal, and a whole range of q-values make Navratilova's mixture exploitation-proof. For example, if the two payoffs in the Lob row of the table in Figure 7.8 are 70 each, then it is easy to calculate that the left kink in a revised Figure 7.9 would be at $q = 1/3$ and the right kink at $q = 5/7$. For any q in the range from 1/3 to 5/7, Evert's best response is Lob, and we get an unusual equilibrium in which Evert plays a pure strategy and Navratilova mixes. Further, Navratilova's equilibrium mixture probabilities are indeterminate within the range from $q = 1/3$ to $q = 5/7$.

Second, if Evert's payoffs from Lob against Navratilova's DL and CC are lower than those of Figure 7.8 by just the right amounts (or those of the other two strategies are higher by just the right amounts), all three lines can meet in one point. For example, if the payoffs of Evert's Lob are 66 and 56 against Navratilova's DL and CC, respectively, instead of 70 and 60, then for $q = 0.6$, Evert's expected payoff from the Lob becomes $66 \times 0.6 + 56 \times 0.4 = 39.6 + 22.6 = 62$, the same as that from DL and CC when $q = 0.6$. Then Evert is indifferent among all three of her strategies when $q = 0.6$ and is willing to mix among all three.

In this special case, Evert's equilibrium mixture probabilities are not fully determinate. Rather, a whole range of mixtures, including some where all three strategies are used, can do the job of keeping Navratilova indifferent between her DL and CC and therefore willing to mix. However, Navratilova must use the mixture with $q = 0.6$. If she does not, Evert's best response will be to switch to one of her pure strategies, and this will work to Navratilova's detriment. We do not dwell on the determination of the precise range over which Evert's equilibrium mixtures can vary, because this case can only arise for exceptional combinations of the payoff numbers and is therefore relatively unimportant.

Note that Evert's payoffs from using her Lob against Navratilova's DL and CC could be even lower than the values that make all three lines intersect at one point (for example, if the payoffs from Lob were 75 and 30 instead of 70 and 60 as in Figure 7.8). Then Lob is never the best response for Evert even though it is not dominated by either DL or CC. This case of Lob being dominated by a *mixture* of DL and CC is explained in the online appendix to this chapter.

7 MIXING WHEN BOTH PLAYERS HAVE THREE STRATEGIES

When we consider games in which both players have three pure strategies and are considering mixing among all three, we need two variables to specify each mix.[12] The row player's p-mix would put probability p_1 on his first pure strategy and probability p_2 on his second pure strategy. Then the probability of using the third pure strategy must equal 1 minus the sum of the probabilities of the other two. The same would be true for the column player's q-mix. So when both players have three strategies, we cannot find a mixed-strategy equilibrium without doing two-variable algebra. In many cases, however, such algebra is still manageable.

A. Full Mixture of All Strategies

Consider a simplified representation of a penalty kick in soccer. Suppose a right-footed kicker has just three pure strategies: kick to the left, right, or center. (Left and right refer to the goalie's left or right. For a right-footed kicker, the most natural motion would send the ball to the goalie's right.) Then he can mix among these strategies, with probabilities denoted by p_L, p_R, and p_C, respectively. Any two of them can be taken to be the independent variables and the third expressed in terms of them. If p_L and p_R are made the two independent-choice variables, then $p_C = 1 - p_L - p_R$. The goalie also has three pure strategies—namely, move to the kicker's left (the goalie's own right), move to the kicker's right, or continue to stand in the center—and can mix among them with probabilities q_L, q_R, and q_C, two of which can be chosen independently.

As in Section 6.A, a best-response diagram for this game would require more than two dimensions. [Four, to be exact. The goalie would choose his two independent variables, say (q_L, q_R), as his best response to the kicker's two, (p_L, p_R), and vice versa.] Instead, we again use the principle of the opponent's indifference to focus on the mixture probabilities for one player at a time. Each player's probabilities should be such that the other player is indifferent among all the pure strategies that constitute his mixture. This gives us a set of equations that can be solved for the mixture probabilities. In the soccer example, the kicker's (p_L, p_R) would satisfy two equations expressing the requirement that the goalie's expected payoff from using his left should equal that from using his right and that the goalie's expected payoff from using his right should equal that from using his center. (Then the equality of expected payoffs from left and center follows automatically and is not a separate equation.) With more pure strategies,

[12] More generally, if a player has N pure strategies, then her mix has $(N - 1)$ independent variables, or "degrees of freedom of choice."

		GOALIE		
		Left	Center	Right
KICKER	Left	45, 55	90, 10	90, 10
	Center	85, 15	0, 100	85, 15
	Right	95, 5	95, 5	60, 40

FIGURE 7.10 Soccer Penalty Kick Game

the number of the probabilities to be solved for and the number of equations that they must satisfy also increase.

Figure 7.10 shows the game table for the interaction between Kicker and Goalie, with success percentages as payoffs for each player. (Unlike the evidence we present on European soccer later in this chapter, these are not real data but similar rounded numbers to simplify calculations.) Because the kicker wants to maximize the percentage probability that he successfully scores a goal and the goalie wants to minimize the probability that he lets the goal through, this is a zero-sum game. For example, if the kicker kicks to his left while the goalie moves to the kicker's left (the top-left-corner cell), we suppose that the kicker still succeeds (in scoring) 45% of the time and the goalie therefore succeeds (in saving a goal) 55% of the time. But if the kicker kicks to his right and the goalie goes to the kicker's left, then the kicker has a 90% chance of scoring; we suppose a 10% probability that he might kick wide or too high so the goalie is still "successful" 10% of the time. You can experiment with different payoff numbers that you think might be more appropriate.

It is easy to verify that the game has no equilibrium in pure strategies. So suppose the kicker is mixing with probabilities p_L, p_R, and $p_C = 1 - p_L - p_R$. For each of the goalie's pure strategies, this mixture yields the goalie the following payoffs:

$$\begin{aligned}
&\text{Left:} && 55p_L + 15p_C + 5p_R = 55p_L + 15(1 - p_L - p_R) + 5p_R\\
&\text{Center:} && 10p_L + 100p_C + 5p_R = 10p_L + 100(1 - p_L - p_R) + 5p_R\\
&\text{Right:} && 10p_L + 15p_C + 40p_R = 10p_L + 15(1 - p_L - p_R) + 40p_R.
\end{aligned}$$

The opponent's indifference rule says that the kicker should choose p_L and p_R so that all three of these expressions are equal in equilibrium.

Equating the Left and Right expressions and simplifying, we have $45p_L = 35p_R$, or $p_R = (9/7)p_L$. Next, equate the Center and Right expressions and simplify by using the link between p_L and p_R just obtained. This gives

$$10p_L + 100\,[1 - p_L - (9p_L/7)] + 5(9p_L/7) = 10p_L + 15[1 - p_L - (9p_L/7)] + 40(9p_L/7),$$

or $[85 + 120(9/7)]\; p_L = 85$, which yields $p_L = 0.355$.

Then we get $p_R = 0.355(9/7) = 0.457$, and finally $p_C = 1 - 0.355 - 0.457 = 0.188$. The goalie's payoff from any of his pure strategies against this mixture can then be calculated by using any of the preceding three payoff lines; the result is 24.6.

The goalie's mixture probabilities can be found by writing down and solving the equations for the kicker's indifference among his three pure strategies against the goalie's mixture. We will do this in detail for a slight variant of the same game in Section 7.B, so we omit the details here and just give you the answer: $q_L = 0.325$, $q_R = 0.561$, and $q_C = 0.113$. The kicker's payoff from any of his pure strategies when played against the goalie's equilibrium mixture is 75.4. That answer is, of course, consistent with the goalie's payoff of 24.6 that we calculated before.

Now we can interpret the findings. The kicker does better with his pure Right than his pure Left, both when the goalie guesses correctly ($60 > 45$) and when he guesses incorrectly ($95 > 90$). (Presumably the kicker is left-footed and can kick harder to his right.) Therefore, the kicker chooses Right with greater probability and, to counter that, the goalie chooses Right with the highest probability, too. However, the kicker should not and does not choose his pure-strategy Right; if he did so, the goalie would then choose his own pure-strategy Right, too, and the kicker's payoff would be only 60, less than the 75.4 that he gets in the mixed-strategy equilibrium.

B. Equilibrium Mixtures with Some Strategies Unused

In the preceding equilibrium, the probabilities of using Center in the mix are quite low for each player. The (Center, Center) combination would result in a sure save and the kicker would get a really low payoff—namely, 0. Therefore, the kicker puts a low probability on this choice. But then the goalie also should put a low probability on it, concentrating on countering the kicker's more likely choices. But if the kicker gets a sufficiently high payoff from choosing Center when the goalie chooses Left or Right, then the kicker will choose Center with some positive probability. If the kicker's payoffs in the Center row were lower, he might then choose Center with zero probability; if so, the goalie would similarly put zero probability on Center. The game would reduce to one with just two basic pure strategies, Left and Right, for each player.

We show such a variant of the soccer game in Figure 7.11. The only difference in payoffs between this variant and the original game of Figure 7.10 is that the kicker's payoffs from (Center, Left) and (Center, Right) have been lowered even further, from 85 to 70. This might be because this kicker has the habit of kicking too high and therefore missing the goal when aiming for the center. Let us try to calculate the equilibrium here by using the same methods as in Section 7.A. This time we do it from the goalie's perspective: we try

		GOALIE		
		Left	Center	Right
KICKER	Left	45, 55	90, 10	90, 10
	Center	70, 30	0, 100	70, 30
	Right	95, 5	95, 5	60, 40

FIGURE 7.11 Variant of Soccer Penalty Kick Game

to find his mixture probabilities q_L, q_R, and q_C by using the condition that the kicker should be indifferent among all three of his pure strategies when played against this mixture.

The kicker's payoffs from his pure strategies are

$$\begin{aligned}
\text{Left:}\quad & 45q_L + 90q_C + 90q_R = 45q_L + 90(1 - q_L - q_R) + 90q_R \\
& = 45q_L + 90(1 - q_L) \\
\text{Center:}\quad & 70q_L + 0q_C + 70q_R = 70q_L + 70q_R \\
\text{Right:}\quad & 95q_L + 95q_C + 60q_R = 95q_L + 95(1 - q_L - q_R) + 60\,q_R \\
& = 95(1 - q_R) + 60q_R.
\end{aligned}$$

Equating the Left and Right expressions and simplifying, we have $90 - 45q_L = 95 - 35q_R$, or $35q_R = 5 + 45q_L$. Next, equate the Left and Center expressions and simplify to get $90 - 45q_L = 70q_L + 70q_R$, or $115q_L + 70q_R = 90$. Substituting for q_R from the first of these equations (after multiplying through by 2 to get $70q_R = 10 + 90q_L$) into the second yields $205q_L = 80$, or $q_L = 0.390$. Then, using this value for q_L in either of the equations gives $q_R = 0.644$. Finally, we use both of these values to obtain $q_C = 1 - 0.390 - 0.644 = -0.034$. Because probabilities cannot be negative, something has obviously gone wrong.

To understand what happens in this example, start by noting that Center is now a poorer strategy for the kicker than it was in the original version of the game, where his probability of choosing it was already quite low. But the logic of the opponent's indifference, expressed in the equations that led to the solution, means that the kicker has to be kept willing to use this poor strategy. That can happen only if the goalie is using his best counter to the kicker's Center—namely, the goalie's own Center—sufficiently infrequently. And in this example, that logic has to be carried so far that the goalie's probability of Center has to become negative.

As pure algebra, the solution that we derived may be fine, but it violates the requirement of probability theory and real-life randomization that probabilities be nonnegative. The best that can be done in reality is to push the goalie's probability of choosing Center as low as possible—namely, to zero. But that leaves the kicker unwilling to use his own Center. In other words, we get a situation in which each player is not using one of his pure strategies in his mixture—that is, each is using it with zero probability.

Can there then be an equilibrium in which each player is mixing between his two remaining strategies—namely, Left and Right? If we regard this reduced two-by-two game in its own right, we can easily find its mixed-strategy equilibrium. With all the practice that you have had so far, it is safe to leave the details to you and to state the result:

Kicker's mixture probabilities: $p_L = 0.4375$, $p_R = 0.5625$
Goalie's mixture probabilities: $q_L = 0.3750$, $q_R = 0.6250$
Kicker's expected payoff (success percentage): 73.13
Goalie's expected payoff (success percentage): 26.87.

We found this result by simply removing the two players' Center strategies from consideration on intuitive grounds. But we must check that it is a genuine equilibrium of the full three-by-three game. That is, we must check that neither player finds it desirable to bring in his third strategy, given the mixture of two strategies chosen by the other player.

When the goalie is choosing this particular mixture, the kicker's payoff from pure Center is $0.375 \times 70 + 0.625 \times 70 = 70$. This payoff is less than the 73.13 that he gets from either of his pure Left or pure Right or any mixture between the two, so the kicker does not want to bring his Center strategy into play. When the kicker is choosing the two-strategy mixture with the preceding probabilities, the goalie's payoff from pure Center is $0.4375 \times 10 + 0.5625 \times 5 = 7.2$. This number is (well) below the 26.87 that the goalie would get using his pure Left or pure Right or any mixture of the two. Thus, the goalie does not want to bring his Center strategy into play either. The equilibrium that we found for the two-by-two game is indeed an equilibrium of the three-by-three game.

To allow for the possibility that some strategies may go unused in an equilibrium mixture, we must modify or extend the "opponent's indifference" principle. Each player's equilibrium mix should be such that the other player is indifferent among all the strategies *that are actually used in his equilibrium mix*. The other player is not indifferent between these and his unused strategies; he prefers the ones used to the ones unused. In other words, against the opponent's equilibrium mix, all of the strategies used in your own equilibrium mix should give you the same expected payoff, which in turn should be higher than what you would get from any of your unused strategies.

Which strategies will go unused in equilibrium? Answering that requires much trial and error as in our calculation above, or leaving it all to a computer program, and once you have understood the concept, it is safe to do the latter. For the general theory of mixed-strategy equilibria when players can have any number of possible strategies, see the online appendix to this chapter.

8 HOW TO USE MIXED STRATEGIES IN PRACTICE

There are several important things to remember when finding or using a mixed strategy in a zero-sum game. First, to use a mixed strategy effectively in such a game, a player needs to do more than calculate the equilibrium percentages with which to use each of her actions. Indeed, in our tennis-point game, Evert cannot simply play DL seven-tenths of the time and CC three-tenths of the time by mechanically rotating seven shots down the line and three shots crosscourt. Why not? Because mixing your strategies is supposed to help you benefit from the element of surprise against your opponent. If you use a recognizable pattern of plays, your opponent is sure to discover it and exploit it to her advantage.

The lack of a pattern means that, after any history of choices, the probability of choosing DL or CC on the next turn is the same as it always was. If a run of several successive DLs happens by chance, there is no sense in which CC is now "due" on the next turn. In practice, many people mistakenly think otherwise, and therefore they alternate their choices too much compared with what a truly random sequence of choices would require: they produce too few runs of identical successive choices. However, detecting a pattern from observed actions is a tricky statistical exercise that the opponents may not be able to perform while playing the game. As we will see in Section 9, analysis of data from grand-slam tennis finals found that servers alternated their serves too much, but receivers were not able to detect and exploit this departure from true randomization.

The importance of avoiding predictability is clearest in ongoing interactions of a zero-sum nature. Because of the diametrically opposed interests of the players in such games, your opponent always benefits from exploiting your choice of action to the greatest degree possible. Thus, if you play the same game against each other on a regular basis, she will always be on the lookout for ways to break the code that you are using to randomize your moves. If she can do so, she has a chance to improve her payoffs in future plays of the game. But even in single-meet (sometimes called one-shot) zero-sum games, mixing remains beneficial because of the benefit of tactical surprise.

Daniel Harrington, a winner of the World Series of Poker and author with Bill Robertie of an excellent series of books on how to play Texas Hold 'em tournaments, notes the importance of randomizing your strategy in poker in order to prevent opponents from reading what cards you're holding and exploiting your behavior.[13] Because humans often have trouble being unpredictable, he

[13] Poker is a game of incomplete information because each player holds private information about her cards. While we do not analyze the details of such games until Chapter 8, they may involve mixed-strategy equilibria (called *semiseparating equilibria*) where the random mixtures are specifically designed to prevent other players from using your actions to infer your private information.

gives the following advice about how to implement a mixture between the pure strategies of calling and raising:

> It's hard to remember exactly what you did the last four or five times a given situation appeared, but fortunately you don't have to. Just use the little random number generator that you carry around with you all day. What's that? You didn't know you had one? It's the second hand on your watch. If you know that you want to raise 80 percent of the time with a premium pair in early position and call the rest, just glance down at your watch and note the position of the second hand. Since 80 percent of 60 is 48, if the second hand is between 0 and 48, you raise, and if it's between 48 and 60 you just call. The nice thing about this method is that even if someone knew exactly what you were doing, they still couldn't read you![14]

Of course, in using the second hand of a watch to implement a mixed strategy, it is important that your watch not be so accurate and synchronized that your opponent can use the same watch and figure out what you are going to do!

So far, we have assumed that you are interested in implementing a mixed strategy in order to avoid possible exploitation by your opponent. But if your opponent is not playing his equilibrium strategy, you may want to try to exploit his mistake. A simple example is illustrated using an episode of *The Simpsons* in which Bart and Lisa play a game of rock-paper-scissors with each other. (In Exercise S10, we give a full description of this three-by-three game, and you will derive each player's equilibrium mixture.) Just before they choose their strategies, Bart thinks to himself, "Good ol' Rock. Nothing beats Rock," while Lisa thinks to herself, "Poor Bart. He always plays Rock." Clearly, Lisa's best response is the pure strategy Paper against this naive opponent; she need not use her equilibrium mix.

We have observed a more subtle example of exploitation when pairs of students play a best-of-100 version of the tennis game in this chapter. As with professional tennis players, our students often switch strategies too often, apparently thinking that playing five DLs in a row doesn't look "random" enough. To exploit this behavior, a Navratilova player could predict that after playing three DLs in a row, an Evert player is likely to switch to CC, and she can exploit this by switching to CC herself. She should do this more often than if she were randomizing independently each round, but ideally not so often that the Evert player notices and starts learning to repeat her strategy in longer runs.

Finally, players must understand and accept the fact that the use of mixed strategies guards you against exploitation and gives the best possible expected payoff against an opponent who is making her best choices, but that it is only a

[14] Daniel Harrington and Bill Robertie, *Harrington on Hold 'em: Expert Strategies for No-Limit Tournaments, Volume 1: Strategic Play* (Henderson, Nev.: Two Plus Two Publishing, 2004), p. 53.

probabilistic average. On particular occasions, you can get poor outcomes. For example, the long pass on third down with a yard to go, intended to keep the defense honest, may fail on any specific occasion. If you use a mixed strategy in a situation in which you are responsible to a higher authority, therefore, you may need to plan ahead for this possibility. You may need to justify your use of such a strategy ahead of time to your coach or your boss, for example. They need to understand why you have adopted your mixture and why you expect it to yield you the best possible payoff on average, even though it might yield an occasional low payoff as well. Even such advance planning may not work to protect your "reputation," though, and you should prepare yourself for criticism in the face of a bad outcome.

9 EVIDENCE ON MIXING

A. Zero-Sum Games

Early researchers who performed laboratory experiments were generally dismissive of mixed strategies. To quote Douglas Davis and Charles Holt, "Subjects in experiments are rarely (if ever) observed flipping coins, and when told ex post that the equilibrium involves randomization, subjects have expressed surprise and skepticism."[15] When the predicted equilibrium entails mixing two or more pure strategies, experimental results do show some subjects in the group pursuing one of the pure strategies and others pursuing another, but this does not constitute true mixing by an individual player. When subjects play zero-sum games repeatedly, individual players often choose different pure strategies over time. But they seem to mistake alternation for randomization—that is, they switch their choices more often than true randomization would require.

Later research has found somewhat better evidence for mixing in zero-sum games. When laboratory subjects are allowed to acquire a lot of experience, they do appear to learn mixing in zero-sum games. However, departures from equilibrium predictions remain significant. Averaged across all subjects, the empirical probabilities are usually rather close to those predicted by equilibrium, but many individual subjects play proportions far from those predicted by equilibrium. To quote Colin Camerer, "The overall picture is that mixed equilibria do not provide bad guesses about how people behave, on average."[16]

[15] Douglas D. Davis and Charles A. Holt, *Experimental Economics* (Princeton: Princeton University Press, 1993), p. 99.

[16] For a detailed account and discussion, see Chapter 3 of Colin F. Camerer, *Behavioral Game Theory* (Princeton: Princeton University Press, 2003). The quote is from p. 468 of this book.

An instance of randomization in practice comes from Malaya in the late 1940s.[17] The British army escorted convoys of food trucks to protect the trucks from communist terrorist attacks. The terrorists could either launch a large-scale attack or create a smaller sniping incident intended to frighten the truck drivers and keep them from serving again. The British escort could be either concentrated or dispersed throughout the convoy. For the army, concentration was better to counter a full-scale attack, and dispersal was better against sniping. For the terrorists, a full-scale attack was better if the army escort was dispersed, and sniping was better if the escort was concentrated. This zero-sum game has only a mixed-strategy equilibrium. The escort commander, who had never heard of game theory, made his decision as follows. Each morning, as the convoy was forming, he took a blade of grass and concealed it in one of his hands, holding both hands behind his back. Then he asked one of his troops to guess which hand held the blade, and he chose the form of the convoy according to whether the man guessed correctly. Although the precise payoff numbers are difficult to judge and therefore we cannot say whether 50–50 was the right mixture, the officer had correctly figured out the need for true randomization and the importance of using a fresh randomization procedure every day to avoid falling into a pattern or making too much alternation between the choices.

The best evidence in support of mixed strategies in zero-sum games comes from sports, especially from professional sports, in which players accumulate a great deal of experience in such games, and their intrinsic desire to win is buttressed by large financial gains from winning.

Mark Walker and John Wooders examined the serve-and-return play of top-level players at Wimbledon.[18] They model this interaction as a game with two players, the server and the receiver, in which each player has two pure strategies. The server can serve to the receiver's forehand or backhand, and the receiver can guess to which side the serve will go and move that way. Because serves are so fast at the top levels of men's singles, the receiver cannot react after observing the actual direction of the serve; rather, the receiver must move in anticipation of the serve's direction. Thus, this game has simultaneous moves. Further, because the receiver wants to guess correctly and the server wants to wrong-foot the receiver, this interaction has a mixed-strategy equilibrium. It is impossible to observe the receiver's strategy on a videotape (on which foot is he resting his weight?), so one cannot easily reconstruct the entire matrix of payoffs to test whether players are mixing according to the equilibrium predictions. However, an important prediction of the theory can be tested by calculating the server's frequency of winning the point for each of his possible serving strategies.

[17] R. S. Beresford and M. H. Peston, "A Mixed Strategy in Action," *Operations Research*, vol. 6, no. 4 (December 1955), pp. 173–76.

[18] Mark Walker and John Wooders, "Minimax Play at Wimbledon," *American Economic Review*, vol. 91, no. 5 (December 2001), pp. 1521–38.

If the tennis players are using their equilibrium mixtures in the serve-and-return game, the server should win the point with the same probability whether he serves to the receiver's forehand or backhand. An actual tennis match contains a hundred or more points played by the same two players; thus there is enough data to test whether this implication holds for each match. Walker and Wooders tabulated the results of serves in 10 matches. Each match contains four kinds of serve-and-return combinations: A serving to B and vice versa, combined with service from the right or the left side of the court (Deuce or Ad side). Thus, they had data on 40 serving situations and found that in 39 of them the server's success rates with forehand and backhand serves were equal to within acceptable limits of statistical error.

The top-level players must have had enough general experience playing the game, as well as particular experience playing against the specific opponents, to have learned the general principle of mixing and the correct proportions to mix against the specific opponents. However, in one respect the servers' choices departed from true mixing. To achieve the necessary unpredictability, there should be no pattern of any kind in a sequence of serves: the choice of side for each serve should be independent of what has gone before. As we said in reference to the practice of mixed strategies, players can alternate too much, not realizing that alternation is a pattern just as much as repeating the same action a few times would be a pattern. And indeed, the data show that the tennis servers alternated too much. But the data also indicate that this departure from true mixing was not enough for the opponents to pick up and exploit.

As we showed in Section 8, penalty kicks in soccer are another excellent context in which to study mixed strategies. The advantage to analyzing penalty kicks is that one can actually observe the strategies of both the kicker and the goalkeeper: not only where the kicker aims but also which direction the keeper dives. This means one can compute the actual mixing probabilities and compare them to the theoretical prediction. The disadvantage, relative to tennis, is that no two players ever face each other more than a few times in a season. Instead of analyzing specific matchups of players, one must aggregate across all kickers and shooters in order to get enough data. Two studies using exactly this kind of data find firm support for predictions of the theory.

Using a large data set from professional soccer leagues in Europe, Ignacio Palacios-Huerta constructed the payoff table of the kicker's average success probabilities shown in Figure 7.12.[19] Because the data include both right- and left-footed kickers, and therefore the natural direction of kicking differs between them, they refer to any kicker's natural side as "Right." (Kickers usually kick with the inside of the foot. A right-footed kicker naturally kicks to the goalie's right

[19] See "Professionals Play Minimax," by Ignacio Palacios-Huerta, *Review of Economics Studies*, vol. 70, no. 20 (2003), pp. 395–415.

		GOALIE	
		Left	Right
KICKER	Left	58	95
	Right	93	70

FIGURE 7.12 Soccer Penalty Kick Success Probabilities in European Major Leagues

and a left-footed kicker to the goalie's left.) The choices are Left and Right for each player. When the goalie chooses Right, it means covering the kicker's natural side.

Using the opponent's indifference property, it is easy to calculate that the kicker *should* choose Left 38.3% of the time and Right 61.7% of the time. This mixture achieves a success rate of 79.6% no matter what the goalie chooses. The goalie *should* choose the probabilities of covering her Left and Right to be 41.7 and 58.3, respectively; this mixture holds the kicker down to a success rate of 79.6%.

What actually happens? Kickers choose Left 40.0% of the time, and goalies choose Left 41.3% of the time. These values are startlingly close to the theoretical predictions. The chosen mixtures are almost exploitation proof. The kicker's mix achieves a success rate of 79.0% against the goalie's Left and 80% against the goalie's Right. The goalie's mix holds kickers down to 79.3% if they choose Left and 79.7% if they choose Right.

In an earlier paper, Pierre-André Chiappori, Timothy Groseclose, and Steven Levitt used similar data and found similar results.[20] They also analyzed the whole sequence of choices of each kicker and goalie and did not even find too much alternation. One reason for this last result could be that most penalty kicks take place as isolated incidents across many games by contrast with the rapidly repeated points in tennis, so players may find it easier to ignore what happened on the previous kick. Nevertheless, these findings suggest that behavior in soccer penalty kicks is even closer to true mixing than behavior in the tennis serve-and-return game.

With such strong empirical confirmation of the theory, one might ask whether the mixed-strategy skills that players learn in soccer carry over to other game contexts. One study indicated that the answer is yes (Spanish professional soccer players played exactly according to the equilibrium predictions in

[20] Pierre-André Chiappori, Timothy Groseclose, and Steven Levitt, "Testing Mixed Strategy Equilibria When Players are Heterogeneous: The Case of Penalty Kicks in Soccer," *American Economic Review*, vol. 92, no. 4 (September 2002), pp. 1138–51.

laboratory experiments with two-by-two and four-by-four zero-sum matrix games). But a second study failed to replicate these results. That study examined American Major League Soccer players as well as participants in the World Series of Poker (who, as noted in Section 8 above, also have professional reasons to prevent exploitation by mixing), finding that the professionals' behavior in abstract matrix games was just as far from equilibrium as that of students. Consistent with the results on professional chess players we discussed in Chapter 3, experience leads professional players to mix according to equilibrium theory in their jobs, but this experience does not automatically lead players to equilibrium in new and unfamiliar games.[21]

B. Non-Zero-Sum Games

Laboratory experiments on games with mixed strategies in non-zero-sum games yield even more negative results than experiments involving mixing in zero-sum games. This is not surprising. As we have seen, in such games the property that each player's equilibrium mixture keeps her opponent indifferent among her pure strategies is a logical property of the equilibrium. Unlike in zero-sum games, in general each player in a non-zero-sum game has no positive or purposive reason to keep the other players indifferent. Then the reasoning underlying the mixture calculations is more difficult for players to comprehend and learn. This shows up in their behavior.

In a group of experimental subjects playing a non-zero-sum game, we may see some pursuing one pure strategy and others pursuing another. This type of mixing in the population, although it does not fit the theory of mixed-strategy equilibria, does have an interesting evolutionary interpretation, which we examine in Chapter 12.

As we saw in Section 5.B above, each player's mixture probabilities should not change when the player's own payoffs change. But in fact they do: players tend to choose an action more when their own payoff to that action increases.[22] The players do change their actions from one round to the next in repeated trials with different partners, but not in accordance with equilibrium predictions.

The overall conclusion is that you should interpret and use mixed-strategy equilibria in non-zero-sum games with, at best, considerable caution.

[21] The first study referenced is Ignacio Palacios-Huerta and Oskar Volij, "Experientia Docet: Professionals Play Minimax in Laboratory Experiments," *Econometrica*, vol. 76, no. 1 (January 2008), pp. 71–115. The second is Steven D. Levitt, John A. List, and David H. Reiley, "What Happens in the Field Stays in the Field: Exploring Whether Professionals Play Minimax in Laboratory Experiments," *Econometrica*, vol. 78, no. 4 (July 2010), pp. 1413–34.

[22] Jack Ochs, "Games with Unique Mixed-Strategy Equilibria: An Experimental Study," *Games and Economic Behavior*, vol. 10, no. 1 (July 1995), pp. 202–17.

SUMMARY

Zero-sum games in which one player prefers a coincidence of actions and the other prefers the opposite often have no Nash equilibrium in pure strategies. In these games, each player wants to be unpredictable and thus uses a mixed strategy that specifies a probability distribution over her set of pure strategies. Each player's equilibrium mixture probabilities are calculated using the *opponent's indifference property*, namely that the opponent should get equal *expected payoffs* from all her pure strategies when facing the first player's equilibrium mix. Best-response-curve diagrams can be used to show all mixed-strategy (as well as pure-strategy) equilibria of a game.

Non-zero-sum games can also have mixed-strategy equilibria that can be calculated from the opponent's indifference property and illustrated using best-response curves. But here the motivation for keeping the opponent indifferent is weaker or missing; therefore such equilibria have less appeal and are often unstable.

Mixed strategies are a special case of continuous strategies but have additional matters that deserve separate study. Mixed-strategy equilibria can be interpreted as outcomes in which each player has correct beliefs about the probabilities with which the other player chooses from among her underlying pure actions. And mixed-strategy equilibria may have some counterintuitive properties when payoffs for players change.

If one player has three pure strategies and the other has only two, the player with three available strategies will generally use only two in her equilibrium mix. If both players have three (or more) pure strategies, equilibrium mixtures may put positive probability on all pure strategies or only a subset. All strategies that are actively used in the mixture yield equal expected payoff against the opponent's equilibrium mix; all the unused ones yield lower expected payoff. In these large games, equilibrium mixtures may also be indeterminate in some exceptional cases.

When using mixed strategies, players should remember that their system of randomization should not be predictable in any way. Most important, they should avoid excessive alternation of actions. Laboratory experiments show only weak support for the use of mixed strategies. But mixed-strategy equilibria give good predictions in many zero-sum situations in sports played by experienced professionals.

KEY TERMS

expected payoff (216) **opponent's indifference property** (218)

SOLVED EXERCISES

S1. Consider the following game:

		COLIN	
		Safe	Risky
ROWENA	Safe	4, 4	4, 1
	Risky	1, 4	6, 6

(a) Which game does this most resemble: tennis, assurance, or chicken? Explain.

(b) Find all of this game's Nash equilibria.

S2. The following table illustrates the money payoffs associated with a two-person simultaneous-move game:

		COLIN	
		Left	Right
ROWENA	Up	1, 16	4, 6
	Down	2, 20	3, 40

(a) Find the Nash equilibrium in mixed strategies for this game.

(b) What are the players' expected payoffs in this equilibrium?

(c) Rowena and Colin jointly get the most money when Rowena plays Down. However, in the equilibrium, she does not always play Down. Why not? Can you think of ways in which a more cooperative outcome can be sustained?

S3. Recall Exercise S7 from Chapter 4, about an old lady looking for help crossing the street and two players simultaneously deciding whether to offer help. If you did that exercise, you also found all of the pure-strategy Nash equilibria of the game. Now find the mixed-strategy equilibrium of the game.

S4. Revisit the tennis game in Section 2.A of this chapter. Recall that the mixed-strategy Nash equilibrium found in that section had Evert playing DL with probability 0.7, while Navratilova played DL with probability 0.6. Now suppose that Evert injures herself later in the match, so her DL shots are much slower and easier for Navratilova to defend. The payoffs are now given by the following table:

		NAVRATILOVA	
		DL	CC
EVERT	DL	30, 70	60, 40
	CC	90, 10	20, 80

(a) Relative to the game before her injury (see Figure 7.1), the strategy DL seems much less attractive to Evert than before. Would you expect Evert to play DL more, less, or the same amount in a new mixed-strategy equilibrium? Explain.

(b) Find each player's equilibrium mixture for this game. What is the expected value of the game to Evert?

(c) How do the equilibrium mixtures found in part (b) compare with those of the original game and with your answer to part (a)? Explain why each mixture has or has not changed.

S5. Exercise S7 in Chapter 6 introduced a simplified version of baseball, and part (e) pointed out that the simultaneous-move game has no Nash equilibrium in pure strategies. This is because pitchers and batters have conflicting goals. Pitchers want to get the ball *past* batters, but batters want to *connect* with pitched balls. The game table is as follows:

		PITCHER	
		Throw fastball	Throw curve
BATTER	Anticipate fastball	0.30, 0.70	0.20, 0.80
	Anticipate curve	0.15, 0.85	0.35, 0.65

(a) Find the mixed-strategy Nash equilibrium to this simplified baseball game.

(b) What is each player's expected payoff for the game?

(c) Now suppose that the pitcher tries to improve his expected payoff in the mixed-strategy equilibrium by slowing down his fastball, thereby making it more similar to a curve ball. This changes the payoff to the hitter in the "anticipate fastball/throw fastball" cell from 0.30 to 0.25, and the pitcher's payoff adjusts accordingly. Can this modification improve the pitcher's expected payoff as desired? Explain your answer carefully and show your work. Also, explain *why* slowing the fastball can or cannot improve the pitcher's expected payoff in the game.

S6. Undeterred by their experiences with chicken so far (see Section 4.B), James and Dean decide to increase the excitement (and the stakes) by starting their cars farther apart. This way they can keep the crowd in suspense longer, and they'll be able to accelerate to even higher speeds before they may or may not be involved in a much more serious collision. The new game table thus has a higher penalty for collision.

		DEAN	
		Swerve	Straight
JAMES	Swerve	0, 0	−1, 1
	Straight	1, −1	−10, −10

(a) What is the mixed-strategy Nash equilibrium for this more dangerous version of chicken? Do James and Dean play Straight more or less often than in the game shown in Figure 7.4?

(b) What is the expected payoff to each player in the mixed-strategy equilibrium found in part (a)?

(c) James and Dean decide to play the chicken game repeatedly (say, in front of different crowds of reckless youths). Moreover, because they don't want to collide, they collude and alternate between the two pure-strategy equilibria. Assuming they play an even number of games, what is the average payoff to each of them when they collude in this way? Is this better or worse than they can expect from playing the mixed-strategy equilibrium? Why?

(d) After several weeks of not playing chicken as in part (c), James and Dean agree to play again. However, each of them has entirely forgotten which pure-strategy Nash equilibrium they played last time and neither realizes this until they're revving their engines moments before starting the game. Instead of playing the mixed-strategy Nash equilibrium, each of them tosses a separate coin to decide which strategy to play. What is the expected payoff to James and Dean when each mixes 50–50 in this way? How does this compare with their expected payoffs when they play their equilibrium mixtures? Explain why these payoffs are the same or different from those found in part (c).

S7. Section 2.B illustrates how to graph best-response curves for the tennis-point game. Section 4.B notes that when there are multiple equilibria, they can be identified from multiple intersections of the best-response curves. For the battle-of-the-sexes game in Figure 4.12

from Chapter 4, graph the best responses of Harry and Sally on a p-q coordinate plane. Label all of the Nash equilibria.

S8. Consider the following game:

		COLIN	
		Yes	No
ROWENA	Yes	x, x	0, 1
	No	1, 0	1, 1

(a) For what values of x does this game have a unique Nash equilibrium? What is that equilibrium?

(b) For what values of x does this game have a mixed-strategy Nash equilibrium? With what probability, expressed in terms of x, does each player play Yes in this mixed-strategy equilibrium?

(c) For the values of x found in part (b), is the game an example of an assurance game, a game of chicken, or a game similar to tennis? Explain.

(d) Let $x = 3$. Graph the best-response curves of Rowena and Colin on a p-q coordinate plane. Label all the Nash equilibria in pure and mixed strategies.

(e) Let $x = 1$. Graph the best-response curves of Rowena and Colin on a p-q coordinate plane. Label all the Nash equilibria in pure and mixed strategies.

S9. Consider the following game:

		PROFESSOR PLUM		
		Revolver	Knife	Wrench
MRS. PEACOCK	Conservatory	1, 3	2, −2	0, 6
	Ballroom	3, 1	1, 4	5, 0

(a) Graph the expected payoffs from each of Professor Plum's strategies as a function of Mrs. Peacock's p-mix.

(b) Over what range of p does Revolver yield a higher expected payoff for Professor Plum than Knife?

(c) Over what range of p does Revolver yield a higher expected payoff than Wrench?

(d) Which pure strategies will Professor Plum use in his equilibrium mixture? Why?

(e) What is the mixed-strategy Nash equilibrium of this game?

S10. Many of you will be familiar with the children's game rock-paper-scissors. In rock-paper-scissors, two people simultaneously choose either "rock," "paper," or "scissors," usually by putting their hands into the shape of one of the three choices. The game is scored as follows. A player choosing Scissors beats a player choosing Paper (because scissors cut paper). A player choosing Paper beats a player choosing Rock (because paper covers rock). A player choosing Rock beats a player choosing Scissors (because rock breaks scissors). If two players choose the same object, they tie. Suppose that each individual play of the game is worth 10 points. The following matrix shows the possible outcomes in the game:

		LISA		
		Rock	Scissors	Paper
BART	Rock	0, 0	10, −10	−10, 10
	Scissors	−10, 10	0, 0	10, −10
	Paper	10, −10	−10, 10	0, 0

(a) Derive the mixed-strategy equilibrium of the rock-paper-scissors game.

(b) Suppose that Lisa announced that she would use a mixture in which her probability of choosing Rock would be 40%, her probability of choosing Scissors would be 30%, and her probability of choosing Paper would be 30%. What is Bart's best response to this strategy choice by Player 2? Explain why your answer makes sense, given your knowledge of mixed strategies.

S11. Recall the game between ice-cream vendors on a beach from Exercise U6 in Chapter 6. In that game, we found two asymmetric pure-strategy equilibria. There is also a symmetric mixed-strategy equilibrium to the game.

(a) Write down the five-by-five table for the game.

(b) Eliminate dominated strategies, and explain why they should not be used in the equilibrium.

(c) Use your answer to part (b) to help you find the mixed-strategy equilibrium to the game.

S12. Suppose that the soccer penalty-kick game of Section 7.A in this chapter is expanded to include a total of six distinct strategies for the kicker: to shoot high and to the left (HL), low and to the left (LL), high and in the center (HC), low and in the center (LC), high right (HR), and low right (LR). The goalkeeper continues to have three strategies: to move to the kicker's left (L) or right (R) or to stay in the center (C). The players' success percentages are shown in the following table:

		GOALIE		
		L	C	R
KICKER	HL	0.50, 0.50	0.85, 0.15	0.85, 0.15
	LL	0.40, 0.60	0.95, 0.05	0.95, 0.05
	HC	0.85, 0.15	0, 0	0.85, 0.15
	LC	0.70, 0.30	0, 0	0.70, 0.30
	HR	0.85, 0.15	0.85, 0.15	0.50, 0.50
	LR	0.95, 0.05	0.95, 0.05	0.40, 0.60

In this problem, you will verify that the mixed-strategy equilibrium of this game entails the goalie using L and R each 42.2% of the time and C 15.6% of the time, while the kicker uses LL and LR each 37.8% of the time and HC 24.4% of the time.

(a) Given the goalie's proposed mixed strategy, compute the expected payoff to the kicker for each of her six pure strategies. (Use only three significant digits to keep things simple.)

(b) Use your answer to part (a) to explain why the kicker's proposed mixed strategy is a best response to the goalie's proposed mixed strategy.

(c) Given the kicker's proposed mixed strategy, compute the expected payoff to the goalie for each of her three pure strategies. (Again, use only three significant digits to keep things simple.)

(d) Use your answer to part (a) to explain why the goalie's proposed mixed strategy is a best response to the kicker's proposed mixed strategy.

(e) Using your previous answers, explain why the proposed strategies are indeed a Nash equilibrium.

(f) Compute the equilibrium payoff to the kicker.

S13. **(Optional)** In Section 5.B, we demonstrated for the assurance game that changing Sally's payoffs does not change her equilibrium mixing proportions—only Harry's payoffs determine her equilibrium mixture. In this exercise, you will prove this as a general result for the mixed-strategy equilibria of all two-by-two games. Consider a general two-by-two non-zero-sum game with the payoff table shown below:

		COLIN	
		Left	Right
ROW	Up	a, A	b, B
	Down	c, C	d, D

(a) Suppose the game has a mixed-strategy equilibrium. As a function of the payoffs in the table, solve for the probability that Rowena plays Up in equilibrium.

(b) Solve for the probability that Colin plays Left in equilibrium.

(c) Explain how your results show that each player's equilibrium mixtures depend only on the other player's payoffs.

(d) What conditions must be satisfied by the payoffs in order to guarantee that the game does indeed have a mixed-strategy equilibrium?

S14. **(Optional)** Recall Exercise S13 of Chapter 4, which was based on the bar scene from the film *A Beautiful Mind*. Here we consider the mixed-strategy equilibria of that game when played by $n > 2$ young men.

(a) Begin by considering the symmetric case in which each of the n young men independently goes after the solitary blonde with some probability P. This probability is determined by the condition that each young man should be indifferent between the pure strategies Blonde and Brunette, given that everyone else is mixing. What is the condition that guarantees the indifference of each player? What is the equilibrium value of P in this game?

(b) There are also some asymmetric mixed-strategy equilibria in this game. In these equilibria, $m < n$ young men each go for the blonde with probability Q, and the remaining $n - m$ young men go after the brunettes. What is the condition that guarantees that each of the m young men is indifferent, given what everyone else is doing? What condition must hold so that the remaining $n - m$ players don't want to switch from the pure strategy of choosing a brunette? What is the equilibrium value of Q in the asymmetric equilibrium?

UNSOLVED EXERCISES

U1. In football the offense can either run the ball or pass the ball, whereas the defense can either anticipate (and prepare for) a run or anticipate (and prepare for) a pass. Assume that the expected payoffs (in yards) for the two teams on any given down are as follows:

		DEFENSE	
		Anticipate Run	Anticipate Pass
OFFENSE	Run	1, −1	5, −5
	Pass	9, −9	−3, 3

(a) Show that this game has no pure-strategy Nash equilibrium.
(b) Find the unique mixed-strategy Nash equilibrium to this game.
(c) Explain why the mixture used by the offense is different from the mixture used by the defense.
(d) How many yards is the offense expected to gain per down in equilibrium?

U2. On the eve of a problem-set due date, a professor receives an e-mail from one of her students who claims to be stuck on one of the problems after working on it for more than an hour. The professor would rather help the student if he has sincerely been working, but she would rather not render aid if the student is just fishing for hints. Given the timing of the request, she could simply pretend not to have read the e-mail until later. Obviously, the student would rather receive help whether or not he has been working on the problem. But if help isn't coming, he would rather be working instead of slacking, since the problem set *is* due the next day. Assume the payoffs are as follows:

		STUDENT	
		Work and ask for help	Slack and fish for hints
PROFESSOR	Help student	3, 3	−1, 4
	Ignore e-mail	−2, 1	0, 0

(a) What is the mixed-strategy Nash equilibrium to this game?
(b) What is the expected payoff to each of the players?

U3. Exercise S12 in Chapter 4 introduced the game "Evens or Odds," which has no Nash equilibrium in pure strategies. It does have an equilibrium in mixed strategies.

(a) If Anne plays 1 (that is, she puts in one finger) with probability p, what is the expected payoff to Bruce from playing 1, in terms of p? What is his expected payoff from playing 2?

(b) What level of p will make Bruce indifferent between playing 1 and playing 2?

(c) If Bruce plays 1 with probability q, what level of q will make Anne indifferent between playing 1 and playing 2?

(d) Write the mixed-strategy equilibrium of this game. What is the expected payoff of the game to each player?

U4. Return again to the tennis rivals Evert and Navratilova, discussed in Section 2.A. Months later, they meet again in a new tournament. Evert has healed from her injury (see Exercise S4), but during that same time Navratilova has worked very hard on improving her defense against DL serves. The payoffs are now as follows:

		NAVRATILOVA	
		DL	CC
EVERT	DL	25, 75	80, 20
	CC	90, 10	20, 80

(a) Find each player's equilibrium mixture for the game above.

(b) What happened to Evert's p-mixture compared to the game presented in Section 2.A? Why?

(c) What is the expected value of the game to Evert? Why is it different from the expected value of the original game in Section 2.A?

U5. Section 4.A of this chapter discussed mixing in the battle-of-the-sexes game between Harry and Sally.

(a) What do you expect to happen to the equilibrium values of p and q found in the chapter if Sally decides she really likes Local Latte a lot more than Starbucks, so that the payoffs in the (Local Latte, Local Latte) cell are now (1, 3)? Explain your reasoning.

(b) Now find the new mixed-strategy equilibrium values of p and q. How do they compare with those of the original game?

(c) What is the expected payoff to each player in the new mixed-strategy equilibrium?

(d) Do you think Harry and Sally might play the mixed-strategy equilibrium in this new version of the game? Explain why or why not.

U6. Consider the following variant of chicken, in which James's payoff from being "tough" when Dean is "chicken" is 2, rather than 1:

		DEAN	
		Swerve	Straight
JAMES	Swerve	0, 0	−1, 1
	Straight	2, −1	−2, −2

(a) Find the mixed-strategy equilibrium in this game, including the expected payoffs for the players.

(b) Compare the results with those of the original game in Section 4.B of this chapter. Is Dean's probability of playing Straight (being tough) higher now than before? What about James's probability of playing Straight?

(c) What has happened to the two players' expected payoffs? Are these differences in the equilibrium outcomes paradoxical in light of the new payoff structure? Explain how your findings can be understood in light of the opponent's indifference principle.

U7. For the chicken game in Figure 4.13 from Chapter 4, graph the best responses of James and Dean on a p-q coordinate plane. Label all of the Nash equilibria.

U8. (a) Find all pure-strategy Nash equilibria of the following game:

		COLIN			
		L	M	N	R
ROWENA	Up	1, 1	2, 2	3, 4	9, 3
	Down	2, 5	3, 3	1, 2	7, 1

(b) Now find a mixed-strategy equilibrium of the game. What are the players' expected payoffs in the equilibrium?

U9. Consider a revised version of the game from Exercise S9:

		PROFESSOR PLUM		
		Revolver	Knife	Wrench
MRS. PEACOCK	Conservatory	1, 3	2, −2	0, 6
	Ballroom	3, 2	1, 4	5, 0

(a) Graph the expected payoffs from each of Professor Plum's strategies as a function of Mrs. Peacock's p-mix.

(b) Which strategies will Professor Plum use in his equilibrium mixture? Why?

(c) What is the mixed-strategy Nash equilibrium of this game?

(d) Note that this game is only slightly different from the game in Exercise S9. How are the two games different? Explain why you intuitively think the equilibrium outcome has changed from Exercise S9.

U10. Consider a modified version of rock-paper-scissors in which Bart gets a bonus when he wins with Rock. If Bart picks Rock while Lisa picks Scissors, Bart wins twice as many points as when either player wins in any other way. The new payoff matrix is:

		LISA		
		Rock	Scissors	Paper
BART	Rock	0, 0	20, −20	−10, 10
	Scissors	−10, 10	0, 0	10, −10
	Paper	10, −10	−10, 10	0, 0

(a) What is the mixed-strategy equilibrium in this version of the game?

(b) Compare your answer here with your answer for the mixed-strategy equilibrium in Exercise S10. How can you explain the differences in the equilibrium strategy choices?

U11. Consider the following game:

		MACARTHUR		
		Air	Sea	Land
PATTON	Air	0, 3	2, 0	1, 7
	Sea	2, 4	0, 6	2, 0
	Land	1, 3	2, 4	0, 3

(a) Does this game have a pure-strategy Nash equilibrium? If so, what is it?

(b) Find a mixed-strategy equilibrium to this game.

(c) Actually, this game has two mixed-strategy equilibria. Find the one you didn't find in part (b). (Hint: In one of these equilibria, one of the players plays a mixed strategy, whereas the other plays a pure strategy.)

U12. The recalcitrant James and Dean are playing their more dangerous variant of chicken again (see Exercise S6). They've noticed that their payoff for being perceived as "tough" varies with the size of the crowd. The larger the crowd, the more glory and praise each receives from driving straight when his opponent swerves. Smaller crowds, of course, have the opposite effect. Let $k > 0$ be the payoff for appearing "tough." The game may now be represented as:

		DEAN	
		Swerve	Straight
JAMES	Swerve	0, 0	-1, k
	Straight	k, -1	-10, -10

(a) Expressed in terms of k, with what probability does each driver play Swerve in the mixed-strategy Nash equilibrium? Do James and Dean play Swerve more or less often as k increases?

(b) In terms of k, what is the expected value of the game to each player in the mixed-strategy Nash equilibrium found in part (a)?

(c) At what value of k do both James and Dean mix 50–50 in the mixed-strategy equilibrium?

(d) How large must k be for the average payoff to be positive under the alternating scheme discussed in part (c) of Exercise S6?

U13. **(Optional)** Recall the game from Exercise S11 in Chapter 4, where Larry, Moe, and Curly can choose to buy tickets toward a prize worth \$30. We found six pure-strategy Nash equilibria in that game. In this problem, you will find a symmetric equilibrium in mixed strategies.

(a) Eliminate the weakly dominated strategy for each player. Explain why a player would never use this weakly dominated strategy in his equilibrium mixture.

(b) Find the equilibrium in mixed strategies.

U14. **(Optional)** Exercises S4 and U4 demonstrate that in zero-sum games such as the Evert-Navratilova tennis rivalry, changes in a player's payoffs can sometimes lead to unexpected or unintuitive changes to her equilibrium mixture. But what happens to the expected value of the game? Consider the following general form of a two-player zero-sum game:

		COLIN	
		L	R
ROWENA	U	a, −a	b, −b
	D	c, −c	d, −d

Assume that there is no Nash equilibrium in pure strategies, and assume that *a*, *b*, *c*, and *d* are all greater than or equal to 0. Can an *increase* in any one of *a*, *b*, *c*, or *d* lead to a *lower* expected value of the game for Rowena? If not, prove why not. If so, provide an example.

Appendix: Probability and Expected Utility

To calculate the expected payoffs and mixed-strategy equilibria of games in this chapter, we had to do some simple manipulation of probabilities. Some simple rules govern calculations involving probabilities. Many of you may be familiar with them, but we give a brief statement and explanation of the basics here by way of reminder or remediation, as appropriate. We also state how to calculate expected values of random numerical values.

THE BASIC ALGEBRA OF PROBABILITIES

The basic intuition about the probability of an event comes from thinking about the frequency with which this event occurs by chance among a larger set of possibilities. Usually, any one element of this larger set is just as likely to occur by chance as any other, so finding the probability of the event in which we are interested is simply a matter of counting the elements corresponding to that event and dividing by the total number of elements in the whole large set.[23]

In any standard deck of 52 playing cards, for instance, there are four suits (clubs, diamonds, hearts, and spades) and 13 cards in each suit (ace through 10 and the face cards—jack, queen, king). We can ask a variety of questions about the likelihood that a card of a particular suit or value—or suit *and* value—might be drawn from this deck of cards: How likely are we to draw a spade? How likely are we to draw a black card? How likely are we to draw a 10? How likely are we to draw the queen of spades? and so on. We would need to know something about the calculation and manipulation of probabilities to answer such questions. If we had two decks of cards, one with blue backs and one with green backs, we

[23] When we say "by chance," we simply mean that a systematic order cannot be detected in the outcome or that it cannot be determined by using available scientific methods of prediction and calculation. Actually, the motions of coins and dice are fully determined by laws of physics, and highly skilled people can manipulate decks of cards, but for all practical purposes, coin tosses, rolls of dice, or card shuffles are devices of chance that can be used to generate random outcomes. However, randomness can be harder to achieve than you think. For example, a perfect shuffle, where a deck of cards is divided exactly in half and then interleaved by dropping cards one at a time alternately from each, may seem a good way to destroy the initial order of the deck. But Cornell mathematician Persi Diaconis has shown that, after eight of the shuffles, the original order is fully restored. For slightly imperfect shuffles that people carry out in reality, he finds that some order persists through six, but randomness suddenly appears on the seventh! See "How to Win at Poker, and Other Science Lessons," *The Economist*, October 12, 1996. For an interesting discussion of such topics, see Deborah J. Bennett, *Randomness* (Cambridge, Mass.: Harvard University Press, 1998), chs. 6–9.

could ask even more complex questions ("How likely are we to draw one card from each deck and have them both be the jack of diamonds?"), but we would still use the algebra of probabilities to answer them.

In general, a **probability** measures the likelihood of a particular event or set of events occurring. The likelihood that you draw a spade from a deck of cards is just the probability of the event "drawing a spade." Here the large set has 52 elements—the total number of equally likely possibilities—and the event "drawing a spade" corresponds to a subset of 13 particular elements. Thus, you have 13 chances out of the 52 to get a spade, which makes the probability of getting a spade in a single draw equal to $13/52 = 1/4 = 25\%$. To see this another way, consider the fact that there are four suits of 13 cards each, so your chance of drawing a card from any particular suit is one out of four, or 25%. If you made a large number of such draws (each time from a complete deck), then out of 52 times you will not always draw exactly 13 spades; by chance you may draw a few more or a few less. But the chance averages out over different such occasions—over different sets of 52 draws. Then the probability of 25% is the average of the frequencies of spades drawn in a large number of observations.[24]

The algebra of probabilities simply develops such ideas in general terms and obtains formulas that you can then apply mechanically instead of having to do the thinking from scratch every time. We will organize our discussion of these probability formulas around the types of questions that one might ask when drawing cards from a standard deck (or two: blue backed and green backed).[25] This method will allow us to provide both specific and general formulas for you to use later. You can use the card-drawing analogy to help you reason out other questions about probabilities that you encounter in other contexts. One other point to note: In ordinary language, it is customary to write probabilities as percentages, but the algebra requires that they be written as fractions or decimals; thus instead of 25%, the mathematics works with 13/52, or 0.25. We will use one or the other, depending on the occasion; be aware that they mean the same thing.

A. The Addition Rule

The first questions that we ask are: If we were to draw one card from the blue deck, how likely are we to draw a spade? And how likely are we to draw a card that is not a spade? We already know that the probability of drawing a spade is 25% because we determined that earlier. But what is the probability of drawing

[24] Bennett, *Randomness*, chs. 4 and 5, offers several examples of such calculations of probabilities.

[25] If you want a more detailed exposition of the following addition and multiplication rules, as well as more exercises to practice these rules, we recommend David Freeman, Robert Pisani, and Robert Purves, *Statistics*, 4th ed. (New York: W. W. Norton & Company, 2007), chs. 13 and 14.

a card that is not a spade? It is the same likelihood of drawing a club or a diamond or a heart instead of a spade. It should be clear that the probability in question should be larger than any of the individual probabilities of which it is formed; in fact, the probability is 13/52 (clubs) + 13/52 (diamonds) + 13/52 (hearts) = 0.75. The *or* in our verbal interpretation of the question is the clue that the probabilities should be added together, because we want to know the chances of drawing a card from any of those three suits.

We could more easily have found our answer to the second question by noting that not getting a spade is what happens the other 75% of the time. Thus, the probability of drawing "not a spade" is 75% (100% − 25%) or, more formally, $1 - 0.25 = 0.75$. As is often the case with probability calculations, the same result can be obtained here by two different routes, entailing different ways of thinking about the event for which we are trying to find the probability. We will see other examples of this later in this appendix, where it will become clear that the different methods of calculation can sometimes require vastly different amounts of effort. As you develop experience, you will discover and remember the easy ways or shortcuts. In the meantime, be comforted that each of the different routes, when correctly followed, leads to the same final answer.

To generalize our preceding calculation, we note that, if you divide the set of events, X, in which you are interested into some number of subsets, $Y, Z, \ldots$, none of which overlap (in mathematical terminology, such subsets are said to be **disjoint**), then the probabilities of each subset occurring must sum to the probability of the full set of events; if that full set of events includes all possible outcomes, then its probability is 1. In other words, if the occurrence of X requires the occurrence of *any one* of several disjoint $Y, Z, \ldots$, then the probability of X is the sum of the separate probabilities of $Y, Z, \ldots$. Using Prob(X) to denote the probability that X occurs and remembering the caveats on X (that it requires any one of $Y, Z, \ldots$) and on $Y, Z, \ldots$ (that they must be disjoint), we can write the **addition rule** in mathematical notation as $\text{Prob}(X) = \text{Prob}(Y) + \text{Prob}(Z) + \cdots$.

EXERCISE Use the addition rule to find the probability of drawing two cards, one from each deck, such that the two cards have identical faces.

B. The Multiplication Rule

Now we ask: What is the likelihood that when we draw two cards, one from each deck, both of them will be spades? This event occurs if we draw a spade from the blue deck *and* a spade from the green deck. The switch from *or* to *and* in our interpretation of what we are looking for indicates a switch in mathematical operations from addition to multiplication. Thus, the probability of two spades, one from each deck, is the product of the probabilities of drawing a spade from each deck, or $(13/52) \times (13/52) = 1/16 = 0.0625$, or 6.25%. Not surprisingly, we are much less likely to get two spades than we were in the previous section to get one spade. (Always check to make sure that your calculations accord in this way with your intuition regarding the outcome.)

In much the same way as the addition rule requires events to be disjoint, the multiplication rule requires them to be independent: if we break down a set of events, X, into some number of subsets Y, Z, . . . , those subsets are independent if the occurrence of one does not affect the probability of the other. Our events—a spade from the blue deck and a spade from the green deck—satisfy this condition of independence; that is, drawing a spade from the blue deck does nothing to alter the probability of getting a spade from the green deck. If we were drawing both cards from the same deck, however, then after we had drawn a spade (with a probability of 13/52), the probability of drawing another spade would no longer be 13/52 (in fact, it would be 12/51); drawing one spade and then a second spade from the *same* deck are not **independent events**.

The formal statement of the **multiplication rule** tells us that, if the occurrence of X requires the simultaneous occurrence of *all* the several independent Y, Z, . . . , then the probability of X is the *product* of the separate probabilities of Y, Z, . . . : $\text{Prob}(X) = \text{Prob}(Y) \times \text{Prob}(Z) \times \cdots$.

> **EXERCISE** Use the multiplication rule to find the probability of drawing two cards, one from each deck, and getting a red card from the blue deck and a face card from the green deck.

C. Expected Values

If a numerical magnitude (such as money winnings or rainfall) is subject to chance and can take on any one of n possible values X_1, X_2, . . . , X_n with respective probabilities p_1, p_2, . . . , p_n, then the **expected value** is defined as the weighted average of all its possible values using the probabilities as weights; that is, as $p_1X_1 + p_2X_2 + \cdots + p_nX_n$. For example, suppose you bet on the toss of two fair coins. You win \$5 if both coins come up heads, \$1 if one shows heads and the other tails, and nothing if both come up tails. Using the rules for manipulating probabilities discussed earlier in this section, you can see that the probabilities of these events are, respectively, 0.25, 0.50, and 0.25. Therefore, your expected winnings are $(0.25 \times \$5) + (0.50 \times \$1) + (0.25 \times \$0) = \$1.75$.

In game theory, the numerical magnitudes that we need to average in this way are payoffs, measured in numerical ratings, or money, or, as we will see later in the appendix to Chapter 8, utilities. We will refer to the expected values in each context appropriately, for example, as *expected payoffs* or *expected utilities.*

SUMMARY

The *probability* of an event is the likelihood of its occurrence by chance from among a larger set of possibilities. Probabilities can be combined by using some rules. The *addition rule* says that the probability of any one of a number of *disjoint* events occurring is the sum of the probabilities of these events. According to the *multiplication rule,* the probability that all of a number of *independent events* will occur is the product of the probabilities of these events. Probability-weighted averages are used to compute *expected payoffs* in games.

KEY TERMS

addition rule (265)
disjoint (265)
expected value (266)
independent events (266)
multiplication rule (266)
probability (264)