2. *Evaluate:* Compute point-value representations of $A(x)$ and $B(x)$ of length $2n$ by applying the FFT of order $2n$ on each polynomial. These representations contain the values of the two polynomials at the $(2n)$th roots of unity.
3. *Pointwise multiply:* Compute a point-value representation for the polynomial $C(x) = A(x)B(x)$ by multiplying these values together pointwise. This representation contains the value of $C(x)$ at each $(2n)$th root of unity.
4. *Interpolate:* Create the coefficient representation of the polynomial $C(x)$ by applying the FFT on $2n$ point-value pairs to compute the inverse DFT.

Steps (1) and (3) take time $\Theta(n)$, and steps (2) and (4) take time $\Theta(n \lg n)$. Thus, once we show how to use the FFT, we will have proven the following.

Theorem 30.2
We can multiply two polynomials of degree-bound n in time $\Theta(n \lg n)$, with both the input and output representations in coefficient form. ■

Exercises

30.1-1
Multiply the polynomials $A(x) = 7x^3 - x^2 + x - 10$ and $B(x) = 8x^3 - 6x + 3$ using equations (30.1) and (30.2).

30.1-2
Another way to evaluate a polynomial $A(x)$ of degree-bound n at a given point x_0 is to divide $A(x)$ by the polynomial $(x - x_0)$, obtaining a quotient polynomial $q(x)$ of degree-bound $n - 1$ and a remainder r, such that

$$A(x) = q(x)(x - x_0) + r \ .$$

Clearly, $A(x_0) = r$. Show how to compute the remainder r and the coefficients of $q(x)$ in time $\Theta(n)$ from x_0 and the coefficients of A.

30.1-3
Derive a point-value representation for $A^{\text{rev}}(x) = \sum_{j=0}^{n-1} a_{n-1-j} x^j$ from a point-value representation for $A(x) = \sum_{j=0}^{n-1} a_j x^j$, assuming that none of the points is 0.

30.1-4
Prove that n distinct point-value pairs are necessary to uniquely specify a polynomial of degree-bound n, that is, if fewer than n distinct point-value pairs are given, they fail to specify a unique polynomial of degree-bound n. (*Hint:* Using Theorem 30.1, what can you say about a set of $n - 1$ point-value pairs to which you add one more arbitrarily chosen point-value pair?)

30.1-5

Show how to use equation (30.5) to interpolate in time $\Theta(n^2)$. (*Hint:* First compute the coefficient representation of the polynomial $\prod_j (x - x_j)$ and then divide by $(x - x_k)$ as necessary for the numerator of each term; see Exercise 30.1-2. You can compute each of the n denominators in time $O(n)$.)

30.1-6

Explain what is wrong with the "obvious" approach to polynomial division using a point-value representation, i.e., dividing the corresponding y values. Discuss separately the case in which the division comes out exactly and the case in which it doesn't.

30.1-7

Consider two sets A and B, each having n integers in the range from 0 to $10n$. We wish to compute the ***Cartesian sum*** of A and B, defined by

$$C = \{x + y : x \in A \text{ and } y \in B\} .$$

Note that the integers in C are in the range from 0 to $20n$. We want to find the elements of C and the number of times each element of C is realized as a sum of elements in A and B. Show how to solve the problem in $O(n \lg n)$ time. (*Hint:* Represent A and B as polynomials of degree at most $10n$.)

30.2 The DFT and FFT

In Section 30.1, we claimed that if we use complex roots of unity, we can evaluate and interpolate polynomials in $\Theta(n \lg n)$ time. In this section, we define complex roots of unity and study their properties, define the DFT, and then show how the FFT computes the DFT and its inverse in $\Theta(n \lg n)$ time.

Complex roots of unity

A ***complex nth root of unity*** is a complex number ω such that

$$\omega^n = 1 .$$

There are exactly n complex nth roots of unity: $e^{2\pi ik/n}$ for $k = 0, 1, \ldots, n-1$. To interpret this formula, we use the definition of the exponential of a complex number:

$$e^{iu} = \cos(u) + i \sin(u) .$$

Figure 30.2 shows that the n complex roots of unity are equally spaced around the circle of unit radius centered at the origin of the complex plane. The value

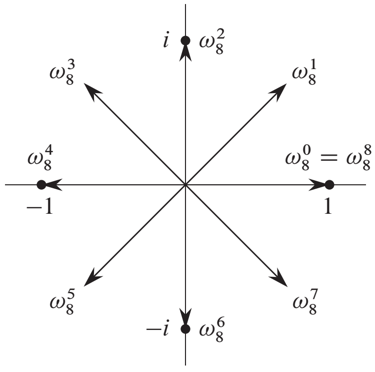


Figure 30.2 The values of $\omega_8^0, \omega_8^1, \ldots, \omega_8^7$ in the complex plane, where $\omega_8 = e^{2\pi i/8}$ is the principal 8th root of unity.

$$\omega_n = e^{2\pi i/n} \tag{30.6}$$

is the ***principal nth root of unity***;[2] all other complex nth roots of unity are powers of ω_n.

The n complex nth roots of unity,

$$\omega_n^0, \omega_n^1, \ldots, \omega_n^{n-1} ,$$

form a group under multiplication (see Section 31.3). This group has the same structure as the additive group $(\mathbb{Z}_n, +)$ modulo n, since $\omega_n^n = \omega_n^0 = 1$ implies that $\omega_n^j \omega_n^k = \omega_n^{j+k} = \omega_n^{(j+k) \bmod n}$. Similarly, $\omega_n^{-1} = \omega_n^{n-1}$. The following lemmas furnish some essential properties of the complex nth roots of unity.

Lemma 30.3 (Cancellation lemma)
For any integers $n \geq 0$, $k \geq 0$, and $d > 0$,

$$\omega_{dn}^{dk} = \omega_n^k . \tag{30.7}$$

Proof The lemma follows directly from equation (30.6), since

$$\begin{aligned}
\omega_{dn}^{dk} &= \left(e^{2\pi i/dn}\right)^{dk} \\
&= \left(e^{2\pi i/n}\right)^{k} \\
&= \omega_n^k .
\end{aligned}$$

■

[2]Many other authors define ω_n differently: $\omega_n = e^{-2\pi i/n}$. This alternative definition tends to be used for signal-processing applications. The underlying mathematics is substantially the same with either definition of ω_n.

Corollary 30.4
For any even integer $n > 0$,

$$\omega_n^{n/2} = \omega_2 = -1 .$$

Proof The proof is left as Exercise 30.2-1. ■

Lemma 30.5 (Halving lemma)
If $n > 0$ is even, then the squares of the n complex nth roots of unity are the $n/2$ complex $(n/2)$th roots of unity.

Proof By the cancellation lemma, we have $(\omega_n^k)^2 = \omega_{n/2}^k$, for any nonnegative integer k. Note that if we square all of the complex nth roots of unity, then we obtain each $(n/2)$th root of unity exactly twice, since

$$\begin{aligned}(\omega_n^{k+n/2})^2 &= \omega_n^{2k+n} \\ &= \omega_n^{2k}\omega_n^n \\ &= \omega_n^{2k} \\ &= (\omega_n^k)^2 .\end{aligned}$$

Thus, ω_n^k and $\omega_n^{k+n/2}$ have the same square. We could also have used Corollary 30.4 to prove this property, since $\omega_n^{n/2} = -1$ implies $\omega_n^{k+n/2} = -\omega_n^k$, and thus $(\omega_n^{k+n/2})^2 = (\omega_n^k)^2$. ■

As we shall see, the halving lemma is essential to our divide-and-conquer approach for converting between coefficient and point-value representations of polynomials, since it guarantees that the recursive subproblems are only half as large.

Lemma 30.6 (Summation lemma)
For any integer $n \geq 1$ and nonzero integer k not divisible by n,

$$\sum_{j=0}^{n-1} \left(\omega_n^k\right)^j = 0 .$$

Proof Equation (A.5) applies to complex values as well as to reals, and so we have

$$\begin{aligned}
\sum_{j=0}^{n-1} \left(\omega_n^k\right)^j &= \frac{(\omega_n^k)^n - 1}{\omega_n^k - 1} \\
&= \frac{(\omega_n^n)^k - 1}{\omega_n^k - 1} \\
&= \frac{(1)^k - 1}{\omega_n^k - 1} \\
&= 0 .
\end{aligned}$$

Because we require that k is not divisible by n, and because $\omega_n^k = 1$ only when k is divisible by n, we ensure that the denominator is not 0. ■

The DFT

Recall that we wish to evaluate a polynomial

$$A(x) = \sum_{j=0}^{n-1} a_j x^j$$

of degree-bound n at $\omega_n^0, \omega_n^1, \omega_n^2, \ldots, \omega_n^{n-1}$ (that is, at the n complex nth roots of unity).[3] We assume that A is given in coefficient form: $a = (a_0, a_1, \ldots, a_{n-1})$. Let us define the results y_k, for $k = 0, 1, \ldots, n-1$, by

$$\begin{aligned}
y_k &= A(\omega_n^k) \\
&= \sum_{j=0}^{n-1} a_j \omega_n^{kj} . \qquad (30.8)
\end{aligned}$$

The vector $y = (y_0, y_1, \ldots, y_{n-1})$ is the ***discrete Fourier transform (DFT)*** of the coefficient vector $a = (a_0, a_1, \ldots, a_{n-1})$. We also write $y = \mathrm{DFT}_n(a)$.

The FFT

By using a method known as the ***fast Fourier transform (FFT)***, which takes advantage of the special properties of the complex roots of unity, we can compute $\mathrm{DFT}_n(a)$ in time $\Theta(n \lg n)$, as opposed to the $\Theta(n^2)$ time of the straightforward method. We assume throughout that n is an exact power of 2. Although strategies

[3]The length n is actually what we referred to as $2n$ in Section 30.1, since we double the degree-bound of the given polynomials prior to evaluation. In the context of polynomial multiplication, therefore, we are actually working with complex $(2n)$th roots of unity.

for dealing with non-power-of-2 sizes are known, they are beyond the scope of this book.

The FFT method employs a divide-and-conquer strategy, using the even-indexed and odd-indexed coefficients of $A(x)$ separately to define the two new polynomials $A^{[0]}(x)$ and $A^{[1]}(x)$ of degree-bound $n/2$:

$$\begin{aligned} A^{[0]}(x) &= a_0 + a_2x + a_4x^2 + \cdots + a_{n-2}x^{n/2-1} , \\ A^{[1]}(x) &= a_1 + a_3x + a_5x^2 + \cdots + a_{n-1}x^{n/2-1} . \end{aligned}$$

Note that $A^{[0]}$ contains all the even-indexed coefficients of A (the binary representation of the index ends in 0) and $A^{[1]}$ contains all the odd-indexed coefficients (the binary representation of the index ends in 1). It follows that

$$A(x) = A^{[0]}(x^2) + xA^{[1]}(x^2) , \tag{30.9}$$

so that the problem of evaluating $A(x)$ at $\omega_n^0, \omega_n^1, \ldots, \omega_n^{n-1}$ reduces to

1. evaluating the degree-bound $n/2$ polynomials $A^{[0]}(x)$ and $A^{[1]}(x)$ at the points

   $$(\omega_n^0)^2, (\omega_n^1)^2, \ldots, (\omega_n^{n-1})^2 , \tag{30.10}$$

   and then
2. combining the results according to equation (30.9).

By the halving lemma, the list of values (30.10) consists not of n distinct values but only of the $n/2$ complex $(n/2)$th roots of unity, with each root occurring exactly twice. Therefore, we recursively evaluate the polynomials $A^{[0]}$ and $A^{[1]}$ of degree-bound $n/2$ at the $n/2$ complex $(n/2)$th roots of unity. These subproblems have exactly the same form as the original problem, but are half the size. We have now successfully divided an n-element DFT_n computation into two $n/2$-element $\text{DFT}_{n/2}$ computations. This decomposition is the basis for the following recursive FFT algorithm, which computes the DFT of an n-element vector $a = (a_0, a_1, \ldots, a_{n-1})$, where n is a power of 2.

```
RECURSIVE-FFT($a$)
 1  $n = a.length$                    // $n$ is a power of 2
 2  if $n == 1$
 3      return $a$
 4  $\omega_n = e^{2\pi i/n}$
 5  $\omega = 1$
 6  $a^{[0]} = (a_0, a_2, \ldots, a_{n-2})$
 7  $a^{[1]} = (a_1, a_3, \ldots, a_{n-1})$
 8  $y^{[0]} = $ RECURSIVE-FFT($a^{[0]}$)
 9  $y^{[1]} = $ RECURSIVE-FFT($a^{[1]}$)
10  for $k = 0$ to $n/2 - 1$
11      $y_k = y_k^{[0]} + \omega\, y_k^{[1]}$
12      $y_{k+(n/2)} = y_k^{[0]} - \omega\, y_k^{[1]}$
13      $\omega = \omega\, \omega_n$
14  return $y$                        // $y$ is assumed to be a column vector
```

The RECURSIVE-FFT procedure works as follows. Lines 2–3 represent the basis of the recursion; the DFT of one element is the element itself, since in this case

$$\begin{aligned} y_0 &= a_0\, \omega_1^0 \\ &= a_0 \cdot 1 \\ &= a_0 \ . \end{aligned}$$

Lines 6–7 define the coefficient vectors for the polynomials $A^{[0]}$ and $A^{[1]}$. Lines 4, 5, and 13 guarantee that ω is updated properly so that whenever lines 11–12 are executed, we have $\omega = \omega_n^k$. (Keeping a running value of ω from iteration to iteration saves time over computing ω_n^k from scratch each time through the **for** loop.) Lines 8–9 perform the recursive $\mathrm{DFT}_{n/2}$ computations, setting, for $k = 0, 1, \ldots, n/2 - 1$,

$$\begin{aligned} y_k^{[0]} &= A^{[0]}(\omega_{n/2}^k) \ , \\ y_k^{[1]} &= A^{[1]}(\omega_{n/2}^k) \ , \end{aligned}$$

or, since $\omega_{n/2}^k = \omega_n^{2k}$ by the cancellation lemma,

$$\begin{aligned} y_k^{[0]} &= A^{[0]}(\omega_n^{2k}) \ , \\ y_k^{[1]} &= A^{[1]}(\omega_n^{2k}) \ . \end{aligned}$$

Lines 11–12 combine the results of the recursive $\text{DFT}_{n/2}$ calculations. For $y_0, y_1, \ldots, y_{n/2-1}$, line 11 yields

$$\begin{aligned} y_k &= y_k^{[0]} + \omega_n^k y_k^{[1]} \\ &= A^{[0]}(\omega_n^{2k}) + \omega_n^k A^{[1]}(\omega_n^{2k}) \\ &= A(\omega_n^k) && \text{(by equation (30.9))} . \end{aligned}$$

For $y_{n/2}, y_{n/2+1}, \ldots, y_{n-1}$, letting $k = 0, 1, \ldots, n/2 - 1$, line 12 yields

$$\begin{aligned} y_{k+(n/2)} &= y_k^{[0]} - \omega_n^k y_k^{[1]} \\ &= y_k^{[0]} + \omega_n^{k+(n/2)} y_k^{[1]} && (\text{since } \omega_n^{k+(n/2)} = -\omega_n^k) \\ &= A^{[0]}(\omega_n^{2k}) + \omega_n^{k+(n/2)} A^{[1]}(\omega_n^{2k}) \\ &= A^{[0]}(\omega_n^{2k+n}) + \omega_n^{k+(n/2)} A^{[1]}(\omega_n^{2k+n}) && (\text{since } \omega_n^{2k+n} = \omega_n^{2k}) \\ &= A(\omega_n^{k+(n/2)}) && \text{(by equation (30.9))} . \end{aligned}$$

Thus, the vector y returned by RECURSIVE-FFT is indeed the DFT of the input vector a.

Lines 11 and 12 multiply each value $y_k^{[1]}$ by ω_n^k, for $k = 0, 1, \ldots, n/2 - 1$. Line 11 adds this product to $y_k^{[0]}$, and line 12 subtracts it. Because we use each factor ω_n^k in both its positive and negative forms, we call the factors ω_n^k ***twiddle factors***.

To determine the running time of procedure RECURSIVE-FFT, we note that exclusive of the recursive calls, each invocation takes time $\Theta(n)$, where n is the length of the input vector. The recurrence for the running time is therefore

$$\begin{aligned} T(n) &= 2T(n/2) + \Theta(n) \\ &= \Theta(n \lg n) . \end{aligned}$$

Thus, we can evaluate a polynomial of degree-bound n at the complex nth roots of unity in time $\Theta(n \lg n)$ using the fast Fourier transform.

Interpolation at the complex roots of unity

We now complete the polynomial multiplication scheme by showing how to interpolate the complex roots of unity by a polynomial, which enables us to convert from point-value form back to coefficient form. We interpolate by writing the DFT as a matrix equation and then looking at the form of the matrix inverse.

From equation (30.4), we can write the DFT as the matrix product $y = V_n a$, where V_n is a Vandermonde matrix containing the appropriate powers of ω_n:

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \omega_n^3 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \omega_n^6 & \cdots & \omega_n^{2(n-1)} \\ 1 & \omega_n^3 & \omega_n^6 & \omega_n^9 & \cdots & \omega_n^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \omega_n^{3(n-1)} & \cdots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{pmatrix} .$$

The (k, j) entry of V_n is ω_n^{kj}, for $j, k = 0, 1, \ldots, n-1$. The exponents of the entries of V_n form a multiplication table.

For the inverse operation, which we write as $a = \text{DFT}_n^{-1}(y)$, we proceed by multiplying y by the matrix V_n^{-1}, the inverse of V_n.

Theorem 30.7
For $j, k = 0, 1, \ldots, n-1$, the (j, k) entry of V_n^{-1} is ω_n^{-kj}/n.

Proof We show that $V_n^{-1} V_n = I_n$, the $n \times n$ identity matrix. Consider the (j, j') entry of $V_n^{-1} V_n$:

$$\begin{aligned} [V_n^{-1} V_n]_{jj'} &= \sum_{k=0}^{n-1} (\omega_n^{-kj}/n)(\omega_n^{kj'}) \\ &= \sum_{k=0}^{n-1} \omega_n^{k(j'-j)}/n . \end{aligned}$$

This summation equals 1 if $j' = j$, and it is 0 otherwise by the summation lemma (Lemma 30.6). Note that we rely on $-(n-1) \le j' - j \le n-1$, so that $j' - j$ is not divisible by n, in order for the summation lemma to apply. ■

Given the inverse matrix V_n^{-1}, we have that $\text{DFT}_n^{-1}(y)$ is given by

$$a_j = \frac{1}{n} \sum_{k=0}^{n-1} y_k \omega_n^{-kj} \qquad (30.11)$$

for $j = 0, 1, \ldots, n-1$. By comparing equations (30.8) and (30.11), we see that by modifying the FFT algorithm to switch the roles of a and y, replace ω_n by ω_n^{-1}, and divide each element of the result by n, we compute the inverse DFT (see Exercise 30.2-4). Thus, we can compute DFT_n^{-1} in $\Theta(n \lg n)$ time as well.

We see that, by using the FFT and the inverse FFT, we can transform a polynomial of degree-bound n back and forth between its coefficient representation and a point-value representation in time $\Theta(n \lg n)$. In the context of polynomial multiplication, we have shown the following.

Theorem 30.8 (Convolution theorem)
For any two vectors a and b of length n, where n is a power of 2,

$$a \otimes b = \mathrm{DFT}_{2n}^{-1}(\mathrm{DFT}_{2n}(a) \cdot \mathrm{DFT}_{2n}(b)) ,$$

where the vectors a and b are padded with 0s to length $2n$ and $\cdot$ denotes the componentwise product of two $2n$-element vectors. ■

Exercises

30.2-1
Prove Corollary 30.4.

30.2-2
Compute the DFT of the vector $(0, 1, 2, 3)$.

30.2-3
Do Exercise 30.1-1 by using the $\Theta(n \lg n)$-time scheme.

30.2-4
Write pseudocode to compute DFT_n^{-1} in $\Theta(n \lg n)$ time.

30.2-5
Describe the generalization of the FFT procedure to the case in which n is a power of 3. Give a recurrence for the running time, and solve the recurrence.

30.2-6 ★
Suppose that instead of performing an n-element FFT over the field of complex numbers (where n is even), we use the ring $\mathbb{Z}_m$ of integers modulo m, where $m = 2^{tn/2} + 1$ and t is an arbitrary positive integer. Use $\omega = 2^t$ instead of ω_n as a principal nth root of unity, modulo m. Prove that the DFT and the inverse DFT are well defined in this system.

30.2-7
Given a list of values $z_0, z_1, \ldots, z_{n-1}$ (possibly with repetitions), show how to find the coefficients of a polynomial $P(x)$ of degree-bound $n + 1$ that has zeros only at $z_0, z_1, \ldots, z_{n-1}$ (possibly with repetitions). Your procedure should run in time $O(n \lg^2 n)$. (*Hint:* The polynomial $P(x)$ has a zero at z_j if and only if $P(x)$ is a multiple of $(x - z_j)$.)

30.2-8 ★
The ***chirp transform*** of a vector $a = (a_0, a_1, \ldots, a_{n-1})$ is the vector $y = (y_0, y_1, \ldots, y_{n-1})$, where $y_k = \sum_{j=0}^{n-1} a_j z^{kj}$ and z is any complex number. The

DFT is therefore a special case of the chirp transform, obtained by taking $z = \omega_n$. Show how to evaluate the chirp transform in time $O(n \lg n)$ for any complex number z. (*Hint:* Use the equation

$$y_k = z^{k^2/2} \sum_{j=0}^{n-1} \left(a_j z^{j^2/2}\right) \left(z^{-(k-j)^2/2}\right)$$

to view the chirp transform as a convolution.)

30.3 Efficient FFT implementations

Since the practical applications of the DFT, such as signal processing, demand the utmost speed, this section examines two efficient FFT implementations. First, we shall examine an iterative version of the FFT algorithm that runs in $\Theta(n \lg n)$ time but can have a lower constant hidden in the Θ-notation than the recursive version in Section 30.2. (Depending on the exact implementation, the recursive version may use the hardware cache more efficiently.) Then, we shall use the insights that led us to the iterative implementation to design an efficient parallel FFT circuit.

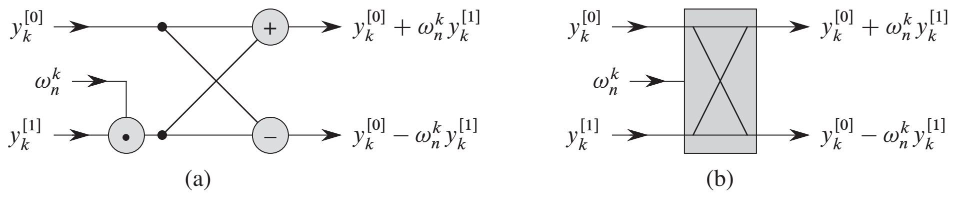
An iterative FFT implementation

We first note that the **for** loop of lines 10–13 of RECURSIVE-FFT involves computing the value $\omega_n^k \, y_k^{[1]}$ twice. In compiler terminology, we call such a value a ***common subexpression***. We can change the loop to compute it only once, storing it in a temporary variable t.

```
for $k = 0$ to $n/2 - 1$
    $t = \omega \, y_k^{[1]}$
    $y_k = y_k^{[0]} + t$
    $y_{k+(n/2)} = y_k^{[0]} - t$
    $\omega = \omega \, \omega_n$
```

The operation in this loop, multiplying the twiddle factor $\omega = \omega_n^k$ by $y_k^{[1]}$, storing the product into t, and adding and subtracting t from $y_k^{[0]}$, is known as a ***butterfly operation*** and is shown schematically in Figure 30.3.

We now show how to make the FFT algorithm iterative rather than recursive in structure. In Figure 30.4, we have arranged the input vectors to the recursive calls in an invocation of RECURSIVE-FFT in a tree structure, where the initial call is for $n = 8$. The tree has one node for each call of the procedure, labeled



$y_k^{[0]}$
ω_n^k
$y_k^{[1]}$
$+$
$-$
$y_k^{[0]} + \omega_n^k y_k^{[1]}$
$y_k^{[0]} - \omega_n^k y_k^{[1]}$
(a)
(b)

Figure 30.3 A butterfly operation. **(a)** The two input values enter from the left, the twiddle factor ω_n^k is multiplied by $y_k^{[1]}$, and the sum and difference are output on the right. **(b)** A simplified drawing of a butterfly operation. We will use this representation in a parallel FFT circuit.

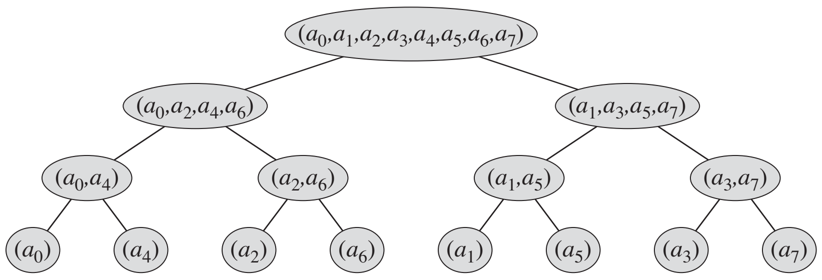


Figure 30.4 The tree of input vectors to the recursive calls of the RECURSIVE-FFT procedure. The initial invocation is for $n = 8$.

by the corresponding input vector. Each RECURSIVE-FFT invocation makes two recursive calls, unless it has received a 1-element vector. The first call appears in the left child, and the second call appears in the right child.

Looking at the tree, we observe that if we could arrange the elements of the initial vector a into the order in which they appear in the leaves, we could trace the execution of the RECURSIVE-FFT procedure, but bottom up instead of top down. First, we take the elements in pairs, compute the DFT of each pair using one butterfly operation, and replace the pair with its DFT. The vector then holds $n/2$ 2-element DFTs. Next, we take these $n/2$ DFTs in pairs and compute the DFT of the four vector elements they come from by executing two butterfly operations, replacing two 2-element DFTs with one 4-element DFT. The vector then holds $n/4$ 4-element DFTs. We continue in this manner until the vector holds two $(n/2)$-element DFTs, which we combine using $n/2$ butterfly operations into the final n-element DFT.

To turn this bottom-up approach into code, we use an array $A[0 \mathinner{..} n-1]$ that initially holds the elements of the input vector a in the order in which they appear in the leaves of the tree of Figure 30.4. (We shall show later how to determine this order, which is known as a bit-reversal permutation.) Because we have to combine DFTs on each level of the tree, we introduce a variable s to count the levels, ranging from 1 (at the bottom, when we are combining pairs to form 2-element DFTs) to $\lg n$ (at the top, when we are combining two $(n/2)$-element DFTs to produce the final result). The algorithm therefore has the following structure:

```
1  for $s = 1$ to $\lg n$
2      for $k = 0$ to $n - 1$ by $2^s$
3          combine the two $2^{s-1}$-element DFTs in
               $A[k \mathinner{..} k + 2^{s-1} - 1]$ and $A[k + 2^{s-1} \mathinner{..} k + 2^s - 1]$
               into one $2^s$-element DFT in $A[k \mathinner{..} k + 2^s - 1]$
```

We can express the body of the loop (line 3) as more precise pseudocode. We copy the **for** loop from the RECURSIVE-FFT procedure, identifying $y^{[0]}$ with $A[k \mathinner{..} k + 2^{s-1} - 1]$ and $y^{[1]}$ with $A[k + 2^{s-1} \mathinner{..} k + 2^s - 1]$. The twiddle factor used in each butterfly operation depends on the value of s; it is a power of ω_m, where $m = 2^s$. (We introduce the variable m solely for the sake of readability.) We introduce another temporary variable u that allows us to perform the butterfly operation in place. When we replace line 3 of the overall structure by the loop body, we get the following pseudocode, which forms the basis of the parallel implementation we shall present later. The code first calls the auxiliary procedure BIT-REVERSE-COPY(a, A) to copy vector a into array A in the initial order in which we need the values.

```
ITERATIVE-FFT$(a)$
BIT-REVERSE-COPY$(a, A)$
$n = a.length$                  // $n$ is a power of 2
for $s = 1$ to $\lg n$
    $m = 2^s$
    $\omega_m = e^{2\pi i/m}$
    for $k = 0$ to $n - 1$ by $m$
        $\omega = 1$
        for $j = 0$ to $m/2 - 1$
            $t = \omega\, A[k + j + m/2]$
            $u = A[k + j]$
            $A[k + j] = u + t$
            $A[k + j + m/2] = u - t$
            $\omega = \omega\, \omega_m$
return $A$
```

How does BIT-REVERSE-COPY get the elements of the input vector a into the desired order in the array A? The order in which the leaves appear in Figure 30.4

is a ***bit-reversal permutation***. That is, if we let $\text{rev}(k)$ be the $\lg n$-bit integer formed by reversing the bits of the binary representation of k, then we want to place vector element a_k in array position $A[\text{rev}(k)]$. In Figure 30.4, for example, the leaves appear in the order $0, 4, 2, 6, 1, 5, 3, 7$; this sequence in binary is $000, 100, 010, 110, 001, 101, 011, 111$, and when we reverse the bits of each value we get the sequence $000, 001, 010, 011, 100, 101, 110, 111$. To see that we want a bit-reversal permutation in general, we note that at the top level of the tree, indices whose low-order bit is 0 go into the left subtree and indices whose low-order bit is 1 go into the right subtree. Stripping off the low-order bit at each level, we continue this process down the tree, until we get the order given by the bit-reversal permutation at the leaves.

Since we can easily compute the function $\text{rev}(k)$, the BIT-REVERSE-COPY procedure is simple:

```
BIT-REVERSE-COPY$(a, A)$
1  $n = a.length$
2  for $k = 0$ to $n - 1$
3      $A[\text{rev}(k)] = a_k$
```

The iterative FFT implementation runs in time $\Theta(n \lg n)$. The call to BIT-REVERSE-COPY(a, A) certainly runs in $O(n \lg n)$ time, since we iterate n times and can reverse an integer between 0 and $n - 1$, with $\lg n$ bits, in $O(\lg n)$ time. (In practice, because we usually know the initial value of n in advance, we would probably code a table mapping k to $\text{rev}(k)$, making BIT-REVERSE-COPY run in $\Theta(n)$ time with a low hidden constant. Alternatively, we could use the clever amortized reverse binary counter scheme described in Problem 17-1.) To complete the proof that ITERATIVE-FFT runs in time $\Theta(n \lg n)$, we show that $L(n)$, the number of times the body of the innermost loop (lines 8–13) executes, is $\Theta(n \lg n)$. The **for** loop of lines 6–13 iterates $n/m = n/2^s$ times for each value of s, and the innermost loop of lines 8–13 iterates $m/2 = 2^{s-1}$ times. Thus,

$$\begin{aligned} L(n) &= \sum_{s=1}^{\lg n} \frac{n}{2^s} \cdot 2^{s-1} \\ &= \sum_{s=1}^{\lg n} \frac{n}{2} \\ &= \Theta(n \lg n) . \end{aligned}$$

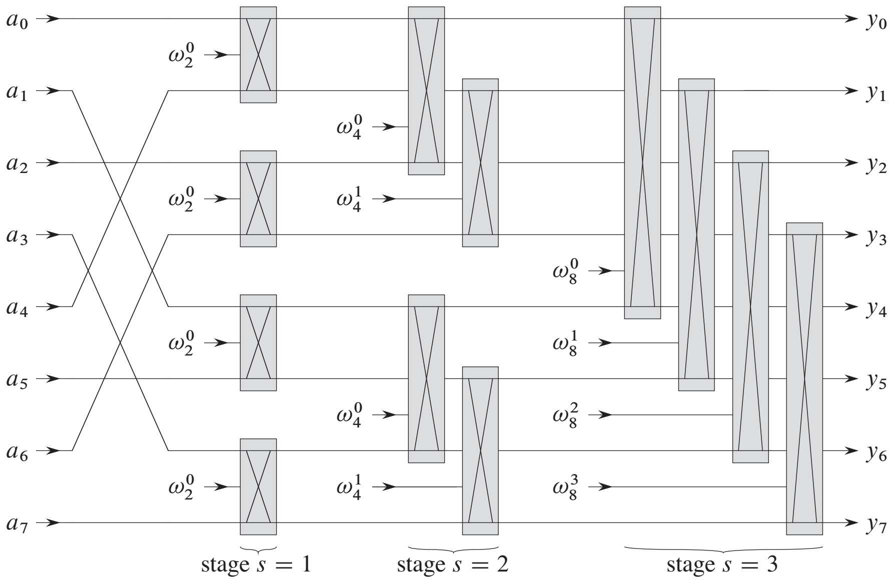


Figure 30.5 A circuit that computes the FFT in parallel, here shown on $n = 8$ inputs. Each butterfly operation takes as input the values on two wires, along with a twiddle factor, and it produces as outputs the values on two wires. The stages of butterflies are labeled to correspond to iterations of the outermost loop of the ITERATIVE-FFT procedure. Only the top and bottom wires passing through a butterfly interact with it; wires that pass through the middle of a butterfly do not affect that butterfly, nor are their values changed by that butterfly. For example, the top butterfly in stage 2 has nothing to do with wire 1 (the wire whose output is labeled y_1); its inputs and outputs are only on wires 0 and 2 (labeled y_0 and y_2, respectively). This circuit has depth $\Theta(\lg n)$ and performs $\Theta(n \lg n)$ butterfly operations altogether.

A parallel FFT circuit

We can exploit many of the properties that allowed us to implement an efficient iterative FFT algorithm to produce an efficient parallel algorithm for the FFT. We will express the parallel FFT algorithm as a circuit. Figure 30.5 shows a parallel FFT circuit, which computes the FFT on n inputs, for $n = 8$. The circuit begins with a bit-reverse permutation of the inputs, followed by $\lg n$ stages, each stage consisting of $n/2$ butterflies executed in parallel. The ***depth*** of the circuit—the maximum number of computational elements between any output and any input that can reach it—is therefore $\Theta(\lg n)$.

The leftmost part of the parallel FFT circuit performs the bit-reverse permutation, and the remainder mimics the iterative ITERATIVE-FFT procedure. Because each iteration of the outermost **for** loop performs $n/2$ independent butterfly operations, the circuit performs them in parallel. The value of s in each iteration within

ITERATIVE-FFT corresponds to a stage of butterflies shown in Figure 30.5. For $s = 1, 2, \ldots, \lg n$, stage s consists of $n/2^s$ groups of butterflies (corresponding to each value of k in ITERATIVE-FFT), with 2^{s-1} butterflies per group (corresponding to each value of j in ITERATIVE-FFT). The butterflies shown in Figure 30.5 correspond to the butterfly operations of the innermost loop (lines 9–12 of ITERATIVE-FFT). Note also that the twiddle factors used in the butterflies correspond to those used in ITERATIVE-FFT: in stage s, we use $\omega_m^0, \omega_m^1, \ldots, \omega_m^{m/2-1}$, where $m = 2^s$.

Exercises

30.3-1

Show how ITERATIVE-FFT computes the DFT of the input vector $(0, 2, 3, -1, 4, 5, 7, 9)$.

30.3-2

Show how to implement an FFT algorithm with the bit-reversal permutation occurring at the end, rather than at the beginning, of the computation. (*Hint:* Consider the inverse DFT.)

30.3-3

How many times does ITERATIVE-FFT compute twiddle factors in each stage? Rewrite ITERATIVE-FFT to compute twiddle factors only 2^{s-1} times in stage s.

30.3-4 ★

Suppose that the adders within the butterfly operations of the FFT circuit sometimes fail in such a manner that they always produce a zero output, independent of their inputs. Suppose that exactly one adder has failed, but that you don't know which one. Describe how you can identify the failed adder by supplying inputs to the overall FFT circuit and observing the outputs. How efficient is your method?

Problems

30-1 Divide-and-conquer multiplication

a. Show how to multiply two linear polynomials $ax + b$ and $cx + d$ using only three multiplications. (*Hint:* One of the multiplications is $(a + b) \cdot (c + d)$.)

b. Give two divide-and-conquer algorithms for multiplying two polynomials of degree-bound n in $\Theta(n^{\lg 3})$ time. The first algorithm should divide the input polynomial coefficients into a high half and a low half, and the second algorithm should divide them according to whether their index is odd or even.

c. Show how to multiply two n-bit integers in $O(n^{\lg 3})$ steps, where each step operates on at most a constant number of 1-bit values.

30-2 Toeplitz matrices

A ***Toeplitz matrix*** is an $n \times n$ matrix $A = (a_{ij})$ such that $a_{ij} = a_{i-1,j-1}$ for $i = 2, 3, \ldots, n$ and $j = 2, 3, \ldots, n$.

a. Is the sum of two Toeplitz matrices necessarily Toeplitz? What about the product?

b. Describe how to represent a Toeplitz matrix so that you can add two $n \times n$ Toeplitz matrices in $O(n)$ time.

c. Give an $O(n \lg n)$-time algorithm for multiplying an $n \times n$ Toeplitz matrix by a vector of length n. Use your representation from part (b).

d. Give an efficient algorithm for multiplying two $n \times n$ Toeplitz matrices. Analyze its running time.

30-3 Multidimensional fast Fourier transform

We can generalize the 1-dimensional discrete Fourier transform defined by equation (30.8) to d dimensions. The input is a d-dimensional array $A = (a_{j_1, j_2, \ldots, j_d})$ whose dimensions are $n_1, n_2, \ldots, n_d$, where $n_1 n_2 \cdots n_d = n$. We define the d-dimensional discrete Fourier transform by the equation

$$y_{k_1,k_2,\ldots,k_d} = \sum_{j_1=0}^{n_1-1} \sum_{j_2=0}^{n_2-1} \cdots \sum_{j_d=0}^{n_d-1} a_{j_1,j_2,\ldots,j_d}\, \omega_{n_1}^{j_1 k_1} \omega_{n_2}^{j_2 k_2} \cdots \omega_{n_d}^{j_d k_d}$$

for $0 \le k_1 < n_1, 0 \le k_2 < n_2, \ldots, 0 \le k_d < n_d$.

a. Show that we can compute a d-dimensional DFT by computing 1-dimensional DFTs on each dimension in turn. That is, we first compute n/n_1 separate 1-dimensional DFTs along dimension 1. Then, using the result of the DFTs along dimension 1 as the input, we compute n/n_2 separate 1-dimensional DFTs along dimension 2. Using this result as the input, we compute n/n_3 separate 1-dimensional DFTs along dimension 3, and so on, through dimension d.

b. Show that the ordering of dimensions does not matter, so that we can compute a d-dimensional DFT by computing the 1-dimensional DFTs in any order of the d dimensions.

c. Show that if we compute each 1-dimensional DFT by computing the fast Fourier transform, the total time to compute a d-dimensional DFT is $O(n \lg n)$, independent of d.

30-4 Evaluating all derivatives of a polynomial at a point

Given a polynomial $A(x)$ of degree-bound n, we define its tth derivative by

$$A^{(t)}(x) = \begin{cases} A(x) & \text{if } t = 0 \, , \\ \frac{d}{dx} A^{(t-1)}(x) & \text{if } 1 \le t \le n-1 \, , \\ 0 & \text{if } t \ge n \, . \end{cases}$$

From the coefficient representation $(a_0, a_1, \ldots, a_{n-1})$ of $A(x)$ and a given point x_0, we wish to determine $A^{(t)}(x_0)$ for $t = 0, 1, \ldots, n-1$.

a. Given coefficients $b_0, b_1, \ldots, b_{n-1}$ such that

$$A(x) = \sum_{j=0}^{n-1} b_j (x - x_0)^j \, ,$$

show how to compute $A^{(t)}(x_0)$, for $t = 0, 1, \ldots, n-1$, in $O(n)$ time.

b. Explain how to find $b_0, b_1, \ldots, b_{n-1}$ in $O(n \lg n)$ time, given $A(x_0 + \omega_n^k)$ for $k = 0, 1, \ldots, n-1$.

c. Prove that

$$A(x_0 + \omega_n^k) = \sum_{r=0}^{n-1} \left(\frac{\omega_n^{kr}}{r!} \sum_{j=0}^{n-1} f(j) g(r-j) \right) \, ,$$

where $f(j) = a_j \cdot j!$ and

$$g(l) = \begin{cases} x_0^{-l}/(-l)! & \text{if } -(n-1) \le l \le 0 \, , \\ 0 & \text{if } 1 \le l \le n-1 \, . \end{cases}$$

d. Explain how to evaluate $A(x_0 + \omega_n^k)$ for $k = 0, 1, \ldots, n-1$ in $O(n \lg n)$ time. Conclude that we can evaluate all nontrivial derivatives of $A(x)$ at x_0 in $O(n \lg n)$ time.

30-5 Polynomial evaluation at multiple points

We have seen how to evaluate a polynomial of degree-bound n at a single point in $O(n)$ time using Horner's rule. We have also discovered how to evaluate such a polynomial at all n complex roots of unity in $O(n \lg n)$ time using the FFT. We shall now show how to evaluate a polynomial of degree-bound n at n arbitrary points in $O(n \lg^2 n)$ time.

To do so, we shall assume that we can compute the polynomial remainder when one such polynomial is divided by another in $O(n \lg n)$ time, a result that we state without proof. For example, the remainder of $3x^3 + x^2 - 3x + 1$ when divided by $x^2 + x + 2$ is

$$(3x^3 + x^2 - 3x + 1) \bmod (x^2 + x + 2) = -7x + 5 .$$

Given the coefficient representation of a polynomial $A(x) = \sum_{k=0}^{n-1} a_k x^k$ and n points $x_0, x_1, \ldots, x_{n-1}$, we wish to compute the n values $A(x_0), A(x_1), \ldots, A(x_{n-1})$. For $0 \le i \le j \le n-1$, define the polynomials $P_{ij}(x) = \prod_{k=i}^{j}(x - x_k)$ and $Q_{ij}(x) = A(x) \bmod P_{ij}(x)$. Note that $Q_{ij}(x)$ has degree at most $j - i$.

a. Prove that $A(x) \bmod (x - z) = A(z)$ for any point z.

b. Prove that $Q_{kk}(x) = A(x_k)$ and that $Q_{0,n-1}(x) = A(x)$.

c. Prove that for $i \le k \le j$, we have $Q_{ik}(x) = Q_{ij}(x) \bmod P_{ik}(x)$ and $Q_{kj}(x) = Q_{ij}(x) \bmod P_{kj}(x)$.

d. Give an $O(n \lg^2 n)$-time algorithm to evaluate $A(x_0), A(x_1), \ldots, A(x_{n-1})$.

30-6 FFT using modular arithmetic

As defined, the discrete Fourier transform requires us to compute with complex numbers, which can result in a loss of precision due to round-off errors. For some problems, the answer is known to contain only integers, and by using a variant of the FFT based on modular arithmetic, we can guarantee that the answer is calculated exactly. An example of such a problem is that of multiplying two polynomials with integer coefficients. Exercise 30.2-6 gives one approach, using a modulus of length $\Omega(n)$ bits to handle a DFT on n points. This problem gives another approach, which uses a modulus of the more reasonable length $O(\lg n)$; it requires that you understand the material of Chapter 31. Let n be a power of 2.

a. Suppose that we search for the smallest k such that $p = kn + 1$ is prime. Give a simple heuristic argument why we might expect k to be approximately $\ln n$. (The value of k might be much larger or smaller, but we can reasonably expect to examine $O(\lg n)$ candidate values of k on average.) How does the expected length of p compare to the length of n?

Let g be a generator of $\mathbb{Z}_p^*$, and let $w = g^k \bmod p$.

b. Argue that the DFT and the inverse DFT are well-defined inverse operations modulo p, where w is used as a principal nth root of unity.

c. Show how to make the FFT and its inverse work modulo p in time $O(n \lg n)$, where operations on words of $O(\lg n)$ bits take unit time. Assume that the algorithm is given p and w.

d. Compute the DFT modulo $p = 17$ of the vector $(0, 5, 3, 7, 7, 2, 1, 6)$. Note that $g = 3$ is a generator of $\mathbb{Z}_{17}^*$.

Chapter notes

Van Loan's book [343] provides an outstanding treatment of the fast Fourier transform. Press, Teukolsky, Vetterling, and Flannery [283, 284] have a good description of the fast Fourier transform and its applications. For an excellent introduction to signal processing, a popular FFT application area, see the texts by Oppenheim and Schafer [266] and Oppenheim and Willsky [267]. The Oppenheim and Schafer book also shows how to handle cases in which n is not an integer power of 2.

Fourier analysis is not limited to 1-dimensional data. It is widely used in image processing to analyze data in 2 or more dimensions. The books by Gonzalez and Woods [146] and Pratt [281] discuss multidimensional Fourier transforms and their use in image processing, and books by Tolimieri, An, and Lu [338] and Van Loan [343] discuss the mathematics of multidimensional fast Fourier transforms.

Cooley and Tukey [76] are widely credited with devising the FFT in the 1960s. The FFT had in fact been discovered many times previously, but its importance was not fully realized before the advent of modern digital computers. Although Press, Teukolsky, Vetterling, and Flannery attribute the origins of the method to Runge and König in 1924, an article by Heideman, Johnson, and Burrus [163] traces the history of the FFT as far back as C. F. Gauss in 1805.

Frigo and Johnson [117] developed a fast and flexible implementation of the FFT, called FFTW ("fastest Fourier transform in the West"). FFTW is designed for situations requiring multiple DFT computations on the same problem size. Before actually computing the DFTs, FFTW executes a "planner," which, by a series of trial runs, determines how best to decompose the FFT computation for the given problem size on the host machine. FFTW adapts to use the hardware cache efficiently, and once subproblems are small enough, FFTW solves them with optimized, straight-line code. Furthermore, FFTW has the unusual advantage of taking $\Theta(n \lg n)$ time for any problem size n, even when n is a large prime.

Although the standard Fourier transform assumes that the input represents points that are uniformly spaced in the time domain, other techniques can approximate the FFT on “nonequispaced” data. The article by Ware [348] provides an overview.

31 Number-Theoretic Algorithms

Number theory was once viewed as a beautiful but largely useless subject in pure mathematics. Today number-theoretic algorithms are used widely, due in large part to the invention of cryptographic schemes based on large prime numbers. These schemes are feasible because we can find large primes easily, and they are secure because we do not know how to factor the product of large primes (or solve related problems, such as computing discrete logarithms) efficiently. This chapter presents some of the number theory and related algorithms that underlie such applications.

Section 31.1 introduces basic concepts of number theory, such as divisibility, modular equivalence, and unique factorization. Section 31.2 studies one of the world's oldest algorithms: Euclid's algorithm for computing the greatest common divisor of two integers. Section 31.3 reviews concepts of modular arithmetic. Section 31.4 then studies the set of multiples of a given number a, modulo n, and shows how to find all solutions to the equation $ax \equiv b \pmod{n}$ by using Euclid's algorithm. The Chinese remainder theorem is presented in Section 31.5. Section 31.6 considers powers of a given number a, modulo n, and presents a repeated-squaring algorithm for efficiently computing $a^b \bmod n$, given a, b, and n. This operation is at the heart of efficient primality testing and of much modern cryptography. Section 31.7 then describes the RSA public-key cryptosystem. Section 31.8 examines a randomized primality test. We can use this test to find large primes efficiently, which we need to do in order to create keys for the RSA cryptosystem. Finally, Section 31.9 reviews a simple but effective heuristic for factoring small integers. It is a curious fact that factoring is one problem people may wish to be intractable, since the security of RSA depends on the difficulty of factoring large integers.

Size of inputs and cost of arithmetic computations

Because we shall be working with large integers, we need to adjust how we think about the size of an input and about the cost of elementary arithmetic operations.

In this chapter, a "large input" typically means an input containing "large integers" rather than an input containing "many integers" (as for sorting). Thus, we shall measure the size of an input in terms of the *number of bits* required to represent that input, not just the number of integers in the input. An algorithm with integer inputs $a_1, a_2, \ldots, a_k$ is a ***polynomial-time algorithm*** if it runs in time polynomial in $\lg a_1, \lg a_2, \ldots, \lg a_k$, that is, polynomial in the lengths of its binary-encoded inputs.

In most of this book, we have found it convenient to think of the elementary arithmetic operations (multiplications, divisions, or computing remainders) as primitive operations that take one unit of time. By counting the number of such arithmetic operations that an algorithm performs, we have a basis for making a reasonable estimate of the algorithm's actual running time on a computer. Elementary operations can be time-consuming, however, when their inputs are large. It thus becomes convenient to measure how many ***bit operations*** a number-theoretic algorithm requires. In this model, multiplying two β-bit integers by the ordinary method uses $\Theta(\beta^2)$ bit operations. Similarly, we can divide a β-bit integer by a shorter integer or take the remainder of a β-bit integer when divided by a shorter integer in time $\Theta(\beta^2)$ by simple algorithms. (See Exercise 31.1-12.) Faster methods are known. For example, a simple divide-and-conquer method for multiplying two β-bit integers has a running time of $\Theta(\beta^{\lg 3})$, and the fastest known method has a running time of $\Theta(\beta \lg \beta \lg \lg \beta)$. For practical purposes, however, the $\Theta(\beta^2)$ algorithm is often best, and we shall use this bound as a basis for our analyses.

We shall generally analyze algorithms in this chapter in terms of both the number of arithmetic operations and the number of bit operations they require.

31.1 Elementary number-theoretic notions

This section provides a brief review of notions from elementary number theory concerning the set $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ of integers and the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of natural numbers.

Divisibility and divisors

The notion of one integer being divisible by another is key to the theory of numbers. The notation $d \mid a$ (read "d ***divides*** a") means that $a = kd$ for some integer k. Every integer divides 0. If $a > 0$ and $d \mid a$, then $|d| \leq |a|$. If $d \mid a$, then we also say that a is a ***multiple*** of d. If d does not divide a, we write $d \nmid a$.

If $d \mid a$ and $d \geq 0$, we say that d is a ***divisor*** of a. Note that $d \mid a$ if and only if $-d \mid a$, so that no generality is lost by defining the divisors to be nonnegative, with the understanding that the negative of any divisor of a also divides a. A divisor of a nonzero integer a is at least 1 but not greater than $|a|$. For example, the divisors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24.

Every positive integer a is divisible by the ***trivial divisors*** 1 and a. The nontrivial divisors of a are the ***factors*** of a. For example, the factors of 20 are 2, 4, 5, and 10.

Prime and composite numbers

An integer $a > 1$ whose only divisors are the trivial divisors 1 and a is a ***prime number*** or, more simply, a ***prime***. Primes have many special properties and play a critical role in number theory. The first 20 primes, in order, are

$$2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31,\ 37,\ 41,\ 43,\ 47,\ 53,\ 59,\ 61,\ 67,\ 71\ .$$

Exercise 31.1-2 asks you to prove that there are infinitely many primes. An integer $a > 1$ that is not prime is a ***composite number*** or, more simply, a ***composite***. For example, 39 is composite because $3 \mid 39$. We call the integer 1 a ***unit***, and it is neither prime nor composite. Similarly, the integer 0 and all negative integers are neither prime nor composite.

The division theorem, remainders, and modular equivalence

Given an integer n, we can partition the integers into those that are multiples of n and those that are not multiples of n. Much number theory is based upon refining this partition by classifying the nonmultiples of n according to their remainders when divided by n. The following theorem provides the basis for this refinement. We omit the proof (but see, for example, Niven and Zuckerman [265]).

Theorem 31.1 (Division theorem)
For any integer a and any positive integer n, there exist unique integers q and r such that $0 \le r < n$ and $a = qn + r$. ∎

The value $q = \lfloor a/n \rfloor$ is the ***quotient*** of the division. The value $r = a \bmod n$ is the ***remainder*** (or ***residue***) of the division. We have that $n \mid a$ if and only if $a \bmod n = 0$.

We can partition the integers into n equivalence classes according to their remainders modulo n. The ***equivalence class modulo n*** containing an integer a is

$$[a]_n = \{a + kn : k \in \mathbb{Z}\}\ .$$

For example, $[3]_7 = \{\ldots, -11, -4, 3, 10, 17, \ldots\}$; we can also denote this set by $[-4]_7$ and $[10]_7$. Using the notation defined on page 54, we can say that writing $a \in [b]_n$ is the same as writing $a \equiv b \pmod{n}$. The set of all such equivalence classes is

$$\mathbb{Z}_n = \{[a]_n : 0 \le a \le n-1\} \ . \tag{31.1}$$

When you see the definition

$$\mathbb{Z}_n = \{0, 1, \ldots, n-1\} \ , \tag{31.2}$$

you should read it as equivalent to equation (31.1) with the understanding that 0 represents $[0]_n$, 1 represents $[1]_n$, and so on; each class is represented by its smallest nonnegative element. You should keep the underlying equivalence classes in mind, however. For example, if we refer to -1 as a member of $\mathbb{Z}_n$, we are really referring to $[n-1]_n$, since $-1 \equiv n-1 \pmod{n}$.

Common divisors and greatest common divisors

If d is a divisor of a and d is also a divisor of b, then d is a ***common divisor*** of a and b. For example, the divisors of 30 are 1, 2, 3, 5, 6, 10, 15, and 30, and so the common divisors of 24 and 30 are 1, 2, 3, and 6. Note that 1 is a common divisor of any two integers.

An important property of common divisors is that

$$d \mid a \text{ and } d \mid b \text{ implies } d \mid (a+b) \text{ and } d \mid (a-b) \ . \tag{31.3}$$

More generally, we have that

$$d \mid a \text{ and } d \mid b \text{ implies } d \mid (ax+by) \tag{31.4}$$

for any integers x and y. Also, if $a \mid b$, then either $|a| \le |b|$ or $b = 0$, which implies that

$$a \mid b \text{ and } b \mid a \text{ implies } a = \pm b \ . \tag{31.5}$$

The ***greatest common divisor*** of two integers a and b, not both zero, is the largest of the common divisors of a and b; we denote it by $\gcd(a,b)$. For example, $\gcd(24,30) = 6$, $\gcd(5,7) = 1$, and $\gcd(0,9) = 9$. If a and b are both nonzero, then $\gcd(a,b)$ is an integer between 1 and $\min(|a|,|b|)$. We define $\gcd(0,0)$ to be 0; this definition is necessary to make standard properties of the gcd function (such as equation (31.9) below) universally valid.

The following are elementary properties of the gcd function:

$$\gcd(a,b) = \gcd(b,a) \ , \tag{31.6}$$

$$\gcd(a,b) = \gcd(-a,b) \ , \tag{31.7}$$

$$\gcd(a,b) = \gcd(|a|,|b|) \ , \tag{31.8}$$

$$\gcd(a,0) = |a| \ , \tag{31.9}$$

$$\gcd(a,ka) = |a| \qquad \text{for any } k \in \mathbb{Z} \ . \tag{31.10}$$

The following theorem provides an alternative and useful characterization of $\gcd(a,b)$.

Theorem 31.2
If a and b are any integers, not both zero, then $\gcd(a, b)$ is the smallest positive element of the set $\{ax + by : x, y \in \mathbb{Z}\}$ of linear combinations of a and b.

Proof Let s be the smallest positive such linear combination of a and b, and let $s = ax + by$ for some $x, y \in \mathbb{Z}$. Let $q = \lfloor a/s \rfloor$. Equation (3.8) then implies

$$\begin{aligned} a \bmod s &= a - qs \\ &= a - q(ax + by) \\ &= a\,(1 - qx) + b\,(-qy) \ , \end{aligned}$$

and so $a \bmod s$ is a linear combination of a and b as well. But, since $0 \le a \bmod s < s$, we have that $a \bmod s = 0$, because s is the smallest positive such linear combination. Therefore, we have that $s \mid a$ and, by analogous reasoning, $s \mid b$. Thus, s is a common divisor of a and b, and so $\gcd(a, b) \ge s$. Equation (31.4) implies that $\gcd(a, b) \mid s$, since $\gcd(a, b)$ divides both a and b and s is a linear combination of a and b. But $\gcd(a, b) \mid s$ and $s > 0$ imply that $\gcd(a, b) \le s$. Combining $\gcd(a, b) \ge s$ and $\gcd(a, b) \le s$ yields $\gcd(a, b) = s$. We conclude that s is the greatest common divisor of a and b. ■

Corollary 31.3
For any integers a and b, if $d \mid a$ and $d \mid b$, then $d \mid \gcd(a, b)$.

Proof This corollary follows from equation (31.4), because $\gcd(a, b)$ is a linear combination of a and b by Theorem 31.2. ■

Corollary 31.4
For all integers a and b and any nonnegative integer n,

$$\gcd(an, bn) = n \gcd(a, b) \ .$$

Proof If $n = 0$, the corollary is trivial. If $n > 0$, then $\gcd(an, bn)$ is the smallest positive element of the set $\{anx + bny : x, y \in \mathbb{Z}\}$, which is n times the smallest positive element of the set $\{ax + by : x, y \in \mathbb{Z}\}$. ■

Corollary 31.5
For all positive integers n, a, and b, if $n \mid ab$ and $\gcd(a, n) = 1$, then $n \mid b$.

Proof We leave the proof as Exercise 31.1-5. ■

Relatively prime integers

Two integers a and b are ***relatively prime*** if their only common divisor is 1, that is, if $\gcd(a, b) = 1$. For example, 8 and 15 are relatively prime, since the divisors of 8 are 1, 2, 4, and 8, and the divisors of 15 are 1, 3, 5, and 15. The following theorem states that if two integers are each relatively prime to an integer p, then their product is relatively prime to p.

Theorem 31.6
For any integers a, b, and p, if both $\gcd(a, p) = 1$ and $\gcd(b, p) = 1$, then $\gcd(ab, p) = 1$.

Proof It follows from Theorem 31.2 that there exist integers x, y, x', and y' such that

$$\begin{aligned} ax + py &= 1 , \\ bx' + py' &= 1 . \end{aligned}$$

Multiplying these equations and rearranging, we have

$$ab(xx') + p(ybx' + y'ax + pyy') = 1 .$$

Since 1 is thus a positive linear combination of ab and p, an appeal to Theorem 31.2 completes the proof. ■

Integers n_1, n_2, …, n_k are ***pairwise relatively prime*** if, whenever $i \neq j$, we have $\gcd(n_i, n_j) = 1$.

Unique factorization

An elementary but important fact about divisibility by primes is the following.

Theorem 31.7
For all primes p and all integers a and b, if $p \mid ab$, then $p \mid a$ or $p \mid b$ (or both).

Proof Assume for the purpose of contradiction that $p \mid ab$, but that $p \nmid a$ and $p \nmid b$. Thus, $\gcd(a, p) = 1$ and $\gcd(b, p) = 1$, since the only divisors of p are 1 and p, and we assume that p divides neither a nor b. Theorem 31.6 then implies that $\gcd(ab, p) = 1$, contradicting our assumption that $p \mid ab$, since $p \mid ab$ implies $\gcd(ab, p) = p$. This contradiction completes the proof. ■

A consequence of Theorem 31.7 is that we can uniquely factor any composite integer into a product of primes.

Theorem 31.8 (Unique factorization)
There is exactly one way to write any composite integer a as a product of the form

$$a = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} ,$$

where the p_i are prime, $p_1 < p_2 < \cdots < p_r$, and the e_i are positive integers.

Proof We leave the proof as Exercise 31.1-11. ■

As an example, the number 6000 is uniquely factored into primes as $2^4 \cdot 3 \cdot 5^3$.

Exercises

31.1-1
Prove that if $a > b > 0$ and $c = a + b$, then $c \bmod a = b$.

31.1-2
Prove that there are infinitely many primes. (*Hint:* Show that none of the primes $p_1, p_2, \ldots, p_k$ divide $(p_1 p_2 \cdots p_k) + 1$.)

31.1-3
Prove that if $a \mid b$ and $b \mid c$, then $a \mid c$.

31.1-4
Prove that if p is prime and $0 < k < p$, then $\gcd(k, p) = 1$.

31.1-5
Prove Corollary 31.5.

31.1-6
Prove that if p is prime and $0 < k < p$, then $p \mid \binom{p}{k}$. Conclude that for all integers a and b and all primes p,

$$(a + b)^p \equiv a^p + b^p \pmod{p} .$$

31.1-7
Prove that if a and b are any positive integers such that $a \mid b$, then

$$(x \bmod b) \bmod a = x \bmod a$$

for any x. Prove, under the same assumptions, that

$$x \equiv y \pmod{b} \text{ implies } x \equiv y \pmod{a}$$

for any integers x and y.

31.1-8
For any integer $k > 0$, an integer n is a ***kth power*** if there exists an integer a such that $a^k = n$. Furthermore, $n > 1$ is a ***nontrivial power*** if it is a kth power for some integer $k > 1$. Show how to determine whether a given β-bit integer n is a nontrivial power in time polynomial in β.

31.1-9
Prove equations (31.6)–(31.10).

31.1-10
Show that the gcd operator is associative. That is, prove that for all integers a, b, and c,

$$\gcd(a, \gcd(b, c)) = \gcd(\gcd(a, b), c) .$$

31.1-11 ★
Prove Theorem 31.8.

31.1-12
Give efficient algorithms for the operations of dividing a β-bit integer by a shorter integer and of taking the remainder of a β-bit integer when divided by a shorter integer. Your algorithms should run in time $\Theta(\beta^2)$.

31.1-13
Give an efficient algorithm to convert a given β-bit (binary) integer to a decimal representation. Argue that if multiplication or division of integers whose length is at most β takes time $M(\beta)$, then we can convert binary to decimal in time $\Theta(M(\beta) \lg \beta)$. (*Hint:* Use a divide-and-conquer approach, obtaining the top and bottom halves of the result with separate recursions.)

31.2 Greatest common divisor

In this section, we describe Euclid's algorithm for efficiently computing the greatest common divisor of two integers. When we analyze the running time, we shall see a surprising connection with the Fibonacci numbers, which yield a worst-case input for Euclid's algorithm.

We restrict ourselves in this section to nonnegative integers. This restriction is justified by equation (31.8), which states that $\gcd(a, b) = \gcd(|a|, |b|)$.

In principle, we can compute $\gcd(a,b)$ for positive integers a and b from the prime factorizations of a and b. Indeed, if

$$a = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} , \tag{31.11}$$
$$b = p_1^{f_1} p_2^{f_2} \cdots p_r^{f_r} , \tag{31.12}$$

with zero exponents being used to make the set of primes $p_1, p_2, \ldots, p_r$ the same for both a and b, then, as Exercise 31.2-1 asks you to show,

$$\gcd(a,b) = p_1^{\min(e_1,f_1)} p_2^{\min(e_2,f_2)} \cdots p_r^{\min(e_r,f_r)} . \tag{31.13}$$

As we shall show in Section 31.9, however, the best algorithms to date for factoring do not run in polynomial time. Thus, this approach to computing greatest common divisors seems unlikely to yield an efficient algorithm.

Euclid's algorithm for computing greatest common divisors relies on the following theorem.

Theorem 31.9 (GCD recursion theorem)
For any nonnegative integer a and any positive integer b,

$$\gcd(a,b) = \gcd(b, a \bmod b) .$$

Proof We shall show that $\gcd(a,b)$ and $\gcd(b, a \bmod b)$ divide each other, so that by equation (31.5) they must be equal (since they are both nonnegative).

We first show that $\gcd(a,b) \mid \gcd(b, a \bmod b)$. If we let $d = \gcd(a,b)$, then $d \mid a$ and $d \mid b$. By equation (3.8), $a \bmod b = a - qb$, where $q = \lfloor a/b \rfloor$. Since $a \bmod b$ is thus a linear combination of a and b, equation (31.4) implies that $d \mid (a \bmod b)$. Therefore, since $d \mid b$ and $d \mid (a \bmod b)$, Corollary 31.3 implies that $d \mid \gcd(b, a \bmod b)$ or, equivalently, that

$$\gcd(a,b) \mid \gcd(b, a \bmod b). \tag{31.14}$$

Showing that $\gcd(b, a \bmod b) \mid \gcd(a,b)$ is almost the same. If we now let $d = \gcd(b, a \bmod b)$, then $d \mid b$ and $d \mid (a \bmod b)$. Since $a = qb + (a \bmod b)$, where $q = \lfloor a/b \rfloor$, we have that a is a linear combination of b and $(a \bmod b)$. By equation (31.4), we conclude that $d \mid a$. Since $d \mid b$ and $d \mid a$, we have that $d \mid \gcd(a,b)$ by Corollary 31.3 or, equivalently, that

$$\gcd(b, a \bmod b) \mid \gcd(a,b). \tag{31.15}$$

Using equation (31.5) to combine equations (31.14) and (31.15) completes the proof. ■

Euclid's algorithm

The *Elements* of Euclid (circa 300 B.C.) describes the following gcd algorithm, although it may be of even earlier origin. We express Euclid's algorithm as a recursive program based directly on Theorem 31.9. The inputs a and b are arbitrary nonnegative integers.

```
EUCLID($a, b$)
1  if $b == 0$
2      return $a$
3  else return EUCLID($b, a \bmod b$)
```

As an example of the running of EUCLID, consider the computation of gcd(30, 21):

$$\begin{aligned}\text{EUCLID}(30, 21) &= \text{EUCLID}(21, 9) \\ &= \text{EUCLID}(9, 3) \\ &= \text{EUCLID}(3, 0) \\ &= 3 .\end{aligned}$$

This computation calls EUCLID recursively three times.

The correctness of EUCLID follows from Theorem 31.9 and the property that if the algorithm returns a in line 2, then $b = 0$, so that equation (31.9) implies that $\gcd(a, b) = \gcd(a, 0) = a$. The algorithm cannot recurse indefinitely, since the second argument strictly decreases in each recursive call and is always nonnegative. Therefore, EUCLID always terminates with the correct answer.

The running time of Euclid's algorithm

We analyze the worst-case running time of EUCLID as a function of the size of a and b. We assume with no loss of generality that $a > b \geq 0$. To justify this assumption, observe that if $b > a \geq 0$, then EUCLID(a, b) immediately makes the recursive call EUCLID(b, a). That is, if the first argument is less than the second argument, EUCLID spends one recursive call swapping its arguments and then proceeds. Similarly, if $b = a > 0$, the procedure terminates after one recursive call, since $a \bmod b = 0$.

The overall running time of EUCLID is proportional to the number of recursive calls it makes. Our analysis makes use of the Fibonacci numbers F_k, defined by the recurrence (3.22).

Lemma 31.10
If $a > b \geq 1$ and the call EUCLID(a, b) performs $k \geq 1$ recursive calls, then $a \geq F_{k+2}$ and $b \geq F_{k+1}$.

Proof The proof proceeds by induction on k. For the basis of the induction, let $k = 1$. Then, $b \geq 1 = F_2$, and since $a > b$, we must have $a \geq 2 = F_3$. Since $b > (a \bmod b)$, in each recursive call the first argument is strictly larger than the second; the assumption that $a > b$ therefore holds for each recursive call.

Assume inductively that the lemma holds if $k - 1$ recursive calls are made; we shall then prove that the lemma holds for k recursive calls. Since $k > 0$, we have $b > 0$, and $\text{EUCLID}(a, b)$ calls $\text{EUCLID}(b, a \bmod b)$ recursively, which in turn makes $k - 1$ recursive calls. The inductive hypothesis then implies that $b \geq F_{k+1}$ (thus proving part of the lemma), and $a \bmod b \geq F_k$. We have

$$\begin{aligned} b + (a \bmod b) &= b + (a - b\lfloor a/b \rfloor) \\ &\leq a , \end{aligned}$$

since $a > b > 0$ implies $\lfloor a/b \rfloor \geq 1$. Thus,

$$\begin{aligned} a &\geq b + (a \bmod b) \\ &\geq F_{k+1} + F_k \\ &= F_{k+2} . \end{aligned}$$

■

The following theorem is an immediate corollary of this lemma.

Theorem 31.11 (Lamé's theorem)
For any integer $k \geq 1$, if $a > b \geq 1$ and $b < F_{k+1}$, then the call $\text{EUCLID}(a, b)$ makes fewer than k recursive calls. ■

We can show that the upper bound of Theorem 31.11 is the best possible by showing that the call $\text{EUCLID}(F_{k+1}, F_k)$ makes exactly $k - 1$ recursive calls when $k \geq 2$. We use induction on k. For the base case, $k = 2$, and the call $\text{EUCLID}(F_3, F_2)$ makes exactly one recursive call, to $\text{EUCLID}(1, 0)$. (We have to start at $k = 2$, because when $k = 1$ we do not have $F_2 > F_1$.) For the inductive step, assume that $\text{EUCLID}(F_k, F_{k-1})$ makes exactly $k - 2$ recursive calls. For $k > 2$, we have $F_k > F_{k-1} > 0$ and $F_{k+1} = F_k + F_{k-1}$, and so by Exercise 31.1-1, we have $F_{k+1} \bmod F_k = F_{k-1}$. Thus, we have

$$\begin{aligned} \gcd(F_{k+1}, F_k) &= \gcd(F_k, F_{k+1} \bmod F_k) \\ &= \gcd(F_k, F_{k-1}) . \end{aligned}$$

Therefore, the call $\text{EUCLID}(F_{k+1}, F_k)$ recurses one time more than the call $\text{EUCLID}(F_k, F_{k-1})$, or exactly $k - 1$ times, meeting the upper bound of Theorem 31.11.

Since F_k is approximately $\phi^k/\sqrt{5}$, where ϕ is the golden ratio $(1 + \sqrt{5})/2$ defined by equation (3.24), the number of recursive calls in EUCLID is $O(\lg b)$. (See

a	b	$\lfloor a/b \rfloor$	d	x	y
99	78	1	3	−11	14
78	21	3	3	3	−11
21	15	1	3	−2	3
15	6	2	3	1	−2
6	3	2	3	0	1
3	0	—	3	1	0

Figure 31.1 How EXTENDED-EUCLID computes gcd(99, 78). Each line shows one level of the recursion: the values of the inputs a and b, the computed value $\lfloor a/b \rfloor$, and the values d, x, and y returned. The triple (d, x, y) returned becomes the triple (d', x', y') used at the next higher level of recursion. The call EXTENDED-EUCLID(99, 78) returns $(3, -11, 14)$, so that $\gcd(99, 78) = 3 = 99 \cdot (-11) + 78 \cdot 14$.

Exercise 31.2-5 for a tighter bound.) Therefore, if we call EUCLID on two β-bit numbers, then it performs $O(\beta)$ arithmetic operations and $O(\beta^3)$ bit operations (assuming that multiplication and division of β-bit numbers take $O(\beta^2)$ bit operations). Problem 31-2 asks you to show an $O(\beta^2)$ bound on the number of bit operations.

The extended form of Euclid's algorithm

We now rewrite Euclid's algorithm to compute additional useful information. Specifically, we extend the algorithm to compute the integer coefficients x and y such that

$$d = \gcd(a, b) = ax + by . \tag{31.16}$$

Note that x and y may be zero or negative. We shall find these coefficients useful later for computing modular multiplicative inverses. The procedure EXTENDED-EUCLID takes as input a pair of nonnegative integers and returns a triple of the form (d, x, y) that satisfies equation (31.16).

```
EXTENDED-EUCLID$(a, b)$
1  if $b$ == 0
2      return $(a, 1, 0)$
3  else $(d', x', y') = $ EXTENDED-EUCLID$(b, a \bmod b)$
4      $(d, x, y) = (d', y', x' - \lfloor a/b \rfloor y')$
5      return $(d, x, y)$
```

Figure 31.1 illustrates how EXTENDED-EUCLID computes gcd(99, 78).

The EXTENDED-EUCLID procedure is a variation of the EUCLID procedure. Line 1 is equivalent to the test "b == 0" in line 1 of EUCLID. If $b = 0$, then

EXTENDED-EUCLID returns not only $d = a$ in line 2, but also the coefficients $x = 1$ and $y = 0$, so that $a = ax + by$. If $b \neq 0$, EXTENDED-EUCLID first computes (d', x', y') such that $d' = \gcd(b, a \bmod b)$ and

$$d' = bx' + (a \bmod b)y' . \tag{31.17}$$

As for EUCLID, we have in this case $d = \gcd(a, b) = d' = \gcd(b, a \bmod b)$. To obtain x and y such that $d = ax + by$, we start by rewriting equation (31.17) using the equation $d = d'$ and equation (3.8):

$$\begin{aligned} d &= bx' + (a - b\lfloor a/b \rfloor)y' \\ &= ay' + b(x' - \lfloor a/b \rfloor y') . \end{aligned}$$

Thus, choosing $x = y'$ and $y = x' - \lfloor a/b \rfloor y'$ satisfies the equation $d = ax + by$, proving the correctness of EXTENDED-EUCLID.

Since the number of recursive calls made in EUCLID is equal to the number of recursive calls made in EXTENDED-EUCLID, the running times of EUCLID and EXTENDED-EUCLID are the same, to within a constant factor. That is, for $a > b > 0$, the number of recursive calls is $O(\lg b)$.

Exercises

31.2-1
Prove that equations (31.11) and (31.12) imply equation (31.13).

31.2-2
Compute the values (d, x, y) that the call EXTENDED-EUCLID(899, 493) returns.

31.2-3
Prove that for all integers a, k, and n,

$$\gcd(a, n) = \gcd(a + kn, n) .$$

31.2-4
Rewrite EUCLID in an iterative form that uses only a constant amount of memory (that is, stores only a constant number of integer values).

31.2-5
If $a > b \geq 0$, show that the call EUCLID(a, b) makes at most $1 + \log_\phi b$ recursive calls. Improve this bound to $1 + \log_\phi(b/\gcd(a, b))$.

31.2-6
What does EXTENDED-EUCLID(F_{k+1}, F_k) return? Prove your answer correct.

31.2-7

Define the gcd function for more than two arguments by the recursive equation $\gcd(a_0, a_1, \ldots, a_n) = \gcd(a_0, \gcd(a_1, a_2, \ldots, a_n))$. Show that the gcd function returns the same answer independent of the order in which its arguments are specified. Also show how to find integers $x_0, x_1, \ldots, x_n$ such that $\gcd(a_0, a_1, \ldots, a_n) = a_0x_0 + a_1x_1 + \cdots + a_nx_n$. Show that the number of divisions performed by your algorithm is $O(n + \lg(\max\{a_0, a_1, \ldots, a_n\}))$.

31.2-8

Define $\mathrm{lcm}(a_1, a_2, \ldots, a_n)$ to be the ***least common multiple*** of the n integers $a_1, a_2, \ldots, a_n$, that is, the smallest nonnegative integer that is a multiple of each a_i. Show how to compute $\mathrm{lcm}(a_1, a_2, \ldots, a_n)$ efficiently using the (two-argument) gcd operation as a subroutine.

31.2-9

Prove that n_1, n_2, n_3, and n_4 are pairwise relatively prime if and only if

$$\gcd(n_1n_2, n_3n_4) = \gcd(n_1n_3, n_2n_4) = 1 .$$

More generally, show that $n_1, n_2, \ldots, n_k$ are pairwise relatively prime if and only if a set of $\lceil \lg k \rceil$ pairs of numbers derived from the n_i are relatively prime.

31.3 Modular arithmetic

Informally, we can think of modular arithmetic as arithmetic as usual over the integers, except that if we are working modulo n, then every result x is replaced by the element of $\{0, 1, \ldots, n-1\}$ that is equivalent to x, modulo n (that is, x is replaced by $x \bmod n$). This informal model suffices if we stick to the operations of addition, subtraction, and multiplication. A more formal model for modular arithmetic, which we now give, is best described within the framework of group theory.

Finite groups

A ***group*** $(S, \oplus)$ is a set S together with a binary operation $\oplus$ defined on S for which the following properties hold:

1. **Closure:** For all $a, b \in S$, we have $a \oplus b \in S$.
2. **Identity:** There exists an element $e \in S$, called the ***identity*** of the group, such that $e \oplus a = a \oplus e = a$ for all $a \in S$.
3. **Associativity:** For all $a, b, c \in S$, we have $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

4. **Inverses:** For each $a \in S$, there exists a unique element $b \in S$, called the ***inverse*** of a, such that $a \oplus b = b \oplus a = e$.

As an example, consider the familiar group $(\mathbb{Z}, +)$ of the integers $\mathbb{Z}$ under the operation of addition: 0 is the identity, and the inverse of a is $-a$. If a group $(S, \oplus)$ satisfies the ***commutative law*** $a \oplus b = b \oplus a$ for all $a, b \in S$, then it is an ***abelian group***. If a group $(S, \oplus)$ satisfies $|S| < \infty$, then it is a ***finite group***.

The groups defined by modular addition and multiplication

We can form two finite abelian groups by using addition and multiplication modulo n, where n is a positive integer. These groups are based on the equivalence classes of the integers modulo n, defined in Section 31.1.

To define a group on $\mathbb{Z}_n$, we need to have suitable binary operations, which we obtain by redefining the ordinary operations of addition and multiplication. We can easily define addition and multiplication operations for $\mathbb{Z}_n$, because the equivalence class of two integers uniquely determines the equivalence class of their sum or product. That is, if $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$, then

$$\begin{aligned} a + b &\equiv a' + b' \pmod{n} , \\ ab &\equiv a'b' \pmod{n} . \end{aligned}$$

Thus, we define addition and multiplication modulo n, denoted $+_n$ and $\cdot_n$, by

$$\begin{aligned} [a]_n +_n [b]_n &= [a+b]_n , \\ [a]_n \cdot_n [b]_n &= [ab]_n . \end{aligned} \tag{31.18}$$

(We can define subtraction similarly on $\mathbb{Z}_n$ by $[a]_n -_n [b]_n = [a-b]_n$, but division is more complicated, as we shall see.) These facts justify the common and convenient practice of using the smallest nonnegative element of each equivalence class as its representative when performing computations in $\mathbb{Z}_n$. We add, subtract, and multiply as usual on the representatives, but we replace each result x by the representative of its class, that is, by $x \bmod n$.

Using this definition of addition modulo n, we define the ***additive group modulo n*** as $(\mathbb{Z}_n, +_n)$. The size of the additive group modulo n is $|\mathbb{Z}_n| = n$. Figure 31.2(a) gives the operation table for the group $(\mathbb{Z}_6, +_6)$.

Theorem 31.12
The system $(\mathbb{Z}_n, +_n)$ is a finite abelian group.

Proof Equation (31.18) shows that $(\mathbb{Z}_n, +_n)$ is closed. Associativity and commutativity of $+_n$ follow from the associativity and commutativity of $+$:

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

(a)

$\cdot_{15}$	1	2	4	7	8	11	13	14
1	1	2	4	7	8	11	13	14
2	2	4	8	14	1	7	11	13
4	4	8	1	13	2	14	7	11
7	7	14	13	4	11	2	1	8
8	8	1	2	11	4	13	14	7
11	11	7	14	2	13	1	8	4
13	13	11	7	1	14	8	4	2
14	14	13	11	8	7	4	2	1

(b)

Figure 31.2 Two finite groups. Equivalence classes are denoted by their representative elements. **(a)** The group $(\mathbb{Z}_6, +_6)$. **(b)** The group $(\mathbb{Z}_{15}^*, \cdot_{15})$.

$$\begin{aligned}
([a]_n +_n [b]_n) +_n [c]_n &= [a+b]_n +_n [c]_n \\
&= [(a+b)+c]_n \\
&= [a+(b+c)]_n \\
&= [a]_n +_n [b+c]_n \\
&= [a]_n +_n ([b]_n +_n [c]_n) ,
\end{aligned}$$

$$\begin{aligned}
[a]_n +_n [b]_n &= [a+b]_n \\
&= [b+a]_n \\
&= [b]_n +_n [a]_n .
\end{aligned}$$

The identity element of $(\mathbb{Z}_n, +_n)$ is 0 (that is, $[0]_n$). The (additive) inverse of an element a (that is, of $[a]_n$) is the element $-a$ (that is, $[-a]_n$ or $[n-a]_n$), since $[a]_n +_n [-a]_n = [a-a]_n = [0]_n$. ■

Using the definition of multiplication modulo n, we define the ***multiplicative group modulo n*** as $(\mathbb{Z}_n^*, \cdot_n)$. The elements of this group are the set $\mathbb{Z}_n^*$ of elements in $\mathbb{Z}_n$ that are relatively prime to n, so that each one has a unique inverse, modulo n:

$$\mathbb{Z}_n^* = \{[a]_n \in \mathbb{Z}_n : \gcd(a,n) = 1\} .$$

To see that $\mathbb{Z}_n^*$ is well defined, note that for $0 \le a < n$, we have $a \equiv (a + kn) \pmod{n}$ for all integers k. By Exercise 31.2-3, therefore, $\gcd(a,n) = 1$ implies $\gcd(a + kn, n) = 1$ for all integers k. Since $[a]_n = \{a + kn : k \in \mathbb{Z}\}$, the set $\mathbb{Z}_n^*$ is well defined. An example of such a group is

$$\mathbb{Z}_{15}^* = \{1, 2, 4, 7, 8, 11, 13, 14\} ,$$

where the group operation is multiplication modulo 15. (Here we denote an element $[a]_{15}$ as a; for example, we denote $[7]_{15}$ as 7.) Figure 31.2(b) shows the group $(\mathbb{Z}^*_{15}, \cdot_{15})$. For example, $8 \cdot 11 \equiv 13 \pmod{15}$, working in $\mathbb{Z}^*_{15}$. The identity for this group is 1.

Theorem 31.13
The system $(\mathbb{Z}^*_n, \cdot_n)$ is a finite abelian group.

Proof Theorem 31.6 implies that $(\mathbb{Z}^*_n, \cdot_n)$ is closed. Associativity and commutativity can be proved for $\cdot_n$ as they were for $+_n$ in the proof of Theorem 31.12. The identity element is $[1]_n$. To show the existence of inverses, let a be an element of $\mathbb{Z}^*_n$ and let (d, x, y) be returned by EXTENDED-EUCLID(a, n). Then, $d = 1$, since $a \in \mathbb{Z}^*_n$, and

$$ax + ny = 1 \tag{31.19}$$

or, equivalently,

$$ax \equiv 1 \pmod{n} .$$

Thus, $[x]_n$ is a multiplicative inverse of $[a]_n$, modulo n. Furthermore, we claim that $[x]_n \in \mathbb{Z}^*_n$. To see why, equation (31.19) demonstrates that the smallest positive linear combination of x and n must be 1. Therefore, Theorem 31.2 implies that $\gcd(x, n) = 1$. We defer the proof that inverses are uniquely defined until Corollary 31.26. ■

As an example of computing multiplicative inverses, suppose that $a = 5$ and $n = 11$. Then EXTENDED-EUCLID(a, n) returns $(d, x, y) = (1, -2, 1)$, so that $1 = 5 \cdot (-2) + 11 \cdot 1$. Thus, $[-2]_{11}$ (i.e., $[9]_{11}$) is the multiplicative inverse of $[5]_{11}$.

When working with the groups $(\mathbb{Z}_n, +_n)$ and $(\mathbb{Z}^*_n, \cdot_n)$ in the remainder of this chapter, we follow the convenient practice of denoting equivalence classes by their representative elements and denoting the operations $+_n$ and $\cdot_n$ by the usual arithmetic notations $+$ and $\cdot$ (or juxtaposition, so that $ab = a \cdot b$) respectively. Also, equivalences modulo n may also be interpreted as equations in $\mathbb{Z}_n$. For example, the following two statements are equivalent:

$$\begin{aligned} ax &\equiv b \pmod{n} , \\ [a]_n \cdot_n [x]_n &= [b]_n . \end{aligned}$$

As a further convenience, we sometimes refer to a group $(S, \oplus)$ merely as S when the operation $\oplus$ is understood from context. We may thus refer to the groups $(\mathbb{Z}_n, +_n)$ and $(\mathbb{Z}^*_n, \cdot_n)$ as $\mathbb{Z}_n$ and $\mathbb{Z}^*_n$, respectively.

We denote the (multiplicative) inverse of an element a by $(a^{-1} \bmod n)$. Division in $\mathbb{Z}^*_n$ is defined by the equation $a/b \equiv ab^{-1} \pmod{n}$. For example, in $\mathbb{Z}^*_{15}$

we have that $7^{-1} \equiv 13 \pmod{15}$, since $7 \cdot 13 = 91 \equiv 1 \pmod{15}$, so that $4/7 \equiv 4 \cdot 13 \equiv 7 \pmod{15}$.

The size of $\mathbb{Z}_n^*$ is denoted $\phi(n)$. This function, known as ***Euler's phi function***, satisfies the equation

$$\phi(n) = n \prod_{p\,:\,p \text{ is prime and } p \mid n} \left(1 - \frac{1}{p}\right) , \tag{31.20}$$

so that p runs over all the primes dividing n (including n itself, if n is prime). We shall not prove this formula here. Intuitively, we begin with a list of the n remainders $\{0, 1, \ldots, n-1\}$ and then, for each prime p that divides n, cross out every multiple of p in the list. For example, since the prime divisors of 45 are 3 and 5,

$$\begin{aligned} \phi(45) &= 45\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) \\ &= 45\left(\frac{2}{3}\right)\left(\frac{4}{5}\right) \\ &= 24 . \end{aligned}$$

If p is prime, then $\mathbb{Z}_p^* = \{1, 2, \ldots, p-1\}$, and

$$\begin{aligned} \phi(p) &= p\left(1 - \frac{1}{p}\right) \\ &= p - 1 . \end{aligned} \tag{31.21}$$

If n is composite, then $\phi(n) < n - 1$, although it can be shown that

$$\phi(n) > \frac{n}{e^{\gamma} \ln\ln n + \frac{3}{\ln\ln n}} \tag{31.22}$$

for $n \geq 3$, where $\gamma = 0.5772156649\ldots$ is ***Euler's constant***. A somewhat simpler (but looser) lower bound for $n > 5$ is

$$\phi(n) > \frac{n}{6 \ln\ln n} . \tag{31.23}$$

The lower bound (31.22) is essentially the best possible, since

$$\liminf_{n \to \infty} \frac{\phi(n)}{n / \ln\ln n} = e^{-\gamma} . \tag{31.24}$$

Subgroups

If $(S, \oplus)$ is a group, $S' \subseteq S$, and $(S', \oplus)$ is also a group, then $(S', \oplus)$ is a ***subgroup*** of $(S, \oplus)$. For example, the even integers form a subgroup of the integers under the operation of addition. The following theorem provides a useful tool for recognizing subgroups.

Theorem 31.14 (A nonempty closed subset of a finite group is a subgroup)
If $(S, \oplus)$ is a finite group and S' is any nonempty subset of S such that $a \oplus b \in S'$ for all $a, b \in S'$, then $(S', \oplus)$ is a subgroup of $(S, \oplus)$.

Proof We leave the proof as Exercise 31.3-3. ■

For example, the set $\{0, 2, 4, 6\}$ forms a subgroup of $\mathbb{Z}_8$, since it is nonempty and closed under the operation $+$ (that is, it is closed under $+_8$).

The following theorem provides an extremely useful constraint on the size of a subgroup; we omit the proof.

Theorem 31.15 (Lagrange's theorem)
If $(S, \oplus)$ is a finite group and $(S', \oplus)$ is a subgroup of $(S, \oplus)$, then $|S'|$ is a divisor of $|S|$. ■

A subgroup S' of a group S is a ***proper*** subgroup if $S' \neq S$. We shall use the following corollary in our analysis in Section 31.8 of the Miller-Rabin primality test procedure.

Corollary 31.16
If S' is a proper subgroup of a finite group S, then $|S'| \le |S|/2$. ■

Subgroups generated by an element

Theorem 31.14 gives us an easy way to produce a subgroup of a finite group $(S, \oplus)$: choose an element a and take all elements that can be generated from a using the group operation. Specifically, define $a^{(k)}$ for $k \ge 1$ by

$$a^{(k)} = \bigoplus_{i=1}^{k} a = \underbrace{a \oplus a \oplus \cdots \oplus a}_{k} .$$

For example, if we take $a = 2$ in the group $\mathbb{Z}_6$, the sequence $a^{(1)}, a^{(2)}, a^{(3)}, \ldots$ is

$$2, 4, 0, 2, 4, 0, 2, 4, 0, \ldots .$$

In the group $\mathbb{Z}_n$, we have $a^{(k)} = ka \bmod n$, and in the group $\mathbb{Z}_n^*$, we have $a^{(k)} = a^k \bmod n$. We define the ***subgroup generated by a***, denoted $\langle a \rangle$ or $(\langle a \rangle, \oplus)$, by

$$\langle a \rangle = \{a^{(k)} : k \ge 1\} .$$

We say that a ***generates*** the subgroup $\langle a \rangle$ or that a is a ***generator*** of $\langle a \rangle$. Since S is finite, $\langle a \rangle$ is a finite subset of S, possibly including all of S. Since the associativity of $\oplus$ implies

$$a^{(i)} \oplus a^{(j)} = a^{(i+j)} ,$$

$\langle a \rangle$ is closed and therefore, by Theorem 31.14, $\langle a \rangle$ is a subgroup of S. For example, in $\mathbb{Z}_6$, we have

$$\begin{aligned} \langle 0 \rangle &= \{0\} , \\ \langle 1 \rangle &= \{0, 1, 2, 3, 4, 5\} , \\ \langle 2 \rangle &= \{0, 2, 4\} . \end{aligned}$$

Similarly, in $\mathbb{Z}_7^*$, we have

$$\begin{aligned} \langle 1 \rangle &= \{1\} , \\ \langle 2 \rangle &= \{1, 2, 4\} , \\ \langle 3 \rangle &= \{1, 2, 3, 4, 5, 6\} . \end{aligned}$$

The ***order*** of a (in the group S), denoted $\text{ord}(a)$, is defined as the smallest positive integer t such that $a^{(t)} = e$.

Theorem 31.17
For any finite group $(S, \oplus)$ and any $a \in S$, the order of a is equal to the size of the subgroup it generates, or $\text{ord}(a) = |\langle a \rangle|$.

Proof Let $t = \text{ord}(a)$. Since $a^{(t)} = e$ and $a^{(t+k)} = a^{(t)} \oplus a^{(k)} = a^{(k)}$ for $k \geq 1$, if $i > t$, then $a^{(i)} = a^{(j)}$ for some $j < i$. Thus, as we generate elements by a, we see no new elements after $a^{(t)}$. Thus, $\langle a \rangle = \{a^{(1)}, a^{(2)}, \ldots, a^{(t)}\}$, and so $|\langle a \rangle| \leq t$. To show that $|\langle a \rangle| \geq t$, we show that each element of the sequence $a^{(1)}, a^{(2)}, \ldots, a^{(t)}$ is distinct. Suppose for the purpose of contradiction that $a^{(i)} = a^{(j)}$ for some i and j satisfying $1 \leq i < j \leq t$. Then, $a^{(i+k)} = a^{(j+k)}$ for $k \geq 0$. But this equality implies that $a^{(i+(t-j))} = a^{(j+(t-j))} = e$, a contradiction, since $i + (t - j) < t$ but t is the least positive value such that $a^{(t)} = e$. Therefore, each element of the sequence $a^{(1)}, a^{(2)}, \ldots, a^{(t)}$ is distinct, and $|\langle a \rangle| \geq t$. We conclude that $\text{ord}(a) = |\langle a \rangle|$. ■

Corollary 31.18
The sequence $a^{(1)}, a^{(2)}, \ldots$ is periodic with period $t = \text{ord}(a)$; that is, $a^{(i)} = a^{(j)}$ if and only if $i \equiv j \pmod{t}$. ■

Consistent with the above corollary, we define $a^{(0)}$ as e and $a^{(i)}$ as $a^{(i \bmod t)}$, where $t = \text{ord}(a)$, for all integers i.

Corollary 31.19
If $(S, \oplus)$ is a finite group with identity e, then for all $a \in S$,

$$a^{(|S|)} = e .$$

Proof Lagrange's theorem (Theorem 31.15) implies that $\operatorname{ord}(a) \mid |S|$, and so $|S| \equiv 0 \pmod{t}$, where $t = \operatorname{ord}(a)$. Therefore, $a^{(|S|)} = a^{(0)} = e$. ■

Exercises

31.3-1
Draw the group operation tables for the groups $(\mathbb{Z}_4, +_4)$ and $(\mathbb{Z}_5^*, \cdot_5)$. Show that these groups are isomorphic by exhibiting a one-to-one correspondence α between their elements such that $a + b \equiv c \pmod{4}$ if and only if $\alpha(a) \cdot \alpha(b) \equiv \alpha(c) \pmod{5}$.

31.3-2
List all subgroups of $\mathbb{Z}_9$ and of $\mathbb{Z}_{13}^*$.

31.3-3
Prove Theorem 31.14.

31.3-4
Show that if p is prime and e is a positive integer, then

$$\phi(p^e) = p^{e-1}(p-1) .$$

31.3-5
Show that for any integer $n > 1$ and for any $a \in \mathbb{Z}_n^*$, the function $f_a : \mathbb{Z}_n^* \to \mathbb{Z}_n^*$ defined by $f_a(x) = ax \bmod n$ is a permutation of $\mathbb{Z}_n^*$.

31.4 Solving modular linear equations

We now consider the problem of finding solutions to the equation

$$ax \equiv b \pmod{n} , \tag{31.25}$$

where $a > 0$ and $n > 0$. This problem has several applications; for example, we shall use it as part of the procedure for finding keys in the RSA public-key cryptosystem in Section 31.7. We assume that a, b, and n are given, and we wish to find all values of x, modulo n, that satisfy equation (31.25). The equation may have zero, one, or more than one such solution.

Let $\langle a \rangle$ denote the subgroup of $\mathbb{Z}_n$ generated by a. Since $\langle a \rangle = \{a^{(x)} : x > 0\} = \{ax \bmod n : x > 0\}$, equation (31.25) has a solution if and only if $[b] \in \langle a \rangle$. Lagrange's theorem (Theorem 31.15) tells us that $|\langle a \rangle|$ must be a divisor of n. The following theorem gives us a precise characterization of $\langle a \rangle$.

Theorem 31.20
For any positive integers a and n, if $d = \gcd(a, n)$, then

$$\langle a \rangle = \langle d \rangle = \{0, d, 2d, \ldots, ((n/d) - 1)d\} \tag{31.26}$$

in $\mathbb{Z}_n$, and thus

$$|\langle a \rangle| = n/d \ .$$

Proof We begin by showing that $d \in \langle a \rangle$. Recall that EXTENDED-EUCLID(a, n) produces integers x' and y' such that $ax' + ny' = d$. Thus, $ax' \equiv d \pmod{n}$, so that $d \in \langle a \rangle$. In other words, d is a multiple of a in $\mathbb{Z}_n$.

Since $d \in \langle a \rangle$, it follows that every multiple of d belongs to $\langle a \rangle$, because any multiple of a multiple of a is itself a multiple of a. Thus, $\langle a \rangle$ contains every element in $\{0, d, 2d, \ldots, ((n/d) - 1)d\}$. That is, $\langle d \rangle \subseteq \langle a \rangle$.

We now show that $\langle a \rangle \subseteq \langle d \rangle$. If $m \in \langle a \rangle$, then $m = ax \bmod n$ for some integer x, and so $m = ax + ny$ for some integer y. However, $d \mid a$ and $d \mid n$, and so $d \mid m$ by equation (31.4). Therefore, $m \in \langle d \rangle$.

Combining these results, we have that $\langle a \rangle = \langle d \rangle$. To see that $|\langle a \rangle| = n/d$, observe that there are exactly n/d multiples of d between 0 and $n-1$, inclusive. ■

Corollary 31.21
The equation $ax \equiv b \pmod{n}$ is solvable for the unknown x if and only if $d \mid b$, where $d = \gcd(a, n)$.

Proof The equation $ax \equiv b \pmod{n}$ is solvable if and only if $[b] \in \langle a \rangle$, which is the same as saying

$$(b \bmod n) \in \{0, d, 2d, \ldots, ((n/d) - 1)d\} \ ,$$

by Theorem 31.20. If $0 \le b < n$, then $b \in \langle a \rangle$ if and only if $d \mid b$, since the members of $\langle a \rangle$ are precisely the multiples of d. If $b < 0$ or $b \ge n$, the corollary then follows from the observation that $d \mid b$ if and only if $d \mid (b \bmod n)$, since b and $b \bmod n$ differ by a multiple of n, which is itself a multiple of d. ■

Corollary 31.22
The equation $ax \equiv b \pmod{n}$ either has d distinct solutions modulo n, where $d = \gcd(a, n)$, or it has no solutions.

Proof If $ax \equiv b \pmod{n}$ has a solution, then $b \in \langle a \rangle$. By Theorem 31.17, $\operatorname{ord}(a) = |\langle a \rangle|$, and so Corollary 31.18 and Theorem 31.20 imply that the sequence $ai \bmod n$, for $i = 0, 1, \ldots$, is periodic with period $|\langle a \rangle| = n/d$. If $b \in \langle a \rangle$, then b appears exactly d times in the sequence $ai \bmod n$, for $i = 0, 1, \ldots, n-1$, since the length-(n/d) block of values $\langle a \rangle$ repeats exactly d times as i increases from 0 to $n-1$. The indices x of the d positions for which $ax \bmod n = b$ are the solutions of the equation $ax \equiv b \pmod{n}$. ■

Theorem 31.23
Let $d = \gcd(a, n)$, and suppose that $d = ax' + ny'$ for some integers x' and y' (for example, as computed by EXTENDED-EUCLID). If $d \mid b$, then the equation $ax \equiv b \pmod{n}$ has as one of its solutions the value x_0, where

$$x_0 = x'(b/d) \bmod n .$$

Proof We have

$$\begin{aligned} ax_0 &\equiv ax'(b/d) \pmod{n} \\ &\equiv d(b/d) \pmod{n} \qquad \text{(because } ax' \equiv d \pmod{n}\text{)} \\ &\equiv b \pmod{n} , \end{aligned}$$

and thus x_0 is a solution to $ax \equiv b \pmod{n}$. ■

Theorem 31.24
Suppose that the equation $ax \equiv b \pmod{n}$ is solvable (that is, $d \mid b$, where $d = \gcd(a, n)$) and that x_0 is any solution to this equation. Then, this equation has exactly d distinct solutions, modulo n, given by $x_i = x_0 + i(n/d)$ for $i = 0, 1, \ldots, d-1$.

Proof Because $n/d > 0$ and $0 \le i(n/d) < n$ for $i = 0, 1, \ldots, d-1$, the values $x_0, x_1, \ldots, x_{d-1}$ are all distinct, modulo n. Since x_0 is a solution of $ax \equiv b \pmod{n}$, we have $ax_0 \bmod n \equiv b \pmod{n}$. Thus, for $i = 0, 1, \ldots, d-1$, we have

$$\begin{aligned} ax_i \bmod n &= a(x_0 + in/d) \bmod n \\ &= (ax_0 + ain/d) \bmod n \\ &= ax_0 \bmod n \quad \text{(because } d \mid a \text{ implies that } ain/d \text{ is a multiple of } n\text{)} \\ &\equiv b \pmod{n} , \end{aligned}$$

and hence $ax_i \equiv b \pmod{n}$, making x_i a solution, too. By Corollary 31.22, the equation $ax \equiv b \pmod{n}$ has exactly d solutions, so that $x_0, x_1, \ldots, x_{d-1}$ must be all of them. ■

We have now developed the mathematics needed to solve the equation $ax \equiv b \pmod{n}$; the following algorithm prints all solutions to this equation. The inputs a and n are arbitrary positive integers, and b is an arbitrary integer.

```
MODULAR-LINEAR-EQUATION-SOLVER$(a, b, n)$
1  $(d, x', y') = $ EXTENDED-EUCLID$(a, n)$
2  if $d \mid b$
3      $x_0 = x'(b/d) \bmod n$
4      for $i = 0$ to $d - 1$
5          print $(x_0 + i(n/d)) \bmod n$
6  else print “no solutions”
```

As an example of the operation of this procedure, consider the equation $14x \equiv 30 \pmod{100}$ (here, $a = 14$, $b = 30$, and $n = 100$). Calling EXTENDED-EUCLID in line 1, we obtain $(d, x', y') = (2, -7, 1)$. Since $2 \mid 30$, lines 3–5 execute. Line 3 computes $x_0 = (-7)(15) \bmod 100 = 95$. The loop on lines 4–5 prints the two solutions 95 and 45.

The procedure MODULAR-LINEAR-EQUATION-SOLVER works as follows. Line 1 computes $d = \gcd(a, n)$, along with two values x' and y' such that $d = ax' + ny'$, demonstrating that x' is a solution to the equation $ax' \equiv d \pmod{n}$. If d does not divide b, then the equation $ax \equiv b \pmod{n}$ has no solution, by Corollary 31.21. Line 2 checks to see whether $d \mid b$; if not, line 6 reports that there are no solutions. Otherwise, line 3 computes a solution x_0 to $ax \equiv b \pmod{n}$, in accordance with Theorem 31.23. Given one solution, Theorem 31.24 states that adding multiples of (n/d), modulo n, yields the other $d - 1$ solutions. The **for** loop of lines 4–5 prints out all d solutions, beginning with x_0 and spaced n/d apart, modulo n.

MODULAR-LINEAR-EQUATION-SOLVER performs $O(\lg n + \gcd(a, n))$ arithmetic operations, since EXTENDED-EUCLID performs $O(\lg n)$ arithmetic operations, and each iteration of the **for** loop of lines 4–5 performs a constant number of arithmetic operations.

The following corollaries of Theorem 31.24 give specializations of particular interest.

Corollary 31.25
For any $n > 1$, if $\gcd(a, n) = 1$, then the equation $ax \equiv b \pmod{n}$ has a unique solution, modulo n. ■

If $b = 1$, a common case of considerable interest, the x we are looking for is a ***multiplicative inverse*** of a, modulo n.

Corollary 31.26
For any $n > 1$, if $\gcd(a, n) = 1$, then the equation $ax \equiv 1 \pmod{n}$ has a unique solution, modulo n. Otherwise, it has no solution. ■

Thanks to Corollary 31.26, we can use the notation $a^{-1} \bmod n$ to refer to *the* multiplicative inverse of a, modulo n, when a and n are relatively prime. If $\gcd(a, n) = 1$, then the unique solution to the equation $ax \equiv 1 \pmod{n}$ is the integer x returned by EXTENDED-EUCLID, since the equation

$$\gcd(a, n) = 1 = ax + ny$$

implies $ax \equiv 1 \pmod{n}$. Thus, we can compute $a^{-1} \bmod n$ efficiently using EXTENDED-EUCLID.

Exercises

31.4-1
Find all solutions to the equation $35x \equiv 10 \pmod{50}$.

31.4-2
Prove that the equation $ax \equiv ay \pmod{n}$ implies $x \equiv y \pmod{n}$ whenever $\gcd(a, n) = 1$. Show that the condition $\gcd(a, n) = 1$ is necessary by supplying a counterexample with $\gcd(a, n) > 1$.

31.4-3
Consider the following change to line 3 of the procedure MODULAR-LINEAR-EQUATION-SOLVER:

```
3       $x_0 = x'(b/d) \bmod (n/d)$
```

Will this work? Explain why or why not.

31.4-4 ★
Let p be prime and $f(x) \equiv f_0 + f_1 x + \cdots + f_t x^t \pmod{p}$ be a polynomial of degree t, with coefficients f_i drawn from $\mathbb{Z}_p$. We say that $a \in \mathbb{Z}_p$ is a ***zero*** of f if $f(a) \equiv 0 \pmod{p}$. Prove that if a is a zero of f, then $f(x) \equiv (x - a)g(x) \pmod{p}$ for some polynomial $g(x)$ of degree $t - 1$. Prove by induction on t that if p is prime, then a polynomial $f(x)$ of degree t can have at most t distinct zeros modulo p.

31.5 The Chinese remainder theorem

Around A.D. 100, the Chinese mathematician Sun-Tsŭ solved the problem of finding those integers x that leave remainders 2, 3, and 2 when divided by 3, 5, and 7 respectively. One such solution is $x = 23$; all solutions are of the form $23 + 105k$

for arbitrary integers k. The "Chinese remainder theorem" provides a correspondence between a system of equations modulo a set of pairwise relatively prime moduli (for example, 3, 5, and 7) and an equation modulo their product (for example, 105).

The Chinese remainder theorem has two major applications. Let the integer n be factored as $n = n_1 n_2 \cdots n_k$, where the factors n_i are pairwise relatively prime. First, the Chinese remainder theorem is a descriptive "structure theorem" that describes the structure of $\mathbb{Z}_n$ as identical to that of the Cartesian product $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$ with componentwise addition and multiplication modulo n_i in the ith component. Second, this description helps us to design efficient algorithms, since working in each of the systems $\mathbb{Z}_{n_i}$ can be more efficient (in terms of bit operations) than working modulo n.

Theorem 31.27 (Chinese remainder theorem)
Let $n = n_1 n_2 \cdots n_k$, where the n_i are pairwise relatively prime. Consider the correspondence

$$a \leftrightarrow (a_1, a_2, \ldots, a_k) , \tag{31.27}$$

where $a \in \mathbb{Z}_n$, $a_i \in \mathbb{Z}_{n_i}$, and

$$a_i = a \bmod n_i$$

for $i = 1, 2, \ldots, k$. Then, mapping (31.27) is a one-to-one correspondence (bijection) between $\mathbb{Z}_n$ and the Cartesian product $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$. Operations performed on the elements of $\mathbb{Z}_n$ can be equivalently performed on the corresponding k-tuples by performing the operations independently in each coordinate position in the appropriate system. That is, if

$$\begin{aligned} a &\leftrightarrow (a_1, a_2, \ldots, a_k) , \\ b &\leftrightarrow (b_1, b_2, \ldots, b_k) , \end{aligned}$$

then

$$(a + b) \bmod n \leftrightarrow ((a_1 + b_1) \bmod n_1, \ldots, (a_k + b_k) \bmod n_k) , \tag{31.28}$$

$$(a - b) \bmod n \leftrightarrow ((a_1 - b_1) \bmod n_1, \ldots, (a_k - b_k) \bmod n_k) , \tag{31.29}$$

$$(ab) \bmod n \leftrightarrow (a_1 b_1 \bmod n_1, \ldots, a_k b_k \bmod n_k) . \tag{31.30}$$

Proof Transforming between the two representations is fairly straightforward. Going from a to $(a_1, a_2, \ldots, a_k)$ is quite easy and requires only k "mod" operations.

Computing a from inputs $(a_1, a_2, \ldots, a_k)$ is a bit more complicated. We begin by defining $m_i = n/n_i$ for $i = 1, 2, \ldots, k$; thus m_i is the product of all of the n_j's other than n_i: $m_i = n_1 n_2 \cdots n_{i-1} n_{i+1} \cdots n_k$. We next define

$$c_i = m_i(m_i^{-1} \bmod n_i) \tag{31.31}$$

for $i = 1, 2, \ldots, k$. Equation (31.31) is always well defined: since m_i and n_i are relatively prime (by Theorem 31.6), Corollary 31.26 guarantees that $m_i^{-1} \bmod n_i$ exists. Finally, we can compute a as a function of $a_1, a_2, \ldots, a_k$ as follows:

$$a \equiv (a_1c_1 + a_2c_2 + \cdots + a_kc_k) \pmod{n} . \tag{31.32}$$

We now show that equation (31.32) ensures that $a \equiv a_i \pmod{n_i}$ for $i = 1, 2, \ldots, k$. Note that if $j \neq i$, then $m_j \equiv 0 \pmod{n_i}$, which implies that $c_j \equiv m_j \equiv 0 \pmod{n_i}$. Note also that $c_i \equiv 1 \pmod{n_i}$, from equation (31.31). We thus have the appealing and useful correspondence

$$c_i \leftrightarrow (0, 0, \ldots, 0, 1, 0, \ldots, 0) ,$$

a vector that has 0s everywhere except in the ith coordinate, where it has a 1; the c_i thus form a "basis" for the representation, in a certain sense. For each i, therefore, we have

$$\begin{aligned} a &\equiv a_ic_i && \pmod{n_i} \\ &\equiv a_im_i(m_i^{-1} \bmod n_i) && \pmod{n_i} \\ &\equiv a_i && \pmod{n_i} , \end{aligned}$$

which is what we wished to show: our method of computing a from the a_i's produces a result a that satisfies the constraints $a \equiv a_i \pmod{n_i}$ for $i = 1, 2, \ldots, k$. The correspondence is one-to-one, since we can transform in both directions. Finally, equations (31.28)–(31.30) follow directly from Exercise 31.1-7, since $x \bmod n_i = (x \bmod n) \bmod n_i$ for any x and $i = 1, 2, \ldots, k$. ■

We shall use the following corollaries later in this chapter.

Corollary 31.28

If $n_1, n_2, \ldots, n_k$ are pairwise relatively prime and $n = n_1n_2 \cdots n_k$, then for any integers $a_1, a_2, \ldots, a_k$, the set of simultaneous equations

$$x \equiv a_i \pmod{n_i} ,$$

for $i = 1, 2, \ldots, k$, has a unique solution modulo n for the unknown x. ■

Corollary 31.29

If $n_1, n_2, \ldots, n_k$ are pairwise relatively prime and $n = n_1n_2 \cdots n_k$, then for all integers x and a,

$$x \equiv a \pmod{n_i}$$

for $i = 1, 2, \ldots, k$ if and only if

$$x \equiv a \pmod{n} .$$

■

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	40	15	55	30	5	45	20	60	35	10	50	25
1	26	1	41	16	56	31	6	46	21	61	36	11	51
2	52	27	2	42	17	57	32	7	47	22	62	37	12
3	13	53	28	3	43	18	58	33	8	48	23	63	38
4	39	14	54	29	4	44	19	59	34	9	49	24	64

Figure 31.3 An illustration of the Chinese remainder theorem for $n_1 = 5$ and $n_2 = 13$. For this example, $c_1 = 26$ and $c_2 = 40$. In row i, column j is shown the value of a, modulo 65, such that $a \bmod 5 = i$ and $a \bmod 13 = j$. Note that row 0, column 0 contains a 0. Similarly, row 4, column 12 contains a 64 (equivalent to -1). Since $c_1 = 26$, moving down a row increases a by 26. Similarly, $c_2 = 40$ means that moving right by a column increases a by 40. Increasing a by 1 corresponds to moving diagonally downward and to the right, wrapping around from the bottom to the top and from the right to the left.

As an example of the application of the Chinese remainder theorem, suppose we are given the two equations

$$\begin{aligned} a &\equiv 2 \pmod{5} , \\ a &\equiv 3 \pmod{13} , \end{aligned}$$

so that $a_1 = 2$, $n_1 = m_2 = 5$, $a_2 = 3$, and $n_2 = m_1 = 13$, and we wish to compute $a \bmod 65$, since $n = n_1 n_2 = 65$. Because $13^{-1} \equiv 2 \pmod{5}$ and $5^{-1} \equiv 8 \pmod{13}$, we have

$$\begin{aligned} c_1 &= 13(2 \bmod 5) &= 26 , \\ c_2 &= 5(8 \bmod 13) &= 40 , \end{aligned}$$

and

$$\begin{aligned} a &\equiv 2 \cdot 26 + 3 \cdot 40 \pmod{65} \\ &\equiv 52 + 120 \pmod{65} \\ &\equiv 42 \pmod{65} . \end{aligned}$$

See Figure 31.3 for an illustration of the Chinese remainder theorem, modulo 65.

Thus, we can work modulo n by working modulo n directly or by working in the transformed representation using separate modulo n_i computations, as convenient. The computations are entirely equivalent.

Exercises

31.5-1

Find all solutions to the equations $x \equiv 4 \pmod{5}$ and $x \equiv 5 \pmod{11}$.

31.5-2
Find all integers x that leave remainders 1, 2, 3 when divided by 9, 8, 7 respectively.

31.5-3
Argue that, under the definitions of Theorem 31.27, if $\gcd(a, n) = 1$, then

$$(a^{-1} \bmod n) \leftrightarrow ((a_1^{-1} \bmod n_1), (a_2^{-1} \bmod n_2), \ldots, (a_k^{-1} \bmod n_k)) .$$

31.5-4
Under the definitions of Theorem 31.27, prove that for any polynomial f, the number of roots of the equation $f(x) \equiv 0 \pmod{n}$ equals the product of the number of roots of each of the equations $f(x) \equiv 0 \pmod{n_1}$, $f(x) \equiv 0 \pmod{n_2}$, ..., $f(x) \equiv 0 \pmod{n_k}$.

31.6 Powers of an element

Just as we often consider the multiples of a given element a, modulo n, we consider the sequence of powers of a, modulo n, where $a \in \mathbb{Z}_n^*$:

$$a^0, a^1, a^2, a^3, \ldots, \tag{31.33}$$

modulo n. Indexing from 0, the 0th value in this sequence is $a^0 \bmod n = 1$, and the ith value is $a^i \bmod n$. For example, the powers of 3 modulo 7 are

i	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$3^i \bmod 7$	1	3	2	6	4	5	1	3	2	6	4	5	$\cdots$

whereas the powers of 2 modulo 7 are

i	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$2^i \bmod 7$	1	2	4	1	2	4	1	2	4	1	2	4	$\cdots$

In this section, let $\langle a \rangle$ denote the subgroup of $\mathbb{Z}_n^*$ generated by a by repeated multiplication, and let $\text{ord}_n(a)$ (the "order of a, modulo n") denote the order of a in $\mathbb{Z}_n^*$. For example, $\langle 2 \rangle = \{1, 2, 4\}$ in $\mathbb{Z}_7^*$, and $\text{ord}_7(2) = 3$. Using the definition of the Euler phi function $\phi(n)$ as the size of $\mathbb{Z}_n^*$ (see Section 31.3), we now translate Corollary 31.19 into the notation of $\mathbb{Z}_n^*$ to obtain Euler's theorem and specialize it to $\mathbb{Z}_p^*$, where p is prime, to obtain Fermat's theorem.

Theorem 31.30 (Euler's theorem)
For any integer $n > 1$,

$$a^{\phi(n)} \equiv 1 \pmod{n} \text{ for all } a \in \mathbb{Z}_n^* .$$ ■

Theorem 31.31 (Fermat's theorem)
If p is prime, then

$$a^{p-1} \equiv 1 \pmod{p} \text{ for all } a \in \mathbb{Z}_p^* .$$

Proof By equation (31.21), $\phi(p) = p - 1$ if p is prime. ■

Fermat's theorem applies to every element in $\mathbb{Z}_p$ except 0, since $0 \notin \mathbb{Z}_p^*$. For all $a \in \mathbb{Z}_p$, however, we have $a^p \equiv a \pmod{p}$ if p is prime.

If $\text{ord}_n(g) = |\mathbb{Z}_n^*|$, then every element in $\mathbb{Z}_n^*$ is a power of g, modulo n, and g is a ***primitive root*** or a ***generator*** of $\mathbb{Z}_n^*$. For example, 3 is a primitive root, modulo 7, but 2 is not a primitive root, modulo 7. If $\mathbb{Z}_n^*$ possesses a primitive root, the group $\mathbb{Z}_n^*$ is ***cyclic***. We omit the proof of the following theorem, which is proven by Niven and Zuckerman [265].

Theorem 31.32
The values of $n > 1$ for which $\mathbb{Z}_n^*$ is cyclic are 2, 4, p^e, and $2p^e$, for all primes $p > 2$ and all positive integers e. ■

If g is a primitive root of $\mathbb{Z}_n^*$ and a is any element of $\mathbb{Z}_n^*$, then there exists a z such that $g^z \equiv a \pmod{n}$. This z is a ***discrete logarithm*** or an ***index*** of a, modulo n, to the base g; we denote this value as $\text{ind}_{n,g}(a)$.

Theorem 31.33 (Discrete logarithm theorem)
If g is a primitive root of $\mathbb{Z}_n^*$, then the equation $g^x \equiv g^y \pmod{n}$ holds if and only if the equation $x \equiv y \pmod{\phi(n)}$ holds.

Proof Suppose first that $x \equiv y \pmod{\phi(n)}$. Then, $x = y + k\phi(n)$ for some integer k. Therefore,

$$\begin{aligned}
g^x &\equiv g^{y+k\phi(n)} && \pmod{n} \\
&\equiv g^y \cdot (g^{\phi(n)})^k && \pmod{n} \\
&\equiv g^y \cdot 1^k && \pmod{n} \qquad \text{(by Euler's theorem)} \\
&\equiv g^y && \pmod{n} .
\end{aligned}$$

Conversely, suppose that $g^x \equiv g^y \pmod{n}$. Because the sequence of powers of g generates every element of $\langle g \rangle$ and $|\langle g \rangle| = \phi(n)$, Corollary 31.18 implies that the sequence of powers of g is periodic with period $\phi(n)$. Therefore, if $g^x \equiv g^y \pmod{n}$, then we must have $x \equiv y \pmod{\phi(n)}$. ■

We now turn our attention to the square roots of 1, modulo a prime power. The following theorem will be useful in our development of a primality-testing algorithm in Section 31.8.

Theorem 31.34
If p is an odd prime and $e \geq 1$, then the equation

$$x^2 \equiv 1 \pmod{p^e} \tag{31.34}$$

has only two solutions, namely $x = 1$ and $x = -1$.

Proof Equation (31.34) is equivalent to

$$p^e \mid (x-1)(x+1) .$$

Since $p > 2$, we can have $p \mid (x-1)$ or $p \mid (x+1)$, but not both. (Otherwise, by property (31.3), p would also divide their difference $(x+1) - (x-1) = 2$.) If $p \nmid (x-1)$, then $\gcd(p^e, x-1) = 1$, and by Corollary 31.5, we would have $p^e \mid (x+1)$. That is, $x \equiv -1 \pmod{p^e}$. Symmetrically, if $p \nmid (x+1)$, then $\gcd(p^e, x+1) = 1$, and Corollary 31.5 implies that $p^e \mid (x-1)$, so that $x \equiv 1 \pmod{p^e}$. Therefore, either $x \equiv -1 \pmod{p^e}$ or $x \equiv 1 \pmod{p^e}$. ■

A number x is a ***nontrivial square root of 1, modulo n***, if it satisfies the equation $x^2 \equiv 1 \pmod{n}$ but x is equivalent to neither of the two "trivial" square roots: 1 or -1, modulo n. For example, 6 is a nontrivial square root of 1, modulo 35. We shall use the following corollary to Theorem 31.34 in the correctness proof in Section 31.8 for the Miller-Rabin primality-testing procedure.

Corollary 31.35
If there exists a nontrivial square root of 1, modulo n, then n is composite.

Proof By the contrapositive of Theorem 31.34, if there exists a nontrivial square root of 1, modulo n, then n cannot be an odd prime or a power of an odd prime. If $x^2 \equiv 1 \pmod{2}$, then $x \equiv 1 \pmod{2}$, and so all square roots of 1, modulo 2, are trivial. Thus, n cannot be prime. Finally, we must have $n > 1$ for a nontrivial square root of 1 to exist. Therefore, n must be composite. ■

Raising to powers with repeated squaring

A frequently occurring operation in number-theoretic computations is raising one number to a power modulo another number, also known as ***modular exponentiation***. More precisely, we would like an efficient way to compute $a^b \bmod n$, where a and b are nonnegative integers and n is a positive integer. Modular exponentiation is an essential operation in many primality-testing routines and in the RSA public-key cryptosystem. The method of ***repeated squaring*** solves this problem efficiently using the binary representation of b.

Let $\langle b_k, b_{k-1}, \ldots, b_1, b_0 \rangle$ be the binary representation of b. (That is, the binary representation is $k+1$ bits long, b_k is the most significant bit, and b_0 is the least

i	9	8	7	6	5	4	3	2	1	0
b_i	1	0	0	0	1	1	0	0	0	0
c	1	2	4	8	17	35	70	140	280	560
d	7	49	157	526	160	241	298	166	67	1

Figure 31.4 The results of MODULAR-EXPONENTIATION when computing $a^b \pmod n$, where $a = 7$, $b = 560 = \langle 1000110000 \rangle$, and $n = 561$. The values are shown after each execution of the **for** loop. The final result is 1.

significant bit.) The following procedure computes $a^c \bmod n$ as c is increased by doublings and incrementations from 0 to b.

```
MODULAR-EXPONENTIATION($a, b, n$)
 1  $c = 0$
 2  $d = 1$
 3  let $\langle b_k, b_{k-1}, \ldots, b_0 \rangle$ be the binary representation of $b$
 4  for $i = k$ downto 0
 5      $c = 2c$
 6      $d = (d \cdot d) \bmod n$
 7      if $b_i == 1$
 8          $c = c + 1$
 9          $d = (d \cdot a) \bmod n$
10  return $d$
```

The essential use of squaring in line 6 of each iteration explains the name "repeated squaring." As an example, for $a = 7$, $b = 560$, and $n = 561$, the algorithm computes the sequence of values modulo 561 shown in Figure 31.4; the sequence of exponents used appears in the row of the table labeled by c.

The variable c is not really needed by the algorithm but is included for the following two-part loop invariant:

Just prior to each iteration of the **for** loop of lines 4–9,

1. The value of c is the same as the prefix $\langle b_k, b_{k-1}, \ldots, b_{i+1} \rangle$ of the binary representation of b, and
2. $d = a^c \bmod n$.

We use this loop invariant as follows:

Initialization: Initially, $i = k$, so that the prefix $\langle b_k, b_{k-1}, \ldots, b_{i+1} \rangle$ is empty, which corresponds to $c = 0$. Moreover, $d = 1 = a^0 \bmod n$.

Maintenance: Let c' and d' denote the values of c and d at the end of an iteration of the **for** loop, and thus the values prior to the next iteration. Each iteration updates $c' = 2c$ (if $b_i = 0$) or $c' = 2c + 1$ (if $b_i = 1$), so that c will be correct prior to the next iteration. If $b_i = 0$, then $d' = d^2 \bmod n = (a^c)^2 \bmod n = a^{2c} \bmod n = a^{c'} \bmod n$. If $b_i = 1$, then $d' = d^2 a \bmod n = (a^c)^2 a \bmod n = a^{2c+1} \bmod n = a^{c'} \bmod n$. In either case, $d = a^c \bmod n$ prior to the next iteration.

Termination: At termination, $i = -1$. Thus, $c = b$, since c has the value of the prefix $\langle b_k, b_{k-1}, \ldots, b_0 \rangle$ of b's binary representation. Hence $d = a^c \bmod n = a^b \bmod n$.

If the inputs a, b, and n are β-bit numbers, then the total number of arithmetic operations required is $O(\beta)$ and the total number of bit operations required is $O(\beta^3)$.

Exercises

31.6-1
Draw a table showing the order of every element in $\mathbb{Z}_{11}^*$. Pick the smallest primitive root g and compute a table giving $\text{ind}_{11,g}(x)$ for all $x \in \mathbb{Z}_{11}^*$.

31.6-2
Give a modular exponentiation algorithm that examines the bits of b from right to left instead of left to right.

31.6-3
Assuming that you know $\phi(n)$, explain how to compute $a^{-1} \bmod n$ for any $a \in \mathbb{Z}_n^*$ using the procedure MODULAR-EXPONENTIATION.

31.7 The RSA public-key cryptosystem

With a public-key cryptosystem, we can encrypt messages sent between two communicating parties so that an eavesdropper who overhears the encrypted messages will not be able to decode them. A public-key cryptosystem also enables a party to append an unforgeable "digital signature" to the end of an electronic message. Such a signature is the electronic version of a handwritten signature on a paper document. It can be easily checked by anyone, forged by no one, yet loses its validity if any bit of the message is altered. It therefore provides authentication of both the identity of the signer and the contents of the signed message. It is the perfect tool for electronically signed business contracts, electronic checks, electronic purchase orders, and other electronic communications that parties wish to authenticate.

The RSA public-key cryptosystem relies on the dramatic difference between the ease of finding large prime numbers and the difficulty of factoring the product of two large prime numbers. Section 31.8 describes an efficient procedure for finding large prime numbers, and Section 31.9 discusses the problem of factoring large integers.

Public-key cryptosystems

In a public-key cryptosystem, each participant has both a ***public key*** and a ***secret key***. Each key is a piece of information. For example, in the RSA cryptosystem, each key consists of a pair of integers. The participants "Alice" and "Bob" are traditionally used in cryptography examples; we denote their public and secret keys as P_A, S_A for Alice and P_B, S_B for Bob.

Each participant creates his or her own public and secret keys. Secret keys are kept secret, but public keys can be revealed to anyone or even published. In fact, it is often convenient to assume that everyone's public key is available in a public directory, so that any participant can easily obtain the public key of any other participant.

The public and secret keys specify functions that can be applied to any message. Let $\mathcal{D}$ denote the set of permissible messages. For example, $\mathcal{D}$ might be the set of all finite-length bit sequences. In the simplest, and original, formulation of public-key cryptography, we require that the public and secret keys specify one-to-one functions from $\mathcal{D}$ to itself. We denote the function corresponding to Alice's public key P_A by $P_A()$ and the function corresponding to her secret key S_A by $S_A()$. The functions $P_A()$ and $S_A()$ are thus permutations of $\mathcal{D}$. We assume that the functions $P_A()$ and $S_A()$ are efficiently computable given the corresponding key P_A or S_A.

The public and secret keys for any participant are a "matched pair" in that they specify functions that are inverses of each other. That is,

$$M = S_A(P_A(M)) , \tag{31.35}$$

$$M = P_A(S_A(M)) \tag{31.36}$$

for any message $M \in \mathcal{D}$. Transforming M with the two keys P_A and S_A successively, in either order, yields the message M back.

In a public-key cryptosystem, we require that no one but Alice be able to compute the function $S_A()$ in any practical amount of time. This assumption is crucial to keeping encrypted mail sent to Alice private and to knowing that Alice's digital signatures are authentic. Alice must keep S_A secret; if she does not, she loses her uniqueness and the cryptosystem cannot provide her with unique capabilities. The assumption that only Alice can compute $S_A()$ must hold even though everyone

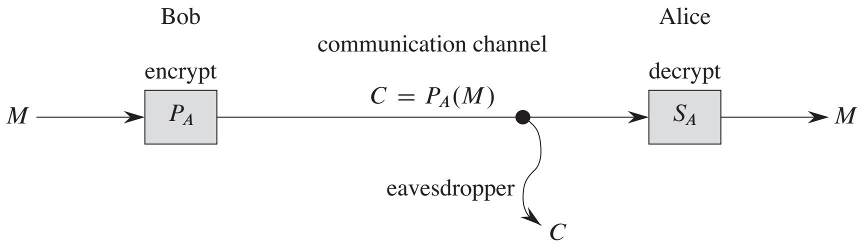


Figure 31.5 Encryption in a public key system. Bob encrypts the message M using Alice's public key P_A and transmits the resulting ciphertext $C = P_A(M)$ over a communication channel to Alice. An eavesdropper who captures the transmitted ciphertext gains no information about M. Alice receives C and decrypts it using her secret key to obtain the original message $M = S_A(C)$.

knows P_A and can compute $P_A()$, the inverse function to $S_A()$, efficiently. In order to design a workable public-key cryptosystem, we must figure out how to create a system in which we can reveal a transformation $P_A()$ without thereby revealing how to compute the corresponding inverse transformation $S_A()$. This task appears formidable, but we shall see how to accomplish it.

In a public-key cryptosystem, encryption works as shown in Figure 31.5. Suppose Bob wishes to send Alice a message M encrypted so that it will look like unintelligible gibberish to an eavesdropper. The scenario for sending the message goes as follows.

- Bob obtains Alice's public key P_A (from a public directory or directly from Alice).
- Bob computes the ***ciphertext*** $C = P_A(M)$ corresponding to the message M and sends C to Alice.
- When Alice receives the ciphertext C, she applies her secret key S_A to retrieve the original message: $S_A(C) = S_A(P_A(M)) = M$.

Because $S_A()$ and $P_A()$ are inverse functions, Alice can compute M from C. Because only Alice is able to compute $S_A()$, Alice is the only one who can compute M from C. Because Bob encrypts M using $P_A()$, only Alice can understand the transmitted message.

We can just as easily implement digital signatures within our formulation of a public-key cryptosystem. (There are other ways of approaching the problem of constructing digital signatures, but we shall not go into them here.) Suppose now that Alice wishes to send Bob a digitally signed response M'. Figure 31.6 shows how the digital-signature scenario proceeds.

- Alice computes her ***digital signature*** σ for the message M' using her secret key S_A and the equation $\sigma = S_A(M')$.

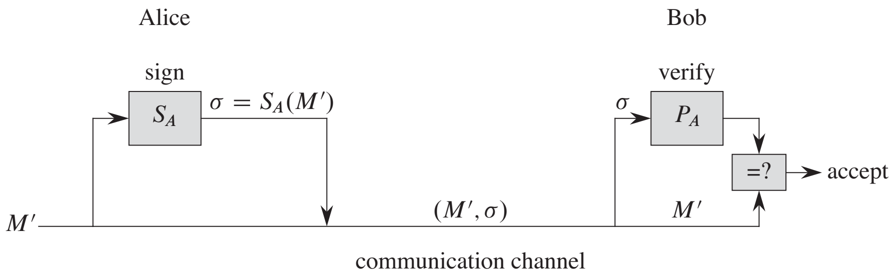


Figure 31.6 Digital signatures in a public-key system. Alice signs the message M' by appending her digital signature $\sigma = S_A(M')$ to it. She transmits the message/signature pair (M', σ) to Bob, who verifies it by checking the equation $M' = P_A(\sigma)$. If the equation holds, he accepts (M', σ) as a message that Alice has signed.

- Alice sends the message/signature pair (M', σ) to Bob.
- When Bob receives (M', σ), he can verify that it originated from Alice by using Alice's public key to verify the equation $M' = P_A(\sigma)$. (Presumably, M' contains Alice's name, so Bob knows whose public key to use.) If the equation holds, then Bob concludes that the message M' was actually signed by Alice. If the equation fails to hold, Bob concludes either that the message M' or the digital signature σ was corrupted by transmission errors or that the pair (M', σ) is an attempted forgery.

Because a digital signature provides both authentication of the signer's identity and authentication of the contents of the signed message, it is analogous to a handwritten signature at the end of a written document.

A digital signature must be verifiable by anyone who has access to the signer's public key. A signed message can be verified by one party and then passed on to other parties who can also verify the signature. For example, the message might be an electronic check from Alice to Bob. After Bob verifies Alice's signature on the check, he can give the check to his bank, who can then also verify the signature and effect the appropriate funds transfer.

A signed message is not necessarily encrypted; the message can be "in the clear" and not protected from disclosure. By composing the above protocols for encryption and for signatures, we can create messages that are both signed and encrypted. The signer first appends his or her digital signature to the message and then encrypts the resulting message/signature pair with the public key of the intended recipient. The recipient decrypts the received message with his or her secret key to obtain both the original message and its digital signature. The recipient can then verify the signature using the public key of the signer. The corresponding combined process using paper-based systems would be to sign the paper document and then seal the document inside a paper envelope that is opened only by the intended recipient.

The RSA cryptosystem

In the ***RSA public-key cryptosystem***, a participant creates his or her public and secret keys with the following procedure:

1. Select at random two large prime numbers p and q such that $p \neq q$. The primes p and q might be, say, 1024 bits each.
2. Compute $n = pq$.
3. Select a small odd integer e that is relatively prime to $\phi(n)$, which, by equation (31.20), equals $(p-1)(q-1)$.
4. Compute d as the multiplicative inverse of e, modulo $\phi(n)$. (Corollary 31.26 guarantees that d exists and is uniquely defined. We can use the technique of Section 31.4 to compute d, given e and $\phi(n)$.)
5. Publish the pair $P = (e, n)$ as the participant's ***RSA public key***.
6. Keep secret the pair $S = (d, n)$ as the participant's ***RSA secret key***.

For this scheme, the domain $\mathcal{D}$ is the set $\mathbb{Z}_n$. To transform a message M associated with a public key $P = (e, n)$, compute

$$P(M) = M^e \bmod n \ . \tag{31.37}$$

To transform a ciphertext C associated with a secret key $S = (d, n)$, compute

$$S(C) = C^d \bmod n \ . \tag{31.38}$$

These equations apply to both encryption and signatures. To create a signature, the signer applies his or her secret key to the message to be signed, rather than to a ciphertext. To verify a signature, the public key of the signer is applied to it, rather than to a message to be encrypted.

We can implement the public-key and secret-key operations using the procedure MODULAR-EXPONENTIATION described in Section 31.6. To analyze the running time of these operations, assume that the public key (e, n) and secret key (d, n) satisfy $\lg e = O(1)$, $\lg d \leq \beta$, and $\lg n \leq \beta$. Then, applying a public key requires $O(1)$ modular multiplications and uses $O(\beta^2)$ bit operations. Applying a secret key requires $O(\beta)$ modular multiplications, using $O(\beta^3)$ bit operations.

Theorem 31.36 (Correctness of RSA)
The RSA equations (31.37) and (31.38) define inverse transformations of $\mathbb{Z}_n$ satisfying equations (31.35) and (31.36).

Proof From equations (31.37) and (31.38), we have that for any $M \in \mathbb{Z}_n$,

$$P(S(M)) = S(P(M)) = M^{ed} \pmod{n} .$$

Since e and d are multiplicative inverses modulo $\phi(n) = (p-1)(q-1)$,

$$ed = 1 + k(p-1)(q-1)$$

for some integer k. But then, if $M \not\equiv 0 \pmod{p}$, we have

$$\begin{aligned} M^{ed} &\equiv M(M^{p-1})^{k(q-1)} && \pmod{p} \\ &\equiv M((M \bmod p)^{p-1})^{k(q-1)} && \pmod{p} \\ &\equiv M(1)^{k(q-1)} && \pmod{p} \quad \text{(by Theorem 31.31)} \\ &\equiv M && \pmod{p} . \end{aligned}$$

Also, $M^{ed} \equiv M \pmod{p}$ if $M \equiv 0 \pmod{p}$. Thus,

$$M^{ed} \equiv M \pmod{p}$$

for all M. Similarly,

$$M^{ed} \equiv M \pmod{q}$$

for all M. Thus, by Corollary 31.29 to the Chinese remainder theorem,

$$M^{ed} \equiv M \pmod{n}$$

for all M. ■

The security of the RSA cryptosystem rests in large part on the difficulty of factoring large integers. If an adversary can factor the modulus n in a public key, then the adversary can derive the secret key from the public key, using the knowledge of the factors p and q in the same way that the creator of the public key used them. Therefore, if factoring large integers is easy, then breaking the RSA cryptosystem is easy. The converse statement, that if factoring large integers is hard, then breaking RSA is hard, is unproven. After two decades of research, however, no easier method has been found to break the RSA public-key cryptosystem than to factor the modulus n. And as we shall see in Section 31.9, factoring large integers is surprisingly difficult. By randomly selecting and multiplying together two 1024-bit primes, we can create a public key that cannot be "broken" in any feasible amount of time with current technology. In the absence of a fundamental breakthrough in the design of number-theoretic algorithms, and when implemented with care following recommended standards, the RSA cryptosystem is capable of providing a high degree of security in applications.

In order to achieve security with the RSA cryptosystem, however, we should use integers that are quite long—hundreds or even more than one thousand bits