Each player has a token on one of the vertices of G. At the start of the game, The Doctor's token is on the source vertex s, and River's token is on the sink vertex t. The players alternate turns, with The Doctor moving first. On each of his turns, the Doctor moves his token forward along a directed edge; on each of her turns, River moves her token *backward* along a directed edge.

If the two tokens ever meet on the same vertex, River wins the game. ("Hello, Sweetie!") If the Doctor's token reaches t or River's token reaches s before the two tokens meet, then the Doctor wins the game.

Describe and analyze an algorithm to determine who wins this game, assuming both players play perfectly. That is, if the Doctor can win *no matter how River moves*, then your algorithm should output "Doctor", and if River can win *no matter how the Doctor moves*, your algorithm should output "River". (Why are these the only two possibilities?) The input to your algorithm is the graph G.

♣♥24. Let $x = x_1 x_2 \ldots x_n$ be a given n-character string over some finite alphabet Σ, and let A be a deterministic finite-state machine with m states over the same alphabet.

(a) Describe and analyze an algorithm to compute the length of the longest subsequence of x that is accepted by A. For example, if A accepts the language $(\texttt{AR})^*$ and $x = \texttt{ABRACADABRA}$, your algorithm should output the number 4, which is the length of the subsequence ARAR.

(b) Describe and analyze an algorithm to compute the length of the shortest supersequence of x that is accepted by A. For example, if A accepts the language $(\texttt{ABCDR})^*$ and $x = \texttt{ABRACADABRA}$, your algorithm should output the number 25, which is the length of the supersequence ABCDRABCDRABCDRABCDRABCDR.

Analyze your algorithms in terms of the length n of the input string, the number m of states in the finite-state machine, and the size of the alphabet Σ.

25. Not *every* dynamic programming algorithm can be modeled as finding an optimal path through a directed acyclic graph, but every dynamic programming algorithm does process some underlying dependency graph in postorder.

(a) Suppose we are given a directed acyclic graph G where every node stores a numerical search key. Describe and analyze an algorithm to find the largest binary search tree that is a subgraph of G.

(b) Suppose we are given a directed acyclic graph G and two vertices s and t. Describe an algorithm to compute the number of directed paths in G from s to t. (Assume that any arithmetic operation requires $O(1)$ time.)

(c) Let G be a directed acyclic graph with the following features:

- G has a single source s and several sinks $t_1, t_2, \ldots, t_k$.
- Each edge $v \to w$ has an associated weight $p(v \to w)$ between 0 and 1.
- For each non-sink vertex v, the total weight of all edges leaving v is 1; that is, $\sum_w p(v \to w) = 1$.

The weights $p(v \to w)$ define a random walk in G from the source s to some sink t_i; after reaching any non-sink vertex v, the walk follows edge $v \to w$ with probability $p(v \to w)$. All probabilities are mutually independent. Describe and analyze an algorithm to compute the probability that this random walk reaches sink t_i, for every index i. (Assume that each arithmetic operation takes only $O(1)$ time.)

We must all hang together, gentlemen,
or else we shall most assuredly hang separately.

— Benjamin Franklin, at the signing of the
Declaration of Independence (July 4, 1776)

I remember seeking advice from someone—who could it have been?—about whether this work was worth submitting for publication; the reasoning it uses is so very simple.... Fortunately he advised me to go ahead, and many years passed before another of my publications became as well-known as this very simple one.

— Joseph Kruskal, describing his shortest-spanning-subtree algorithm (1997)

Clean ALL the things!

— Allie Brosh, "This is Why I'll Never be an Adult",
Hyperbole and a Half, June 17, 2010.

7

Minimum Spanning Trees

Suppose we are given a connected, undirected, *weighted* graph. This is a graph $G = (V, E)$ together with a function $w\colon E \to \mathbb{R}$ that assigns a real *weight* $w(e)$ to each edge e, which may be positive, negative, or zero. This chapter describes several algorithms to find the ***minimum spanning tree*** of G, that is, the spanning tree T that minimizes the function

$$w(T) := \sum_{e \in T} w(e).$$

See Figure 7.1 for an example.

7.1 Distinct Edge Weights

An annoying subtlety in the problem statement is that weighted graphs can have more than one spanning tree with the same minimum weight; in particular,

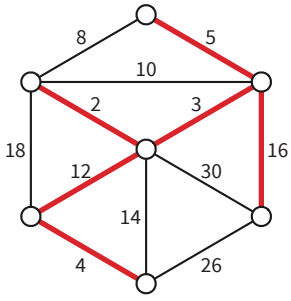


Figure 7.1. A weighted graph and its minimum spanning tree.

if every edge in G has weight 1, then *every* spanning tree of G is a minimum spanning tree, with weight $V-1$. This ambiguity complicates the development of our algorithms; everything would be much simpler if we could simply *assume* that minimum spanning trees are unique.

Fortunately, there is an easy condition that implies the uniqueness we want.

Lemma 7.1. *If all edge weights in a connected graph G are distinct, then G has a unique minimum spanning tree.*[1]

Proof: Let G be an arbitrary connected graph with two minimum spanning trees T and T'; we need to prove that some pair of edges in G have the same weight. The proof is essentially a greedy exchange argument.

Each of our spanning trees must contain an edge that the other tree omits. Let e be a minimum-weight edge in $T \setminus T'$, and let e' be a minimum-weight edge in $T' \setminus T$ (breaking ties arbitrarily). Without loss of generality, suppose $w(e) \le w(e')$.

The subgraph $T' \cup \{e\}$ contains exactly one cycle C, which passes through the the edge e. Let e'' be *any* edge of this cycle that is *not* in T. At least one such edge must exist, because T is a tree. (We may or may not have $e'' = e'$.) Because $e \in T$, we immediately have $e'' \neq e$ and therefore $e'' \in T' \setminus T$. It follows that $w(e'') \ge w(e') \ge w(e)$.

Now consider the spanning tree $T'' = T' + e - e''$. (This new tree T'' *might* be equal to T.) We immediately have $w(T'') = w(T') + w(e) - w(e'') \le w(T')$. But T' is a *minimum* spanning tree, so we must have $w(T'') = w(T')$; in other words, T'' is also a minimum spanning tree. We conclude that $w(e) = w(e'')$, which completes the proof. □

If we already have an algorithm that assumes distinct edge weights, we can still run it on graphs where some edges have equal weights, as long as we have

[1]The converse of this lemma is false; a connected graph with repeated edge weights can still have a unique minimum spanning tree. As a trivial example, suppose G is a tree!

a consistent method for breaking ties. One such method uses the following algorithm in place of simple weight comparisons. ShorterEdge takes as input four integers i, j, k, l, representing four (not necessarily distinct) vertices, and decides which of the two edges (i, j) and (k, l) has "smaller" weight. (Because the input graph undirected, the pairs (i, j) and (j, i) represent the same edge.)

```
ShorterEdge(i, j, k, l)
  if w(i, j) < w(k, l)                then return (i, j)
  if w(i, j) > w(k, l)                then return (k, l)
  if min(i, j) < min(k, l)            then return (i, j)
  if min(i, j) > min(k, l)            then return (k, l)
  if max(i, j) < max(k, l)            then return (i, j)
  ⟨⟨if max(i,j) > max(k,l) ⟩⟩          return (k, l)
```

In light of Lemma 7.1 and this tie-breaking rule, we will safely assume for the rest of this chapter that edge weights are *always* distinct, and therefore minimum spanning trees are *always* unique. In particular, we can freely discuss ***the*** minimum spanning tree with no confusion.

7.2 The Only Minimum Spanning Tree Algorithm

There are many algorithms to compute minimum spanning trees, but almost all of them are instances of the following generic strategy. The situation is similar to graph traversal, where several different algorithms are all variants of the generic traversal algorithm whatever-first search.

The generic minimum spanning tree algorithm maintains an acyclic subgraph F of the input graph G, which we will call the *intermediate spanning forest*. At all times, F satisfies two invariants:

- F is a subgraph of the minimum spanning tree of G.
- Every component of F is a minimum spanning tree of its vertices.

(In fact, the second invariant implies the first.) Initially, F consists of V one-node trees. The generic algorithm merges trees in F together by adding certain edges between them. When the algorithm halts, F consists of a single spanning tree; our invariants imply that this must be the minimum spanning tree of G. Obviously, we have to be careful about *which* edges we add to the evolving forest, since not every edge is in the minimum spanning tree.

At any stage of its evolution, the intermediate spanning forest F induces two special types of edges.

- An edge is ***useless*** if it is not an edge of F, but both its endpoints are in the same component of F.
- An edge is ***safe*** if it is the minimum-weight edge with with exactly one endpoint in some component of F.

The same edge could be safe for two different components of F. Some edges of $G \setminus F$ are neither safe nor useless; we call these edges *undecided*.

All minimum spanning tree algorithms are based on two simple observations. The first observation was proved by Robert Prim in 1957 (although it is implicit in several earlier algorithms), and the second is immediate.

Lemma 7.2 (Prim). *The minimum spanning tree of G contains every safe edge.*

Proof: In fact we prove the following stronger statement: For *any* subset S of the vertices of G, the minimum spanning tree of G contains the minimum-weight edge with exactly one endpoint in S. Like the previous lemma, we prove this claim using a greedy exchange argument.

Let S be an arbitrary subset of vertices of G, and let e be the lightest edge with exactly one endpoint in S. (Our assumption that all edge weights are distinct implies that e is unique.) Let T be an arbitrary spanning tree that does *not* contain e; we need to prove that T is *not* the minimum spanning tree of G.

Because T is connected, it contains a path from one endpoint of e to the other. Because this path starts at a vertex of S and ends at a vertex not in S, it must contain at least one edge with exactly one endpoint in S; let e' be *any* such edge. Because T is acyclic, removing e' from T yields a spanning *forest* with exactly two components, one containing each endpoint of e. Thus, adding e to this forest gives us a new spanning tree $T' = T - e' + e$. The definition of e implies $w(e') > w(e)$, which implies that T' has smaller total weight than T. Thus, T is not the minimum spanning tree of G, which completes the proof. □

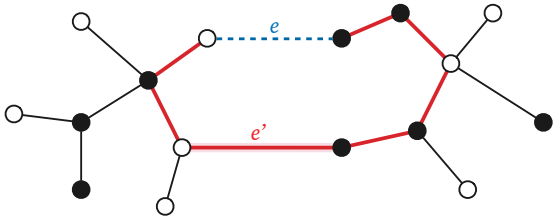


Figure 7.2. Every safe edge is in the minimum spanning tree. Black vertices are in the subset S.

Lemma 7.3. *The minimum spanning tree contains no useless edge.*

Proof: Adding any useless edge to F would introduce a cycle. □

Our generic minimum spanning tree algorithm repeatedly adds safe edges to the evolving forest F. Whenever we add new edges to F, some undecided edges become safe, and others become useless. (Once an edge becomes useless, it stays useless forever.) To specify a particular algorithm, we must specify *which* safe edge(s) to add in each iteration, and we must describe how to find those edges.

7.3 Borůvka's Algorithm

The oldest and arguably simplest minimum spanning tree algorithm was discovered by the Czech mathematician Otakar Borůvka in 1926, about a year after Jindřich Saxel asked him how to construct an electrical network connecting several cities using the least amount of wire.[2] The algorithm was rediscovered by Gustav Choquet in 1938, rediscovered again by a team of Polish mathematicians led by Józef Łukaszewicz in 1951, and rediscovered again by George Sollin in 1961. Although Sollin never published his rediscovery, it was carefully described and credited in one of the first textbooks on graph algorithms; as a result, this algorithm is sometimes called "Sollin's algorithm".

The Borůvka / Choquet / Florek-Łukaziewicz-Perkal-Steinhaus-Zubrzycki / Prim / Sollin / Brosh[3] algorithm can be summarized in one line:

> BORŮVKA: Add ***ALL*** the safe edges and recurse.

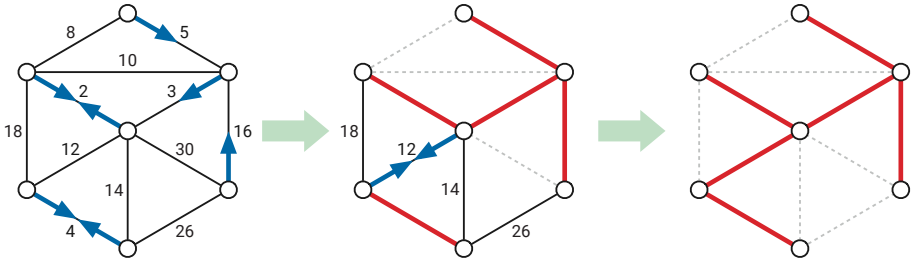


Figure 7.3. Borůvka's algorithm run on the example graph. Thick red edges are in F; dashed edges are useless. Arrows point along each component's safe edge. The algorithm ends after just two iterations.

Here is Borůvka's algorithm in more detail. The algorithm calls the COUNTANDLABEL algorithm from Chapter 5 (on page 202) to count the components of F and label each vertex v with an integer $comp(v)$ indicating its component.

```
BORŮVKA(V, E):
  F = (V, ∅)
  count ← COUNTANDLABEL(F)
  while count > 1
      ADDALLSAFEEDGES(E, F, count)
      count ← COUNTANDLABEL(F)
  return F
```

[2] Saxel was an employee of the West Moravian Power Company, described by Borůvka as "very talented and hard-working", who was later executed by the Nazis as a person of Jewish descent.

[3] Go read everything in *Hyperbole and a Half*. And then go buy the book. And an extra copy for your cat. What's that? You don't have a cat? What kind of a monster are you? Go get a cat, and then buy it an extra copy of *Hyperbole and a Half*.

It remains only to describe how to identify and add all the safe edges to F. Suppose F has more than one component, since otherwise we're already done. The following subroutine computes an array $safe[1..V]$ of safe edges, where $safe[i]$ is the minimum-weight edge with one endpoint in the ith component of F, by a brute force examination of every edge in G. For each edge uv, if u and v are in the same component, then uv is either useless or already an edge in F. Otherwise, we compare the weight of uv to the weights of $safe[comp(u)]$ and $safe[comp(v)]$ and update the array entries if necessary. Once we have identified all the safe edges, we add each edge $safe[i]$ to F.

```
ADDALLSAFEEDGES(E, F, count):
  for i ← 1 to count
      safe[i] ← NULL
  for each edge uv ∈ E
      if comp(u) ≠ comp(v)
          if safe[comp(u)] = NULL or w(uv) < w(safe[comp(u)])
              safe[comp(u)] ← uv
          if safe[comp(v)] = NULL or w(uv) < w(safe[comp(v)])
              safe[label(v)] ← uv
  for i ← 1 to count
      add safe[i] to F
```

Each call to COUNTANDLABEL runs in $O(V)$ time, because the forest F has at most $V-1$ edges. *AddAllSafeEdges* runs in $O(V+E)$ time, because we spend constant time on each vertex, each edge of G, and each component of F. Because the input graph is connected, we have $V \leq E+1$. It follows that each iteration of the while loop of BORŮVKA takes $O(E)$ time.

Each iteration reduces the number of components of F by at least a factor of two—in the worst case, the components of F coalesce in pairs. Because F initially has V components, the while loop iterates at most $O(\log V)$ times. We conclude that the overall running time of Borůvka's algorithm is $\boldsymbol{O(E \log V)}$.

This is the MST Algorithm You Want

Despite its relatively obscure origin, early algorithms researchers were aware of Borůvka's algorithm, but dismissed it as being "too complicated". As a result, despite its simplicity and efficiency, most algorithms and data structures textbooks unfortunately never even mention Borůvka's algorithm. This omission is a serious mistake; Borůvka's algorithm has several distinct advantages over other classical MST algorithms.

- Borůvka's algorithm often runs faster than the $O(E \log V)$ worst-case running time. The number of components in F can drop by significantly more than a factor of 2 in a single iteration, reducing the number of iterations below the worst-case $\lceil \log_2 V \rceil$.

- A slight reformulation of Borůvka's algorithm (actually closer to Borůvka's original presentation) actually runs in $O(E)$ time for a broad class of interesting graphs, including graphs that can be drawn in the plane without edge crossings. In contrast, the time analysis for the other two algorithms applies to *all* graphs.
- Borůvka's algorithm allows for significant parallelism; in each iteration, each component of F can be handled in a separate independent thread. This implicit parallelism allows for even faster performance on multicore or distributed systems. In contrast, the other two classical MST algorithms are intrinsically serial.
- There are several more recent minimum-spanning-tree algorithms that are faster even in the worst case than the classical algorithms described here. *All* of these faster algorithms are generalizations of Borůvka's algorithm.

In short, if you ever need to implement a minimum-spanning-tree algorithm, use Borůvka. On the other hand, if you want to *prove things about* minimum spanning trees effectively, you really need to know the next two algorithms as well.

7.4 Jarník's ("Prim's") Algorithm

The next oldest minimum spanning tree algorithm was first described by the Czech mathematician Vojtěch Jarník in a 1929 letter to Borůvka; Jarník published his discovery the following year. The algorithm was independently rediscovered by Joseph Kruskal in 1956, (arguably) by Robert Prim in 1957, and finally by Edsger Dijkstra in 1958. Both Prim and Dijkstra (eventually) knew of and even cited Kruskal's paper, but since Kruskal also described two other minimum-spanning-tree algorithms in the same paper, *this* algorithm is usually called "Prim's algorithm", or sometimes "the Prim/Dijkstra algorithm", even though by 1958 Dijkstra already had another algorithm (inappropriately) named after him.

In Jarník's algorithm, the intermediate forest F has only one nontrivial component T; all the other components are isolated vertices. Initially, T consists of a single arbitrary vertex of the graph. The algorithm repeats the following step until T spans the whole graph:

> JARNÍK: Repeatedly add T's safe edge to T.

To implement Jarník's algorithm, we keep all the edges adjacent to T in a priority queue. When we pull the minimum-weight edge out of the priority queue, we first check whether both of its endpoints are in T. If not, we add the edge to T and then add the new neighboring edges to the priority queue. In other words, Jarník's algorithm is a variant of "best first search", as described at

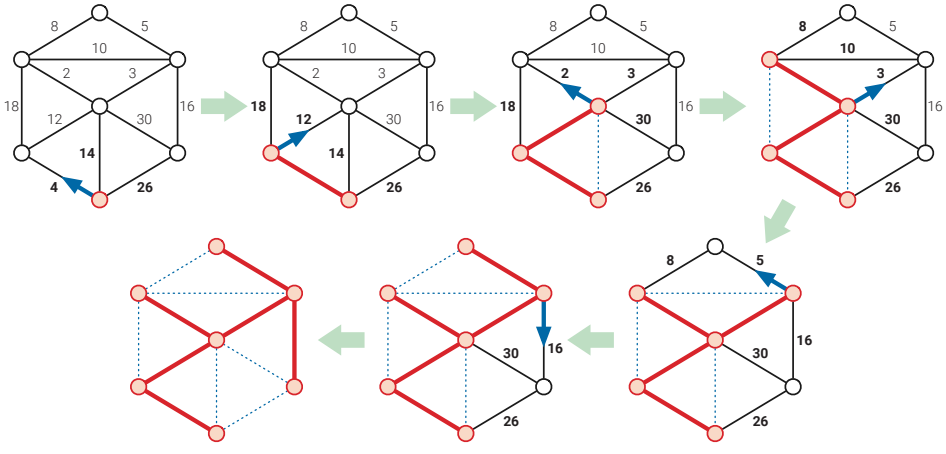


Figure 7.4. Jarník's algorithm run on the example graph, starting with the bottom vertex. At each stage, thick red edges are in T, an arrow points along T's safe edge; and dashed edges are useless.

the end of Chapter 5! If we implement the underlying priority queue using a standard binary heap, Jarník's algorithm runs in $O(E \log E) = \boldsymbol{O(E \log V)}$ time.

♥Improving Jarník's Algorithm

We can improve Jarník's algorithm using a more complex priority queue data structure called a ***Fibonacci heap***, first described by Michael Fredman and Robert Tarjan in 1984. Just like binary heaps, Fibonacci heaps support the standard priority queue operations INSERT, EXTRACTMIN, and DECREASEKEY. However, unlike standard binary heaps, which require $O(\log n)$ time for every operation, Fibonacci heaps support INSERT and DECREASEKEY in constant *amortized* time. The amortized cost of EXTRACTMIN is still $O(\log n)$.[4]

To apply this faster data structure, we keep the *vertices* of G in the priority queue instead of edges, where the priority of each vertex v is either the minimum-weight edge between v and the evolving tree T, or ∞ if there is no such edge. We can INSERT all the vertices into the priority queue at the beginning of the algorithm; then, whenever we add a new edge to T, we may need to decrease the priorities of some neighboring vertices.

To make the description easier, we break the algorithm into two parts. JARNÍKINIT initializes the priority queue; JARNÍKLOOP is the main algorithm.

[4]Amortized time is an accounting trick that allows us to ignore infrequent fluctuations in the time for a single data structure operation. A Fibonacci heap can execute any intermixed sequence of I INSERTS, D DECREASEKEYS, and X EXTRACTMINS in $O(I + D + X \log n)$ time, in the worst case. So the *average* INSERT and the *average* DECREASEKEY each take constant time, and the *average* EXTRACTMIN takes $O(\log n)$ time; however, some individual operations may take longer in the worst case. Amortization uses *statistical* averaging over the sequence of operations; there is no assumption of randomness here, either in the input data or in the algorithm.

The input consists of the vertices and edges of the graph, along with the start vertex s. For each vertex v, we maintain both its priority $priority(v)$ and the incident edge $edge(v)$ such that $w(edge(v)) = priority(v)$.

```
JarníkInit(V, E, s):
  for each vertex v ∈ V \ {s}
      if vs ∈ E
          edge(v) ← vs
          priority(v) ← w(vs)
      else
          edge(v) ← Null
          priority(v) ← ∞
      Insert(v)
```

```
JarníkLoop(V, E, s):
  T ← ({s}, ∅)
  for i ← 1 to |V| − 1
      v ← ExtractMin
      add v and edge(v) to T
      for each neighbor u of v
          if u ∉ T and priority(u) > w(uv)
              edge(u) ← uv
              DecreaseKey(u, w(uv))
```

```
Jarník(V, E, s):
  JarníkInit(V, E, s)
  JarníkLoop(V, E, s)
```

Figure 7.5. Jarník's minimum spanning tree algorithm, ready to be used with a Fibonacci heap

The operations Insert and ExtractMin are each called $O(V)$ times once for each vertex except s, and DecreaseKey is called $O(E)$ times, at most twice for each edge. Thus, if we use a Fibonacci heap, the improved algorithm runs in $\boldsymbol{O(E + V \log V)}$ ***time***, which is faster than Borůvka's algorithm unless $E = O(V)$.

In practice, however, this improvement is rarely faster than the naive implementation using a binary heap, unless the graph is extremely large and dense. The Fibonacci heap algorithms are quite complex, and the hidden constants in both the running time and space are significant—not outrageous, but certainly bigger than the hidden constant 1 in the $O(\log n)$ time bound for binary heap operations.

7.5 Kruskal's Algorithm

The last minimum spanning tree algorithm we'll consider was first described by Joseph Kruskal in 1956, in the same paper where he rediscovered Jarnik's algorithm. Kruskal was motivated by "a typewritten translation (of obscure origin)" of Borůvka's original paper that had been "floating around" the Princeton math department. Kruskal found Borůvka's algorithm "unnecessarily elaborate".[5] The same algorithm was rediscovered in 1957 by Loberman and Weinberger, but somehow avoided being renamed after them.

[5]To be fair, Borůvka's first paper *was* unnecessarily elaborate, in part because it was written for mathematicians in the formal language of (linear) algebra, rather than in the language of graphs. Borůvka's followup paper, also published in 1927 but in an electrotechnical journal, was written in plain language for a much broader audience, essentially its current modern form. Kruskal was apparently unaware of Borůvka's second paper. Stupid Iron Curtain.

Like our earlier minimum-spanning tree algorithms, Kruskal's algorithm has a memorable one-line description:

> KRUSKAL: Scan all edges by increasing weight; if an edge is safe, add it to F.

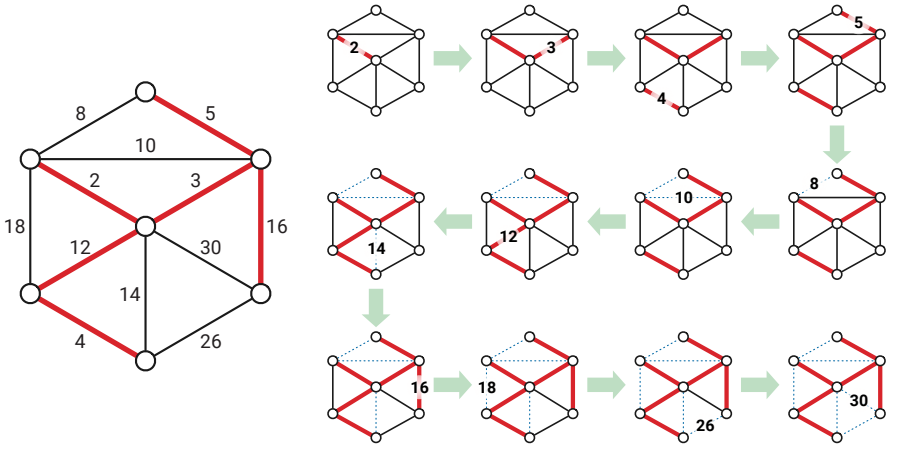


Figure 7.6. Kruskal's algorithm run on the example graph. Thick red edges are in F; thin dashed edges are useless.

The simplest method to scan the edges in increases weight order is to sort the edges by weight, in $O(E \log E)$ time, and then use a simple for-loop over the sorted edge list. As we will see shortly, this preliminary sorting dominates the running time of the algorithm.

Because we examine the edges in order from lightest to heaviest, any edge we examine is safe if and only if its endpoints are in different components of the forest F. Suppose we encounter an edge e joins two components A and B but is not safe. Then there mut be a lighter edge e' with exactly one endpoint in A. But this is impossible, because (inductively) every previously examined edge has both endpoints in the same component of F.

Just as in Borůvka's algorithm, each vertex of F needs to "know" which component of F contains it. Unlike Borůvka's algorithm, however, we do not recompute all component labels from scratch every time we add an edge. Instead, when two components are joined by an edge, the smaller component inherits the label of the larger component; that is, we traverse the smaller component (via whatever-first search). This traversal requires $O(1)$ time for each vertex in the smaller component. Each time the component label of a vertex changes, the component of F containing that vertex grows by at least a factor of 2; thus, each vertex label changes at most $O(\log V)$ times. It follows that the *total* time spent updating vertex labels is only $O(V \log V)$.

More generally, Kruskal's algorithm maintains a partition of the the vertices of G into disjoint subsets (in our case, the components of F), using a data structure that supports the following operations:

- MakeSet(v) — Create a set containing only the vertex v.
- Find(v) — Return an identifier unique to the set containing v.
- Union(u, v) — Replace the sets containing u and v with their union. (This operation decreases the number of sets.)

Here's a complete description of Kruskal's algorithm in terms of these operations:

```
Kruskal(V, E):
  sort E by increasing weight
  F ← (V, ∅)
  for each vertex v ∈ V
      MakeSet(v)
  for i ← 1 to |E|
      uv ← ith lightest edge in E
      if Find(u) ≠ Find(v)
          Union(u, v)
          add uv to F
  return F
```

After the initial sort, the algorithm performs exactly V MakeSet operations (one for each vertex), $2E$ Find operations (two for each edge), and $V-1$ Union operations (one for each edge in the minimum spanning tree). We just described a disjoint-set data structure for which MakeSet and Find require $O(1)$ time, and Union runs in $O(\log V)$ *amortized* time. Using this implementation, the total time spent maintaining the set partition is $O(E + V \log V)$.[6]

But recall that we already need $O(E \log E) = O(E \log V)$ time just to sort the edges. Because this is larger than the time spent maintaining the Union-Find data structure, the overall running time of Kruskal's algorithm is $\boldsymbol{O(E \log V)}$, exactly the same as Borůvka's algorithm, or Jarník's algorithm with a normal (non-Fibonacci) heap.

Exercises

1. Let $G = (V, E)$ be an arbitrary connected graph with weighted edges.
   (a) Prove that for any cycle in G, the minimum spanning tree of G *excludes* the maximum-weight edge in that cycle.

[6] A different disjoint-set data structure, which uses a strategy called *union-by-rank with path compression*, performs each Union or Find in $O(\alpha(V))$ amortized time, where α is the almost-but-not-quite-constant *inverse-Ackerman function*. If you don't feel like using Wikipedia, just think of $\alpha(V)$ as 4. Using this implementation, the total time spent maintaining the set partition is $O(E\alpha(V))$, which is slightly faster when V is large and E is very close to V.

(b) Prove or disprove: The minimum spanning tree of G includes the minimum-weight edge in *every* cycle in G.

2. Throughout this chapter, we assumed that no two edges in the input graph have equal weights, which implies that the minimum spanning tree is unique. In fact, a weaker condition on the edge weights implies MST uniqueness.

   (a) Describe an edge-weighted graph that has a unique minimum spanning tree, even though two edges have equal weights.

   (b) Prove that an edge-weighted graph G has a *unique* minimum spanning tree if and only if the following conditions hold:

   - For any partition of the vertices of G into two subsets, the minimum-weight edge with one endpoint in each subset is unique.
   - The maximum-weight edge in any cycle of G is unique.

   (c) Describe and analyze an algorithm to determine whether or not a graph has a unique minimum spanning tree.

3. Most classical minimum-spanning-tree algorithms use the notions of "safe" and "useless" edges described in the text, but there is an alternate formulation. Let G be a weighted undirected graph, where the edge weights are distinct. We say that an edge e is ***dangerous*** if it is the longest edge in some cycle in G, and ***useful*** if it does not lie in any cycle in G.

   (a) Prove that the minimum spanning tree of G contains every useful edge.

   (b) Prove that the minimum spanning tree of G does not contain any dangerous edge.

   (c) Describe and analyze an efficient implementation of the following algorithm, first described by Kruskal in the same 1956 paper where he proposed "Kruskal's algorithm". Examine the edges of G in *decreasing* order; if an edge is dangerous, remove it from G. *[Hint: It won't be as fast as Kruskal's usual algorithm.]*

4. (a) Describe and analyze an algorithm to compute the *maximum*-weight spanning tree of a given edge-weighted graph.

   (b) A *feedback edge set* of an undirected graph G is a subset F of the edges such that every cycle in G contains at least one edge in F. In other words, removing every edge in F makes the graph G acyclic. Describe and analyze a fast algorithm to compute the minimum weight feedback edge set of of a given edge-weighted graph.

5. Suppose we are given both an undirected graph G with weighted edges and a minimum spanning tree T of G.

(a) Describe an algorithm to update the minimum spanning tree when the weight of a single edge e is decreased.

(b) Describe an algorithm to update the minimum spanning tree when the weight of a single edge e is increased.

In both cases, the input to your algorithm is the edge e and its new weight; your algorithms should modify T so that it is still a minimum spanning tree. *[Hint: Consider the cases $e \in T$ and $e \notin T$ separately.]*

6. (a) Describe and analyze an algorithm to find the *second smallest spanning tree* of a given graph G, that is, the spanning tree of G with smallest total weight except for the minimum spanning tree.

♥(b) Describe and analyze an efficient algorithm to compute, given a weighted undirected graph G and an integer k, the k spanning trees of G with smallest weight.

7. A graph $G = (V, E)$ is *dense* if $E = \Theta(V^2)$. Describe a modification of Jarník's minimum-spanning tree algorithm that runs in $O(V^2)$ time (independent of E) when the input graph is dense, using only simple data structures (and in particular, *without* using a Fibonacci heap).

8. Minimum-spanning tree algorithms are often formulated using an operation called *edge contraction*. To contract the edge uv, we insert a new node, redirect any edge incident to u or v (except uv) to this new node, and then delete u and v. After contraction, there may be multiple parallel edges between the new node and other nodes in the graph; we remove all but the lightest edge between any two nodes.

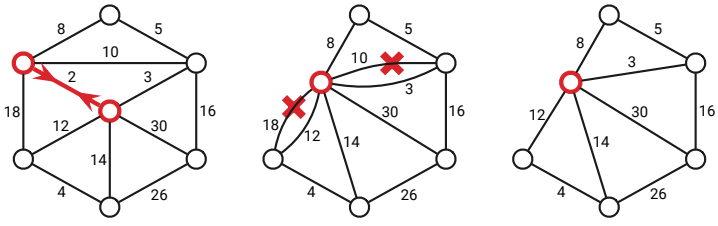


Figure 7.7. Contracting an edge and removing redundant parallel edges.

The three classical minimum-spanning tree algorithms described in this chapter can all be expressed cleanly in terms of contraction as follows. All three algorithms start by making a clean copy G' of the input graph G and then repeatedly contract safe edges in G'; the minimum spanning tree consists of the contracted edges.

- Borůvka: Mark the lightest edge leaving each vertex, contract all marked edges, and recurse.

- JARNÍK: Repeatedly contract the lightest edge incident to some fixed root vertex.
- KRUSKAL: Repeatedly contract the lightest edge in the graph.

(a) Describe an algorithm to execute a single pass of Borůvka's contraction algorithm in $O(V + E)$ time. The input graph is represented in an adjacency list.

(b) Consider an algorithm that first performs k passes of Borůvka's contraction algorithm, and then runs Jarník's algorithm (*with* a Fibonacci heap) on the resulting contracted graph.

i. What is the running time of this hybrid algorithm, as a function of V, E, and k?

ii. For which value of k is this running time minimized? What is the resulting running time?

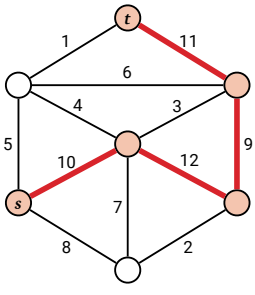
(c) Call a family of graphs *nice* if it has the following properties:

- A nice graph with n vertices has only $O(n)$ edges.
- Contracting an edge of a nice graph yields another nice graph.

For example, planar graphs—graphs that can be drawn in the plane with no crossing edges—are nice. Euler's formula implies that any planar graph with V vertices has at most $3V - 6$ edges, and contracting any edge of a planar graph leaves a smaller planar graph.

Prove that Borůvka's contraction algorithm computes the minimum spanning tree of any nice graph in $O(V)$ time.

9. Consider a path between two vertices s and t in a undirected weighted graph G. The *width* of this path is the minimum weight of any edge in the path. The *bottleneck distance* between s and t is the width of the widest path from s to t. (If there are no paths from s to t, the bottleneck distance is $-\infty$; on the other hand, the bottleneck distance from s to itself is ∞.)



The bottleneck distance between s and t is 9.

(a) Prove that the *maximum* spanning tree of G contains widest paths between *every* pair of vertices.

(b) Describe an algorithm to solve the following problem in $O(V + E)$ time: Given a undirected weighted graph G, two vertices s and t, and a weight W, is the bottleneck distance between s and t at most W?

(c) Suppose B is the bottleneck distance between s and t.

i. Prove that deleting any edge with weight less than B does not change the bottleneck distance between s and t.

ii. Prove that *contracting* any edge with weight *greater* than B does not change the bottleneck distance between s and t. (If contraction creates parallel edges, delete all but the *heaviest* edge between each pair of nodes.)

(d) Describe an algorithm to compute a minimum-bottleneck path between s and t in $O(V + E)$ time. *[Hint: Start by finding the median-weight edge in G.]*

♣10. (This problem assumes familiarity with standard implementations of disjoint-set data structures.)

Consider the following variant of Borůvka's algorithm. Instead of counting and labeling components of F to find safe edges, we use a standard disjoint set data structure. Each component of F is represented by an up-tree; each vertex v stores a pointer $parent(v)$ to its parent in the up-tree containing v. Each leader vertex $\bar{v}$ also maintains an edge $safe(\bar{v})$, which is (eventually) the lightest edge with one endpoint in $\bar{v}$'s component of F.

```
BORŮVKA(V, E):
  F = ∅
  for each vertex v ∈ V
      parent(v) ← v
  while FINDSAFEEDGES(V, E)
      ADDSAFEEDGES(V, E, F)
  return F
```

```
FINDSAFEEDGES(V, E):
  for each vertex v ∈ V
      safe(v) ← NULL
  found ← FALSE
  for each edge uv ∈ E
      ū ← FIND(u); v̄ ← FIND(v)
      if ū ≠ v̄
          if w(uv) < w(safe(ū))
              safe(ū) ← uv
          if w(uv) < w(safe(v̄))
              safe(v̄) ← uv
          found ← TRUE
  return found
```

```
ADDSAFEEDGES(V, E, F):
  for each vertex v ∈ V
      if safe(v) ≠ NULL
          xy ← safe(v)
          if FIND(x) ≠ FIND(y)
              UNION(x, y)
              add xy to F
```

Prove that if FIND uses path compression, then each call to FINDSAFEEDGES and ADDSAFEEDGES requires only $O(V + E)$ time. *[Hint: It doesn't matter how UNION is implemented! What is the depth of the up-trees when FINDSAFEEDGES ends?]*

I study my Bible as I gather apples. First I shake the whole tree, that the ripest might fall. Then I climb the tree and shake each limb, and then each branch and then each twig, and then I look under each leaf.

— attributed to Martin Luther (c. 1500)

Life is an unfoldment, and the further we travel the more truth we can comprehend. To understand the things that are at our door is the best preparation for understanding those that lie beyond.

— attributed to Hypatia of Alexandria (c. 400) by Elbert Hubbard in *Little Journeys to the Homes of Great Teachers* (1908)

Your mind will answer most questions if you learn to relax and wait for the answer. Like one of those thinking machines, you feed in your question, sit back, and wait …

— William S. Burroughs. *Naked Lunch* (1959)

The methods given in this paper require no foresight or ingenuity, and hence deserve to be called algorithms.

— Edward R. Moore, "The Shortest Path Through a Maze" (1959)

8

Shortest Paths

Suppose we are given a weighted *directed* graph $G = (V, E, w)$ with two special vertices, and we want to find the shortest path from a *source* vertex s to a *target* vertex t. That is, we want to find the directed path P starting at s and ending at t that minimizes the function

$$w(P) := \sum_{u \to v \in P} w(u \to v).$$

For example, if I want to answer the question "What's the fastest way to drive from my old apartment in Champaign, Illinois to my wife's old apartment in Columbus, Ohio?", I might use a graph whose vertices are cities, edges are roads, weights are driving times, s is Champaign, and t is Columbus.[1] The graph is directed, because driving times along the same road might be different

[1]West on Church, north on Prospect, east on I-74, south on I-465, east on Airport Expressway, north on I-65, east on I-70, north on Grandview, east on 5th, north on Olentangy River, east on Dodridge, north on High, west on Kelso, south on Neil. Depending on traffic. We live in Urbana now.

in different directions. (At one time, there was a speed trap on I-70 just east of the Indiana/Ohio border, but only for eastbound traffic.)

8.1 Shortest Path Trees

Almost every algorithm known for computing shortest paths from one vertex to another actually solves (large portions of) the following more general ***single source shortest path*** or ***SSSP*** problem: Find shortest paths from the source vertex s to *every* other vertex in the graph. This problem is usually solved by finding a ***shortest path tree*** rooted at s that contains all the desired shortest paths.

It's not hard to see that if shortest paths are unique, then they form a tree, because any subpath of a shortest path is itself a shortest path. If there are multiple shortest paths to some vertices, we can always choose one shortest path to each vertex so that the union of the paths is a tree. If there are shortest paths to two vertices u and v that diverge, then meet, then diverge again, we can modify one of the paths without changing its length so that the two paths only diverge once.

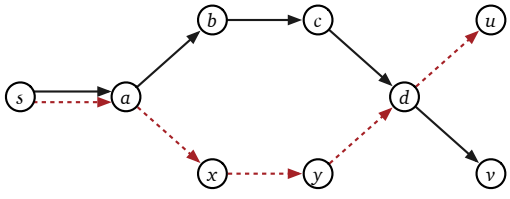


Figure 8.1. If $s{\to}a{\to}b{\to}c{\to}d{\to}v$ (solid) and $s{\to}a{\to}x{\to}y{\to}d{\to}u$ (dashed) are shortest paths, then $s{\to}a{\to}b{\to}c{\to}d{\to}u$ (along the top) is also a shortest path.

Although they are both optimal spanning trees, shortest-path trees and minimum spanning trees are very different creatures. Shortest-path trees are rooted and directed; minimum spanning trees are unrooted and undirected. Shortest-path trees are most naturally defined for directed graphs; minimum spanning trees are more naturally defined for undirected graphs. If edge weights are distinct, there is only one minimum spanning tree, but every source vertex induces a different shortest-path tree; moreover, it is possible for *every* shortest path tree to use a different set of edges from the minimum spanning tree.

♥8.2 Negative Edges

For most shortest-path problems, where the edge weights correspond to distance or length or time, it is natural to assume that all edge weights are non-negative, or even positive. However, for many applications of shortest-path algorithms, it is natural to consider negative edges. For example, the weight of an edge might

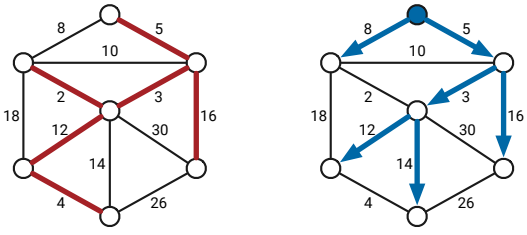


Figure 8.2. A minimum spanning tree and a shortest path tree of the same undirected graph.

represent the *cost* of moving from one vertex to another, so negative-weight edges represent transitions with negative cost, or equivalently, transitions that earn a profit.

Negative edges are a thorn in the side of most shortest-path problems, because the presence of a negative *cycle* might imply that there is no shortest path. To be precise, a shortest path from s to t exists if and only if there is at least one path from s to t, but there is no path from s to t that touches a negative cycle. For *any* path from s to t that touches a negative cycle, there is a shorter path from s to t that goes around the cycle one more time.[2] Thus, if at least one path from s to t touches a negative cycle, there is no shortest path from s to t.

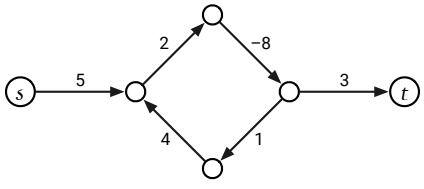


Figure 8.3. There is no shortest walk from s to t.

In part because we need to consider negative wedge weights, this chapter explicitly considers *only* directed graphs. All of the algorithms described here also work for undirected graphs with essentially trivial modifications, *if and only if* negative edges are prohibited. Correctly handling negative edges in undirected graphs is considerably more subtle. We cannot simply replace every undirected edge with a pair of directed edges, because this would transform any negative edge into a short negative cycle. Subpaths of an *undirected* shortest path that contains a negative edge are *not* necessarily shortest paths; consequently, the set of all undirected shortest paths from a single source vertex may not define a tree, even if shortest paths are unique.

[2]Technically, we should be discussing shortest *walks* here, rather than shortest *paths*, but the abuse of terminology is standard. If s can reach t, there must be a shortest simple path from s to t; it's just NP-hard to compute (when there are negative cycles). On the other hand, if there is a shortest *walk* from s to t, that walk must be a simple path, and therefore must be the shortest simple path from s to t. Blerg.

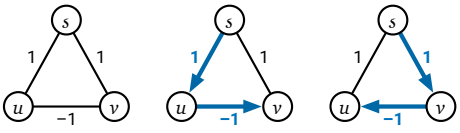


Figure 8.4. An undirected graph where shortest paths from s are unique but do not define a tree.

A complete treatment of undirected graphs with negative edges is beyond the scope of this book. I will only mention, for people who want to follow up via Google, that a *single* shortest path in an undirected graph with negative edges can be computed in $O(VE + V^2 \log V)$ time, by a reduction to maximum weighted matching.

8.3 The Only SSSP Algorithm

Just like graph traversal and minimum spanning trees, many different SSSP algorithms can be described as special cases of a single generic algorithm, first proposed by Lester Ford in 1956 and independently described by George Dantzig in 1957 and again by George Minty in 1958.[3] Each vertex v in the graph stores two values, which (inductively) describe a *tentative* shortest path from s to v.

- $dist(v)$ is the length of the tentative shortest $s \rightsquigarrow v$ path, or ∞ if there is no such path.
- $pred(v)$ is the predecessor of v in the tentative shortest $s \rightsquigarrow v$ path, or NULL if there is no such vertex.

The predecessor pointers automatically define a tentative shortest-path *tree* rooted at s; these pointers are exactly the same as the parent pointers in our generic graph traversal algorithm. At the beginning of the algorithm, we initialize the distances and predecessors as follows:

```
INITSSSP(s):
  dist(s) ← 0
  pred(s) ← NULL
  for all vertices v ≠ s
      dist(v) ← ∞
      pred(v) ← NULL
```

During the execution of the algorithm, an edge $u \to v$ is ***tense*** if $dist(u) + w(u \to v) < dist(v)$. If $u \to v$ is tense, the tentative shortest path $s \rightsquigarrow v$ is clearly incorrect, because the path $s \rightsquigarrow u \to v$ is shorter. We can correct (or at least improve) this obvious overestimate by ***relaxing*** the edge as follows:

[3]Specifically, Dantzig showed that the shortest path problem can be phrased as a linear programming problem, and then described an interpretation of his simplex method in terms of the original graph. His description is (morally) equivalent to Ford's relaxation strategy.

```
RELAX(u→v):
  dist(v) ← dist(u) + w(u→v)
  pred(v) ← u
```

Now that everything is set up, Ford's generic algorithm has a simple one-line description:

Repeatedly relax tense edges, until there are no more tense edges.

```
FORDSSSP(s):
  INITSSSP(s)
  while there is at least one tense edge
      RELAX any tense edge
```

If FORDSSSP eventually terminates (because there are no more tense edges), then the predecessor pointers correctly define a shortest-path tree, and each value $dist(v)$ is the actual shortest-path distance from s to v. In particular, if s cannot reach v, then $dist(v) = \infty$, and if any negative cycle is reachable from s, then the algorithm never terminates.

The correctness of Ford's generic algorithm follows from the following series of simpler claims:

1. At any moment during the execution of the algorithm, for every vertex v, the distance $dist(v)$ is either ∞ or the length of a walk from s to v. This claim can be proved by induction on the number of relaxations.
2. If the graph has no negative cycles, then $dist(v)$ is either ∞ or the length of some *simple path* from s to v. Specifically, if $dist(v)$ is the length of a walk from s to v that contains a directed cycle, that cycle must have negative length. This claim implies that if G has no negative cycles, the relaxation algorithm eventually halts, because there are only a finite number of simple paths in G.
3. If no edge in G is tense, then for every vertex v, the distance $dist(v)$ is the length of the predecessor path $s \to \cdots pred(pred(v)) \to pred(v) \to v$. Specifically, if v violates this condition but its predecessor $pred(v)$ does not, the edge $pred(v) \to v$ is tense.
4. If no edge in G is tense, then for every vertex v, the path of predecessor edges $s \to \cdots \to pred(pred(v)) \to pred(v) \to v$ is in fact a shortest path from s to v. Specifically, if v violates this condition but its predecessor u *in some shortest path* does not, the edge $u \to v$ is tense. This claim also implies that if G has a negative cycle, then some edge is *always* tense, so the generic algorithm never halts.

So far I haven't said anything about how to find tense edges, or which tense edge to relax if there is more than one. Just as with graph traversal, there are

several different instantiations of Ford's generic relaxation algorithm. Unlike graph traversal, however, the efficiency and correctness of each search strategy depends on the structure of the input graph.

In the rest of this chapter, we discuss the four most common instantiations of Ford's algorithm, each of which is the best choice for a different class of input graphs. I'll leave the remaining details of the generic correctness proof as exercises, and instead give (more informative) correctness proofs for each of these four specific algorithms.

8.4 Unweighted Graphs: Breadth-First Search

In the simplest special case of the shortest path problem, all edges have weight 1, and the length of a path is just the number of edges. This special case can be solved by a species of our generic graph-traversal algorithm called ***breadth-first search***. Breadth-first search is often attributed to Edward Moore, who described it in 1957 (as "Algorithm A") as the first published method to find the shortest path through a maze.[4] Especially in the context of VLSI wiring and robot path planning, breadth-first search is sometimes attributed to Chin Yang Lee, who described several applications of Moore's "Algorithm A" (with proper credit to Moore) in 1961. However, in 1945, more than a decade before Moore considered mazes, Konrad Zuse described an implementation of breadth-first search, as a method to count and label the components of a disconnected graph.[6]

[4]Moore was motivated by a weakness in Claude Shannon's maze-solving robot "Theseus", which Shannon designed and constructed in 1950. (Theseus used a memoized version of depth-first search, implemented using electromechanical relays; this was almost certainly the first *implementation* of depth-first search in graphs.) According to Moore, "When this machine was used with a maze which had more than one solution, a visitor asked why it had not been built to always find the shortest path. Shannon and I each attempted to find economical methods of doing this by machine. He found several methods suitable for analog computation,[5] and I obtained these algorithms."

[5]Analog methods for computing shortest paths through mazes have been proposed using ball bearings, fluid/plasma flow, chemical reaction waves, chemotaxis, resistor networks, electric circuits with LEDs, memristor networks, glow discharge in microfluidic chips, growing plants, slime mold, amoebas, ants, bees, nematodes, and tourists.

[6]Konrad Zuse was one of the early pioneers of computing; he designed and built his first programmable computer (later dubbed the Z1) in the late 1930s from metal strips and rods in his parents' living room; the Z1 and its original blueprints were destroyed by a British air raid in 1944. Zuse's 1945 PhD thesis describes the very first high-level programming language, called *Plankalkül*. The first complete example of a *Plankalkül* program in Zuse's thesis is an implementation of breadth-first search to count components, along with a pseudocode explanation and an illustrated step-by-step trace of the algorithm's execution on a disconnected graph with eight vertices. Due to the collapse of the Nazi government, Zuse was unable to submit his PhD thesis, and *Plankalkül* remained unpublished until 1972. The first *Plankalkül* compiler was finally implemented in 1975 by Joachim Hohmann.