Chapter 4

Forward Kinematics

The **forward kinematics** of a robot refers to the calculation of the position and orientation of its end-effector frame from its joint coordinates θ. Figure 4.1 illustrates the forward kinematics problem for a 3R planar open chain. The link lengths are L_1, L_2, and L_3. Choose a fixed frame $\{0\}$ with origin located at the base joint as shown, and assume an end-effector frame $\{4\}$ has been attached to the tip of the third link. The Cartesian position (x, y) and orientation ϕ of the end-effector frame as functions of the joint angles $(\theta_1, \theta_2, \theta_3)$ are then given by

$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3), \tag{4.1}$$

$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3), \tag{4.2}$$

$$\phi = \theta_1 + \theta_2 + \theta_3. \tag{4.3}$$

If one is only interested in the (x, y) position of the end-effector, the robot's task space is then taken to be the x–y-plane, and the forward kinematics would consist of Equations (4.1) and (4.2) only. If the end-effector's position and orientation both matter, the forward kinematics would consist of the three equations (4.1)–(4.3).

While the above analysis can be done using only basic trigonometry, it is not difficult to imagine that for more general spatial chains the analysis can become considerably more complicated. A more systematic method of deriving the forward kinematics might involve attaching reference frames to each link; in Figure 4.1 the three link reference frames are respectively labeled $\{1\}$, $\{2\}$, and $\{3\}$. The forward kinematics can then be written as a product of four homogeneous transformation matrices:

$$T_{04} = T_{01} T_{12} T_{23} T_{34}, \tag{4.4}$$

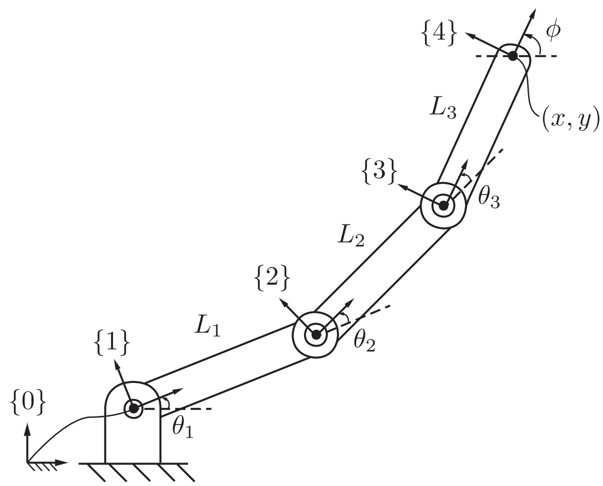

Figure 4.1: Forward kinematics of a 3R planar open chain. For each frame, the $\hat{x}$- and $\hat{y}$-axis is shown; the $\hat{z}$-axes are parallel and out of the page.

where

$$T_{01} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad T_{12} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & L_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$T_{23} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & L_2 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad T_{34} = \begin{bmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.5}$$

Observe that T_{34} is constant and that each remaining $T_{i-1,i}$ depends only on the joint variable θ_i.

As an alternative to this approach, let us define M to be the position and orientation of frame {4} when all joint angles are set to zero (the "home" or "zero" position of the robot). Then

$$M = \begin{bmatrix} 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.6}$$

Now consider each revolute joint axis to be a zero-pitch screw axis. If θ_1 and θ_2 are held at their zero position then the screw axis corresponding to rotating about joint 3 can be expressed in the $\{0\}$ frame as

$$\mathcal{S}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -(L_1 + L_2) \\ 0 \end{bmatrix}.$$

You should be able to confirm this by simple visual inspection of Figure 4.1. When the arm is stretched out straight to the right at its zero configuration, imagine a turntable rotating with an angular velocity of $\omega_3 = 1$ rad/s about the axis of joint 3. The linear velocity v_3 of the point on the turntable at the origin of $\{0\}$ is in the $-\hat{\mathrm{y}}_0$-direction at a rate of $L_1 + L_2$ units/s. Algebraically, $v_3 = -\omega_3 \times q_3$, where q_3 is any point on the axis of joint 3 expressed in $\{0\}$, e.g., $q_3 = (L_1 + L_2, 0, 0)$.

The screw axis $\mathcal{S}_3$ can be expressed in $se(3)$ matrix form as

$$[\mathcal{S}_3] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -(L_1 + L_2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, for any θ_3, the matrix exponential representation for screw motions from the previous chapter allows us to write

$$T_{04} = e^{[\mathcal{S}_3]\theta_3} M \qquad (\text{for } \theta_1 = \theta_2 = 0). \tag{4.7}$$

Now, for $\theta_1 = 0$ and any fixed (but arbitrary) θ_3, rotation about joint 2 can be viewed as applying a screw motion to the rigid (link 2)/(link 3) pair, i.e.,

$$T_{04} = e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M \qquad (\text{for } \theta_1 = 0), \tag{4.8}$$

where $[\mathcal{S}_3]$ and M are as defined previously, and

$$[\mathcal{S}_2] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.9}$$

Finally, keeping θ_2 and θ_3 fixed, rotation about joint 1 can be viewed as applying a screw motion to the entire rigid three-link assembly. We can therefore write, for arbitrary values of $(\theta_1, \theta_2, \theta_3)$,

$$T_{04} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M, \tag{4.10}$$

where

$$[\mathcal{S}_1] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.11}$$

Thus the forward kinematics can be expressed as a product of matrix exponentials, each corresponding to a screw motion. Note that this latter derivation of the forward kinematics does not use any link reference frames; only $\{0\}$ and M must be defined.

In this chapter we consider the forward kinematics of general open chains. One widely used representation for the forward kinematics of open chains relies on the **Denavit–Hartenberg parameters** (D–H parameters), and this representation uses Equation (4.4). Another representation relies on the **product of exponentials** (PoE) formula, which corresponds to Equation (4.10). The advantage of the D–H representation is that it requires the minimum number of parameters to describe the robot's kinematics: for an n-joint robot, it uses $3n$ numbers to describe the robot's structure and n numbers to describe the joint values. The PoE representation is not minimal (it requires $6n$ numbers to describe the n screw axes, in addition to the n joint values), but it has advantages over the D–H representation (e.g., no link frames are necessary) and it is our preferred choice of forward kinematics representation. The D–H representation, and its relationship to the PoE representation, is described in Appendix C.

4.1 Product of Exponentials Formula

To use the PoE formula, it is only necessary to assign a stationary frame {s} (e.g., at the fixed base of the robot or anywhere else that is convenient for defining a reference frame) and a frame {b} at the end-effector, described by M when the robot is at its zero position. It is common to define a frame at each link, though, typically at the joint axis; these are needed for the D–H representation and they are useful for displaying a graphic rendering of a geometric model of the robot and for defining the mass properties of the link, which we will need starting in Chapter 8. Thus when we are defining the kinematics of an n-joint robot, we may either (1) minimally use the frames {s} and {b} if we are only

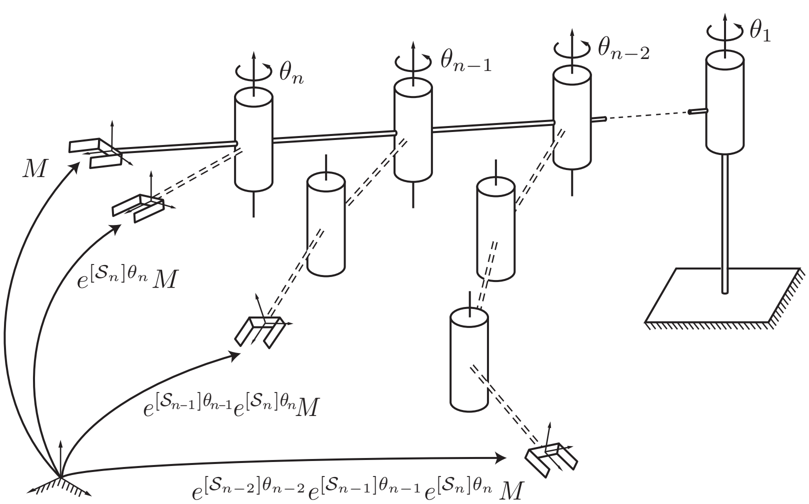

Figure 4.2: Illustration of the PoE formula for an n-link spatial open chain.

interested in the kinematics, or (2) refer to {s} as frame {0}, use frames $\{i\}$ for $i = 1, \ldots, n$ (the frames for links i at joints i), and use one more frame $\{n+1\}$ (corresponding to {b}) at the end-effector. The frame $\{n + 1\}$ (i.e., {b}) is fixed relative to $\{n\}$, but it is at a more convenient location to represent the configuration of the end-effector. In some cases we dispense with frame $\{n+1\}$ and simply refer to $\{n\}$ as the end-effector frame {b}.

4.1.1 First Formulation: Screw Axes in the Base Frame

The key concept behind the PoE formula is to regard each joint as applying a screw motion to all the outward links. To illustrate this consider a general spatial open chain like the one shown in Figure 4.2, consisting of n one-dof joints that are connected serially. To apply the PoE formula, you must choose a fixed base frame {s} and an end-effector frame {b} attached to the last link. Place the robot in its zero position by setting all joint values to zero, with the direction of positive displacement (rotation for revolute joints, translation for prismatic joints) for each joint specified. Let $M \in SE(3)$ denote the configuration of the end-effector frame relative to the fixed base frame when the robot is in its zero position.

Now suppose that joint n is displaced to some joint value θ_n. The end-effector frame M then undergoes a displacement of the form

$$T = e^{[\mathcal{S}_n]\theta_n} M, \tag{4.12}$$

where $T \in SE(3)$ is the new configuration of the end-effector frame and $\mathcal{S}_n = (\omega_n, v_n)$ is the screw axis of joint n as expressed in the fixed base frame. If joint n is revolute (corresponding to a screw motion of zero pitch) then $\omega_n \in \mathbb{R}^3$ is a unit vector in the positive direction of joint axis n; $v_n = -\omega_n \times q_n$, with q_n any arbitrary point on joint axis n as written in coordinates in the fixed base frame; and θ_n is the joint angle. If joint n is prismatic then $\omega_n = 0$, $v_n \in \mathbb{R}^3$ is a unit vector in the direction of positive translation, and θ_n represents the prismatic extension/retraction.

If we assume that joint $n-1$ is also allowed to vary then this has the effect of applying a screw motion to link $n-1$ (and by extension to link n, since link n is connected to link $n-1$ via joint n). The end-effector frame thus undergoes a displacement of the form

$$T = e^{[\mathcal{S}_{n-1}]\theta_{n-1}} \left(e^{[\mathcal{S}_n]\theta_n} M \right). \tag{4.13}$$

Continuing with this reasoning and now allowing all the joints $(\theta_1, \ldots, \theta_n)$ to vary, it follows that

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} e^{[\mathcal{S}_n]\theta_n} M. \tag{4.14}$$

This is the product of exponentials formula describing the forward kinematics of an n-dof open chain. Specifically, we call Equation (4.14) the **space form** of the product of exponentials formula, referring to the fact that the screw axes are expressed in the fixed space frame.

To summarize, to calculate the forward kinematics of an open chain using the space form of the PoE formula (4.14), we need the following elements:

(a) the end-effector configuration $M \in SE(3)$ when the robot is at its home position;

(b) the screw axes $\mathcal{S}_1, \ldots, \mathcal{S}_n$ expressed in the fixed base frame, corresponding to the joint motions when the robot is at its home position;

(c) the joint variables $\theta_1, \ldots, \theta_n$.

Unlike the D–H representation, no link reference frames need to be defined. Further advantages will come to light when we examine the velocity kinematics in the next chapter.

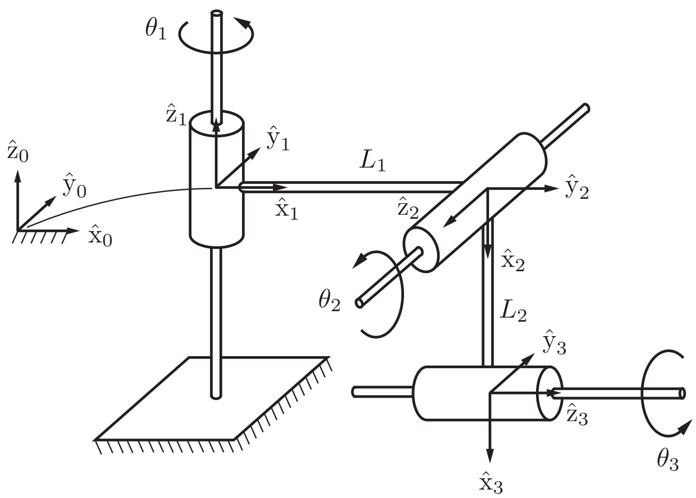

Figure 4.3: A 3R spatial open chain.

4.1.2 Examples

We now derive the forward kinematics for some common spatial open chains using the PoE formula.

Example 4.1 (3R spatial open chain)**.** Consider the 3R open chain of Figure 4.3, shown in its home position (all joint variables set equal to zero). Choose the fixed frame $\{0\}$ and end-effector frame $\{3\}$ as indicated in the figure, and express all vectors and homogeneous transformations in terms of the fixed frame. The forward kinematics has the form

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M,$$

where $M \in SE(3)$ is the end-effector frame configuration when the robot is in its zero position. By inspection M can be obtained as

$$M = \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The screw axis $\mathcal{S}_1 = (\omega_1, v_1)$ for joint axis 1 is then given by $\omega_1 = (0, 0, 1)$ and $v_1 = (0, 0, 0)$ (the fixed frame origin (0,0,0) is a convenient choice for the point q_1 lying on joint axis 1). To determine the screw axis $\mathcal{S}_2$ for joint axis 2, observe that joint axis 2 points in the $-\hat{\mathrm{y}}_0$-direction, so that $\omega_2 = (0,-1,0)$. Choose $q_2 = (L_1, 0, 0)$, in which case $v_2 = -\omega_2 \times q_2 = (0, 0, -L_1)$. Finally, to determine the screw axis $\mathcal{S}_3$ for joint axis 3, note that $\omega_3 = (1,0,0)$. Choosing $q_3 = (0,0,-L_2)$, it follows that $v_3 = -\omega_3 \times q_3 = (0, -L_2, 0)$.

In summary, we have the following 4×4 matrix representations for the three joint screw axes $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$:

$$
\begin{aligned}
[\mathcal{S}_1] &= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
[\mathcal{S}_2] &= \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
[\mathcal{S}_3] &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

It will be more convenient to list the screw axes in the following tabular form:

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,-1,0)$	$(0,0,-L_1)$
3	$(1,0,0)$	$(0,L_2,0)$

Example 4.2 (3R planar open chain)**.** For the robot in Figure 4.1, we expressed the end-effector home configuration M (Equation (4.6)) and the screw axes $\mathcal{S}_i$ as follows:

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,0,1)$	$(0,-L_1,0)$
3	$(0,0,1)$	$(0,-(L_1+L_2),0)$

Since the motion is in the $\hat{\mathrm{x}}$–$\hat{\mathrm{y}}$-plane, we could equivalently write each screw axis $\mathcal{S}_i$ as a 3-vector (ω_z, v_x, v_y):

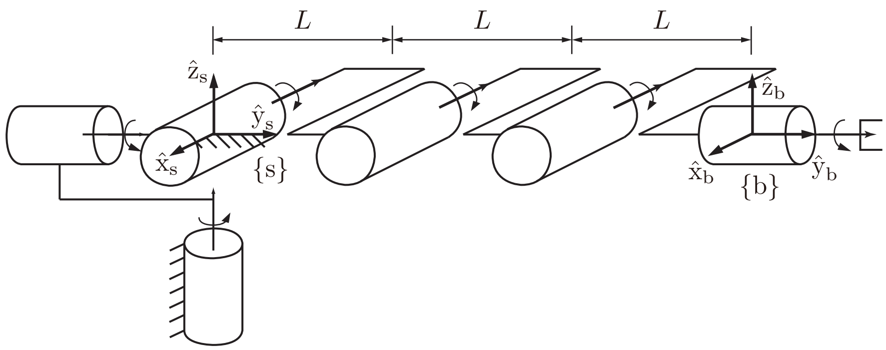

Figure 4.4: PoE forward kinematics for the 6R open chain.

i	ω_i	v_i
1	1	$(0, 0)$
2	1	$(0, -L_1)$
3	1	$(0, -(L_1 + L_2))$

and M as an element of $SE(2)$:

$$M = \begin{bmatrix} 1 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In this case, the forward kinematics would use the simplified matrix exponential for planar motions (Exercise 3.49).

Example 4.3 (6R spatial open chain)**.** We now derive the forward kinematics of the 6R open chain of Figure 4.4. Six-dof arms play an important role in robotics because they have the minimum number of joints that allows the end-effector to move a rigid body in all its degrees of freedom, subject only to limits on the robot's workspace. For this reason, six-dof robot arms are sometimes called general purpose manipulators.

The zero position and the direction of positive rotation for each joint axis are as shown in the figure. A fixed frame {s} and end-effector frame {b} are also assigned as shown. The end-effector frame M in the zero position is then

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.15}$$

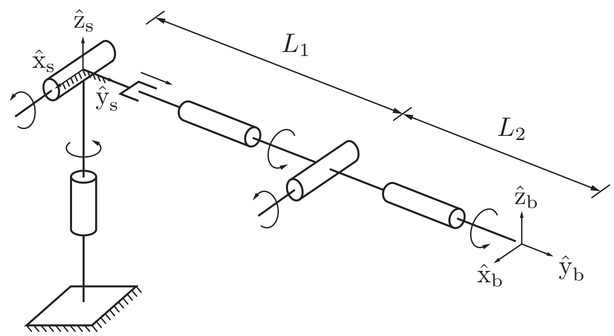

Figure 4.5: The RRPRRR spatial open chain.

The screw axis for joint 1 is in the direction $\omega_1 = (0, 0, 1)$. The most convenient choice for point q_1 lying on joint axis 1 is the origin, so that $v_1 = (0, 0, 0)$. The screw axis for joint 2 is in the ŷ-direction of the fixed frame, so $\omega_2 = (0, 1, 0)$. Choosing $q_2 = (0, 0, 0)$, we have $v_2 = (0, 0, 0)$. The screw axis for joint 3 is in the direction $\omega_3 = (-1, 0, 0)$. Choosing $q_3 = (0, 0, 0)$ leads to $v_3 = (0, 0, 0)$. The screw axis for joint 4 is in the direction $\omega_4 = (-1, 0, 0)$. Choosing $q_4 = (0, L, 0)$ leads to $v_4 = (0, 0, L)$. The screw axis for joint 5 is in the direction $\omega_5 = (-1, 0, 0)$; choosing $q_5 = (0, 2L, 0)$ leads to $v_5 = (0, 0, 2L)$. The screw axis for joint 6 is in the direction $\omega_6 = (0, 1, 0)$; choosing $q_6 = (0, 0, 0)$ leads to $v_6 = (0, 0, 0)$. In summary, the screw axes $\mathcal{S}_i = (\omega_i, v_i)$, $i = 1, \ldots, 6$, are as follows:

i	ω_i	v_i
1	$(0, 0, 1)$	$(0, 0, 0)$
2	$(0, 1, 0)$	$(0, 0, 0)$
3	$(-1, 0, 0)$	$(0, 0, 0)$
4	$(-1, 0, 0)$	$(0, 0, L)$
5	$(-1, 0, 0)$	$(0, 0, 2L)$
6	$(0, 1, 0)$	$(0, 0, 0)$

Example 4.4 (An RRPRRR spatial open chain)**.** In this example we consider the six-degree-of-freedom RRPRRR spatial open chain of Figure 4.5. The end-effector frame in the zero position is given by

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The screw axes $\mathcal{S}_i = (\omega_i, v_i)$ are listed in the following table:

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(1,0,0)$	$(0,0,0)$
3	$(0,0,0)$	$(0,1,0)$
4	$(0,1,0)$	$(0,0,0)$
5	$(1,0,0)$	$(0,0,-L_1)$
6	$(0,1,0)$	$(0,0,0)$

Note that the third joint is prismatic, so that $\omega_3 = 0$ and v_3 is a unit vector in the direction of positive translation.

Example 4.5 (Universal Robots' UR5 6R robot arm)**.** Universal Robots' UR5 6R robot arm is shown in Figure 4.6. Each joint is directly driven by a brushless motor combined with 100:1 zero-backlash harmonic drive gearing, which greatly increases the torque available at the joint while reducing its maximum speed. Figure 4.6 shows the screw axes $\mathcal{S}_1, \ldots, \mathcal{S}_6$ when the robot is at its zero position. The end-effector frame {b} in the zero position is given by

$$M = \begin{bmatrix} -1 & 0 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & W_1 + W_2 \\ 0 & 1 & 0 & H_1 - H_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The screw axes $\mathcal{S}_i = (\omega_i, v_i)$ are listed in the following table:

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,1,0)$	$(-H_1,0,0)$
3	$(0,1,0)$	$(-H_1,0,L_1)$
4	$(0,1,0)$	$(-H_1,0,L_1+L_2)$
5	$(0,0,-1)$	$(-W_1,L_1+L_2,0)$
6	$(0,1,0)$	$(H_2-H_1,0,L_1+L_2)$

As an example of the forward kinematics, set $\theta_2 = -\pi/2$ and $\theta_5 = \pi/2$, with all other joint angles equal to zero. Then the configuration of the end-effector is

$$\begin{aligned} T(\theta) &= e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} e^{[\mathcal{S}_4]\theta_4} e^{[\mathcal{S}_5]\theta_5} e^{[\mathcal{S}_6]\theta_6} M \\ &= I e^{-[\mathcal{S}_2]\pi/2} I^2 e^{[\mathcal{S}_5]\pi/2} I M \\ &= e^{-[\mathcal{S}_2]\pi/2} e^{[\mathcal{S}_5]\pi/2} M \end{aligned}$$

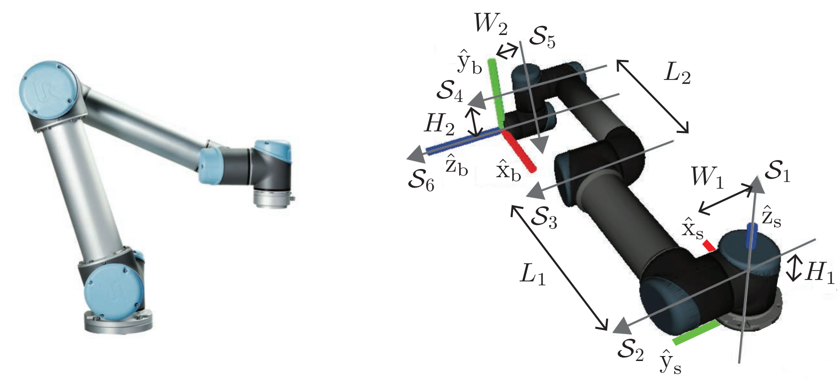

Figure 4.6: (Left) Universal Robots' UR5 6R robot arm. (Right) Shown at its zero position. Positive rotations about the axes indicated are given by the usual right-hand rule. W_1 is the distance along the $\hat{\mathrm{y}}_\mathrm{s}$-direction between the anti-parallel axes of joints 1 and 5. $W_1 = 109$ mm, $W_2 = 82$ mm, $L_1 = 425$ mm, $L_2 = 392$ mm, $H_1 = 89$ mm, $H_2 = 95$ mm.

since $e^0 = I$. Evaluating, we get

$$e^{-[\mathcal{S}_2]\pi/2} = \begin{bmatrix} 0 & 0 & -1 & 0.089 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0.089 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad e^{[\mathcal{S}_5]\pi/2} = \begin{bmatrix} 0 & 1 & 0 & 0.708 \\ -1 & 0 & 0 & 0.926 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where the linear units are meters, and

$$T(\theta) = e^{-[\mathcal{S}_2]\pi/2} e^{[\mathcal{S}_5]\pi/2} M = \begin{bmatrix} 0 & -1 & 0 & 0.095 \\ 1 & 0 & 0 & 0.109 \\ 0 & 0 & 1 & 0.988 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

as shown in Figure 4.7.

4.1.3 Second Formulation: Screw Axes in the End-Effector Frame

The matrix identity $e^{M^{-1}PM} = M^{-1}e^P M$ (Proposition 3.10) can also be expressed as $Me^{M^{-1}PM} = e^P M$. Beginning with the rightmost term of the previously derived product of exponentials formula, if we repeatedly apply this

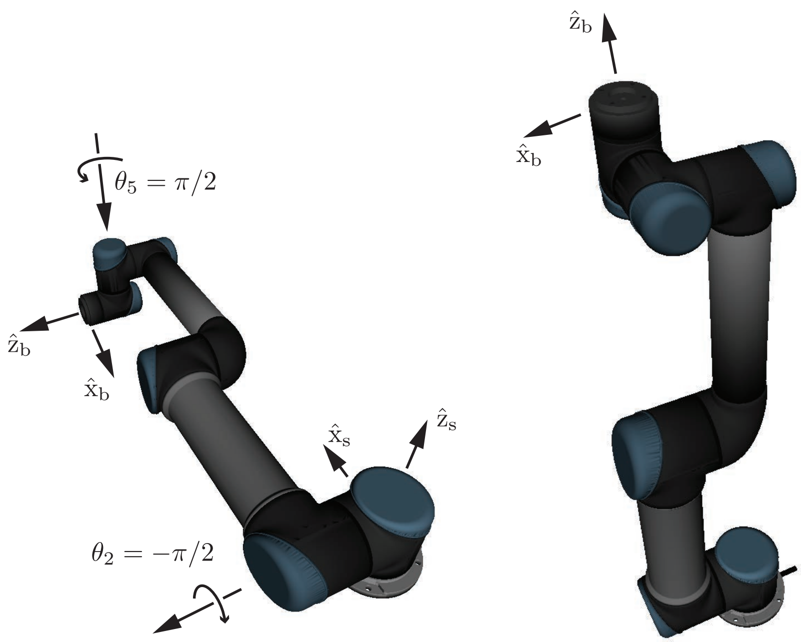

Figure 4.7: (Left) The UR5 at its home position, with the axes of joints 2 and 5 indicated. (Right) The UR5 at joint angles $\theta = (\theta_1, \ldots, \theta_6) = (0, -\pi/2, 0, 0, \pi/2, 0)$.

identity then after n iterations we obtain

$$\begin{aligned} T(\theta) &= e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_n]\theta_n} M \\ &= e^{[\mathcal{S}_1]\theta_1} \cdots M e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\ &= e^{[\mathcal{S}_1]\theta_1} \cdots M e^{M^{-1}[\mathcal{S}_{n-1}]M\theta_{n-1}} e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\ &= M e^{M^{-1}[\mathcal{S}_1]M\theta_1} \cdots e^{M^{-1}[\mathcal{S}_{n-1}]M\theta_{n-1}} e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\ &= M e^{[\mathcal{B}_1]\theta_1} \cdots e^{[\mathcal{B}_{n-1}]\theta_{n-1}} e^{[\mathcal{B}_n]\theta_n}, \end{aligned} \tag{4.16}$$

where each $[\mathcal{B}_i]$ is given by $M^{-1}[\mathcal{S}_i]M$, i.e., $\mathcal{B}_i = [\mathrm{Ad}_{M^{-1}}]\mathcal{S}_i$, $i = 1, \ldots, n$. Equation (4.16) is an alternative form of the product of exponentials formula, representing the joint axes as screw axes $\mathcal{B}_i$ in the end-effector (body) frame when the robot is at its zero position. We call Equation (4.16) the **body form** of the product of exponentials formula.

It is worth thinking about the order of the transformations expressed in the space-form PoE formula (Equation (4.14)) and in the body-form formula

(Equation (4.16)). In the space form, M is first transformed by the most distal joint, progressively moving inward to more proximal joints. Note that the fixed space-frame representation of the screw axis for a more proximal joint is not affected by the joint displacement at a distal joint (e.g., joint 3's displacement does not affect joint 2's screw axis representation in the space frame). In the body form, M is first transformed by the first joint, progressively moving outward to more distal joints. The body-frame representation of the screw axis for a more distal joint is not affected by the joint displacement at a proximal joint (e.g., joint 2's displacement does not affect joint 3's screw axis representation in the body frame.) Therefore, it makes sense that we need to determine the screw axes only at the robot's zero position: any $\mathcal{S}_i$ is unaffected by more distal transformations, and any $\mathcal{B}_i$ is unaffected by more proximal transformations.

Example 4.6 (6R spatial open chain)**.** We now express the forward kinematics of the 6R open chain of Figure 4.4 in the second form,

$$T(\theta) = Me^{[\mathcal{B}_1]\theta_1} e^{[\mathcal{B}_2]\theta_2} \cdots e^{[\mathcal{B}_6]\theta_6}.$$

Assume the same fixed and end-effector frames and zero position as found previously; M is still the same as in Equation (4.15), obtained as the end-effector frame as seen from the fixed frame with the chain in its zero position. The screw axis for each joint axis, expressed with respect to the end-effector frame in its zero position, is given in the following table:

i	ω_i	v_i
1	$(0,0,1)$	$(-3L,0,0)$
2	$(0,1,0)$	$(0,0,0)$
3	$(-1,0,0)$	$(0,0,-3L)$
4	$(-1,0,0)$	$(0,0,-2L)$
5	$(-1,0,0)$	$(0,0,-L)$
6	$(0,1,0)$	$(0,0,0)$

Example 4.7 (Barrett Technology's WAM 7R robot arm)**.** Barrett Technology's WAM 7R robot arm is shown in Figure 4.8. The extra (seventh) joint means that the robot is **redundant** for the task of positioning its end-effector frame in $SE(3)$; in general, for a given end-effector configuration in the robot's workspace, there is a one-dimensional set of joint variables in the robot's seven-dimensional joint space that achieves that configuration. This extra degree of freedom can be used for obstacle avoidance or to optimize some objective function such as minimizing the motor power needed to hold the end-effector at that configuration.

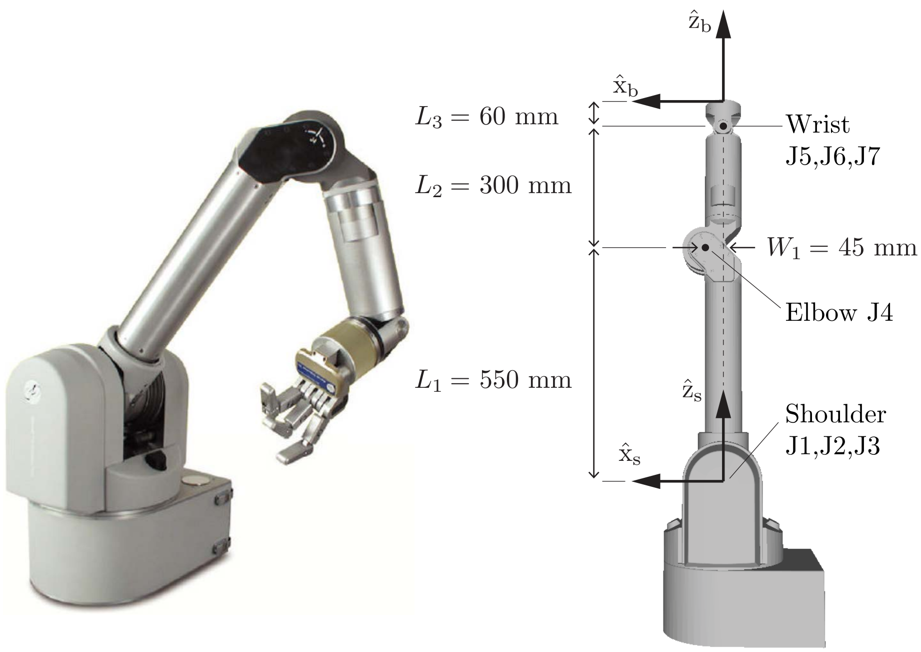

Figure 4.8: Barrett Technology's WAM 7R robot arm at its zero configuration (right). At the zero configuration, axes 1, 3, 5, and 7 are along $\hat{z}_s$ and axes 2, 4, and 6 are aligned with $\hat{y}_s$ out of the page. Positive rotations are given by the right-hand rule. Axes 1, 2, and 3 intersect at the origin of $\{s\}$ and axes 5, 6, and 7 intersect at a point 60mm from $\{b\}$. The zero configuration is singular, as discussed in Section 5.3.

Also, some joints of the WAM are driven by motors placed at the base of the robot, reducing the robot's moving mass. Torques are transferred from the motors to the joints by cables winding around drums at the joints and motors. Because the moving mass is reduced, the motor torque requirements are decreased, allowing low (cable) gear ratios and high speeds. This design is in contrast with that of the UR5, where the motor and harmonic drive gearing for each joint are directly at the joint.

Figure 4.8 illustrates the WAM's end-effector frame screw axes $\mathcal{B}_1, \ldots, \mathcal{B}_7$ when the robot is at its zero position. The end-effector frame $\{b\}$ in the zero position is given by

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_1 + L_2 + L_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The screw axes $\mathcal{B}_i = (\omega_i, v_i)$ are listed in the following table:

i	ω_i	v_i
1	$(0, 0, 1)$	$(0, 0, 0)$
2	$(0, 1, 0)$	$(L_1 + L_2 + L_3, 0, 0)$
3	$(0, 0, 1)$	$(0, 0, 0)$
4	$(0, 1, 0)$	$(L_2 + L_3, 0, W_1)$
5	$(0, 0, 1)$	$(0, 0, 0)$
6	$(0, 1, 0)$	$(L_3, 0, 0)$
7	$(0, 0, 1)$	$(0, 0, 0)$

Figure 4.9 shows the WAM arm with $\theta_2 = 45°$, $\theta_4 = -45°$, $\theta_6 = -90°$ and all other joint angles equal to zero, giving

$$T(\theta) = Me^{[\mathcal{B}_2]\pi/4} e^{-[\mathcal{B}_4]\pi/4} e^{-[\mathcal{B}_6]\pi/2} = \begin{bmatrix} 0 & 0 & -1 & 0.3157 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0.6571 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

4.2 The Universal Robot Description Format

The **Universal Robot Description Format** (URDF) is an XML (eXtensible Markup Language) file format used by the Robot Operating System (ROS) to describe the kinematics, inertial properties, and link geometry of robots. A URDF file describes the joints and links of a robot:

- **Joints.** Joints connect two links: a `parent` link and a `child` link. A few of the possible joint types include prismatic, revolute (including joint limits), continuous (revolute without joint limits), and fixed (a virtual joint that does not permit any motion). Each joint has an `origin` frame that defines the position and orientation of the `child` link frame relative to the `parent` link frame when the joint variable is zero. The `origin` is on the joint's axis. Each joint has an `axis` 3-vector, a unit vector expressed in the `child` link's frame, in the direction of positive rotation for a revolute joint or positive translation for a prismatic joint.

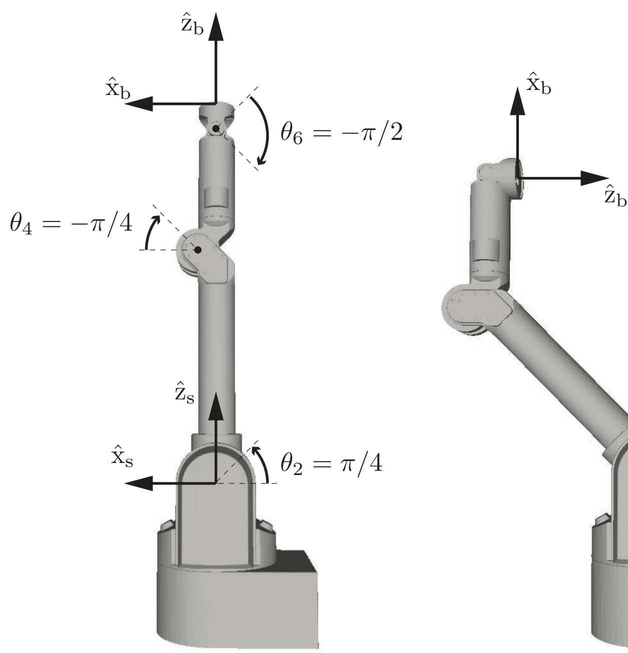

Figure 4.9: (Left) The WAM at its home configuration, with the axes of joints 2, 4, and 6 indicated. (Right) The WAM at $\theta = (\theta_1, \ldots, \theta_7) = (0, \pi/4, 0, -\pi/4, 0, -\pi/2, 0)$.

- **Links.** While the joints fully describe the kinematics of a robot, the links define its mass properties. These start to be needed in Chapter 8, when we begin to study the dynamics of robots. The elements of a link include its `mass`; an `origin` frame that defines the position and orientation of a frame at the link's center of mass relative to the link's joint frame described above; and an `inertia` matrix, relative to the link's center of mass frame, specified by the six elements on or above the diagonal. (As we will see in Chapter 8, the inertia matrix for a rigid body is a 3×3 symmetric positive-definite matrix. Since the inertia matrix is symmetric, it is only necessary to define the terms on and above the diagonal.)

Note that most links have two frames rigidly attached: a first frame at the joint (defined by the joint element that connects the link to its parent) and a second frame at the link's center of mass (defined by the link element).

A URDF file can represent any robot with a tree structure. This includes serial-chain robot arms and robot hands, but not a Stewart platform or other

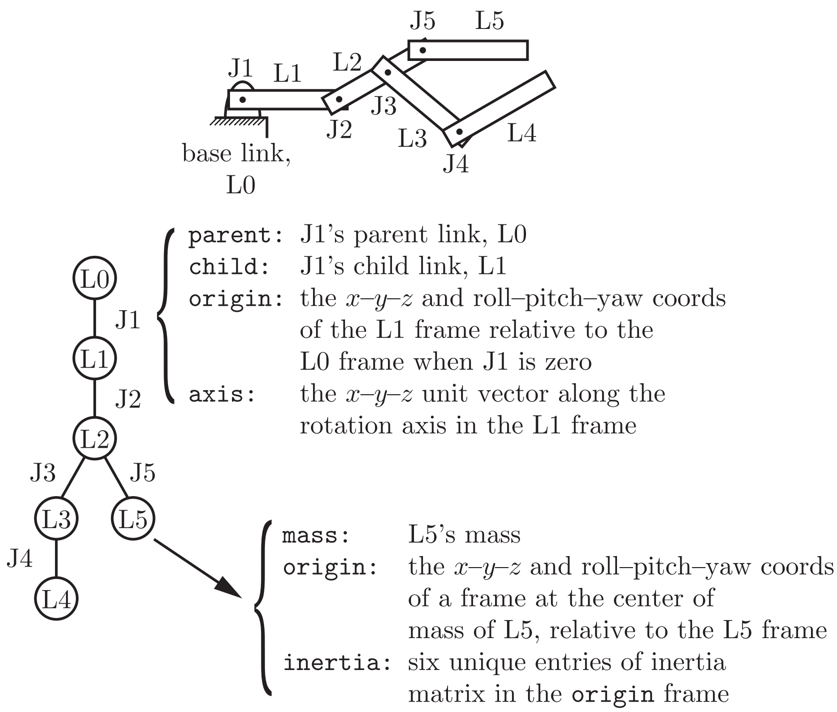

Figure 4.10: A five-link robot represented as a tree, where the nodes of the tree are the links and the edges of the tree are the joints.

mechanisms with closed loops. An example of a robot with a tree structure is shown in Figure 4.10.

The orientation of a frame {b} relative to a frame {a} is represented using roll–pitch–yaw coordinates: first, a roll about the fixed $\hat{x}_a$-axis; then a pitch about the fixed $\hat{y}_a$-axis; then a yaw about the fixed $\hat{z}_a$-axis.

The kinematics and mass properties of the UR5 robot arm (Figure 4.11) are defined in the URDF file below, which demonstrates the syntax of the joint's elements (`parent`, `child`, `origin`, and `axis`) and the link's elements (`mass`, `origin`, and `inertia`). A URDF requires a frame defined at every joint, so we define frames {1} to {6} in addition to the fixed base frame {0} (i.e., {s}) and the end-effector frame {7} (i.e., {b}). Figure 4.11 gives the extra information needed to fully write the URDF.

Although the joint types in the URDF are defined as "continuous," the UR5 joints do in fact have joint limits; they are omitted here for simplicity. The mass

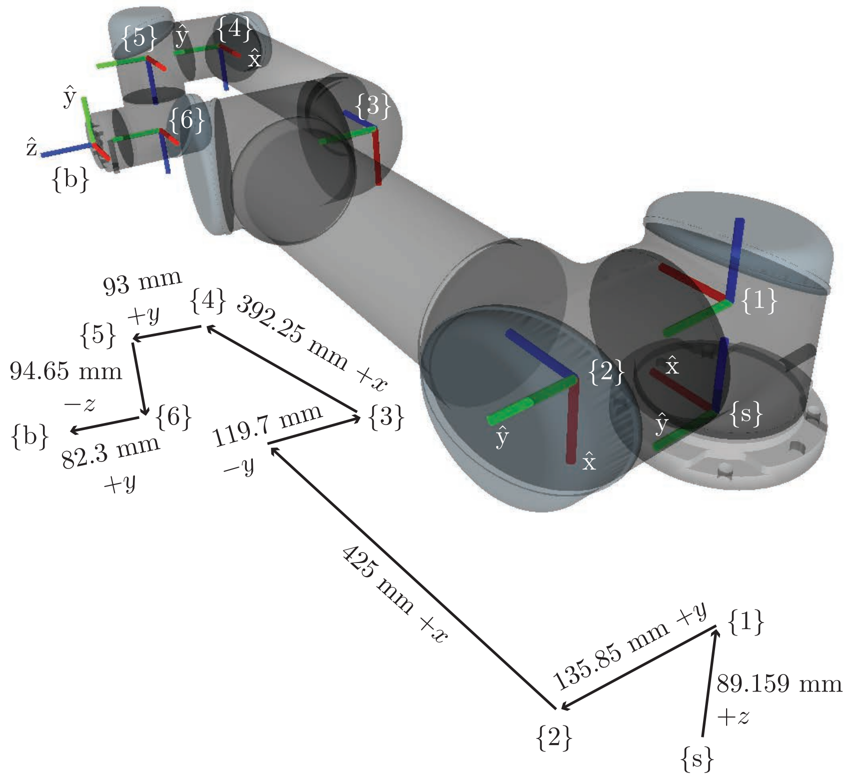

Figure 4.11: The orientations of the frames {s} (also called {0}), {b} (also called {7}), and {1} through {6} are illustrated on the translucent UR5. The frames {s} and {1} are aligned with each other; frames {2} and {3} are aligned with each other; and frames {4}, {5}, and {6} are aligned with each other. Therefore, only the axes of frames {s}, {2}, {4}, and {b} are labeled. Just below the image of the robot is a skeleton indicating how the frames are offset from each other, including distances and directions (expressed in the {s} frame).

and inertial properties listed here are not exact.

The UR5 URDF file (kinematics and inertial properties only).

```
<?xml version="1.0" ?>
<robot name="ur5">

<!-- ********** KINEMATIC PROPERTIES (JOINTS) ********** -->
  <joint name="world_joint" type="fixed">
```

```
    <parent link="world"/>
    <child link="base_link"/>
    <origin rpy="0.0 0.0 0.0" xyz="0.0 0.0 0.0"/>
  </joint>
  <joint name="joint1" type="continuous">
    <parent link="base_link"/>
    <child link="link1"/>
    <origin rpy="0.0 0.0 0.0" xyz="0.0 0.0 0.089159"/>
    <axis xyz="0 0 1"/>
  </joint>
  <joint name="joint2" type="continuous">
    <parent link="link1"/>
    <child link="link2"/>
    <origin rpy="0.0 1.570796325 0.0" xyz="0.0 0.13585 0.0"/>
    <axis xyz="0 1 0"/>
  </joint>
  <joint name="joint3" type="continuous">
    <parent link="link2"/>
    <child link="link3"/>
    <origin rpy="0.0 0.0 0.0" xyz="0.0 -0.1197 0.425"/>
    <axis xyz="0 1 0"/>
  </joint>
  <joint name="joint4" type="continuous">
    <parent link="link3"/>
    <child link="link4"/>
    <origin rpy="0.0 1.570796325 0.0" xyz="0.0 0.0 0.39225"/>
    <axis xyz="0 1 0"/>
  </joint>
  <joint name="joint5" type="continuous">
    <parent link="link4"/>
    <child link="link5"/>
    <origin rpy="0.0 0.0 0.0" xyz="0.0 0.093 0.0"/>
    <axis xyz="0 0 1"/>
  </joint>
  <joint name="joint6" type="continuous">
    <parent link="link5"/>
    <child link="link6"/>
    <origin rpy="0.0 0.0 0.0" xyz="0.0 0.0 0.09465"/>
    <axis xyz="0 1 0"/>
  </joint>
  <joint name="ee_joint" type="fixed">
    <origin rpy="-1.570796325 0 0" xyz="0 0.0823 0"/>
    <parent link="link6"/>
    <child link="ee_link"/>
  </joint>

<!-- ********** INERTIAL PROPERTIES (LINKS) ********** -->
  <link name="world"/>
  <link name="base_link">
    <inertial>
```

```
    <mass value="4.0"/>
    <origin rpy="0 0 0" xyz="0.0 0.0 0.0"/>
    <inertia ixx="0.00443333156" ixy="0.0" ixz="0.0"
             iyy="0.00443333156" iyz="0.0" izz="0.0072"/>
  </inertial>
</link>
<link name="link1">
  <inertial>
    <mass value="3.7"/>
    <origin rpy="0 0 0" xyz="0.0 0.0 0.0"/>
    <inertia ixx="0.010267495893" ixy="0.0" ixz="0.0"
             iyy="0.010267495893" iyz="0.0" izz="0.00666"/>
  </inertial>
</link>
<link name="link2">
  <inertial>
    <mass value="8.393"/>
    <origin rpy="0 0 0" xyz="0.0 0.0 0.28"/>
    <inertia ixx="0.22689067591" ixy="0.0" ixz="0.0"
             iyy="0.22689067591" iyz="0.0" izz="0.0151074"/>
  </inertial>
</link>
<link name="link3">
  <inertial>
    <mass value="2.275"/>
    <origin rpy="0 0 0" xyz="0.0 0.0 0.25"/>
    <inertia ixx="0.049443313556" ixy="0.0" ixz="0.0"
             iyy="0.049443313556" iyz="0.0" izz="0.004095"/>
  </inertial>
</link>
<link name="link4">
  <inertial>
    <mass value="1.219"/>
    <origin rpy="0 0 0" xyz="0.0 0.0 0.0"/>
    <inertia ixx="0.111172755531" ixy="0.0" ixz="0.0"
             iyy="0.111172755531" iyz="0.0" izz="0.21942"/>
  </inertial>
</link>
<link name="link5">
  <inertial>
    <mass value="1.219"/>
    <origin rpy="0 0 0" xyz="0.0 0.0 0.0"/>
    <inertia ixx="0.111172755531" ixy="0.0" ixz="0.0"
             iyy="0.111172755531" iyz="0.0" izz="0.21942"/>
  </inertial>
</link>
<link name="link6">
  <inertial>
    <mass value="0.1879"/>
    <origin rpy="0 0 0" xyz="0.0 0.0 0.0"/>
```

```
      <inertia ixx="0.0171364731454" ixy="0.0" ixz="0.0"
               iyy="0.0171364731454" iyz="0.0" izz="0.033822"/>
    </inertial>
  </link>
  <link name="ee_link"/>
</robot>
```

Beyond the properties described above, a URDF can describe other properties of a robot, such as its visual appearance (including geometric models of the links) as well as simplified representations of link geometries that can be used for collision detection in motion planning algorithms.

4.3 Summary

- Given an open chain with a fixed reference frame {s} and a reference frame {b} attached to some point on its last link – this frame is denoted the end-effector frame – the forward kinematics is the mapping $T(\theta)$ from the joint values θ to the position and orientation of {b} in {s}.

- In the Denavit–Hartenberg representation the forward kinematics of an open chain is described in terms of the relative displacements between reference frames attached to each link. If the link frames are sequentially labeled $\{0\}, \ldots, \{n+1\}$, where $\{0\}$ is the fixed frame {s}, $\{i\}$ is a frame attached to link i at joint i (with $i = 1, \ldots, n$), and $\{n+1\}$ is the end-effector frame {b} then the forward kinematics is expressed as

$$T_{0,n+1}(\theta) = T_{01}(\theta_1) \cdots T_{n-1,n}(\theta_n) T_{n,n+1}$$

  where θ_i denotes the joint i variable and $T_{n,n+1}$ indicates the (fixed) configuration of the end-effector frame in $\{n\}$. If the end-effector frame {b} is chosen to be coincident with $\{n\}$ then we can dispense with the frame $\{n+1\}$.

- The Denavit–Hartenberg convention requires that reference frames assigned to each link obey a strict convention (see Appendix C). Following this convention, the link frame transformation $T_{i-1,i}$ between link frames $\{i-1\}$ and $\{i\}$ can be parametrized using only four parameters, the Denavit–Hartenberg parameters. Three of these parameters describe the kinematic structure, while the fourth is the joint value. Four numbers is the minimum needed to represent the displacement between two link frames.

- The forward kinematics can also be expressed as the following product of exponentials (the space form),

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_n]\theta_n} M,$$

where $\mathcal{S}_i = (\omega_i, v_i)$ denotes the screw axis associated with positive motion along joint i expressed in fixed-frame $\{s\}$ coordinates, θ_i is the joint-i variable, and $M \in SE(3)$ denotes the position and orientation of the end-effector frame $\{b\}$ when the robot is in its zero position. It is not necessary to define individual link frames; it is only necessary to define M and the screw axes $\mathcal{S}_1, \ldots, \mathcal{S}_n$.

- The product of exponentials formula can also be written in the equivalent body form,

$$T(\theta) = Me^{[\mathcal{B}_1]\theta_1} \cdots e^{[\mathcal{B}_n]\theta_n},$$

where $\mathcal{B}_i = [\mathrm{Ad}_{M^{-1}}]\mathcal{S}_i$, $i = 1, \ldots, n$; $\mathcal{B}_i = (\omega_i, v_i)$ is the screw axis corresponding to joint axis i, expressed in $\{b\}$, with the robot in its zero position.

- The Universal Robot Description Format (URDF) is a file format used by the Robot Operating System and other software for representing the kinematics, inertial properties, visual properties, and other information for general tree-like robot mechanisms, including serial chains. A URDF file includes descriptions of joints, which connect a parent link and a child link and fully specify the kinematics of the robot, as well as descriptions of links, which specify its inertial properties.

4.4 Software

Software functions associated with this chapter are listed in MATLAB format below.

`T = FKinBody(M,Blist,thetalist)`
Computes the end-effector frame given the zero position of the end-effector `M`, the list of joint screws `Blist` expressed in the end-effector frame, and the list of joint values `thetalist`.

`T = FKinSpace(M,Slist,thetalist)`
Computes the end-effector frame given the zero position of the end-effector `M`, the list of joint screws `Slist` expressed in the fixed-space frame, and the list of joint values `thetalist`.

4.5 Notes and References

The literature on robot kinematics is quite extensive, and with very few exceptions most approaches are based on the Denavit–Hartenberg (D–H) parameters originally presented in [34] and summarized in Appendix C. Our approach is based on the Product of Exponentials (PoE) formula first presented by Brockett in [20]. Computational aspects of the PoE formula are discussed in [132].

Appendix C also elucidates in some detail the many advantages of the PoE formula over the D–H parameters, e.g., the elimination of link reference frames, the uniform treatment of revolute and prismatic joints, and the intuitive geometric interpretation of the joint axes as screws. These advantages more than offset the lone advantage of the D–H parameters, namely that they constitute a minimal set. Moreover, it should be noted that when using D–H parameters, there are differing conventions for assigning link frames, e.g., some methods align the joint axis with the $\hat{\mathrm{x}}$-axis rather than the $\hat{\mathrm{z}}$-axis of the link frame as we have done. Both the link frames and the accompanying D–H parameters need to be specified together in order to have a complete description of the robot's forward kinematics.

In summary, unless using a minimal set of parameters to represent a joint's spatial motion is critical, there is no compelling reason to prefer the D–H parameters over the PoE formula. In the next chapter, an even stronger case can be made for preferring the PoE formula to model the forward kinematics.

4.6 Exercises

Exercise 4.1 Familiarize yourself with the functions `FKinBody` and `FKinSpace` in your favorite programming language. Can you make these functions more computationally efficient? If so, indicate how. If not, explain why not.

Exercise 4.2 The RRRP SCARA robot of Figure 4.12 is shown in its zero position. Determine the end-effector zero position configuration M, the screw axes $\mathcal{S}_i$ in $\{0\}$, and the screw axes $\mathcal{B}_i$ in $\{\mathrm{b}\}$. For $\ell_0 = \ell_1 = \ell_2 = 1$ and the joint variable values $\theta = (0, \pi/2, -\pi/2, 1)$, use both the `FKinSpace` and the `FKinBody` functions to find the end-effector configuration $T \in SE(3)$. Confirm that they agree with each other.

Exercise 4.3 Determine the end-effector frame screw axes $\mathcal{B}_i$ for the 3R robot

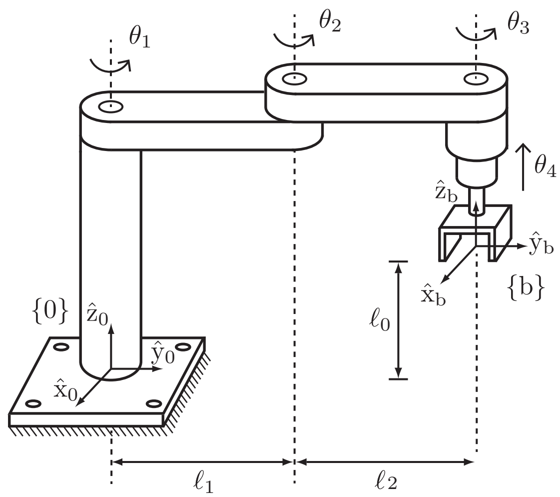

Figure 4.12: An RRRP SCARA robot for performing pick-and-place operations.

in Figure 4.3.

Exercise 4.4 Determine the end-effector frame screw axes $\mathcal{B}_i$ for the RRPRRR robot in Figure 4.5.

Exercise 4.5 Determine the end-effector frame screw axes $\mathcal{B}_i$ for the UR5 robot in Figure 4.6.

Exercise 4.6 Determine the space frame screw axes $\mathcal{S}_i$ for the WAM robot in Figure 4.8.

Exercise 4.7 The PRRRRR spatial open chain of Figure 4.13 is shown in its zero position. Determine the end-effector zero position configuration M, the screw axes $\mathcal{S}_i$ in $\{0\}$, and the screw axes $\mathcal{B}_i$ in $\{b\}$.

Exercise 4.8 The spatial RRRRPR open chain of Figure 4.14 is shown in its zero position, with fixed and end-effector frames chosen as indicated. Determine the end-effector zero position configuration M, the screw axes $\mathcal{S}_i$ in $\{0\}$, and the screw axes $\mathcal{B}_i$ in $\{b\}$.

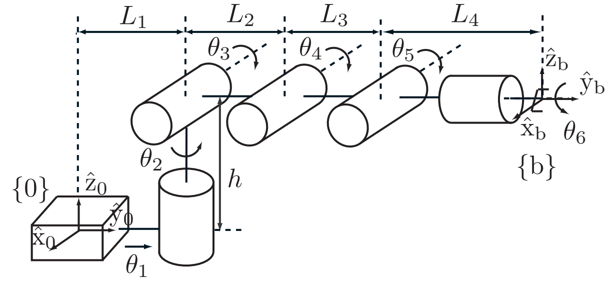

Figure 4.13: A PRRRRR spatial open chain at its zero configuration.

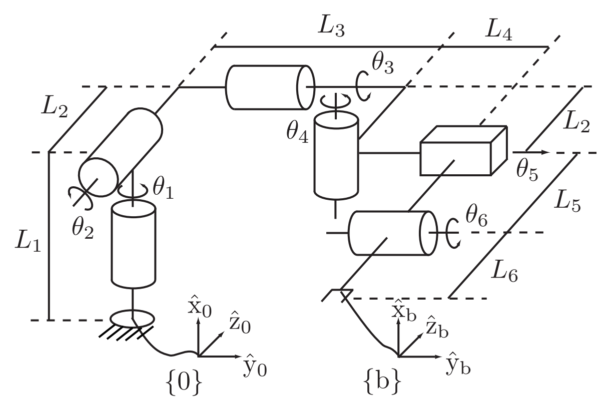

Figure 4.14: A spatial RRRRPR open chain.

Exercise 4.9 The spatial RRPPRR open chain of Figure 4.15 is shown in its zero position. Determine the end-effector zero position configuration M, the screw axes $\mathcal{S}_i$ in {0}, and the screw axes $\mathcal{B}_i$ in {b}.

Exercise 4.10 The URRPR spatial open chain of Figure 4.16 is shown in its zero position. Determine the end-effector zero position configuration M, the screw axes $\mathcal{S}_i$ in {0}, and the screw axes $\mathcal{B}_i$ in {b}.

Exercise 4.11 The spatial RPRRR open chain of Figure 4.17 is shown in its

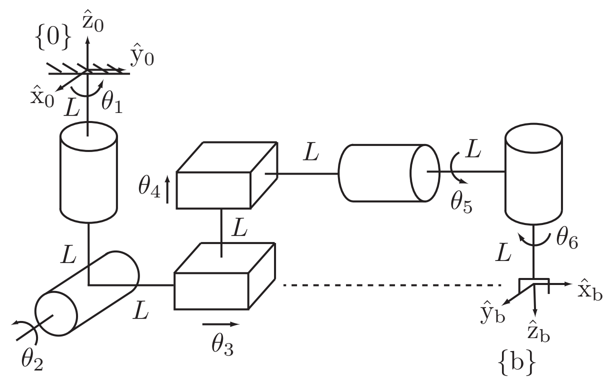

Figure 4.15: A spatial RRPPRR open chain with prescribed fixed and end-effector frames.

zero position. Determine the end-effector zero position configuration M, the screw axes $\mathcal{S}_i$ in $\{0\}$, and the screw axes $\mathcal{B}_i$ in $\{b\}$.

Exercise 4.12 The RRPRRR spatial open chain of Figure 4.18 is shown in its zero position (all joints lie on the same plane). Determine the end-effector zero position configuration M, the screw axes $\mathcal{S}_i$ in $\{0\}$, and the screw axes $\mathcal{B}_i$ in $\{b\}$. Setting $\theta_5 = \pi$ and all other joint variables to zero, find T_{06} and T_{60}.

Exercise 4.13 The spatial RRRPRR open chain of Figure 4.19 is shown in its zero position. Determine the end-effector zero position configuration M, the screw axes $\mathcal{S}_i$ in $\{0\}$, and the screw axes $\mathcal{B}_i$ in $\{b\}$.

Exercise 4.14 The RPH robot of Figure 4.20 is shown in its zero position. Determine the end-effector zero position configuration M, the screw axes $\mathcal{S}_i$ in $\{s\}$, and the screw axes $\mathcal{B}_i$ in $\{b\}$. Use both the `FKinSpace` and the `FKinBody` functions to find the end-effector configuration $T \in SE(3)$ when $\theta = (\pi/2, 3, \pi)$. Confirm that the results agree.

Exercise 4.15 The HRR robot in Figure 4.21 is shown in its zero position. Determine the end-effector zero position configuration M, the screw axes $\mathcal{S}_i$ in $\{0\}$, and the screw axes $\mathcal{B}_i$ in $\{b\}$.

Exercise 4.16 The forward kinematics of a four-dof open chain in its zero

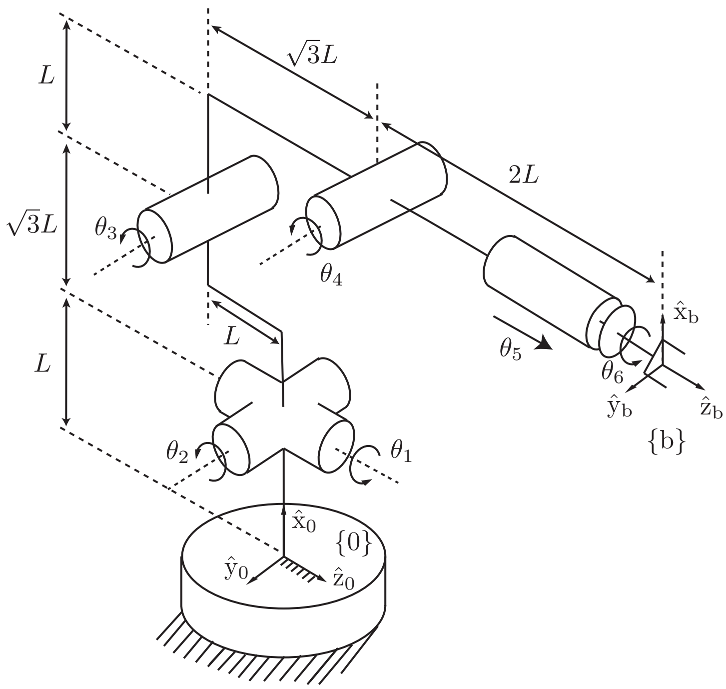

Figure 4.16: A URRPR spatial open-chain robot.

position is written in the following exponential form:

$$T(\theta) = e^{[A_1]\theta_1} e^{[A_2]\theta_2} M e^{[A_3]\theta_3} e^{[A_4]\theta_4}.$$

Suppose that the manipulator's zero position is redefined as follows:

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4).$$

Defining $\theta'_i = \theta_i - \alpha_i, i = 1, \ldots, 4$, the forward kinematics can then be written

$$T_{04}(\theta'_1, \theta'_2, \theta'_3, \theta'_4) = e^{[A'_1]\theta'_1} e^{[A'_2]\theta'_2} M' e^{[A'_3]\theta'_3} e^{[A'_4]\theta'_4}.$$

Find M' and each of the A'_i.

Exercise 4.17 Figure 4.22 shows a snake robot with end-effectors at each end. Reference frames $\{b_1\}$ and $\{b_2\}$ are attached to the two end-effectors, as shown.

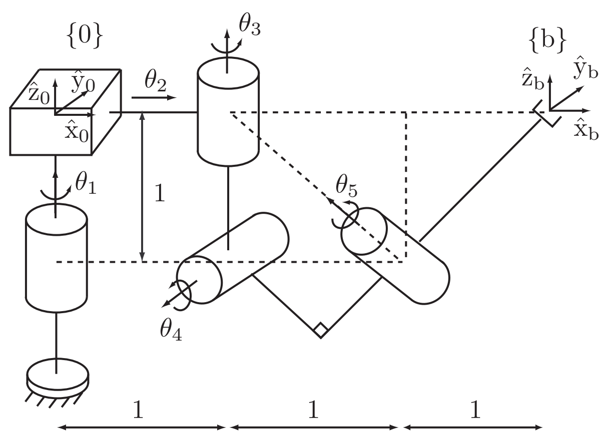

Figure 4.17: An RPRRR spatial open chain.

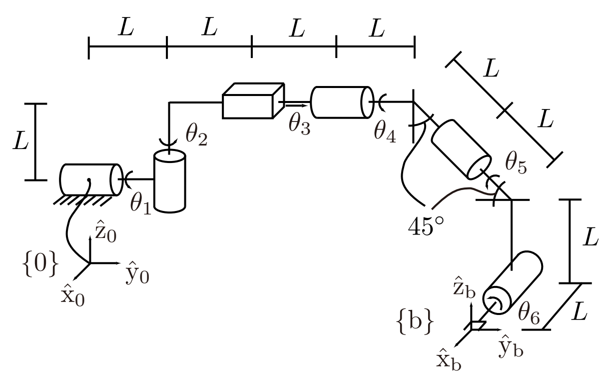

Figure 4.18: An RRPRRR spatial open chain.

(a) Suppose that end-effector 1 is grasping a tree (which can be thought of as "ground") and end-effector 2 is free to move. Assume that the robot is in its zero configuration. Then $T_{b_1 b_2} \in SE(3)$ can be expressed in the following product of exponentials form:

$$T_{b_1 b_2} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_5]\theta_5} M.$$

Find $\mathcal{S}_3, \mathcal{S}_5$, and M.

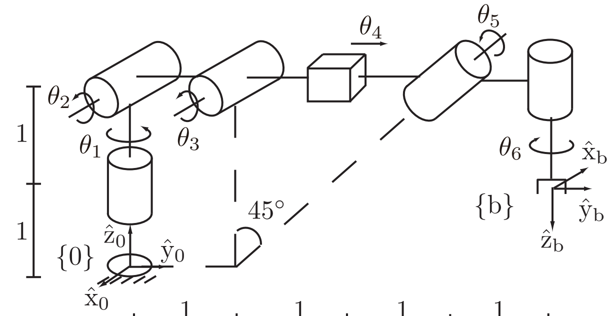

Figure 4.19: A spatial RRRPRR open chain with prescribed fixed and end-effector frames.

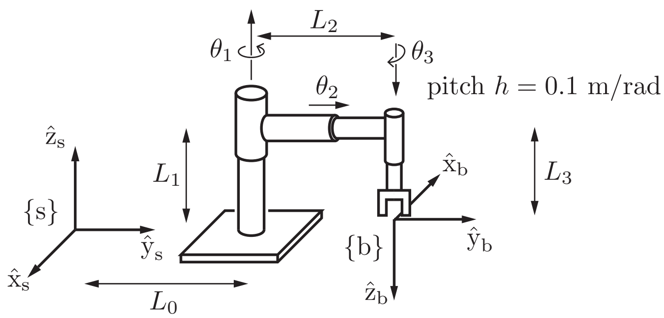

Figure 4.20: An RPH open chain shown at its zero position. All arrows along/about the joint axes are drawn in the positive direction (i.e., in the direction of increasing joint value). The pitch of the screw joint is 0.1 m/rad, i.e., it advances linearly by 0.1 m for every radian rotated. The link lengths are $L_0 = 4$, $L_1 = 3$, $L_2 = 2$, and $L_3 = 1$ (figure not drawn to scale).

(b) Now suppose that end-effector 2 is rigidly grasping a tree and end-effector 1 is free to move. Then $T_{b_2 b_1} \in SE(3)$ can be expressed in the following product of exponentials form:

$$T_{b_2 b_1} = e^{[\mathcal{A}_5]\theta_5} e^{[\mathcal{A}_4]\theta_4} e^{[\mathcal{A}_3]\theta_3} N e^{[\mathcal{A}_2]\theta_2} e^{[\mathcal{A}_1]\theta_1}.$$

Find $\mathcal{A}_2, \mathcal{A}_4$, and N.

Exercise 4.18 The two identical PUPR open chains of Figure 4.23 are shown

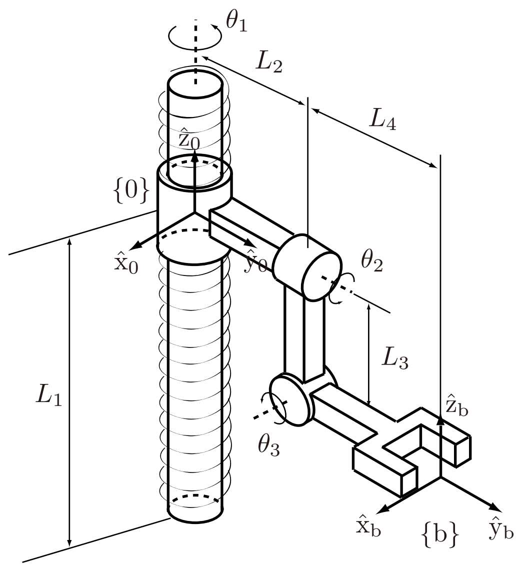

Figure 4.21: HRR robot. The pitch of the screw joint is denoted by h.

in their zero position.

(a) In terms of the given fixed frame {A} and end-effector frame {a}, the forward kinematics for the robot on the left (robot A) can be expressed in the following product of exponentials form:

$$T_{Aa} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_5]\theta_5} M_a.$$

Find $\mathcal{S}_2$ and $\mathcal{S}_4$.

(b) Suppose that the end-effector of robot A is inserted into the end-effector of robot B in such a way that the origins of the end-effectors coincide; the two robots then form a single-loop closed chain. Then the configuration space of the single-loop closed chain can be expressed in the form

$$M = e^{-[\mathcal{B}_5]\phi_5} e^{-[\mathcal{B}_4]\phi_4} e^{-[\mathcal{B}_3]\phi_3} e^{-[\mathcal{B}_2]\phi_2} e^{-[\mathcal{B}_1]\phi_1} e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} e^{[\mathcal{S}_4]\theta_4} e^{[\mathcal{S}_5]\theta_5}$$

for some constant $M \in SE(3)$ and $\mathcal{B}_i = (\omega_i, v_i)$, for $i = 1, \ldots, 5$. Find $\mathcal{B}_3$, $\mathcal{B}_5$, and M. (Hint: Given any $A \in \mathbb{R}^{n\times n}$, $(e^A)^{-1} = e^{-A}$).

Exercise 4.19 The RRPRR spatial open chain of Figure 4.24 is shown in its zero position.

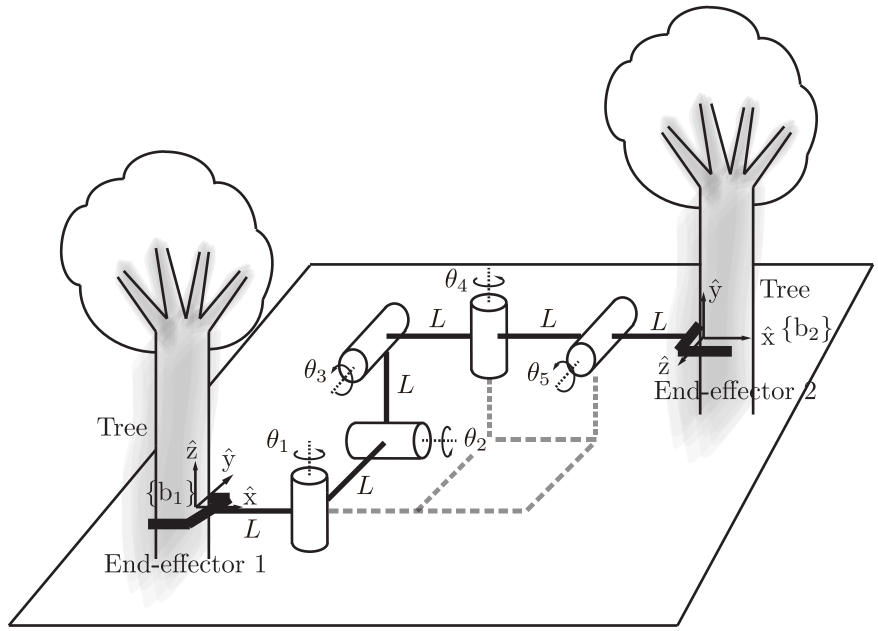

Figure 4.22: Snake robot.

(a) The forward kinematics can be expressed in the form

$$T_{sb} = M_1 e^{[\mathcal{A}_1]\theta_1} M_2 e^{[\mathcal{A}_2]\theta_2} \cdots M_5 e^{[\mathcal{A}_5]\theta_5}.$$

Find $M_2, M_3, \mathcal{A}_2$, and $\mathcal{A}_3$. (Hint: Appendix C may be helpful.)

(b) Expressing the forward kinematics in the form

$$T_{sb} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_5]\theta_5} M,$$

find M and $\mathcal{S}_1, \ldots, \mathcal{S}_5$ in terms of the quantities $M_1, \ldots, M_5$, $\mathcal{A}_1, \ldots, \mathcal{A}_5$ appearing in (a).

Exercise 4.20 The spatial PRRPRR open chain of Figure 4.25 is shown in its zero position, with space and end-effector frames chosen as indicated. Derive its forward kinematics in the form

$$T_{0n} = e^{[S_1]\theta_1} e^{[S_2]\theta_2} e^{[S_3]\theta_3} e^{[S_4]\theta_4} e^{[S_5]\theta_5} M e^{[S_6]\theta_6},$$

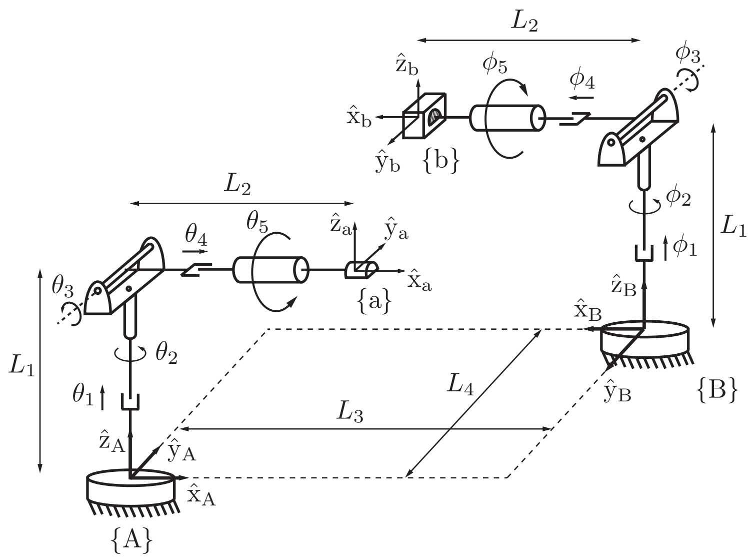

Figure 4.23: Two PUPR open chains.

where $M \in SE(3)$.

Exercise 4.21 (Refer to Appendix C.) For each $T \in SE(3)$ below, find, if they exist, values for the four parameters (α, a, d, ϕ) that satisfy

$$T = \mathrm{Rot}(\hat{\mathrm{x}}, \alpha)\,\mathrm{Trans}(\hat{\mathrm{x}}, a)\,\mathrm{Trans}(\hat{\mathrm{z}}, d)\,\mathrm{Rot}(\hat{\mathrm{z}}, \phi).$$

(a) $T = \begin{bmatrix} 0 & 1 & 1 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(b) $T = \begin{bmatrix} \cos\beta & \sin\beta & 0 & 1 \\ \sin\beta & -\cos\beta & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(c) $T = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

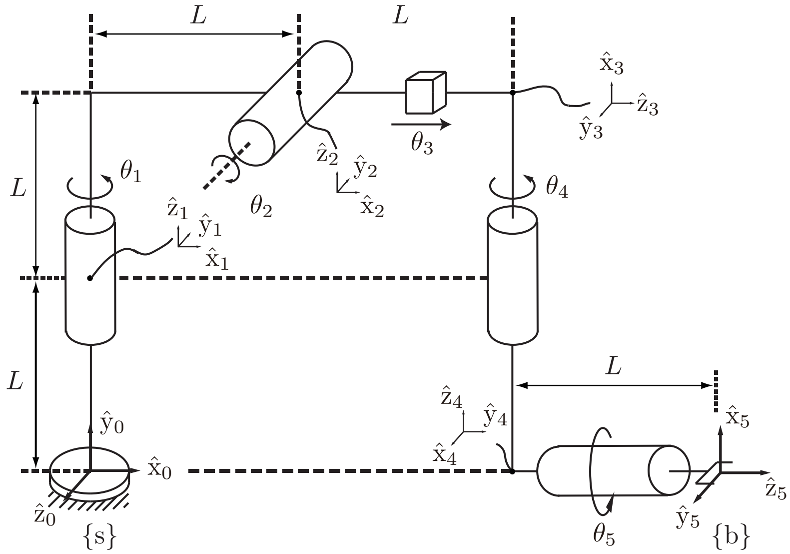

Figure 4.24: A spatial RRPRR open chain.

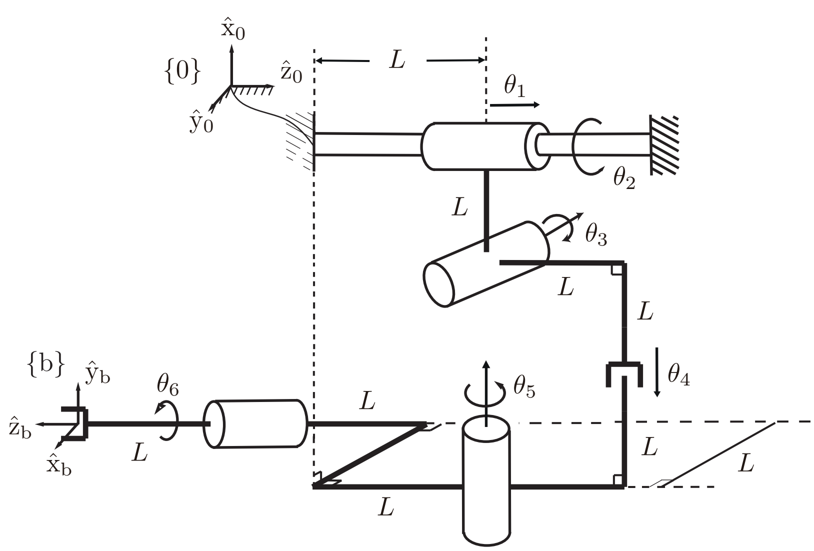

Figure 4.25: A spatial PRRPRR open chain.