is the event point at which a and b become consecutive in the total preorder. If p is on sweep line z, then $q = p$. If p is not on sweep line z, then q is to the left of p. In either case, the order given by T is correct just before encountering q. (Here is where we use the lexicographic order in which the algorithm processes event points. Because p is the lowest of the leftmost intersection points, even if p is on sweep line z and some other intersection point p' is on z, event point $q = p$ is processed before the other intersection p' can interfere with the total preorder T. Moreover, even if p is the left endpoint of one segment, say a, and the right endpoint of the other segment, say b, because left endpoint events occur before right endpoint events, segment b is in T upon first encountering segment a.) Either event point q is processed by ANY-SEGMENTS-INTERSECT or it is not processed.

If q is processed by ANY-SEGMENTS-INTERSECT, only two possible actions may occur:

1. Either a or b is inserted into T, and the other segment is above or below it in the total preorder. Lines 4–7 detect this case.
2. Segments a and b are already in T, and a segment between them in the total preorder is deleted, making a and b become consecutive. Lines 8–11 detect this case.

In either case, we find the intersection p and ANY-SEGMENTS-INTERSECT returns TRUE.

If event point q is not processed by ANY-SEGMENTS-INTERSECT, the procedure must have returned before processing all event points. This situation could have occurred only if ANY-SEGMENTS-INTERSECT had already found an intersection and returned TRUE.

Thus, if there is an intersection, ANY-SEGMENTS-INTERSECT returns TRUE. As we have already seen, if ANY-SEGMENTS-INTERSECT returns TRUE, there is an intersection. Therefore, ANY-SEGMENTS-INTERSECT always returns a correct answer.

Running time

If set S contains n segments, then ANY-SEGMENTS-INTERSECT runs in time $O(n \lg n)$. Line 1 takes $O(1)$ time. Line 2 takes $O(n \lg n)$ time, using merge sort or heapsort. The **for** loop of lines 3–11 iterates at most once per event point, and so with $2n$ event points, the loop iterates at most $2n$ times. Each iteration takes $O(\lg n)$ time, since each red-black-tree operation takes $O(\lg n)$ time and, using the method of Section 33.1, each intersection test takes $O(1)$ time. The total time is thus $O(n \lg n)$.

Exercises

33.2-1
Show that a set of n line segments may contain $\Theta(n^2)$ intersections.

33.2-2
Given two segments a and b that are comparable at x, show how to determine in $O(1)$ time which of $a \succcurlyeq_x b$ or $b \succcurlyeq_x a$ holds. Assume that neither segment is vertical. (*Hint:* If a and b do not intersect, you can just use cross products. If a and b intersect—which you can of course determine using only cross products—you can still use only addition, subtraction, and multiplication, avoiding division. Of course, in the application of the $\succcurlyeq_x$ relation used here, if a and b intersect, we can just stop and declare that we have found an intersection.)

33.2-3
Professor Mason suggests that we modify ANY-SEGMENTS-INTERSECT so that instead of returning upon finding an intersection, it prints the segments that intersect and continues on to the next iteration of the **for** loop. The professor calls the resulting procedure PRINT-INTERSECTING-SEGMENTS and claims that it prints all intersections, from left to right, as they occur in the set of line segments. Professor Dixon disagrees, claiming that Professor Mason's idea is incorrect. Which professor is right? Will PRINT-INTERSECTING-SEGMENTS always find the leftmost intersection first? Will it always find all the intersections?

33.2-4
Give an $O(n \lg n)$-time algorithm to determine whether an n-vertex polygon is simple.

33.2-5
Give an $O(n \lg n)$-time algorithm to determine whether two simple polygons with a total of n vertices intersect.

33.2-6
A ***disk*** consists of a circle plus its interior and is represented by its center point and radius. Two disks intersect if they have any point in common. Give an $O(n \lg n)$-time algorithm to determine whether any two disks in a set of n intersect.

33.2-7
Given a set of n line segments containing a total of k intersections, show how to output all k intersections in $O((n + k) \lg n)$ time.

33.2-8
Argue that ANY-SEGMENTS-INTERSECT works correctly even if three or more segments intersect at the same point.

33.2-9
Show that ANY-SEGMENTS-INTERSECT works correctly in the presence of vertical segments if we treat the bottom endpoint of a vertical segment as if it were a left endpoint and the top endpoint as if it were a right endpoint. How does your answer to Exercise 33.2-2 change if we allow vertical segments?

33.3 Finding the convex hull

The ***convex hull*** of a set Q of points, denoted by $\mathrm{CH}(Q)$, is the smallest convex polygon P for which each point in Q is either on the boundary of P or in its interior. (See Exercise 33.1-5 for a precise definition of a convex polygon.) We implicitly assume that all points in the set Q are unique and that Q contains at least three points which are not colinear. Intuitively, we can think of each point in Q as being a nail sticking out from a board. The convex hull is then the shape formed by a tight rubber band that surrounds all the nails. Figure 33.6 shows a set of points and its convex hull.

In this section, we shall present two algorithms that compute the convex hull of a set of n points. Both algorithms output the vertices of the convex hull in counterclockwise order. The first, known as Graham's scan, runs in $O(n \lg n)$ time. The second, called Jarvis's march, runs in $O(nh)$ time, where h is the number of vertices of the convex hull. As Figure 33.6 illustrates, every vertex of $\mathrm{CH}(Q)$ is a

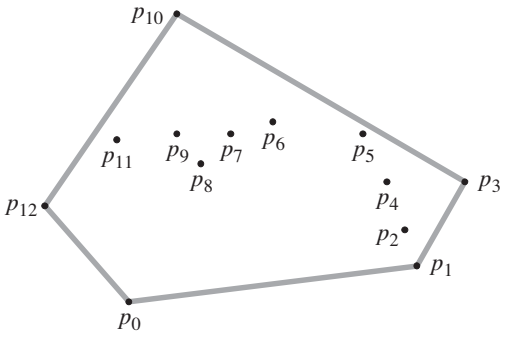

Figure 33.6 A set of points $Q = \{p_0, p_1, \ldots, p_{12}\}$ with its convex hull $\mathrm{CH}(Q)$ in gray.

point in Q. Both algorithms exploit this property, deciding which vertices in Q to keep as vertices of the convex hull and which vertices in Q to reject.

We can compute convex hulls in $O(n \lg n)$ time by any one of several methods. Both Graham's scan and Jarvis's march use a technique called "rotational sweep," processing vertices in the order of the polar angles they form with a reference vertex. Other methods include the following:

- In the ***incremental method***, we first sort the points from left to right, yielding a sequence $\langle p_1, p_2, \ldots, p_n \rangle$. At the ith stage, we update the convex hull of the $i - 1$ leftmost points, $\text{CH}(\{p_1, p_2, \ldots, p_{i-1}\})$, according to the ith point from the left, thus forming $\text{CH}(\{p_1, p_2, \ldots, p_i\})$. Exercise 33.3-6 asks you how to implement this method to take a total of $O(n \lg n)$ time.
- In the ***divide-and-conquer method***, we divide the set of n points in $\Theta(n)$ time into two subsets, one containing the leftmost $\lceil n/2 \rceil$ points and one containing the rightmost $\lfloor n/2 \rfloor$ points, recursively compute the convex hulls of the subsets, and then, by means of a clever method, combine the hulls in $O(n)$ time. The running time is described by the familiar recurrence $T(n) = 2T(n/2) + O(n)$, and so the divide-and-conquer method runs in $O(n \lg n)$ time.
- The ***prune-and-search method*** is similar to the worst-case linear-time median algorithm of Section 9.3. With this method, we find the upper portion (or "upper chain") of the convex hull by repeatedly throwing out a constant fraction of the remaining points until only the upper chain of the convex hull remains. We then do the same for the lower chain. This method is asymptotically the fastest: if the convex hull contains h vertices, it runs in only $O(n \lg h)$ time.

Computing the convex hull of a set of points is an interesting problem in its own right. Moreover, algorithms for some other computational-geometry problems start by computing a convex hull. Consider, for example, the two-dimensional ***farthest-pair problem***: we are given a set of n points in the plane and wish to find the two points whose distance from each other is maximum. As Exercise 33.3-3 asks you to prove, these two points must be vertices of the convex hull. Although we won't prove it here, we can find the farthest pair of vertices of an n-vertex convex polygon in $O(n)$ time. Thus, by computing the convex hull of the n input points in $O(n \lg n)$ time and then finding the farthest pair of the resulting convex-polygon vertices, we can find the farthest pair of points in any set of n points in $O(n \lg n)$ time.

Graham's scan

Graham's scan solves the convex-hull problem by maintaining a stack S of candidate points. It pushes each point of the input set Q onto the stack one time,

and it eventually pops from the stack each point that is not a vertex of $\mathrm{CH}(Q)$. When the algorithm terminates, stack S contains exactly the vertices of $\mathrm{CH}(Q)$, in counterclockwise order of their appearance on the boundary.

The procedure GRAHAM-SCAN takes as input a set Q of points, where $|Q| \geq 3$. It calls the functions TOP(S), which returns the point on top of stack S without changing S, and NEXT-TO-TOP(S), which returns the point one entry below the top of stack S without changing S. As we shall prove in a moment, the stack S returned by GRAHAM-SCAN contains, from bottom to top, exactly the vertices of $\mathrm{CH}(Q)$ in counterclockwise order.

```
GRAHAM-SCAN(Q)
 1  let p_0 be the point in Q with the minimum y-coordinate,
        or the leftmost such point in case of a tie
 2  let ⟨p_1, p_2, ..., p_m⟩ be the remaining points in Q,
        sorted by polar angle in counterclockwise order around p_0
        (if more than one point has the same angle, remove all but
        the one that is farthest from p_0)
 3  let S be an empty stack
 4  PUSH(p_0, S)
 5  PUSH(p_1, S)
 6  PUSH(p_2, S)
 7  for i = 3 to m
 8      while the angle formed by points NEXT-TO-TOP(S), TOP(S),
              and p_i makes a nonleft turn
 9          POP(S)
10      PUSH(p_i, S)
11  return S
```

Figure 33.7 illustrates the progress of GRAHAM-SCAN. Line 1 chooses point p_0 as the point with the lowest y-coordinate, picking the leftmost such point in case of a tie. Since there is no point in Q that is below p_0 and any other points with the same y-coordinate are to its right, p_0 must be a vertex of $\mathrm{CH}(Q)$. Line 2 sorts the remaining points of Q by polar angle relative to p_0, using the same method—comparing cross products—as in Exercise 33.1-3. If two or more points have the same polar angle relative to p_0, all but the farthest such point are convex combinations of p_0 and the farthest point, and so we remove them entirely from consideration. We let m denote the number of points other than p_0 that remain. The polar angle, measured in radians, of each point in Q relative to p_0 is in the half-open interval $[0, \pi)$. Since the points are sorted according to polar angles, they are sorted in counterclockwise order relative to p_0. We designate this sorted sequence of points by $\langle p_1, p_2, \ldots, p_m \rangle$. Note that points p_1 and p_m are vertices

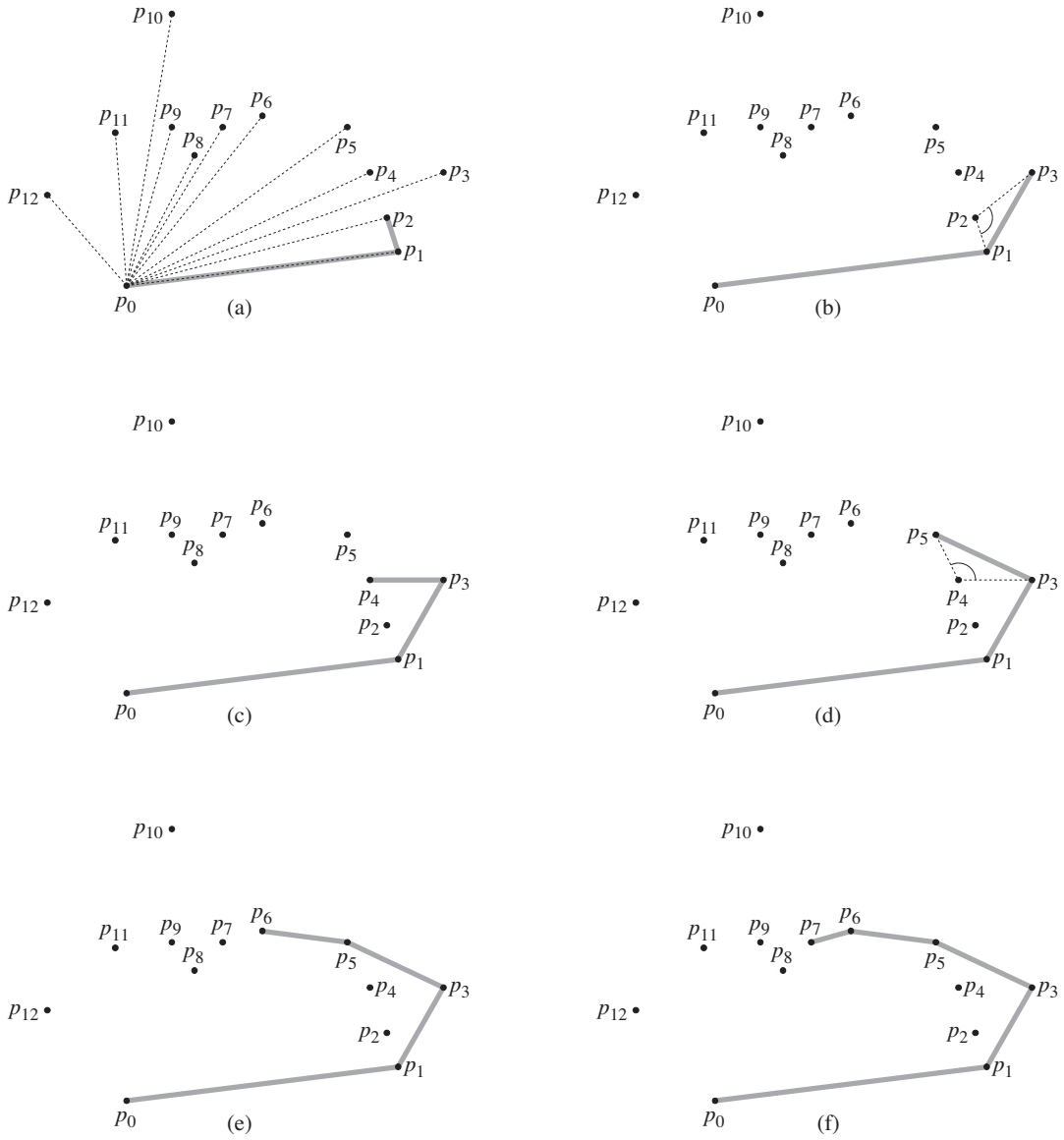

Figure 33.7 The execution of GRAHAM-SCAN on the set Q of Figure 33.6. The current convex hull contained in stack S is shown in gray at each step. **(a)** The sequence $\langle p_1, p_2, \ldots, p_{12} \rangle$ of points numbered in order of increasing polar angle relative to p_0, and the initial stack S containing p_0, p_1, and p_2. **(b)–(k)** Stack S after each iteration of the **for** loop of lines 7–10. Dashed lines show nonleft turns, which cause points to be popped from the stack. In part (h), for example, the right turn at angle $\angle p_7 p_8 p_9$ causes p_8 to be popped, and then the right turn at angle $\angle p_6 p_7 p_9$ causes p_7 to be popped.

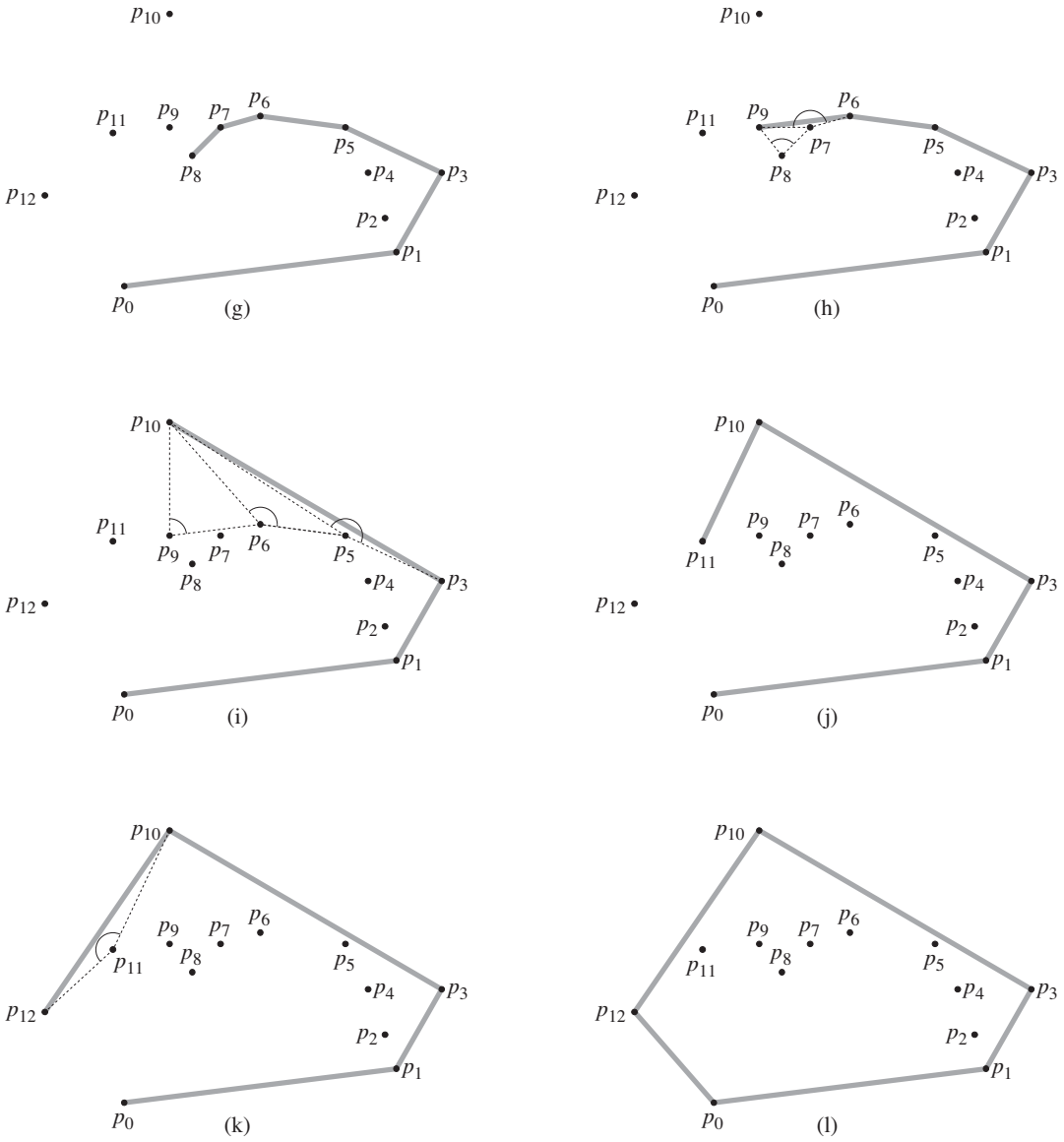

Figure 33.7, continued **(l)** The convex hull returned by the procedure, which matches that of Figure 33.6.

of CH(Q) (see Exercise 33.3-1). Figure 33.7(a) shows the points of Figure 33.6 sequentially numbered in order of increasing polar angle relative to p_0.

The remainder of the procedure uses the stack S. Lines 3–6 initialize the stack to contain, from bottom to top, the first three points p_0, p_1, and p_2. Figure 33.7(a) shows the initial stack S. The **for** loop of lines 7–10 iterates once for each point in the subsequence $\langle p_3, p_4, \ldots, p_m \rangle$. We shall see that after processing point p_i, stack S contains, from bottom to top, the vertices of CH($\{p_0, p_1, \ldots, p_i\}$) in counterclockwise order. The **while** loop of lines 8–9 removes points from the stack if we find them not to be vertices of the convex hull. When we traverse the convex hull counterclockwise, we should make a left turn at each vertex. Thus, each time the **while** loop finds a vertex at which we make a nonleft turn, we pop the vertex from the stack. (By checking for a nonleft turn, rather than just a right turn, this test precludes the possibility of a straight angle at a vertex of the resulting convex hull. We want no straight angles, since no vertex of a convex polygon may be a convex combination of other vertices of the polygon.) After we pop all vertices that have nonleft turns when heading toward point p_i, we push p_i onto the stack. Figures 33.7(b)–(k) show the state of the stack S after each iteration of the **for** loop. Finally, GRAHAM-SCAN returns the stack S in line 11. Figure 33.7(l) shows the corresponding convex hull.

The following theorem formally proves the correctness of GRAHAM-SCAN.

Theorem 33.1 (Correctness of Graham's scan)
If GRAHAM-SCAN executes on a set Q of points, where $|Q| \geq 3$, then at termination, the stack S consists of, from bottom to top, exactly the vertices of CH(Q) in counterclockwise order.

Proof After line 2, we have the sequence of points $\langle p_1, p_2, \ldots, p_m \rangle$. Let us define, for $i = 2, 3, \ldots, m$, the subset of points $Q_i = \{p_0, p_1, \ldots, p_i\}$. The points in $Q - Q_m$ are those that were removed because they had the same polar angle relative to p_0 as some point in Q_m; these points are not in CH(Q), and so CH(Q_m) = CH(Q). Thus, it suffices to show that when GRAHAM-SCAN terminates, the stack S consists of the vertices of CH(Q_m) in counterclockwise order, when listed from bottom to top. Note that just as p_0, p_1, and p_m are vertices of CH(Q), the points p_0, p_1, and p_i are all vertices of CH(Q_i).

The proof uses the following loop invariant:

> At the start of each iteration of the **for** loop of lines 7–10, stack S consists of, from bottom to top, exactly the vertices of CH(Q_{i-1}) in counterclockwise order.

Initialization: The invariant holds the first time we execute line 7, since at that time, stack S consists of exactly the vertices of $Q_2 = Q_{i-1}$, and this set of three

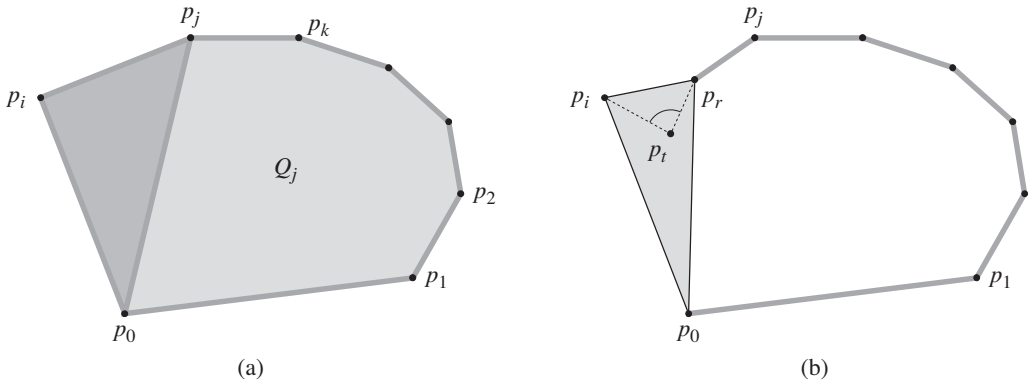

Figure 33.8 The proof of correctness of GRAHAM-SCAN. **(a)** Because p_i's polar angle relative to p_0 is greater than p_j's polar angle, and because the angle $\angle p_k p_j p_i$ makes a left turn, adding p_i to $\mathrm{CH}(Q_j)$ gives exactly the vertices of $\mathrm{CH}(Q_j \cup \{p_i\})$. **(b)** If the angle $\angle p_r p_t p_i$ makes a nonleft turn, then p_t is either in the interior of the triangle formed by p_0, p_r, and p_i or on a side of the triangle, which means that it cannot be a vertex of $\mathrm{CH}(Q_i)$.

vertices forms its own convex hull. Moreover, they appear in counterclockwise order from bottom to top.

Maintenance: Entering an iteration of the **for** loop, the top point on stack S is p_{i-1}, which was pushed at the end of the previous iteration (or before the first iteration, when $i = 3$). Let p_j be the top point on S after executing the while loop of lines 8–9 but before line 10 pushes p_i, and let p_k be the point just below p_j on S. At the moment that p_j is the top point on S and we have not yet pushed p_i, stack S contains exactly the same points it contained after iteration j of the **for** loop. By the loop invariant, therefore, S contains exactly the vertices of $\mathrm{CH}(Q_j)$ at that moment, and they appear in counterclockwise order from bottom to top.

Let us continue to focus on this moment just before pushing p_i. We know that p_i's polar angle relative to p_0 is greater than p_j's polar angle and that the angle $\angle p_k p_j p_i$ makes a left turn (otherwise we would have popped p_j). Therefore, because S contains exactly the vertices of $\mathrm{CH}(Q_j)$, we see from Figure 33.8(a) that once we push p_i, stack S will contain exactly the vertices of $\mathrm{CH}(Q_j \cup \{p_i\})$, still in counterclockwise order from bottom to top.

We now show that $\mathrm{CH}(Q_j \cup \{p_i\})$ is the same set of points as $\mathrm{CH}(Q_i)$. Consider any point p_t that was popped during iteration i of the **for** loop, and let p_r be the point just below p_t on stack S at the time p_t was popped (p_r might be p_j). The angle $\angle p_r p_t p_i$ makes a nonleft turn, and the polar angle of p_t relative to p_0 is greater than the polar angle of p_r. As Figure 33.8(b) shows, p_t must

be either in the interior of the triangle formed by p_0, p_r, and p_i or on a side of this triangle (but it is not a vertex of the triangle). Clearly, since p_t is within a triangle formed by three other points of Q_i, it cannot be a vertex of $\mathrm{CH}(Q_i)$. Since p_t is not a vertex of $\mathrm{CH}(Q_i)$, we have that

$$\mathrm{CH}(Q_i - \{p_t\}) = \mathrm{CH}(Q_i) \; . \tag{33.1}$$

Let P_i be the set of points that were popped during iteration i of the **for** loop. Since the equality (33.1) applies for all points in P_i, we can apply it repeatedly to show that $\mathrm{CH}(Q_i - P_i) = \mathrm{CH}(Q_i)$. But $Q_i - P_i = Q_j \cup \{p_i\}$, and so we conclude that $\mathrm{CH}(Q_j \cup \{p_i\}) = \mathrm{CH}(Q_i - P_i) = \mathrm{CH}(Q_i)$.

We have shown that once we push p_i, stack S contains exactly the vertices of $\mathrm{CH}(Q_i)$ in counterclockwise order from bottom to top. Incrementing i will then cause the loop invariant to hold for the next iteration.

Termination: When the loop terminates, we have $i = m + 1$, and so the loop invariant implies that stack S consists of exactly the vertices of $\mathrm{CH}(Q_m)$, which is $\mathrm{CH}(Q)$, in counterclockwise order from bottom to top. This completes the proof. ■

We now show that the running time of GRAHAM-SCAN is $O(n \lg n)$, where $n = |Q|$. Line 1 takes $\Theta(n)$ time. Line 2 takes $O(n \lg n)$ time, using merge sort or heapsort to sort the polar angles and the cross-product method of Section 33.1 to compare angles. (We can remove all but the farthest point with the same polar angle in total of $O(n)$ time over all n points.) Lines 3–6 take $O(1)$ time. Because $m \le n - 1$, the **for** loop of lines 7–10 executes at most $n - 3$ times. Since PUSH takes $O(1)$ time, each iteration takes $O(1)$ time exclusive of the time spent in the **while** loop of lines 8–9, and thus overall the **for** loop takes $O(n)$ time exclusive of the nested **while** loop.

We use aggregate analysis to show that the **while** loop takes $O(n)$ time overall. For $i = 0, 1, \ldots, m$, we push each point p_i onto stack S exactly once. As in the analysis of the MULTIPOP procedure of Section 17.1, we observe that we can pop at most the number of items that we push. At least three points — p_0, p_1, and p_m — are never popped from the stack, so that in fact at most $m - 2$ POP operations are performed in total. Each iteration of the **while** loop performs one POP, and so there are at most $m - 2$ iterations of the **while** loop altogether. Since the test in line 8 takes $O(1)$ time, each call of POP takes $O(1)$ time, and $m \le n - 1$, the total time taken by the **while** loop is $O(n)$. Thus, the running time of GRAHAM-SCAN is $O(n \lg n)$.

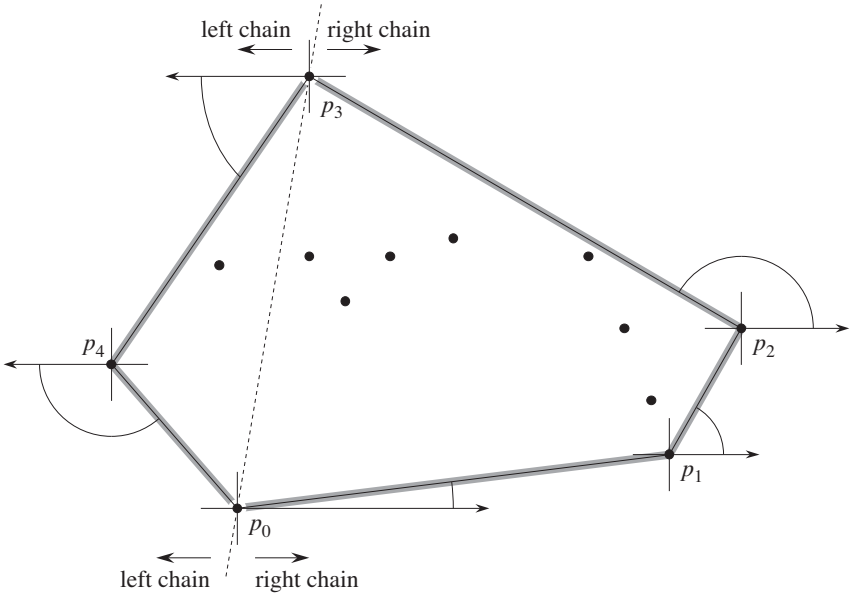

Figure 33.9 The operation of Jarvis's march. We choose the first vertex as the lowest point p_0. The next vertex, p_1, has the smallest polar angle of any point with respect to p_0. Then, p_2 has the smallest polar angle with respect to p_1. The right chain goes as high as the highest point p_3. Then, we construct the left chain by finding smallest polar angles with respect to the negative x-axis.

Jarvis's march

Jarvis's march computes the convex hull of a set Q of points by a technique known as ***package wrapping*** (or ***gift wrapping***). The algorithm runs in time $O(nh)$, where h is the number of vertices of $\mathrm{CH}(Q)$. When h is $o(\lg n)$, Jarvis's march is asymptotically faster than Graham's scan.

Intuitively, Jarvis's march simulates wrapping a taut piece of paper around the set Q. We start by taping the end of the paper to the lowest point in the set, that is, to the same point p_0 with which we start Graham's scan. We know that this point must be a vertex of the convex hull. We pull the paper to the right to make it taut, and then we pull it higher until it touches a point. This point must also be a vertex of the convex hull. Keeping the paper taut, we continue in this way around the set of vertices until we come back to our original point p_0.

More formally, Jarvis's march builds a sequence $H = \langle p_0, p_1, \ldots, p_{h-1} \rangle$ of the vertices of $\mathrm{CH}(Q)$. We start with p_0. As Figure 33.9 shows, the next vertex p_1 in the convex hull has the smallest polar angle with respect to p_0. (In case of ties, we choose the point farthest from p_0.) Similarly, p_2 has the smallest polar angle

with respect to p_1, and so on. When we reach the highest vertex, say p_k (breaking ties by choosing the farthest such vertex), we have constructed, as Figure 33.9 shows, the ***right chain*** of $\text{CH}(Q)$. To construct the ***left chain***, we start at p_k and choose p_{k+1} as the point with the smallest polar angle with respect to p_k, but *from the negative x-axis*. We continue on, forming the left chain by taking polar angles from the negative x-axis, until we come back to our original vertex p_0.

We could implement Jarvis's march in one conceptual sweep around the convex hull, that is, without separately constructing the right and left chains. Such implementations typically keep track of the angle of the last convex-hull side chosen and require the sequence of angles of hull sides to be strictly increasing (in the range of 0 to 2π radians). The advantage of constructing separate chains is that we need not explicitly compute angles; the techniques of Section 33.1 suffice to compare angles.

If implemented properly, Jarvis's march has a running time of $O(nh)$. For each of the h vertices of $\text{CH}(Q)$, we find the vertex with the minimum polar angle. Each comparison between polar angles takes $O(1)$ time, using the techniques of Section 33.1. As Section 9.1 shows, we can compute the minimum of n values in $O(n)$ time if each comparison takes $O(1)$ time. Thus, Jarvis's march takes $O(nh)$ time.

Exercises

33.3-1

Prove that in the procedure GRAHAM-SCAN, points p_1 and p_m must be vertices of $\text{CH}(Q)$.

33.3-2

Consider a model of computation that supports addition, comparison, and multiplication and for which there is a lower bound of $\Omega(n \lg n)$ to sort n numbers. Prove that $\Omega(n \lg n)$ is a lower bound for computing, in order, the vertices of the convex hull of a set of n points in such a model.

33.3-3

Given a set of points Q, prove that the pair of points farthest from each other must be vertices of $\text{CH}(Q)$.

33.3-4

For a given polygon P and a point q on its boundary, the ***shadow*** of q is the set of points r such that the segment $\overline{qr}$ is entirely on the boundary or in the interior of P. As Figure 33.10 illustrates, a polygon P is ***star-shaped*** if there exists a point p in the interior of P that is in the shadow of every point on the boundary of P. The set of all such points p is called the ***kernel*** of P. Given an n-vertex,

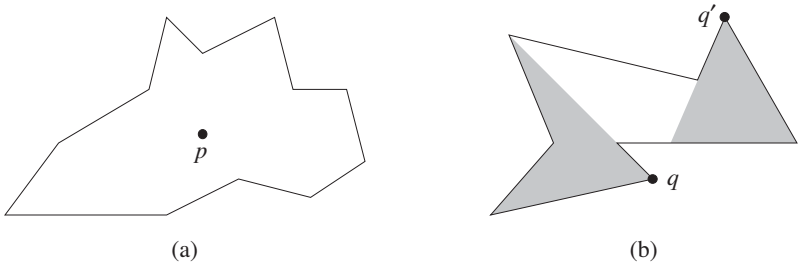

Figure 33.10 The definition of a star-shaped polygon, for use in Exercise 33.3-4. **(a)** A star-shaped polygon. The segment from point p to any point q on the boundary intersects the boundary only at q. **(b)** A non-star-shaped polygon. The shaded region on the left is the shadow of q, and the shaded region on the right is the shadow of q'. Since these regions are disjoint, the kernel is empty.

star-shaped polygon P specified by its vertices in counterclockwise order, show how to compute $\mathrm{CH}(P)$ in $O(n)$ time.

33.3-5

In the ***on-line convex-hull problem***, we are given the set Q of n points one point at a time. After receiving each point, we compute the convex hull of the points seen so far. Obviously, we could run Graham's scan once for each point, with a total running time of $O(n^2 \lg n)$. Show how to solve the on-line convex-hull problem in a total of $O(n^2)$ time.

33.3-6 ⋆

Show how to implement the incremental method for computing the convex hull of n points so that it runs in $O(n \lg n)$ time.

33.4 Finding the closest pair of points

We now consider the problem of finding the closest pair of points in a set Q of $n \geq 2$ points. "Closest" refers to the usual euclidean distance: the distance between points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Two points in set Q may be coincident, in which case the distance between them is zero. This problem has applications in, for example, traffic-control systems. A system for controlling air or sea traffic might need to identify the two closest vehicles in order to detect potential collisions.

A brute-force closest-pair algorithm simply looks at all the $\binom{n}{2} = \Theta(n^2)$ pairs of points. In this section, we shall describe a divide-and-conquer algorithm for

this problem, whose running time is described by the familiar recurrence $T(n) = 2T(n/2) + O(n)$. Thus, this algorithm uses only $O(n \lg n)$ time.

The divide-and-conquer algorithm

Each recursive invocation of the algorithm takes as input a subset $P \subseteq Q$ and arrays X and Y, each of which contains all the points of the input subset P. The points in array X are sorted so that their x-coordinates are monotonically increasing. Similarly, array Y is sorted by monotonically increasing y-coordinate. Note that in order to attain the $O(n \lg n)$ time bound, we cannot afford to sort in each recursive call; if we did, the recurrence for the running time would be $T(n) = 2T(n/2) + O(n \lg n)$, whose solution is $T(n) = O(n \lg^2 n)$. (Use the version of the master method given in Exercise 4.6-2.) We shall see a little later how to use "presorting" to maintain this sorted property without actually sorting in each recursive call.

A given recursive invocation with inputs P, X, and Y first checks whether $|P| \leq 3$. If so, the invocation simply performs the brute-force method described above: try all $\binom{|P|}{2}$ pairs of points and return the closest pair. If $|P| > 3$, the recursive invocation carries out the divide-and-conquer paradigm as follows.

Divide: Find a vertical line l that bisects the point set P into two sets P_L and P_R such that $|P_L| = \lceil |P|/2 \rceil$, $|P_R| = \lfloor |P|/2 \rfloor$, all points in P_L are on or to the left of line l, and all points in P_R are on or to the right of l. Divide the array X into arrays X_L and X_R, which contain the points of P_L and P_R respectively, sorted by monotonically increasing x-coordinate. Similarly, divide the array Y into arrays Y_L and Y_R, which contain the points of P_L and P_R respectively, sorted by monotonically increasing y-coordinate.

Conquer: Having divided P into P_L and P_R, make two recursive calls, one to find the closest pair of points in P_L and the other to find the closest pair of points in P_R. The inputs to the first call are the subset P_L and arrays X_L and Y_L; the second call receives the inputs P_R, X_R, and Y_R. Let the closest-pair distances returned for P_L and P_R be δ_L and δ_R, respectively, and let $\delta = \min(\delta_L, \delta_R)$.

Combine: The closest pair is either the pair with distance δ found by one of the recursive calls, or it is a pair of points with one point in P_L and the other in P_R. The algorithm determines whether there is a pair with one point in P_L and the other point in P_R and whose distance is less than δ. Observe that if a pair of points has distance less than δ, both points of the pair must be within δ units of line l. Thus, as Figure 33.11(a) shows, they both must reside in the 2δ-wide vertical strip centered at line l. To find such a pair, if one exists, we do the following:

1. Create an array Y', which is the array Y with all points not in the 2δ-wide vertical strip removed. The array Y' is sorted by y-coordinate, just as Y is.
2. For each point p in the array Y', try to find points in Y' that are within δ units of p. As we shall see shortly, only the 7 points in Y' that follow p need be considered. Compute the distance from p to each of these 7 points, and keep track of the closest-pair distance δ' found over all pairs of points in Y'.
3. If $\delta' < \delta$, then the vertical strip does indeed contain a closer pair than the recursive calls found. Return this pair and its distance δ'. Otherwise, return the closest pair and its distance δ found by the recursive calls.

The above description omits some implementation details that are necessary to achieve the $O(n \lg n)$ running time. After proving the correctness of the algorithm, we shall show how to implement the algorithm to achieve the desired time bound.

Correctness

The correctness of this closest-pair algorithm is obvious, except for two aspects. First, by bottoming out the recursion when $|P| \leq 3$, we ensure that we never try to solve a subproblem consisting of only one point. The second aspect is that we need only check the 7 points following each point p in array Y'; we shall now prove this property.

Suppose that at some level of the recursion, the closest pair of points is $p_L \in P_L$ and $p_R \in P_R$. Thus, the distance δ' between p_L and p_R is strictly less than δ. Point p_L must be on or to the left of line l and less than δ units away. Similarly, p_R is on or to the right of l and less than δ units away. Moreover, p_L and p_R are within δ units of each other vertically. Thus, as Figure 33.11(a) shows, p_L and p_R are within a $\delta \times 2\delta$ rectangle centered at line l. (There may be other points within this rectangle as well.)

We next show that at most 8 points of P can reside within this $\delta \times 2\delta$ rectangle. Consider the $\delta \times \delta$ square forming the left half of this rectangle. Since all points within P_L are at least δ units apart, at most 4 points can reside within this square; Figure 33.11(b) shows how. Similarly, at most 4 points in P_R can reside within the $\delta \times \delta$ square forming the right half of the rectangle. Thus, at most 8 points of P can reside within the $\delta \times 2\delta$ rectangle. (Note that since points on line l may be in either P_L or P_R, there may be up to 4 points on l. This limit is achieved if there are two pairs of coincident points such that each pair consists of one point from P_L and one point from P_R, one pair is at the intersection of l and the top of the rectangle, and the other pair is where l intersects the bottom of the rectangle.)

Having shown that at most 8 points of P can reside within the rectangle, we can easily see why we need to check only the 7 points following each point in the array Y'. Still assuming that the closest pair is p_L and p_R, let us assume without

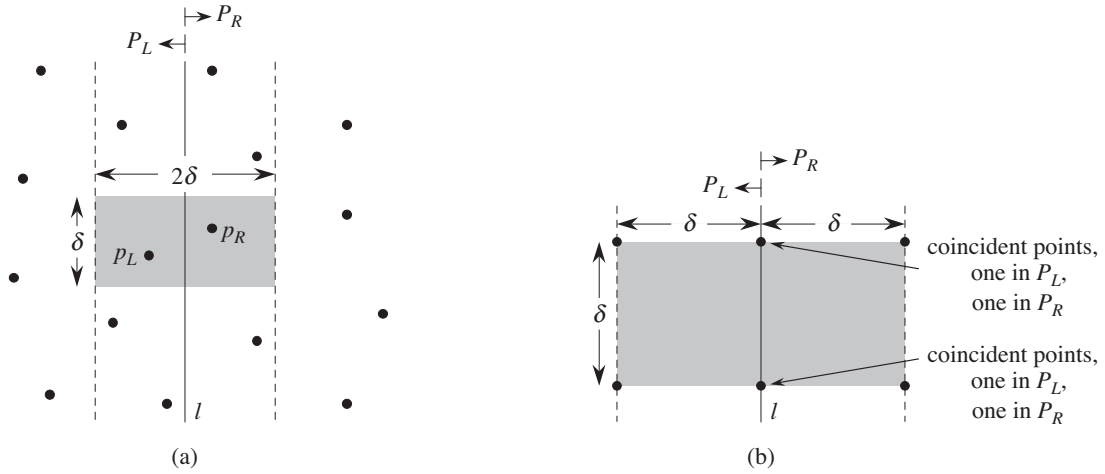

Figure 33.11 Key concepts in the proof that the closest-pair algorithm needs to check only 7 points following each point in the array Y'. **(a)** If $p_L \in P_L$ and $p_R \in P_R$ are less than δ units apart, they must reside within a $\delta \times 2\delta$ rectangle centered at line l. **(b)** How 4 points that are pairwise at least δ units apart can all reside within a $\delta \times \delta$ square. On the left are 4 points in P_L, and on the right are 4 points in P_R. The $\delta \times 2\delta$ rectangle can contain 8 points if the points shown on line l are actually pairs of coincident points with one point in P_L and one in P_R.

loss of generality that p_L precedes p_R in array Y'. Then, even if p_L occurs as early as possible in Y' and p_R occurs as late as possible, p_R is in one of the 7 positions following p_L. Thus, we have shown the correctness of the closest-pair algorithm.

Implementation and running time

As we have noted, our goal is to have the recurrence for the running time be $T(n) = 2T(n/2) + O(n)$, where $T(n)$ is the running time for a set of n points. The main difficulty comes from ensuring that the arrays X_L, X_R, Y_L, and Y_R, which are passed to recursive calls, are sorted by the proper coordinate and also that the array Y' is sorted by y-coordinate. (Note that if the array X that is received by a recursive call is already sorted, then we can easily divide set P into P_L and P_R in linear time.)

The key observation is that in each call, we wish to form a sorted subset of a sorted array. For example, a particular invocation receives the subset P and the array Y, sorted by y-coordinate. Having partitioned P into P_L and P_R, it needs to form the arrays Y_L and Y_R, which are sorted by y-coordinate, in linear time. We can view the method as the opposite of the MERGE procedure from merge sort in

Section 2.3.1: we are splitting a sorted array into two sorted arrays. The following pseudocode gives the idea.

```
1  let Y_L[1..Y.length] and Y_R[1..Y.length] be new arrays
2  Y_L.length = Y_R.length = 0
3  for i = 1 to Y.length
4      if Y[i] ∈ P_L
5          Y_L.length = Y_L.length + 1
6          Y_L[Y_L.length] = Y[i]
7      else Y_R.length = Y_R.length + 1
8          Y_R[Y_R.length] = Y[i]
```

We simply examine the points in array Y in order. If a point $Y[i]$ is in P_L, we append it to the end of array Y_L; otherwise, we append it to the end of array Y_R. Similar pseudocode works for forming arrays X_L, X_R, and Y'.

The only remaining question is how to get the points sorted in the first place. We ***presort*** them; that is, we sort them once and for all *before* the first recursive call. We pass these sorted arrays into the first recursive call, and from there we whittle them down through the recursive calls as necessary. Presorting adds an additional $O(n \lg n)$ term to the running time, but now each step of the recursion takes linear time exclusive of the recursive calls. Thus, if we let $T(n)$ be the running time of each recursive step and $T'(n)$ be the running time of the entire algorithm, we get $T'(n) = T(n) + O(n \lg n)$ and

$$T(n) = \begin{cases} 2T(n/2) + O(n) & \text{if } n > 3\,, \\ O(1) & \text{if } n \le 3\,. \end{cases}$$

Thus, $T(n) = O(n \lg n)$ and $T'(n) = O(n \lg n)$.

Exercises

33.4-1

Professor Williams comes up with a scheme that allows the closest-pair algorithm to check only 5 points following each point in array Y'. The idea is always to place points on line l into set P_L. Then, there cannot be pairs of coincident points on line l with one point in P_L and one in P_R. Thus, at most 6 points can reside in the $\delta \times 2\delta$ rectangle. What is the flaw in the professor's scheme?

33.4-2

Show that it actually suffices to check only the points in the 5 array positions following each point in the array Y'.

33.4-3

We can define the distance between two points in ways other than euclidean. In the plane, the ***L_m-distance*** between points p_1 and p_2 is given by the expression $(|x_1 - x_2|^m + |y_1 - y_2|^m)^{1/m}$. Euclidean distance, therefore, is L_2-distance. Modify the closest-pair algorithm to use the L_1-distance, which is also known as the ***Manhattan distance***.

33.4-4

Given two points p_1 and p_2 in the plane, the L_∞-distance between them is given by $\max(|x_1 - x_2|, |y_1 - y_2|)$. Modify the closest-pair algorithm to use the L_∞-distance.

33.4-5

Suppose that $\Omega(n)$ of the points given to the closest-pair algorithm are covertical. Show how to determine the sets P_L and P_R and how to determine whether each point of Y is in P_L or P_R so that the running time for the closest-pair algorithm remains $O(n \lg n)$.

33.4-6

Suggest a change to the closest-pair algorithm that avoids presorting the Y array but leaves the running time as $O(n \lg n)$. (*Hint:* Merge sorted arrays Y_L and Y_R to form the sorted array Y.)

Problems

33-1 Convex layers

Given a set Q of points in the plane, we define the ***convex layers*** of Q inductively. The first convex layer of Q consists of those points in Q that are vertices of $\mathrm{CH}(Q)$. For $i > 1$, define Q_i to consist of the points of Q with all points in convex layers $1, 2, \ldots, i-1$ removed. Then, the ith convex layer of Q is $\mathrm{CH}(Q_i)$ if $Q_i \neq \emptyset$ and is undefined otherwise.

a. Give an $O(n^2)$-time algorithm to find the convex layers of a set of n points.

b. Prove that $\Omega(n \lg n)$ time is required to compute the convex layers of a set of n points with any model of computation that requires $\Omega(n \lg n)$ time to sort n real numbers.

33-2 Maximal layers

Let Q be a set of n points in the plane. We say that point (x, y) ***dominates*** point (x', y') if $x \geq x'$ and $y \geq y'$. A point in Q that is dominated by no other points in Q is said to be ***maximal***. Note that Q may contain many maximal points, which can be organized into ***maximal layers*** as follows. The first maximal layer L_1 is the set of maximal points of Q. For $i > 1$, the ith maximal layer L_i is the set of maximal points in $Q - \bigcup_{j=1}^{i-1} L_j$.

Suppose that Q has k nonempty maximal layers, and let y_i be the y-coordinate of the leftmost point in L_i for $i = 1, 2, \ldots, k$. For now, assume that no two points in Q have the same x- or y-coordinate.

a. Show that $y_1 > y_2 > \cdots > y_k$.

Consider a point (x, y) that is to the left of any point in Q and for which y is distinct from the y-coordinate of any point in Q. Let $Q' = Q \cup \{(x, y)\}$.

b. Let j be the minimum index such that $y_j < y$, unless $y < y_k$, in which case we let $j = k + 1$. Show that the maximal layers of Q' are as follows:

- If $j \leq k$, then the maximal layers of Q' are the same as the maximal layers of Q, except that L_j also includes (x, y) as its new leftmost point.
- If $j = k + 1$, then the first k maximal layers of Q' are the same as for Q, but in addition, Q' has a nonempty $(k + 1)$st maximal layer: $L_{k+1} = \{(x, y)\}$.

c. Describe an $O(n \lg n)$-time algorithm to compute the maximal layers of a set Q of n points. (*Hint:* Move a sweep line from right to left.)

d. Do any difficulties arise if we now allow input points to have the same x- or y-coordinate? Suggest a way to resolve such problems.

33-3 Ghostbusters and ghosts

A group of n Ghostbusters is battling n ghosts. Each Ghostbuster carries a proton pack, which shoots a stream at a ghost, eradicating it. A stream goes in a straight line and terminates when it hits the ghost. The Ghostbusters decide upon the following strategy. They will pair off with the ghosts, forming n Ghostbuster-ghost pairs, and then simultaneously each Ghostbuster will shoot a stream at his chosen ghost. As we all know, it is *very* dangerous to let streams cross, and so the Ghostbusters must choose pairings for which no streams will cross.

Assume that the position of each Ghostbuster and each ghost is a fixed point in the plane and that no three positions are colinear.

a. Argue that there exists a line passing through one Ghostbuster and one ghost such that the number of Ghostbusters on one side of the line equals the number of ghosts on the same side. Describe how to find such a line in $O(n \lg n)$ time.

b. Give an $O(n^2 \lg n)$-time algorithm to pair Ghostbusters with ghosts in such a way that no streams cross.

33-4 Picking up sticks

Professor Charon has a set of n sticks, which are piled up in some configuration. Each stick is specified by its endpoints, and each endpoint is an ordered triple giving its (x, y, z) coordinates. No stick is vertical. He wishes to pick up all the sticks, one at a time, subject to the condition that he may pick up a stick only if there is no other stick on top of it.

a. Give a procedure that takes two sticks a and b and reports whether a is above, below, or unrelated to b.

b. Describe an efficient algorithm that determines whether it is possible to pick up all the sticks, and if so, provides a legal order in which to pick them up.

33-5 Sparse-hulled distributions

Consider the problem of computing the convex hull of a set of points in the plane that have been drawn according to some known random distribution. Sometimes, the number of points, or size, of the convex hull of n points drawn from such a distribution has expectation $O(n^{1-\epsilon})$ for some constant $\epsilon > 0$. We call such a distribution ***sparse-hulled***. Sparse-hulled distributions include the following:

- Points drawn uniformly from a unit-radius disk. The convex hull has expected size $\Theta(n^{1/3})$.
- Points drawn uniformly from the interior of a convex polygon with k sides, for any constant k. The convex hull has expected size $\Theta(\lg n)$.
- Points drawn according to a two-dimensional normal distribution. The convex hull has expected size $\Theta(\sqrt{\lg n})$.

a. Given two convex polygons with n_1 and n_2 vertices respectively, show how to compute the convex hull of all $n_1 + n_2$ points in $O(n_1 + n_2)$ time. (The polygons may overlap.)

b. Show how to compute the convex hull of a set of n points drawn independently according to a sparse-hulled distribution in $O(n)$ average-case time. (*Hint:* Recursively find the convex hulls of the first $n/2$ points and the second $n/2$ points, and then combine the results.)

Chapter notes

This chapter barely scratches the surface of computational-geometry algorithms and techniques. Books on computational geometry include those by Preparata and Shamos [282], Edelsbrunner [99], and O'Rourke [269].

Although geometry has been studied since antiquity, the development of algorithms for geometric problems is relatively new. Preparata and Shamos note that the earliest notion of the complexity of a problem was given by E. Lemoine in 1902. He was studying euclidean constructions—those using a compass and a ruler—and devised a set of five primitives: placing one leg of the compass on a given point, placing one leg of the compass on a given line, drawing a circle, passing the ruler's edge through a given point, and drawing a line. Lemoine was interested in the number of primitives needed to effect a given construction; he called this amount the "simplicity" of the construction.

The algorithm of Section 33.2, which determines whether any segments intersect, is due to Shamos and Hoey [313].

The original version of Graham's scan is given by Graham [150]. The package-wrapping algorithm is due to Jarvis [189]. Using a decision-tree model of computation, Yao [359] proved a worst-case lower bound of $\Omega(n \lg n)$ for the running time of any convex-hull algorithm. When the number of vertices h of the convex hull is taken into account, the prune-and-search algorithm of Kirkpatrick and Seidel [206], which takes $O(n \lg h)$ time, is asymptotically optimal.

The $O(n \lg n)$-time divide-and-conquer algorithm for finding the closest pair of points is by Shamos and appears in Preparata and Shamos [282]. Preparata and Shamos also show that the algorithm is asymptotically optimal in a decision-tree model.

34 NP-Completeness

Almost all the algorithms we have studied thus far have been ***polynomial-time algorithms***: on inputs of size n, their worst-case running time is $O(n^k)$ for some constant k. You might wonder whether *all* problems can be solved in polynomial time. The answer is no. For example, there are problems, such as Turing's famous "Halting Problem," that cannot be solved by any computer, no matter how much time we allow. There are also problems that can be solved, but not in time $O(n^k)$ for any constant k. Generally, we think of problems that are solvable by polynomial-time algorithms as being tractable, or easy, and problems that require superpolynomial time as being intractable, or hard.

The subject of this chapter, however, is an interesting class of problems, called the "NP-complete" problems, whose status is unknown. No polynomial-time algorithm has yet been discovered for an NP-complete problem, nor has anyone yet been able to prove that no polynomial-time algorithm can exist for any one of them. This so-called P $\neq$ NP question has been one of the deepest, most perplexing open research problems in theoretical computer science since it was first posed in 1971.

Several NP-complete problems are particularly tantalizing because they seem on the surface to be similar to problems that we know how to solve in polynomial time. In each of the following pairs of problems, one is solvable in polynomial time and the other is NP-complete, but the difference between problems appears to be slight:

Shortest vs. longest simple paths: In Chapter 24, we saw that even with negative edge weights, we can find *shortest* paths from a single source in a directed graph $G = (V, E)$ in $O(VE)$ time. Finding a *longest* simple path between two vertices is difficult, however. Merely determining whether a graph contains a simple path with at least a given number of edges is NP-complete.

Euler tour vs. hamiltonian cycle: An ***Euler tour*** of a connected, directed graph $G = (V, E)$ is a cycle that traverses each *edge* of G exactly once, although it is allowed to visit each vertex more than once. By Problem 22-3, we can determine whether a graph has an Euler tour in only $O(E)$ time and, in fact,

we can find the edges of the Euler tour in $O(E)$ time. A ***hamiltonian cycle*** of a directed graph $G = (V, E)$ is a simple cycle that contains each *vertex* in V. Determining whether a directed graph has a hamiltonian cycle is NP-complete. (Later in this chapter, we shall prove that determining whether an *undirected* graph has a hamiltonian cycle is NP-complete.)

2-CNF satisfiability vs. 3-CNF satisfiability: A boolean formula contains variables whose values are 0 or 1; boolean connectives such as $\wedge$ (AND), $\vee$ (OR), and $\neg$ (NOT); and parentheses. A boolean formula is ***satisfiable*** if there exists some assignment of the values 0 and 1 to its variables that causes it to evaluate to 1. We shall define terms more formally later in this chapter, but informally, a boolean formula is in ***k-conjunctive normal form***, or k-CNF, if it is the AND of clauses of ORs of exactly k variables or their negations. For example, the boolean formula $(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_3) \wedge (\neg x_2 \vee \neg x_3)$ is in 2-CNF. (It has the satisfying assignment $x_1 = 1, x_2 = 0, x_3 = 1$.) Although we can determine in polynomial time whether a 2-CNF formula is satisfiable, we shall see later in this chapter that determining whether a 3-CNF formula is satisfiable is NP-complete.

NP-completeness and the classes P and NP

Throughout this chapter, we shall refer to three classes of problems: P, NP, and NPC, the latter class being the NP-complete problems. We describe them informally here, and we shall define them more formally later on.

The class P consists of those problems that are solvable in polynomial time. More specifically, they are problems that can be solved in time $O(n^k)$ for some constant k, where n is the size of the input to the problem. Most of the problems examined in previous chapters are in P.

The class NP consists of those problems that are "verifiable" in polynomial time. What do we mean by a problem being verifiable? If we were somehow given a "certificate" of a solution, then we could verify that the certificate is correct in time polynomial in the size of the input to the problem. For example, in the hamiltonian-cycle problem, given a directed graph $G = (V, E)$, a certificate would be a sequence $\langle v_1, v_2, v_3, \ldots, v_{|V|} \rangle$ of $|V|$ vertices. We could easily check in polynomial time that $(v_i, v_{i+1}) \in E$ for $i = 1, 2, 3, \ldots, |V| - 1$ and that $(v_{|V|}, v_1) \in E$ as well. As another example, for 3-CNF satisfiability, a certificate would be an assignment of values to variables. We could check in polynomial time that this assignment satisfies the boolean formula.

Any problem in P is also in NP, since if a problem is in P then we can solve it in polynomial time without even being supplied a certificate. We shall formalize this notion later in this chapter, but for now we can believe that P $\subseteq$ NP. The open question is whether or not P is a proper subset of NP.

Informally, a problem is in the class NPC—and we refer to it as being ***NP-complete***—if it is in NP and is as "hard" as any problem in NP. We shall formally define what it means to be as hard as any problem in NP later in this chapter. In the meantime, we will state without proof that if *any* NP-complete problem can be solved in polynomial time, then *every* problem in NP has a polynomial-time algorithm. Most theoretical computer scientists believe that the NP-complete problems are intractable, since given the wide range of NP-complete problems that have been studied to date—without anyone having discovered a polynomial-time solution to any of them—it would be truly astounding if all of them could be solved in polynomial time. Yet, given the effort devoted thus far to proving that NP-complete problems are intractable—without a conclusive outcome—we cannot rule out the possibility that the NP-complete problems are in fact solvable in polynomial time.

To become a good algorithm designer, you must understand the rudiments of the theory of NP-completeness. If you can establish a problem as NP-complete, you provide good evidence for its intractability. As an engineer, you would then do better to spend your time developing an approximation algorithm (see Chapter 35) or solving a tractable special case, rather than searching for a fast algorithm that solves the problem exactly. Moreover, many natural and interesting problems that on the surface seem no harder than sorting, graph searching, or network flow are in fact NP-complete. Therefore, you should become familiar with this remarkable class of problems.

Overview of showing problems to be NP-complete

The techniques we use to show that a particular problem is NP-complete differ fundamentally from the techniques used throughout most of this book to design and analyze algorithms. When we demonstrate that a problem is NP-complete, we are making a statement about how hard it is (or at least how hard we think it is), rather than about how easy it is. We are not trying to prove the existence of an efficient algorithm, but instead that no efficient algorithm is likely to exist. In this way, NP-completeness proofs bear some similarity to the proof in Section 8.1 of an $\Omega(n \lg n)$-time lower bound for any comparison sort algorithm; the specific techniques used for showing NP-completeness differ from the decision-tree method used in Section 8.1, however.

We rely on three key concepts in showing a problem to be NP-complete:

Decision problems vs. optimization problems

Many problems of interest are ***optimization problems***, in which each feasible (i.e., "legal") solution has an associated value, and we wish to find a feasible solution with the best value. For example, in a problem that we call SHORTEST-PATH,

we are given an undirected graph G and vertices u and v, and we wish to find a path from u to v that uses the fewest edges. In other words, SHORTEST-PATH is the single-pair shortest-path problem in an unweighted, undirected graph. NP-completeness applies directly not to optimization problems, however, but to ***decision problems***, in which the answer is simply "yes" or "no" (or, more formally, "1" or "0").

Although NP-complete problems are confined to the realm of decision problems, we can take advantage of a convenient relationship between optimization problems and decision problems. We usually can cast a given optimization problem as a related decision problem by imposing a bound on the value to be optimized. For example, a decision problem related to SHORTEST-PATH is PATH: given a directed graph G, vertices u and v, and an integer k, does a path exist from u to v consisting of at most k edges?

The relationship between an optimization problem and its related decision problem works in our favor when we try to show that the optimization problem is "hard." That is because the decision problem is in a sense "easier," or at least "no harder." As a specific example, we can solve PATH by solving SHORTEST-PATH and then comparing the number of edges in the shortest path found to the value of the decision-problem parameter k. In other words, if an optimization problem is easy, its related decision problem is easy as well. Stated in a way that has more relevance to NP-completeness, if we can provide evidence that a decision problem is hard, we also provide evidence that its related optimization problem is hard. Thus, even though it restricts attention to decision problems, the theory of NP-completeness often has implications for optimization problems as well.

Reductions

The above notion of showing that one problem is no harder or no easier than another applies even when both problems are decision problems. We take advantage of this idea in almost every NP-completeness proof, as follows. Let us consider a decision problem A, which we would like to solve in polynomial time. We call the input to a particular problem an ***instance*** of that problem; for example, in PATH, an instance would be a particular graph G, particular vertices u and v of G, and a particular integer k. Now suppose that we already know how to solve a different decision problem B in polynomial time. Finally, suppose that we have a procedure that transforms any instance α of A into some instance β of B with the following characteristics:

- The transformation takes polynomial time.
- The answers are the same. That is, the answer for α is "yes" if and only if the answer for β is also "yes."

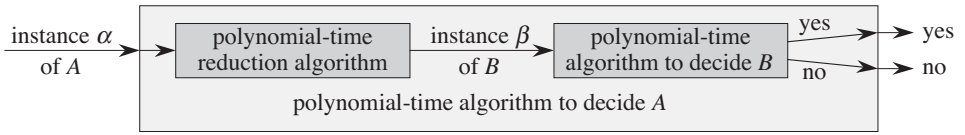

Figure 34.1 How to use a polynomial-time reduction algorithm to solve a decision problem A in polynomial time, given a polynomial-time decision algorithm for another problem B. In polynomial time, we transform an instance α of A into an instance β of B, we solve B in polynomial time, and we use the answer for β as the answer for α.

We call such a procedure a polynomial-time ***reduction algorithm*** and, as Figure 34.1 shows, it provides us a way to solve problem A in polynomial time:

1. Given an instance α of problem A, use a polynomial-time reduction algorithm to transform it to an instance β of problem B.
2. Run the polynomial-time decision algorithm for B on the instance β.
3. Use the answer for β as the answer for α.

As long as each of these steps takes polynomial time, all three together do also, and so we have a way to decide on α in polynomial time. In other words, by "reducing" solving problem A to solving problem B, we use the "easiness" of B to prove the "easiness" of A.

Recalling that NP-completeness is about showing how hard a problem is rather than how easy it is, we use polynomial-time reductions in the opposite way to show that a problem is NP-complete. Let us take the idea a step further, and show how we could use polynomial-time reductions to show that no polynomial-time algorithm can exist for a particular problem B. Suppose we have a decision problem A for which we already know that no polynomial-time algorithm can exist. (Let us not concern ourselves for now with how to find such a problem A.) Suppose further that we have a polynomial-time reduction transforming instances of A to instances of B. Now we can use a simple proof by contradiction to show that no polynomial-time algorithm can exist for B. Suppose otherwise; i.e., suppose that B has a polynomial-time algorithm. Then, using the method shown in Figure 34.1, we would have a way to solve problem A in polynomial time, which contradicts our assumption that there is no polynomial-time algorithm for A.

For NP-completeness, we cannot assume that there is absolutely no polynomial-time algorithm for problem A. The proof methodology is similar, however, in that we prove that problem B is NP-complete on the assumption that problem A is also NP-complete.

A first NP-complete problem

Because the technique of reduction relies on having a problem already known to be NP-complete in order to prove a different problem NP-complete, we need a "first" NP-complete problem. The problem we shall use is the circuit-satisfiability problem, in which we are given a boolean combinational circuit composed of AND, OR, and NOT gates, and we wish to know whether there exists some set of boolean inputs to this circuit that causes its output to be 1. We shall prove that this first problem is NP-complete in Section 34.3.

Chapter outline

This chapter studies the aspects of NP-completeness that bear most directly on the analysis of algorithms. In Section 34.1, we formalize our notion of "problem" and define the complexity class P of polynomial-time solvable decision problems. We also see how these notions fit into the framework of formal-language theory. Section 34.2 defines the class NP of decision problems whose solutions are verifiable in polynomial time. It also formally poses the $P \neq NP$ question.

Section 34.3 shows we can relate problems via polynomial-time "reductions." It defines NP-completeness and sketches a proof that one problem, called "circuit satisfiability," is NP-complete. Having found one NP-complete problem, we show in Section 34.4 how to prove other problems to be NP-complete much more simply by the methodology of reductions. We illustrate this methodology by showing that two formula-satisfiability problems are NP-complete. With additional reductions, we show in Section 34.5 a variety of other problems to be NP-complete.

34.1 Polynomial time

We begin our study of NP-completeness by formalizing our notion of polynomial-time solvable problems. We generally regard these problems as tractable, but for philosophical, not mathematical, reasons. We can offer three supporting arguments.

First, although we may reasonably regard a problem that requires time $\Theta(n^{100})$ to be intractable, very few practical problems require time on the order of such a high-degree polynomial. The polynomial-time computable problems encountered in practice typically require much less time. Experience has shown that once the first polynomial-time algorithm for a problem has been discovered, more efficient algorithms often follow. Even if the current best algorithm for a problem has a running time of $\Theta(n^{100})$, an algorithm with a much better running time will likely soon be discovered.

Second, for many reasonable models of computation, a problem that can be solved in polynomial time in one model can be solved in polynomial time in another. For example, the class of problems solvable in polynomial time by the serial random-access machine used throughout most of this book is the same as the class of problems solvable in polynomial time on abstract Turing machines.[1] It is also the same as the class of problems solvable in polynomial time on a parallel computer when the number of processors grows polynomially with the input size.

Third, the class of polynomial-time solvable problems has nice closure properties, since polynomials are closed under addition, multiplication, and composition. For example, if the output of one polynomial-time algorithm is fed into the input of another, the composite algorithm is polynomial. Exercise 34.1-5 asks you to show that if an algorithm makes a constant number of calls to polynomial-time subroutines and performs an additional amount of work that also takes polynomial time, then the running time of the composite algorithm is polynomial.

Abstract problems

To understand the class of polynomial-time solvable problems, we must first have a formal notion of what a "problem" is. We define an ***abstract problem*** Q to be a binary relation on a set I of problem ***instances*** and a set S of problem ***solutions***. For example, an instance for SHORTEST-PATH is a triple consisting of a graph and two vertices. A solution is a sequence of vertices in the graph, with perhaps the empty sequence denoting that no path exists. The problem SHORTEST-PATH itself is the relation that associates each instance of a graph and two vertices with a shortest path in the graph that connects the two vertices. Since shortest paths are not necessarily unique, a given problem instance may have more than one solution.

This formulation of an abstract problem is more general than we need for our purposes. As we saw above, the theory of NP-completeness restricts attention to ***decision problems***: those having a yes/no solution. In this case, we can view an abstract decision problem as a function that maps the instance set I to the solution set $\{0, 1\}$. For example, a decision problem related to SHORTEST-PATH is the problem PATH that we saw earlier. If $i = \langle G, u, v, k \rangle$ is an instance of the decision problem PATH, then $\text{PATH}(i) = 1$ (yes) if a shortest path from u to v has at most k edges, and $\text{PATH}(i) = 0$ (no) otherwise. Many abstract problems are not decision problems, but rather ***optimization problems***, which require some value to be minimized or maximized. As we saw above, however, we can usually recast an optimization problem as a decision problem that is no harder.

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[1] See Hopcroft and Ullman [180] or Lewis and Papadimitriou [236] for a thorough treatment of the Turing-machine model.

Encodings

In order for a computer program to solve an abstract problem, we must represent problem instances in a way that the program understands. An ***encoding*** of a set S of abstract objects is a mapping e from S to the set of binary strings.[2] For example, we are all familiar with encoding the natural numbers $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$ as the strings $\{0, 1, 10, 11, 100, \ldots\}$. Using this encoding, $e(17) = 10001$. If you have looked at computer representations of keyboard characters, you probably have seen the ASCII code, where, for example, the encoding of `A` is 1000001. We can encode a compound object as a binary string by combining the representations of its constituent parts. Polygons, graphs, functions, ordered pairs, programs—all can be encoded as binary strings.

Thus, a computer algorithm that "solves" some abstract decision problem actually takes an encoding of a problem instance as input. We call a problem whose instance set is the set of binary strings a ***concrete problem***. We say that an algorithm ***solves*** a concrete problem in time $O(T(n))$ if, when it is provided a problem instance i of length $n = |i|$, the algorithm can produce the solution in $O(T(n))$ time.[3] A concrete problem is ***polynomial-time solvable***, therefore, if there exists an algorithm to solve it in time $O(n^k)$ for some constant k.

We can now formally define the ***complexity class*** **P** as the set of concrete decision problems that are polynomial-time solvable.

We can use encodings to map abstract problems to concrete problems. Given an abstract decision problem Q mapping an instance set I to $\{0, 1\}$, an encoding $e : I \to \{0, 1\}^*$ can induce a related concrete decision problem, which we denote by $e(Q)$.[4] If the solution to an abstract-problem instance $i \in I$ is $Q(i) \in \{0, 1\}$, then the solution to the concrete-problem instance $e(i) \in \{0, 1\}^*$ is also $Q(i)$. As a technicality, some binary strings might represent no meaningful abstract-problem instance. For convenience, we shall assume that any such string maps arbitrarily to 0. Thus, the concrete problem produces the same solutions as the abstract problem on binary-string instances that represent the encodings of abstract-problem instances.

We would like to extend the definition of polynomial-time solvability from concrete problems to abstract problems by using encodings as the bridge, but we would

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[2] The codomain of e need not be *binary* strings; any set of strings over a finite alphabet having at least 2 symbols will do.

[3] We assume that the algorithm's output is separate from its input. Because it takes at least one time step to produce each bit of the output and the algorithm takes $O(T(n))$ time steps, the size of the output is $O(T(n))$.

[4] We denote by $\{0, 1\}^*$ the set of all strings composed of symbols from the set $\{0, 1\}$.

like the definition to be independent of any particular encoding. That is, the efficiency of solving a problem should not depend on how the problem is encoded. Unfortunately, it depends quite heavily on the encoding. For example, suppose that an integer k is to be provided as the sole input to an algorithm, and suppose that the running time of the algorithm is $\Theta(k)$. If the integer k is provided in ***unary***—a string of k 1s—then the running time of the algorithm is $O(n)$ on length-n inputs, which is polynomial time. If we use the more natural binary representation of the integer k, however, then the input length is $n = \lfloor \lg k \rfloor + 1$. In this case, the running time of the algorithm is $\Theta(k) = \Theta(2^n)$, which is exponential in the size of the input. Thus, depending on the encoding, the algorithm runs in either polynomial or superpolynomial time.

How we encode an abstract problem matters quite a bit to how we understand polynomial time. We cannot really talk about solving an abstract problem without first specifying an encoding. Nevertheless, in practice, if we rule out "expensive" encodings such as unary ones, the actual encoding of a problem makes little difference to whether the problem can be solved in polynomial time. For example, representing integers in base 3 instead of binary has no effect on whether a problem is solvable in polynomial time, since we can convert an integer represented in base 3 to an integer represented in base 2 in polynomial time.

We say that a function $f : \{0,1\}^* \to \{0,1\}^*$ is ***polynomial-time computable*** if there exists a polynomial-time algorithm A that, given any input $x \in \{0,1\}^*$, produces as output $f(x)$. For some set I of problem instances, we say that two encodings e_1 and e_2 are ***polynomially related*** if there exist two polynomial-time computable functions f_{12} and f_{21} such that for any $i \in I$, we have $f_{12}(e_1(i)) = e_2(i)$ and $f_{21}(e_2(i)) = e_1(i)$.[5] That is, a polynomial-time algorithm can compute the encoding $e_2(i)$ from the encoding $e_1(i)$, and vice versa. If two encodings e_1 and e_2 of an abstract problem are polynomially related, whether the problem is polynomial-time solvable or not is independent of which encoding we use, as the following lemma shows.

Lemma 34.1

Let Q be an abstract decision problem on an instance set I, and let e_1 and e_2 be polynomially related encodings on I. Then, $e_1(Q) \in \mathrm{P}$ if and only if $e_2(Q) \in \mathrm{P}$.

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[5]Technically, we also require the functions f_{12} and f_{21} to "map noninstances to noninstances." A ***noninstance*** of an encoding e is a string $x \in \{0,1\}^*$ such that there is no instance i for which $e(i) = x$. We require that $f_{12}(x) = y$ for every noninstance x of encoding e_1, where y is some noninstance of e_2, and that $f_{21}(x') = y'$ for every noninstance x' of e_2, where y' is some noninstance of e_1.

Proof We need only prove the forward direction, since the backward direction is symmetric. Suppose, therefore, that $e_1(Q)$ can be solved in time $O(n^k)$ for some constant k. Further, suppose that for any problem instance i, the encoding $e_1(i)$ can be computed from the encoding $e_2(i)$ in time $O(n^c)$ for some constant c, where $n = |e_2(i)|$. To solve problem $e_2(Q)$, on input $e_2(i)$, we first compute $e_1(i)$ and then run the algorithm for $e_1(Q)$ on $e_1(i)$. How long does this take? Converting encodings takes time $O(n^c)$, and therefore $|e_1(i)| = O(n^c)$, since the output of a serial computer cannot be longer than its running time. Solving the problem on $e_1(i)$ takes time $O(|e_1(i)|^k) = O(n^{ck})$, which is polynomial since both c and k are constants. ■

Thus, whether an abstract problem has its instances encoded in binary or base 3 does not affect its "complexity," that is, whether it is polynomial-time solvable or not; but if instances are encoded in unary, its complexity may change. In order to be able to converse in an encoding-independent fashion, we shall generally assume that problem instances are encoded in any reasonable, concise fashion, unless we specifically say otherwise. To be precise, we shall assume that the encoding of an integer is polynomially related to its binary representation, and that the encoding of a finite set is polynomially related to its encoding as a list of its elements, enclosed in braces and separated by commas. (ASCII is one such encoding scheme.) With such a "standard" encoding in hand, we can derive reasonable encodings of other mathematical objects, such as tuples, graphs, and formulas. To denote the standard encoding of an object, we shall enclose the object in angle braces. Thus, $\langle G \rangle$ denotes the standard encoding of a graph G.

As long as we implicitly use an encoding that is polynomially related to this standard encoding, we can talk directly about abstract problems without reference to any particular encoding, knowing that the choice of encoding has no effect on whether the abstract problem is polynomial-time solvable. Henceforth, we shall generally assume that all problem instances are binary strings encoded using the standard encoding, unless we explicitly specify the contrary. We shall also typically neglect the distinction between abstract and concrete problems. You should watch out for problems that arise in practice, however, in which a standard encoding is not obvious and the encoding does make a difference.

A formal-language framework

By focusing on decision problems, we can take advantage of the machinery of formal-language theory. Let's review some definitions from that theory. An ***alphabet*** Σ is a finite set of symbols. A ***language*** L over Σ is any set of strings made up of symbols from Σ. For example, if $\Sigma = \{0, 1\}$, the set $L = \{10, 11, 101, 111, 1011, 1101, 10001, \ldots\}$ is the language of binary represen-

tations of prime numbers. We denote the ***empty string*** by ε, the ***empty language*** by $\emptyset$, and the language of all strings over Σ by Σ^*. For example, if $\Sigma = \{0, 1\}$, then $\Sigma^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\}$ is the set of all binary strings. Every language L over Σ is a subset of Σ^*.

We can perform a variety of operations on languages. Set-theoretic operations, such as ***union*** and ***intersection***, follow directly from the set-theoretic definitions. We define the ***complement*** of L by $\overline{L} = \Sigma^* - L$. The ***concatenation*** $L_1 L_2$ of two languages L_1 and L_2 is the language

$$L = \{x_1 x_2 : x_1 \in L_1 \text{ and } x_2 \in L_2\} .$$

The ***closure*** or ***Kleene star*** of a language L is the language

$$L^* = \{\varepsilon\} \cup L \cup L^2 \cup L^3 \cup \cdots ,$$

where L^k is the language obtained by concatenating L to itself k times.

From the point of view of language theory, the set of instances for any decision problem Q is simply the set Σ^*, where $\Sigma = \{0, 1\}$. Since Q is entirely characterized by those problem instances that produce a 1 (yes) answer, we can view Q as a language L over $\Sigma = \{0, 1\}$, where

$$L = \{x \in \Sigma^* : Q(x) = 1\} .$$

For example, the decision problem PATH has the corresponding language

$$\begin{aligned} \text{PATH} = \{\langle G, u, v, k\rangle : \ & G = (V, E) \text{ is an undirected graph,} \\ & u, v \in V, \\ & k \geq 0 \text{ is an integer, and} \\ & \text{there exists a path from } u \text{ to } v \text{ in } G \\ & \text{consisting of at most } k \text{ edges}\} . \end{aligned}$$

(Where convenient, we shall sometimes use the same name—PATH in this case—to refer to both a decision problem and its corresponding language.)

The formal-language framework allows us to express concisely the relation between decision problems and algorithms that solve them. We say that an algorithm A ***accepts*** a string $x \in \{0, 1\}^*$ if, given input x, the algorithm's output $A(x)$ is 1. The language ***accepted*** by an algorithm A is the set of strings $L = \{x \in \{0, 1\}^* : A(x) = 1\}$, that is, the set of strings that the algorithm accepts. An algorithm A ***rejects*** a string x if $A(x) = 0$.

Even if language L is accepted by an algorithm A, the algorithm will not necessarily reject a string $x \notin L$ provided as input to it. For example, the algorithm may loop forever. A language L is ***decided*** by an algorithm A if every binary string in L is accepted by A and every binary string not in L is rejected by A. A language L is ***accepted in polynomial time*** by an algorithm A if it is accepted by A and if in addition there exists a constant k such that for any length-n string $x \in L$,

algorithm A accepts x in time $O(n^k)$. A language L is ***decided in polynomial time*** by an algorithm A if there exists a constant k such that for any length-n string $x \in \{0,1\}^*$, the algorithm correctly decides whether $x \in L$ in time $O(n^k)$. Thus, to accept a language, an algorithm need only produce an answer when provided a string in L, but to decide a language, it must correctly accept or reject every string in $\{0,1\}^*$.

As an example, the language PATH can be accepted in polynomial time. One polynomial-time accepting algorithm verifies that G encodes an undirected graph, verifies that u and v are vertices in G, uses breadth-first search to compute a shortest path from u to v in G, and then compares the number of edges on the shortest path obtained with k. If G encodes an undirected graph and the path found from u to v has at most k edges, the algorithm outputs 1 and halts. Otherwise, the algorithm runs forever. This algorithm does not decide PATH, however, since it does not explicitly output 0 for instances in which a shortest path has more than k edges. A decision algorithm for PATH must explicitly reject binary strings that do not belong to PATH. For a decision problem such as PATH, such a decision algorithm is easy to design: instead of running forever when there is not a path from u to v with at most k edges, it outputs 0 and halts. (It must also output 0 and halt if the input encoding is faulty.) For other problems, such as Turing's Halting Problem, there exists an accepting algorithm, but no decision algorithm exists.

We can informally define a ***complexity class*** as a set of languages, membership in which is determined by a ***complexity measure***, such as running time, of an algorithm that determines whether a given string x belongs to language L. The actual definition of a complexity class is somewhat more technical.[6]

Using this language-theoretic framework, we can provide an alternative definition of the complexity class P:

$$\mathrm{P} = \{L \subseteq \{0,1\}^* : \text{there exists an algorithm } A \text{ that decides } L \text{ in polynomial time}\} .$$

In fact, P is also the class of languages that can be accepted in polynomial time.

Theorem 34.2
$\mathrm{P} = \{L : L \text{ is accepted by a polynomial-time algorithm}\}$.

Proof Because the class of languages decided by polynomial-time algorithms is a subset of the class of languages accepted by polynomial-time algorithms, we need only show that if L is accepted by a polynomial-time algorithm, it is decided by a polynomial-time algorithm. Let L be the language accepted by some

[6]For more on complexity classes, see the seminal paper by Hartmanis and Stearns [162].

polynomial-time algorithm A. We shall use a classic "simulation" argument to construct another polynomial-time algorithm A' that decides L. Because A accepts L in time $O(n^k)$ for some constant k, there also exists a constant c such that A accepts L in at most cn^k steps. For any input string x, the algorithm A' simulates cn^k steps of A. After simulating cn^k steps, algorithm A' inspects the behavior of A. If A has accepted x, then A' accepts x by outputting a 1. If A has not accepted x, then A' rejects x by outputting a 0. The overhead of A' simulating A does not increase the running time by more than a polynomial factor, and thus A' is a polynomial-time algorithm that decides L. ■

Note that the proof of Theorem 34.2 is nonconstructive. For a given language $L \in \text{P}$, we may not actually know a bound on the running time for the algorithm A that accepts L. Nevertheless, we know that such a bound exists, and therefore, that an algorithm A' exists that can check the bound, even though we may not be able to find the algorithm A' easily.

Exercises

34.1-1

Define the optimization problem LONGEST-PATH-LENGTH as the relation that associates each instance of an undirected graph and two vertices with the number of edges in a longest simple path between the two vertices. Define the decision problem LONGEST-PATH $= \{\langle G, u, v, k\rangle : G = (V, E)$ is an undirected graph, $u, v \in V$, $k \geq 0$ is an integer, and there exists a simple path from u to v in G consisting of at least k edges$\}$. Show that the optimization problem LONGEST-PATH-LENGTH can be solved in polynomial time if and only if LONGEST-PATH $\in$ P.

34.1-2

Give a formal definition for the problem of finding the longest simple cycle in an undirected graph. Give a related decision problem. Give the language corresponding to the decision problem.

34.1-3

Give a formal encoding of directed graphs as binary strings using an adjacency-matrix representation. Do the same using an adjacency-list representation. Argue that the two representations are polynomially related.

34.1-4

Is the dynamic-programming algorithm for the 0-1 knapsack problem that is asked for in Exercise 16.2-2 a polynomial-time algorithm? Explain your answer.

34.1-5
Show that if an algorithm makes at most a constant number of calls to polynomial-time subroutines and performs an additional amount of work that also takes polynomial time, then it runs in polynomial time. Also show that a polynomial number of calls to polynomial-time subroutines may result in an exponential-time algorithm.

34.1-6
Show that the class P, viewed as a set of languages, is closed under union, intersection, concatenation, complement, and Kleene star. That is, if $L_1, L_2 \in \mathrm{P}$, then $L_1 \cup L_2 \in \mathrm{P}$, $L_1 \cap L_2 \in \mathrm{P}$, $L_1 L_2 \in \mathrm{P}$, $\overline{L}_1 \in \mathrm{P}$, and $L_1^* \in \mathrm{P}$.

34.2 Polynomial-time verification

We now look at algorithms that verify membership in languages. For example, suppose that for a given instance $\langle G, u, v, k \rangle$ of the decision problem PATH, we are also given a path p from u to v. We can easily check whether p is a path in G and whether the length of p is at most k, and if so, we can view p as a "certificate" that the instance indeed belongs to PATH. For the decision problem PATH, this certificate doesn't seem to buy us much. After all, PATH belongs to P—in fact, we can solve PATH in linear time—and so verifying membership from a given certificate takes as long as solving the problem from scratch. We shall now examine a problem for which we know of no polynomial-time decision algorithm and yet, given a certificate, verification is easy.

Hamiltonian cycles

The problem of finding a hamiltonian cycle in an undirected graph has been studied for over a hundred years. Formally, a ***hamiltonian cycle*** of an undirected graph $G = (V, E)$ is a simple cycle that contains each vertex in V. A graph that contains a hamiltonian cycle is said to be ***hamiltonian***; otherwise, it is ***nonhamiltonian***. The name honors W. R. Hamilton, who described a mathematical game on the dodecahedron (Figure 34.2(a)) in which one player sticks five pins in any five consecutive vertices and the other player must complete the path to form a cycle

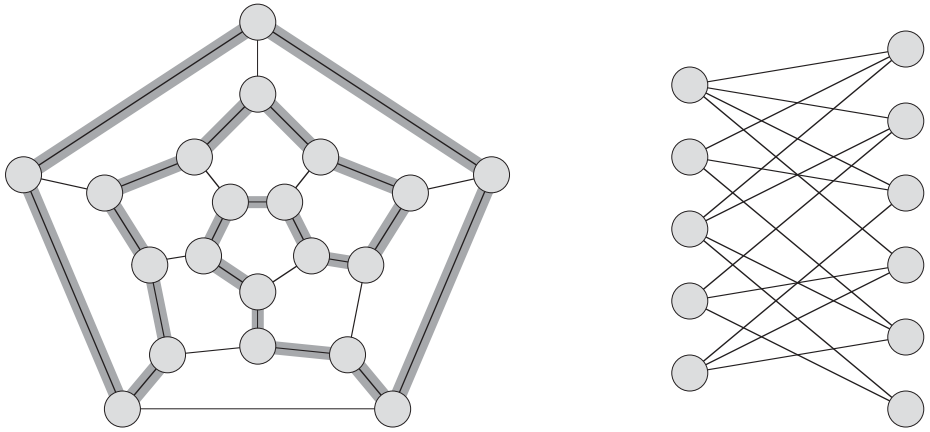

Figure 34.2 **(a)** A graph representing the vertices, edges, and faces of a dodecahedron, with a hamiltonian cycle shown by shaded edges. **(b)** A bipartite graph with an odd number of vertices. Any such graph is nonhamiltonian.

containing all the vertices.[7] The dodecahedron is hamiltonian, and Figure 34.2(a) shows one hamiltonian cycle. Not all graphs are hamiltonian, however. For example, Figure 34.2(b) shows a bipartite graph with an odd number of vertices. Exercise 34.2-2 asks you to show that all such graphs are nonhamiltonian.

We can define the ***hamiltonian-cycle problem***, "Does a graph G have a hamiltonian cycle?" as a formal language:

$$\text{HAM-CYCLE} = \{\langle G \rangle : G \text{ is a hamiltonian graph}\} .$$

How might an algorithm decide the language HAM-CYCLE? Given a problem instance $\langle G \rangle$, one possible decision algorithm lists all permutations of the vertices of G and then checks each permutation to see if it is a hamiltonian path. What is the running time of this algorithm? If we use the "reasonable" encoding of a graph as its adjacency matrix, the number m of vertices in the graph is $\Omega(\sqrt{n})$, where $n = |\langle G \rangle|$ is the length of the encoding of G. There are $m!$ possible permutations

[7] In a letter dated 17 October 1856 to his friend John T. Graves, Hamilton [157, p. 624] wrote, "I have found that some young persons have been much amused by trying a new mathematical game which the Icosion furnishes, one person sticking five pins in any five consecutive points ... and the other player then aiming to insert, which by the theory in this letter can always be done, fifteen other pins, in cyclical succession, so as to cover all the other points, and to end in immediate proximity to the pin wherewith his antagonist had begun."

of the vertices, and therefore the running time is $\Omega(m!) = \Omega(\sqrt{n}\,!) = \Omega(2^{\sqrt{n}})$, which is not $O(n^k)$ for any constant k. Thus, this naive algorithm does not run in polynomial time. In fact, the hamiltonian-cycle problem is NP-complete, as we shall prove in Section 34.5.

Verification algorithms

Consider a slightly easier problem. Suppose that a friend tells you that a given graph G is hamiltonian, and then offers to prove it by giving you the vertices in order along the hamiltonian cycle. It would certainly be easy enough to verify the proof: simply verify that the provided cycle is hamiltonian by checking whether it is a permutation of the vertices of V and whether each of the consecutive edges along the cycle actually exists in the graph. You could certainly implement this verification algorithm to run in $O(n^2)$ time, where n is the length of the encoding of G. Thus, a proof that a hamiltonian cycle exists in a graph can be verified in polynomial time.

We define a ***verification algorithm*** as being a two-argument algorithm A, where one argument is an ordinary input string x and the other is a binary string y called a ***certificate***. A two-argument algorithm A ***verifies*** an input string x if there exists a certificate y such that $A(x, y) = 1$. The ***language verified*** by a verification algorithm A is

$$L = \{x \in \{0,1\}^* : \text{ there exists } y \in \{0,1\}^* \text{ such that } A(x, y) = 1\} .$$

Intuitively, an algorithm A verifies a language L if for any string $x \in L$, there exists a certificate y that A can use to prove that $x \in L$. Moreover, for any string $x \notin L$, there must be no certificate proving that $x \in L$. For example, in the hamiltonian-cycle problem, the certificate is the list of vertices in some hamiltonian cycle. If a graph is hamiltonian, the hamiltonian cycle itself offers enough information to verify this fact. Conversely, if a graph is not hamiltonian, there can be no list of vertices that fools the verification algorithm into believing that the graph is hamiltonian, since the verification algorithm carefully checks the proposed "cycle" to be sure.

The complexity class NP

The ***complexity class*** **NP** is the class of languages that can be verified by a polynomial-time algorithm.[8] More precisely, a language L belongs to NP if and only if there exist a two-input polynomial-time algorithm A and a constant c such that

$$L = \{x \in \{0,1\}^* : \text{ there exists a certificate } y \text{ with } |y| = O(|x|^c) \text{ such that } A(x, y) = 1\} .$$

We say that algorithm A ***verifies*** language L ***in polynomial time***.

From our earlier discussion on the hamiltonian-cycle problem, we now see that HAM-CYCLE $\in$ NP. (It is always nice to know that an important set is nonempty.) Moreover, if $L \in$ P, then $L \in$ NP, since if there is a polynomial-time algorithm to decide L, the algorithm can be easily converted to a two-argument verification algorithm that simply ignores any certificate and accepts exactly those input strings it determines to be in L. Thus, P $\subseteq$ NP.

It is unknown whether P = NP, but most researchers believe that P and NP are not the same class. Intuitively, the class P consists of problems that can be solved quickly. The class NP consists of problems for which a solution can be verified quickly. You may have learned from experience that it is often more difficult to solve a problem from scratch than to verify a clearly presented solution, especially when working under time constraints. Theoretical computer scientists generally believe that this analogy extends to the classes P and NP, and thus that NP includes languages that are not in P.

There is more compelling, though not conclusive, evidence that P $\neq$ NP—the existence of languages that are "NP-complete." We shall study this class in Section 34.3.

Many other fundamental questions beyond the P $\neq$ NP question remain unresolved. Figure 34.3 shows some possible scenarios. Despite much work by many researchers, no one even knows whether the class NP is closed under complement. That is, does $L \in$ NP imply $\overline{L} \in$ NP? We can define the ***complexity class*** **co-NP** as the set of languages L such that $\overline{L} \in$ NP. We can restate the question of whether NP is closed under complement as whether NP = co-NP. Since P is closed under complement (Exercise 34.1-6), it follows from Exercise 34.2-9 that P $\subseteq$ NP $\cap$ co-NP. Once again, however, no one knows whether P = NP $\cap$ co-NP or whether there is some language in NP $\cap$ co-NP $-$ P.

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[8] The name "NP" stands for "nondeterministic polynomial time." The class NP was originally studied in the context of nondeterminism, but this book uses the somewhat simpler yet equivalent notion of verification. Hopcroft and Ullman [180] give a good presentation of NP-completeness in terms of nondeterministic models of computation.

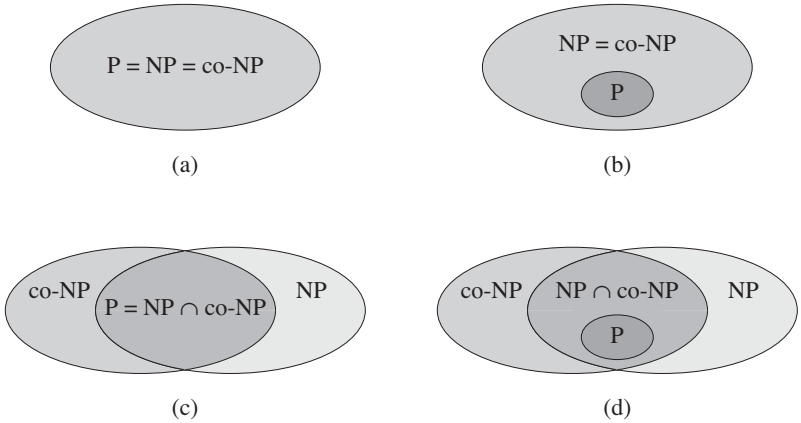


Figure 34.3 Four possibilities for relationships among complexity classes. In each diagram, one region enclosing another indicates a proper-subset relation. **(a)** P = NP = co-NP. Most researchers regard this possibility as the most unlikely. **(b)** If NP is closed under complement, then NP = co-NP, but it need not be the case that P = NP. **(c)** P = NP∩co-NP, but NP is not closed under complement. **(d)** NP ≠ co-NP and P ≠ NP ∩ co-NP. Most researchers regard this possibility as the most likely.

Thus, our understanding of the precise relationship between P and NP is woefully incomplete. Nevertheless, even though we might not be able to prove that a particular problem is intractable, if we can prove that it is NP-complete, then we have gained valuable information about it.

Exercises

34.2-1

Consider the language GRAPH-ISOMORPHISM $= \{\langle G_1, G_2\rangle : G_1 \text{ and } G_2 \text{ are isomorphic graphs}\}$. Prove that GRAPH-ISOMORPHISM $\in$ NP by describing a polynomial-time algorithm to verify the language.

34.2-2

Prove that if G is an undirected bipartite graph with an odd number of vertices, then G is nonhamiltonian.

34.2-3

Show that if HAM-CYCLE $\in$ P, then the problem of listing the vertices of a hamiltonian cycle, in order, is polynomial-time solvable.

34.2-4
Prove that the class NP of languages is closed under union, intersection, concatenation, and Kleene star. Discuss the closure of NP under complement.

34.2-5
Show that any language in NP can be decided by an algorithm running in time $2^{O(n^k)}$ for some constant k.

34.2-6
A ***hamiltonian path*** in a graph is a simple path that visits every vertex exactly once. Show that the language HAM-PATH $= \{\langle G, u, v\rangle$: there is a hamiltonian path from u to v in graph $G\}$ belongs to NP.

34.2-7
Show that the hamiltonian-path problem from Exercise 34.2-6 can be solved in polynomial time on directed acyclic graphs. Give an efficient algorithm for the problem.

34.2-8
Let ϕ be a boolean formula constructed from the boolean input variables $x_1, x_2, \ldots, x_k$, negations ($\neg$), ANDs ($\wedge$), ORs ($\vee$), and parentheses. The formula ϕ is a ***tautology*** if it evaluates to 1 for every assignment of 1 and 0 to the input variables. Define TAUTOLOGY as the language of boolean formulas that are tautologies. Show that TAUTOLOGY $\in$ co-NP.

34.2-9
Prove that P $\subseteq$ co-NP.

34.2-10
Prove that if NP $\neq$ co-NP, then P $\neq$ NP.

34.2-11
Let G be a connected, undirected graph with at least 3 vertices, and let G^3 be the graph obtained by connecting all pairs of vertices that are connected by a path in G of length at most 3. Prove that G^3 is hamiltonian. (*Hint:* Construct a spanning tree for G, and use an inductive argument.)

34.3 NP-completeness and reducibility

Perhaps the most compelling reason why theoretical computer scientists believe that $\mathrm{P} \neq \mathrm{NP}$ comes from the existence of the class of "NP-complete" problems. This class has the intriguing property that if *any* NP-complete problem can be solved in polynomial time, then *every* problem in NP has a polynomial-time solution, that is, $\mathrm{P} = \mathrm{NP}$. Despite years of study, though, no polynomial-time algorithm has ever been discovered for any NP-complete problem.

The language HAM-CYCLE is one NP-complete problem. If we could decide HAM-CYCLE in polynomial time, then we could solve every problem in NP in polynomial time. In fact, if $\mathrm{NP} - \mathrm{P}$ should turn out to be nonempty, we could say with certainty that HAM-CYCLE $\in \mathrm{NP} - \mathrm{P}$.

The NP-complete languages are, in a sense, the "hardest" languages in NP. In this section, we shall show how to compare the relative "hardness" of languages using a precise notion called "polynomial-time reducibility." Then we formally define the NP-complete languages, and we finish by sketching a proof that one such language, called CIRCUIT-SAT, is NP-complete. In Sections 34.4 and 34.5, we shall use the notion of reducibility to show that many other problems are NP-complete.

Reducibility

Intuitively, a problem Q can be reduced to another problem Q' if any instance of Q can be "easily rephrased" as an instance of Q', the solution to which provides a solution to the instance of Q. For example, the problem of solving linear equations in an indeterminate x reduces to the problem of solving quadratic equations. Given an instance $ax + b = 0$, we transform it to $0x^2 + ax + b = 0$, whose solution provides a solution to $ax + b = 0$. Thus, if a problem Q reduces to another problem Q', then Q is, in a sense, "no harder to solve" than Q'.

Returning to our formal-language framework for decision problems, we say that a language L_1 is ***polynomial-time reducible*** to a language L_2, written $L_1 \leq_{\mathrm{P}} L_2$, if there exists a polynomial-time computable function $f : \{0,1\}^* \to \{0,1\}^*$ such that for all $x \in \{0,1\}^*$,

$$x \in L_1 \text{ if and only if } f(x) \in L_2 \,. \tag{34.1}$$

We call the function f the ***reduction function***, and a polynomial-time algorithm F that computes f is a ***reduction algorithm***.

Figure 34.4 illustrates the idea of a polynomial-time reduction from a language L_1 to another language L_2. Each language is a subset of $\{0,1\}^*$. The reduction function f provides a polynomial-time mapping such that if $x \in L_1$,

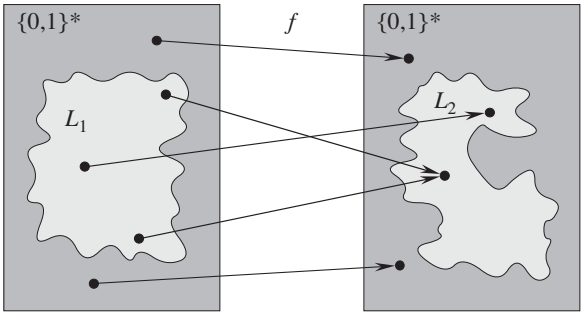


Figure 34.4 An illustration of a polynomial-time reduction from a language L_1 to a language L_2 via a reduction function f. For any input $x \in \{0,1\}^*$, the question of whether $x \in L_1$ has the same answer as the question of whether $f(x) \in L_2$.

then $f(x) \in L_2$. Moreover, if $x \notin L_1$, then $f(x) \notin L_2$. Thus, the reduction function maps any instance x of the decision problem represented by the language L_1 to an instance $f(x)$ of the problem represented by L_2. Providing an answer to whether $f(x) \in L_2$ directly provides the answer to whether $x \in L_1$.

Polynomial-time reductions give us a powerful tool for proving that various languages belong to P.

Lemma 34.3
If $L_1, L_2 \subseteq \{0,1\}^*$ are languages such that $L_1 \leq_P L_2$, then $L_2 \in$ P implies $L_1 \in$ P.

Proof Let A_2 be a polynomial-time algorithm that decides L_2, and let F be a polynomial-time reduction algorithm that computes the reduction function f. We shall construct a polynomial-time algorithm A_1 that decides L_1.

Figure 34.5 illustrates how we construct A_1. For a given input $x \in \{0,1\}^*$, algorithm A_1 uses F to transform x into $f(x)$, and then it uses A_2 to test whether $f(x) \in L_2$. Algorithm A_1 takes the output from algorithm A_2 and produces that answer as its own output.

The correctness of A_1 follows from condition (34.1). The algorithm runs in polynomial time, since both F and A_2 run in polynomial time (see Exercise 34.1-5). ■

NP-completeness

Polynomial-time reductions provide a formal means for showing that one problem is at least as hard as another, to within a polynomial-time factor. That is, if $L_1 \leq_P L_2$, then L_1 is not more than a polynomial factor harder than L_2, which is

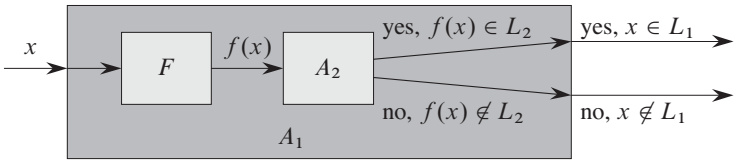


Figure 34.5 The proof of Lemma 34.3. The algorithm F is a reduction algorithm that computes the reduction function f from L_1 to L_2 in polynomial time, and A_2 is a polynomial-time algorithm that decides L_2. Algorithm A_1 decides whether $x \in L_1$ by using F to transform any input x into $f(x)$ and then using A_2 to decide whether $f(x) \in L_2$.

why the "less than or equal to" notation for reduction is mnemonic. We can now define the set of NP-complete languages, which are the hardest problems in NP.

A language $L \subseteq \{0,1\}^*$ is ***NP-complete*** if

1. $L \in$ NP, and
2. $L' \leq_P L$ for every $L' \in$ NP.

If a language L satisfies property 2, but not necessarily property 1, we say that L is ***NP-hard***. We also define NPC to be the class of NP-complete languages.

As the following theorem shows, NP-completeness is at the crux of deciding whether P is in fact equal to NP.

Theorem 34.4
If any NP-complete problem is polynomial-time solvable, then P = NP. Equivalently, if any problem in NP is not polynomial-time solvable, then no NP-complete problem is polynomial-time solvable.

Proof Suppose that $L \in$ P and also that $L \in$ NPC. For any $L' \in$ NP, we have $L' \leq_P L$ by property 2 of the definition of NP-completeness. Thus, by Lemma 34.3, we also have that $L' \in$ P, which proves the first statement of the theorem.

To prove the second statement, note that it is the contrapositive of the first statement. ■

It is for this reason that research into the P $\neq$ NP question centers around the NP-complete problems. Most theoretical computer scientists believe that P $\neq$ NP, which leads to the relationships among P, NP, and NPC shown in Figure 34.6. But, for all we know, someone may yet come up with a polynomial-time algorithm for an NP-complete problem, thus proving that P = NP. Nevertheless, since no polynomial-time algorithm for any NP-complete problem has yet been discov-

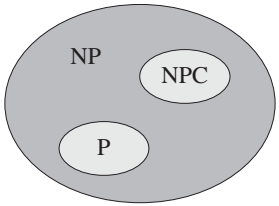


Figure 34.6 How most theoretical computer scientists view the relationships among P, NP, and NPC. Both P and NPC are wholly contained within NP, and $\text{P} \cap \text{NPC} = \emptyset$.

ered, a proof that a problem is NP-complete provides excellent evidence that it is intractable.

Circuit satisfiability

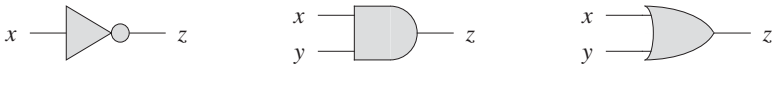
We have defined the notion of an NP-complete problem, but up to this point, we have not actually proved that any problem is NP-complete. Once we prove that at least one problem is NP-complete, we can use polynomial-time reducibility as a tool to prove other problems to be NP-complete. Thus, we now focus on demonstrating the existence of an NP-complete problem: the circuit-satisfiability problem.

Unfortunately, the formal proof that the circuit-satisfiability problem is NP-complete requires technical detail beyond the scope of this text. Instead, we shall informally describe a proof that relies on a basic understanding of boolean combinational circuits.

Boolean combinational circuits are built from boolean combinational elements that are interconnected by wires. A ***boolean combinational element*** is any circuit element that has a constant number of boolean inputs and outputs and that performs a well-defined function. Boolean values are drawn from the set $\{0, 1\}$, where 0 represents FALSE and 1 represents TRUE.

The boolean combinational elements that we use in the circuit-satisfiability problem compute simple boolean functions, and they are known as ***logic gates***. Figure 34.7 shows the three basic logic gates that we use in the circuit-satisfiability problem: the ***NOT gate*** (or ***inverter***), the ***AND gate***, and the ***OR gate***. The NOT gate takes a single binary ***input*** x, whose value is either 0 or 1, and produces a binary ***output*** z whose value is opposite that of the input value. Each of the other two gates takes two binary inputs x and y and produces a single binary output z.

We can describe the operation of each gate, and of any boolean combinational element, by a ***truth table***, shown under each gate in Figure 34.7. A truth table gives the outputs of the combinational element for each possible setting of the inputs. For



x	$\neg x$
0	1
1	0

(a)

x	y	$x \wedge y$
0	0	0
0	1	0
1	0	0
1	1	1

(b)

x	y	$x \vee y$
0	0	0
0	1	1
1	0	1
1	1	1

(c)

Figure 34.7 Three basic logic gates, with binary inputs and outputs. Under each gate is the truth table that describes the gate's operation. **(a)** The NOT gate. **(b)** The AND gate. **(c)** The OR gate.

example, the truth table for the OR gate tells us that when the inputs are $x = 0$ and $y = 1$, the output value is $z = 1$. We use the symbols $\neg$ to denote the NOT function, $\wedge$ to denote the AND function, and $\vee$ to denote the OR function. Thus, for example, $0 \vee 1 = 1$.

We can generalize AND and OR gates to take more than two inputs. An AND gate's output is 1 if all of its inputs are 1, and its output is 0 otherwise. An OR gate's output is 1 if any of its inputs are 1, and its output is 0 otherwise.

A ***boolean combinational circuit*** consists of one or more boolean combinational elements interconnected by ***wires***. A wire can connect the output of one element to the input of another, thereby providing the output value of the first element as an input value of the second. Figure 34.8 shows two similar boolean combinational circuits, differing in only one gate. Part (a) of the figure also shows the values on the individual wires, given the input $\langle x_1 = 1, x_2 = 1, x_3 = 0 \rangle$. Although a single wire may have no more than one combinational-element output connected to it, it can feed several element inputs. The number of element inputs fed by a wire is called the ***fan-out*** of the wire. If no element output is connected to a wire, the wire is a ***circuit input***, accepting input values from an external source. If no element input is connected to a wire, the wire is a ***circuit output***, providing the results of the circuit's computation to the outside world. (An internal wire can also fan out to a circuit output.) For the purpose of defining the circuit-satisfiability problem, we limit the number of circuit outputs to 1, though in actual hardware design, a boolean combinational circuit may have multiple outputs.

Boolean combinational circuits contain no cycles. In other words, suppose we create a directed graph $G = (V, E)$ with one vertex for each combinational element and with k directed edges for each wire whose fan-out is k; the graph contains a directed edge (u, v) if a wire connects the output of element u to an input of element v. Then G must be acyclic.

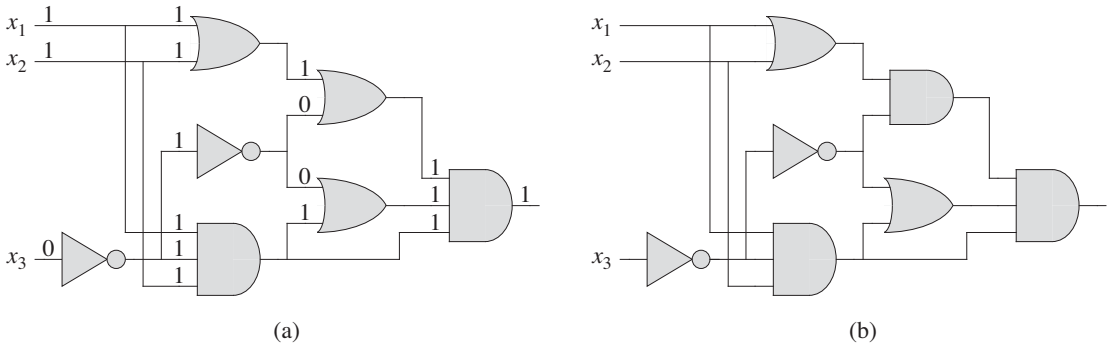


Figure 34.8 Two instances of the circuit-satisfiability problem. **(a)** The assignment $\langle x_1 = 1, x_2 = 1, x_3 = 0\rangle$ to the inputs of this circuit causes the output of the circuit to be 1. The circuit is therefore satisfiable. **(b)** No assignment to the inputs of this circuit can cause the output of the circuit to be 1. The circuit is therefore unsatisfiable.

A ***truth assignment*** for a boolean combinational circuit is a set of boolean input values. We say that a one-output boolean combinational circuit is ***satisfiable*** if it has a ***satisfying assignment***: a truth assignment that causes the output of the circuit to be 1. For example, the circuit in Figure 34.8(a) has the satisfying assignment $\langle x_1 = 1, x_2 = 1, x_3 = 0\rangle$, and so it is satisfiable. As Exercise 34.3-1 asks you to show, no assignment of values to x_1, x_2, and x_3 causes the circuit in Figure 34.8(b) to produce a 1 output; it always produces 0, and so it is unsatisfiable.

The ***circuit-satisfiability problem*** is, "Given a boolean combinational circuit composed of AND, OR, and NOT gates, is it satisfiable?" In order to pose this question formally, however, we must agree on a standard encoding for circuits. The ***size*** of a boolean combinational circuit is the number of boolean combinational elements plus the number of wires in the circuit. We could devise a graphlike encoding that maps any given circuit C into a binary string $\langle C\rangle$ whose length is polynomial in the size of the circuit itself. As a formal language, we can therefore define

$$\text{CIRCUIT-SAT} = \{\langle C\rangle : C \text{ is a satisfiable boolean combinational circuit}\} .$$

The circuit-satisfiability problem arises in the area of computer-aided hardware optimization. If a subcircuit always produces 0, that subcircuit is unnecessary; the designer can replace it by a simpler subcircuit that omits all logic gates and provides the constant 0 value as its output. You can see why we would like to have a polynomial-time algorithm for this problem.

Given a circuit C, we might attempt to determine whether it is satisfiable by simply checking all possible assignments to the inputs. Unfortunately, if the circuit has k inputs, then we would have to check up to 2^k possible assignments. When

the size of C is polynomial in k, checking each one takes $\Omega(2^k)$ time, which is superpolynomial in the size of the circuit.[9] In fact, as we have claimed, there is strong evidence that no polynomial-time algorithm exists that solves the circuit-satisfiability problem because circuit satisfiability is NP-complete. We break the proof of this fact into two parts, based on the two parts of the definition of NP-completeness.

Lemma 34.5
The circuit-satisfiability problem belongs to the class NP.

Proof We shall provide a two-input, polynomial-time algorithm A that can verify CIRCUIT-SAT. One of the inputs to A is (a standard encoding of) a boolean combinational circuit C. The other input is a certificate corresponding to an assignment of boolean values to the wires in C. (See Exercise 34.3-4 for a smaller certificate.)

We construct the algorithm A as follows. For each logic gate in the circuit, it checks that the value provided by the certificate on the output wire is correctly computed as a function of the values on the input wires. Then, if the output of the entire circuit is 1, the algorithm outputs 1, since the values assigned to the inputs of C provide a satisfying assignment. Otherwise, A outputs 0.

Whenever a satisfiable circuit C is input to algorithm A, there exists a certificate whose length is polynomial in the size of C and that causes A to output a 1. Whenever an unsatisfiable circuit is input, no certificate can fool A into believing that the circuit is satisfiable. Algorithm A runs in polynomial time: with a good implementation, linear time suffices. Thus, we can verify CIRCUIT-SAT in polynomial time, and CIRCUIT-SAT $\in$ NP. ■

The second part of proving that CIRCUIT-SAT is NP-complete is to show that the language is NP-hard. That is, we must show that every language in NP is polynomial-time reducible to CIRCUIT-SAT. The actual proof of this fact is full of technical intricacies, and so we shall settle for a sketch of the proof based on some understanding of the workings of computer hardware.

A computer program is stored in the computer memory as a sequence of instructions. A typical instruction encodes an operation to be performed, addresses of operands in memory, and an address where the result is to be stored. A special memory location, called the ***program counter***, keeps track of which instruc-

[9] On the other hand, if the size of the circuit C is $\Theta(2^k)$, then an algorithm whose running time is $O(2^k)$ has a running time that is polynomial in the circuit size. Even if P $\neq$ NP, this situation would not contradict the NP-completeness of the problem; the existence of a polynomial-time algorithm for a special case does not imply that there is a polynomial-time algorithm for all cases.

tion is to be executed next. The program counter automatically increments upon fetching each instruction, thereby causing the computer to execute instructions sequentially. The execution of an instruction can cause a value to be written to the program counter, however, which alters the normal sequential execution and allows the computer to loop and perform conditional branches.

At any point during the execution of a program, the computer's memory holds the entire state of the computation. (We take the memory to include the program itself, the program counter, working storage, and any of the various bits of state that a computer maintains for bookkeeping.) We call any particular state of computer memory a ***configuration***. We can view the execution of an instruction as mapping one configuration to another. The computer hardware that accomplishes this mapping can be implemented as a boolean combinational circuit, which we denote by M in the proof of the following lemma.

Lemma 34.6
The circuit-satisfiability problem is NP-hard.

Proof Let L be any language in NP. We shall describe a polynomial-time algorithm F computing a reduction function f that maps every binary string x to a circuit $C = f(x)$ such that $x \in L$ if and only if $C \in$ CIRCUIT-SAT.

Since $L \in$ NP, there must exist an algorithm A that verifies L in polynomial time. The algorithm F that we shall construct uses the two-input algorithm A to compute the reduction function f.

Let $T(n)$ denote the worst-case running time of algorithm A on length-n input strings, and let $k \geq 1$ be a constant such that $T(n) = O(n^k)$ and the length of the certificate is $O(n^k)$. (The running time of A is actually a polynomial in the total input size, which includes both an input string and a certificate, but since the length of the certificate is polynomial in the length n of the input string, the running time is polynomial in n.)

The basic idea of the proof is to represent the computation of A as a sequence of configurations. As Figure 34.9 illustrates, we can break each configuration into parts consisting of the program for A, the program counter and auxiliary machine state, the input x, the certificate y, and working storage. The combinational circuit M, which implements the computer hardware, maps each configuration c_i to the next configuration c_{i+1}, starting from the initial configuration c_0. Algorithm A writes its output—0 or 1—to some designated location by the time it finishes executing, and if we assume that thereafter A halts, the value never changes. Thus, if the algorithm runs for at most $T(n)$ steps, the output appears as one of the bits in $c_{T(n)}$.

The reduction algorithm F constructs a single combinational circuit that computes all configurations produced by a given initial configuration. The idea is to

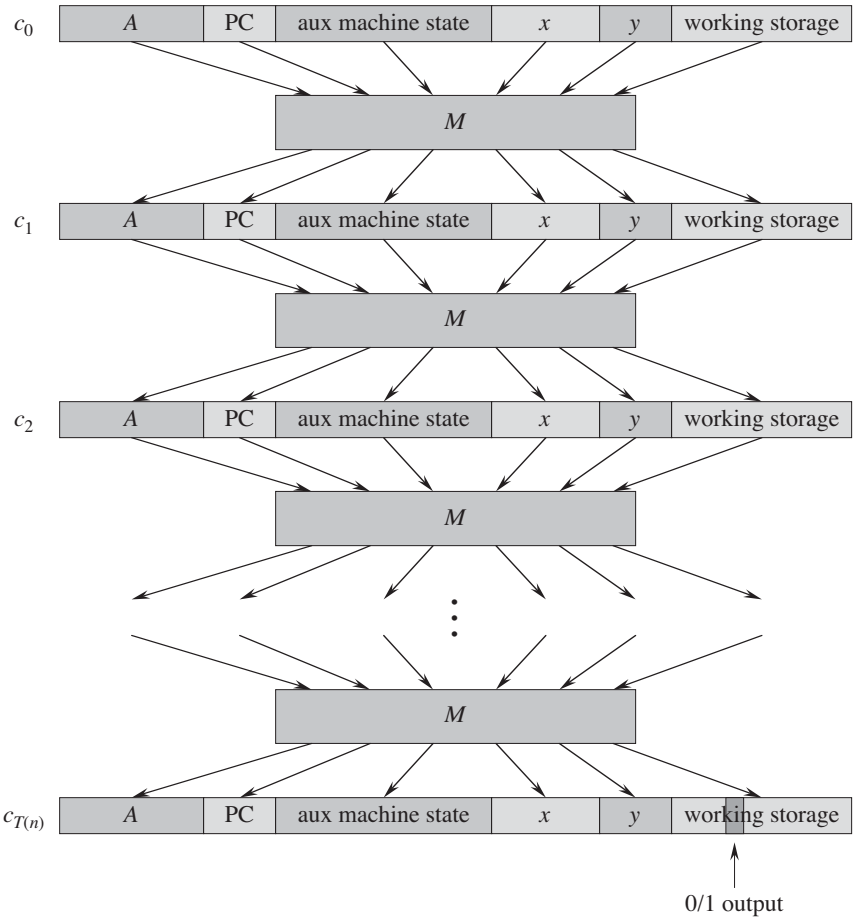


Figure 34.9 The sequence of configurations produced by an algorithm A running on an input x and certificate y. Each configuration represents the state of the computer for one step of the computation and, besides A, x, and y, includes the program counter (PC), auxiliary machine state, and working storage. Except for the certificate y, the initial configuration c_0 is constant. A boolean combinational circuit M maps each configuration to the next configuration. The output is a distinguished bit in the working storage.

paste together $T(n)$ copies of the circuit M. The output of the ith circuit, which produces configuration c_i, feeds directly into the input of the $(i+1)$st circuit. Thus, the configurations, rather than being stored in the computer's memory, simply reside as values on the wires connecting copies of M.

Recall what the polynomial-time reduction algorithm F must do. Given an input x, it must compute a circuit $C = f(x)$ that is satisfiable if and only if there exists a certificate y such that $A(x, y) = 1$. When F obtains an input x, it first computes $n = |x|$ and constructs a combinational circuit C' consisting of $T(n)$ copies of M. The input to C' is an initial configuration corresponding to a computation on $A(x, y)$, and the output is the configuration $c_{T(n)}$.

Algorithm F modifies circuit C' slightly to construct the circuit $C = f(x)$. First, it wires the inputs to C' corresponding to the program for A, the initial program counter, the input x, and the initial state of memory directly to these known values. Thus, the only remaining inputs to the circuit correspond to the certificate y. Second, it ignores all outputs from C', except for the one bit of $c_{T(n)}$ corresponding to the output of A. This circuit C, so constructed, computes $C(y) = A(x, y)$ for any input y of length $O(n^k)$. The reduction algorithm F, when provided an input string x, computes such a circuit C and outputs it.

We need to prove two properties. First, we must show that F correctly computes a reduction function f. That is, we must show that C is satisfiable if and only if there exists a certificate y such that $A(x, y) = 1$. Second, we must show that F runs in polynomial time.

To show that F correctly computes a reduction function, let us suppose that there exists a certificate y of length $O(n^k)$ such that $A(x, y) = 1$. Then, if we apply the bits of y to the inputs of C, the output of C is $C(y) = A(x, y) = 1$. Thus, if a certificate exists, then C is satisfiable. For the other direction, suppose that C is satisfiable. Hence, there exists an input y to C such that $C(y) = 1$, from which we conclude that $A(x, y) = 1$. Thus, F correctly computes a reduction function.

To complete the proof sketch, we need only show that F runs in time polynomial in $n = |x|$. The first observation we make is that the number of bits required to represent a configuration is polynomial in n. The program for A itself has constant size, independent of the length of its input x. The length of the input x is n, and the length of the certificate y is $O(n^k)$. Since the algorithm runs for at most $O(n^k)$ steps, the amount of working storage required by A is polynomial in n as well. (We assume that this memory is contiguous; Exercise 34.3-5 asks you to extend the argument to the situation in which the locations accessed by A are scattered across a much larger region of memory and the particular pattern of scattering can differ for each input x.)

The combinational circuit M implementing the computer hardware has size polynomial in the length of a configuration, which is $O(n^k)$; hence, the size of M is polynomial in n. (Most of this circuitry implements the logic of the memory

system.) The circuit C consists of at most $t = O(n^k)$ copies of M, and hence it has size polynomial in n. The reduction algorithm F can construct C from x in polynomial time, since each step of the construction takes polynomial time. ■

The language CIRCUIT-SAT is therefore at least as hard as any language in NP, and since it belongs to NP, it is NP-complete.

Theorem 34.7
The circuit-satisfiability problem is NP-complete.

Proof Immediate from Lemmas 34.5 and 34.6 and from the definition of NP-completeness. ■

Exercises

34.3-1
Verify that the circuit in Figure 34.8(b) is unsatisfiable.

34.3-2
Show that the $\leq_P$ relation is a transitive relation on languages. That is, show that if $L_1 \leq_P L_2$ and $L_2 \leq_P L_3$, then $L_1 \leq_P L_3$.

34.3-3
Prove that $L \leq_P \overline{L}$ if and only if $\overline{L} \leq_P L$.

34.3-4
Show that we could have used a satisfying assignment as a certificate in an alternative proof of Lemma 34.5. Which certificate makes for an easier proof?

34.3-5
The proof of Lemma 34.6 assumes that the working storage for algorithm A occupies a contiguous region of polynomial size. Where in the proof do we exploit this assumption? Argue that this assumption does not involve any loss of generality.

34.3-6
A language L is ***complete*** for a language class C with respect to polynomial-time reductions if $L \in C$ and $L' \leq_P L$ for all $L' \in C$. Show that $\emptyset$ and $\{0,1\}^*$ are the only languages in P that are not complete for P with respect to polynomial-time reductions.

34.3-7
Show that, with respect to polynomial-time reductions (see Exercise 34.3-6), L is complete for NP if and only if $\overline{L}$ is complete for co-NP.

34.3-8
The reduction algorithm F in the proof of Lemma 34.6 constructs the circuit $C = f(x)$ based on knowledge of x, A, and k. Professor Sartre observes that the string x is input to F, but only the existence of A, k, and the constant factor implicit in the $O(n^k)$ running time is known to F (since the language L belongs to NP), not their actual values. Thus, the professor concludes that F can't possibly construct the circuit C and that the language CIRCUIT-SAT is not necessarily NP-hard. Explain the flaw in the professor's reasoning.

34.4 NP-completeness proofs

We proved that the circuit-satisfiability problem is NP-complete by a direct proof that $L \leq_P$ CIRCUIT-SAT for every language $L \in$ NP. In this section, we shall show how to prove that languages are NP-complete without directly reducing *every* language in NP to the given language. We shall illustrate this methodology by proving that various formula-satisfiability problems are NP-complete. Section 34.5 provides many more examples of the methodology.

The following lemma is the basis of our method for showing that a language is NP-complete.

Lemma 34.8
If L is a language such that $L' \leq_P L$ for some $L' \in$ NPC, then L is NP-hard. If, in addition, $L \in$ NP, then $L \in$ NPC.

Proof Since L' is NP-complete, for all $L'' \in$ NP, we have $L'' \leq_P L'$. By supposition, $L' \leq_P L$, and thus by transitivity (Exercise 34.3-2), we have $L'' \leq_P L$, which shows that L is NP-hard. If $L \in$ NP, we also have $L \in$ NPC. ■

In other words, by reducing a known NP-complete language L' to L, we implicitly reduce every language in NP to L. Thus, Lemma 34.8 gives us a method for proving that a language L is NP-complete:

1. Prove $L \in$ NP.
2. Select a known NP-complete language L'.

3. Describe an algorithm that computes a function f mapping every instance $x \in \{0,1\}^*$ of L' to an instance $f(x)$ of L.
4. Prove that the function f satisfies $x \in L'$ if and only if $f(x) \in L$ for all $x \in \{0,1\}^*$.
5. Prove that the algorithm computing f runs in polynomial time.

(Steps 2–5 show that L is NP-hard.) This methodology of reducing from a single known NP-complete language is far simpler than the more complicated process of showing directly how to reduce from every language in NP. Proving CIRCUIT-SAT $\in$ NPC has given us a "foot in the door." Because we know that the circuit-satisfiability problem is NP-complete, we now can prove much more easily that other problems are NP-complete. Moreover, as we develop a catalog of known NP-complete problems, we will have more and more choices for languages from which to reduce.

Formula satisfiability

We illustrate the reduction methodology by giving an NP-completeness proof for the problem of determining whether a boolean formula, not a circuit, is satisfiable. This problem has the historical honor of being the first problem ever shown to be NP-complete.

We formulate the ***(formula) satisfiability*** problem in terms of the language SAT as follows. An instance of SAT is a boolean formula ϕ composed of

1. n boolean variables: $x_1, x_2, \ldots, x_n$;
2. m boolean connectives: any boolean function with one or two inputs and one output, such as $\wedge$ (AND), $\vee$ (OR), $\neg$ (NOT), $\rightarrow$ (implication), $\leftrightarrow$ (if and only if); and
3. parentheses. (Without loss of generality, we assume that there are no redundant parentheses, i.e., a formula contains at most one pair of parentheses per boolean connective.)

We can easily encode a boolean formula ϕ in a length that is polynomial in $n + m$. As in boolean combinational circuits, a ***truth assignment*** for a boolean formula ϕ is a set of values for the variables of ϕ, and a ***satisfying assignment*** is a truth assignment that causes it to evaluate to 1. A formula with a satisfying assignment is a ***satisfiable*** formula. The satisfiability problem asks whether a given boolean formula is satisfiable; in formal-language terms,

$$\text{SAT} = \{\langle\phi\rangle : \phi \text{ is a satisfiable boolean formula}\} .$$

As an example, the formula

$$\phi = ((x_1 \to x_2) \vee \neg((\neg x_1 \leftrightarrow x_3) \vee x_4)) \wedge \neg x_2$$

has the satisfying assignment $\langle x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 1\rangle$, since

$$\begin{aligned} \phi &= ((0 \to 0) \vee \neg((\neg 0 \leftrightarrow 1) \vee 1)) \wedge \neg 0 \\ &= (1 \vee \neg(1 \vee 1)) \wedge 1 \\ &= (1 \vee 0) \wedge 1 \\ &= 1 \,, \end{aligned} \tag{34.2}$$

and thus this formula ϕ belongs to SAT.

The naive algorithm to determine whether an arbitrary boolean formula is satisfiable does not run in polynomial time. A formula with n variables has 2^n possible assignments. If the length of $\langle\phi\rangle$ is polynomial in n, then checking every assignment requires $\Omega(2^n)$ time, which is superpolynomial in the length of $\langle\phi\rangle$. As the following theorem shows, a polynomial-time algorithm is unlikely to exist.

Theorem 34.9
Satisfiability of boolean formulas is NP-complete.

Proof We start by arguing that SAT $\in$ NP. Then we prove that SAT is NP-hard by showing that CIRCUIT-SAT $\leq_P$ SAT; by Lemma 34.8, this will prove the theorem.

To show that SAT belongs to NP, we show that a certificate consisting of a satisfying assignment for an input formula ϕ can be verified in polynomial time. The verifying algorithm simply replaces each variable in the formula with its corresponding value and then evaluates the expression, much as we did in equation (34.2) above. This task is easy to do in polynomial time. If the expression evaluates to 1, then the algorithm has verified that the formula is satisfiable. Thus, the first condition of Lemma 34.8 for NP-completeness holds.

To prove that SAT is NP-hard, we show that CIRCUIT-SAT $\leq_P$ SAT. In other words, we need to show how to reduce any instance of circuit satisfiability to an instance of formula satisfiability in polynomial time. We can use induction to express any boolean combinational circuit as a boolean formula. We simply look at the gate that produces the circuit output and inductively express each of the gate's inputs as formulas. We then obtain the formula for the circuit by writing an expression that applies the gate's function to its inputs' formulas.

Unfortunately, this straightforward method does not amount to a polynomial-time reduction. As Exercise 34.4-1 asks you to show, shared subformulas—which arise from gates whose output wires have fan-out of 2 or more—can cause the size of the generated formula to grow exponentially. Thus, the reduction algorithm must be somewhat more clever.

Figure 34.10 illustrates how we overcome this problem, using as an example the circuit from Figure 34.8(a). For each wire x_i in the circuit C, the formula ϕ

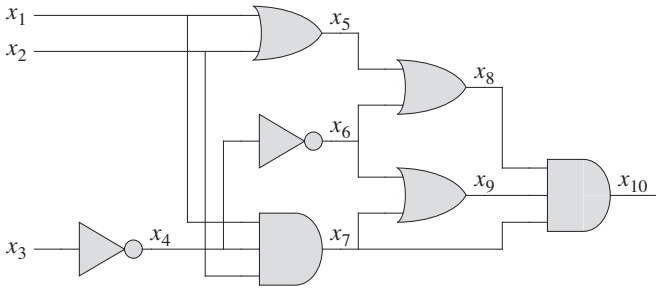


Figure 34.10 Reducing circuit satisfiability to formula satisfiability. The formula produced by the reduction algorithm has a variable for each wire in the circuit.

has a variable x_i. We can now express how each gate operates as a small formula involving the variables of its incident wires. For example, the operation of the output AND gate is $x_{10} \leftrightarrow (x_7 \land x_8 \land x_9)$. We call each of these small formulas a ***clause***.

The formula ϕ produced by the reduction algorithm is the AND of the circuit-output variable with the conjunction of clauses describing the operation of each gate. For the circuit in the figure, the formula is

$$\begin{aligned}\phi \;=\; & x_{10} \land (x_4 \leftrightarrow \neg x_3) \\ & \land (x_5 \leftrightarrow (x_1 \lor x_2)) \\ & \land (x_6 \leftrightarrow \neg x_4) \\ & \land (x_7 \leftrightarrow (x_1 \land x_2 \land x_4)) \\ & \land (x_8 \leftrightarrow (x_5 \lor x_6)) \\ & \land (x_9 \leftrightarrow (x_6 \lor x_7)) \\ & \land (x_{10} \leftrightarrow (x_7 \land x_8 \land x_9)) \;.\end{aligned}$$

Given a circuit C, it is straightforward to produce such a formula ϕ in polynomial time.

Why is the circuit C satisfiable exactly when the formula ϕ is satisfiable? If C has a satisfying assignment, then each wire of the circuit has a well-defined value, and the output of the circuit is 1. Therefore, when we assign wire values to variables in ϕ, each clause of ϕ evaluates to 1, and thus the conjunction of all evaluates to 1. Conversely, if some assignment causes ϕ to evaluate to 1, the circuit C is satisfiable by an analogous argument. Thus, we have shown that CIRCUIT-SAT $\leq_{\mathrm{P}}$ SAT, which completes the proof. ■

3-CNF satisfiability

We can prove many problems NP-complete by reducing from formula satisfiability. The reduction algorithm must handle any input formula, though, and this requirement can lead to a huge number of cases that we must consider. We often prefer to reduce from a restricted language of boolean formulas, so that we need to consider fewer cases. Of course, we must not restrict the language so much that it becomes polynomial-time solvable. One convenient language is 3-CNF satisfiability, or 3-CNF-SAT.

We define 3-CNF satisfiability using the following terms. A ***literal*** in a boolean formula is an occurrence of a variable or its negation. A boolean formula is in ***conjunctive normal form***, or ***CNF***, if it is expressed as an AND of ***clauses***, each of which is the OR of one or more literals. A boolean formula is in ***3-conjunctive normal form***, or ***3-CNF***, if each clause has exactly three distinct literals.

For example, the boolean formula

$$(x_1 \vee \neg x_1 \vee \neg x_2) \wedge (x_3 \vee x_2 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4)$$

is in 3-CNF. The first of its three clauses is $(x_1 \vee \neg x_1 \vee \neg x_2)$, which contains the three literals x_1, $\neg x_1$, and $\neg x_2$.

In 3-CNF-SAT, we are asked whether a given boolean formula ϕ in 3-CNF is satisfiable. The following theorem shows that a polynomial-time algorithm that can determine the satisfiability of boolean formulas is unlikely to exist, even when they are expressed in this simple normal form.

Theorem 34.10
Satisfiability of boolean formulas in 3-conjunctive normal form is NP-complete.

Proof The argument we used in the proof of Theorem 34.9 to show that SAT $\in$ NP applies equally well here to show that 3-CNF-SAT $\in$ NP. By Lemma 34.8, therefore, we need only show that SAT $\leq_P$ 3-CNF-SAT.

We break the reduction algorithm into three basic steps. Each step progressively transforms the input formula ϕ closer to the desired 3-conjunctive normal form.

The first step is similar to the one used to prove CIRCUIT-SAT $\leq_P$ SAT in Theorem 34.9. First, we construct a binary "parse" tree for the input formula ϕ, with literals as leaves and connectives as internal nodes. Figure 34.11 shows such a parse tree for the formula

$$\phi = ((x_1 \rightarrow x_2) \vee \neg((\neg x_1 \leftrightarrow x_3) \vee x_4)) \wedge \neg x_2 \,. \tag{34.3}$$

Should the input formula contain a clause such as the OR of several literals, we use associativity to parenthesize the expression fully so that every internal node in the resulting tree has 1 or 2 children. We can now think of the binary parse tree as a circuit for computing the function.

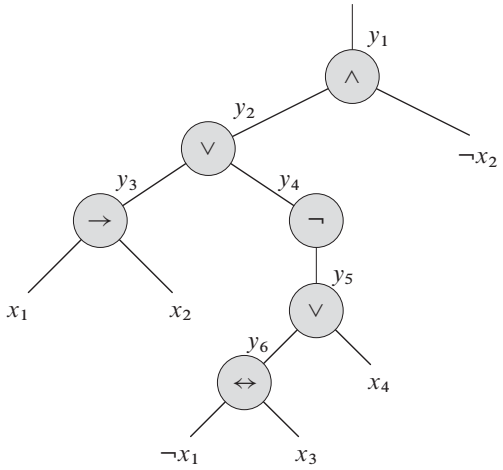


Figure 34.11 The tree corresponding to the formula $\phi = ((x_1 \rightarrow x_2) \vee \neg((\neg x_1 \leftrightarrow x_3) \vee x_4)) \wedge \neg x_2$.

Mimicking the reduction in the proof of Theorem 34.9, we introduce a variable y_i for the output of each internal node. Then, we rewrite the original formula ϕ as the AND of the root variable and a conjunction of clauses describing the operation of each node. For the formula (34.3), the resulting expression is

$$\begin{aligned}\phi' \;=\; & y_1 \wedge (y_1 \leftrightarrow (y_2 \wedge \neg x_2)) \\ & \wedge\; (y_2 \leftrightarrow (y_3 \vee y_4)) \\ & \wedge\; (y_3 \leftrightarrow (x_1 \rightarrow x_2)) \\ & \wedge\; (y_4 \leftrightarrow \neg y_5) \\ & \wedge\; (y_5 \leftrightarrow (y_6 \vee x_4)) \\ & \wedge\; (y_6 \leftrightarrow (\neg x_1 \leftrightarrow x_3))\ .\end{aligned}$$

Observe that the formula ϕ' thus obtained is a conjunction of clauses ϕ'_i, each of which has at most 3 literals. The only requirement that we might fail to meet is that each clause has to be an OR of 3 literals.

The second step of the reduction converts each clause ϕ'_i into conjunctive normal form. We construct a truth table for ϕ'_i by evaluating all possible assignments to its variables. Each row of the truth table consists of a possible assignment of the variables of the clause, together with the value of the clause under that assignment. Using the truth-table entries that evaluate to 0, we build a formula in ***disjunctive normal form*** (or ***DNF***)—an OR of ANDs—that is equivalent to $\neg\phi'_i$. We then negate this formula and convert it into a CNF formula ϕ''_i by using ***DeMorgan's***

y_1	y_2	x_2	$(y_1 \leftrightarrow (y_2 \wedge \neg x_2))$
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	1
0	0	0	1

Figure 34.12 The truth table for the clause $(y_1 \leftrightarrow (y_2 \wedge \neg x_2))$.

laws for propositional logic,

$$\begin{aligned}\neg(a \wedge b) &= \neg a \vee \neg b\ ,\\ \neg(a \vee b) &= \neg a \wedge \neg b\ ,\end{aligned}$$

to complement all literals, change ORs into ANDs, and change ANDs into ORs.

In our example, we convert the clause $\phi'_1 = (y_1 \leftrightarrow (y_2 \wedge \neg x_2))$ into CNF as follows. The truth table for ϕ'_1 appears in Figure 34.12. The DNF formula equivalent to $\neg\phi'_1$ is

$$(y_1 \wedge y_2 \wedge x_2) \vee (y_1 \wedge \neg y_2 \wedge x_2) \vee (y_1 \wedge \neg y_2 \wedge \neg x_2) \vee (\neg y_1 \wedge y_2 \wedge \neg x_2)\ .$$

Negating and applying DeMorgan's laws, we get the CNF formula

$$\begin{aligned}\phi''_1 = {} & (\neg y_1 \vee \neg y_2 \vee \neg x_2) \wedge (\neg y_1 \vee y_2 \vee \neg x_2)\\ & \wedge (\neg y_1 \vee y_2 \vee x_2) \wedge (y_1 \vee \neg y_2 \vee x_2)\ ,\end{aligned}$$

which is equivalent to the original clause ϕ'_1.

At this point, we have converted each clause ϕ'_i of the formula ϕ' into a CNF formula ϕ''_i, and thus ϕ' is equivalent to the CNF formula ϕ'' consisting of the conjunction of the ϕ''_i. Moreover, each clause of ϕ'' has at most 3 literals.

The third and final step of the reduction further transforms the formula so that each clause has *exactly* 3 distinct literals. We construct the final 3-CNF formula ϕ''' from the clauses of the CNF formula ϕ''. The formula ϕ''' also uses two auxiliary variables that we shall call p and q. For each clause C_i of ϕ'', we include the following clauses in ϕ''':

- If C_i has 3 distinct literals, then simply include C_i as a clause of ϕ'''.
- If C_i has 2 distinct literals, that is, if $C_i = (l_1 \vee l_2)$, where l_1 and l_2 are literals, then include $(l_1 \vee l_2 \vee p) \wedge (l_1 \vee l_2 \vee \neg p)$ as clauses of ϕ'''. The literals p and $\neg p$ merely fulfill the syntactic requirement that each clause of ϕ''' has