19.2 Mergeable-heap operations

The mergeable-heap operations on Fibonacci heaps delay work as long as possible. The various operations have performance trade-offs. For example, we insert a node by adding it to the root list, which takes just constant time. If we were to start with an empty Fibonacci heap and then insert k nodes, the Fibonacci heap would consist of just a root list of k nodes. The trade-off is that if we then perform an EXTRACT-MIN operation on Fibonacci heap H, after removing the node that $H.min$ points to, we would have to look through each of the remaining $k-1$ nodes in the root list to find the new minimum node. As long as we have to go through the entire root list during the EXTRACT-MIN operation, we also consolidate nodes into min-heap-ordered trees to reduce the size of the root list. We shall see that, no matter what the root list looks like before a EXTRACT-MIN operation, afterward each node in the root list has a degree that is unique within the root list, which leads to a root list of size at most $D(n)+1$.

Creating a new Fibonacci heap

To make an empty Fibonacci heap, the MAKE-FIB-HEAP procedure allocates and returns the Fibonacci heap object H, where $H.n = 0$ and $H.min = \text{NIL}$; there are no trees in H. Because $t(H) = 0$ and $m(H) = 0$, the potential of the empty Fibonacci heap is $\Phi(H) = 0$. The amortized cost of MAKE-FIB-HEAP is thus equal to its $O(1)$ actual cost.

Inserting a node

The following procedure inserts node x into Fibonacci heap H, assuming that the node has already been allocated and that $x.key$ has already been filled in.

```
FIB-HEAP-INSERT(H, x)
 1  x.degree = 0
 2  x.p = NIL
 3  x.child = NIL
 4  x.mark = FALSE
 5  if H.min == NIL
 6       create a root list for H containing just x
 7       H.min = x
 8  else insert x into H's root list
 9       if x.key < H.min.key
10            H.min = x
11  H.n = H.n + 1
```

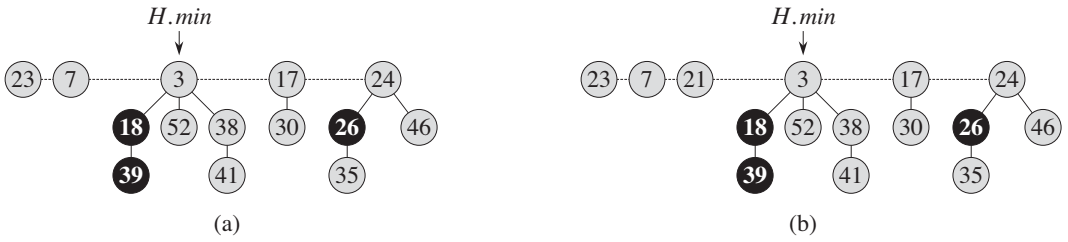

Figure 19.3 Inserting a node into a Fibonacci heap. **(a)** A Fibonacci heap H. **(b)** Fibonacci heap H after inserting the node with key 21. The node becomes its own min-heap-ordered tree and is then added to the root list, becoming the left sibling of the root.

Lines 1–4 initialize some of the structural attributes of node x. Line 5 tests to see whether Fibonacci heap H is empty. If it is, then lines 6–7 make x be the only node in H's root list and set $H.min$ to point to x. Otherwise, lines 8–10 insert x into H's root list and update $H.min$ if necessary. Finally, line 11 increments $H.n$ to reflect the addition of the new node. Figure 19.3 shows a node with key 21 inserted into the Fibonacci heap of Figure 19.2.

To determine the amortized cost of FIB-HEAP-INSERT, let H be the input Fibonacci heap and H' be the resulting Fibonacci heap. Then, $t(H') = t(H) + 1$ and $m(H') = m(H)$, and the increase in potential is

$$((t(H) + 1) + 2\,m(H)) - (t(H) + 2\,m(H)) = 1\ .$$

Since the actual cost is $O(1)$, the amortized cost is $O(1) + 1 = O(1)$.

Finding the minimum node

The minimum node of a Fibonacci heap H is given by the pointer $H.min$, so we can find the minimum node in $O(1)$ actual time. Because the potential of H does not change, the amortized cost of this operation is equal to its $O(1)$ actual cost.

Uniting two Fibonacci heaps

The following procedure unites Fibonacci heaps H_1 and H_2, destroying H_1 and H_2 in the process. It simply concatenates the root lists of H_1 and H_2 and then determines the new minimum node. Afterward, the objects representing H_1 and H_2 will never be used again.

```
FIB-HEAP-UNION(H1, H2)
1  H = MAKE-FIB-HEAP()
2  H.min = H1.min
3  concatenate the root list of H2 with the root list of H
4  if (H1.min == NIL) or (H2.min ≠ NIL and H2.min.key < H1.min.key)
5       H.min = H2.min
6  H.n = H1.n + H2.n
7  return H
```

Lines 1–3 concatenate the root lists of H_1 and H_2 into a new root list H. Lines 2, 4, and 5 set the minimum node of H, and line 6 sets $H.n$ to the total number of nodes. Line 7 returns the resulting Fibonacci heap H. As in the FIB-HEAP-INSERT procedure, all roots remain roots.

The change in potential is

$$\begin{aligned}\Phi(H) - (\Phi(H_1) + \Phi(H_2)) &= (t(H) + 2\,m(H)) - ((t(H_1) + 2\,m(H_1)) + (t(H_2) + 2\,m(H_2))) \\ &= 0\,,\end{aligned}$$

because $t(H) = t(H_1) + t(H_2)$ and $m(H) = m(H_1) + m(H_2)$. The amortized cost of FIB-HEAP-UNION is therefore equal to its $O(1)$ actual cost.

Extracting the minimum node

The process of extracting the minimum node is the most complicated of the operations presented in this section. It is also where the delayed work of consolidating trees in the root list finally occurs. The following pseudocode extracts the minimum node. The code assumes for convenience that when a node is removed from a linked list, pointers remaining in the list are updated, but pointers in the extracted node are left unchanged. It also calls the auxiliary procedure CONSOLIDATE, which we shall see shortly.

```
FIB-HEAP-EXTRACT-MIN(H)
 1  z = H.min
 2  if z ≠ NIL
 3       for each child x of z
 4            add x to the root list of H
 5            x.p = NIL
 6       remove z from the root list of H
 7       if z == z.right
 8            H.min = NIL
 9       else H.min = z.right
10            CONSOLIDATE(H)
11       H.n = H.n − 1
12  return z
```

As Figure 19.4 illustrates, FIB-HEAP-EXTRACT-MIN works by first making a root out of each of the minimum node's children and removing the minimum node from the root list. It then consolidates the root list by linking roots of equal degree until at most one root remains of each degree.

We start in line 1 by saving a pointer z to the minimum node; the procedure returns this pointer at the end. If z is NIL, then Fibonacci heap H is already empty and we are done. Otherwise, we delete node z from H by making all of z's children roots of H in lines 3–5 (putting them into the root list) and removing z from the root list in line 6. If z is its own right sibling after line 6, then z was the only node on the root list and it had no children, so all that remains is to make the Fibonacci heap empty in line 8 before returning z. Otherwise, we set the pointer $H.min$ into the root list to point to a root other than z (in this case, z's right sibling), which is not necessarily going to be the new minimum node when FIB-HEAP-EXTRACT-MIN is done. Figure 19.4(b) shows the Fibonacci heap of Figure 19.4(a) after executing line 9.

The next step, in which we reduce the number of trees in the Fibonacci heap, is ***consolidating*** the root list of H, which the call CONSOLIDATE(H) accomplishes. Consolidating the root list consists of repeatedly executing the following steps until every root in the root list has a distinct *degree* value:

1. Find two roots x and y in the root list with the same degree. Without loss of generality, let $x.key \leq y.key$.
2. ***Link*** y to x: remove y from the root list, and make y a child of x by calling the FIB-HEAP-LINK procedure. This procedure increments the attribute $x.degree$ and clears the mark on y.

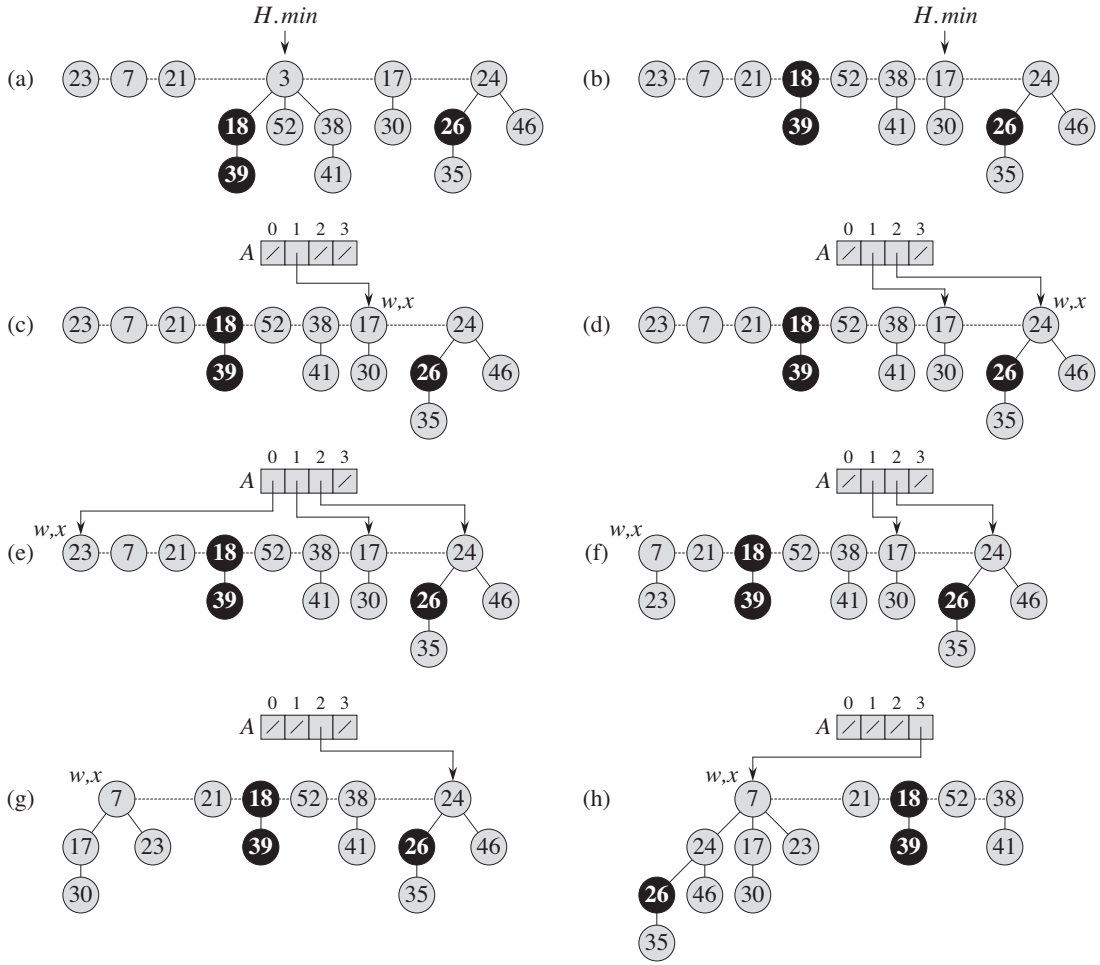

Figure 19.4 The action of FIB-HEAP-EXTRACT-MIN. **(a)** A Fibonacci heap H. **(b)** The situation after removing the minimum node z from the root list and adding its children to the root list. **(c)–(e)** The array A and the trees after each of the first three iterations of the **for** loop of lines 4–14 of the procedure CONSOLIDATE. The procedure processes the root list by starting at the node pointed to by $H.min$ and following *right* pointers. Each part shows the values of w and x at the end of an iteration. **(f)–(h)** The next iteration of the **for** loop, with the values of w and x shown at the end of each iteration of the **while** loop of lines 7–13. Part (f) shows the situation after the first time through the **while** loop. The node with key 23 has been linked to the node with key 7, which x now points to. In part (g), the node with key 17 has been linked to the node with key 7, which x still points to. In part (h), the node with key 24 has been linked to the node with key 7. Since no node was previously pointed to by $A[3]$, at the end of the **for** loop iteration, $A[3]$ is set to point to the root of the resulting tree.

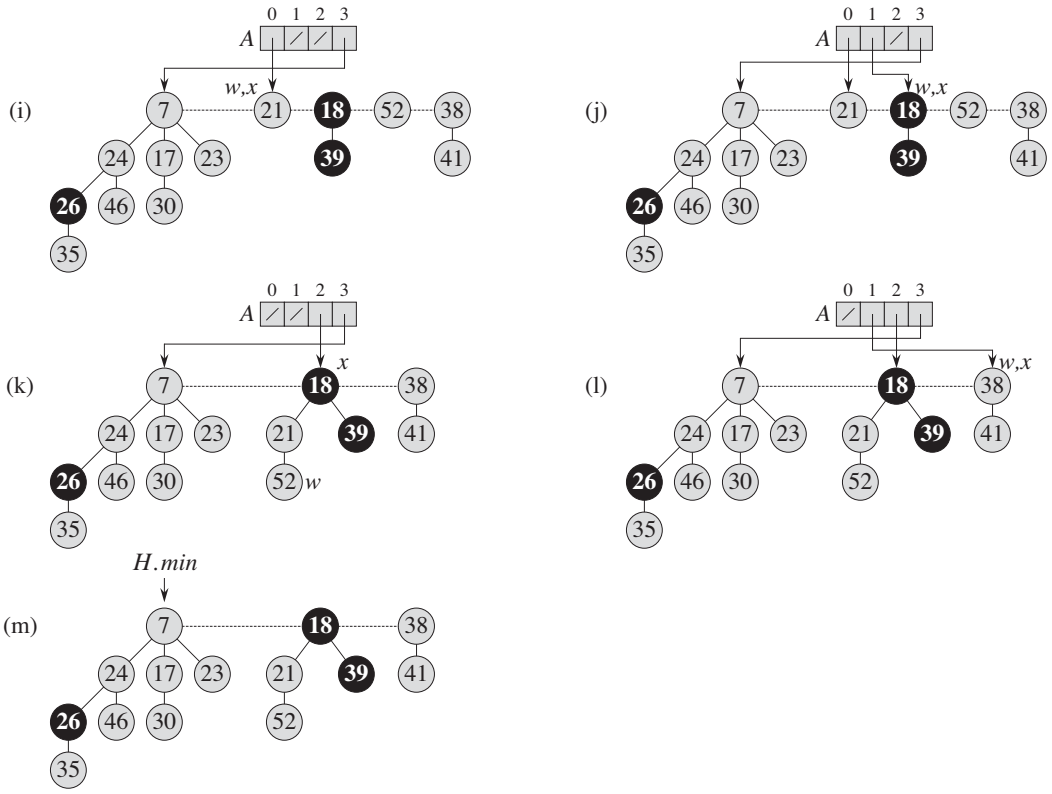

Figure 19.4, continued **(i)–(l)** The situation after each of the next four iterations of the **for** loop. **(m)** Fibonacci heap H after reconstructing the root list from the array A and determining the new $H.min$ pointer.

The procedure CONSOLIDATE uses an auxiliary array $A[0\,..\,D(H.n)]$ to keep track of roots according to their degrees. If $A[i] = y$, then y is currently a root with $y.degree = i$. Of course, in order to allocate the array we have to know how to calculate the upper bound $D(H.n)$ on the maximum degree, but we will see how to do so in Section 19.4.

```
CONSOLIDATE(H)
 1  let A[0..D(H.n)] be a new array
 2  for i = 0 to D(H.n)
 3      A[i] = NIL
 4  for each node w in the root list of H
 5      x = w
 6      d = x.degree
 7      while A[d] ≠ NIL
 8          y = A[d]          // another node with the same degree as x
 9          if x.key > y.key
10              exchange x with y
11          FIB-HEAP-LINK(H, y, x)
12          A[d] = NIL
13          d = d + 1
14      A[d] = x
15  H.min = NIL
16  for i = 0 to D(H.n)
17      if A[i] ≠ NIL
18          if H.min == NIL
19              create a root list for H containing just A[i]
20              H.min = A[i]
21          else insert A[i] into H's root list
22              if A[i].key < H.min.key
23                  H.min = A[i]
```

```
FIB-HEAP-LINK(H, y, x)
1  remove y from the root list of H
2  make y a child of x, incrementing x.degree
3  y.mark = FALSE
```

In detail, the CONSOLIDATE procedure works as follows. Lines 1–3 allocate and initialize the array A by making each entry NIL. The **for** loop of lines 4–14 processes each root w in the root list. As we link roots together, w may be linked to some other node and no longer be a root. Nevertheless, w is always in a tree rooted at some node x, which may or may not be w itself. Because we want at most one root with each degree, we look in the array A to see whether it contains a root y with the same degree as x. If it does, then we link the roots x and y but guaranteeing that x remains a root after linking. That is, we link y to x after first exchanging the pointers to the two roots if y's key is smaller than x's key. After we link y to x, the degree of x has increased by 1, and so we continue this process, linking x and another root whose degree equals x's new degree, until no other root

that we have processed has the same degree as x. We then set the appropriate entry of A to point to x, so that as we process roots later on, we have recorded that x is the unique root of its degree that we have already processed. When this **for** loop terminates, at most one root of each degree will remain, and the array A will point to each remaining root.

The **while** loop of lines 7–13 repeatedly links the root x of the tree containing node w to another tree whose root has the same degree as x, until no other root has the same degree. This **while** loop maintains the following invariant:

At the start of each iteration of the **while** loop, $d = x.degree$.

We use this loop invariant as follows:

Initialization: Line 6 ensures that the loop invariant holds the first time we enter the loop.

Maintenance: In each iteration of the **while** loop, $A[d]$ points to some root y. Because $d = x.degree = y.degree$, we want to link x and y. Whichever of x and y has the smaller key becomes the parent of the other as a result of the link operation, and so lines 9–10 exchange the pointers to x and y if necessary. Next, we link y to x by the call FIB-HEAP-LINK(H, y, x) in line 11. This call increments $x.degree$ but leaves $y.degree$ as d. Node y is no longer a root, and so line 12 removes the pointer to it in array A. Because the call of FIB-HEAP-LINK increments the value of $x.degree$, line 13 restores the invariant that $d = x.degree$.

Termination: We repeat the **while** loop until $A[d] =$ NIL, in which case there is no other root with the same degree as x.

After the **while** loop terminates, we set $A[d]$ to x in line 14 and perform the next iteration of the **for** loop.

Figures 19.4(c)–(e) show the array A and the resulting trees after the first three iterations of the **for** loop of lines 4–14. In the next iteration of the **for** loop, three links occur; their results are shown in Figures 19.4(f)–(h). Figures 19.4(i)–(l) show the result of the next four iterations of the **for** loop.

All that remains is to clean up. Once the **for** loop of lines 4–14 completes, line 15 empties the root list, and lines 16–23 reconstruct it from the array A. The resulting Fibonacci heap appears in Figure 19.4(m). After consolidating the root list, FIB-HEAP-EXTRACT-MIN finishes up by decrementing $H.n$ in line 11 and returning a pointer to the deleted node z in line 12.

We are now ready to show that the amortized cost of extracting the minimum node of an n-node Fibonacci heap is $O(D(n))$. Let H denote the Fibonacci heap just prior to the FIB-HEAP-EXTRACT-MIN operation.

We start by accounting for the actual cost of extracting the minimum node. An $O(D(n))$ contribution comes from FIB-HEAP-EXTRACT-MIN processing at

most $D(n)$ children of the minimum node and from the work in lines 2–3 and 16–23 of CONSOLIDATE. It remains to analyze the contribution from the **for** loop of lines 4–14 in CONSOLIDATE, for which we use an aggregate analysis. The size of the root list upon calling CONSOLIDATE is at most $D(n) + t(H) - 1$, since it consists of the original $t(H)$ root-list nodes, minus the extracted root node, plus the children of the extracted node, which number at most $D(n)$. Within a given iteration of the **for** loop of lines 4–14, the number of iterations of the **while** loop of lines 7–13 depends on the root list. But we know that every time through the **while** loop, one of the roots is linked to another, and thus the total number of iterations of the **while** loop over all iterations of the **for** loop is at most the number of roots in the root list. Hence, the total amount of work performed in the **for** loop is at most proportional to $D(n) + t(H)$. Thus, the total actual work in extracting the minimum node is $O(D(n) + t(H))$.

The potential before extracting the minimum node is $t(H) + 2\,m(H)$, and the potential afterward is at most $(D(n) + 1) + 2\,m(H)$, since at most $D(n) + 1$ roots remain and no nodes become marked during the operation. The amortized cost is thus at most

$$\begin{aligned} & O(D(n) + t(H)) + ((D(n) + 1) + 2\,m(H)) - (t(H) + 2\,m(H)) \\ & \quad = \quad O(D(n)) + O(t(H)) - t(H) \\ & \quad = \quad O(D(n)) \,, \end{aligned}$$

since we can scale up the units of potential to dominate the constant hidden in $O(t(H))$. Intuitively, the cost of performing each link is paid for by the reduction in potential due to the link's reducing the number of roots by one. We shall see in Section 19.4 that $D(n) = O(\lg n)$, so that the amortized cost of extracting the minimum node is $O(\lg n)$.

Exercises

19.2-1

Show the Fibonacci heap that results from calling FIB-HEAP-EXTRACT-MIN on the Fibonacci heap shown in Figure 19.4(m).

19.3 Decreasing a key and deleting a node

In this section, we show how to decrease the key of a node in a Fibonacci heap in $O(1)$ amortized time and how to delete any node from an n-node Fibonacci heap in $O(D(n))$ amortized time. In Section 19.4, we will show that the maxi-

mum degree $D(n)$ is $O(\lg n)$, which will imply that FIB-HEAP-EXTRACT-MIN and FIB-HEAP-DELETE run in $O(\lg n)$ amortized time.

Decreasing a key

In the following pseudocode for the operation FIB-HEAP-DECREASE-KEY, we assume as before that removing a node from a linked list does not change any of the structural attributes in the removed node.

```
FIB-HEAP-DECREASE-KEY(H, x, k)
1  if k > x.key
2       error "new key is greater than current key"
3  x.key = k
4  y = x.p
5  if y ≠ NIL and x.key < y.key
6       CUT(H, x, y)
7       CASCADING-CUT(H, y)
8  if x.key < H.min.key
9       H.min = x

CUT(H, x, y)
1  remove x from the child list of y, decrementing y.degree
2  add x to the root list of H
3  x.p = NIL
4  x.mark = FALSE

CASCADING-CUT(H, y)
1  z = y.p
2  if z ≠ NIL
3       if y.mark == FALSE
4            y.mark = TRUE
5       else CUT(H, y, z)
6            CASCADING-CUT(H, z)
```

The FIB-HEAP-DECREASE-KEY procedure works as follows. Lines 1–3 ensure that the new key is no greater than the current key of x and then assign the new key to x. If x is a root or if $x.key \geq y.key$, where y is x's parent, then no structural changes need occur, since min-heap order has not been violated. Lines 4–5 test for this condition.

If min-heap order has been violated, many changes may occur. We start by ***cutting*** x in line 6. The CUT procedure "cuts" the link between x and its parent y, making x a root.

We use the *mark* attributes to obtain the desired time bounds. They record a little piece of the history of each node. Suppose that the following events have happened to node x:

1. at some time, x was a root,
2. then x was linked to (made the child of) another node,
3. then two children of x were removed by cuts.

As soon as the second child has been lost, we cut x from its parent, making it a new root. The attribute $x.mark$ is TRUE if steps 1 and 2 have occurred and one child of x has been cut. The CUT procedure, therefore, clears $x.mark$ in line 4, since it performs step 1. (We can now see why line 3 of FIB-HEAP-LINK clears $y.mark$: node y is being linked to another node, and so step 2 is being performed. The next time a child of y is cut, $y.mark$ will be set to TRUE.)

We are not yet done, because x might be the second child cut from its parent y since the time that y was linked to another node. Therefore, line 7 of FIB-HEAP-DECREASE-KEY attempts to perform a ***cascading-cut*** operation on y. If y is a root, then the test in line 2 of CASCADING-CUT causes the procedure to just return. If y is unmarked, the procedure marks it in line 4, since its first child has just been cut, and returns. If y is marked, however, it has just lost its second child; y is cut in line 5, and CASCADING-CUT calls itself recursively in line 6 on y's parent z. The CASCADING-CUT procedure recurses its way up the tree until it finds either a root or an unmarked node.

Once all the cascading cuts have occurred, lines 8–9 of FIB-HEAP-DECREASE-KEY finish up by updating $H.min$ if necessary. The only node whose key changed was the node x whose key decreased. Thus, the new minimum node is either the original minimum node or node x.

Figure 19.5 shows the execution of two calls of FIB-HEAP-DECREASE-KEY, starting with the Fibonacci heap shown in Figure 19.5(a). The first call, shown in Figure 19.5(b), involves no cascading cuts. The second call, shown in Figures 19.5(c)–(e), invokes two cascading cuts.

We shall now show that the amortized cost of FIB-HEAP-DECREASE-KEY is only $O(1)$. We start by determining its actual cost. The FIB-HEAP-DECREASE-KEY procedure takes $O(1)$ time, plus the time to perform the cascading cuts. Suppose that a given invocation of FIB-HEAP-DECREASE-KEY results in c calls of CASCADING-CUT (the call made from line 7 of FIB-HEAP-DECREASE-KEY followed by $c - 1$ recursive calls of CASCADING-CUT). Each call of CASCADING-CUT takes $O(1)$ time exclusive of recursive calls. Thus, the actual cost of FIB-HEAP-DECREASE-KEY, including all recursive calls, is $O(c)$.

We next compute the change in potential. Let H denote the Fibonacci heap just prior to the FIB-HEAP-DECREASE-KEY operation. The call to CUT in line 6 of

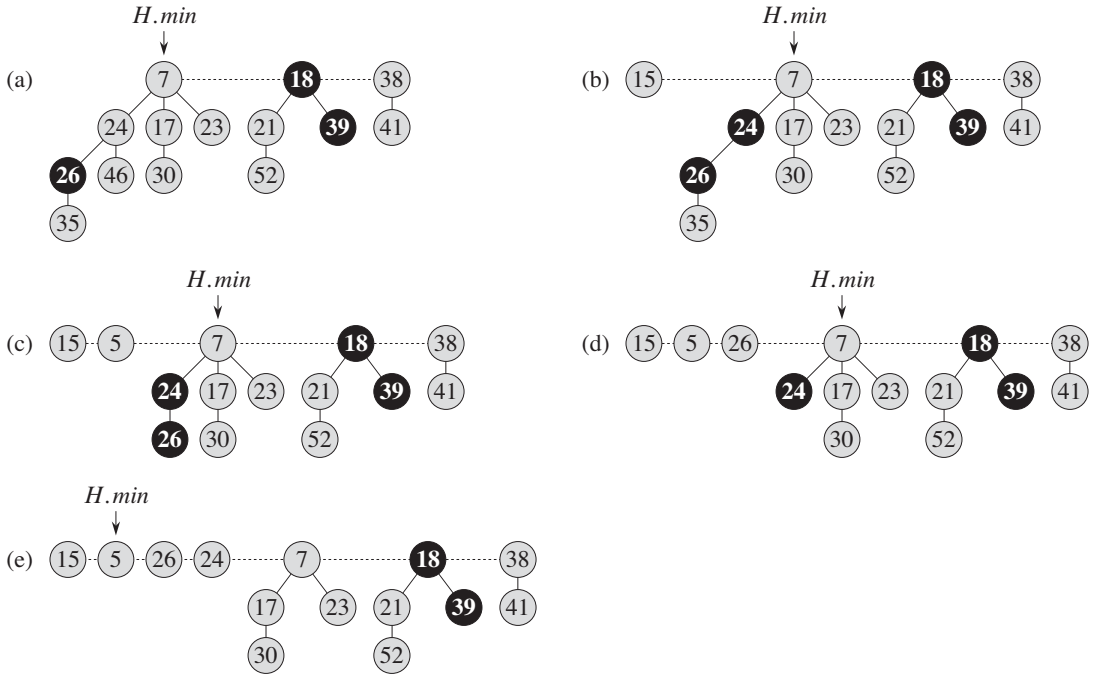

Figure 19.5 Two calls of FIB-HEAP-DECREASE-KEY. **(a)** The initial Fibonacci heap. **(b)** The node with key 46 has its key decreased to 15. The node becomes a root, and its parent (with key 24), which had previously been unmarked, becomes marked. **(c)–(e)** The node with key 35 has its key decreased to 5. In part (c), the node, now with key 5, becomes a root. Its parent, with key 26, is marked, so a cascading cut occurs. The node with key 26 is cut from its parent and made an unmarked root in (d). Another cascading cut occurs, since the node with key 24 is marked as well. This node is cut from its parent and made an unmarked root in part (e). The cascading cuts stop at this point, since the node with key 7 is a root. (Even if this node were not a root, the cascading cuts would stop, since it is unmarked.) Part (e) shows the result of the FIB-HEAP-DECREASE-KEY operation, with $H.min$ pointing to the new minimum node.

FIB-HEAP-DECREASE-KEY creates a new tree rooted at node x and clears x's mark bit (which may have already been FALSE). Each call of CASCADING-CUT, except for the last one, cuts a marked node and clears the mark bit. Afterward, the Fibonacci heap contains $t(H)+c$ trees (the original $t(H)$ trees, $c-1$ trees produced by cascading cuts, and the tree rooted at x) and at most $m(H)-c+2$ marked nodes ($c-1$ were unmarked by cascading cuts and the last call of CASCADING-CUT may have marked a node). The change in potential is therefore at most

$$((t(H)+c)+2(m(H)-c+2))-(t(H)+2\,m(H))=4-c\ .$$

Thus, the amortized cost of FIB-HEAP-DECREASE-KEY is at most

$$O(c) + 4 - c = O(1) ,$$

since we can scale up the units of potential to dominate the constant hidden in $O(c)$.

You can now see why we defined the potential function to include a term that is twice the number of marked nodes. When a marked node y is cut by a cascading cut, its mark bit is cleared, which reduces the potential by 2. One unit of potential pays for the cut and the clearing of the mark bit, and the other unit compensates for the unit increase in potential due to node y becoming a root.

Deleting a node

The following pseudocode deletes a node from an n-node Fibonacci heap in $O(D(n))$ amortized time. We assume that there is no key value of $-\infty$ currently in the Fibonacci heap.

```
FIB-HEAP-DELETE(H, x)
1  FIB-HEAP-DECREASE-KEY(H, x, -∞)
2  FIB-HEAP-EXTRACT-MIN(H)
```

FIB-HEAP-DELETE makes x become the minimum node in the Fibonacci heap by giving it a uniquely small key of $-\infty$. The FIB-HEAP-EXTRACT-MIN procedure then removes node x from the Fibonacci heap. The amortized time of FIB-HEAP-DELETE is the sum of the $O(1)$ amortized time of FIB-HEAP-DECREASE-KEY and the $O(D(n))$ amortized time of FIB-HEAP-EXTRACT-MIN. Since we shall see in Section 19.4 that $D(n) = O(\lg n)$, the amortized time of FIB-HEAP-DELETE is $O(\lg n)$.

Exercises

19.3-1

Suppose that a root x in a Fibonacci heap is marked. Explain how x came to be a marked root. Argue that it doesn't matter to the analysis that x is marked, even though it is not a root that was first linked to another node and then lost one child.

19.3-2

Justify the $O(1)$ amortized time of FIB-HEAP-DECREASE-KEY as an average cost per operation by using aggregate analysis.

19.4 Bounding the maximum degree

To prove that the amortized time of FIB-HEAP-EXTRACT-MIN and FIB-HEAP-DELETE is $O(\lg n)$, we must show that the upper bound $D(n)$ on the degree of any node of an n-node Fibonacci heap is $O(\lg n)$. In particular, we shall show that $D(n) \leq \lfloor \log_\phi n \rfloor$, where ϕ is the golden ratio, defined in equation (3.24) as

$$\phi = (1 + \sqrt{5})/2 = 1.61803\ldots\ .$$

The key to the analysis is as follows. For each node x within a Fibonacci heap, define size(x) to be the number of nodes, including x itself, in the subtree rooted at x. (Note that x need not be in the root list—it can be any node at all.) We shall show that size(x) is exponential in $x.degree$. Bear in mind that $x.degree$ is always maintained as an accurate count of the degree of x.

Lemma 19.1
Let x be any node in a Fibonacci heap, and suppose that $x.degree = k$. Let $y_1, y_2, \ldots, y_k$ denote the children of x in the order in which they were linked to x, from the earliest to the latest. Then, $y_1.degree \geq 0$ and $y_i.degree \geq i - 2$ for $i = 2, 3, \ldots, k$.

Proof Obviously, $y_1.degree \geq 0$.

For $i \geq 2$, we note that when y_i was linked to x, all of $y_1, y_2, \ldots, y_{i-1}$ were children of x, and so we must have had $x.degree \geq i - 1$. Because node y_i is linked to x (by CONSOLIDATE) only if $x.degree = y_i.degree$, we must have also had $y_i.degree \geq i - 1$ at that time. Since then, node y_i has lost at most one child, since it would have been cut from x (by CASCADING-CUT) if it had lost two children. We conclude that $y_i.degree \geq i - 2$. ∎

We finally come to the part of the analysis that explains the name "Fibonacci heaps." Recall from Section 3.2 that for $k = 0, 1, 2, \ldots$, the kth Fibonacci number is defined by the recurrence

$$F_k = \begin{cases} 0 & \text{if } k = 0\,, \\ 1 & \text{if } k = 1\,, \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2\,. \end{cases}$$

The following lemma gives another way to express F_k.

Lemma 19.2
For all integers $k \geq 0$,

$$F_{k+2} = 1 + \sum_{i=0}^{k} F_i \ .$$

Proof The proof is by induction on k. When $k = 0$,

$$\begin{aligned} 1 + \sum_{i=0}^{0} F_i &= 1 + F_0 \\ &= 1 + 0 \\ &= F_2 \ . \end{aligned}$$

We now assume the inductive hypothesis that $F_{k+1} = 1 + \sum_{i=0}^{k-1} F_i$, and we have

$$\begin{aligned} F_{k+2} &= F_k + F_{k+1} \\ &= F_k + \left(1 + \sum_{i=0}^{k-1} F_i\right) \\ &= 1 + \sum_{i=0}^{k} F_i \ . \end{aligned}$$

■

Lemma 19.3
For all integers $k \geq 0$, the $(k+2)$nd Fibonacci number satisfies $F_{k+2} \geq \phi^k$.

Proof The proof is by induction on k. The base cases are for $k = 0$ and $k = 1$. When $k = 0$ we have $F_2 = 1 = \phi^0$, and when $k = 1$ we have $F_3 = 2 > 1.619 > \phi^1$. The inductive step is for $k \geq 2$, and we assume that $F_{i+2} > \phi^i$ for $i = 0, 1, \ldots, k-1$. Recall that ϕ is the positive root of equation (3.23), $x^2 = x+1$. Thus, we have

$$\begin{aligned} F_{k+2} &= F_{k+1} + F_k \\ &\geq \phi^{k-1} + \phi^{k-2} && \text{(by the inductive hypothesis)} \\ &= \phi^{k-2}(\phi + 1) \\ &= \phi^{k-2} \cdot \phi^2 && \text{(by equation (3.23))} \\ &= \phi^k \ . \end{aligned}$$

■

The following lemma and its corollary complete the analysis.

Lemma 19.4
Let x be any node in a Fibonacci heap, and let $k = x.degree$. Then $\text{size}(x) \geq F_{k+2} \geq \phi^k$, where $\phi = (1 + \sqrt{5})/2$.

Proof Let s_k denote the minimum possible size of any node of degree k in any Fibonacci heap. Trivially, $s_0 = 1$ and $s_1 = 2$. The number s_k is at most $\text{size}(x)$ and, because adding children to a node cannot decrease the node's size, the value of s_k increases monotonically with k. Consider some node z, in any Fibonacci heap, such that $z.degree = k$ and $\text{size}(z) = s_k$. Because $s_k \leq \text{size}(x)$, we compute a lower bound on $\text{size}(x)$ by computing a lower bound on s_k. As in Lemma 19.1, let $y_1, y_2, \ldots, y_k$ denote the children of z in the order in which they were linked to z. To bound s_k, we count one for z itself and one for the first child y_1 (for which $\text{size}(y_1) \geq 1$), giving

$$\begin{aligned}
\text{size}(x) &\geq s_k \\
&\geq 2 + \sum_{i=2}^{k} s_{y_i.degree} \\
&\geq 2 + \sum_{i=2}^{k} s_{i-2} ,
\end{aligned}$$

where the last line follows from Lemma 19.1 (so that $y_i.degree \geq i - 2$) and the monotonicity of s_k (so that $s_{y_i.degree} \geq s_{i-2}$).

We now show by induction on k that $s_k \geq F_{k+2}$ for all nonnegative integers k. The bases, for $k = 0$ and $k = 1$, are trivial. For the inductive step, we assume that $k \geq 2$ and that $s_i \geq F_{i+2}$ for $i = 0, 1, \ldots, k - 1$. We have

$$\begin{aligned}
s_k &\geq 2 + \sum_{i=2}^{k} s_{i-2} \\
&\geq 2 + \sum_{i=2}^{k} F_i \\
&= 1 + \sum_{i=0}^{k} F_i \\
&= F_{k+2} && \text{(by Lemma 19.2)} \\
&\geq \phi^k && \text{(by Lemma 19.3) .}
\end{aligned}$$

Thus, we have shown that $\text{size}(x) \geq s_k \geq F_{k+2} \geq \phi^k$. ■

Corollary 19.5
The maximum degree $D(n)$ of any node in an n-node Fibonacci heap is $O(\lg n)$.

Proof Let x be any node in an n-node Fibonacci heap, and let $k = x.degree$. By Lemma 19.4, we have $n \geq \text{size}(x) \geq \phi^k$. Taking base-$\phi$ logarithms gives us $k \leq \log_\phi n$. (In fact, because k is an integer, $k \leq \lfloor \log_\phi n \rfloor$.) The maximum degree $D(n)$ of any node is thus $O(\lg n)$. ■

Exercises

19.4-1
Professor Pinocchio claims that the height of an n-node Fibonacci heap is $O(\lg n)$. Show that the professor is mistaken by exhibiting, for any positive integer n, a sequence of Fibonacci-heap operations that creates a Fibonacci heap consisting of just one tree that is a linear chain of n nodes.

19.4-2
Suppose we generalize the cascading-cut rule to cut a node x from its parent as soon as it loses its kth child, for some integer constant k. (The rule in Section 19.3 uses $k = 2$.) For what values of k is $D(n) = O(\lg n)$?

Problems

19-1 Alternative implementation of deletion
Professor Pisano has proposed the following variant of the FIB-HEAP-DELETE procedure, claiming that it runs faster when the node being deleted is not the node pointed to by $H.min$.

```
PISANO-DELETE(H, x)
1  if x == H.min
2      FIB-HEAP-EXTRACT-MIN(H)
3  else y = x.p
4      if y ≠ NIL
5          CUT(H, x, y)
6          CASCADING-CUT(H, y)
7      add x's child list to the root list of H
8      remove x from the root list of H
```

a. The professor's claim that this procedure runs faster is based partly on the assumption that line 7 can be performed in $O(1)$ actual time. What is wrong with this assumption?

b. Give a good upper bound on the actual time of PISANO-DELETE when x is not $H.min$. Your bound should be in terms of $x.degree$ and the number c of calls to the CASCADING-CUT procedure.

c. Suppose that we call PISANO-DELETE(H, x), and let H' be the Fibonacci heap that results. Assuming that node x is not a root, bound the potential of H' in terms of $x.degree$, c, $t(H)$, and $m(H)$.

d. Conclude that the amortized time for PISANO-DELETE is asymptotically no better than for FIB-HEAP-DELETE, even when $x \neq H.min$.

19-2 Binomial trees and binomial heaps

The ***binomial tree*** B_k is an ordered tree (see Section B.5.2) defined recursively. As shown in Figure 19.6(a), the binomial tree B_0 consists of a single node. The binomial tree B_k consists of two binomial trees B_{k-1} that are linked together so that the root of one is the leftmost child of the root of the other. Figure 19.6(b) shows the binomial trees B_0 through B_4.

a. Show that for the binomial tree B_k,

1. there are 2^k nodes,
2. the height of the tree is k,
3. there are exactly $\binom{k}{i}$ nodes at depth i for $i = 0, 1, \ldots, k$, and
4. the root has degree k, which is greater than that of any other node; moreover, as Figure 19.6(c) shows, if we number the children of the root from left to right by $k-1, k-2, \ldots, 0$, then child i is the root of a subtree B_i.

A ***binomial heap*** H is a set of binomial trees that satisfies the following properties:

1. Each node has a *key* (like a Fibonacci heap).
2. Each binomial tree in H obeys the min-heap property.
3. For any nonnegative integer k, there is at most one binomial tree in H whose root has degree k.

b. Suppose that a binomial heap H has a total of n nodes. Discuss the relationship between the binomial trees that H contains and the binary representation of n. Conclude that H consists of at most $\lfloor \lg n \rfloor + 1$ binomial trees.

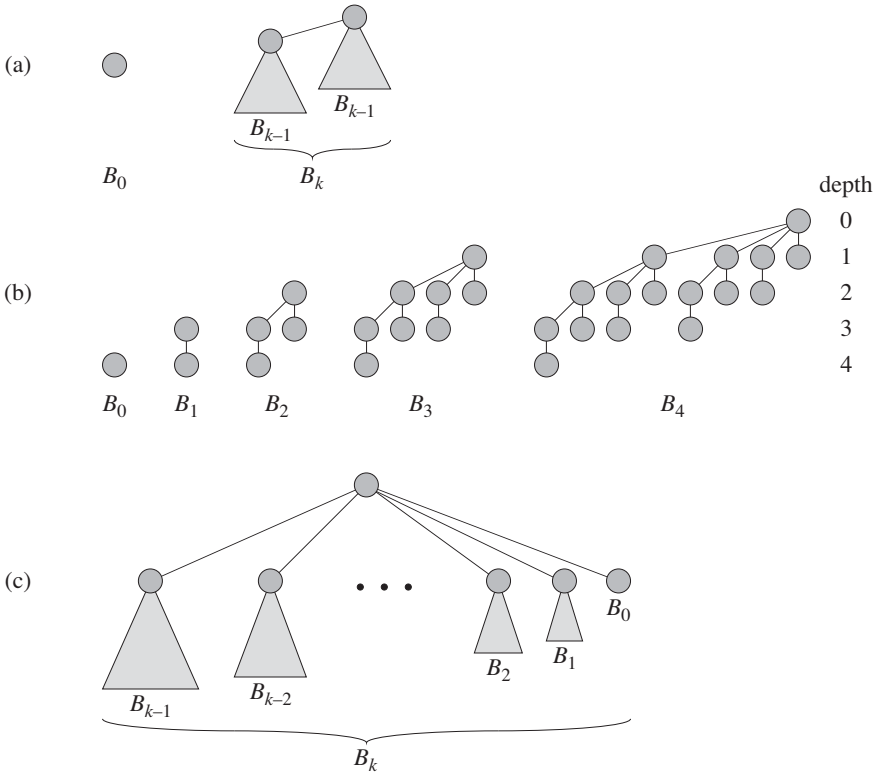

Figure 19.6 **(a)** The recursive definition of the binomial tree B_k. Triangles represent rooted subtrees. **(b)** The binomial trees B_0 through B_4. Node depths in B_4 are shown. **(c)** Another way of looking at the binomial tree B_k.

Suppose that we represent a binomial heap as follows. The left-child, right-sibling scheme of Section 10.4 represents each binomial tree within a binomial heap. Each node contains its key; pointers to its parent, to its leftmost child, and to the sibling immediately to its right (these pointers are NIL when appropriate); and its degree (as in Fibonacci heaps, how many children it has). The roots form a singly linked root list, ordered by the degrees of the roots (from low to high), and we access the binomial heap by a pointer to the first node on the root list.

c. Complete the description of how to represent a binomial heap (i.e., name the attributes, describe when attributes have the value NIL, and define how the root list is organized), and show how to implement the same seven operations on binomial heaps as this chapter implemented on Fibonacci heaps. Each operation should run in $O(\lg n)$ worst-case time, where n is the number of nodes in

the binomial heap (or in the case of the UNION operation, in the two binomial heaps that are being united). The MAKE-HEAP operation should take constant time.

d. Suppose that we were to implement only the mergeable-heap operations on a Fibonacci heap (i.e., we do not implement the DECREASE-KEY or DELETE operations). How would the trees in a Fibonacci heap resemble those in a binomial heap? How would they differ? Show that the maximum degree in an n-node Fibonacci heap would be at most $\lfloor \lg n \rfloor$.

e. Professor McGee has devised a new data structure based on Fibonacci heaps. A McGee heap has the same structure as a Fibonacci heap and supports just the mergeable-heap operations. The implementations of the operations are the same as for Fibonacci heaps, except that insertion and union consolidate the root list as their last step. What are the worst-case running times of operations on McGee heaps?

19-3 More Fibonacci-heap operations

We wish to augment a Fibonacci heap H to support two new operations without changing the amortized running time of any other Fibonacci-heap operations.

a. The operation FIB-HEAP-CHANGE-KEY(H, x, k) changes the key of node x to the value k. Give an efficient implementation of FIB-HEAP-CHANGE-KEY, and analyze the amortized running time of your implementation for the cases in which k is greater than, less than, or equal to $x.key$.

b. Give an efficient implementation of FIB-HEAP-PRUNE(H, r), which deletes $q = \min(r, H.n)$ nodes from H. You may choose any q nodes to delete. Analyze the amortized running time of your implementation. (*Hint:* You may need to modify the data structure and potential function.)

19-4 2-3-4 heaps

Chapter 18 introduced the 2-3-4 tree, in which every internal node (other than possibly the root) has two, three, or four children and all leaves have the same depth. In this problem, we shall implement ***2-3-4 heaps***, which support the mergeable-heap operations.

The 2-3-4 heaps differ from 2-3-4 trees in the following ways. In 2-3-4 heaps, only leaves store keys, and each leaf x stores exactly one key in the attribute $x.key$. The keys in the leaves may appear in any order. Each internal node x contains a value $x.small$ that is equal to the smallest key stored in any leaf in the subtree rooted at x. The root r contains an attribute $r.height$ that gives the height of the

tree. Finally, 2-3-4 heaps are designed to be kept in main memory, so that disk reads and writes are not needed.

Implement the following 2-3-4 heap operations. In parts (a)–(e), each operation should run in $O(\lg n)$ time on a 2-3-4 heap with n elements. The UNION operation in part (f) should run in $O(\lg n)$ time, where n is the number of elements in the two input heaps.

a. MINIMUM, which returns a pointer to the leaf with the smallest key.

b. DECREASE-KEY, which decreases the key of a given leaf x to a given value $k \leq x.key$.

c. INSERT, which inserts leaf x with key k.

d. DELETE, which deletes a given leaf x.

e. EXTRACT-MIN, which extracts the leaf with the smallest key.

f. UNION, which unites two 2-3-4 heaps, returning a single 2-3-4 heap and destroying the input heaps.

Chapter notes

Fredman and Tarjan [114] introduced Fibonacci heaps. Their paper also describes the application of Fibonacci heaps to the problems of single-source shortest paths, all-pairs shortest paths, weighted bipartite matching, and the minimum-spanning-tree problem.

Subsequently, Driscoll, Gabow, Shrairman, and Tarjan [96] developed "relaxed heaps" as an alternative to Fibonacci heaps. They devised two varieties of relaxed heaps. One gives the same amortized time bounds as Fibonacci heaps. The other allows DECREASE-KEY to run in $O(1)$ worst-case (not amortized) time and EXTRACT-MIN and DELETE to run in $O(\lg n)$ worst-case time. Relaxed heaps also have some advantages over Fibonacci heaps in parallel algorithms.

See also the chapter notes for Chapter 6 for other data structures that support fast DECREASE-KEY operations when the sequence of values returned by EXTRACT-MIN calls are monotonically increasing over time and the data are integers in a specific range.

20 van Emde Boas Trees

In previous chapters, we saw data structures that support the operations of a priority queue—binary heaps in Chapter 6, red-black trees in Chapter 13,[1] and Fibonacci heaps in Chapter 19. In each of these data structures, at least one important operation took $O(\lg n)$ time, either worst case or amortized. In fact, because each of these data structures bases its decisions on comparing keys, the $\Omega(n \lg n)$ lower bound for sorting in Section 8.1 tells us that at least one operation will have to take $\Omega(\lg n)$ time. Why? If we could perform both the INSERT and EXTRACT-MIN operations in $o(\lg n)$ time, then we could sort n keys in $o(n \lg n)$ time by first performing n INSERT operations, followed by n EXTRACT-MIN operations.

We saw in Chapter 8, however, that sometimes we can exploit additional information about the keys to sort in $o(n \lg n)$ time. In particular, with counting sort we can sort n keys, each an integer in the range 0 to k, in time $\Theta(n + k)$, which is $\Theta(n)$ when $k = O(n)$.

Since we can circumvent the $\Omega(n \lg n)$ lower bound for sorting when the keys are integers in a bounded range, you might wonder whether we can perform each of the priority-queue operations in $o(\lg n)$ time in a similar scenario. In this chapter, we shall see that we can: van Emde Boas trees support the priority-queue operations, and a few others, each in $O(\lg \lg n)$ worst-case time. The hitch is that the keys must be integers in the range 0 to $n - 1$, with no duplicates allowed.

Specifically, van Emde Boas trees support each of the dynamic set operations listed on page 230—SEARCH, INSERT, DELETE, MINIMUM, MAXIMUM, SUCCESSOR, and PREDECESSOR—in $O(\lg \lg n)$ time. In this chapter, we will omit discussion of satellite data and focus only on storing keys. Because we concentrate on keys and disallow duplicate keys to be stored, instead of describing the SEARCH

[1] Chapter 13 does not explicitly discuss how to implement EXTRACT-MIN and DECREASE-KEY, but we can easily build these operations for any data structure that supports MINIMUM, DELETE, and INSERT.

operation, we will implement the simpler operation MEMBER(S, x), which returns a boolean indicating whether the value x is currently in dynamic set S.

So far, we have used the parameter n for two distinct purposes: the number of elements in the dynamic set, and the range of the possible values. To avoid any further confusion, from here on we will use n to denote the number of elements currently in the set and u as the range of possible values, so that each van Emde Boas tree operation runs in $O(\lg \lg u)$ time. We call the set $\{0, 1, 2, \ldots, u-1\}$ the ***universe*** of values that can be stored and u the ***universe size***. We assume throughout this chapter that u is an exact power of 2, i.e., $u = 2^k$ for some integer $k \geq 1$.

Section 20.1 starts us out by examining some simple approaches that will get us going in the right direction. We enhance these approaches in Section 20.2, introducing proto van Emde Boas structures, which are recursive but do not achieve our goal of $O(\lg \lg u)$-time operations. Section 20.3 modifies proto van Emde Boas structures to develop van Emde Boas trees, and it shows how to implement each operation in $O(\lg \lg u)$ time.

20.1 Preliminary approaches

In this section, we shall examine various approaches for storing a dynamic set. Although none will achieve the $O(\lg \lg u)$ time bounds that we desire, we will gain insights that will help us understand van Emde Boas trees when we see them later in this chapter.

Direct addressing

Direct addressing, as we saw in Section 11.1, provides the simplest approach to storing a dynamic set. Since in this chapter we are concerned only with storing keys, we can simplify the direct-addressing approach to store the dynamic set as a bit vector, as discussed in Exercise 11.1-2. To store a dynamic set of values from the universe $\{0, 1, 2, \ldots, u-1\}$, we maintain an array $A[0\,..\,u-1]$ of u bits. The entry $A[x]$ holds a 1 if the value x is in the dynamic set, and it holds a 0 otherwise. Although we can perform each of the INSERT, DELETE, and MEMBER operations in $O(1)$ time with a bit vector, the remaining operations—MINIMUM, MAXIMUM, SUCCESSOR, and PREDECESSOR—each take $\Theta(u)$ time in the worst case because

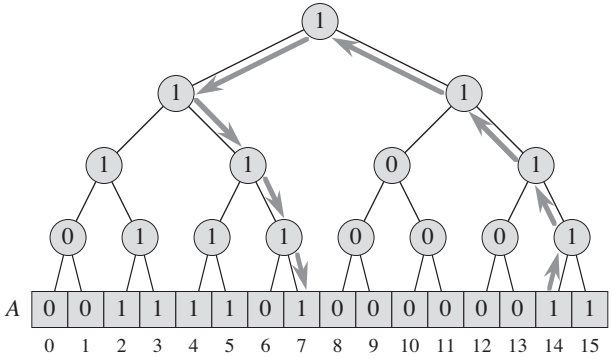

Figure 20.1 A binary tree of bits superimposed on top of a bit vector representing the set $\{2, 3, 4, 5, 7, 14, 15\}$ when $u = 16$. Each internal node contains a 1 if and only if some leaf in its subtree contains a 1. The arrows show the path followed to determine the predecessor of 14 in the set.

we might have to scan through $\Theta(u)$ elements.[2] For example, if a set contains only the values 0 and $u - 1$, then to find the successor of 0, we would have to scan entries 1 through $u - 2$ before finding a 1 in $A[u - 1]$.

Superimposing a binary tree structure

We can short-cut long scans in the bit vector by superimposing a binary tree of bits on top of it. Figure 20.1 shows an example. The entries of the bit vector form the leaves of the binary tree, and each internal node contains a 1 if and only if any leaf in its subtree contains a 1. In other words, the bit stored in an internal node is the logical-or of its two children.

The operations that took $\Theta(u)$ worst-case time with an unadorned bit vector now use the tree structure:

- To find the minimum value in the set, start at the root and head down toward the leaves, always taking the leftmost node containing a 1.
- To find the maximum value in the set, start at the root and head down toward the leaves, always taking the rightmost node containing a 1.

---

[2]We assume throughout this chapter that MINIMUM and MAXIMUM return NIL if the dynamic set is empty and that SUCCESSOR and PREDECESSOR return NIL if the element they are given has no successor or predecessor, respectively.

- To find the successor of x, start at the leaf indexed by x, and head up toward the root until we enter a node from the left and this node has a 1 in its right child z. Then head down through node z, always taking the leftmost node containing a 1 (i.e., find the minimum value in the subtree rooted at the right child z).
- To find the predecessor of x, start at the leaf indexed by x, and head up toward the root until we enter a node from the right and this node has a 1 in its left child z. Then head down through node z, always taking the rightmost node containing a 1 (i.e., find the maximum value in the subtree rooted at the left child z).

Figure 20.1 shows the path taken to find the predecessor, 7, of the value 14.

We also augment the INSERT and DELETE operations appropriately. When inserting a value, we store a 1 in each node on the simple path from the appropriate leaf up to the root. When deleting a value, we go from the appropriate leaf up to the root, recomputing the bit in each internal node on the path as the logical-or of its two children.

Since the height of the tree is $\lg u$ and each of the above operations makes at most one pass up the tree and at most one pass down, each operation takes $O(\lg u)$ time in the worst case.

This approach is only marginally better than just using a red-black tree. We can still perform the MEMBER operation in $O(1)$ time, whereas searching a red-black tree takes $O(\lg n)$ time. Then again, if the number n of elements stored is much smaller than the size u of the universe, a red-black tree would be faster for all the other operations.

Superimposing a tree of constant height

What happens if we superimpose a tree with greater degree? Let us assume that the size of the universe is $u = 2^{2k}$ for some integer k, so that $\sqrt{u}$ is an integer. Instead of superimposing a binary tree on top of the bit vector, we superimpose a tree of degree $\sqrt{u}$. Figure 20.2(a) shows such a tree for the same bit vector as in Figure 20.1. The height of the resulting tree is always 2.

As before, each internal node stores the logical-or of the bits within its subtree, so that the $\sqrt{u}$ internal nodes at depth 1 summarize each group of $\sqrt{u}$ values. As Figure 20.2(b) demonstrates, we can think of these nodes as an array $summary[0\,..\,\sqrt{u}-1]$, where $summary[i]$ contains a 1 if and only if the subarray $A[i\sqrt{u}\,..\,(i+1)\sqrt{u}-1]$ contains a 1. We call this $\sqrt{u}$-bit subarray of A the ith ***cluster***. For a given value of x, the bit $A[x]$ appears in cluster number $\lfloor x/\sqrt{u} \rfloor$. Now INSERT becomes an $O(1)$-time operation: to insert x, set both $A[x]$ and $summary[\lfloor x/\sqrt{u} \rfloor]$ to 1. We can use the *summary* array to perform

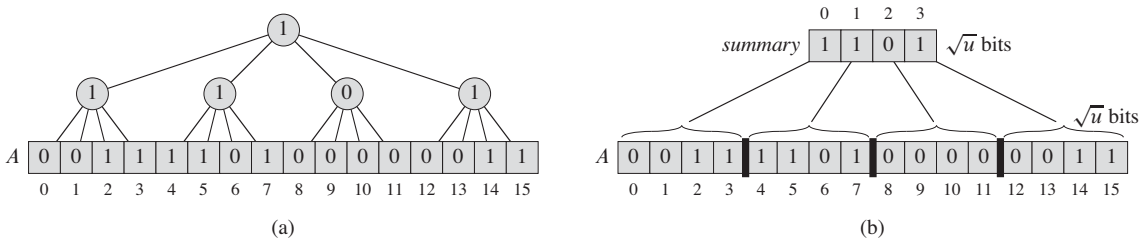

Figure 20.2 **(a)** A tree of degree $\sqrt{u}$ superimposed on top of the same bit vector as in Figure 20.1. Each internal node stores the logical-or of the bits in its subtree. **(b)** A view of the same structure, but with the internal nodes at depth 1 treated as an array $summary[0\,..\,\sqrt{u}-1]$, where $summary[i]$ is the logical-or of the subarray $A[i\sqrt{u}\,..\,(i+1)\sqrt{u}-1]$.

each of the operations MINIMUM, MAXIMUM, SUCCESSOR, PREDECESSOR, and DELETE in $O(\sqrt{u})$ time:

- To find the minimum (maximum) value, find the leftmost (rightmost) entry in *summary* that contains a 1, say $summary[i]$, and then do a linear search within the ith cluster for the leftmost (rightmost) 1.
- To find the successor (predecessor) of x, first search to the right (left) within its cluster. If we find a 1, that position gives the result. Otherwise, let $i = \lfloor x/\sqrt{u} \rfloor$ and search to the right (left) within the *summary* array from index i. The first position that holds a 1 gives the index of a cluster. Search within that cluster for the leftmost (rightmost) 1. That position holds the successor (predecessor).
- To delete the value x, let $i = \lfloor x/\sqrt{u} \rfloor$. Set $A[x]$ to 0 and then set $summary[i]$ to the logical-or of the bits in the ith cluster.

In each of the above operations, we search through at most two clusters of $\sqrt{u}$ bits plus the *summary* array, and so each operation takes $O(\sqrt{u})$ time.

At first glance, it seems as though we have made negative progress. Superimposing a binary tree gave us $O(\lg u)$-time operations, which are asymptotically faster than $O(\sqrt{u})$ time. Using a tree of degree $\sqrt{u}$ will turn out to be a key idea of van Emde Boas trees, however. We continue down this path in the next section.

Exercises

20.1-1

Modify the data structures in this section to support duplicate keys.

20.1-2
Modify the data structures in this section to support keys that have associated satellite data.

20.1-3
Observe that, using the structures in this section, the way we find the successor and predecessor of a value x does not depend on whether x is in the set at the time. Show how to find the successor of x in a binary search tree when x is not stored in the tree.

20.1-4
Suppose that instead of superimposing a tree of degree $\sqrt{u}$, we were to superimpose a tree of degree $u^{1/k}$, where $k > 1$ is a constant. What would be the height of such a tree, and how long would each of the operations take?

20.2 A recursive structure

In this section, we modify the idea of superimposing a tree of degree $\sqrt{u}$ on top of a bit vector. In the previous section, we used a summary structure of size $\sqrt{u}$, with each entry pointing to another stucture of size $\sqrt{u}$. Now, we make the structure recursive, shrinking the universe size by the square root at each level of recursion. Starting with a universe of size u, we make structures holding $\sqrt{u} = u^{1/2}$ items, which themselves hold structures of $u^{1/4}$ items, which hold structures of $u^{1/8}$ items, and so on, down to a base size of 2.

For simplicity, in this section, we assume that $u = 2^{2^k}$ for some integer k, so that $u, u^{1/2}, u^{1/4}, \ldots$ are integers. This restriction would be quite severe in practice, allowing only values of u in the sequence $2, 4, 16, 256, 65536, \ldots$. We shall see in the next section how to relax this assumption and assume only that $u = 2^k$ for some integer k. Since the structure we examine in this section is only a precursor to the true van Emde Boas tree structure, we tolerate this restriction in favor of aiding our understanding.

Recalling that our goal is to achieve running times of $O(\lg \lg u)$ for the operations, let's think about how we might obtain such running times. At the end of Section 4.3, we saw that by changing variables, we could show that the recurrence

$$T(n) = 2T\left(\lfloor \sqrt{n} \rfloor\right) + \lg n \tag{20.1}$$

has the solution $T(n) = O(\lg n \lg \lg n)$. Let's consider a similar, but simpler, recurrence:

$$T(u) = T(\sqrt{u}) + O(1) \, . \tag{20.2}$$

If we use the same technique, changing variables, we can show that recurrence (20.2) has the solution $T(u) = O(\lg \lg u)$. Let $m = \lg u$, so that $u = 2^m$ and we have

$$T(2^m) = T(2^{m/2}) + O(1) .$$

Now we rename $S(m) = T(2^m)$, giving the new recurrence

$$S(m) = S(m/2) + O(1) .$$

By case 2 of the master method, this recurrence has the solution $S(m) = O(\lg m)$. We change back from $S(m)$ to $T(u)$, giving $T(u) = T(2^m) = S(m) = O(\lg m) = O(\lg \lg u)$.

Recurrence (20.2) will guide our search for a data structure. We will design a recursive data structure that shrinks by a factor of $\sqrt{u}$ in each level of its recursion. When an operation traverses this data structure, it will spend a constant amount of time at each level before recursing to the level below. Recurrence (20.2) will then characterize the running time of the operation.

Here is another way to think of how the term $\lg \lg u$ ends up in the solution to recurrence (20.2). As we look at the universe size in each level of the recursive data structure, we see the sequence $u, u^{1/2}, u^{1/4}, u^{1/8}, \ldots$. If we consider how many bits we need to store the universe size at each level, we need $\lg u$ at the top level, and each level needs half the bits of the previous level. In general, if we start with b bits and halve the number of bits at each level, then after $\lg b$ levels, we get down to just one bit. Since $b = \lg u$, we see that after $\lg \lg u$ levels, we have a universe size of 2.

Looking back at the data structure in Figure 20.2, a given value x resides in cluster number $\lfloor x/\sqrt{u} \rfloor$. If we view x as a $\lg u$-bit binary integer, that cluster number, $\lfloor x/\sqrt{u} \rfloor$, is given by the most significant $(\lg u)/2$ bits of x. Within its cluster, x appears in position $x \bmod \sqrt{u}$, which is given by the least significant $(\lg u)/2$ bits of x. We will need to index in this way, and so let us define some functions that will help us do so:

$$\begin{aligned} \text{high}(x) &= \lfloor x/\sqrt{u} \rfloor , \\ \text{low}(x) &= x \bmod \sqrt{u} , \\ \text{index}(x, y) &= x\sqrt{u} + y . \end{aligned}$$

The function $\text{high}(x)$ gives the most significant $(\lg u)/2$ bits of x, producing the number of x's cluster. The function $\text{low}(x)$ gives the least significant $(\lg u)/2$ bits of x and provides x's position within its cluster. The function $\text{index}(x, y)$ builds an element number from x and y, treating x as the most significant $(\lg u)/2$ bits of the element number and y as the least significant $(\lg u)/2$ bits. We have the identity $x = \text{index}(\text{high}(x), \text{low}(x))$. The value of u used by each of these functions will

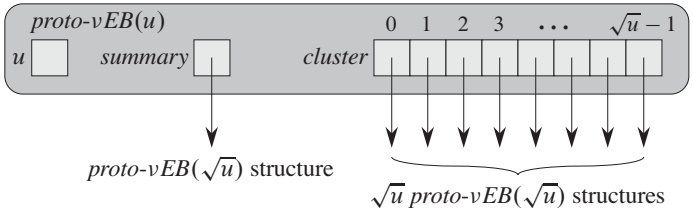

Figure 20.3 The information in a *proto-vEB*(u) structure when $u \geq 4$. The structure contains the universe size u, a pointer *summary* to a *proto-vEB*$(\sqrt{u})$ structure, and an array *cluster*$[0\,..\,\sqrt{u}-1]$ of $\sqrt{u}$ pointers to *proto-vEB*$(\sqrt{u})$ structures.

always be the universe size of the data structure in which we call the function, which changes as we descend into the recursive structure.

20.2.1 Proto van Emde Boas structures

Taking our cue from recurrence (20.2), let us design a recursive data structure to support the operations. Although this data structure will fail to achieve our goal of $O(\lg \lg u)$ time for some operations, it serves as a basis for the van Emde Boas tree structure that we will see in Section 20.3.

For the universe $\{0, 1, 2, \ldots, u-1\}$, we define a ***proto van Emde Boas structure***, or ***proto-vEB structure***, which we denote as *proto-vEB*(u), recursively as follows. Each *proto-vEB*(u) structure contains an attribute u giving its universe size. In addition, it contains the following:

- If $u = 2$, then it is the base size, and it contains an array $A[0\,..\,1]$ of two bits.
- Otherwise, $u = 2^{2^k}$ for some integer $k \geq 1$, so that $u \geq 4$. In addition to the universe size u, the data structure *proto-vEB*(u) contains the following attributes, illustrated in Figure 20.3:
  - a pointer named *summary* to a *proto-vEB*$(\sqrt{u})$ structure and
  - an array *cluster*$[0\,..\,\sqrt{u}-1]$ of $\sqrt{u}$ pointers, each to a *proto-vEB*$(\sqrt{u})$ structure.

The element x, where $0 \leq x < u$, is recursively stored in the cluster numbered high(x) as element low(x) within that cluster.

In the two-level structure of the previous section, each node stores a summary array of size $\sqrt{u}$, in which each entry contains a bit. From the index of each entry, we can compute the starting index of the subarray of size $\sqrt{u}$ that the bit summarizes. In the proto-vEB structure, we use explicit pointers rather than index

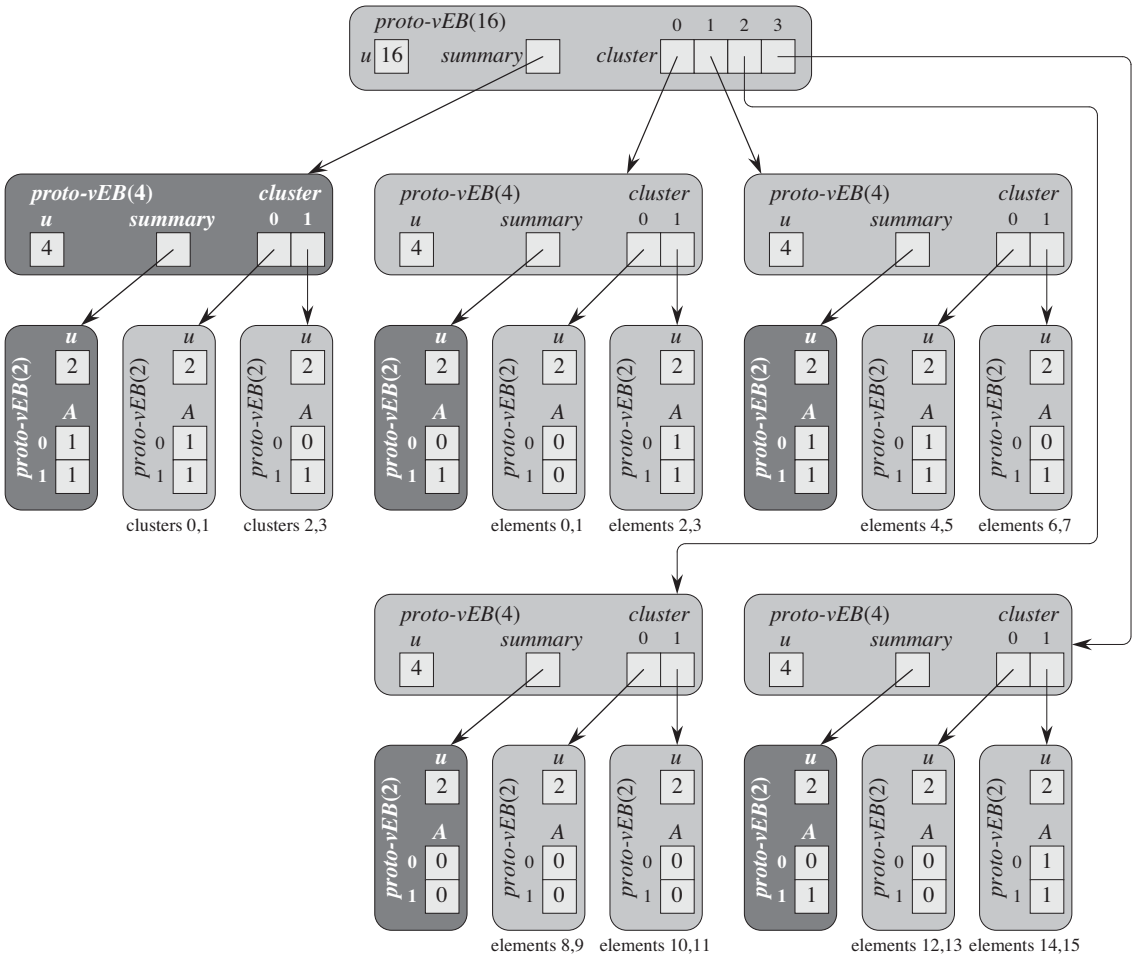

Figure 20.4 A *proto-vEB*(16) structure representing the set $\{2, 3, 4, 5, 7, 14, 15\}$. It points to four *proto-vEB*(4) structures in *cluster*$[0\,..\,3]$, and to a summary structure, which is also a *proto-vEB*(4). Each *proto-vEB*(4) structure points to two *proto-vEB*(2) structures in *cluster*$[0\,..\,1]$, and to a *proto-vEB*(2) summary. Each *proto-vEB*(2) structure contains just an array $A[0\,..\,1]$ of two bits. The *proto-vEB*(2) structures above "elements i, j" store bits i and j of the actual dynamic set, and the *proto-vEB*(2) structures above "clusters i, j" store the summary bits for clusters i and j in the top-level *proto-vEB*(16) structure. For clarity, heavy shading indicates the top level of a proto-vEB structure that stores summary information for its parent structure; such a proto-vEB structure is otherwise identical to any other proto-vEB structure with the same universe size.