1. Introduction

The discipline of game theory was pioneered in the early 20^{th} century by mathematicians Ernst Zermelo (1913) and John von Neumann (1928). The breakthrough came with John von Neumann and Oscar Morgenstern's book, *Theory of Games and Economic Behavior*, published in 1944. This was followed by important work by John Nash (1950-51) and Lloyd Shapley (1953). Game theory had a major influence on the development of several branches of economics (industrial organization, international trade, labor economics, macroeconomics, etc.). Over time the impact of game theory extended to other branches of the social sciences (political science, international relations, philosophy, sociology, anthropology, etc.) as well as to fields outside the social sciences, such as biology, computer science, logic, etc. In 1994 the Nobel Memorial prize in economics was given to three game theorists, John Nash, John Harsanyi and Reinhard Selten, for their theoretical work in game theory which was very influential in economics. At the same time, the US Federal Communications Commission was using game theory to help it design a $7-billion auction of the radio spectrum for personal communication services (naturally, the bidders used game theory too!). The Nobel Memorial prize in economics was awarded to game theorists three more times: in 2005 to Robert Aumann and Thomas Schelling, in 2007 to Leonid Hurwicz, Eric Maskin and Roger Myerson and in 2012 to Lloyd Shapley and Alvin Roth.

Game theory provides a formal language for the representation and analysis of *interactive situations*, that is, situations where several "entities", called *players*, take actions that affect each other. The nature of the players varies depending on the context in which the game theoretic language is invoked: in evolutionary biology (see, for example, John Maynard Smith, 1982) players are non-thinking living organisms;[1] in computer science

[1]Evolutionary game theory has been applied not only to the analysis of animal and insect behavior but also to studying the "most successful strategies" for tumor and cancer cells (see, for example, Gerstung *et al.*, 2011).

(see, for example, Shoham-Leyton-Brown, 2008) players are artificial agents; in behavioral game theory (see, for example, Camerer, 2003) players are "ordinary" human beings, etc. Traditionally, however, game theory has focused on interaction among intelligent, sophisticated and rational individuals. For example, Robert Aumann describes game theory as follows:

> "Briefly put, game and economic theory are concerned with the interactive behavior of *Homo rationalis* – rational man. *Homo rationalis* is the species that always acts both purposefully and logically, has well-defined goals, is motivated solely by the desire to approach these goals as closely as possible, and has the calculating ability required to do so." (Aumann, 1985, p. 35.)

This book is concerned with the traditional interpretation of game theory.

Game theory is divided into two main branches. The first is *cooperative game theory*, which assumes that the players can communicate, form coalitions and sign binding agreements. Cooperative game theory has been used, for example, to analyze voting behavior and other issues in political science and related fields.

We will deal exclusively with the other main branch, namely *non-cooperative game theory*. Non-cooperative game theory models situations where the players are either unable to communicate or are able to communicate but cannot sign binding contracts. An example of the latter situation is the interaction among firms in an industry in an environment where antitrust laws make it illegal for firms to reach agreements concerning prices or production quotas or other forms of collusive behavior.

The book is divided into five parts. The printed version of the book is split into two volumes. **Volume 1** covers the basic concepts and encompasses Chapters 1-7 (Parts I and II), while **Volume 2** is devoted to advanced topics, encompassing Chapters 8-16 (Parts III to V).

Part I deals with games with *ordinal* payoffs, that is, with games where the players' preferences over the possible outcomes are only specified in terms of an ordinal ranking (outcome o is better than outcome o' or o is just as good as o'). Chapter 2 covers strategic-form games, Chapter 3 deals with dynamic games with perfect information and Chapter 4 with general dynamic games with (possibly) imperfect information.

Part II is devoted to games with *cardinal* payoffs, that is, with games where the players' preferences extend to uncertain prospects or lotteries: players are assumed to have a consistent ranking of the set of lotteries over basic outcomes. Chapter 5 reviews the theory of expected utility, Chapter 6 discusses the notion of mixed strategy in strategic-form games and of mixed-strategy Nash equilibrium, while Chapter 7 deals with mixed strategies in dynamic games.

Parts III, IV and V cover a number of advanced topics.

Part III deals with the notions of knowledge, common knowledge and belief. Chapter 8 explains how to model what an individual knows and what she is uncertain about and how to extend the analysis to the interactive knowledge of several individuals (e.g. what Individual 1 knows about what Individual 2 knows about some facts or about the state of knowledge of Individual 1). The chapter ends with the notion of common knowledge. Chapter 9 adds probabilistic beliefs to the knowledge structures of the previous chapter and discusses the notions of Bayesian updating, belief revision, like-mindedness and the possibility of "agreeing to disagree". Chapter 10 uses the interactive knowledge-belief structures of the previous two chapters to model the players' state of mind in a possible play of a given game and studies the implications of common knowledge of rationality in strategic-form games.

Part IV focuses on dynamic (or extensive-form) games and on the issue of how to refine the notion of subgame-perfect equilibrium (which was introduced in Chapters 4 and 7). Chapter 11 introduces a simple notion, called weak sequential equilibrium, which achieves some desirable goals (such as the elimination of strictly dominated choices) but fails to provide a refinement of subgame-perfect equilibrium. Chapter 12 explains the more complex notion of sequential equilibrium, which is extensively used in applications of game theory. That notion, however, leaves much to be desired from a practical point of view (it is typically hard to show that an equilibrium is indeed a sequential equilibrium) and also from a conceptual point of view (it appeals to a topological condition, whose interpretation is not clear). Chapter 13 introduces an intermediate notion, called perfect Bayesian equilibrium, whose conceptual justification is anchored in the so called AGM theory of belief revision, extensively studied in philosophy and computer science, which was pioneered by Carlos Alchourrón (a legal scholar), Peter Gärdenfors (a philosopher) and David Makinson (a logician) in 1985. In Chapter 13 we also provide an alternative characterization of sequential equilibrium based on the notion of perfect Bayesian equilibrium, which is free of topological conditions.

Part V deals with the so-called "theory of games of incomplete information", which was pioneered by John Harsanyi (1967-68). This theory is usually explained using the so-called "type-space" approach suggested by Harsanyi. However, we follow a different approach: the so-called "state-space" approach, which makes use of the interactive knowledge-belief structures developed in Part III. We find this approach both simpler and more elegant. For completeness, in Chapter 16 we explain the commonly used type-based structures and show how to convert a state-space structure into a type-space structure and *vice versa* (the two approaches are equivalent). Chapter 14 deals with situations of incomplete information that involve static (or strategic-form) games, while Chapter 15 deals with situations of incomplete information that involve dynamic (or extensive-form) games.

At the end of each section of each chapter the reader is invited to try the exercises for that section. All the exercises are collected in the penultimate section of the chapter, followed by a section containing complete and detailed answers for each exercise. For each chapter, the set of exercises culminates in a "challenging question", which is more difficult and more time consuming than the other exercises. In game theory, as in mathematics in general, it is essential to test one's understanding of the material by attempting to solve exercises and problems. The reader is encouraged to attempt solving exercises after the introduction of every new concept.

The spacing in this book does not necessarily follow conventional formatting standards. Rather, it is the editor's intention that each step is made plain in order for the student to easily follow along and quickly discover where he/she may grapple with a complete understanding of the material.

I Games with Ordinal Payoffs

2. Ordinal Games in Strategic Form

2.1 Game frames and games

Game theory deals with interactive situations where two or more individuals, called players, make decisions that jointly determine the final outcome. To see an example point your browser to the following video:[1]

`https://www.youtube.com/watch?v=tBtr8-VMj0E`.

In this video each of two players, Sarah and Steve, has to pick one of two balls: inside one ball appears the word 'split' and inside the other the word 'steal' (each player is first asked to secretly check which of the two balls in front of him/her is the split ball and which is the steal ball). They make their decisions simultaneously. The possible outcomes are shown in Figure 2.1, where each row is labeled with a possible choice for Sarah and each column with a possible choice for Steven. Each cell in the table thus corresponds to a possible pair of choices and the resulting outcome is written inside the cell.

		Steven: Split		Steven: Steal	
Sarah	Split	Sarah gets \$50,000	Steven gets \$50,000	Sarah gets nothing	Steven gets \$100,000
	Steal	Sarah gets \$100,000	Steven gets nothing	Sarah gets nothing	Steven gets nothing

Figure 2.1: The *Golden Balls* "game"

What should a rational player do in such a situation? It is tempting to reason as follows.

[1]The video shows an excerpt from *Golden Balls*, a British daytime TV game show. If you search for 'Split or Steal' on youtube.com you will find several instances of this game.

Let us focus on Sarah's decision problem. She realizes that her decision alone is not sufficient to determine the outcome; she has no control over what Steven will choose to do. However, she can envision two scenarios: one where Steven chooses *Steal* and the other where he chooses *Split*.

- If Steven decides to *Steal*, then it does not matter what Sarah does, because she ends up with nothing, no matter what she chooses.
- If Steven picks *Split*, then Sarah will get either $\$50,000$ (if she also picks *Split*) or $\$100,000$ (if she picks *Steal*).

Thus Sarah should choose *Steal*.

The above argument, however, is not valid because it is based on an implicit and unwarranted assumption about how Sarah ranks the outcomes; namely, it assumes that Sarah is selfish and greedy, which may or may not be true. Let us denote the outcomes as follows:

o_1 :	Sarah gets $\$50,000$ and Steven gets $\$50,000$.
o_2 :	Sarah gets nothing and Steven gets $\$100,000$.
o_3 :	Sarah gets $\$100,000$ and Steven gets nothing.
o_4 :	Sarah gets nothing and Steven gets nothing.

Table 2.1: Names for the outcomes shown in Figure 2.1

If, indeed, Sarah is *selfish and greedy* – in the sense that, in evaluating the outcomes, she focuses exclusively on what she herself gets and prefers more money to less – then her ranking of the outcomes is as follows: $o_3 \succ o_1 \succ o_2 \sim o_4$ (which reads 'o_3 is better than o_1, o_1 is better than o_2 and o_2 is just as good as o_4'). But there are other possibilities. For example, Sarah might be *fair-minded* and view the outcome where both get $\$50,000$ as better than all the other outcomes. For instance, her ranking could be $o_1 \succ o_3 \succ o_2 \succ o_4$; according to this ranking, besides valuing fairness, she also displays *benevolence* towards Steven, in the sense that – when comparing the two outcomes where she gets nothing, namely o_2 and o_4 – she prefers the one where at least Steven goes home with some money. If, in fact, Sarah is fair-minded and benevolent, then the logic underlying the above argument would yield the opposite conclusion, namely that she should choose *Split*.

Thus we cannot presume to know the answer to the question "What is the rational choice for Sarah?" if we don't know what her preferences are. It is a common mistake (unfortunately one that even game theorists sometimes make) to reason under the assumption that players are selfish and greedy. This is, typically, an unwarranted assumption. Research in experimental psychology, philosophy and economics has amply demonstrated that many people are strongly motivated by considerations of fairness. Indeed, fairness seems to motivate not only humans but also primates, as shown in the following video:[2]
`http://www.ted.com/talks/frans_de_waal_do_animals_have_morals`.

The situation illustrated in Figure 2.1 is not a game as we have no information about the preferences of the players; we use the expression *game-frame* to refer to it. In the case

[2]Also available at `https://www.youtube.com/watch?v=GcJxRqTs5nk`

where there are only two players and each player has a small number of possible choices (also called *strategies*), a game-frame can be represented – as we did in Figure 2.1 – by means of a table, with as many rows as the number of possible strategies of Player 1 and as many columns as the number of strategies of Player 2; each row is labeled with one strategy of Player 1 and each column with one strategy of Player 2; inside each cell of the table (which corresponds to a pair of strategies, one for Player 1 and one for Player 2) we write the corresponding outcome.

Before presenting the definition of game-frame, we remind the reader of what the Cartesian product of two or more sets is. Let S_1 and S_2 be two sets. Then the Cartesian product of S_1 and S_2, denoted by $S_1 \times S_2$, is the set of *ordered pairs* (x_1, x_2) where x_1 is an element of S_1 ($x_1 \in S_1$) and x_2 is an element of S_2 ($x_2 \in S_2$). For example, if $S_1 = \{a, b, c\}$ and $S_2 = \{D, E\}$ then

$$S_1 \times S_2 = \{(a,D), (a,E), (b,D), (b,E), (c,D), (c,E)\}.$$

The definition extends to the general case of n sets ($n \geq 2$): an element of $S_1 \times S_2 \times ... \times S_n$ is an ordered n-tuple $(x_1, x_2, ..., x_n)$ where, for each $i = 1, \ldots, n$, $x_i \in S_i$.

The definition of game-frame is as follows

Definition 2.1.1 A *game-frame in strategic form* is a list of four items (a quadruple) $\langle I, (S_1, S_2, ..., S_n), O, f \rangle$ where:

- $I = \{1, 2, \ldots, n\}$ is a set of *players* ($n \geq 2$).
- $(S_1, S_2, \ldots, S_n)$ is a list of sets, one for each player. For every Player $i \in I$, S_i is the set of *strategies* (or possible choices) of Player i. We denote by S the Cartesian product of these sets: $S = S_1 \times S_2 \times \cdots \times S_n$; thus an element of S is a list $s = (s_1, s_2, \ldots, s_n)$ consisting of one strategy for each player. We call S the set of *strategy profiles*.
- O is a set of *outcomes*.
- $f : S \to O$ is a function that associates with every strategy profile s an outcome $f(s) \in O$.

Using the notation of Definition 2.1.1, the situation illustrated in Figure 2.1 is the following game-frame in strategic form:

- $I = \{1, 2\}$ (letting Sarah be Player 1 and Steven Player 2),
- $(S_1, S_2) = (\{Split, Steal\}, \{Split, Steal\})$; thus $S_1 = S_2 = \{Split, Steal\}$, so that the set of strategy profiles is
  $S = \{(Split, Split), (Split, Steal), (Steal, Split), (Steal, Steal)\}$,
- O is the set of outcomes listed in Table 2.1,
- f is the following function:

$$\begin{array}{rcccc} s: & (Split, Split) & (Split, Steal) & (Steal, Split) & (Steal, Steal) \\ f(s): & o_1 & o_2 & o_3 & o_4 \end{array}$$

(that is, $f(Split, Split) = o_1$, $f(Split, Steal) = o_2$, etc.).

From a game-frame one obtains a *game* by adding, for each player, her preferences over (or ranking of) the possible outcomes. We use the notation shown in Table 2.2. For example, if M denotes 'Mexican food' and J denotes 'Japanese food', then $M \succ_{Alice} J$ means that Alice prefers Mexican food to Japanese food and $M \sim_{Bob} J$ means that Bob is indifferent between the two.

Notation	**Interpretation**
$o \succsim_i o'$	Player i considers outcome o to be *at least as good as* o' (that is, either better than or just as good as)
$o \succ_i o'$	Player i considers outcome o to be *better than* o' (that is, she prefers o to o')
$o \sim_i o'$	Player i considers outcome o to be *just as good as* o' (that is, she is indifferent between o and o')

Table 2.2: Notation for preference relations

R The "at least as good" relation $\succsim$ is sufficient to capture also strict preference $\succ$ and indifference $\sim$. In fact, starting from $\succsim$, one can define strict preference as follows: $o \succ o'$ if and only if $o \succsim o'$ and $o' \not\succsim o$ and one can define indifference as follows: $o \sim o'$ if and only if $o \succsim o'$ and $o' \succsim o$.

We will assume throughout this book that the "at least as good" relation $\succsim_i$ of Player i – which embodies her preferences over (or ranking of) the outcomes – is complete (for every two outcomes o_1 and o_2, either $o_1 \succsim_i o_2$ or $o_2 \succsim_i o_1$, or both) and transitive (if $o_1 \succsim_i o_2$ and $o_2 \succsim_i o_3$ then $o_1 \succsim_i o_3$).[3]

There are (at least) four ways of representing, or expressing, a complete and transitive preference relation over (or ranking of) a set of outcomes. For example, suppose that $O = \{o_1, o_2, o_3, o_4, o_5\}$ and that we want to represent the following ranking (expressing the preferences of a given individual): o_3 is better than o_5, which is just as good as o_1, o_1 is better than o_4, which, in turn, is better than o_2 (thus, o_3 is the best outcome and o_2 is the worst outcome). We can represent this ranking in one of the following ways:

- As a subset of $O \times O$ (the interpretation of $(o, o') \in O \times O$ is that o is at least as good as o'):

$$\begin{array}{l}
\{(o_1,o_1),(o_1,o_2),(o_1,o_4),(o_1,o_5) \\
(o_2,o_2), \\
(o_3,o_1),(o_3,o_2),(o_3,o_3),(o_3,o_4),(o_3,o_5), \\
(o_4,o_2),(o_4,o_4), \\
(o_5,o_1),(o_5,o_2),(o_5,o_4),(o_5,o_5)\}
\end{array}$$

- By using the notation of Table 2.2: $o_3 \succ o_5 \sim o_1 \succ o_4 \succ o_2$.

[3]Transitivity of the "at least as good" relation implies transitivity of the indifference relation (if $o_1 \sim o_2$ and $o_2 \sim o_3$ then $o_1 \sim o_3$) as well as transitivity of the strict preference relation (not only in the sense that (1) if $o_1 \succ o_2$ and $o_2 \succ o_3$ then $o_1 \succ o_3$, but also that (2) if $o_1 \succ o_2$ and $o_2 \sim o_3$ then $o_1 \succ o_3$ and (3) if $o_1 \sim o_2$ and $o_2 \succ o_3$ then $o_1 \succ o_3$).

- By listing the outcomes in a column, starting with the best at the top and proceeding down to the worst, thus using the convention that if outcome o is listed above outcome o' then o is preferred to o', while if o and o' are written next to each other (on the same row), then they are considered to be just as good:

$$\begin{array}{ll} \text{best} & o_3 \\ & o_1, o_5 \\ & o_4 \\ \text{worst} & o_2 \end{array}$$

- By assigning a number to each outcome, with the convention that *if the number assigned to o is greater than the number assigned to o'* then o is preferred to o', and if two outcomes are assigned the same number then they are considered to be just as good. For example, we could choose the following numbers: $\begin{array}{ccccc} o_1 & o_2 & o_3 & o_4 & o_5 \\ 6 & 1 & 8 & 2 & 6 \end{array}$. Such an assignment of numbers is called a *utility function*. A useful way of thinking of utility is as an "index of satisfaction": the higher the index the better the outcome; however, this suggestion is just to aid memory and should be taken with a grain of salt because a utility function *does not measure anything* and, furthermore, as explained below, the actual numbers used as utility indices are completely arbitrary.[4]

Definition 2.1.2 Given a complete and transitive ranking $\succsim$ of a finite set of outcomes O, a function $U : O \to \mathbb{R}$ (where $\mathbb{R}$ denotes the set of real numbers)[a] is said to be an *ordinal utility function that represents the ranking* $\succsim$ if, for every two outcomes o and o', $U(o) > U(o')$ if and only if $o \succ o'$ and $U(o) = U(o')$ if and only if $o \sim o'$. The number $U(o)$ is called the *utility of outcome o*.[b]

[a] The notation $f : X \to Y$ is used to denote a function which associates with every $x \in X$ an element $y = f(x)$ with $y \in Y$.

[b] Thus, $o \succsim o'$ if and only if $U(o) \geq U(o')$.

R Note that the statement "for Alice the utility of Mexican food is 10" is in itself a meaningless statement; on the other hand, what would be a meaningful statement is "for Alice the utility of Mexican food is 10 and the utility of Japanese food is 5" , because such a statement conveys the information that she prefers Mexican food to Japanese food. However, the two numbers 10 and 5 have no other meaning besides the fact that 10 is greater than 5: for example, we **cannot** infer from these numbers that she considers Mexican food twice as good as Japanese food. The reason for this is that we could have expressed the same fact, namely that she prefers Mexican food to Japanese food, by assigning utility 100 to Mexican food and -25 to Japanese food, or with any other two numbers (as long as the number assigned to Mexican food is larger than the number assigned to Japanese food).

[4] Note that assigning a utility of 1 to an outcome o does not mean that o is the "first choice". Indeed, in this example a utility of 1 is assigned to the worst outcome: o_2 is the worst outcome because it has the lowest utility (which happens to be 1, in this example).

It follows from the above remark that there is an infinite number of utility functions that represent the same ranking. For instance, the following are equivalent ways of representing the ranking $o_3 \succ o_1 \succ o_2 \sim o_4$ (f, g and h are three out of the many possible utility functions):

outcome $\rightarrow$ *utility function* $\downarrow$	o_1	o_2	o_3	o_4
f :	5	2	10	2
g :	0.8	0.7	1	0.7
h :	27	1	100	1

Utility functions are a particularly convenient way of representing preferences. In fact, by using utility functions one can give a more condensed representation of games, as explained in the last paragraph of the following definition.

Definition 2.1.3 An *ordinal game in strategic form* is a quintuple $\langle I, (S_1,\ldots,S_n), O, f, (\succsim_1,\ldots,\succsim_n)\rangle$ where:

- $\langle I, (S_1,\ldots,S_n), O, f\rangle$ is a game-frame in strategic form (Definition 2.1.1) and
- for every Player $i \in I$, $\succsim_i$ is a complete and transitive ranking of the set of outcomes O.

If we replace each ranking $\succsim_i$ with a utility function U_i that represents it, and we assign, to each strategy profile s, Player i's utility of $f(s)$ $\big($recall that $f(s)$ is the outcome associated with $s\big)$ then we obtain a function $\pi_i : S \rightarrow \mathbb{R}$ called Player i's *payoff function*. Thus $\pi_i(s) = U_i(f(s))$.[a] Having done so, we obtain a triple $\langle I, (S_1,\ldots,S_n), (\pi_1,\ldots,\pi_n)\rangle$ called a *reduced-form ordinal game in strategic form* ('reduced-form' because some information is lost, namely the specification of the possible outcomes).

[a]Note that, in this book, the symbol π is not used to denote the irrational number used to compute the circumference and area of a circle, but rather as the Greek letter for 'p' which stands for 'payoff'.

For example, take the game-frame illustrated in Figure 2.1, let Sarah be Player 1 and Steven Player 2 and name the possible outcomes as shown in Table 2.1. Let us add the information that both players are selfish and greedy (that is, Player 1's ranking is $o_3 \succ_1 o_1 \succ_1 o_2 \sim_1 o_4$ and Player 2's ranking is $o_2 \succ_2 o_1 \succ_2 o_3 \sim_2 o_4$) and let us represent their rankings with the following utility functions (note, again, that the choice of numbers 2, 3 and 4 for utilities is arbitrary: any other three numbers would do):

outcome $\rightarrow$ *utility function* $\downarrow$	o_1	o_2	o_3	o_4
U_1 (Player 1):	3	2	4	2
U_2 (Player 2):	3	4	2	2

Then we obtain the reduced-form game shown in Figure 2.2, where in each cell the first number is the payoff of Player 1 and the second number is the payoff of Player 2.

		Player 2 (Steven)	
		Split	Steal
Player 1 (Sarah)	Split	3 3	2 4
	Steal	4 2	2 2

Figure 2.2: One possible game based on the game-frame of Figure 2.1

On the other hand, if we add to the game-frame of Figure 2.1 the information that Player 1 is fair-minded and benevolent (that is, her ranking is $o_1 \succ_1 o_3 \succ_1 o_2 \succ_1 o_4$), while Player 2 is selfish and greedy and represent these rankings with the following utility functions:

outcome → *utility function* ↓	o_1	o_2	o_3	o_4
U_1 (Player 1):	4	2	3	1
U_2 (Player 2):	3	4	2	2

then we obtain the reduced-form game shown in Figure 2.3.

		Player 2 (Steven)	
		Split	Steal
Player 1 (Sarah)	Split	4 3	2 4
	Steal	3 2	1 2

Figure 2.3: Another possible game based on the game-frame of Figure 2.1

In general, a player will act differently in different games, even if they are based on the same game-frame, because her incentives and objectives (as captured by her ranking of the outcomes) will be different. For example, one can argue that in the game of Figure 2.2 a rational Player 1 would choose *Steal*, while in the game of Figure 2.3 the rational choice for Player 1 is *Split*.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 2.9.1 at the end of this chapter.

2.2 Strict and weak dominance

In this section we define two relations on the set of strategies of a player. Before introducing the formal definition, we shall illustrate these notion with an example. The first relation is called "strict dominance". Let us focus our attention on one player, say Player 1, and select two of her strategies, say a and b. We say that a **strictly** dominates b (for Player 1) if, for every possible strategy profile of the other players, strategy a of Player 1, in conjunction with the strategies selected by the other players, yields a payoff for Player 1 which is **greater than** the payoff associated with strategy b (in conjunction with the strategies selected by the other players). For example, consider the following two-player game, where only the payoffs of Player 1 are shown:

		Player 2		
		E	F	G
Player 1	A	3 ...	2 ...	1 ...
	B	2 ...	1 ...	0 ...
	C	3 ...	2 ...	1 ...
	D	2 ...	0 ...	0 ...

Figure 2.4: A game showing only the payoffs of Player 1

In this game for Player 1 strategy A strictly dominates strategy B:

- if Player 2 selects E then A in conjunction with E gives Player 1 a payoff of 3, while B in conjunction with E gives her only a payoff of 2,
- if Player 2 selects F then A in conjunction with F gives Player 1 a payoff of 2, while B in conjunction with F gives her only a payoff of 1,
- if Player 2 selects G then A in conjunction with G gives Player 1 a payoff of 1, while B in conjunction with G gives her only a payoff of 0.

In the game of Figure 2.4 we also have that A strictly dominates D and C strictly dominates D; however, it is not the case that B strictly dominates D because, in conjunction with strategy E of Player 2, B and D yield the same payoff for Player 1.

The second relation is called "weak dominance". The definition is similar to that of strict dominance, but we replace 'greater than' with 'greater than or equal to' while insisting on at least one strict inequality: a **weakly** dominates b (for Player 1) if, for every possible strategy profile of the other players, strategy a of Player 1, in conjunction with the strategies selected by the other players, yields a payoff for Player 1 which is **greater than or equal to** the payoff associated with strategy b (in conjunction with the strategies selected by the other players) and, furthermore, there is at least one strategy profile of the other players against which strategy a gives a larger payoff to Player 1 than strategy b. In the example of Figure 2.4, we have that, while it is not true that B strictly dominates D, it is true that B weakly dominates D:

- if Player 2 selects E, then B in conjunction with E gives Player 1 the same payoff as D in conjunction with E (namely 2),
- if Player 2 selects F, then B in conjunction with F gives Player 1 a payoff of 1, while D in conjunction with F gives her only a payoff of 0,
- if Player 2 selects G then B in conjunction with G gives Player 1 the same payoff as D in conjunction with G (namely 0).

In order to give the definitions in full generality we need to introduce some notation. Recall that S denotes the set of strategy profiles, that is, an element s of S is an ordered list of strategies $s = (s_1, ..., s_n)$, one for each player. We will often want to focus on one player, say Player i, and view s as a pair consisting of the strategy of Player i and the remaining strategies of all the other players. For example, suppose that there are three players and the strategy sets are as follows: $S_1 = \{a,b,c\}$, $S_2 = \{d,e\}$ and $S_3 = \{f,g\}$. Then one possible strategy profile is $s = (b,d,g)$ (thus $s_1 = b$, $s_2 = d$ and $s_3 = g$). If we focus on, say, Player 2 then we will denote by s_{-2} the sub-profile consisting of the strategies of the players *other than* 2: in this case $s_{-2} = (b,g)$. This gives us an alternative way of denoting s, namely as (s_2, s_{-2}). Continuing our example where $s = (b,d,g)$, letting $s_{-2} = (b,g)$, we can denote s also by (d, s_{-2}) and we can write the result of replacing Player 2's strategy d with her strategy e in s by (e, s_{-2}); thus $(d, s_{-2}) = (b,d,g)$ while $(e, s_{-2}) = (b,e,g)$. In general, given a Player i, we denote by S_{-i} the set of strategy profiles of the players other than i (that is, S_{-i} is the Cartesian product of the strategy sets of the other players; in the above example we have that $S_{-2} = S_1 \times S_3 = \{a,b,c\} \times \{f,g\} = \{(a,f),(a,g),(b,f),(b,g),(c,f),(c,g)\}$. We denote an element of S_{-i} by s_{-i}.

Definition 2.2.1 Given an ordinal game in strategic form, let i be a Player and a and b two of her strategies ($a, b \in S_i$). We say that, for Player i,

- *a strictly dominates b* (or *b is strictly dominated by a*) if, in every situation (that is, no matter what the other players do), a gives Player i a payoff which is *greater* than the payoff that b gives. Formally: for every $s_{-i} \in S_{-i}$, $\pi_i(a, s_{-i}) > \pi_i(b, s_{-i})$.[a]
- *a weakly dominates b* (or *b is weakly dominated by a*) if, in every situation, a gives Player i a payoff which is *greater than or equal to* the payoff that b gives and, furthermore, there is at least one situation where a gives a greater payoff than b. Formally: for every $s_{-i} \in S_{-i}$, $\pi_i(a, s_{-i}) \geq \pi_i(b, s_{-i})$ and there exists an $\bar{s}_{-i} \in S_{-i}$ such that $\pi_i(a, \bar{s}_{-i}) > \pi_i(b, \bar{s}_{-i})$.[b]
- *a is equivalent to b* if, in every situation, a and b give Player i the *same* payoff. Formally: for every $s_{-i} \in S_{-i}$, $\pi_i(a, s_{-i}) = \pi_i(b, s_{-i})$.[c]

[a] Or, stated in terms of rankings instead of payoffs, $f(a, s_{-i}) \succ_i f(b, s_{-i})$ for every $s_{-i} \in S_{-i}$.

[b] Or, stated in terms of rankings, $f(a, s_{-i}) \succsim_i f(b, s_{-i})$, for every $s_{-i} \in S_{-i}$, and there exists an $\bar{s}_{-i} \in S_{-i}$ such that $f(a, \bar{s}_{-i}) \succ_i f(b, \bar{s}_{-i})$.

[c] Or, stated in terms of rankings, $f(a, s_{-i}) \sim_i f(b, s_{-i})$, for every $s_{-i} \in S_{-i}$.

For example, in the game of Figure 2.5 (which reproduces Figure 2.4), we have that

- A strictly dominates B.
- A and C are equivalent.
- A strictly dominates D.
- B is strictly dominated by C.
- B weakly (but not strictly) dominates D.
- C strictly dominates D.

Player 2

Player 1		*E*	*F*	*G*
	A	3 ...	2 ...	1 ...
	B	2 ...	1 ...	0 ...
	C	3 ...	2 ...	1 ...
	D	2 ...	0 ...	0 ...

Figure 2.5: Copy of Figure 2.4

R Note that if strategy a strictly dominates strategy b then it also satisfies the conditions for weak dominance, that is, 'a strictly dominates b' implies 'a weakly dominates b'. Throughout the book the expression 'a *weakly* dominates b' will be interpreted as 'a dominates b *weakly but not strictly*'.

The expression 'a dominates b' can be understood as 'a is better than b'. The next term we define is 'dominant' which can be understood as 'best'. Thus one cannot meaningfully say "a dominates" because one needs to name another strategy that is dominated by a; for example, one would have to say "a dominates b". On the other hand, one *can* meaningfully say "a is dominant" because it is like saying "a is best", which means "a is better than every other strategy".

Definition 2.2.2 Given an ordinal game in strategic form, let i be a Player and a one of her strategies ($a \in S_i$). We say that, for Player i,

- a is a *strictly dominant* strategy if a strictly dominates every other strategy of Player i.
- a is a *weakly dominant* strategy if, for every other strategy x of Player i, one of the following is true: either (1) a weakly dominates x or (2) a is equivalent to x.

For example, in the game shown in Figure 2.5, A and C are both weakly dominant strategies for Player 1. Note that if a player has two or more strategies that are weakly dominant, then any two of those strategies must be equivalent. On the other hand, there can be at most one strictly dominant strategy.

R The reader should convince herself/himself that the definition of weakly dominant strategy given in Definition 2.2.2 is equivalent to the following: $a \in S_i$ is a weakly dominant strategy for Player i if and only if, for every $s_{-i} \in S_{-i}$, $\pi_i(a, s_{-i}) \geq \pi_i(s_i, s_{-i})$ for every $s_i \in S_i$.[5]

In accordance with the convention established earlier, the expression 'a is a weakly dominant strategy' will have the default interpretation 'a is a *weakly but not strictly dominant strategy*'.

Note: if you claim that, for some player, "strategy x is (weakly or strictly) dominated" then you ought to name another strategy of that player that dominates x. Saying "x is dominated" is akin to saying "x is worse": worse than what? On the other hand, claiming that strategy x is weakly dominant is akin to claiming that it is best, that is, better than, or just as good as, any other strategy.

Definition 2.2.3 Given an ordinal game in strategic form, let $s = (s_1, \ldots, s_n)$ be a strategy profile. We say that

- s is a *strictly dominant-strategy equilibrium* if, for every Player i, s_i is a strictly dominant strategy.
- s is a *weakly dominant-strategy equilibrium* if, for every Player i, s_i is a weakly dominant strategy and, furthermore, for at least one Player j, s_j is not a strictly dominant strategy.

If we refer to a strategy profile as a dominant-strategy equilibrium, without qualifying it as weak or strict, then the default interpretation will be 'weak'.

In the game of Figure 2.6 (which reproduces Figure 2.2), *Steal* is a weakly dominant strategy for each player and thus (*Steal*,*Steal*) is a weakly dominant-strategy equilibrium.

		Player 2 (Steven)			
		Split		Steal	
Player 1 (Sarah)	Split	3	3	2	4
	Steal	4	2	2	2

Figure 2.6: Copy of Figure 2.2

		Player 2 (Steven)			
		Split		Steal	
Player 1 (Sarah)	Split	4	3	2	4
	Steal	3	2	1	2

Figure 2.7: Copy of Figure 2.3

In the game of Figure 2.7 (which reproduces Figure 2.3), *Split* is a strictly dominant strategy for Player 1, while *Steal* is a weakly (but not strictly) dominant strategy for Player 2 and thus (*Split*,*Steal*) is a weakly dominant-strategy equilibrium.

[5] Or, stated in terms of rankings, for every $s_{-i} \in S_{-i}$, $f(a, s_{-i}) \succsim_i f(s_i, s_{-i})$ for every $s_i \in S_i$..

The *Prisoner's Dilemma* is an example of a game with a strictly dominant-strategy equilibrium. For a detailed account of the history of this game and an in-depth analysis of it see `http://plato.stanford.edu/entries/prisoner-dilemma` or `http://en.wikipedia.org/wiki/Prisoner's_dilemma`.

An instance of the Prisoner's Dilemma is the following situation. Doug and Ed work for the same company and the annual party is approaching. They know that they are the only two candidates for the best-worker-of-the-year prize and at the moment they are tied; however, only one person can be awarded the prize and thus, unless one of them manages to outperform the other, nobody will receive the prize. Each chooses between exerting *Normal* effort or *Extra* effort (that is, work overtime) before the party. The corresponding game-frame is shown in Figure 2.8.

		Player 2 (Ed) Normal effort	Extra effort
Player 1 (Doug)	Normal effort	o_1	o_2
	Extra effort	o_3	o_4

o_1 : nobody gets the prize and nobody sacrifices family time

o_2 : Ed gets the prize and sacrifices family time, Doug does not

o_3 : Doug gets the prize and sacrifices family time, Ed does not

o_4 : nobody gets the prize and both sacrifice family time

Figure 2.8: The Prisoner's Dilemma game-frame

Suppose that both Doug and Ed are willing to sacrifice family time to get the prize, but otherwise value family time; furthermore, they are envious of each other, in the sense that they prefer nobody getting the prize to the other person's getting the prize (even at the personal cost of sacrificing family time). That is, their rankings are as follows: $o_3 \succ_{Doug} o_1 \succ_{Doug} o_4 \succ_{Doug} o_2$ and $o_2 \succ_{Ed} o_1 \succ_{Ed} o_4 \succ_{Ed} o_3$. Using utility functions with values from the set $\{0,1,2,3\}$ we can represent the game in reduced form as shown in Figure 2.9. In this game exerting extra effort is a strictly dominant strategy for every player; thus (*Extra effort, Extra effort*) is a strictly dominant-strategy equilibrium.

Definition 2.2.4 Given an ordinal game in strategic form, let o and o' be two outcomes. We say that o is *strictly Pareto superior* to o' if every player prefers o to o' (that is, if $o \succ_i o'$, for every Player i). We say that o is *weakly Pareto superior* to o' if every player considers o to be at least as good as o' and at least one player prefers o to o' (that is, if $o \succsim_i o'$, for every Player i and there is a Player j such that $o \succ_j o'$).
In reduced-form games, this definition can be extended to strategy profiles as follows. If s and s' are two strategy profiles, then s is *strictly Pareto superior* to s' if $\pi_i(s) > \pi_i(s')$ for every Player i and s is *weakly Pareto superior* to s' if $\pi_i(s) \geq \pi_i(s')$ for every Player i and, furthermore, there is a Player j such that $\pi_j(s) > \pi_j(s')$.

		Player 2 (Ed) Normal effort	Player 2 (Ed) Extra effort
Player 1 (Doug)	Normal effort	2 2	0 3
	Extra effort	3 0	1 1

Figure 2.9: The Prisoner's Dilemma game

For example, in the Prisoner's Dilemma game of Figure 2.9, outcome o_1 is strictly Pareto superior to o_4 or, in terms of strategy profiles, (*Normal effort*, *Normal effort*) is strictly Pareto superior to (*Extra effort*, *Extra effort*).

When a player has a strictly dominant strategy, it would be irrational for that player to choose any other strategy, since she would be guaranteed a lower payoff in every possible situation (that is, no matter what the other players do). Thus in the Prisoner's Dilemma individual rationality leads to (*Extra effort, Extra effort*) despite the fact that both players would be better off if they both chose *Normal effort*. It is obvious that if the players could reach a *binding* agreement to exert normal effort then they would do so; however, the underlying assumption in non-cooperative game theory is that such agreements are not possible (e.g. because of lack of communication or because such agreements are illegal or cannot be enforced in a court of law, etc.). Any non-binding agreement to choose *Normal effort* would not be viable: if one player expects the other player to stick to the agreement, then he will gain by cheating and choosing *Extra effort* (on the other hand, if a player does not believe that the other player will honor the agreement then he will gain by deviating from the agreement herself). The Prisoner's Dilemma game is often used to illustrate a conflict between individual rationality and collective rationality: (*Extra effort, Extra effort*) is the individually rational outcome while (*Normal effort, Normal effort*) would be the collectively rational one.

> Test your understanding of the concepts introduced in this section, by going through the exercises in Section 2.9.2 at the end of this chapter.

2.3 Second-price auction

The *second-price auction*, or *Vickrey auction*, is an example of a game that has a weakly dominant-strategy equilibrium. It is a "sealed-bid" auction where bidders submit bids without knowing the bids of the other participants in the auction. The object which is auctioned is then assigned to the bidder who submits the highest bid (the winner), but the winner pays not her own bid but rather the second-highest bid, that is the highest bid among the bids that remain after removing the winner's own bid. Tie-breaking rules must be specified for selecting the winner when the highest bid is submitted by two or more bidders (in which case the winner ends up paying her own bid, because the second-highest bid is equal to the winner's bid). We first illustrate this auction with an example:

Two oil companies bid for the right to drill a field. The possible bids are \$10 million, \$20 million, ..., \$50 million. In case of ties the winner is Player 2 (this was decided earlier by tossing a coin). Let us take the point of view of Player 1. Suppose that Player 1 ordered a geological survey and, based on the report, concludes that the oil field would generate a profit of \$30 million. Suppose also that Player 1 is indifferent between any two outcomes where the oil field is given to Player 2 and prefers to get the oil field herself if and only if it has to pay not more than \$30 million for it; furthermore, getting the oil field for \$30 million is just as good as not getting it. Then we can take as utility function for Player 1 the net gain to Player 1 from the oil field (defined as profits from oil extraction minus the price paid for access to the oil field) if Player 1 wins, and zero otherwise.

		Player 2				
		\$10M	\$20M	\$30M	\$40M	\$50M
	\$10M	**0**	**0**	**0**	**0**	**0**
Player	\$20M	**20**	**0**	**0**	**0**	**0**
1	\$30M	**20**	**10**	**0**	**0**	**0**
(value \$30M)	\$40M	**20**	**10**	**0**	**0**	**0**
	\$50M	**20**	**10**	**0**	**–10**	**0**

Figure 2.10: A second-price auction where, in case of ties, the winner is Player 2

In Figure 2.10 we have written inside each cell only the payoff of Player 1. For example, why is Player 1's payoff 20 when it bids \$30M and Player 2 bids \$10M? Since Player 1's bid is higher than Player 2's bid, Player 1 is the winner and thus the drilling rights are assigned to Player 1; hence Player 1 obtains something worth \$30M and pays, not its own bid of \$30M, but the bid of Player 2, namely \$10M; it follows that Player 1's net gain is \$(30 – 10)M = \$20M.

It can be verified that for Player 1 submitting a bid equal to the value it assigns to the object (namely, a bid of \$30M) is a weakly dominant strategy: it always gives Player 1 the largest of the payoffs that are possible, given the bid of the other player. This does not imply that it is the only weakly dominant strategy; indeed, in this example bidding \$40M is also a weakly dominant strategy for Player 1 (in fact, it is equivalent to bidding \$30M).

Now we can describe the second-price auction in more general terms. Let $n \geq 2$ be the number of bidders. We assume that all non-negative numbers are allowed as bids and that the tie-breaking rule favors the player with the lowest index among those who submit the highest bid: for example, if the highest bid is \$250 and it is submitted by Players 5, 8 and 10, then the winner is Player 5. We shall denote the possible outcomes as pairs (i, p), where i is the winner and p is the price that the winner has to pay. Finally we denote by b_i the bid of Player i. We start by describing the case where there are only two bidders and then generalize to the case of an arbitrary number of bidders. We denote the set of non-negative numbers by $[0, \infty)$.

The case where n = 2: in this case we have that $I = \{1,2\}$, $S_1 = S_2 = [0,\infty)$, $O = \{(i,p) : i \in \{1,2\}, p \in [0,\infty)\}$ and $f : S \to O$ is given by

$$f((b_1,b_2)) = \begin{cases} (1,b_2) & \text{if } b_1 \geqslant b_2 \\ (2,b_1) & \text{if } b_1 < b_2 \end{cases}.$$

The case where $n \geq 2$: in the general case the second-price auction is the following game-frame:

- $I = \{1,\ldots,n\}$.
- $S_i = [0,\infty)$ for every $i = 1,\ldots,n$. We denote an element of S_i by b_i.
- $O = \{(i,p) : i \in I, p \in [0,\infty)\}$.
- $f : S \to O$ is defined as follows. Let $H(b_1,\ldots,b_n) \subseteq I$ be the set of bidders who submit the highest bid: $H(b_1,\ldots,b_n) = \left\{i \in I : b_i \geq b_j \text{ for all } j \in I\right\}$ and let $\hat{\imath}(b_1,\ldots,b_n)$ be the smallest number in the set $H(b_1,\ldots,b_n)$, that is, the winner of the auction. Finally, let $b^{\max}$ denote the maximum bid and $b^{second}(b_1,\ldots,b_n)$ the second-highest bid,[6] that is,

$$b^{\max}(b_1,\ldots,b_n) = Max\{b_1,...,b_n\}$$

$$b^{second}(b_1,\ldots,b_n) = Max(\{b_1,\ldots,b_n\} \setminus \{b^{\max}(b_1,\ldots,b_n)\}).$$

Then $f(b_1,\ldots,b_n) = \left(\hat{\imath}(b_1\ldots,b_n)\ ,\ b^{second}(b_1,\ldots,b_n)\right)$.

How much should a player bid in a second-price auction? Since what we have described is a game-frame and not a game, we cannot answer the question unless we specify the player's preferences over the set of outcomes O. Let us say that Player i in a second-price auction is *selfish and greedy* if she only cares about whether or not she wins and – conditional on winning – prefers to pay less; furthermore, she prefers winning to not winning if and only if she has to pay less than the true value of the object for her, which we denote by v_i, and is indifferent between not winning and winning if she has to pay v_i. Thus the ranking of a selfish and greedy player is as follows (together with everything that follows from transitivity):

$$\begin{array}{ll} (i,p) \succ_i (i,p') & \text{if and only if } p < p' \\ (i,p) \succ_i (j,p') & \text{for all } j \neq i \text{ and for all } p', \text{ if and only if } p < v_i \\ (i,v_i) \sim_i (j,p') & \text{for all } j \neq i \text{ and for all } p' \\ (j,p) \sim_i (k,p') & \text{for all } j \neq i, k \neq i \text{ and for all } p \text{ and } p'. \end{array}$$

An ordinal utility function that represents these preferences is:[7]

$$U_i(j,p) = \begin{cases} v_i - p & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

[6] For example, if $n = 5, b_1 = \$10, b_2 = \$14, b_3 = \$8, b_4 = \$14$ and $b_5 = \$14$ then $H(\$10,\$14,\$8,\$14,\$14) = \{2,4,5\}$, $\hat{\imath}(\$10,\$14,\$8,\$14,\$14) = 2$, $b^{\max}(\$10,\$14,\$8,\$14,\$14) = \$14$ and $b^{second}(\$10,\$14,\$8,\$14,\$14) = \$14$.

[7] Of course there are many more. For example, also the following utility function represents those preferences: $U_i(j,p) = \begin{cases} 2^{(v_i-p)} & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$

Using this utility function we get the following payoff function for Player i:

$$\pi_i(b_1,...,b_n) = \begin{cases} v_i - b^{second}(b_1,...,b_n) & \text{if } i = \hat{i}(b_1,...,b_n) \\ 0 & \text{if } i \neq \hat{i}(b_1,...,b_n) \end{cases}$$

We can now state the following theorem. The proof is given in Section 2.8.

Theorem 2.3.1 — Vickrey, 1961. In a second-price auction, if Player i is selfish and greedy then it is a weakly dominant strategy for Player i to bid her true value, that is, to choose $b_i = v_i$.

Note that, for a player who is not selfish and greedy, Theorem 2.3.1 is not true. For example, if a player has the same preferences as above for the case where she wins, but, conditional on not winning, prefers the other player to pay as much as possible (she is spiteful) or as little as possible (she is generous), then bidding her true value is no longer a dominant strategy.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 2.9.3 at the end of this chapter.

2.4 The pivotal mechanism

An article in the *Davis Enterprise* (the local newspaper in Davis, California) on January 12, 2001 started with the following paragraph:

> "By consensus, the Davis City Council agreed Wednesday to order a communitywide public opinion poll to gauge how much Davis residents would be willing to pay for a park tax and a public safety tax."

Opinion polls of this type are worthwhile only if there are reasons to believe that the people who are interviewed will respond honestly. But will they? If I would like more parks and believe that the final tax I will have to pay is independent of the amount I state in the interview, I would have an incentive to overstate my willingness to pay, hoping to swing the decision towards building a new park. On the other hand, if I fear that the final tax will be affected by the amount I report, then I might have an incentive to understate my willingness to pay.

The *pivotal mechanism*, or *Clarke mechanism*, is a game designed to give the participants an incentive to report their true willingness to pay.

A public project, say to build a park, is under consideration. The cost of the project is \$$C$. There are n individuals in the community. If the project is carried out, individual i $(i = 1,\dots,n)$ will have to pay \$$c_i$ (with $c_1 + c_2 + \cdots + c_n = C$); these amounts are specified as part of the project. Note that we allow for the possibility that some individuals might have to contribute a larger share of the total cost C than others (e.g. because they live closer to the projected park and would therefore benefit more from it). Individual i has an initial wealth of \$$\overline{m}_i > 0$. If the project is carried out, individual i receives benefits from it that she considers equivalent to receiving $\$v_i$. Note that for some individual i, v_i could be negative, that is, the individual could be harmed by the project (e.g. because she likes peace and quiet and a park close to her home would bring extra traffic and noise). We assume that individual i has the following utility-of-wealth function:

$$U_i(\$m) = \begin{cases} m & \text{if the project is not carried out} \\ m + v_i & \text{if the project is carried out} \end{cases}$$

The socially efficient decision is to carry out the project if and only if $\sum_{i=1}^{n} v_i > C$ (recall that $\sum$ is the summation sign: $\sum_{i=1}^{n} v_i$ is a short-hand for $v_1 + v_2 + ... + v_n$).
For example, suppose that $n = 2$, $\overline{m}_1 = 50$, $\overline{m}_2 = 60$, $v_1 = 19$, $v_2 = -15$, $C = 6$, $c_1 = 6$, $c_2 = 0$. In this case $\sum_{i=1}^{n} v_i = 19 - 15 = 4 < C = 6$ hence the project should not be carried out. To see this consider the following table:

	If the project is not carried out	If the project is carried out
Utility of Individual 1	50	$50 + 19 - 6 = 63$
Utility of Individual 2	60	$60 - 15 = 45$

If the project is carried out, Individual 1 has a utility gain of 13, while Individual 2 has a utility loss of 15. Since the loss is greater than the gain, we have a Pareto inefficient situation. Individual 2 could propose the following alternative to Individual 1: let us not carry out the project and I will pay you \$14. Then Individual 1's wealth and utility would be $50 + 14 = 64$ and Individual 2's wealth and utility would be $60 - 14 = 46$ and thus they would both be better off.

Thus Pareto efficiency requires that the project be carried out if and only if $\sum_{i=1}^{n} v_i > C$. This would be a simple decision for the government if it knew the v_i's. But, typically, these values are private information to the individuals. Can the government find a way to induce the individuals to reveal their true valuations? It seems that in general the answer is No: those who gain from the project would have an incentive to overstate their potential gains, while those who suffer would have an incentive to overstate their potential losses.

Influenced by Vickrey's work on second-price auctions, Clarke suggested the following procedure or game. Each individual i is asked to submit a number w_i which will be interpreted as the gross benefit (if positive) or harm (if negative) that individual i associates with the project. Note that, in principle, individual i can lie and report a value w_i which is different from the true value v_i. Then the decision will be:

$$\text{Carry out the project?} \begin{cases} Yes & if \sum_{j=1}^{n} w_j > C \\ No & if \sum_{j=1}^{n} w_j \leq C \end{cases}$$

However, this is not the end of the story. Each individual will be classified as either not pivotal or pivotal.

$$\text{Individual } i \text{ is } \textbf{not} \text{ pivotal if } \begin{cases} either & \left(\sum_{j=1}^{n} w_j > C \;\; and \;\; \sum_{j\neq i} w_j > \sum_{j\neq i} c_j\right) \\ or & \left(\sum_{j=1}^{n} w_j \leq C \;\; and \;\; \sum_{j\neq i} w_j \leq \sum_{j\neq i} c_j\right) \end{cases}$$

and she is pivotal otherwise. In other words, individual i is pivotal if the decision about the project that would be made in the restricted society resulting from removing individual i is different from the decision that is made when she is included. If an individual is not pivotal then she has to pay no taxes. If individual i is pivotal then she has to pay a tax in the amount of

$$\left|\sum_{j\neq i} w_j - \sum_{j\neq i} c_j\right|, \text{ that is, the absolute value of } \sum_{j\neq i} w_j - \sum_{j\neq i} c_j$$

(recall that the absolute value of a is equal to a, if a is positive, and to $-a$, if a is a negative; for instance, $|4| = 4$ and $|-4| = -(-4) = 4$).
For example, let $n = 3,\ C = 10,\ c_1 = 3,\ c_2 = 2,\ c_3 = 5$.
Suppose that they state the following benefits/losses (which may or may not be the true ones): $w_1 = -1, w_2 = 8, w_3 = 3$.
Then $\sum_{i=1}^{3} w_i = 10 = C$.
Thus the project will not be carried out. Who is pivotal? The answer is provided in Figure 2.11.

Individual	Σw_j (including i)	Σc_j (including i)	Decision	$\Sigma w_j \quad j \neq i$ (without i)	$\Sigma c_j \quad j \neq i$ (without i)	Decision	Pivotal?	Tax
1	10	10	No	8 + 3 = 11	2 + 5 = 7	Yes	Yes	11 – 7 = 4
2	10	10	No	–1 + 3 = 2	3 + 5 = 8	No	No	0
3	10	10	No	–1 + 8 = 7	3 + 2 = 5	Yes	Yes	7 – 5 = 2

Figure 2.11: Example of pivotal mechanism

It may seem that, since it involves paying a tax, being pivotal is a bad thing and one should try to avoid it. It is certainly possible for individual i to make sure that she is not pivotal: all she has to do is to report $w_i = c_i$; in fact, if $\sum_{j\neq i} w_j > \sum_{j\neq i} c_j$ then adding c_i to both sides yields $\sum_{j=1}^{n} w_j > C$ and if $\sum_{j\neq i} w_j \leq \sum_{j\neq i} c_j$ then adding c_i to both sides yields $\sum_{j=1}^{n} w_j \leq C$. It is not true, however, that it is best to avoid being pivotal. The following example shows that one can gain by being truthful even if it involves being pivotal and thus having to pay a tax. Let $n = 4, C = 15, c_1 = 5, c_2 = 0, c_3 = 5$ and $c_4 = 5$.

Suppose that $\overline{m}_1 = 40$ and $v_1 = 25$.
Imagine that you are Individual 1 and, for whatever reason, you expect the following reports by the other individuals: $w_2 = -40, w_3 = 15$ and $w_4 = 20$.
If you report $w_1 = c_1 = 5$ then you ensure that you are not pivotal.
In this case $\sum_{j=1}^{4} w_j = 5 - 40 + 15 + 20 = 0 < C = 15$ and thus the project is not carried out and your utility is equal to $\overline{m}_1 = 40$. If you report truthfully, that is, you report $w_1 = v_1 = 25$ then $\sum_{j=1}^{4} w_j = 25 - 40 + 15 + 20 = 20 > C = 15$ and the project is carried out; furthermore, you are pivotal and have to pay a tax t_1 equal to

$$\left| \sum_{j=2}^{4} w_j - \sum_{j=2}^{4} c_j \right| = |(-40 + 15 + 20) - (0 + 5 + 5)| = |-15| = 15$$

and your utility will be $\overline{m}_1 + v_1 - c_1 - t_1 = 40 + 25 - 5 - 15 = 45$; hence you are better off. Indeed, the following theorem states that no individual can ever gain by lying.

The proof of Theorem 2.4.1 is given in Section 2.8.

Theorem 2.4.1 — Clarke, 1971. In the pivotal mechanism (under the assumed preferences) truthful revelation (that is, stating $w_i = v_i$) is a weakly dominant strategy for every Player i.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 2.9.4 at the end of this chapter.

2.5 Iterated deletion procedures

If in a game a player has a (weakly or strictly) dominant strategy then the player ought to choose that strategy: in the case of strict dominance, choosing any other strategy guarantees that the player will do worse and in the case of weak dominance, no other strategy can give a better outcome, no matter what the other players do. Unfortunately, games that have a dominant-strategy equilibrium are not very common. What should a player do when she does not have a dominant strategy? We shall consider two iterative deletion procedures that can help solve some games.

2.5.1 IDSDS

The Iterated Deletion of Strictly Dominated Strategies (IDSDS) is the following procedure or algorithm. Given a finite ordinal strategic-form game G, let G^1 be the game obtained by removing from G, for every Player i, those strategies of Player i (if any) that are strictly dominated in G by some other strategy; let G^2 be the game obtained by removing from G^1, for every Player i, those strategies of Player i (if any) that are strictly dominated in G^1 by some other strategy, and so on. Let G^∞ be the output of this procedure. Since the initial game G is finite, G^∞ will be obtained in a finite number of steps.

Figure 2.12 illustrates this procedure. If G^∞contains a single strategy profile (this is **not** the case in the example of Figure 2.12), then we call that strategy profile the *iterated strictly dominant-strategy equilibrium*. If G^∞ contains two or more strategy profiles then we refer to those strategy profiles merely as the *output of the IDSDS procedure*. For example, in the game of Figure 2.12 the output of the IDSDS procedure is the set of strategy profiles $\{(A,e),(A,f),(B,e),(B,f)\}$.

What is the significance of the output of the IDSDS procedure? Consider game G of Figure 2.12. Since, for Player 2, h is strictly dominated by g, if Player 2 is rational she will not play h. Thus, if Player 1 believes that Player 2 is rational then he believes that Player 2 will not play h, that is, he restricts attention to game G^1; since, in G^1, D is strictly dominated by C for Player 1, if Player 1 is rational he will not play D. It follows that if Player 2 believes that Player 1 is rational and that Player 1 believes that Player 2 is rational, then Player 2 restricts attention to game G^2 where rationality requires that Player 2 not play g, etc. It will be shown in a later chapter that if there is common knowledge of rationality,[8] then only strategy profiles that survive the IDSDS procedure can be played; the converse is also true: any strategy profile that survives the IDSDS procedure is compatible with common knowledge of rationality.

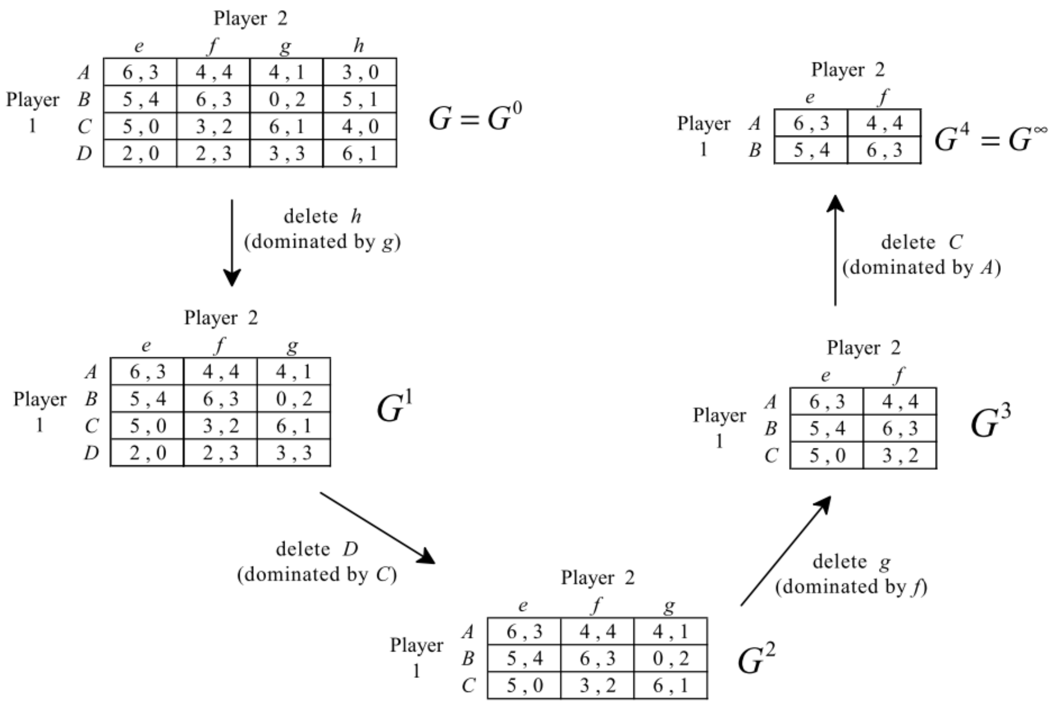

Figure 2.12: An example of the IDSDS procedure

R In finite games, the order in which strictly dominated strategies are deleted is irrelevant, in the sense that any sequence of deletions of strictly dominated strategies leads to the same output.

[8]An event E is commonly known if everybody knows E and everybody knows that everybody knows E and everybody knows that everybody knows that everybody knows E, and so on.

2.5.2 IDWDS

The Iterated Deletion of Weakly Dominated Strategies (IDWDS) is a weakening of IDSDS in that it allows the deletion also of *weakly* dominated strategies. However, this procedure has to be defined carefully, since in this case the order of deletion *can* matter. To see this, consider the game shown in Figure 2.13.

Player 1 \ Player 2	L	R
A	4 , 0	0 , 0
T	3 , 2	2 , 2
M	1 , 1	0 , 0
B	0 , 0	1 , 1

Figure 2.13: A strategic-form game with ordinal payoffs

Since M is strictly dominated by T for Player 1, we can delete it and obtain the reduced game shown in Figure 2.14

Player 1 \ Player 2	L	R
A	4 , 0	0 , 0
T	3 , 2	2 , 2
B	0 , 0	1 , 1

Figure 2.14: The game of Figure 2.13 after deletion of strategy M

Now L is weakly dominated by R for Player 2. Deleting L we are left with the reduced game shown in Figure 2.15.

Player 1 \ Player 2	R
A	0 , 0
T	2 , 2
B	1 , 1

Figure 2.15: The game of Figure 2.14 after deletion of strategy L

Now A and B are strictly dominated by T. Deleting them we are left with $\boxed{(T,R)}$, with corresponding payoffs (2,2).

Alternatively, going back to the game of Figure 2.13, we could note that B is strictly dominated by T; deleting B we are left with

		Player 2	
		L	R
Player 1	A	4 , 0	0 , 0
	T	3 , 2	2 , 2
	M	1 , 1	0 , 0

Figure 2.16: The game of Figure 2.13 after deletion of strategy B

Now R is weakly dominated by L for Player 2. Deleting R we are left with the reduced game shown in Figure 2.17.

		Player 2
		L
Player 1	A	4 , 0
	T	3 , 2
	M	1 , 1

Figure 2.17: The game of Figure 2.16 after deletion of strategy R

Now T and M are strictly dominated by A and deleting them leads to $\boxed{(A,L)}$ with corresponding payoffs (4,0). Since one order of deletion leads to (T,R) with payoffs (2,2) and the other to (A,L) with payoffs (4,0), the procedure is not well defined.

Definition 2.5.1 — IDWDS. In order to avoid the problem illustrated above, the IDWDS procedure is defined as follows: *at every step identify, for every player, all the strategies that are weakly (or strictly) dominated and then delete all such strategies in that step*. If the output of the IDWDS procedure is a single strategy profile then we call that strategy profile the *iterated weakly dominant-strategy equilibrium* (otherwise we just use the expression 'output of the IDWDS procedure').

For example, the IDWDS procedure when applied to the game of Figure 2.13 leads to the set of strategy profiles shown in Figure 2.18.[9]

Player 2

Player 1		L	R
	A	4 , 0	0 , 0
	T	3 , 2	2 , 2

Figure 2.18: The output of the IDWDS procedure applied to the game of Figure 2.13

Hence the game of Figure 2.13 does not have an iterated weakly dominant-strategy equilibrium.

The interpretation of the output of the IDWDS procedure is not as simple as that of the IDSDS procedure: certainly common knowledge of rationality is not sufficient. In order to delete weakly dominated strategies one needs to appeal not only to rationality but also to some notion of caution: a player should not completely rule out any of her opponents' strategies. However, this notion of caution is in direct conflict with the process of deletion of strategies. In this book we shall not address the issue of how to justify the IDWDS procedure.

> Test your understanding of the concepts introduced in this section, by going through the exercises in Section 2.9.5 at the end of this chapter.

2.6 Nash equilibrium

Games where either the IDSDS procedure or the IDWDS procedure leads to a unique strategy profile are not very common. How can one then "solve" games that are not solved by either procedure? The notion of Nash equilibrium offers a more general alternative. We first define Nash equilibrium for a two-player game.

Definition 2.6.1 Given an ordinal game in strategic form with two players, a strategy profile $s^* = (s_1^*, s_2^*) \in S_1 \times S_2$ is a Nash equilibrium if the following two conditions are satisfied:

1. for every $s_1 \in S_1$, $\pi_1(s_1^*, s_2^*) \geq \pi_1(s_1, s_2^*)$ (or stated in terms of outcomes and preferences, $f(s_1^*, s_2^*) \succsim_1 f(s_1, s_2^*)$), and
2. for every $s_2 \in S_2$, $\pi_2(s_1^*, s_2^*) \geq \pi_2(s_1^*, s_2)$ (or, $f(s_1^*, s_2^*) \succsim_2 f(s_1^*, s_2)$).

[9]Note that, in general, the output of the IDWDS procedure is a subset of the output of the IDSDS procedure (not necessarily a proper subset). The game of Figure 2.13 happens to be a game where the two procedures yield the same outcome.

For example, in the game of Figure 2.19 there are two Nash equilibria: (T,L) and (B,C).

		Player 2		
		L	C	R
Player 1	T	3 , 2	0 , 0	1 , 1
	M	3 , 0	1 , 5	4 , 4
	B	1 , 0	2 , 3	3 , 0

Figure 2.19: A strategic-form game with ordinal payoffs

There are several possible interpretations of this definition:

"No regret" interpretation: s^* is a Nash equilibrium if there is no player who, after observing the opponent's choice, regrets his own choice (in the sense that he could have done better with a different strategy of his, given the observed strategy of the opponent).

"Self-enforcing agreement" interpretation: imagine that the players are able to communicate before playing the game and reach a non-binding agreement expressed as a strategy profile s^*; then no player will have an incentive to deviate from the agreement (if she believes that the other player will follow the agreement) if and only if s^* is a Nash equilibrium.

"Viable recommendation" interpretation: imagine that a third party makes a public recommendation to each player on what strategy to play; then no player will have an incentive to deviate from the recommendation (if she believes that the other players will follow the recommendation) if and only if the recommended strategy profile is a Nash equilibrium.

"Transparency of reason" interpretation: if players are all "equally rational" and Player 2 reaches the conclusion that she should play y, then Player 1 must be able to duplicate Player 2's reasoning process and come to the same conclusion; it follows that Player 1's choice of strategy is not rational unless it is a strategy x that is optimal against y. A similar argument applies to Player 2's choice of strategy (y must be optimal against x) and thus (x,y) is a Nash equilibrium.

It should be clear that all of the above interpretations are mere rewording of the formal definition of Nash equilibrium in terms of the inequalities given in Definition 2.6.1.

The generalization of Definition 2.6.1 to games with more than two players is straightforward.

Definition 2.6.2 Given an ordinal game in strategic form with n players, a strategy profile $s^* \in S$ is a Nash equilibrium if the following n inequalities are satisfied: for every Player $i = 1, \ldots, n$,

$$\pi_i(s^*) \geq \pi_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*) \text{ for all } s_i \in S_i.$$

The reader should convince himself/herself that a (weak or strict) dominant strategy equilibrium is a Nash equilibrium and the same is true of a (weak or strict) iterated dominant-strategy equilibrium.

Definition 2.6.3 Consider an ordinal game in strategic form, a Player i and a strategy profile $\bar{s}_{-i} \in S_{-i}$ of the players other than i. A strategy $s_i \in S_i$ of Player i is a *best reply* (or *best response*) to $\bar{s}_{-i}$ if $\pi_i(s_i, \bar{s}_{-i}) \geq \pi_i(s_i', \bar{s}_{-i})$, for every $s_i' \in S_i$.

For example, in the game of Figure 2.20, for Player 1 there are two best replies to L, namely M and T, while the unique best reply to C is B and the unique best reply to R is M; for Player 2 the best reply to T is L, the best reply to M is C and the best reply to B is C.

		Player 2		
		L	C	R
Player 1	T	3 , 2	0 , 0	1 , 1
	M	3 , 0	1 , 5	4 , 4
	B	1 , 0	2 , 3	3 , 0

Figure 2.20: A strategic-form game with ordinal payoffs

R Using the notion of best reply, an alternative definition of Nash equilibrium is as follows: $\bar{s} \in S$ is a Nash equilibrium if and only if, for every Player i, $\bar{s}_i \in S_i$ is a best reply to $\bar{s}_{-i} \in S_{-i}$.

A quick way to find the Nash equilibria of a two-player game is as follows: in each column of the table underline the largest payoff of Player 1 in that column (if there are several instances, underline them all) and in each row underline the largest payoff of Player 2 in that row; if a cell has both payoffs underlined then the corresponding strategy profile is a Nash equilibrium. Underlining of the maximum payoff of Player 1 in a given column identifies the best reply of Player 1 to the strategy of Player 2 that labels that column and similarly for Player 2. This procedure is illustrated in Figure 2.21, where there is a unique Nash equilibrium, namely (B,E).

Player 2

Player 1		E	F	G	H
	A	$\underline{4}$, 0	3 , 2	2 , $\underline{3}$	$\underline{4}$, 1
	B	$\underline{4}$, $\underline{2}$	2 , 1	1 , $\underline{2}$	0 , $\underline{2}$
	C	3 , $\underline{6}$	$\underline{5}$, 5	$\underline{3}$, 1	$\underline{5}$, 0
	D	2 , $\underline{3}$	3 , 2	1 , 2	3 , $\underline{3}$

Figure 2.21: A strategic-form game with ordinal payoffs

Exercise 2.3 in Section 2.9.1 explains how to represent a three-player game by means of a set of tables. In a three-player game the procedure for finding the Nash equilibria is the same, with the necessary adaptation for Player 3: in each cell underline the payoff of Player 3 if and only if her payoff is the largest of all her payoffs in the same cell across different tables. This is illustrated in Figure 2.22, where there is a unique Nash equilibrium, namely (B,R,W).

Player 2

Player 1		L	R
	T	0 , 0 , $\underline{0}$	2 , $\underline{8}$, $\underline{6}$
	B	$\underline{5}$, 3 , $\underline{2}$	$\underline{3}$, $\underline{4}$, $\underline{2}$

Player 3 chooses W

Player 2

	L	R
T	0 , 0 , $\underline{0}$	$\underline{1}$, $\underline{2}$, 5
B	$\underline{1}$, $\underline{6}$, 1	0 , 0 , 1

Player 3 chooses E

Figure 2.22: A three-player game with ordinal payoffs

Unfortunately, when the game has too many players or too many strategies – and it is thus impossible or impractical to represent it as a set of tables – there is no quick procedure for finding the Nash equilibria: one must simply apply the definition of Nash equilibrium. This is illustrated in the following example.

■ **Example 2.1** There are 50 players. A benefactor asks them to simultaneously and secretly write on a piece of paper a request, which must be a multiple of \$10 up to a maximum of \$100 (thus the possible strategies of each player are $\$10, \$20, \ldots, \$90, \$100$). He will then proceed as follows: if not more than 10% of the players ask for \$100 then he will grant every player's request, otherwise every player will get nothing. Assume that every player is selfish and greedy (only cares about how much money she gets and prefers more money to less). What are the Nash equilibria of this game? There are several:

- every strategy profile where 7 or more players request \$100 is a Nash equilibrium (everybody gets nothing and no player can get a positive amount by unilaterally changing her request, since there will still be more than 10% requesting \$100; on the other hand, convince yourself that a strategy profile where exactly 6 players request \$100 is not a Nash equilibrium),
- every strategy profile where exactly 5 players request \$100 and the remaining players request \$90 is a Nash equilibrium.

Any other strategy profile is not a Nash equilibrium: (1) if fewer than 5 players request \$100, then a player who requested less than \$100 can increase her payoff by switching to a request of \$100, (2) if exactly 5 players request \$100 and among the remaining players there is one who is not requesting \$90, then that player can increase her payoff by increasing her request to \$90. ■

We conclude this section by noting that, since so far we have restricted attention to ordinal games, there is no guarantee that an arbitrary game will have at least one Nash equilibrium. An example of a game that has no Nash equilibria is the *Matching Pennies* game. This is a simultaneous two-player game where each player has a coin and decides whether to show the Heads face or the Tails face. If both choose H or both choose T then Player 1 wins, otherwise Player 2 wins. Each player strictly prefers the outcome where she herself wins to the alternative outcome. The game is illustrated in Figure 2.23.

		Player 2	
		H	T
Player 1	H	1 0	0 1
	T	0 1	1 0

Figure 2.23: The matching pennies game

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 2.9.6 at the end of this chapter.

2.7 Games with infinite strategy sets

Games where the strategy set of one or more players is infinite cannot be represented using a table or set of tables. However, all the concepts introduced in this chapter can still be applied. In this section we will focus on the notion of Nash equilibrium. We start with an example.

■ **Example 2.2** There are two players. Each player has to write down a real number (not necessarily an integer) greater than or equal to 1; thus the strategy sets are $S_1 = S_2 = [1, \infty)$. Payoffs are as follows (π_1 is the payoff of Player 1, π_2 the payoff of Player 2, x is the number written by Player 1 and y the number written by Player 2):

$$\pi_1(x,y) = \begin{cases} x-1 & \text{if } x < y \\ 0 & \text{if } x \geq y \end{cases} \quad \text{and} \quad \pi_2(x,y) = \begin{cases} y-1 & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$$

What are the Nash equilibria of this game? ■

There is only one Nash equilibrium, namely (1,1) with payoffs (0,0). First of all, we must show that (1,1) is indeed a Nash equilibrium.
If Player 1 switched to some $x > 1$ then her payoff would remain 0: $\pi_1(x,1) = 0$, for all $x \in [1,\infty)$ and the same is true for Player 2 if he unilaterally switched to some $y > 1$: $\pi_2(1,y) = 0$, for all $y \in [1,\infty)$.
Now we show that no other pair (x,y) is a Nash equilibrium.
Consider first an arbitrary pair (x,y) with $x = y > 1$. Then $\pi_1(x,y) = 0$, but if Player 1 switched to an $\hat{x}$ strictly between 1 and x ($1 < \hat{x} < x$) her payoff would be $\pi_1(\hat{x},y) = \hat{x} - 1 > 0$ (recall that, by hypothesis, $x = y$).
Now consider an arbitrary (x,y) with $x < y$. Then $\pi_1(x,y) = x - 1$, but if Player 1 switched to an $\hat{x}$ strictly between x and y ($x < \hat{x} < y$) her payoff would be $\pi_1(\hat{x},y) = \hat{x} - 1 > x - 1$.
The argument for ruling out pairs (x,y) with $y < x$ is similar.
Note the interesting fact that, for Player 1, $x = 1$ is a weakly dominated strategy: indeed it is weakly dominated by any other strategy: $x = 1$ guarantees a payoff of 0 for Player 1, while any $\hat{x} > 1$ would yield a positive payoff to Player 1 in some cases (against any $y > \hat{x}$) and 0 in the remaining cases. The same is true for Player 2. Thus in this game there is *a unique Nash equilibrium where the strategy of each player is weakly dominated*!

[Note: the rest of this section makes use of calculus. The reader who is not familiar with calculus should skip this part.]

We conclude this section with an example based on the analysis of competition among firms proposed by Augustine Cournot in a book published in 1838. In fact, Cournot is the one who invented what we now call 'Nash equilibrium', although his analysis was restricted to a small class of games. Consider $n \geq 2$ firms which produce an identical product. Let q_i be the quantity produced by Firm i ($i = 1, \ldots n$). For Firm i the cost of producing q_i units of output is $c_i q_i$, where c_i is a positive constant. For simplicity we will restrict attention to the case of two firms ($n = 2$) and identical cost functions: $c_1 = c_2 = c$. Let Q be total industry output, that is, $Q = q_1 + q_2$. The price at which each firm can sell each unit of output is given by the inverse demand function $P = a - bQ$ where a and b are positive constants. Cournot assumed that each firm was only interested in its own profit and preferred higher profit to lower profit (that is, each firm is "selfish and greedy").

The profit function of Firm 1 is given by

$$\pi_1(q_1,q_2) = Pq_1 - cq_1 = [a - b(q_1+q_2)]\,q_1 - cq_1 = (a-c)q_1 - b(q_1)^2 - bq_1q_2.$$

Similarly, the profit function of Firm 2 is given by

$$\pi_2(q_1,q_2) = (a-c)q_2 - b(q_2)^2 - bq_1q_2$$

Cournot defined an equilibrium as a pair $(\overline{q}_1, \overline{q}_2)$ that satisfies the following two inequalities:

$$\pi_1(\overline{q}_1,\overline{q}_2) \geq \pi_1(q_1,\overline{q}_2), \text{ for every } q_1 \geq 0 \qquad (\clubsuit)$$

$$\pi_2(\overline{q}_1,\overline{q}_2) \geq \pi_2(\overline{q}_1,q_2) \text{ for every } q_2 \geq 0. \qquad (\blacklozenge)$$

Of course, this is the same as saying that $(\overline{q}_1, \overline{q}_2)$ is a Nash equilibrium of the game where the players are the two firms, the strategy sets are $S_1 = S_2 = [0,\infty)$ and the payoff functions are the profit functions. How do we find a Nash equilibrium? First of all, note that the profit functions are differentiable. Secondly note that ($\clubsuit$) says that, having fixed the value of q_2 at $\overline{q}_2$, the function $\pi_1(q_1,\overline{q}_2)$ – viewed as a function of q_1 alone – is maximized at the point $q_1 = \overline{q}_1$. A necessary condition for this (if $\overline{q}_1 > 0$) is that the derivative of this function be zero at the point $q_1 = \overline{q}_1$, that is, it must be that $\frac{\partial \pi_1}{\partial q_1}(\overline{q}_1,\overline{q}_2) = 0$. This condition is also sufficient since the second derivative of this function is always negative ($\frac{\partial^2 \pi_1}{\partial q_1^2}(q_1,q_2) = -2b$ for every (q_1,q_2)). Similarly, by ($\blacklozenge$), it must be that $\frac{\partial \pi_2}{\partial q_2}(\overline{q}_1,\overline{q}_2) = 0$. Thus the Nash equilibrium is found by solving the system of two equations

$$\begin{cases} \frac{\partial \pi_1}{\partial q_1}(q_1,q_2) = a - c - 2bq_1 - bq_2 = 0 \\ \frac{\partial \pi_2}{\partial q_2}(q_1,q_2) = a - c - 2bq_2 - bq_1 = 0 \end{cases}$$

The solution is $\overline{q}_1 = \overline{q}_2 = \frac{a-c}{3b}$. The corresponding price is $\overline{P} = a - b\left(2\frac{a-c}{3b}\right) = \frac{a+2c}{3}$ and the corresponding profits are $\pi_1(\frac{a-c}{3b}, \frac{a-c}{3b}) = \pi_2(\frac{a-c}{3b}, \frac{a-c}{3b}) = \frac{(a-c)^2}{9b}$.
For example, if $a = 25$, $b = 2$, $c = 1$ then the Nash equilibrium is given by (4,4) with corresponding profits of 32 for each firm. The analysis can easily be extended to the case of more than two firms.

The reader who is interested in further exploring the topic of competition among firms can consult any textbook on Industrial Organization.

> Test your understanding of the concepts introduced in this section, by going through the exercises in Section 2.9.7 at the end of this chapter.

2.8 Proofs of theorems

Theorem [Vickrey, 1961] In a second-price auction, if Player i is selfish and greedy then it is a weakly dominant strategy for Player i to bid her true value, that is, to choose $b_i = v_i$.

Proof. In order to make the notation simpler and the argument more transparent, we give the proof for the case where $n = 2$. We shall prove that bidding v_1 is a weakly dominant strategy for Player 1 (the proof for Player 2 is similar). Assume that Player 1 is selfish and greedy. Then we can take her payoff function to be as follows:

$$\pi_1(b_1, b_2) = \begin{cases} v_1 - b_2 & \text{if } b_1 \geq b_2 \\ 0 & \text{if } b_1 < b_2 \end{cases}$$

We need to show that, whatever bid Player 2 submits, Player 1 cannot get a higher payoff by submitting a bid different from v_1. Two cases are possible (recall that b_2 denotes the actual bid of Player 2, which is unknown to Player 1).

Case 1: $b_2 \leq v_1$. In this case, bidding v_1 makes Player 1 the winner and his payoff is $v_1 - b_2 \geq 0$. Consider a different bid b_1. If $b_1 \geq b_2$ then Player 1 is still the winner and his payoff is still $v_1 - b_2 \geq 0$; thus such a bid is as good as (hence not better than) v_1. If $b_1 < b_2$ then the winner is Player 2 and Player 1 gets a payoff of 0. Thus such a bid is also not better than v_1.

Case 2: $b_2 > v_1$. In this case, bidding v_1 makes Player 2 the winner and thus Player 1 gets a payoff of 0. Any other bid $b_1 < b_2$ gives the same outcome and payoff. On the other hand, any bid $b_1 \geq b_2$ makes Player 1 the winner, giving him a payoff of $v_1 - b_2 < 0$, thus making Player 1 worse off than with a bid of v_1.

■

Theorem [Clarke, 1971] In the pivotal mechanism (under the assumed preferences) truthful revelation (that is, stating $w_i = v_i$) is a weakly dominant strategy for every Player i.

Proof. Consider an individual i and possible statements w_j for $j \neq i$. Several cases are possible.

Case 1: $\sum_{j\neq i} w_j > \sum_{j\neq i} c_j$ and $v_i + \sum_{j\neq i} w_j > c_i + \sum_{j\neq i} c_j = C$. Then

	decision	i's tax	i's utility
if i states v_i	Yes	0	$\overline{m}_i + v_i - c_i$
if i states w_i such that $w_i + \sum_{j\neq i} w_j > C$	Yes	0	$\overline{m}_i + v_i - c_i$
if i states w_i such that $w_i + \sum_{j\neq i} w_j \leq C$	No	$\sum_{j\neq i} w_j - \sum_{j\neq i} c_j$	$\overline{m}_i - \left(\sum_{j\neq i} w_j - \sum_{j\neq i} c_j\right)$

Individual i cannot gain by lying if and only if

$$\overline{m}_i + v_i - c_i \geq \overline{m}_i - \left(\sum_{j\neq i} w_j - \sum_{j\neq i} c_j\right), \quad \text{i.e. if and only if} \quad v_i + \sum_{j\neq i} w_j \geq C,$$

which is true by our hypothesis.

Case 2: $\sum_{j\neq i} w_j > \sum_{j\neq i} c_j$ and $v_i + \sum_{j\neq i} w_j \leq c_i + \sum_{j\neq i} c_j = C$. Then

	decision	i's tax	i's utility
if i states v_i	No	$\sum_{j\neq i} w_j - \sum_{j\neq i} c_j$	$\overline{m}_i - \left(\sum_{j\neq i} w_j - \sum_{j\neq i} c_j\right)$
if i states w_i such that $w_i + \sum_{j\neq i} w_j \leq C$	No	$\sum_{j\neq i} w_j - \sum_{j\neq i} c_j$	$\overline{m}_i - \left(\sum_{j\neq i} w_j - \sum_{j\neq i} c_j\right)$
if i states w_i such that $w_i + \sum_{j\neq i} w_j > C$	Yes	0	$\overline{m}_i + v_i - c_i$

Individual i cannot gain by lying if and only if $\overline{m}_i - \left(\sum_{j\neq i} w_j - \sum_{j\neq i} c_j\right) \geq \overline{m}_i + v_i - c_i$, i.e. if and only if $v_i + \sum_{j\neq i} w_j \leq C$, which is true by our hypothesis.

Case 3: $\sum_{j\neq i} w_j \leq \sum_{j\neq i} c_j$ and $v_i + \sum_{j\neq i} w_j \leq c_i + \sum_{j\neq i} c_j = C$. Then

	decision	i's tax	i's utility
if i states v_i	No	0	$\overline{m}_i$
if i states w_i such that $w_i + \sum_{j\neq i} w_j \leq C$	No	0	$\overline{m}_i$
if i states w_i such that $w_i + \sum_{j\neq i} w_j > C$	Yes	$\left(\sum_{j\neq i} c_j - \sum_{j\neq i} w_j\right)$ (recall that $\sum_{j\neq i} w_j \leq \sum_{j\neq i} c_j$)	$\overline{m}_i + v_i - c_i - \left(\sum_{j\neq i} c_j - \sum_{j\neq i} w_j\right)$

Individual i cannot gain by lying if and only if $\overline{m}_i \geq \overline{m}_i + v_i - c_i - \left(\sum_{j\neq i} c_j - \sum_{j\neq i} w_j\right)$, i.e. if and only if $v_i + \sum_{j\neq i} w_j \leq C$, which is true by our hypothesis.

Case 4: $\sum_{j\neq i} w_j \leq \sum_{j\neq i} c_j$ and $v_i + \sum_{j\neq i} w_j > c_i + \sum_{j\neq i} c_j = C$. Then

	decision	i's tax	i's utility
if i states v_i	Yes	$\left(\sum_{j\neq i} c_j - \sum_{j\neq i} w_j\right)$ (recall that $\sum_{j\neq i} w_j \leq \sum_{j\neq i} c_j$)	$\overline{m}_i + v_i - c_i - \left(\sum_{j\neq i} w_j - \sum_{j\neq i} c_j\right)$
if i states w_i such that $w_i + \sum_{j\neq i} w_j > C$	Yes	$\left(\sum_{j\neq i} c_j - \sum_{j\neq i} w_j\right)$ (recall that $\sum_{j\neq i} w_j \leq \sum_{j\neq i} c_j$)	$\overline{m}_i + v_i - c_i - \left(\sum_{j\neq i} w_j - \sum_{j\neq i} c_j\right)$
if i states w_i such that $w_i + \sum_{j\neq i} w_j \leq C$	No	0	$\overline{m}_i$

Individual i cannot gain by lying if and only if $\overline{m}_i + v_i - c_i - \left(\sum_{j\neq i} w_j - \sum_{j\neq i} c_j\right) \geq \overline{m}_i$, i.e. if and only if $v_i + \sum_{j\neq i} w_j \geq C$, which is true by our hypothesis.

Since we have covered all the possible cases, the proof is complete. ■

2.9 Exercises

2.9.1 Exercises for Section 2.1: Game frames and games

The answers to the following exercises are in Section 2.10 at the end of this chapter.

Exercise 2.1 Antonia and Bob cannot decide where to go to dinner. Antonia proposes the following procedure: she will write on a piece of paper either the number 2 or the number 4 or the number 6, while Bob will write on his piece of paper either the number 1 or 3 or 5. They will write their numbers secretly and independently. They then will show each other what they wrote and choose a restaurant according to the following rule: if the sum of the two numbers is 5 or less, they will go to a Mexican restaurant, if the sum is 7 they will go to an Italian restaurant and if the number is 9 or more they will go to a Japanese restaurant.

(a) Let Antonia be Player 1 and Bob Player 2. Represent this situation as a game frame, first by writing out each element of the quadruple of Definition 2.1.1 and then by using a table (label the rows with Antonia's strategies and the columns with Bob's strategies, so that we can think of Antonia as choosing the row and Bob as choosing the column).

(b) Suppose that Antonia and Bob have the following preferences (where M stands for 'Mexican', I for 'Italian' and J for 'Japanese'):
for Antonia: $M \succ_{Antonia} I \succ_{Antonia} J$; for Bob: $I \succ_{Bob} M \succ_{Bob} J$.
Using utility function with values 1, 2 and 3 represent the corresponding reduced-form game as a table.

■

Exercise 2.2 Consider the following two-player game-frame where each player is given a set of cards and each card has a number on it. The players are Antonia (Player 1) and Bob (Player 2). Antonia's cards have the following numbers (one number on each card): 2, 4 and 6, whereas Bob's cards are marked 0, 1 and 2 (thus different numbers from the previous exercise). Antonia chooses one of her own cards and Bob chooses one of his own cards: this is done without knowing the other player's choice. The outcome depends on the sum of the points of the chosen cards. If the sum of the points on the two chosen cards is greater than or equal to 5, Antonia gets \$10 minus that sum; otherwise (that is, if the sum is less than 5) she gets nothing; furthermore, if the sum of points is an odd number, Bob gets as many dollars as that sum; if the sum of points turns out to be an even number and is less than or equal to 6, Bob gets \$2; otherwise he gets nothing. (The money comes from a third party.)

(a) Represent the game-frame described above by means of a table. As in the previous exercise, assign the rows to Antonia and the columns to Bob.

(b) Using the game-frame of part **(a)** obtain a reduced-form game by adding the information that each player is selfish and greedy. This means that each player only cares about how much money he/she gets and prefers more money to less.

■

Exercise 2.3 Alice (Player 1), Bob (Player 2), and Charlie (Player 3) play the following simultaneous game. They are sitting in different rooms facing a keyboard with only one key and each has to decide whether or not to press the key. Alice wins if the number of people who press the key is odd (that is, all three of them or only Alice or only Bob or only Charlie), Bob wins if exactly two people (he may be one of them) press the key and Charlie wins if nobody presses the key.

(a) Represent this situation as a game-frame. Note that we can represent a three-player game with a *set of tables*: Player 1 chooses the row, Player 2 chooses the column and Player 3 chooses the table (that is, we label the rows with Player 1's strategies, the columns with Player 2's strategies and the tables with Player 3's strategies).

(b) Using the game-frame of part **(a)** obtain a reduced-form game by adding the information that each player prefers winning to not winning and is indifferent between any two outcomes where he/she does not win. For each player use a utility function with values from the set $\{0,1\}$.

(c) Using the game-frame of part **(a)** obtain a reduced-form game by adding the information that (1) each player prefers winning to not winning, (2) Alice is indifferent between any two outcomes where she does not win, (3) conditional on not winning, Bob prefers if Charlie wins rather than Alice, (4) conditional on not winning, Charlie prefers if Bob wins rather than Alice. For each player use a utility function with values from the set $\{0,1,2\}$.

■

2.9.2 Exercises for Section 2.2: Strict/weak dominance

The answers to the following exercises are in Section 2.10 at the end of this chapter.

Exercise 2.4 There are two players. Each player is given an unmarked envelope and asked to put in it either nothing or \$300 of his own money or \$600 of his own money. A referee collects the envelopes, opens them, gathers all the money, then adds 50% of that amount (using his own money) and divides the total into two equal parts which he then distributes to the players.

(a) Represent this game frame with two alternative tables: the first table showing in each cell the amount of money distributed to Player 1 and the amount of money distributed to Player 2, the second table showing the change in wealth of each player (money received minus contribution).

(b) Suppose that Player 1 has some animosity towards the referee and ranks the outcomes in terms of how much money the referee loses (the more, the better), while Player 2 is selfish and greedy and ranks the outcomes in terms of her own net gain. Represent the corresponding game using a table.

(c) Is there a strictly dominant-strategy equilibrium?

■

Exercise 2.5 Consider again the game of Part **(b)** of Exercise 2.1.

(a) Determine, for each player, whether the player has *strictly* dominated strategies.

(b) Determine, for each player, whether the player has *weakly* dominated strategies.

■

Exercise 2.6 There are three players. Each player is given an unmarked envelope and asked to put in it either nothing or \$3 of his own money or \$6 of his own money. A referee collects the envelopes, opens them, gathers all the money and then doubles the amount (using his own money) and divides the total into three equal parts which he then distributes to the players.

For example, if Players 1 and 2 put nothing and Player 3 puts \$6, then the referee adds another \$6 so that the total becomes \$12, divides this sum into three equal parts and gives \$4 to each player.

Each player is selfish and greedy, in the sense that he ranks the outcomes exclusively in terms of his net change in wealth (what he gets from the referee minus what he contributed).

(a) Represent this game by means of a set of tables. (Do not treat the referee as a player.)

(b) For each player and each pair of strategies determine if one of the two dominates the other and specify if it is weak or strict dominance.

(c) Is there a strictly dominant-strategy equilibrium?

■

2.9.3 Exercises for Section 2.3: Second price auction

The answers to the following exercises are in Section 2.10 at the end of this chapter.

Exercise 2.7 For the second-price auction partially illustrated in Figure 2.10 – reproduced below (recall that the numbers are the payoffs of Player 1 only) – complete the representation by adding the payoffs of Player 2, assuming that Player 2 assigns a value of \$50M to the field and, like Player 1, ranks the outcomes in terms of the net gain from the oil field (defined as profits minus the price paid, if Player 2 wins, and zero otherwise).

		Player 2				
		\$10*M*	\$20*M*	\$30*M*	\$40*M*	\$50*M*
	\$10*M*	0	0	0	0	0
Player 1	\$20*M*	20	0	0	0	0
	\$30*M*	20	10	0	0	0
	\$40*M*	20	10	0	0	0
	\$50*M*	20	10	0	−10	0

■

Exercise 2.8 Consider the following "third-price" auction. There are $n \geq 3$ bidders. A single object is auctioned and Player i values the object \$$v_i$, with $v_i > 0$. The bids are simultaneous and secret.

The utility of Player i is: 0 if she does not win and $(v_i - p)$ if she wins and pays \$$p$.

Every non-negative number is an admissible bid. Let b_i denote the bid of Player i.

The winner is the highest bidder. In case of ties the bidder with the lowest index among those who submitted the highest bid wins (e.g. if the highest bid is \$120 and it is submitted by players 6, 12 and 15, then the winner is Player 6). The losers don't get anything and don't pay anything. The winner gets the object and pays the **third** highest bid, which is defined as follows.

Let i be the winner and fix a Player j such that

$$b_j = \max\left(\{b_1, ..., b_n\} \setminus \{b_i\}\right)$$

[note: if

$$\max\left(\{b_1, ..., b_n\} \setminus \{b_i\}\right)$$

contains more than one element, then we pick any one of them]. Then the third price is defined as

$$\max\left(\{b_1, ..., b_n\} \setminus \{b_i, b_j\}\right).$$

For example, if $n = 3$ and the bids are $b_1 = 30$, $b_2 = 40$ and $b_3 = 40$ then the winner is Player 2 and she pays \$30. If $b_1 = b_2 = b_3 = 50$ then the winner is Player 1 and she pays \$50. For simplicity, let us restrict attention to the case where $n = 3$ and $v_1 > v_2 > v_3 > 0$. Does Player 1 have a weakly dominant strategy in this auction? ■

2.9.4 Exercises for Section 2.4: The pivotal mechanism

The answers to the following exercises are in Section 2.10 at the end of this chapter.

Exercise 2.9 The pivotal mechanism is used to decide whether a new park should be built. There are 5 individuals. According to the proposed project, the cost of the park would be allocated as follows:

Individual	1	2	3	4	5
Share of cost	$c_1 = \$30$	$c_2 = \$25$	$c_3 = \$25$	$c_4 = \$15$	$c_5 = \$5$

For every individual $i = 1,\ldots,5$, let v_i be the perceived gross benefit (if positive; perceived gross loss, if negative) from having the park built. The v_i's are as follows:

Individual	1	2	3	4	5
Gross benefit	$v_1 = \$60$	$v_2 = \$15$	$v_3 = \$55$	$v_4 = -\$25$	$v_5 = -\$20$

(Thus the net benefit (loss) to individual i is $v_i - c_i$). Individual i has the following utility of wealth function (where m_i denotes the wealth of individual i):

$$U_i(\$m_i) = \begin{cases} m_i & \text{if the project is not carried out} \\ m_i + v_i & \text{if the project is carried out} \end{cases}$$

Let $\overline{m}_i$ be the initial endowment of money of individual i and assume that $\overline{m}_i$ is large enough that it exceeds c_i plus any tax that the individual might have to pay.

(a) What is the Pareto-efficient decision: to build the park or not?

Assume that the pivotal mechanism is used, so that each individual i is asked to state a number w_i which is going to be interpreted as the gross benefit to individual i from carrying out the project. There are no restrictions on the number w_i: it can be positive, negative or zero. Suppose that the individuals make the following announcements:

Individual	1	2	3	4	5
Stated benefit	$w_1 = \$70$	$w_2 = \$10$	$w_3 = \$65$	$w_4 = -\$30$	$w_5 = -\$5$

(b) Would the park be built based on the above announcements?

(c) Using the above announcements and the rules of the pivotal mechanism, fill in the following table:

Individual	1	2	3	4	5
Pivotal?					
Tax					

(d) As you know, in the pivotal mechanism each individual has a dominant strategy. If all the individuals played their dominant strategies, would the park be built?

(e) Assuming that all the individuals play their dominant strategies, find out who is pivotal and what tax (if any) each individual has to pay?

(f) Show that if every other individual reports his/her true benefit, then it is best for Individual 1 to also report his/her true benefit.

■

2.9.5 Exercises for Section 2.5: Iterated deletion procedures

The answers to the following exercises are in Section 2.10 at the end of this chapter.

Exercise 2.10 Consider again the game of Part **(b)** of Exercise 2.1.

(a) Apply the IDSDS procedure (Iterated Deletion of Strictly Dominated Strategies).

(b) Apply the IDWDS procedure (Iterated Deletion of Weakly Dominated Strategies).

■

Exercise 2.11 Apply the IDSDS procedure to the game shown in Figure 2.24. Is there a strict iterated dominant-strategy equilibrium? ■

		Player 2		
		d	e	f
Player 1	a	8 , 6	0 , 9	3 , 8
	b	3 , 2	2 , 1	4 , 3
	c	2 , 8	1 , 5	3 , 1

Figure 2.24: A strategic-form game with ordinal payoffs

Exercise 2.12 Consider the following game. There is a club with three members: Ann, Bob and Carla. They have to choose which of the three is going to be president next year. Currently, Ann is the president. Each member is both a candidate and a voter. Voting is as follows: each member votes for one candidate (voting for oneself is allowed); if two or more people vote for the same candidate then that person is chosen as the next president; if there is complete disagreement, in the sense that there is exactly one vote for each candidate, then the person for whom Ann voted is selected as the next president.

(a) Represent this voting procedure as a game frame, indicating inside each cell of each table which candidate is elected.

(b) Assume that the players' preferences are as follows: $Ann \succ_{Ann} Carla \succ_{Ann} Bob$, $Carla \succ_{Bob} Bob \succ_{Bob} Ann$, $Bob \succ_{Carla} Ann \succ_{Carla} Carla$. Using utility values 0, 1 and 2, convert the game frame into a game.

(c) Apply the IDWDS to the game of Part **(b)**. Is there an iterated weakly dominant-strategy equilibrium?

(d) Does the extra power given to Ann (in the form of tie-breaking in case of complete disagreement) benefit Ann?

■

Exercise 2.13 Consider the game shown in Figure 2.25.

(a) Apply the IDSDS procedure. Is there an iterated strictly dominant-strategy equilibrium?

(b) Apply the IDWDS procedure. Is there an iterated weakly dominant-strategy equilibrium?

■

		Player 2		
		D	*E*	*F*
Player 1	*a*	2 3	2 2	3 1
	b	2 0	3 1	1 0
	c	1 4	2 0	0 4

Figure 2.25: A strategic-form game with ordinal payoffs

2.9.6 Exercises for Section 2.6: Nash equilibrium

The answers to the following exercises are in Section 2.10 at the end of this chapter.

Exercise 2.14 Find the Nash equilibria of the game of Exercise 2.2. ■

Exercise 2.15 Find the Nash equilibria of the games of Exercise 2.3 **(b)** and **(c)**. ■

Exercise 2.16 Find the Nash equilibria of the game of Exercise 2.4 **(b)**. ■

Exercise 2.17 Find the Nash equilibria of the game of Exercise 2.6. ■

Exercise 2.18 Find the Nash equilibria of the game of Exercise 2.7. ■

Exercise 2.19 Find a Nash equilibrium of the game of Exercise 2.8 for the case where

$$n = 3 \qquad \text{and} \qquad v_1 > v_2 > v_3 > 0$$

(there are several Nash equilibria: you don't need to find them all). ■

Exercise 2.20 Find the Nash equilibria of the game of Exercise 2.12 **(b)**. ■

Exercise 2.21 Find the Nash equilibria of the game of Exercise 2.13. ■

2.9.7 Exercises for Section 2.7: Games with infinite strategy sets

The answers to the following exercises are in Section 2.10 at the end of this chapter.

Exercise 2.22 Consider a simultaneous n-player game where each Player i chooses an effort level $a_i \in [0,1]$. The payoff to Player i is given by

$$\pi_i(a_1, \ldots, a_n) = 4 \min\{a_1, \ldots, a_n\} - 2a_i$$

(interpretation: efforts are complementary and each player's cost per unit of effort is 2).

(a) Find all the Nash equilibria and prove that they are indeed Nash equilibria.
(b) Are any of the Nash equilibria Pareto efficient?
(c) Find a Nash equilibrium where each player gets a payoff of 1.

■

Exercise 2.23 — ★★★ Challenging Question ★★★.

The Mondevil Corporation operates a chemical plant, which is located on the banks of the Sacramento river. Downstream from the chemical plant is a group of fisheries. The Mondevil plant emits by-products that pollute the river, causing harm to the fisheries. The profit Mondevil obtains from operating the chemical plant is $\$\Pi > 0$.

The harm inflicted on the fisheries due to water pollution is equal to $\$L > 0$ of lost profit [without pollution the fisheries' profit is $\$A$, while with pollution it is $\$(A-L)$]. Suppose that the fisheries collectively sue the Mondevil Corporation. It is easily verified in court that Mondevil's plant pollutes the river. However, the values of Π and L cannot be verified by the court, although they are commonly known to the litigants.

Suppose that the court requires the Mondevil attorney (Player 1) and the fisheries' attorney (Player 2) to play the following litigation game. Player 1 is asked to announce a number $x \geq 0$, which the court interprets as a claim about the plant's profits. Player 2 is asked to announce a number $y \geq 0$, which the court interprets as the fisheries' claim about their profit loss. The announcements are made simultaneously and independently.

Then the court uses Posner's nuisance rule to make its decision (R. Posner, *Economic Analysis of Law*, 9th edition, 1997). According to the rule, if $y > x$, then Mondevil must shut down its chemical plant. If $x \geq y$, then the court allows Mondevil to operate the plant, but the court also requires Mondevil to pay the fisheries the amount y. Note that the court cannot force the attorneys to tell the truth: in fact, it would not be able to tell whether or not the lawyers were reporting truthfully. Assume that the attorneys want to maximize the payoff (profits) of their clients.

(a) Represent this situation as a strategic-form game by describing the strategy set of each player and the payoff functions.

(b) Is it a dominant strategy for the Mondevil attorney to make a truthful announcement (i.e. to choose $x = \Pi$)? [Prove your claim.]

(c) Is it a dominant strategy for the fisheries' attorney to make a truthful announcement (i.e. to choose $y = L$)? [Prove your claim.]

(d) For the case where $\Pi > L$ (recall that Π and L denote the true amounts), find **all** the Nash equilibria of the litigation game. [Prove that what you claim to be Nash equilibria are indeed Nash equilibria and that there are no other Nash equilibria.]

(e) For the case where $\Pi < L$ (recall that Π and L denote the true amounts), find **all** the Nash equilibria of the litigation game. [Prove that what you claim to be Nash equilibria are indeed Nash equilibria and that there are no other Nash equilibria.]

(f) Does the court rule give rise to a Pareto efficient outcome? [Assume that the players end up playing a Nash equilibrium.]

■

2.10 Solutions to exercises

Solution to Exercise 2.1.

(a) $I = \{1,2\}$, $S_1 = \{2,4,6\}$, $S_2 = \{1,3,5\}$, $O = \{M,I,J\}$
(where M stands for 'Mexican', I for 'Italian' and J for 'Japanese').
The set of strategy profiles is
$S = \{(2,1),\ (2,3),\ (2,5),\ (4,1),\ (4,3),\ (4,5),\ (6,1),\ (6,3),\ (6,5)\}$;
the outcome function is:
$f(2,1) = f(2,3) = f(4,1) = M$,
$f(2,5) = f(4,3) = f(6,1) = I$ and
$f(4,5) = f(6,3) = f(6,5) = J$.
The representation as a table is shown in Figure 2.26.

Player 2 (Bob)

Player 1 (Antonia)	1	3	5
2	M	M	I
4	M	I	J
6	I	J	J

Figure 2.26: The game-frame for Exercise 2.1 **(a)**

(b) Using values 1, 2 and 3, the utility functions are as follows, where U_1 is the utility function of Player 1 (Antonia) and U_2 is the utility function of Player 2 (Bob):

$$\begin{pmatrix} & M & I & J \\ U_1: & 3 & 2 & 1 \\ U_2: & 2 & 3 & 1 \end{pmatrix}$$

The reduced-form game is shown in Figure 2.27. □

Player 2 (Bob)

Player 1 (Antonia)	1		3		5	
2	3	2	3	2	2	3
4	3	2	2	3	1	1
6	2	3	1	1	1	1

Figure 2.27: The game for Exercise 2.1 **(b)**

Solution to Exercise 2.2.

(a) The game-frame is shown in Figure 2.28.

(b) When the outcomes are sums of money and Player i is selfish and greedy, we can take the following as Player i's utility function: $U_i(\$x) = x$ (other utility functions would do too: the only requirement is that the utility of a larger sum of money is larger than the utility of a smaller sum of money). Thus the reduced-form game is shown in Figure 2.29. □

		Player 2 (Bob)		
		0	1	2
Player 1 (Antonia)	2	Antonia gets nothing / Bob gets \$2	Antonia gets nothing / Bob gets \$3	Antonia gets nothing / Bob gets \$2
	4	Antonia gets nothing / Bob gets \$2	Antonia gets \$5 / Bob gets \$5	Antonia gets \$4 / Bob gets \$2
	6	Antonia gets \$4 / Bob gets \$2	Antonia gets \$3 / Bob gets \$7	Antonia gets \$2 / Bob gets nothing

Figure 2.28: The game-frame for Exercise 2.2 **(a)**

		Player 2 (Bob)					
		0		1		2	
Player 1 (Antonia)	2	0	2	0	3	0	2
	4	0	2	5	5	4	2
	6	4	2	3	7	2	0

Figure 2.29: The game for Exercise 2.2 **(b)**

Solution to Exercise 2.3.

(a) The game-frame is shown in Figure 2.30.

(b) The reduced-form game is shown in Figure 2.31.

(c) The reduced-form game is shown in Figure 2.32. For Alice we chose 1 and 0 as utilities, but one could also use 2 and 1 or 2 and 0. □

P = Press

Charlie: P

Alice \ Bob	P	not P
P	Alice wins	Bob wins
not P	Bob wins	Alice wins

Charlie: not P

Alice \ Bob	P	not P
P	Bob wins	Alice wins
not P	Alice wins	Charlie wins

Figure 2.30: **The game-frame for Exercise 2.3 (a)**

Charlie: P

Alice \ Bob	P			not P		
P	1	0	0	0	1	0
not P	0	1	0	1	0	0

Charlie: not P

Alice \ Bob	P			not P		
P	0	1	0	1	0	0
not P	1	0	0	0	0	1

Figure 2.31: **The reduced-form game for Exercise 2.3 (b)**

Charlie: P

Alice \ Bob	P			not P		
P	1	0	0	0	2	1
not P	0	2	1	1	0	0

Charlie: not P

Alice \ Bob	P			not P		
P	0	2	1	1	0	0
not P	1	0	0	0	1	2

Figure 2.32: **The reduced-form game for Exercise 2.3 (c)**

Solution to Exercise 2.4.

(a) The tables are shown in Figure 2.33.

(b) For Player 1 we can take as his payoff the total money lost by the referee and for Player 2 her own net gain as shown in Figure 2.34.

(c) For Player 1 contributing \$600 is a strictly dominant strategy and for Player 2 contributing \$0 is a strictly dominant strategy. Thus (\$600,\$0) is the strictly dominant-strategy equilibrium. ☐

Distributed money

Player 1 \ Player 2	0		300		600	
0	0	0	225	225	450	450
300	225	225	450	450	675	675
600	450	450	675	675	900	900

Net amounts

Player 1 \ Player 2	0		300		600	
0	0	0	225	–75	450	–150
300	–75	225	150	150	375	75
600	–150	450	75	375	300	300

Figure 2.33: The tables for Exercise 2.4 **(a)**

Player 1 \ Player 2	0		300		600	
0	0	0	150	–75	300	–150
300	150	225	300	150	450	75
600	300	450	450	375	600	300

Figure 2.34: The game for Exercise 2.4 **(b)**

Solution to Exercise 2.5. The game under consideration is shown in Figure 2.35.

(a) For Player 1, 6 is strictly dominated by 2. There is no other strategy which is strictly dominated. Player 2 does not have any strictly dominated strategies.

(b) For Player 1, 6 is weakly dominated by 4 (and also by 2, since strict dominance implies weak dominance); 4 is weakly dominated by 2. Player 2 does not have any weakly dominated strategies. ☐

Player 1 (Antonia) \ Player 2 (Bob)	1	3	5
2	3 , 2	3 , 2	2 , 3
4	3 , 2	2 , 3	1 , 1
6	2 , 3	1 , 1	1 , 1

Figure 2.35: The game for Exercise 2.5

Solution to Exercise 2.6.

(a) The game under consideration is shown in Figure 2.36.

(b) For Player 1, 0 strictly dominates 3 and 6, 3 strictly dominates 6 (the same is true for every player). Thus 0 is a strictly dominant strategy.

(c) The strictly dominant-strategy equilibrium is (0,0,0) (everybody contributes 0). □

Player 3: 0

Player 1 \ Player 2	0	3	6
0	0 0 0	2 –1 2	4 –2 4
3	–1 2 2	1 1 4	3 0 6
6	–2 4 4	0 3 6	2 2 8

Player 3: 3

Player 1 \ Player 2	0	3	6
0	2 2 –1	4 1 1	6 0 3
3	1 4 1	3 3 3	5 2 5
6	0 6 3	2 5 5	4 4 7

Player 3: 6

Player 1 \ Player 2	0	3	6
0	4 4 –2	6 3 0	8 2 2
3	3 6 0	5 5 2	7 4 4
6	2 8 2	4 7 4	6 6 6

Figure 2.36: The game for Exercise 2.6

Solution to Exercise 2.7. The game under consideration is shown in Figure 2.37. □

Player 1 (value \$30M)	\$10M	\$20M	\$30M	\$40M	\$50M
\$10M	0 , 40	0 , 40	0 , 40	0 , 40	0 , 40
\$20M	20 , 0	0 , 30	0 , 30	0 , 30	0 , 30
\$30M	20 , 0	10 , 0	0 , 20	0 , 20	0 , 20
\$40M	20 , 0	10 , 0	0 , 0	0 , 10	0 , 10
\$50M	20 , 0	10 , 0	0 , 0	–10 , 0	0 , 0

Figure 2.37: The game for Exercise 2.7

Solution to Exercise 2.8. No. Suppose, by contradiction, that $\hat{b}_1$ is a weakly dominant strategy for Player 1. It cannot be that $\hat{b}_1 > v_1$, because when $b_2 = b_3 = \hat{b}_1$ Player 1 wins and pays $\hat{b}_1$, thereby obtaining a payoff of $v_1 - \hat{b}_1 < 0$, whereas bidding 0 would give him a payoff of 0.
It cannot be that $\hat{b}_1 = v_1$ because when $b_2 > \hat{b}_1$ and $b_3 < v_1$ the auction is won by Player 2 and Player 1 gets a payoff of 0, while a bid of Player 1 greater than b_2 would make him the winner with a payoff of $v_1 - b_3 > 0$.
Similarly, it cannot be that $\hat{b}_1 < v_1$ because when $b_2 > v_1$ and $b_3 < v_1$ then with $\hat{b}_1$ the auction is won by Player 2 and Player 1 gets a payoff of 0, while a bid greater than b_2 would make him the winner with a payoff of $v_1 - b_3 > 0$. □

Solution to Exercise 2.9.

(a) Since $\sum_{i=1}^{5} v_i = 85 < \sum_{i=1}^{5} c_i = 100$ the Pareto efficient decision is not to build the park.

(b) Since $\sum_{i=1}^{5} w_i = 120 > \sum_{i=1}^{5} c_i = 100$ the park would be built.

(c) Individuals 1 and 3 are pivotal and each of them has to pay a tax of \$30. The other individuals are not pivotal and thus are not taxed.

(d) For each individual i it is a dominant strategy to report v_i and thus, by Part **(a)**, the decision will be the Pareto efficient one, namely not to build the park.

(e) When every individual reports truthfully, Individuals 4 and 5 are pivotal and Individual 4 has to pay a tax of \$25, while individual 5 has to pay a tax of \$10. The others are not pivotal and do not have to pay a tax.

(f) Assume that all the other individuals report truthfully; then if Individual 1 reports truthfully, he is not pivotal, the project is not carried out and his utility is $\overline{m}_1$. Any other w_1 that leads to the same decision (not to build the park) gives him the same utility.
If, on the other hand, he chooses a w_1 that leads to a decision to build the park, then Individual 1 will become pivotal and will have to pay a tax $t_1 = 45$ with a utility of $\overline{m}_1 + v_1 - c_1 - t_1 = \overline{m}_1 + 60 - 30 - 45 = \overline{m}_1 - 15$, so that he would be worse off relative to reporting truthfully. □

Solution to Exercise 2.10. The game under consideration is shown in Figure 2.38.

Player 1 (Antonia) \ Player 2 (Bob)	1	3	5
2	3, 2	3, 2	2, 3
4	3, 2	2, 3	1, 1
6	2, 3	1, 1	1, 1

Figure 2.38: The game for Exercise 2.10

(a) The first step of the procedure eliminates 6 for Player 1. After this step the procedure stops and thus the output is as shown in Figure 2.39.

		Player 2 (Bob)		
		1	3	5
Player 1 (Antonia)	2	3 2	3 2	2 3
	4	3 2	2 3	1 1

Figure 2.39: The output of the IDWDS procedure applied to the game of Figure 2.38

(b) The first step of the procedure eliminates 4 and 6 for Player 1 and nothing for Player 2. The second step of the procedure eliminates 1 and 3 for Player 2. Thus the output is the strategy profile (2,5), which constitutes the iterated weakly dominant-strategy equilibrium of this game. □

Solution to Exercise 2.11. The game under consideration is shown in Figure 2.40.

		Player 2		
		d	e	f
Player 1	a	8 , 6	0 , 9	3 , 8
	b	3 , 2	2 , 1	4 , 3
	c	2 , 8	1 , 5	3 , 1

Figure 2.40: The game for Exercise 2.11

In this game c is strictly dominated by b;
- after deleting c, d becomes strictly dominated by f;
- after deleting d, a becomes strictly dominated by b;
- after deleting a, e becomes strictly dominated by f;
- deletion of e leads to only one strategy profile, namely (b, f).
Thus (b, f) is the iterated strictly dominant-strategy equilibrium. □

Solution to Exercise 2.12.

(a) The game-frame under consideration is shown in Figure 2.41.

Carla votes for A

ANN \ BOB	A	B	C
A	A	A	A
B	A	B	B
C	A	C	C

Carla votes for B

ANN \ BOB	A	B	C
A	A	B	A
B	B	B	B
C	C	B	C

Carla votes for C

ANN \ BOB	A	B	C
A	A	A	C
B	B	B	C
C	C	C	C

Figure 2.41: The game-frame for Exercise 2.12 **(a)**

(b) The game under consideration is shown in Figure 2.42.

Carla votes for A

ANN \ BOB	A	B	C
A	2,0,1	2,0,1	2,0,1
B	2,0,1	0,1,2	0,1,2
C	2,0,1	1,2,0	1,2,0

Carla votes for B

ANN \ BOB	A	B	C
A	2,0,1	0,1,2	2,0,1
B	0,1,2	0,1,2	0,1,2
C	1,2,0	0,1,2	1,2,0

Carla votes for C

ANN \ BOB	A	B	C
A	2,0,1	2,0,1	1,2,0
B	0,1,2	0,1,2	1,2,0
C	1,2,0	1,2,0	1,2,0

Figure 2.42: The game for Exercise 2.12 **(b)**

(c) For Ann, both B and C are weakly dominated by A; for Bob, A is weakly dominated by C; for Carla, C is weakly dominated by B.
Thus in the first step of the IDWDS we delete B and C for Ann, A for Bob and C for Carla.
Hence the game reduces to the Figure 2.43. In this game, for Bob, C is weakly dominated by B and for Carla, A is weakly dominated by B.
Thus in the second and final step of the IDWDS we delete C for Bob and A for Carla and we are left with a unique strategy profile, namely (A,B,B), that is, Ann votes for herself and Bob and Carla vote for Bob. This is the iterated weakly dominant-strategy equilibrium.

(d) The elected candidate is Bob, who is Ann's least favorite; thus the extra power given to Ann (tie breaking in case of total disagreement) turns out to be detrimental for Ann! □

		CARLA: A	CARLA: B
BOB	B	0 , 1	1 , 2
	C	0 , 1	0 , 1

Figure 2.43: The reduced game for Exercise 2.12 **(c)**

Solution to Exercise 2.13. The game under consideration is shown in Figure 2.44.

		Player 2: D		E		F	
Player 1	a	2	3	2	2	3	1
	b	2	0	3	1	1	0
	c	1	4	2	0	0	4

Figure 2.44: The game for Exercise 2.13

(a) The output of the IDSDS is shown in Figure 2.45 (first delete c and then F). Thus there is no iterated strictly dominant-strategy equilibrium.

		Player 2: D		E	
Player 1	a	2	3	2	2
	b	2	0	3	1

Figure 2.45: The output of the IDSDS procedure applied to the game of Figure 2.44

(b) The output of the IDWDS is (b,E) (in the first step delete c and F, the latter because it is weakly dominated by D; in the second step delete a and in the third step delete D). Thus (b,E) is the iterated weakly dominant-strategy equilibrium. □

Solution to Exercise 2.14. The game under consideration is shown in Figure 2.46. There is only one Nash equilibrium, namely (4,1) with payoffs (5,5). □

		Player 2 (Bob)		
		0	1	2
Player 1 (Antonia)	2	0 2	0 3	0 2
	4	0 2	5 5	4 2
	6	4 2	3 7	2 0

Figure 2.46: The game for Exercise 2.14

Solution to Exercise 2.15. The game of Exercise 2.3 **(b)** is shown in Figure 2.47. This game has only one Nash equilibrium, namely *(not P, P, not P)*.

P = Press

		Bob	
		P	not P
Alice	P	1 0 0	0 1 0
	not P	0 1 0	1 0 0

Charlie: P

		Bob	
		P	not P
Alice	P	0 1 0	1 0 0
	not P	1 0 0	0 0 1

Charlie: not P

Figure 2.47: The first game for Exercise 2.15

The game of Exercise 2.3 **(c)** is shown in Figure 2.48. This game does not have any Nash equilibria. □

P = Press

		Bob					
		P			not P		
Alice	P	1	0	0	0	2	1
	not P	0	2	1	1	0	0

Charlie: P

		Bob					
		P			not P		
Alice	P	0	2	1	1	0	0
	not P	1	0	0	0	1	2

Charlie: not P

Figure 2.48: The second game for Exercise 2.15

Solution to Exercise 2.16. The game under consideration is shown in Figure 2.49. This game has only one Nash equilibrium, namely (600,0). □

		Player 2					
		0		300		600	
Player 1	0	0	0	150	−75	300	−150
	300	150	225	300	150	450	75
	600	300	450	450	375	600	300

Figure 2.49: The game for Exercise 2.16

Solution to Exercise 2.17. The game under consideration is shown in Figure 2.50. This game has only one Nash equilibrium, namely (0,0,0). □

Player 3: 0

Player 1 \ Player 2	0	3	6
0	0 0 0	2 –1 2	4 –2 4
3	–1 2 2	1 1 4	3 0 6
6	–2 4 4	0 3 6	2 2 8

Player 3: 3

Player 1 \ Player 2	0	3	6
0	2 2 –1	4 1 1	6 0 3
3	1 4 1	3 3 3	5 2 5
6	0 6 3	2 5 5	4 4 7

Player 3: 6

Player 1 \ Player 2	0	3	6
0	4 4 –2	6 3 0	8 2 2
3	3 6 0	5 5 2	7 4 4
6	2 8 2	4 7 4	6 6 6

Figure 2.50: The game for Exercise 2.17

Solution to Exercise 2.18. The game under consideration is shown in Figure 2.51. This game has 15 Nash equilibria:
(10,30), (10,40), (10,50), (20,30), (20,40), (20,50), (30,30), (30,40), (30,50), (40,40), (40,50), (50,10), (50,20), (50,30), (50,50). □

Player 1 (value \$30M)	\$10M	\$20M	\$30M	\$40M	\$50M
\$10M	0 , 40	0 , 40	0 , 40	0 , 40	0 , 40
\$20M	20 , 0	0 , 30	0 , 30	0 , 30	0 , 30
\$30M	20 , 0	10 , 0	0 , 20	0 , 20	0 , 20
\$40M	20 , 0	10 , 0	0 , 0	0 , 10	0 , 10
\$50M	20 , 0	10 , 0	0 , 0	–10 , 0	0 , 0

Figure 2.51: The game for Exercise 2.18

Solution to Exercise 2.19. A Nash equilibrium is $b_1 = b_2 = b_3 = v_1$ (with payoffs (0,0,0)). Convince yourself that this is indeed a Nash equilibrium.
There are many more Nash equilibria: for example, any triple (b_1, b_2, b_3) with $b_2 = b_3 = v_1$ and $b_1 > v_1$ is a Nash equilibrium (with payoffs (0,0,0)) and so is any triple (b_1, b_2, b_3) with $b_2 = b_3 = v_2$ and $b_1 \geq v_2$ (with payoffs $(v_1 - v_2, 0, 0)$). □

Solution to Exercise 2.20. The game under consideration is shown in Figure 2.52. There are 5 Nash equilibria: $(A,A,A), (B,B,B), (C,C,C), (A,C,A)$ and (A,B,B). □

ANN \ BOB	A	B	C
A	2,0,1	2,0,1	2,0,1
B	2,0,1	0,1,2	0,1,2
C	2,0,1	1,2,0	1,2,0

Carla votes for A

ANN \ BOB	A	B	C
A	2,0,1	0,1,2	2,0,1
B	0,1,2	0,1,2	0,1,2
C	1,2,0	0,1,2	1,2,0

Carla votes for B

ANN \ BOB	A	B	C
A	2,0,1	2,0,1	1,2,0
B	0,1,2	0,1,2	1,2,0
C	1,2,0	1,2,0	1,2,0

Carla votes for C

Figure 2.52: The game for Exercise 2.20

Solution to Exercise 2.21. The game under consideration is shown in Figure 2.53. There are 2 Nash equilibria: (a, D) and (b, E). □

Player 1 \ Player 2	*D*		*E*		*F*	
a	2	3	2	2	3	1
b	2	0	3	1	1	0
c	1	4	2	0	0	4

Figure 2.53: The game for Exercise 2.21

Solution to Exercise 2.22.

(a) For every $e \in [0,1]$, $(e,e,...,e)$ is a Nash equilibrium.
The payoff of Player i is $\pi_i(e,e,...,e) = 2e$; if player i increases her effort to $a > e$ (of course, this can only happen if $e < 1$), then her payoff decreases to $4e - 2a$ and if she decreases her effort to $a < e$ (of course, this can only happen if $e > 0$), then her payoff decreases to $2a$.
There is no Nash equilibrium where two players choose different levels of effort.
Proof: suppose there is an equilibrium $(a_1, a_2, \dots, a_n)$ where $a_i \neq a_j$ for two players i and j.
Let $a_{\min} = \min\{a_1, \dots, a_n\}$ and let k be a player such that $a_k > a_{\min}$ (such a player exists by our supposition).
Then the payoff to player k is $\pi_k = 4a_{\min} - 2a_k$ and if she reduced her effort to $a_{\min}$ her payoff would increase to $2a_{\min}$.

(b) Any symmetric equilibrium with $e < 1$ is Pareto inefficient, because all the players would be better off if they collectively switched to $(1,1,\dots,1)$. On the other hand, the symmetric equilibrium $(1,1,\dots,1)$ is Pareto efficient.

(c) The symmetric equilibrium $\left(\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right)$. □

Solution to Exercise 2.23.

(a) The strategy sets are $S_1 = S_2 = [0, \infty)$. The payoff functions are as follows:[10]

$$\pi_1(x,y) = \begin{cases} \Pi - y & \text{if } x \geq y \\ 0 & \text{if } y > x \end{cases} \quad \text{and} \quad \pi_2(x,y) = \begin{cases} A - L + y & \text{if } x \geq y \\ A & \text{if } y > x \end{cases}$$

(b) Yes, for player 1 choosing $x = \Pi$ is a weakly dominant strategy.
Proof. Consider an arbitrary y. We must show that $x = \Pi$ gives at least a high a payoff against y as any other x. Three cases are possible.

Case 1: $y < \Pi$. In this case $x = \Pi$ or any other x such that $x \geq y$ yields $\pi_1 = \Pi - y > 0$, while any $x < y$ yields $\pi_1 = 0$.

Case 2: $y = \Pi$. In this case 1's payoff is zero no matter what x he chooses.

Case 3: $y > \Pi$. In this case $x = \Pi$ or any other x such that $x < y$ yields $\pi_1 = 0$, while any $x \geq y$ yields $\pi_1 = \Pi - y < 0$.

(c) No, choosing $y = L$ is not a dominant strategy for Player 2. For example, if $x > L$ then choosing $y = L$ yields $\pi_2 = A$ while choosing a y such that $L < y \leq x$ yields $\pi_2 = A - L + y > A$.

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[10] We have chosen to use accounting profits as payoffs. Alternatively, one could take as payoffs the changes in profits relative to the initial situation, namely

$$\pi_1(x,y) = \begin{cases} -y & \text{if } x \geq y \\ -\Pi & \text{if } y > x \end{cases} \quad \text{and} \quad \pi_2(x,y) = \begin{cases} y & \text{if } x \geq y \\ L & \text{if } y > x. \end{cases}$$

The answers are the same, whichever choice of payoffs one makes.

(d) **Suppose that** $\Pi > L$. If (x,y) is a Nash equilibrium **with** $x \geq y$ then it must be that $y \leq \Pi$ (otherwise Player 1 could increase its payoff by reducing x below y) and $y \geq L$ (otherwise Player 2 would be better off by increasing y above x).
Thus it must be $L \leq y \leq \Pi$, which is possible, given our assumption.
However, it cannot be that $x > y$, because Player 2 would be getting a higher payoff by increasing y to x.
Thus it must be $x \leq y$, which (together with our hypothesis that $x \geq y$) implies that $x = y$. Thus the following are Nash equilibria:

all the pairs (x,y) with $L \leq y \leq \Pi$ and $x = y$.

Now consider pairs (x,y) **with** $x < y$. Then it cannot be that $y < \Pi$, because Player 1 could increase its payoff by increasing x to y. Thus it must be $y \geq \Pi$ (hence, by our supposition that $\Pi > L$, $y > L$). Furthermore, it must be that $x \leq L$ (otherwise Player 2 could increase its profits by reducing y to (or below) x. Thus

(x,y) with $x < y$ is a Nash equilibrium if and only if $x \leq L$ and $y \geq \Pi$.

(e) **Suppose that** $\Pi < L$. For the same reasons given above, an equilibrium with $x \geq y$ requires $L \leq y \leq \Pi$. However, this is not possible given that $\Pi < L$. Hence,

there is no Nash equilibrium (x,y) with $x \geq y$.

Thus we must restrict attention to pairs (x,y) **with** $x < y$. As explained above, it must be that $y \geq \Pi$ and $x \leq L$. Thus,

(x,y) with $x < y$ is a Nash equilibrium if and only if $\Pi \leq y$ and $x \leq L$.

(f) Pareto efficiency requires that the chemical plant be shut down if $\Pi < L$ and that it remain operational if $\Pi > L$.
Now, when $\Pi < L$ all the equilibria have $x < y$ which leads to shut-down, hence a Pareto efficient outcome.
When $\Pi > L$, there are two types of equilibria: one where $x = y$ and the plant remains operational (a Pareto efficient outcome) and the other where $x < y$ in which case the plant shuts down, yielding a Pareto inefficient outcome. □