7. Extensive-form Games

7.1 Behavioral strategies in dynamic games

The definition of dynamic (or extensive-form) game-frame with cardinal payoffs is just like the definition of extensive-form game-frame with ordinal payoffs (Definition 4.1.1, Chapter 4), the only difference being that we postulate von Neumann-Morgenstern preferences instead of merely ordinal preferences.

In Chapter 6 we generalized the notion of strategic-form game-frame by allowing for lotteries (rather than just simple outcomes) to be associated with strategy profiles. One can do the same with extensive-form game-frames. For example, Figure 7.1 shows an extensive-form game-frame where associated with each terminal node (denoted by z_i, $i = 1,2,...,5$) is either a basic outcome (denoted by o_j, $j = 1,2,...,5$) or a lottery (probability distribution) over the set of basic outcomes $\{o_1, o_2, o_3, o_4, o_5\}$.

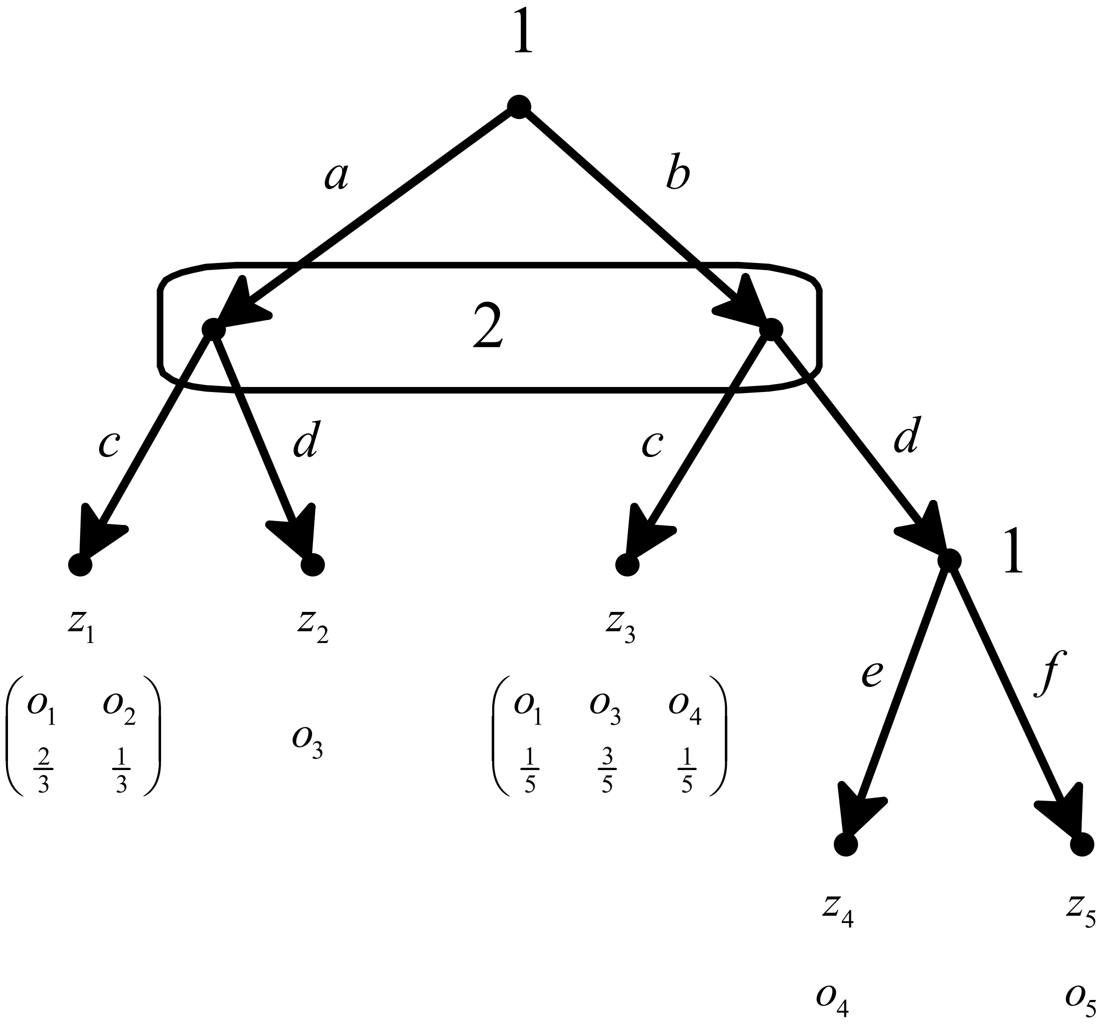


Figure 7.1: An extensive-form frame with probabilistic outcomes. The z_i's are terminal nodes and the o_i's are basic outcomes

In Figure 7.1 $\{z_1, z_2, ..., z_5\}$ is the set of terminal nodes and $\{o_1, o_2, ..., o_5\}$ is the set of basic outcomes.
Associated with z_1 is the lottery $\begin{pmatrix} o_1 & o_2 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$, while the lottery associated with z_3 is $\begin{pmatrix} o_1 & o_3 & o_4 \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{pmatrix}$, etc.

However, as we saw at the end of Chapter 4, in extensive forms one can explicitly represent random events by means of chance moves (also called moves of Nature). Thus an alternative representation of the extensive-form frame of Figure 7.1 is the extensive form shown in Figure 7.2.

We can continue to use the definition of extensive-form frame given in Chapter 4, but from now on we will allow for the possibility of chance moves.

The notion of strategy remains, of course, unchanged: a strategy for a player is a list of choices, one for every information set of that player (Definition 4.2.1, Chapter 4). For example, the set of strategies for Player 1 in the extensive frame of Figure 7.2 is $S_1 = \{(a,e),(a,f),(b,e),(b,f)\}$. Thus mixed strategies can easily be introduced also in extensive frames. For example, in the extensive frame of Figure 7.2, the set of mixed strategies for Player 1, denoted by Σ_1, is the set of probability distributions over S_1:

$$\Sigma_1 = \left\{ \begin{pmatrix} (a,e) & (a,f) & (b,e) & (b,f) \\ p & q & r & 1-p-q-r \end{pmatrix} : p,q,r \in [0,1] \text{ and } p+q+r \leqslant 1 \right\}.$$

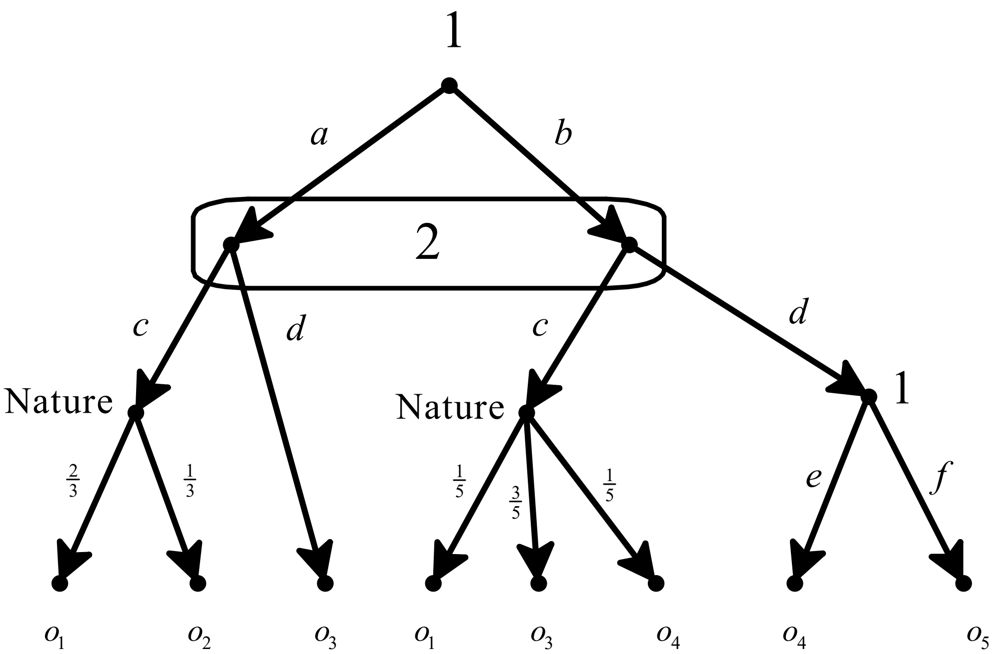


Figure 7.2: An alternative representation of the extensive frame of Figure 7.1. The terminal nodes have not been labeled. The o_i's are basic outcomes.

However, it turns out that in extensive forms with perfect recall one can use simpler objects than mixed strategies, namely behavioral strategies.

Definition 7.1.1 A *behavioral strategy* for a player in an extensive form is a list of probability distributions, one for every information set of that player; each probability distribution is over the set of choices at the corresponding information set.

For example, the set of behavioral strategies for Player 1 in the extensive frame of Figure 7.2 is:

$$\left\{ \left(\begin{array}{cc|cc} a & b & e & f \\ p & 1-p & q & 1-q \end{array} \right) : p,q \in [0,1] \right\}$$

.

A behavioral strategy is a simpler object than a mixed strategy: in this example, specifying a *behavioral strategy* for Player 1 requires specifying the values of two parameters (p and q), while specifying a *mixed strategy* requires specifying the values of three parameters (p, r and q). Can one then use behavioral strategies rather than mixed strategies? The answer is affirmative, as Theorem 7.1.1 below states.

First we illustrate with an example based on the extensive form of Figure 7.3 (the z_i's are terminal nodes and the outcomes have been omitted).

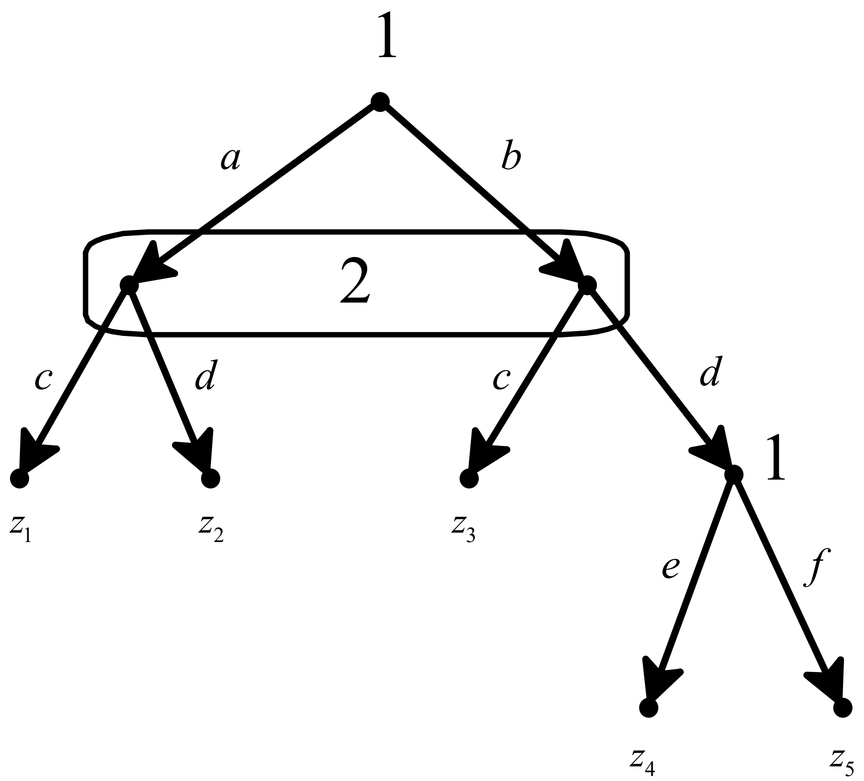


Figure 7.3: An extensive frame with the outcomes omitted. The z_i's are terminal nodes.

Consider the mixed-strategy profile $\sigma = (\sigma_1, \sigma_2)$ with

$$\sigma_1 = \begin{pmatrix} (a,e) & (a,f) & (b,e) & (b,f) \\ \frac{1}{12} & \frac{4}{12} & \frac{2}{12} & \frac{5}{12} \end{pmatrix} \quad \text{and} \quad \sigma_2 = \begin{pmatrix} c & d \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

We can compute the probability of reaching terminal node z_i, denoted by $P(z_i)$, as follows:

$$P(z_1) = \sigma_1((a,e))\,\sigma_2(c) + \sigma_1((a,f))\,\sigma_2(c) = \tfrac{1}{12}\left(\tfrac{1}{3}\right) + \tfrac{4}{12}\left(\tfrac{1}{3}\right) = \tfrac{5}{36}$$
$$P(z_2) = \sigma_1((a,e))\,\sigma_2(d) + \sigma_1((a,f))\,\sigma_2(d) = \tfrac{1}{12}\left(\tfrac{2}{3}\right) + \tfrac{4}{12}\left(\tfrac{2}{3}\right) = \tfrac{10}{36}$$
$$P(z_3) = \sigma_1((b,e))\,\sigma_2(c) + \sigma_1((b,f))\,\sigma_2(c) = \tfrac{2}{12}\left(\tfrac{1}{3}\right) + \tfrac{5}{12}\left(\tfrac{1}{3}\right) = \tfrac{7}{36}$$
$$P(z_4) = \sigma_1((b,e))\,\sigma_2(d) = \tfrac{2}{12}\left(\tfrac{2}{3}\right) = \tfrac{4}{36}$$
$$P(z_5) = \sigma_1((b,f))\,\sigma_2(d) = \tfrac{5}{12}\left(\tfrac{2}{3}\right) = \tfrac{10}{36}.$$

That is, the mixed-strategy profile $\sigma = (\sigma_1, \sigma_2)$ gives rise to the following probability distribution over terminal nodes:

$$\begin{pmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \\ \frac{5}{36} & \frac{10}{36} & \frac{7}{36} & \frac{4}{36} & \frac{10}{36} \end{pmatrix}.$$

Now consider the following behavioral strategy of Player 1:

$$\left(\begin{array}{cc|cc} a & b & e & f \\ \frac{5}{12} & \frac{7}{12} & \frac{2}{7} & \frac{5}{7} \end{array}\right).$$

What probability distribution over the set of terminal nodes would it induce in conjunction with Player 2's mixed strategy $\sigma_2 = \begin{pmatrix} c & d \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$? The calculations are simple:[1]

$$\begin{aligned}
P(z_1) &= P(a)\,\sigma_2(c) = \tfrac{5}{12}\left(\tfrac{1}{3}\right) = \tfrac{5}{36}, \\
P(z_2) &= P(a)\,\sigma_2(d) = \tfrac{5}{12}\left(\tfrac{2}{3}\right) = \tfrac{10}{36}, \\
P(z_3) &= P(b)\,\sigma_2(c) = \tfrac{7}{12}\left(\tfrac{1}{3}\right) = \tfrac{7}{36}, \\
P(z_4) &= P(b)\,\sigma_2(d)\,P(e) = \tfrac{7}{12}\left(\tfrac{2}{3}\right)\left(\tfrac{2}{7}\right) = \tfrac{4}{36}, \\
P(z_5) &= P(b)\,\sigma_2(d)\,P(f) = \tfrac{7}{12}\left(\tfrac{2}{3}\right)\left(\tfrac{5}{7}\right) = \tfrac{10}{36}.
\end{aligned}$$

Thus, against $\sigma_2 = \begin{pmatrix} c & d \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$, Player 1's behavioral strategy $\left(\begin{array}{cc|cc} a & b & e & f \\ \frac{5}{12} & \frac{7}{12} & \frac{2}{7} & \frac{5}{7} \end{array}\right)$ and her mixed strategy $\begin{pmatrix} (a,e) & (a,f) & (b,e) & (b,f) \\ \frac{1}{12} & \frac{4}{12} & \frac{2}{12} & \frac{5}{12} \end{pmatrix}$ are equivalent, in the sense that they give rise to the same probability distribution over terminal nodes, namely

$$\begin{pmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \\ \frac{5}{36} & \frac{10}{36} & \frac{7}{36} & \frac{4}{36} & \frac{10}{36} \end{pmatrix}.$$

[1] $P(x)$ denotes the probability of choice x for Player 1, according to the given behavioral strategy

Theorem 7.1.1 — Kuhn, 1953. In extensive forms *with perfect recall*, behavioral strategies and mixed strategies are equivalent, in the sense that, for every mixed strategy there is a behavioral strategy that gives rise to the same probability distribution over terminal nodes.[a]

[a] A more precise statement is as follows. Consider an extensive form with perfect recall and a Player i. Let x_{-i} be an arbitrary profile of strategies of the players other than i, where, for every $j \neq i$, x_j is either a mixed or a behavioral strategy of Player j. Then, for every mixed strategy σ_i of Player i there is a behavioral strategy b_i of Player i such that (σ_i, x_{-i}) and (b_i, x_{-i}) give rise to the same probability distribution over the set of terminal nodes.

Without perfect recall, Theorem 7.1.1 does not hold. To see this, consider the one-player extensive form shown in Figure 7.4 and the mixed strategy

$$\begin{pmatrix} (a,c) & (a,d) & (b,c) & (b,d) \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

which induces the probability distribution $\begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$ on the set of terminal nodes.

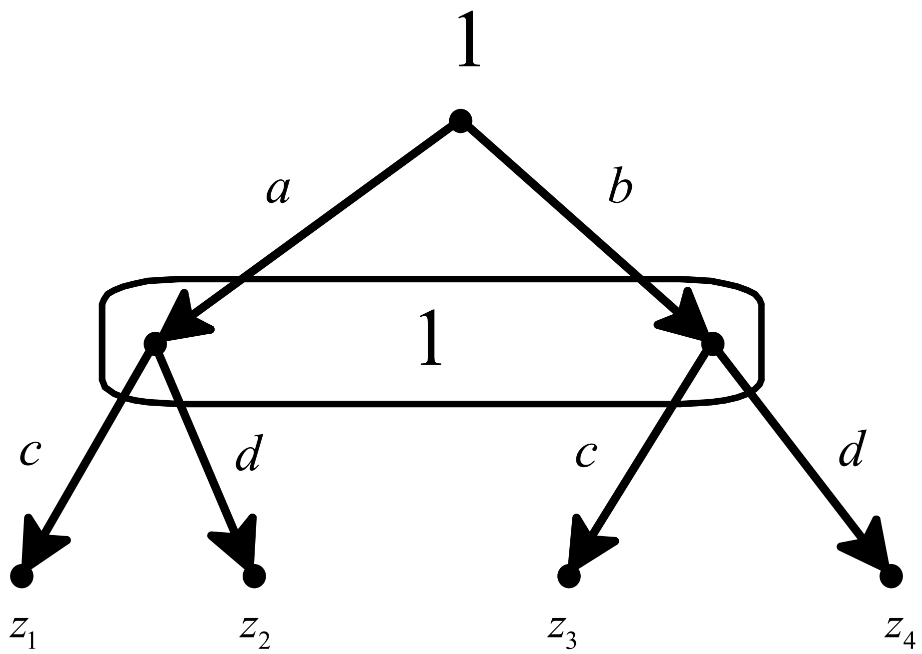


Figure 7.4: A one-player extensive frame without perfect recall.

Consider an arbitrary behavioral strategy

$$\left(\begin{array}{cc|cc} a & b & c & d \\ p & 1-p & q & 1-q \end{array}\right),$$

whose corresponding probability distribution over the set of terminal nodes is

$$\left(\begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \\ pq & p(1-q) & (1-p)q & (1-p)(1-q) \end{array}\right).$$

In order to have $P(z_2) = 0$ it must be that either $p = 0$ or $q = 1$.
If $p = 0$ then $P(z_1) = 0$ and if $q = 1$ then $P(z_4) = 0$.
Thus the probability distribution

$$\left(\begin{array}{cccc} z_1 & z_2 & z_3 & z_4 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array}\right)$$

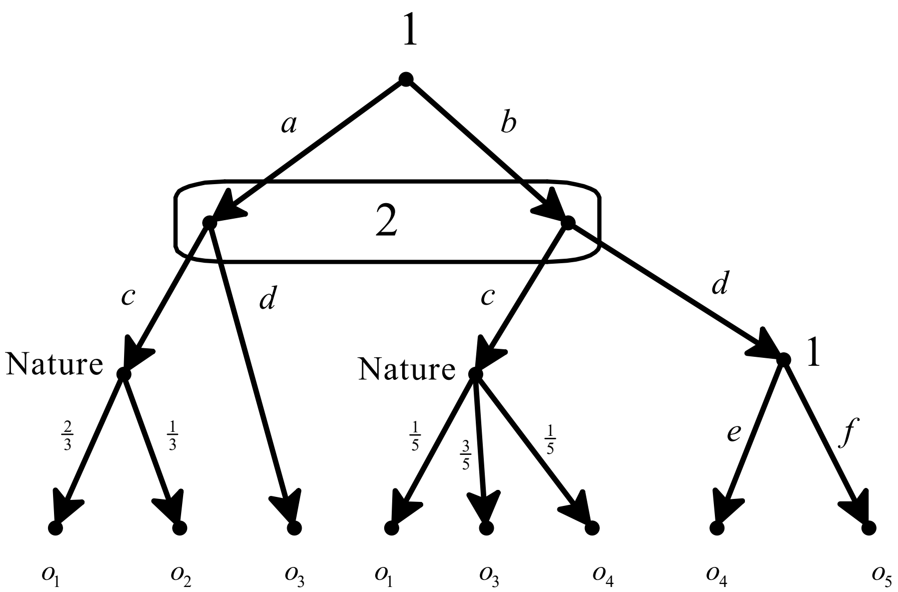
cannot be achieved with a behavioral strategy.

Since the focus of this book is on extensive-form games with perfect recall, by appealing to Theorem 7.1.1, from now on we can restrict attention to behavioral strategies.

As usual, one goes from a frame to a game by adding preferences over outcomes. Let O be the set of basic outcomes (recall that with every terminal node is associated a basic outcome) and $\mathscr{L}(O)$ the set of lotteries (probability distributions) over O.

Definition 7.1.2 An *extensive-form game with cardinal payoffs* is an extensive frame (with, possibly, chance moves) together with a von Neumann-Morgenstern ranking $\succsim_i$ of the set of lotteries $\mathscr{L}(O)$, for every Player i .

As usual, it is convenient to represent a von Neumann-Morgenstern ranking by means of a von Neumann-Morgenstern utility function and replace the outcomes with a vector of utilities, one for every player.



For example, consider the extensive form above, which reproduces Figure 7.2, where the set of basic outcomes is $O = \{o_1, o_2, o_3, o_4, o_5\}$ and suppose that Player 1 has a von Neumann-Morgenstern ranking of $\mathscr{L}(O)$ that is represented by the following von Neumann-Morgenstern utility function:

$$\begin{array}{rccccc} \text{outcome:} & o_1 & o_2 & o_3 & o_4 & o_5 \\ U_1: & 5 & 2 & 0 & 1 & 3 \end{array}$$

Suppose also that Player 2 has preferences represented by the von Neumann-Morgenstern

utility function

$$\begin{array}{rccccc} outcome: & o_1 & o_2 & o_3 & o_4 & o_5 \\ U_2: & 3 & 6 & 4 & 5 & 0 \end{array}$$

Then from the extensive frame of Figure 7.2 we obtain the extensive-form game with cardinal payoffs shown in Figure 7.5.

Since the expected utility of lottery $\begin{pmatrix} o_1 & o_2 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$ is 4 for both players, and the expected utility of lottery $\begin{pmatrix} o_1 & o_3 & o_4 \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{pmatrix}$ is 1.2 for Player 1 and 4 for Player 2, we can simplify the game by replacing the first move of Nature with the payoff vector (4,4) and the second move of Nature with the payoff vector (1.2, 4). The simplified game is shown in Figure 7.6.

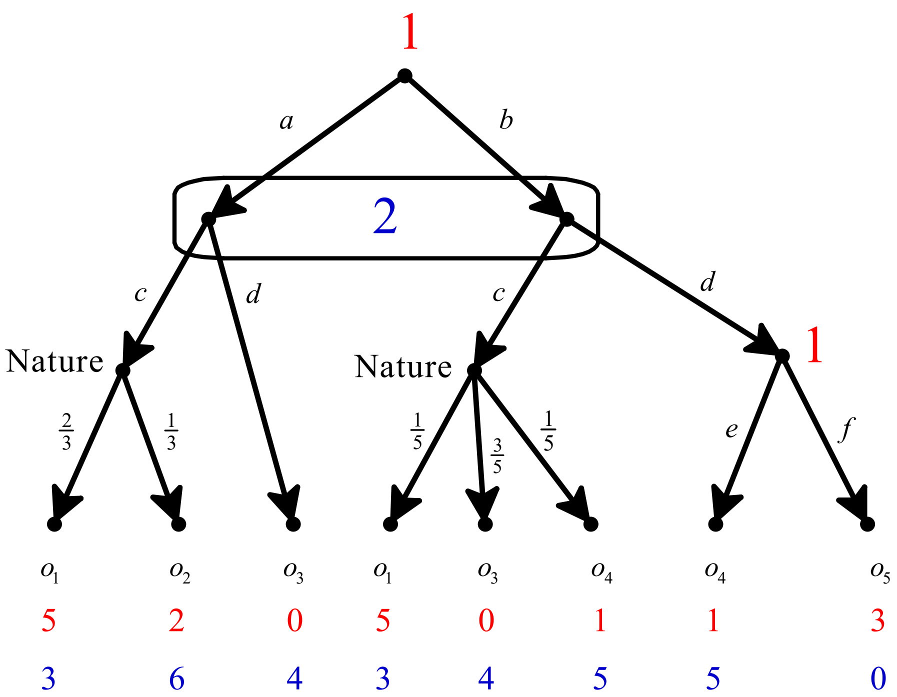


Figure 7.5: An extensive game based on the frame of Figure 7.2. The terminal nodes have not been labeled. The o_i's are basic outcomes.

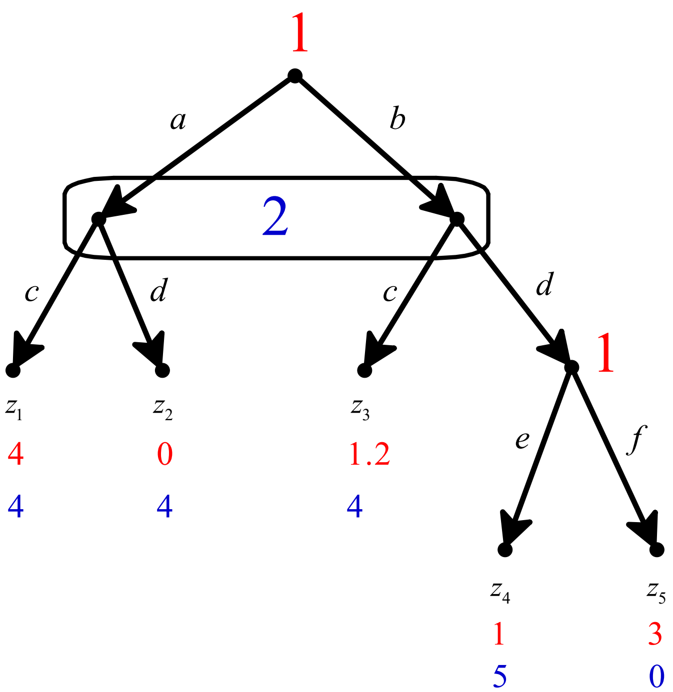


Figure 7.6: A simplified version of the game of Figure 7.5. The z_i's are terminal nodes. Note that this is a game based on the frame of Figure 7.3.

Given an extensive game with cardinal payoffs, associated with every behavioral strategy profile is a lottery over basic outcomes and thus – using a von Neumann-Morgenstern utility function for each player – a payoff for each player. For example, the behavioral strategy profile

$$\left[\begin{pmatrix} a & b & \Big| & e & f \\ \frac{5}{12} & \frac{7}{12} & \Big| & \frac{2}{7} & \frac{5}{7} \end{pmatrix}, \begin{pmatrix} c & d \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}\right]$$

for the extensive game of Figure 7.5 gives rise to the lottery

$$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 & o_5 \\ \frac{71}{540} & \frac{25}{540} & \frac{213}{540} & \frac{81}{540} & \frac{150}{540} \end{pmatrix}$$

(for instance, the probability of basic outcome o_1 is calculated as follows:

$$P(o_1) = P(a)P(c)\tfrac{2}{3} + P(b)P(c)\tfrac{1}{5} = \tfrac{5}{12}\tfrac{1}{3}\tfrac{2}{3} + \tfrac{7}{12}\tfrac{1}{3}\tfrac{1}{5} = \tfrac{71}{540}).$$

Using the utility function postulated above for Player 1, namely

$$\begin{array}{rccccc} outcome: & o_1 & o_2 & o_3 & o_4 & o_5 \\ U_1: & 5 & 2 & 0 & 1 & 3 \end{array}$$

we get a corresponding payoff for Player 1equal to

$$\tfrac{71}{540}5 + \tfrac{25}{540}2 + \tfrac{213}{540}0 + \tfrac{81}{540}1 + \tfrac{150}{540}3 = \tfrac{936}{540} = 1.733.$$

An alternative way of computing this payoff is by using the simplified game of Figure 7.6 where the behavioral strategy profile

$$\left[\begin{pmatrix} a & b & \Big| & e & f \\ \frac{5}{12} & \frac{7}{12} & \Big| & \frac{2}{7} & \frac{5}{7} \end{pmatrix}, \begin{pmatrix} c & d \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}\right]$$

yields the probability distribution over terminal nodes

$$\begin{pmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \\ \frac{5}{36} & \frac{10}{36} & \frac{7}{36} & \frac{4}{36} & \frac{10}{36} \end{pmatrix},$$

which, in turn, yields the probability distribution

$$\begin{pmatrix} 4 & 0 & 1.2 & 1 & 3 \\ \frac{5}{36} & \frac{10}{36} & \frac{7}{36} & \frac{4}{36} & \frac{10}{36} \end{pmatrix}$$

over utilities for Player 1. From the latter we get that the expected payoff for Player 1 is

$$\tfrac{5}{36}4 + \tfrac{10}{36}0 + \tfrac{7}{36}1.2 + \tfrac{4}{36}1 + \tfrac{10}{36}3 = \tfrac{936}{540} = 1.733.$$

The calculations for Player 2 are similar (see Exercise 7.3).

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 7.4.1 at the end of this chapter.

7.2 Subgame-perfect equilibrium revisited

The notion of subgame-perfect equilibrium was introduced in Chapter 4 (Definition 4.4.1) for extensive-form games with ordinal payoffs.

When payoffs are *ordinal*, a subgame-perfect equilibrium may fail to exist because either the entire game or a proper subgame does not have any Nash equilibria.

In the case of finite extensive-form games with *cardinal* payoffs, a subgame-perfect equilibrium always exists, because – by Nash's theorem (Theorem 6.2.1, Chapter 6) – every finite game has at least one Nash equilibrium in mixed strategies.

Thus, in the case of cardinal payoffs, the subgame-perfect equilibrium algorithm (Definition 4.4.2, Chapter 4) never halts and the output of the algorithm is a subgame-perfect equilibrium.

We shall illustrate this with the extensive-form game shown in Figure 7.7.

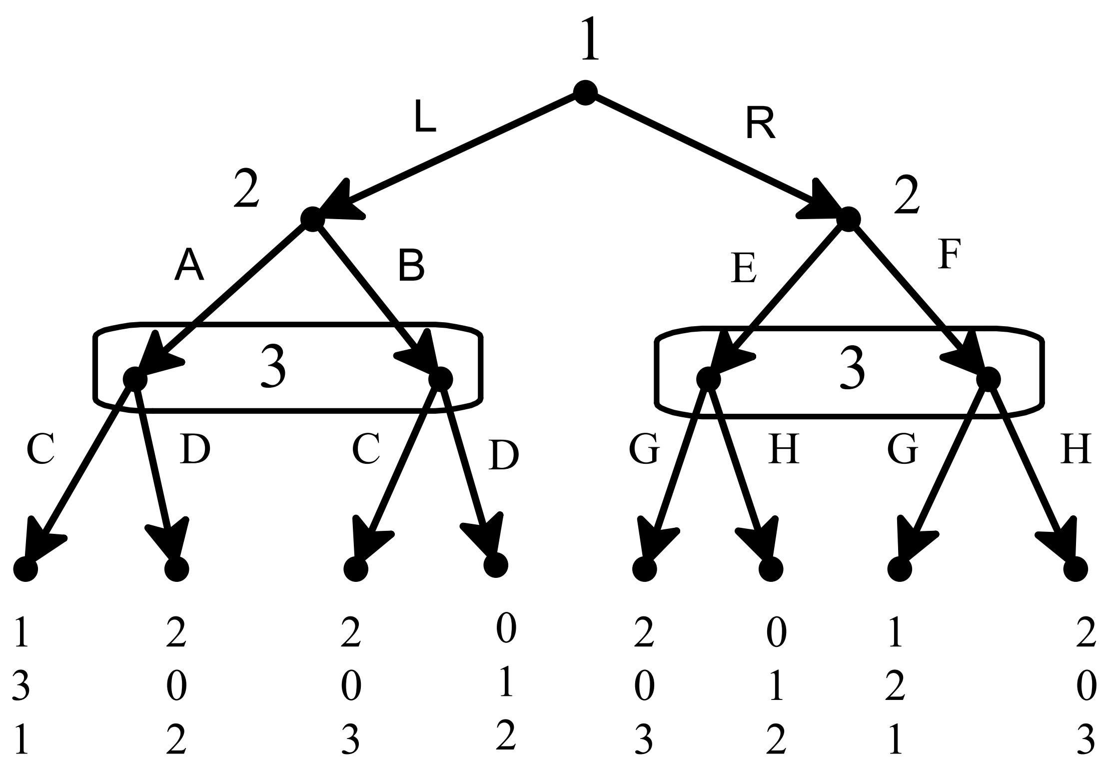


Figure 7.7: An extensive-form game with cardinal payoffs.

Let us apply the subgame-perfect equilibrium algorithm to this game. We start with the proper subgame that begins at Player 2's decision node on the left, whose strategic form is shown in Figure 7.8. Note that this subgame has no pure-strategy Nash equilibria. Thus if payoffs were merely ordinal payoffs the algorithm would halt and we would conclude that the game of Figure 7.7 has no subgame-perfect equilibria. However, we will assume that payoffs are cardinal (that is, that they are von Neumann-Morgenstern utilities).

		Player 3	
		C	*D*
Player 2	*A*	3 , 1	0 , 2
	B	0 , 3	1 , 2

Figure 7.8: The strategic form of the proper subgame on the left in the game of Figure 7.7.

To find the mixed-strategy Nash equilibrium of the game of Figure 7.8, let p be the probability of A and q the probability of C.
- Then we need q to be such that $3q = 1 - q$, that is, $q = \frac{1}{4}$,
- and p to be such that $p + 3(1-p) = 2$, that is, $p = \frac{1}{2}$.

Thus the Nash equilibrium of this proper subgame is:

$$\left[\begin{pmatrix} A & B \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} C & D \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \right],$$

yielding the following payoffs:

for Player 1: $\frac{1}{2}\frac{1}{4}1 + \frac{1}{2}\frac{3}{4}2 + \frac{1}{2}\frac{1}{4}2 + \frac{1}{2}\frac{3}{4}0 = 1.125$

for Player 2: $\frac{1}{2}\frac{1}{4}3 + \frac{1}{2}\frac{3}{4}0 + \frac{1}{2}\frac{1}{4}0 + \frac{1}{2}\frac{3}{4}1 = 0.75$

for Player 3: $\frac{1}{2}\frac{1}{4}1 + \frac{1}{2}\frac{3}{4}2 + \frac{1}{2}\frac{1}{4}3 + \frac{1}{2}\frac{3}{4}2 = 2.$

Thus we can simplify the game of Figure 7.7 as shown in Figure 7.9.

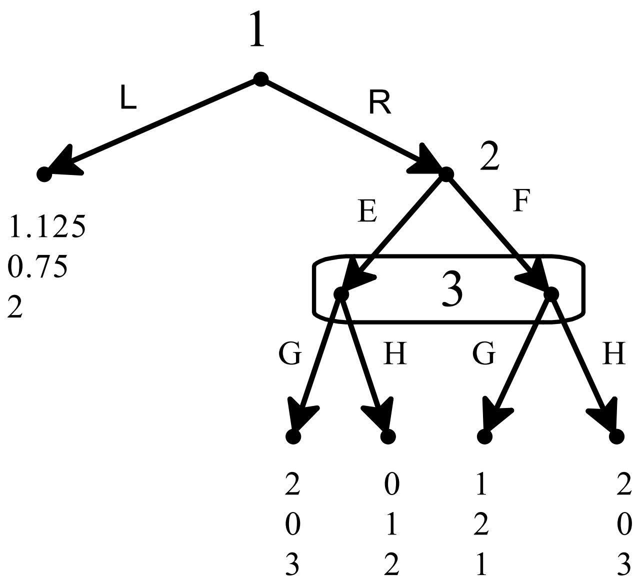


Figure 7.9: The game of Figure 7.7 after replacing the proper subgame on the left with the payoffs associated with its Nash equilibrium.

Now consider the proper subgame of the game of Figure 7.9 (the subgame that starts at Player 2's node). Its strategic form is shown in Figure 7.10.

		Player 3: G	Player 3: H
Player 2	E	0 , 3	1 , 2
	F	2 , 1	0 , 3

Figure 7.10: The strategic form of the proper subgame of the game of Figure 7.9.

Again, there is no pure-strategy Nash equilibrium. To find the mixed-strategy equilibrium let p be the probability of E and q the probability of G.
- Then we need q to be such that $1-q=2q$, that is, $q=\frac{1}{3}$,
- and p to be such that $3p+1-p=2p+3(1-p)$, that is, $p=\frac{2}{3}$.
- Hence the Nash equilibrium is $\left[\begin{pmatrix} E & F \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \begin{pmatrix} G & H \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}\right]$ yielding the following payoffs:

for Player 1: $\frac{2}{3}\left(\frac{1}{3}\right)(2)+\frac{2}{3}\left(\frac{2}{3}\right)(0)+\frac{1}{3}\left(\frac{1}{3}\right)(1)+\frac{1}{3}\left(\frac{2}{3}\right)(2)=1.$

for Player 2: $\frac{2}{3}\left(\frac{1}{3}\right)(0)+\frac{2}{3}\left(\frac{2}{3}\right)(1)+\frac{1}{3}\left(\frac{1}{3}\right)(2)+\frac{1}{3}\left(\frac{2}{3}\right)(0)=0.67.$

for Player 3: $\frac{2}{3}\left(\frac{1}{3}\right)(3)+\frac{2}{3}\left(\frac{2}{3}\right)(2)+\frac{1}{3}\left(\frac{1}{3}\right)(1)+\frac{1}{3}\left(\frac{2}{3}\right)(3)=2.33.$

Thus we can simplify the game of Figure 7.9 as shown in Figure 7.11, where the optimal choice for Player 1 is L.

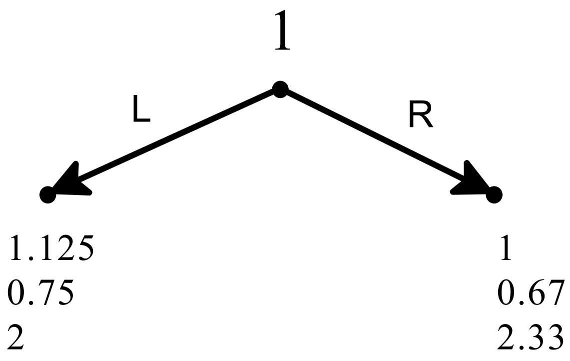


Figure 7.11: The game of Figure 7.9 after replacing the proper subgame with the payoffs associated with the Nash equilibrium.

Hence the subgame-perfect equilibrium of the game of Figure 7.7 (expressed in terms of behavioral strategies) is:

$$\left[\begin{pmatrix} L & R \\ 1 & 0 \end{pmatrix}, \left(\begin{array}{cc|cc} A & B & E & F \\ \frac{1}{2} & \frac{1}{2} & \frac{2}{3} & \frac{1}{3} \end{array}\right), \left(\begin{array}{cc|cc} C & D & G & H \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{3} & \frac{2}{3} \end{array}\right)\right]$$

We conclude this section with the following theorem, which is a corollary of Theorem 6.2.1 (Chapter 6).

Theorem 7.2.1 Every finite extensive-form game with cardinal payoffs has at least one subgame-perfect equilibrium in mixed strategies.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 7.4.2 at the end of this chapter.

7.3 Problems with the notion of subgame-perfect equilibrium

The notion of subgame-perfect equilibrium is a refinement of Nash equilibrium. As explained in Chapter 3, in the context of perfect-information games, the notion of subgame-perfect equilibrium eliminates some "unreasonable" Nash equilibria that involve incredible threats. However, not every subgame-perfect equilibrium can be viewed as a "rational solution". To see this, consider the extensive-form game shown in Figure 7.12. This game has no proper subgames and thus the set of subgame-perfect equilibria coincides with the set of Nash equilibria. The pure-strategy Nash equilibria of this game are (a, f, c), (a, e, c), (b, e, c) and (b, f, d). It can be argued that neither (a, f, c) nor (b, f, d) can be considered "rational solutions".

Consider first the Nash equilibrium (a, f, c). Player 2's plan to play f is rational only in the very limited sense that, given that Player 1 plays a, what Player 2 plans to do is irrelevant because it cannot affect anybody's payoff; thus f is as good as e. However, if we take Player 2's strategy as a "serious" plan specifying what Player 2 would actually do if she had to move, then – given that Player 3 plays c – e would give Player 2 a payoff of 2, while f would only give a payoff of 1. Thus e seems to be a better strategy than f, if Player 2 takes the contingency "seriously".

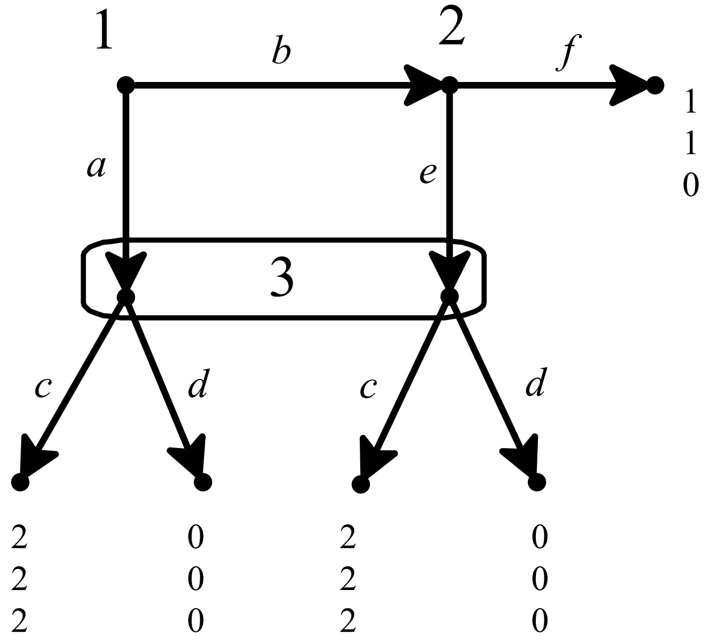


Figure 7.12: An extensive-form game showing the insufficiency of the notion of subgame-perfect equilibrium.

Consider now the Nash equilibrium (b, f, d) and focus on Player 3. As before, Player 3's plan to play d is rational only in the very limited sense that, given that Player 1 plays a and Player 2 plays f, what Player 3 plans to do is irrelevant, so that c is as good as d. However, if Player 3 did find himself having to play, it would not be rational for him to play d, since d is a strictly dominated choice: no matter whether he is making his choice at the left node or at the right node of his information set, c gives him a higher payoff than d. How can it be then that d can be part of a Nash equilibrium? The answer is that d is strictly dominated *conditional on Player 3's information set being reached* but not as a plan formulated before the play of the game starts. In other words, d is strictly dominated *as a choice but not as a strategy*.

The notion of subgame-perfect equilibrium is not strong enough to eliminate "unreasonable" Nash equilibria such as (a, f, c) and (b, f, d) in the game of Figure 7.12 In order to do that we will need a stronger notion. This issue is postponed to Part IV (Chapters 11-13).

7.4 Exercises

7.4.1 Exercises for section 7.1: Behavioral strategies in dynamic games

The answers to the following exercises are in Section 7.5 at the end of this chapter.

Exercise 7.1 What properties must an extensive-form frame satisfy in order for it to be the case that, for a given player, the set of mixed strategies coincides with the set of behavioral strategies? [Assume that there are at least two choices at every information set.] ■

Exercise 7.2 Suppose that, in a given extensive-form frame, Player 1 has four information sets: at one of them she has two choices and at each of the other three she has three choices.

(a) How many parameters are needed to specify a mixed strategy of Player 1?

(b) How many parameters are needed to specify a behavioral strategy of Player 1? ■

Exercise 7.3 From the behavioral strategy profile

$$\left[\left(\begin{array}{cc|cc} a & b & e & f \\ \frac{5}{12} & \frac{7}{12} & \frac{2}{7} & \frac{5}{7} \end{array}\right), \left(\begin{array}{cc} c & d \\ \frac{1}{3} & \frac{2}{3} \end{array}\right)\right]$$

calculate the payoff of Player 2 in two ways:
(1) using the game of Figure 7.5 and
(2) using the simplified game of Figure 7.6 ■

Exercise 7.4 Consider the extensive form of Figure 7.13, where $o_1, \ldots, o_5$ are basic outcomes. Player 1's ranking of O is

$$o_1 \succ_1 o_5 \succ_1 o_4 \succ_1 o_2 \sim_1 o_3;$$

furthermore, she is indifferent between o_5 and the lottery

$$\left(\begin{array}{ccc} o_1 & o_2 & o_3 \\ \frac{6}{8} & \frac{1}{8} & \frac{1}{8} \end{array}\right)$$

and is also indifferent between o_4 and the lottery

$$\left(\begin{array}{cc} o_2 & o_5 \\ \frac{2}{3} & \frac{1}{3} \end{array}\right).$$

Player 2's ranking of O is

$$o_1 \sim_2 o_2 \sim_2 o_4 \succ_2 o_3 \succ_2 o_5;$$

furthermore, he is indifferent between o_3 and the lottery

$$\begin{pmatrix} o_1 & o_2 & o_5 \\ \frac{1}{10} & \frac{1}{10} & \frac{8}{10} \end{pmatrix}.$$

Finally, Player 3's ranking of O is

$$o_2 \succ_3 o_4 \succ_3 o_3 \sim_3 o_5 \succ_3 o_1;$$

furthermore, she is indifferent between o_4 and the lottery

$$\begin{pmatrix} o_1 & o_2 & o_3 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

and is also indifferent between o_3 and the lottery

$$\begin{pmatrix} o_1 & o_2 \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix},$$

Write the corresponding extensive-form game. ■

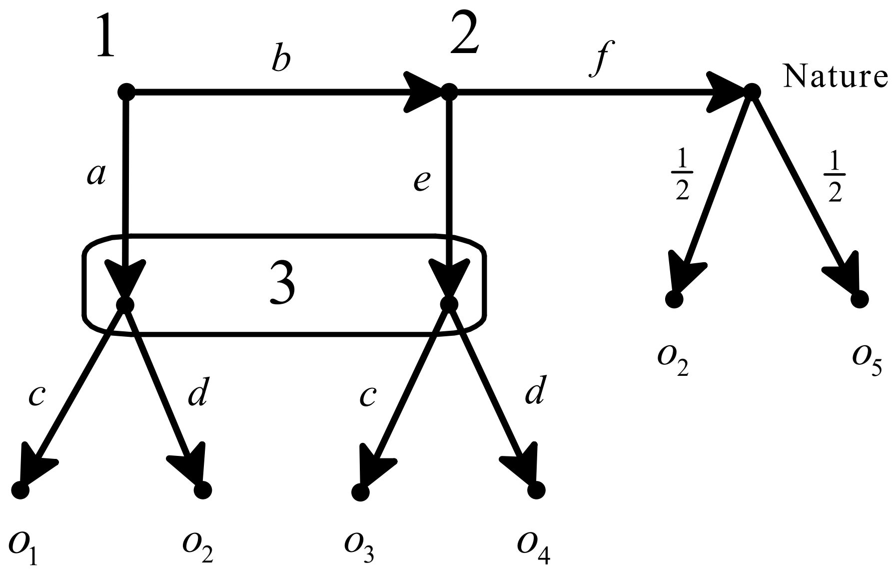


Figure 7.13: The game for Exercise 7.4

7.4.2 Exercises for section 7.2: Subgame-perfect equilibrium revisited

The answers to the following exercises are in Section 7.5 at the end of this chapter.

Exercise 7.5 Consider the extensive-form game with cardinal payoffs shown in Figure 7.14 .

(a) Write the corresponding strategic-form game and find all the pure-strategy Nash equilibria.

(b) Find the subgame-perfect equilibrium. ■

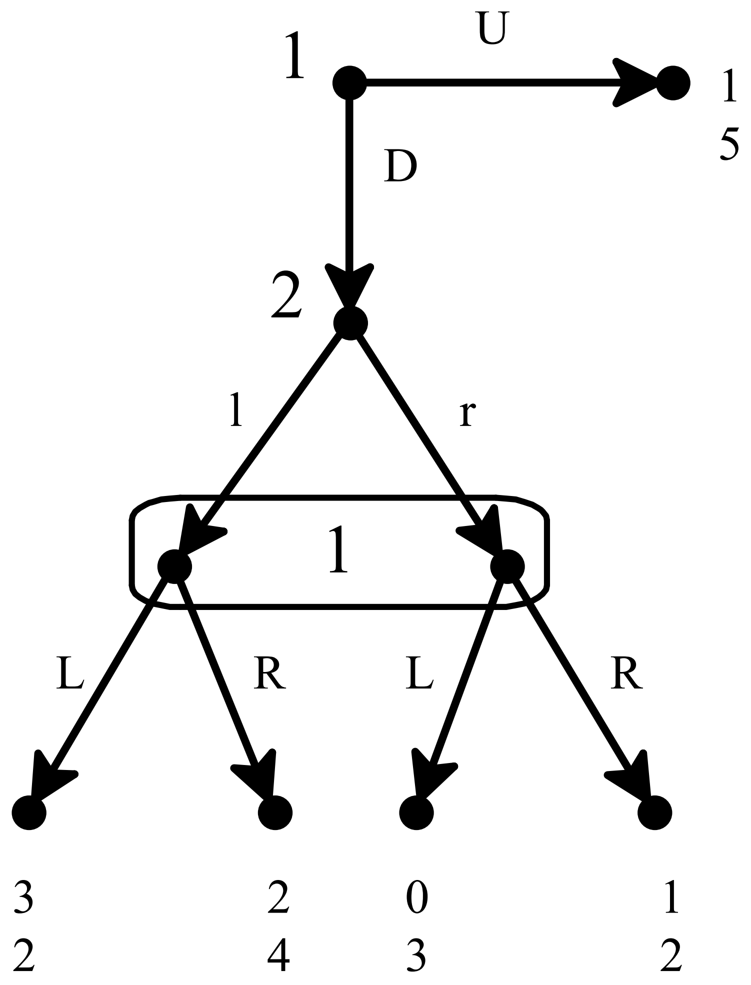


Figure 7.14: The game for Exercise 7.5

Exercise 7.6 Consider the extensive form shown in Figure 7.15 (where the basic outcomes are denoted by xj instead of o_j, $j = 1,\ldots 10$). All the players satisfy the axioms of expected utility. They rank the outcomes as indicated below (as usual, if outcome w is above outcome y then w is strictly preferred to y, and if w and y are written next to each other then the player is indifferent between the two):

$$Player\ 1: \begin{pmatrix} x_7, x_9 \\ x_1, x_2, x_4, x_5 \\ x_{10} \\ x_3, x_6, x_8 \end{pmatrix} \quad Player\ 2: \begin{pmatrix} x_1, x_3 \\ x_4, x_5 \\ x_2, x_7, x_8 \\ x_6 \\ x_9 \end{pmatrix} \quad Player\ 3: \begin{pmatrix} x_2, x_7 \\ x_8 \\ x_1, x_4, x_9 \\ x_3, x_5, x_6 \end{pmatrix}$$

Furthermore, Player 2 is indifferent between x_4 and the lottery $\begin{pmatrix} x_1 & x_2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ and Player 3 is indifferent between x_1 and the lottery $\begin{pmatrix} x_2 & x_5 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

Although the above information is not sufficient to determine the von Neumann-Morgenstern utility functions of the players, it *is* sufficient to compute the subgame-perfect equilibrium. [Hint: apply the IDSDS procedure to the subgame.]
Find the subgame-perfect equilibrium. ■

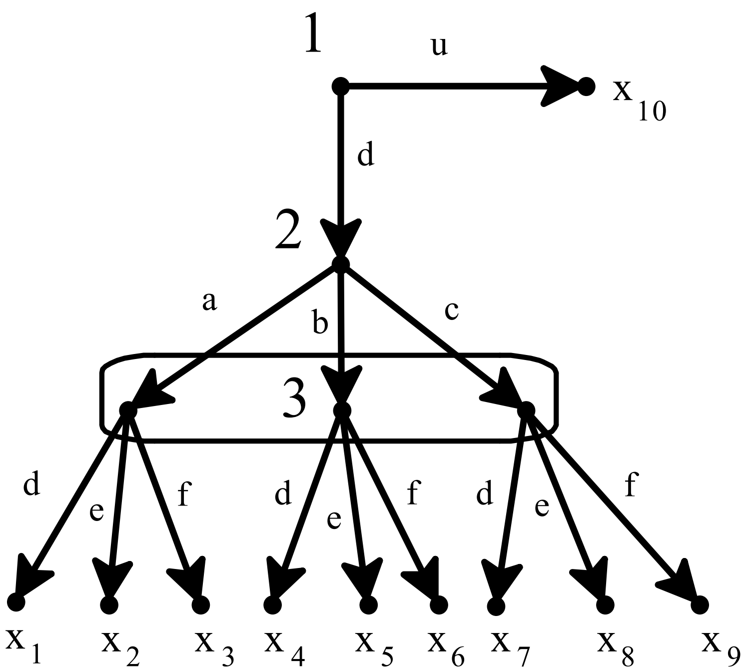


Figure 7.15: The game for Exercise 7.6

Exercise 7.7 Consider the extensive-form game shown in Figure 7.16.

(a) Write the corresponding strategic-form game.

(b) Find all the pure-strategy Nash equilibria.

(c) Find the mixed-strategy subgame-perfect equilibrium.

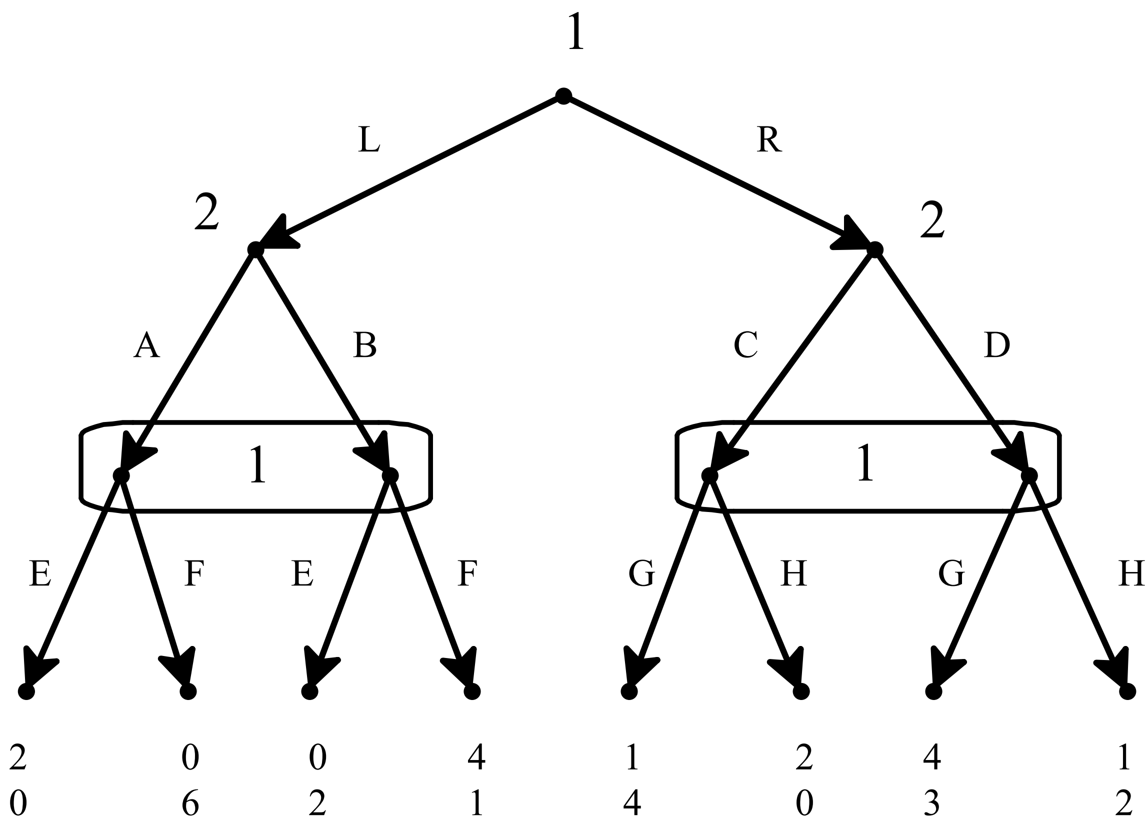


Figure 7.16: The game for Exercise 7.7

Exercise 7.8 Consider the extensive-form game shown in Figure 7.17.

(a) Find all the pure-strategy Nash equilibria. Which ones are also subgame perfect?

(b) (This is a more challenging question) Prove that there is no mixed-strategy Nash equilibrium where Player 1 plays M with probability strictly between 0 and 1.

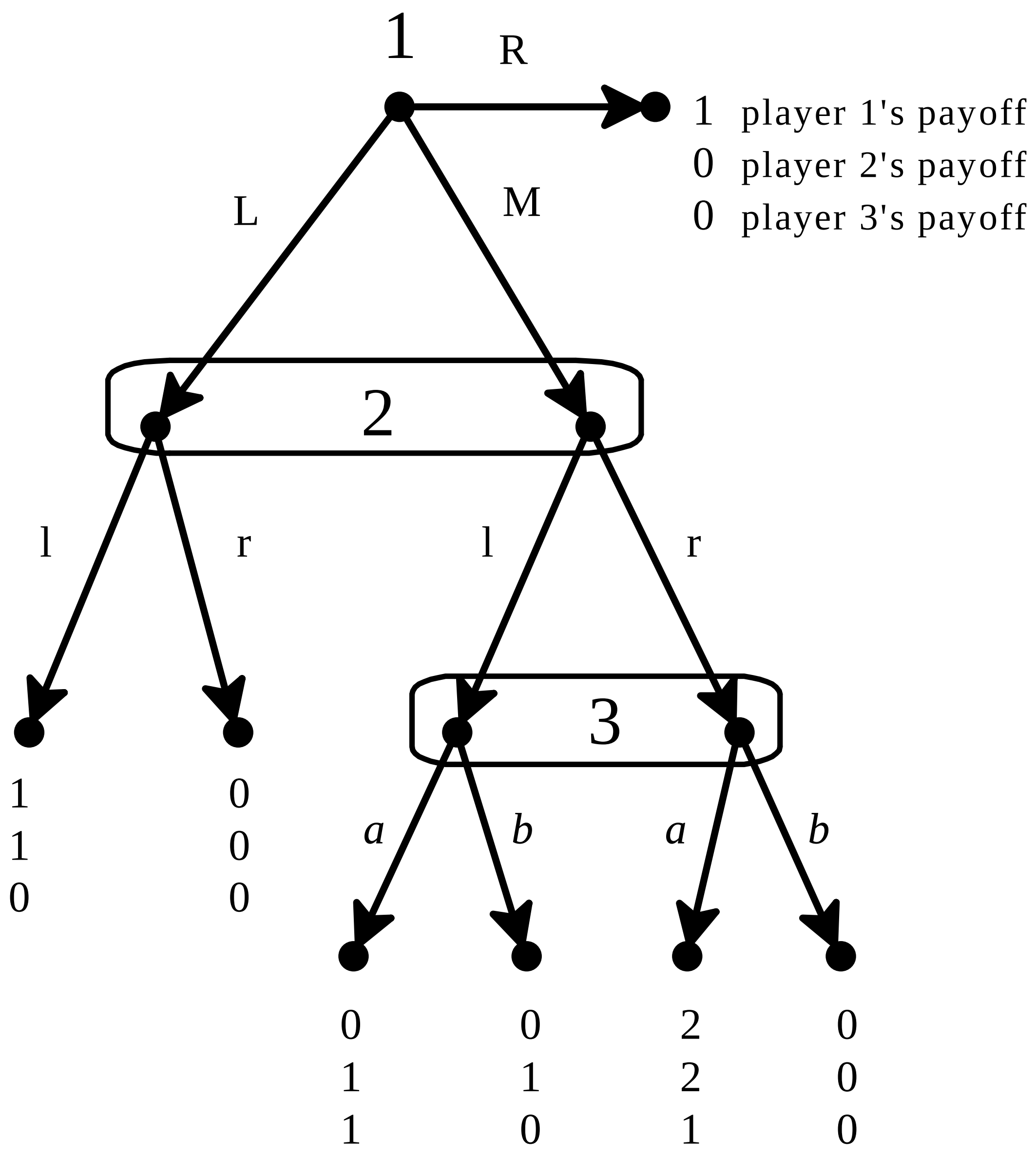


Figure 7.17: The game for Exercise 7.8

Exercise 7.9 — ⋆⋆⋆ Challenging Question ⋆⋆⋆.
You have to go to a part of town where many people have been mugged recently. You consider whether you should leave your wallet at home or carry it with you. Of the four possible outcomes, your most preferred one is having your wallet with you and not being mugged. Being mugged is a very unpleasant experience, so your second favorite alternative is not carrying your wallet and not being mugged (although not having any money with you can be very inconvenient). If, sadly enough, your destiny is to be mugged, then you prefer to have your wallet with you (possibly with not too much money in it!) because you don't want to have to deal with a frustrated mugger.
A typical potential mugger's favorite outcome is the one where you have your wallet with you and he mugs you. His least preferred outcome is the one where he attempts to mug you and you don't have your wallet with you (he risks being caught for nothing). He is indifferent to whether or not you are carrying your wallet if he decides not to mug you. Denote the possible outcomes as shown in Figure 7.18.

(a) What is the ordinal ranking of the outcomes for each player?

Suppose that both players have von Neumann-Morgenstern utility functions. You are indifferent between the following lotteries:

$$L_1 = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ \frac{3}{20} & \frac{14}{20} & \frac{3}{20} & 0 \end{pmatrix} \quad \text{and} \quad L_2 = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix};$$

furthermore, you are indifferent between

$$L_3 = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix} \quad \text{and} \quad L_4 = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

The potential mugger is indifferent between the two lotteries

$$L_5 = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \quad \text{and} \quad L_6 = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ \frac{8}{128} & \frac{67}{128} & \frac{16}{128} & \frac{37}{128} \end{pmatrix}.$$

(b) For each player find the normalized von Neumann-Morgenstern utility function.

You have to decide whether or not to leave your wallet at home. Suppose that, if you leave your wallet at home, with probability p (with $0 < p < 1$) the potential mugger will notice that your pockets are empty and with probability $(1-p)$ he will not notice; in the latter case he will be uncertain as to whether you have your wallet with you or you don't. He will be in the same state of uncertainty if you did take your wallet with you.

(c) Represent this situation as an extensive game with imperfect information.

(d) Write the corresponding normal form.

(e) Find all the subgame-perfect equilibria (including the mixed-strategy ones, if any). (Hint: your answer should distinguish between different values of p).

■

		Potential mugger	
		Not mug	Mug
You	Leave wallet at home	z_1	z_2
	Take wallet with you	z_3	z_4

Figure 7.18: The outcomes for Exercise 7.9

7.5 Solutions to exercises

Solution to Exercise 7.1 It must be the case that the player under consideration has only one information set. □

Solution to Exercise 7.2

(a) 53. The number of pure strategies is $2 \times 3 \times 3 \times 3 = 54$ and thus 53 probabilities are needed to specify a mixed strategy.

(b) 7: one probability for the information set where she has two choices and two probabilities for each of the other three information sets. □

Solution to Exercise 7.3

1. The induced probability distribution on basic outcomes is

$$\begin{pmatrix} o_1 & o_2 & o_3 & o_4 & o_5 \\ \frac{71}{540} & \frac{25}{540} & \frac{213}{540} & \frac{81}{540} & \frac{150}{540} \end{pmatrix}.$$

Thus Player 2's expected utility is

$$\tfrac{71}{540}3 + \tfrac{25}{540}6 + \tfrac{213}{540}4 + \tfrac{81}{540}5 + \tfrac{150}{540}0 = \tfrac{1620}{540} = 3.$$

2. The induced probability distribution on terminal nodes is

$$\begin{pmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \\ \frac{5}{36} & \frac{10}{36} & \frac{7}{36} & \frac{4}{36} & \frac{10}{36} \end{pmatrix}.$$

Thus Player 2's expected payoff is

$$\tfrac{5}{36}4 + \tfrac{10}{36}4 + \tfrac{7}{36}4 + \tfrac{4}{36}5 + \tfrac{10}{36}0 = \tfrac{108}{36} = 3.$$

Not surprisingly, the same number as in Part 1. □

Solution to Exercise 7.4 The normalized von Neumann-Morgenstern utility functions are

$$\left(\begin{array}{c|ccccc} & o_1 & o_2 & o_3 & o_4 & o_5 \\ \hline U_1 & 1 & 0 & 0 & 0.25 & 0.75 \\ U_2 & 1 & 1 & 0.2 & 1 & 0 \\ U_3 & 0 & 1 & 0.4 & 0.6 & 0.4 \end{array}\right)$$

Thus the extensive-form game is shown in Figure 7.19.

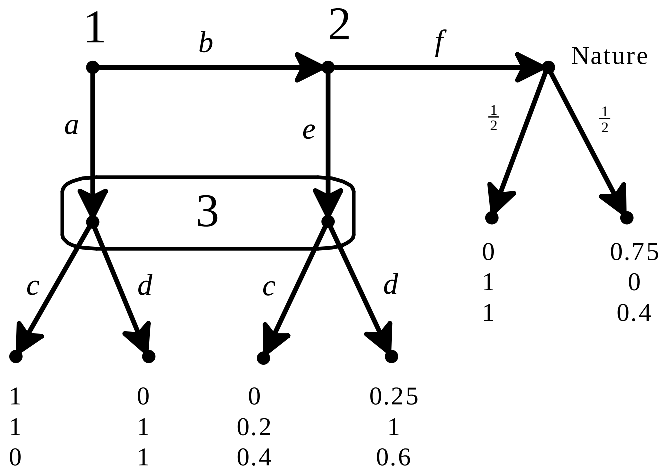


Figure 7.19: The game for Exercise 7.4

Or, in a simplified form obtained by removing the move of Nature, as shown in Figure 7.20. □

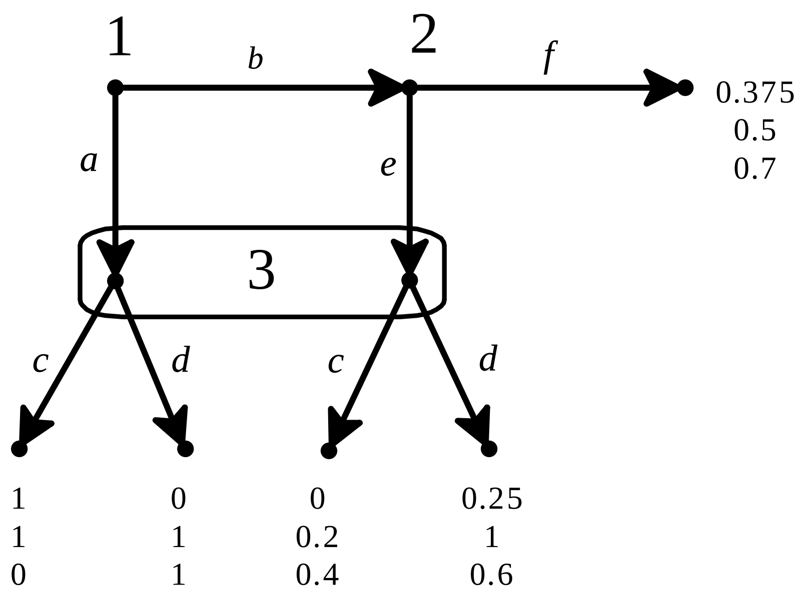


Figure 7.20: The simplified game for Exercise 7.4

Solution to Exercise 7.5

(a) The strategic form is shown in Figure 7.21. The pure-strategy Nash equilibria are (UL, r) and (UR, r).

Player 1 \ Player 2	l		r	
UL	1	5	1	5
UR	1	5	1	5
DL	3	2	0	3
DR	2	4	1	1

Figure 7.21: The strategic form for Part **(a)** of Exercise 7.5

(b) The strategic form of the proper subgame that starts at Player 2's node is as follows:

Player 1 \ Player 2	l	r
L	3 , 2	0 , 3
R	2 , 4	1 , 1

This game has a unique mixed-strategy Nash equilibrium given by

$$\left[\begin{pmatrix} L & R \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}, \begin{pmatrix} l & r \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right], \text{ yielding Player 1 an expected payoff of 1.5.}$$

Thus the unique subgame-perfect equilibrium, expressed as a behavioral-strategy profile, is

$$\left[\left(\begin{array}{cc|cc} U & D & L & R \\ 0 & 1 & \frac{3}{4} & \frac{1}{4} \end{array}\right), \begin{pmatrix} l & r \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right]$$

or, expressed as a mixed-strategy profile,

$$\left[\begin{pmatrix} UL & UR & DL & DR \\ 0 & 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix}, \begin{pmatrix} l & r \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right].$$

□

Solution to Exercise 7.6 There is only one proper subgame starting from Player 2's node; its strategic-form frame is as follows:

		Player 3		
		d	e	f
	a	x_1	x_2	x_3
Player 2	b	x_4	x_5	x_6
	c	x_7	x_8	x_9

For Player 2 strategy c is strictly dominated by strategy b (she prefers x_4 to x_7, and x_5 to x_8 and x_6 to x_9) and for Player 3 strategy f is strictly dominated by strategy d (she prefers x_1 to x_3, and x_4 to x_6 and x_7 to x_9). Thus we can simplify the game as follows:

		Player 3	
		d	e
Player 2	a	x_1	x_2
	b	x_4	x_5

Restricted to these outcomes the payers' rankings are:

$$Player\ 2: \begin{pmatrix} x_1 \\ x_4, x_5 \\ x_2 \end{pmatrix} \qquad Player\ 3: \begin{pmatrix} x_2 \\ x_1, x_4 \\ x_5 \end{pmatrix}.$$

Let U be Player 2's von Neumann-Morgenstern utility function. We can set $U(x_1) = 1$ and $U(x_2) = 0$. Thus, since she is indifferent between x_4 and x_5 and also between x_4 and the lottery $\begin{pmatrix} x_1 & x_2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, $U(x_4) = U(x_5) = \frac{1}{2}$.

Let V be Player 3's von Neumann-Morgenstern utility function. We can set $V(x_2) = 1$ and $V(x_5) = 0$. Thus, since she is indifferent between x_1 and x_4 and also between x_1 and the lottery $\begin{pmatrix} x_2 & x_5 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, $V(x1) = V(x4) = \frac{1}{2}$.

Hence the above game-frame becomes the following game:

		Player 3	
		d	e
Player 2	a	1 , $\frac{1}{2}$	0 , 1
	b	$\frac{1}{2}$, $\frac{1}{2}$	$\frac{1}{2}$, 0

There is no pure-strategy Nash equilibrium. Let p be the probability of a and q the probability of d. Then for a Nash equilibrium we need $q = \frac{1}{2}$ and $p = \frac{1}{2}$.

Hence in the subgame the outcome will be $\begin{pmatrix} x_1 & x_2 & x_4 & x_5 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$.

Since all of these outcomes are better than x_{10} for Player 1, Player 1 will play d. Thus the subgame-perfect equilibrium is

$$\left[\begin{pmatrix} d & u \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} a & b & c \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} d & e & f \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}\right].$$

□

Solution to Exercise 7.7

(a) The strategic form is shown in Figure 7.22.

PLAYER 1 \ PLAYER 2	*AC*	*AD*	*BC*	*BD*
LEG	2 , 0	2 , 0	0 , 2	0 , 2
LEH	2 , 0	2 , 0	0 , 2	0 , 2
LFG	0 , 6	0 , 6	4 , 1	4 , 1
LFH	0 , 6	0 , 6	4 , 1	4 , 1
REG	1 , 4	4 , 3	1 , 4	4 , 3
REH	2 , 0	1 , 2	2 , 0	1 , 2
RFG	1 , 4	4 , 3	1 , 4	4 , 3
RFH	2 , 0	1 , 2	2 , 0	1 , 2

Figure 7.22: The strategic form for Exercise 7.7

(b) There are no pure-strategy Nash equilibria.

(c) First let us solve the subgame on the left, whose strategic form is as follows:

Player 1 \ Player 2	*A*	*B*
E	2 , 0	0 , 2
F	0 , 6	4 , 1

There is no pure-strategy Nash equilibrium. Let us find the mixed-strategy equilibrium. Let p be the probability assigned to E and q the probability assigned to A.
- Then p must be the solution to $6(1-p) = 2p + (1-p)$ and q must be the solution to $2q = 4(1-q)$.
- Thus $p = \frac{5}{7}$ and $q = \frac{2}{3}$.
- The expected payoff of Player 1 is $\frac{4}{3} = 1.33$, while the expected payoff of player 2 is $\frac{12}{7} = 1.714$.

Next we solve the subgame on the right, whose strategic form is as follows:

		Player 2: C	Player 2: D
Player 1	G	1 , 4	4 , 3
	H	2 , 0	1 , 2

There is no pure-strategy Nash equilibrium. Let us find the mixed-strategy equilibrium.

- Let p be the probability assigned to G and q the probability assigned to C.
- Then p must be the solution to $4p = 3p + 2(1-p)$ and q must be the solution to $q + 4(1-q) = 2q + (1-q)$.
- Thus $p = \frac{2}{3}$ and $q = \frac{3}{4}$.
- The expected payoff of Player 1 is $\frac{7}{4} = 1.75$. Thus the game reduces to the the one shown in Figure 7.23, where the optimal choice is R. Hence the subgame-perfect equilibrium is:

$$\left[\left(\begin{array}{cc|cc|cc} L & R & E & F & G & H \\ 0 & 1 & \frac{5}{7} & \frac{2}{7} & \frac{2}{3} & \frac{1}{3} \end{array}\right), \left(\begin{array}{cc|cc} A & B & C & D \\ \frac{2}{3} & \frac{1}{3} & \frac{3}{4} & \frac{1}{4} \end{array}\right)\right] \qquad \square$$

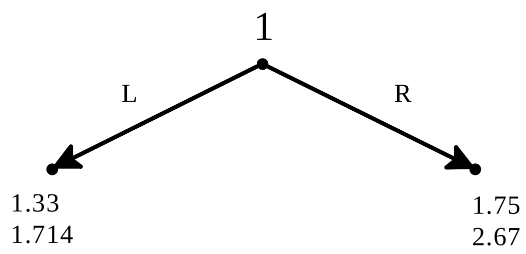


Figure 7.23: The reduced game after eliminating the proper subgames

Solution to Exercise 7.8

(a) The strategic form is shown in Figure 7.24.

Player 3 chooses a

		Player 2: l	Player 2: r
Player 1	R	1 , 0 , 0	1 , 0 , 0
	M	0 , 1 , 1	2 , 2 , 1
	L	1 , 1 , 0	0 , 0 , 0

Player 3 chooses b

		Player 2: l	Player 2: r
Player 1	R	1 , 0 , 0	1 , 0 , 0
	M	0 , 1 , 0	0 , 0 , 0
	L	1 , 1 , 0	0 , 0 , 0

Figure 7.24: The strategic form for Exercise 7.8

The pure-strategy Nash equilibria are highlighted: (R,l,a), (M,r,a), (L,l,a), (R,l,b), (R,r,b) and (L,l,b). They are all subgame perfect because there are no proper subgames.

(b) Since, for Player 3, a strictly dominates b, conditional on his information set being reached, he will have to play a if his information set is reached with positive probability. Now, Player 3's information set is reached with positive probability if and only if Player 1 plays M with positive probability. Thus when $P(M) > 0$ the game essentially reduces to the one shown in Figure 7.25.

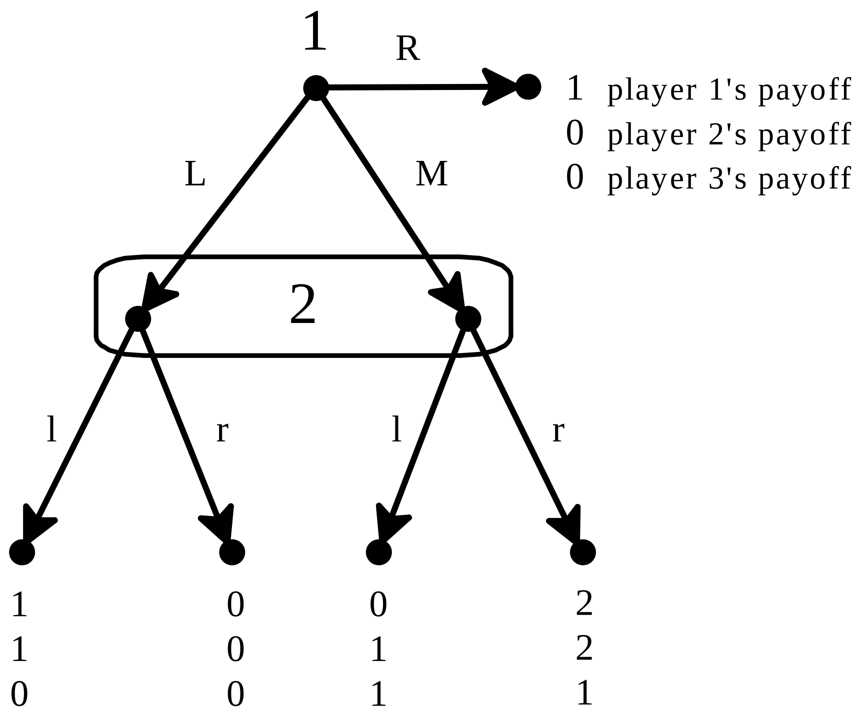


Figure 7.25: The extensive-form game for Part (b) of Exercise 7.8

Now, in order for Player 1 to be willing to assign positive probability to M he must expect a payoff of at least 1 (otherwise R would be better) and the only way he can expect a payoff of at least 1 is if Player 2 plays r with probability at least $\frac{1}{2}$.

- If Player 2 plays r with probability greater than $\frac{1}{2}$, then M gives Player 1 a higher payoff than both L and R and thus he will choose M with probability 1, in which case Player 2 will choose r with probability 1 (and Player 3 will choose a with probability 1) and so we get the pure strategy equilibrium (M, r, a).
- If Player 2 plays r with probability exactly $\frac{1}{2}$ then Player 1 is indifferent between M and R (and can mix between the two), but finds L inferior and must give it probability 0. But then Player 2's best reply to a mixed strategy of Player 1 that assigns positive probability to M and R and zero probability to L is to play r with probability 1 (if his information set is reached it can only be reached at node x_2).
- Thus there cannot be a mixed-strategy equilibrium where Player 1 assigns to M probability p with $0 < p < 1$: it must be either $Pr(M) = 0$ or $Pr(M) = 1$. $\square$

Solution to Exercise 7.9

(a) The rankings are as follows:

$$\text{You: } \begin{pmatrix} \text{best} & z_3 \\ & z_1 \\ & z_4 \\ \text{worst} & z_2 \end{pmatrix}, \qquad \text{Potential Mugger: } \begin{pmatrix} \text{best} & z_4 \\ & z_1, z_3 \\ \text{worst} & z_2 \end{pmatrix}$$

(b) Let U be your utility function. Let $U(z_3) = 1, U(z_1) = a, U(z_4) = b$ and $U(z_2) = 0$, with $0 < b < a < 1$. The expected utilities are as follows: $EU(L_1) = \frac{3}{20}a + \frac{3}{20}$, $EU(L_2) = \frac{1}{2}b$, $EU(L_3) = \frac{1}{3}$ and $EU(L_4) = \frac{1}{2}a$.
From $EU(L_3) = E(L_4)$ we get that $a = \frac{2}{3}$.
Substituting this into the equation $EU(L_1) = EU(L_2)$ gives $b = \frac{1}{2}$.
Thus $U(z_3) = 1$, $U(z_1) = \frac{2}{3}$, $U(z_4) = \frac{1}{2}$ and $U(z_2) = 0$.
Let V be the mugger's utility function. Let $V(z_4) = 1, V(z_1) = V(z_3) = c$ and $V(z_2) = 0$ with $0 < c < 1$. The expected utilities are as follows: $EV(L_5) = \frac{1}{4}(2c+1)$ and $EV(L_6) = \frac{1}{128}(24c+37)$.
Solving $EV(L_5) = EV(L_6)$ gives $c = \frac{1}{8}$.
Thus, $V(z_4) = 1, V(z_1) = V(z_3) = \frac{1}{8}$ and $V(z_2) = 0$.

(c) The extensive game is shown in Figure:7.26.

(d) The strategic form is shown in Figure 7.27 (for the mugger's strategy the first item refers to the left node, the second item to the information set on the right).

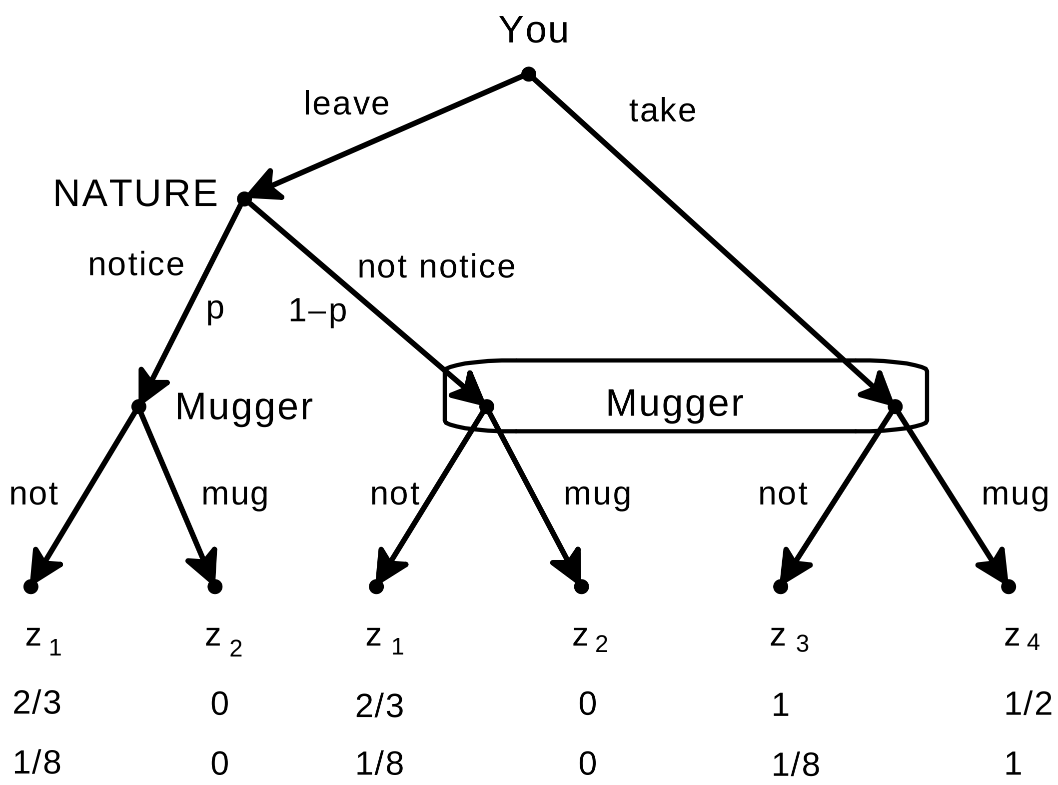


Figure 7.26: The extensive game for the Exercise 7.9

Potential Mugger

You		*NN*		*NM*		*MN*		*MM*	
	L	$\frac{2}{3}$	$\frac{1}{8}$	$\frac{2}{3}p$	$\frac{1}{8}p$	$\frac{2}{3}(1-p)$	$\frac{1}{8}(1-p)$	0	0
	T	1	$\frac{1}{8}$	$\frac{1}{2}$	1	1	$\frac{1}{8}$	$\frac{1}{2}$	1

Figure 7.27: The strategic form for the game of Figure 7.26

(e) At a subgame-perfect equilibrium the mugger will choose not to mug when he notices your empty pockets. Thus the normal form can be simplified as shown in Figure 7.28.

You \ Potential Mugger	NN	NM
L	$\frac{2}{3}$, $\frac{1}{8}$	$\frac{2}{3}p$, $\frac{1}{8}p$
T	1, $\frac{1}{8}$	$\frac{1}{2}$, 1

Figure 7.28: The reduced game for Exercise 7.9

Thus,

- If $p < \frac{3}{4}$ then Take is a strictly dominant strategy for you and therefore there is a unique subgame-perfect equilibrium given by (Take, (Not mug,Mug)).

- If $p = \frac{3}{4}$ then there is a continuum of equilibria where the Mugger chooses (Not mug, Mug) with probability 1 and you choose L with probability q and T with probability $(1-q)$ for any q with $0 \leq q \leq \frac{28}{29}$, obtained from the following condition about the Potential Mugger:

$$\underbrace{\tfrac{3}{32}q + 1 - q}_{\substack{\text{expected payoff} \\ \text{from playing } NM}} \quad \geq \quad \underbrace{\tfrac{1}{8}}_{\substack{\text{payoff from} \\ \text{playing } NN}}$$

- If $p > \frac{3}{4}$ then there is no pure-strategy subgame-perfect equilibrium. Let q be the probability that you choose L and r the probability that the mugger chooses NN. Then the unique mixed strategy equilibrium is given by the solution to:

$$\tfrac{2}{3}r + \tfrac{2}{3}p(1-r) = r + \tfrac{1}{2}(1-r) \quad \text{and} \quad \tfrac{1}{8} = \tfrac{1}{8}pq + (1-q)$$

which is $q = \frac{7}{8-p}$ and $r = \frac{4p-3}{4p-1}$. Thus the unique subgame-perfect equilibrium is:

$$\left(\begin{array}{cc|cccc} L & T & NN & NM & MN & MM \\ \frac{7}{8-p} & \frac{1-p}{8-p} & \frac{4p-3}{4p-1} & \frac{2}{4p-1} & 0 & 0 \end{array} \right).$$

$\square$