

# Answers to Odd-Numbered Exercises

## Chapter 1

### Section 1.1, page 10

1. The solution is  $(x_1, x_2) = (-8, 3)$ , or simply  $(-8, 3)$ .
3.  $(2, 1)$
5. Replace Row 2 by its sum with  $-4$  times Row 3, and then replace Row 1 by its sum with 3 times Row 3.
7. The solution set is empty.
9.  $(16, 21, 14, 4)$       11. Inconsistent
13.  $(5, 3, -1)$       15. Inconsistent
17. Calculations show that the system is inconsistent, so the three lines have no point in common.
19.  $h \neq 2$       21. All  $h$
23. Mark a statement True only if the statement is *always* true. Giving you the answers here would defeat the purpose of the true-false questions, which is to help you learn to read the text carefully. The *Study Guide* will tell you where to look for the answers, but you should not consult it until you have made an honest attempt to find the answers yourself.
25.  $k - 2g + h = 0$
27. The row reduction of 
$$\begin{bmatrix} a & b & f \\ c & d & g \end{bmatrix} \text{ to } \begin{bmatrix} a & b & f \\ 0 & d - b(\frac{c}{a}) & g - f(\frac{c}{a}) \end{bmatrix}$$
 shows that  $d - b(\frac{c}{a})$  must be nonzero, since  $f$  and  $g$  are arbitrary. Otherwise, for some choices of  $f$  and  $g$  the second row could correspond to an equation of the form  $0 = q$ , where  $q$  is nonzero. Thus  $ad \neq bc$ .
29. Swap Row 1 and Row 3; swap Row 1 and Row 3.
31. Replace Row 3 by Row 3 +  $(-4)$ Row 1; replace Row 3 by Row 3 +  $(4)$ Row 1.
33. Review Practice Problem 1 and then *write* a solution. The *Study Guide* has a solution.

### Section 1.2, page 21

1. Reduced echelon form: a and b. Echelon form: d. Not in echelon form: c.
3. 
$$\begin{bmatrix} 1 & 2 & 0 & -8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot cols 1 and 3: 
$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

5. 
$$\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}$$
7. 
$$\begin{cases} x_1 = -5 - 3x_2 \\ x_2 \text{ is free.} \\ x_3 = 3 \end{cases}$$
9. 
$$\begin{cases} x_1 = 3 + 2x_3 \\ x_2 = 3 + 2x_3 \\ x_3 \text{ is free.} \end{cases}$$
11. 
$$\begin{cases} x_1 = \frac{2}{3}x_2 - \frac{4}{3}x_3 \\ x_2 \text{ is free.} \\ x_3 \text{ is free.} \end{cases}$$
13. 
$$\begin{cases} x_1 = 5 + 3x_5 \\ x_2 = 1 + 4x_5 \\ x_3 \text{ is free.} \\ x_4 = 4 - 9x_5 \\ x_5 \text{ is free.} \end{cases}$$

*Note:* The *Study Guide* discusses the common mistake  $x_3 = 0$ .

15. a. Consistent, with many solutions  
b. Consistent, with many solutions
17. All  $h$
19. a. Inconsistent when  $h = 2$  and  $k \neq 8$   
b. Unique solution when  $h \neq 2$   
c. Many solutions when  $h = 2$  and  $k = 8$
21. Read the text carefully, and write your answers before you consult the *Study Guide*. Remember, a statement is true only if it is true in all cases.
23. Since there are four pivots (one in each column of the coefficient matrix), the augmented matrix must reduce to the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 & d \end{bmatrix}$$

and so

$$\begin{aligned} x_1 &= a \\ x_2 &= b \\ x_3 &= c \\ x_4 &= d \end{aligned}$$

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No matter what the values of  $a$ ,  $b$ ,  $c$  and  $d$ , the solution exists and is unique.

25. If the coefficient matrix has a pivot position in every row, then there is a pivot position in the bottom row, and there is no room for a pivot in the augmented column. So, the system is consistent, by Theorem 2.
27. If a linear system is consistent, then the solution is unique if and only if every column in the coefficient matrix is a pivot column; otherwise, there are infinitely many solutions.
29. An underdetermined system always has more variables than equations. There cannot be more basic variables than there are equations, so there must be at least one free variable. Such a variable may be assigned infinitely many different values. If the system is consistent, each different value of a free variable will produce a different solution.
31. Yes, a system of linear equations with more equations than unknowns can be consistent. The following system has a solution ( $x_1 = x_2 = 1$ ):

$$x_1 + x_2 = 2$$

$$x_1 - x_2 = 0$$

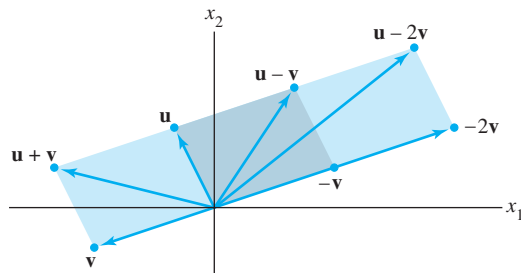
$$3x_1 + 2x_2 = 5$$

33.  $p(t) = 1 + 3t + 2t^2$

### Section 1.3, page 32

1.  $\begin{bmatrix} -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

3.



5.  $x_1 \begin{bmatrix} 3 \\ -2 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 0 \\ -9 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 8 \end{bmatrix},$

$$\begin{bmatrix} 3x_1 \\ -2x_1 \\ 8x_1 \end{bmatrix} + \begin{bmatrix} 5x_2 \\ 0 \\ -9x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 8 \end{bmatrix},$$

$$\begin{bmatrix} 3x_1 + 5x_2 \\ -2x_1 \\ 8x_1 - 9x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 8 \end{bmatrix}$$

$$3x_1 + 5x_2 = 2$$

$$-2x_1 = -3$$

$$8x_1 - 9x_2 = 8$$

Usually the intermediate steps are not displayed.

7.  $\mathbf{a} = \mathbf{u} - 2\mathbf{v}$ ,  $\mathbf{b} = 2\mathbf{u} - 2\mathbf{v}$ ,  $\mathbf{c} = 2\mathbf{u} - 3.5\mathbf{v}$ ,  $\mathbf{d} = 3\mathbf{u} - 4\mathbf{v}$   
Yes, every vector in  $\mathbb{R}^2$  is a linear combination of  $\mathbf{u}$  and  $\mathbf{v}$ .

9.  $x_1 \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

11. No,  $\mathbf{b}$  is not a linear combination of  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ , and  $\mathbf{a}_3$ .

13. No,  $\mathbf{b}$  is not a linear combination of the columns of  $A$ .

15.  $h = 3$

17. Noninteger weights are acceptable, of course, but some simple choices are  $0 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2 = \mathbf{0}$ , and

$$1 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \quad 0 \cdot \mathbf{v}_1 + 1 \cdot \mathbf{v}_2 = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

$$1 \cdot \mathbf{v}_1 + 1 \cdot \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \quad 1 \cdot \mathbf{v}_1 - 1 \cdot \mathbf{v}_2 = \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix}$$

19.  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$  is the set of points on the line through  $\mathbf{v}_1$  and  $\mathbf{0}$ , because  $\mathbf{v}_2$  is a multiple of  $\mathbf{v}_1$ .

21. *Hint:* Show that  $\begin{bmatrix} 2 & 2 & h \\ -1 & 1 & k \end{bmatrix}$  is consistent for all  $h$  and  $k$ . Explain what this calculation shows about  $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ .

23. Before you consult your *Study Guide*, read the entire section carefully. Pay special attention to definitions and theorem statements, and note any remarks that precede or follow them.

25. a. No, three      b. Yes, infinitely many

c.  $\mathbf{a}_1 = 1 \cdot \mathbf{a}_1 + 0 \cdot \mathbf{a}_2 + 0 \cdot \mathbf{a}_3$

27. a.  $5\mathbf{v}_1$  is the output of 5 days of operation of mine #1.

b. The total output is  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2$ , so  $x_1$  and  $x_2$  should satisfy  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \begin{bmatrix} 240 \\ 2824 \end{bmatrix}$ .

c. [M] 1.73 days for mine #1 and 4.70 days for mine #2

29.  $(17/14, -34/14, 16/14) = (17/14, -17/7, 8/7)$

31. a.  $\begin{bmatrix} 10/3 \\ 2 \end{bmatrix}$

b. Add 3.5 g at (0, 1), add 0.5 g at (8, 1), and add 2 g at (2, 4).

33. Review Practice Problem 1 and then write a solution. The *Study Guide* has a solution.

### Section 1.4, page 40

1. The product is not defined because the number of columns (2) in the  $3 \times 2$  matrix does not match the number of entries (3) in the vector.

3. a.  $\mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = -2 \cdot \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} + 3 \cdot \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}$   
 $= \begin{bmatrix} -2 \\ 6 \\ -2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 16 \end{bmatrix}$

$$\begin{aligned} \text{b. } A\mathbf{x} &= \begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \cdot (-2) + 2 \cdot (3) \\ (-3) \cdot (-2) + 1 \cdot (3) \\ 1 \cdot (-2) + 6 \cdot (3) \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 9 \\ 16 \end{bmatrix}. \end{aligned}$$

Show your work here and for Exercises 4–6, but thereafter perform the calculations mentally.

$$5. \quad 2 \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} - 1 \cdot \begin{bmatrix} 2 \\ -3 \end{bmatrix} + 1 \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

$$7. \quad \begin{bmatrix} 4 & -5 & 7 \\ -1 & 3 & -8 \\ 7 & -5 & 0 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$$

$$9. \quad x_1 \begin{bmatrix} 5 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 5 & 1 & -3 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$

$$11. \quad \begin{bmatrix} 1 & 3 & -4 & -2 \\ 1 & 5 & 2 & 4 \\ -3 & -7 & 6 & 12 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11 \\ 3 \\ 0 \end{bmatrix}$$

13. Yes. (Justify your answer.)

15. The equation  $A\mathbf{x} = \mathbf{b}$  is not consistent when  $3b_1 + b_2$  is nonzero. (Show your work.) The set of  $\mathbf{b}$  for which the equation is consistent is a line through the origin—the set of all points  $(b_1, b_2)$  satisfying  $b_2 = -3b_1$ .

17. Only three rows contain a pivot position. The equation  $A\mathbf{x} = \mathbf{b}$  does *not* have a solution for each  $\mathbf{b}$  in  $\mathbb{R}^4$ , by Theorem 4.

19. The work in Exercise 17 shows that statement (d) in Theorem 4 is false. So all four statements in Theorem 4 are false. Thus, not all vectors in  $\mathbb{R}^4$  can be written as a linear combination of the columns of  $A$ . Also, the columns of  $A$  do *not* span  $\mathbb{R}^4$ .

21. The matrix  $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$  does not have a pivot in each row, so the columns of the matrix do not span  $\mathbb{R}^4$ , by Theorem 4. That is,  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  does not span  $\mathbb{R}^4$ .

23. Read the text carefully and try to mark each exercise statement True or False before you consult the *Study Guide*. Several parts of Exercises 29 and 30 are *implications* of the form

“If (statement 1), then (statement 2)”

or equivalently,

“(statement 2), if (statement 1)”

Mark such an implication as True if (statement 2) is true in all cases when (statement 1) is true.

25.  $c_1 = -3, c_2 = -1, c_3 = 2$

27. The matrix equation can be written as  $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 = \mathbf{v}_6$ , where  $c_1 = -3$ ,

$c_2 = 1, c_3 = 2, c_4 = -1, c_5 = 2$ , and

$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 8 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \\ \mathbf{v}_4 = \begin{bmatrix} 9 \\ -2 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 7 \\ -4 \end{bmatrix}, \quad \mathbf{v}_6 = \begin{bmatrix} 11 \\ -11 \end{bmatrix}$$

29. *Hint:* Start with any  $3 \times 3$  matrix  $B$  in echelon form that has three pivot positions.

31. Write your solution before you check the *Study Guide*.

33. *Hint:* How many pivot columns does  $A$  have? Why?

35. Suppose  $\mathbf{y}$  and  $\mathbf{z}$  satisfy  $A\mathbf{y} = \mathbf{z}$ . Then  $5\mathbf{z} = 5A\mathbf{y}$ . By Theorem 5(b),  $5A\mathbf{y} = A(5\mathbf{y})$ . So  $5\mathbf{z} = A(5\mathbf{y})$ , which shows that  $5\mathbf{y}$  is a solution of  $A\mathbf{x} = 5\mathbf{z}$ . Thus the equation  $A\mathbf{x} = 5\mathbf{z}$  is consistent.

37. [M] The columns do not span  $\mathbb{R}^4$ .

39. [M] The columns span  $\mathbb{R}^4$ .

41. [M] Delete column 4 of the matrix in Exercise 39. It is also possible to delete column 3 instead of column 4.

### Section 1.5, page 47

1. The system has a nontrivial solution because there is a free variable,  $x_3$ .

3. The system has a nontrivial solution because there is a free variable,  $x_3$ .

$$5. \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$7. \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

$$9. \quad \mathbf{x} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

11. *Hint:* The system derived from the *reduced* echelon form is

$$\begin{aligned} x_1 - 4x_2 &+ 5x_6 = 0 \\ x_3 &- x_6 = 0 \\ x_5 - 4x_6 &= 0 \\ 0 &= 0 \end{aligned}$$

The basic variables are  $x_1, x_3$ , and  $x_5$ . The remaining variables are free. The *Study Guide* discusses two mistakes that are often made on this type of problem.

$$13. \quad \mathbf{x} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix} = \mathbf{p} + x_3\mathbf{q}. \text{ Geometrically, the}$$

solution set is the line through  $\begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$  parallel to  $\begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix}$ .

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15. Let  $\mathbf{u} = \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}$ ,  $\mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ ,  $\mathbf{p} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$ . The solution of the homogeneous equation is  $\mathbf{x} = x_2\mathbf{u} + x_3\mathbf{v}$ , the plane through the origin spanned by  $\mathbf{u}$  and  $\mathbf{v}$ . The solution set of the nonhomogeneous system is  $\mathbf{x} = \mathbf{p} + x_2\mathbf{u} + x_3\mathbf{v}$ , the plane through  $\mathbf{p}$  parallel to the solution set of the homogeneous equation.

17.  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ . The solution set is the line through  $\begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix}$ , parallel to the line that is the solution set of the homogeneous system in Exercise 5.

19.  $\mathbf{x} = \mathbf{a} + t\mathbf{b}$ , where  $t$  represents a parameter, or  
 $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 3 \end{bmatrix}$ , or  $\begin{cases} x_1 = -2 - 5t \\ x_2 = 3t \end{cases}$

21.  $\mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p}) = \begin{bmatrix} 3 \\ -3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

23. It is important to read the text carefully and write your answers. After that, check the *Study Guide*, if necessary.

25. a.  $A\mathbf{w} = A(\mathbf{p} + \mathbf{v}_h) = A\mathbf{p} + A\mathbf{v}_h = \mathbf{b} + \mathbf{0} = \mathbf{b}$   
 b.  $A\mathbf{v}_h = A(\mathbf{w} - \mathbf{p}) = A\mathbf{w} - A\mathbf{p} = \mathbf{b} - \mathbf{b} = \mathbf{0}$

27. (*Geometric argument using Theorem 6*) Since the equation  $A\mathbf{x} = \mathbf{b}$  is consistent, its solution set is obtained by translating the solution set of  $A\mathbf{x} = \mathbf{0}$ , by Theorem 6. So the solution set of  $A\mathbf{x} = \mathbf{b}$  is a single vector if and only if the solution set of  $A\mathbf{x} = \mathbf{0}$  is a single vector, and that happens if and only if  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution.

(*Proof using free variables*) If  $A\mathbf{x} = \mathbf{b}$  has a solution, then the solution is unique if and only if there are no free variables in the corresponding system of equations, that is, if and only if every column of  $A$  is a pivot column. This happens if and only if the equation  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution.

29. a. When  $A$  is a  $4 \times 4$  matrix with three pivot positions, the equation  $A\mathbf{x} = \mathbf{0}$  has a free variable and hence has nontrivial solutions.

- b. With three pivot positions,  $A$  does not have a pivot position in each of its four rows. By Theorem 4 in Section 1.4, the equation  $A\mathbf{x} = \mathbf{b}$  does not have a solution for every possible  $\mathbf{b}$ . The word “possible” in the exercise means that the only vectors considered in this case are those in  $\mathbb{R}^4$ , because  $A$  has four rows.

31. a. When  $A$  is a  $3 \times 2$  matrix with two pivot positions, each column is a pivot column. So the equation  $A\mathbf{x} = \mathbf{0}$  has no free variables and hence no nontrivial solution.

- b. With two pivot positions and three rows,  $A$  cannot have a pivot in every row. So the equation  $A\mathbf{x} = \mathbf{b}$  cannot have a solution for every possible  $\mathbf{b}$  (in  $\mathbb{R}^3$ ), by Theorem 4 in Section 1.4.

33. Your example should have the property that the sum of the entries in each row is zero. Why?

35. One answer:  $\mathbf{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

37. One answer is  $A = \begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix}$ . The *Study Guide* shows how to analyze the problem in order to construct  $A$ . If  $\mathbf{b}$  is any vector *not* a multiple of the first column of  $A$ , then the solution set of  $A\mathbf{x} = \mathbf{b}$  is empty and thus cannot be formed by translating the solution set of  $A\mathbf{x} = \mathbf{0}$ . This does not contradict Theorem 6, because that theorem applies when the equation  $A\mathbf{x} = \mathbf{b}$  has a nonempty solution set.

39. Suppose  $A\mathbf{v} = \mathbf{0}$  and  $A\mathbf{w} = \mathbf{0}$ . Then, since  $A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w}$  by Theorem 5(a) in Section 1.4,  $A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w} = \mathbf{0} + \mathbf{0} = \mathbf{0}$ . Now, let  $c$  and  $d$  be scalars. Using both parts of Theorem 5,  $A(c\mathbf{v} + d\mathbf{w}) = A(c\mathbf{v}) + A(d\mathbf{w}) = cA\mathbf{v} + dA\mathbf{w} = c\mathbf{0} + d\mathbf{0} = \mathbf{0}$ .

## Section 1.6, page 54

1. The general solution is  $p_G = .875p_S$ , with  $p_S$  free. One equilibrium solution is  $p_S = 1000$  and  $p_G = 875$ . Using fractions, the general solution could be written  $p_G = (7/8)p_S$ , and a natural choice of prices might be  $p_S = 80$  and  $p_G = 70$ . Only the *ratio* of the prices is important. The economic equilibrium is unaffected by a proportional change in prices.

3. a. 

Distribution of Output from:			
F&P	Man.	Ser.	

Output	↓	↓	↓	Input	Purchased by:
.1		.1	.2	→	F&P
.8		.1	.4	→	Man.
.1		.8	.4	→	Ser.

b. 
$$\begin{bmatrix} .9 & -.1 & -.2 & 0 \\ -.8 & .9 & -.4 & 0 \\ -.1 & -.8 & .6 & 0 \end{bmatrix}$$

- c.  $[\mathbf{M}] \ p_{F\&P} \approx 30, p_M \approx 71, p_S = 100$ .

5. a.

Distribution of Output from:

Output	↓	↓	↓	↓	Input	Purchased by:
.20		.35	.10	.20	→	Ag.
.20		.10	.20	.30	→	Man.
.30		.35	.50	.20	→	Ser.
.30		.20	.20	.30	→	Transp.

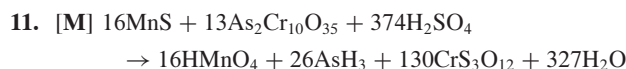
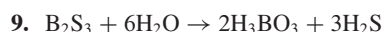
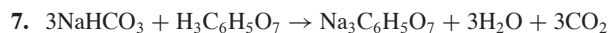
- b. One solution is  $p_A = 7.99, p_M = 8.36, p_S = 14.65$ , and  $p_T = 10.00$ .

- c. Distribution of Output from:  
Ag. Man. Ser. Transp.

Output	↓	↓	↓	↓	Input	Purchased by:
	.20	.35	.10	.20	→	Ag.
	.10	.10	.20	.30	→	Man.
	.40	.35	.50	.20	→	Ser.
	.30	.20	.20	.30	→	Transp.

- d. One solution is  $p_A = 7.81$ ,  $p_M = 7.67$ ,  $p_S = 15.62$ , and  $p_T = 10.00$ .

The campaign has benefited Services the most.



13. a. 
$$\begin{cases} x_1 = x_3 - 40 \\ x_2 = x_3 + 10 \\ x_3 \text{ is free} \\ x_4 = x_6 + 50 \\ x_5 = x_6 + 60 \\ x_6 \text{ is free} \end{cases}$$

b. 
$$\begin{cases} x_2 = 50 \\ x_3 = 40 \\ x_4 = 50 \\ x_5 = 60 \end{cases}$$

15. 
$$\begin{cases} x_1 = 60 + x_6 \\ x_2 = -10 + x_6 \\ x_3 = 90 + x_6 \\ x_4 = x_6 \\ x_5 = 80 + x_6 \\ x_6 \text{ is free} \end{cases}$$

In order for the flow to be nonnegative,  $x_6 \geq 10$

### Section 1.7, page 60

Justify your answers to Exercises 1–22.

1. Lin. indep.      3. Lin. depen.  
 5. Lin. indep.      7. Lin. depen.  
 9. a. No  $h$       b. All  $h$   
 11.  $h = -4$       13. All  $h$   
 15. Lin. depen.    17. Lin. depen.    19. Lin. indep.  
 21. If you consult your *Study Guide* before you make a good effort to answer the true-false questions, you will destroy most of their value.  
 23.  $\begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$   
 25.  $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$   
 27. All four columns of the  $6 \times 4$  matrix  $A$  must be pivot columns. Otherwise, the equation  $A\mathbf{x} = \mathbf{0}$  would have a free variable, in which case the columns of  $A$  would be linearly dependent.

29.  $A$ : Any  $3 \times 2$  matrix with the second column a multiple of the first will have the desired property.

$B$ : Any  $3 \times 2$  matrix with two nonzero columns such that neither column is a multiple of the other will work. In this case, the columns form a linearly independent set, and so the equation  $B\mathbf{x} = \mathbf{0}$  has only the trivial solution.

31.  $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

33. True, by Theorem 7. (The *Study Guide* adds another justification.)

35. True, by Theorem 9.

37. True. A linear dependence relation among  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ , and  $\mathbf{v}_3$  may be extended to a linear dependence relation among  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ ,  $\mathbf{v}_3$ , and  $\mathbf{v}_4$  by placing a zero weight on  $\mathbf{v}_4$ .

39. You should be able to work this important problem without help. Write your solution before you consult the *Study Guide*.

41. [M] Using the pivot columns of  $A$ ,

$$B = \begin{bmatrix} 3 & -4 & 7 \\ -5 & -3 & -11 \\ 4 & 3 & 2 \\ 8 & -7 & 4 \end{bmatrix}$$

Other choices are possible.

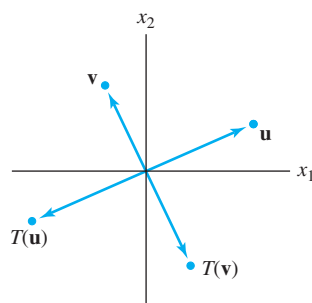
43. [M] Each column of  $A$  that is not a column of  $B$  is in the set spanned by the columns of  $B$ .

### Section 1.8, page 68

1.  $\begin{bmatrix} 2 \\ -6 \end{bmatrix}, \begin{bmatrix} 2a \\ 2b \end{bmatrix}$       3.  $\mathbf{x} = \begin{bmatrix} 7 \\ 6 \\ 3 \end{bmatrix}$ , unique solution  
 5.  $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ , not unique      7.  $a = 5, b = 6$   
 9.  $\mathbf{x} = x_3 \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}$

11. Yes, because the system represented by  $[A \ \mathbf{b}]$  is consistent.

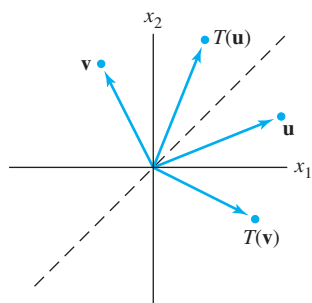
- 13.



A reflection through the origin

## A22 Answers to Odd-Numbered Exercises

15.



A reflection through the line  $x_1 = x_2$ .

17.  $\begin{bmatrix} 8 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \end{bmatrix}, \begin{bmatrix} 5 \\ 11 \end{bmatrix}$       19.  $\begin{bmatrix} 13 \\ 7 \end{bmatrix}, \begin{bmatrix} 2x_1 - x_2 \\ 5x_1 + 6x_2 \end{bmatrix}$

21. Read the text carefully and write your answers before you check the *Study Guide*. Notice that Exercise 21(e) is a sentence of the form

“(statement 1) if and only if (statement 2)”

Mark such a sentence as True if (statement 1) is true whenever (statement 2) is true *and* also (statement 2) is true whenever (statement 1) is true.

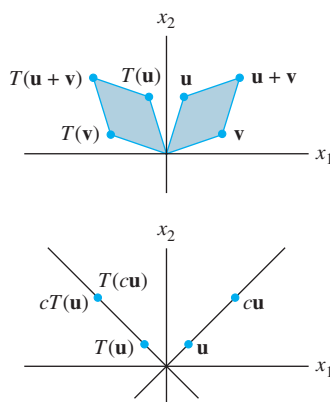
23. a. When  $b = 0$ ,  $f(x) = mx$ . In this case, for all  $x, y$  in  $\mathbb{R}$  and all scalars  $c$  and  $d$ ,
- $$\begin{aligned} f(cx + dy) &= m(cx + dy) = mcx + mdy \\ &= c(mx) + d(my) \\ &= c \cdot f(x) + d \cdot f(y) \end{aligned}$$

This shows that  $f$  is linear.

- b. When  $f(x) = mx + b$ , with  $b$  nonzero,  $f(0) = m(0) + b = b \neq 0$ .
- c. In calculus,  $f$  is called a “linear function” because the graph of  $f$  is a line.

25. *Hint*: Show that the image of a line (that is, the set of images of all points on a line) can be represented by the parametric equation of a line.
27. Any point  $\mathbf{x}$  on the plane  $P$  satisfies the parametric equation  $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$  for some values of  $s$  and  $t$ . By linearity, the image  $T(\mathbf{x})$  satisfies the parametric equation
- $$T(\mathbf{x}) = sT(\mathbf{u}) + tT(\mathbf{v}) \quad (s, t \text{ in } \mathbb{R}) \quad (*)$$
- The set of images is just  $\text{Span}\{T(\mathbf{u}), T(\mathbf{v})\}$ . If  $T(\mathbf{u})$  and  $T(\mathbf{v})$  are linearly independent,  $\text{Span}\{T(\mathbf{u}), T(\mathbf{v})\}$  is a plane through  $T(\mathbf{u})$ ,  $T(\mathbf{v})$ , and  $\mathbf{0}$ . If  $T(\mathbf{u})$  and  $T(\mathbf{v})$  are linearly dependent and not both zero, then  $\text{Span}\{T(\mathbf{u}), T(\mathbf{v})\}$  is a line through  $\mathbf{0}$ . If  $T(\mathbf{u}) = T(\mathbf{v}) = \mathbf{0}$ , then  $\text{Span}\{T(\mathbf{u}), T(\mathbf{v})\}$  is  $\{\mathbf{0}\}$ .

29.



31. *Hint*: Since  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly dependent, you can write a certain equation and work with it.
33. One possibility is to show that  $T$  does not map the zero vector into the zero vector, something that every linear transformation *does* do:  $T(0, 0) = (0, -3, 0)$ .
35. Take  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbb{R}^3$  and let  $c$  and  $d$  be scalars. Then
- $$c\mathbf{u} + d\mathbf{v} = (cu_1 + dv_1, cu_2 + dv_2, cu_3 + dv_3)$$
- The transformation  $T$  is linear because
- $$\begin{aligned} T(c\mathbf{u} + d\mathbf{v}) &= (cu_1 + dv_1, 0, cu_3 + dv_3) \\ &= (cu_1, 0, cu_3) + (dv_1, 0, dv_3) \\ &= c(u_1, 0, u_3) + d(v_1, 0, v_3) \\ &= cT(\mathbf{u}) + dT(\mathbf{v}) \end{aligned}$$
37. [M] All multiples of  $(-1, -1, 1, 0)$
39. [M] Yes. One choice for  $\mathbf{x}$  is  $(1, 2, 0, 0)$ .

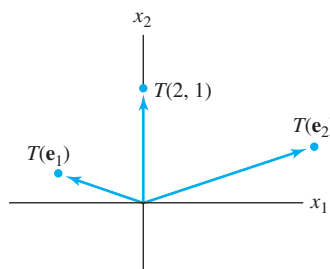
### Section 1.9, page 78

1.  $\begin{bmatrix} 3 & -5 \\ 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}$       3.  $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$       5.  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

7.  $\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$       9.  $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

11. The described transformation  $T$  maps  $\mathbf{e}_1$  into  $-\mathbf{e}_1$  and maps  $\mathbf{e}_2$  into  $-\mathbf{e}_2$ . A rotation through  $\pi$  radians also maps  $\mathbf{e}_1$  into  $-\mathbf{e}_1$  and maps  $\mathbf{e}_2$  into  $-\mathbf{e}_2$ . Since a linear transformation is completely determined by what it does to the columns of the identity matrix, the rotation transformation has the same effect as  $T$  on every vector in  $\mathbb{R}^2$ .

13.



$$15. \begin{bmatrix} 2 & -4 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix} \quad 17. \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$19. \begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix} \quad 21. \mathbf{x} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$

23. Answer the questions before checking the *Study Guide*.

Justify your answers to Exercises 25–28.

25. Not one-to-one and does not map  $\mathbb{R}^4$  onto  $\mathbb{R}^4$

27. Not one-to-one but maps  $\mathbb{R}^3$  onto  $\mathbb{R}^2$

$$29. \begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$$

31.  $n$ . (Explain why, and then check the *Study Guide*.)

33. *Hint*: If  $\mathbf{e}_j$  is the  $j$ th column of  $I_n$ , then  $B\mathbf{e}_j$  is the  $j$ th column of  $B$ .

35. *Hint*: Is it possible that  $m > n$ ? What about  $m < n$ ?

37. [M] No. (Explain why.)

39. [M] No. (Explain why.)

### Section 1.10, page 86

$$1. \text{ a. } x_1 \begin{bmatrix} 110 \\ 4 \\ 20 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 130 \\ 3 \\ 18 \\ 5 \end{bmatrix} = \begin{bmatrix} 295 \\ 9 \\ 48 \\ 8 \end{bmatrix}, \text{ where } x_1 \text{ is the}$$

number of servings of Cheerios and  $x_2$  is the number of servings of 100% Natural Cereal.

$$\text{b. } \begin{bmatrix} 110 & 130 \\ 4 & 3 \\ 20 & 18 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 295 \\ 9 \\ 48 \\ 8 \end{bmatrix}. \text{ Mix 1.5 servings of}$$

Cheerios together with 1 serving of 100% Natural Cereal.

3. a. She should mix .99 serving of Mac and Cheese, 1.54 servings of broccoli, and .79 serving of chicken to get her desired nutritional content.

b. She should mix 1.09 servings of shells and white cheddar, .88 serving of broccoli, and 1.03 servings of chicken to get her desired nutritional content. Notice that this mix contains significantly less broccoli, so she should like it better.

$$5. \mathbf{Ri} = \mathbf{v}, \begin{bmatrix} 11 & -5 & 0 & 0 \\ -5 & 10 & -1 & 0 \\ 0 & -1 & 9 & -2 \\ 0 & 0 & -2 & 10 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 50 \\ -40 \\ 30 \\ -30 \end{bmatrix}$$

$$[\mathbf{M}]: \mathbf{i} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 3.68 \\ -1.90 \\ 2.57 \\ -2.49 \end{bmatrix}$$

$$7. \mathbf{Ri} = \mathbf{v}, \begin{bmatrix} 12 & -7 & 0 & -4 \\ -7 & 15 & -6 & 0 \\ 0 & -6 & 14 & -5 \\ -4 & 0 & -5 & 13 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 40 \\ 30 \\ 20 \\ -10 \end{bmatrix}$$

$$[\mathbf{M}]: \mathbf{i} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 11.43 \\ 10.55 \\ 8.04 \\ 5.84 \end{bmatrix}$$

9.  $\mathbf{x}_{k+1} = M\mathbf{x}_k$  for  $k = 0, 1, 2, \dots$ , where

$$M = \begin{bmatrix} .93 & .05 \\ .07 & .95 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_0 = \begin{bmatrix} 800,000 \\ 500,000 \end{bmatrix}.$$

The population in 2012 (for  $k = 2$ ) is  $\mathbf{x}_2 = \begin{bmatrix} 741,720 \\ 558,280 \end{bmatrix}$ .

$$11. \text{ a. } M = \begin{bmatrix} .98363 & .00167 \\ .01637 & .99833 \end{bmatrix}$$

$$\text{b. } [\mathbf{M}] \mathbf{x}_6 = \begin{bmatrix} 30,754,500 \\ 229,449,000 \end{bmatrix}$$

13. [M]

a. The population of the city decreases. After 7 years, the populations are about equal, but the city population continues to decline. After 20 years, there are only 417,000 persons in the city (417,456 rounded off). However, the changes in population seem to grow smaller each year.

b. The city population is increasing slowly, and the suburban population is decreasing. After 20 years, the city population has grown from 350,000 to about 370,000.

### Chapter 1 Supplementary Exercises, page 88

1. a. F   b. F   c. T   d. F   e. T   f. T  
g. F   h. F   i. T   j. F   k. T   l. F  
m. T   n. T   o. T   p. T   q. F   r. T  
s. F   t. T   u. F   v. F   w. F   x. T  
y. T   z. F

3. a. Any consistent linear system whose echelon form is

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b. Any consistent linear system whose reduced echelon form is  $I_3$ .

c. Any inconsistent linear system of three equations in three variables.

5. a. The solution set: (i) is empty if  $h = 12$  and  $k \neq 2$ ; (ii) contains a unique solution if  $h \neq 12$ ; (iii) contains infinitely many solutions if  $h = 12$  and  $k = 2$ .

b. The solution set is empty if  $k + 3h = 0$ ; otherwise, the solution set contains a unique solution.



## A24 Answers to Odd-Numbered Exercises

7. a. Set  $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -5 \\ 7 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix}$ ,  $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$ , and  $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ . "Determine if  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  span  $\mathbb{R}^3$ ." Solution: No.
- b. Set  $A = \begin{bmatrix} 2 & -4 & -2 \\ -5 & 1 & 1 \\ 7 & -5 & -3 \end{bmatrix}$ . "Determine if the columns of  $A$  span  $\mathbb{R}^3$ ."
- c. Define  $T(\mathbf{x}) = A\mathbf{x}$ . "Determine if  $T$  maps  $\mathbb{R}^3$  onto  $\mathbb{R}^3$ ."
9.  $\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \frac{4}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{7}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  or  $\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 8/3 \\ 4/3 \end{bmatrix} + \begin{bmatrix} 7/3 \\ 14/3 \end{bmatrix}$
10. *Hint*: Construct a "grid" on the  $x_1x_2$ -plane determined by  $\mathbf{a}_1$  and  $\mathbf{a}_2$ .
11. A solution set is a line when the system has one free variable. If the coefficient matrix is  $2 \times 3$ , then two of the columns should be pivot columns. For instance, take  $\begin{bmatrix} 1 & 2 & * \\ 0 & 3 & * \end{bmatrix}$ . Put anything in column 3. The resulting matrix will be in echelon form. Make one row replacement operation on the second row to create a matrix *not* in echelon form, such as  $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 2 \end{bmatrix}$ .
12. *Hint*: How many free variables are in the equation  $A\mathbf{x} = \mathbf{0}$ ?
13.  $E = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$
15. a. If the three vectors are linearly independent, then  $a, c$ , and  $f$  must all be nonzero.  
b. The numbers  $a, \dots, f$  can have any values.
16. *Hint*: List the columns from right to left as  $\mathbf{v}_1, \dots, \mathbf{v}_4$ .
17. *Hint*: Use Theorem 7.
19. Let  $M$  be the line through the origin that is parallel to the line through  $\mathbf{v}_1, \mathbf{v}_2$ , and  $\mathbf{v}_3$ . Then  $\mathbf{v}_2 - \mathbf{v}_1$  and  $\mathbf{v}_3 - \mathbf{v}_1$  are both on  $M$ . So one of these two vectors is a multiple of the other, say  $\mathbf{v}_2 - \mathbf{v}_1 = k(\mathbf{v}_3 - \mathbf{v}_1)$ . This equation produces a linear dependence relation:  $(k-1)\mathbf{v}_1 + \mathbf{v}_2 - k\mathbf{v}_3 = \mathbf{0}$ .  
A second solution: A parametric equation of the line is  $\mathbf{x} = \mathbf{v}_1 + t(\mathbf{v}_2 - \mathbf{v}_1)$ . Since  $\mathbf{v}_3$  is on the line, there is some  $t_0$  such that  $\mathbf{v}_3 = \mathbf{v}_1 + t_0(\mathbf{v}_2 - \mathbf{v}_1) = (1-t_0)\mathbf{v}_1 + t_0\mathbf{v}_2$ . So  $\mathbf{v}_3$  is a linear combination of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , and  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly dependent.
21.  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$       23.  $a = 4/5, b = -3/5$
25. a. The vector lists the numbers of three-, two-, and one-bedroom apartments provided when  $x_1$  floors of plan A are constructed.  
b.  $x_1 \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 3 \\ 9 \end{bmatrix}$

- c. [M] Use 2 floors of plan A and 15 floors of plan B. Or, use 6 floors of plan A, 2 floors of plan B, and 8 floors of plan C. These are the only feasible solutions. There are other mathematical solutions, but they require a negative number (or a fractional number) of floors of one or two of the plans, which makes no physical sense.

## Chapter 2

### Section 2.1, page 100

1.  $\begin{bmatrix} -4 & 0 & 2 \\ -8 & 10 & -4 \end{bmatrix}, \begin{bmatrix} 3 & -5 & 3 \\ -7 & 6 & -7 \end{bmatrix}$ , not defined,  
 $\begin{bmatrix} 1 & 13 \\ -7 & -6 \end{bmatrix}$
3.  $\begin{bmatrix} 1 & 5 \\ -3 & 5 \end{bmatrix}, \begin{bmatrix} 6 & -15 \\ 9 & -6 \end{bmatrix}$
5. a.  $A\mathbf{b}_1 = \begin{bmatrix} -10 \\ 0 \\ 26 \end{bmatrix}, A\mathbf{b}_2 = \begin{bmatrix} 11 \\ 8 \\ -19 \end{bmatrix},$   
 $AB = \begin{bmatrix} -10 & 11 \\ 0 & 8 \\ 26 & -19 \end{bmatrix}$   
b.  $AB = \begin{bmatrix} -1 \cdot 4 + 3(-2) & -1(-2) + 3 \cdot 3 \\ 2 \cdot 4 + 4(-2) & 2(-2) + 4 \cdot 3 \\ 5 \cdot 4 - 3(-2) & 5(-2) - 3 \cdot 3 \end{bmatrix}$   
 $= \begin{bmatrix} -10 & 11 \\ 0 & 8 \\ 26 & -19 \end{bmatrix}$
7.  $3 \times 7$       9.  $k = -2$
11.  $AD = \begin{bmatrix} 5 & 6 & 6 \\ 10 & 12 & 10 \\ 15 & 15 & 12 \end{bmatrix}, DA = \begin{bmatrix} 5 & 10 & 15 \\ 6 & 12 & 15 \\ 6 & 10 & 12 \end{bmatrix}$   
Right-multiplication (that is, multiplication on the right) by  $D$  multiplies each *column* of  $A$  by the corresponding diagonal entry of  $D$ . Left-multiplication by  $D$  multiplies each *row* of  $A$  by the corresponding diagonal entry of  $D$ . The *Study Guide* tells how to make  $AB = BA$ , but you should try this yourself before looking there.
13. *Hint*: One of the two matrices is  $Q$ .
15. Answer the questions before looking in the *Study Guide*.
17.  $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$
19. The third column of  $AB$  is the sum of the first two columns of  $AB$ . Here's why. Write  $B = [\mathbf{b}_1 \ \mathbf{b}_2 \ \mathbf{b}_3]$ . By definition, the third column of  $AB$  is  $A\mathbf{b}_3$ . If  $\mathbf{b}_3 = \mathbf{b}_1 + \mathbf{b}_2$ , then  $A\mathbf{b}_3 = A(\mathbf{b}_1 + \mathbf{b}_2) = A\mathbf{b}_1 + A\mathbf{b}_2$ , by a property of matrix-vector multiplication.
21. The columns of  $A$  are linearly dependent. Why?
23. *Hint*: Suppose  $\mathbf{x}$  satisfies  $A\mathbf{x} = \mathbf{0}$ , and show that  $\mathbf{x}$  must be  $\mathbf{0}$ .



25. *Hint:* Use the results of Exercises 23 and 24, and apply the associative law of multiplication to the product  $CAD$ .

27.  $\mathbf{u}^T \mathbf{v} = \mathbf{v}^T \mathbf{u} = -3a + 2b - 5c,$

$$\mathbf{uv}^T = \begin{bmatrix} -3a & -3b & -3c \\ 2a & 2b & 2c \\ -5a & -5b & -5c \end{bmatrix},$$

$$\mathbf{vu}^T = \begin{bmatrix} -3a & 2a & -5a \\ -3b & 2b & -5b \\ -3c & 2c & -5c \end{bmatrix}$$

29. *Hint:* For Theorem 2(b), show that the  $(i, j)$ -entry of  $A(B + C)$  equals the  $(i, j)$ -entry of  $AB + AC$ .

31. *Hint:* Use the definition of the product  $I_m A$  and the fact that  $I_m \mathbf{x} = \mathbf{x}$  for  $\mathbf{x}$  in  $\mathbb{R}^m$ .

33. *Hint:* First write the  $(i, j)$ -entry of  $(AB)^T$ , which is the  $(j, i)$ -entry of  $AB$ . Then, to compute the  $(i, j)$ -entry in  $B^T A^T$ , use the facts that the entries in row  $i$  of  $B^T$  are  $b_{1i}, \dots, b_{ni}$ , because they come from column  $i$  of  $B$ , and the entries in column  $j$  of  $A^T$  are  $a_{j1}, \dots, a_{jn}$ , because they come from row  $j$  of  $A$ .

35. [M] The answer here depends on the choice of matrix program. For MATLAB, use the `help` command to read about `zeros`, `ones`, `eye`, and `diag`. For the TI-86, study the `dim`, `fill`, and `iden` instructions. The TI-86 does not have a “diagonal” command.

37. [M] Display your results and report your conclusions.

39. [M] Display your results and report your conclusions.

41. The matrices appear to approach the matrix

$$\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

## Section 2.2, page 109

1.  $\begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}$     3.  $-\frac{1}{3} \begin{bmatrix} -3 & -3 \\ 6 & 7 \end{bmatrix}$  or  $\begin{bmatrix} 1 & 1 \\ -2 & -7/3 \end{bmatrix}$

5.  $x_1 = 7$  and  $x_2 = -9$

7.  $\mathbf{a}$  and  $\mathbf{b}$ :  $\begin{bmatrix} -9 \\ 4 \end{bmatrix}, \begin{bmatrix} 11 \\ -5 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \end{bmatrix}$ , and  $\begin{bmatrix} 13 \\ -5 \end{bmatrix}$

9. Write out your answers before checking the *Study Guide*.

11. The proof can be modeled after the proof of Theorem 5.

13.  $AB = AC \Rightarrow A^{-1}AB = A^{-1}AC \Rightarrow IB = IC \Rightarrow B = C$ . No, in general,  $B$  and  $C$  can be different when  $A$  is not invertible. See Exercise 10 in Section 2.1.

15. *Hint:* Apply the elementary matrices used to row reduce  $A$  to  $I$ , to the matrix  $[A, B]$ .

17.  $D = C^{-1}B^{-1}A^{-1}$ . Show that  $D$  works.

19. After you find  $X = CB - A$ , show that  $X$  is a solution.

21. *Hint:* Consider the equation  $A\mathbf{x} = \mathbf{0}$ .

23. *Hint:* If  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution, then there are no free variables in the equation  $A\mathbf{x} = \mathbf{0}$ , and each column of  $A$  is a pivot column.

25. *Hint:* Consider the case  $a = b = 0$ . Then consider the vector  $\begin{bmatrix} -b \\ a \end{bmatrix}$ , and use the fact that  $ad - bc = 0$ .

27. *Hint:* For part (a), interchange  $A$  and  $B$  in the box following Example 6 in Section 2.1, and then replace  $B$  by the identity matrix. For parts (b) and (c), begin by writing

$$A = \begin{bmatrix} \text{row}_1(A) \\ \text{row}_2(A) \\ \text{row}_3(A) \end{bmatrix}$$

29.  $\frac{1}{3} \begin{bmatrix} -9 & 3 \\ -4 & 1 \end{bmatrix}$     31.  $\begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}$

33. The general form of  $A^{-1}$  is

$$A^{-1} = B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & & 0 \\ 0 & -1 & 1 & & \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}.$$

*Hint:* For  $j = 1, \dots, n$ , let  $\mathbf{a}_j$ ,  $\mathbf{b}_j$ , and  $\mathbf{e}_j$  denote the  $j$ th columns of  $A$ ,  $B$ , and  $I$ , respectively. Use the facts that  $\mathbf{a}_j - \mathbf{a}_{j+1} = \mathbf{e}_j$  and  $\mathbf{b}_j = \mathbf{e}_j - \mathbf{e}_{j+1}$  for  $j = 1, \dots, n-1$ , and  $\mathbf{a}_n = \mathbf{b}_n = \mathbf{e}_n$ .

35.  $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ . Find this by row reducing  $[A \quad \mathbf{e}_3]$ .

37.  $C = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}$

39. [M] The deflections are .62, .66, and .52 inches, respectively.

41. [M] .95, 6.19, 11.43, and 3.81 newtons, respectively

## Section 2.3, page 115

The abbreviation IMT (here and in the *Study Guide*) denotes the Invertible Matrix Theorem (Theorem 8).

1. Invertible, by the IMT. Neither column of the matrix is a multiple of the other column, so they are linearly independent. Also, the matrix is invertible by Theorem 4 in Section 2.2 because the determinant is nonzero.

3. Notice that  $A^T$  has a pivot in every column, so by IMT,  $A^T$  is invertible. Hence by IMT,  $A$  is also invertible.

5. Not invertible, by the IMT. Since this matrix has a column of zeros, its columns form a linearly dependent set and hence the matrix is not invertible.

## A26 Answers to Odd-Numbered Exercises

7. Invertible, by the IMT. The matrix row reduces to

$$\begin{bmatrix} -1 & -3 & 0 & 1 \\ 0 & -4 & 8 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and has four pivot positions.

9. [M] The  $4 \times 4$  matrix has four pivot positions, so it is invertible by the IMT.
11. The *Study Guide* will help, but first try to answer the questions based on your careful reading of the text.
13. A square upper triangular matrix is invertible if and only if all the entries on the diagonal are nonzero. Why?

*Note:* The answers below for Exercises 15–29 mention the IMT. In many cases, part or all of an acceptable answer could also be based on results that were used to establish the IMT.

15. No, because statement (h) of the IMT is then false. A  $4 \times 4$  matrix cannot be invertible when its columns do not span  $\mathbb{R}^4$ .
17. If  $A$  has two identical columns, then its columns are linearly dependent. Part (e) of the IMT shows that  $A$  cannot be invertible.
19. By statement (e) of the IMT,  $D$  is invertible. Thus the equation  $D\mathbf{x} = \mathbf{b}$  has a solution for each  $\mathbf{b}$  in  $\mathbb{R}^7$ , by statement (g) of the IMT. Can you say more?
21. The matrix  $C$  cannot be invertible, by Theorem 5 in Section 2.2 or by the paragraph following the IMT. So statement (g) of the IMT is false and so is (h). The columns of  $C$  do not span  $\mathbb{R}^n$ .
23. Statement (g) of the IMT is false for  $F$ , so statement (d) is false, too. That is, the equation  $F\mathbf{x} = \mathbf{0}$  has a nontrivial solution.
25. *Hint:* Use the IMT first.
27. Let  $W$  be the inverse of  $AB$ . Then  $ABW = I$  and  $A(BW) = I$ . Unfortunately, this equation by itself does not prove that  $A$  is invertible. Why not? Finish the proof before you check the *Study Guide*.
29. Since the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  is one-to-one, statement (f) of the IMT is true. Then statement (i) is also true and the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  maps  $\mathbb{R}^n$  onto  $\mathbb{R}^n$ . Also,  $A$  is invertible, which implies that the transformation  $\mathbf{x} \mapsto A\mathbf{x}$  is invertible, by Theorem 9.
31. *Hint:* If the equation  $A\mathbf{x} = \mathbf{b}$  has a solution for each  $\mathbf{b}$ , then  $A$  has a pivot in each row (Theorem 4 in Section 1.4). Could there be free variables in an equation  $A\mathbf{x} = \mathbf{b}$ ?

33. *Hint:* First show that the standard matrix of  $T$  is invertible. Then use a theorem or theorems to show that  $T^{-1}(\mathbf{x}) = B\mathbf{x}$ , where  $B = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}$ .

35. *Hint:* To show that  $T$  is one-to-one, suppose that  $T(\mathbf{u}) = T(\mathbf{v})$  for some vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $\mathbb{R}^n$ . Deduce that  $\mathbf{u} = \mathbf{v}$ . To show that  $T$  is onto, suppose  $\mathbf{y}$  represents an arbitrary vector in  $\mathbb{R}^n$  and use the inverse  $S$  to produce an  $\mathbf{x}$  such that  $T(\mathbf{x}) = \mathbf{y}$ . A second proof can be given using Theorem 9 together with a theorem from Section 1.9.

37. *Hint:* Consider the standard matrices of  $T$  and  $U$ .

39. If  $T$  maps  $\mathbb{R}^n$  onto  $\mathbb{R}^n$ , then the columns of its standard matrix  $A$  span  $\mathbb{R}^n$ , by Theorem 12 in Section 1.9. By the IMT,  $A$  is invertible. Hence, by Theorem 9,  $T$  is invertible, and  $A^{-1}$  is the standard matrix of  $T^{-1}$ . Since  $A^{-1}$  is also invertible, by the IMT, its columns are linearly independent and span  $\mathbb{R}^n$ . Applying Theorem 12 in Section 1.9 to the transformation  $T^{-1}$  shows that  $T^{-1}$  is a one-to-one mapping of  $\mathbb{R}^n$  onto  $\mathbb{R}^n$ .

41. [M]

- a. The exact solution of (3) is  $x_1 = 3.94$  and  $x_2 = .49$ . The exact solution of review (4) is  $x_1 = 2.90$  and  $x_2 = 2.00$ .
- b. When the solution of (4) is used as an approximation for the solution in (3), the error in using the value of 2.90 for  $x_1$  is about 26%, and the error in using 2.0 for  $x_2$  is about 308%.
- c. The condition number of the coefficient matrix is 3363. The percentage change in the solution from (3) to (4) is about 7700 times the percentage change in the right side of the equation. This is the same order of magnitude as the condition number. The condition number gives a rough measure of how sensitive the solution of  $A\mathbf{x} = \mathbf{b}$  can be to changes in  $\mathbf{b}$ . Further information about the condition number is given at the end of Chapter 6 and in Chapter 7.
43. [M]  $\text{cond}(A) \approx 69,000$ , which is between  $10^4$  and  $10^5$ . So about 4 or 5 digits of accuracy may be lost. Several experiments with MATLAB should verify that  $\mathbf{x}$  and  $\mathbf{x}_1$  agree to 11 or 12 digits.
45. [M] Some versions of MATLAB issue a warning when asked to invert a Hilbert matrix of order about 12 or larger using floating-point arithmetic. The product  $AA^{-1}$  should have several off-diagonal entries that are far from being zero. If not, try a larger matrix.

### Section 2.4, page 121

1.  $\begin{bmatrix} A & B \\ EA + C & EB + D \end{bmatrix}$       3.  $\begin{bmatrix} C & D \\ A & B \end{bmatrix}$
5.  $Y = B^{-1}$  (explain why),  $X = -B^{-1}A$ ,  $Z = C$
7.  $X = A^{-1}$  (why?),  $Y = -BA^{-1}$ ,  $Z = 0$  (why?)
9.  $A_{21} = -B_{21}B_{11}^{-1}$ ,  $A_{31} = -B_{31}B_{11}^{-1}$ ,  
 $C_{22} = B_{22} - B_{21}B_{11}^{-1}B_{12}$
11. You can check your answers in the *Study Guide*.
13. *Hint:* Suppose  $A$  is invertible, and let  $A^{-1} = \begin{bmatrix} D & E \\ F & G \end{bmatrix}$ . Show that  $BD = I$  and  $CG = I$ . This implies that  $B$  and

$C$  are invertible. (Explain why!) Conversely, suppose  $B$  and  $C$  are invertible. To prove that  $A$  is invertible, guess what  $A^{-1}$  must be and check that it works.

$$15. G_{k+1} = \begin{bmatrix} X_k & \mathbf{x}_{k+1} \end{bmatrix} \begin{bmatrix} X_k^T \\ \mathbf{x}_{k+1}^T \end{bmatrix} = X_k X_k^T + \mathbf{x}_{k+1} \mathbf{x}_{k+1}^T \\ = G_k + \mathbf{x}_{k+1} \mathbf{x}_{k+1}^T$$

Only the outer product matrix  $\mathbf{x}_{k+1} \mathbf{x}_{k+1}^T$  needs to be computed (and then added to  $G_k$ ).

17. The inverse of  $\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$  is  $\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}$ . Similarly,  $\begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}$  has an inverse. From equation (7), one obtains

$$\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix} \quad (*)$$

If  $A$  is invertible, then the matrix on the right side of  $(*)$  is a product of invertible matrices and hence is invertible. By Exercise 13,  $A_{11}$  and  $S$  must be invertible.

19.  $W(s) = I_m - C(A - sI_n)^{-1}B$ . This is the Schur complement of  $A - sI_n$  in the system matrix.

$$21. \text{ a. } A^2 = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \\ = \begin{bmatrix} 1+0 & 0+0 \\ 2-2 & 0+(-1)^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{ b. } M^2 = \begin{bmatrix} A & 0 \\ I & -A \end{bmatrix} \begin{bmatrix} A & 0 \\ I & -A \end{bmatrix} \\ = \begin{bmatrix} A^2+0 & 0+0 \\ A-A & 0+(-A)^2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

23. If  $A_1$  and  $B_1$  are  $(k+1) \times (k+1)$  and lower triangular, then write  $A_1 = \begin{bmatrix} a & \mathbf{0}^T \\ \mathbf{v} & A \end{bmatrix}$  and  $B_1 = \begin{bmatrix} b & \mathbf{0}^T \\ \mathbf{w} & B \end{bmatrix}$ , where  $A$  and  $B$  are  $k \times k$  and lower triangular,  $\mathbf{v}$  and  $\mathbf{w}$  are in  $\mathbb{R}^k$ , and  $a$  and  $b$  are suitable scalars. Assume that the product of  $k \times k$  lower triangular matrices is lower triangular, and compute the product  $A_1 B_1$ . What do you conclude?

25. Use Exercise 13 to find the inverse of a matrix of the form  $B = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix}$ , where  $B_{11}$  is  $p \times p$ ,  $B_{22}$  is  $q \times q$ , and  $B$  is invertible. Partition the matrix  $A$ , and apply your result twice to find that

$$A^{-1} = \begin{bmatrix} -5 & 2 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & -5/2 & 7/2 \end{bmatrix}$$

27. a., b. The commands to be used in these exercises will depend on the matrix program.

- c. The algebra needed comes from the block matrix equation

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$$

where  $\mathbf{x}_1$  and  $\mathbf{b}_1$  are in  $\mathbb{R}^{20}$  and  $\mathbf{x}_2$  and  $\mathbf{b}_2$  are in  $\mathbb{R}^{30}$ . Then  $A_{11}\mathbf{x}_1 = \mathbf{b}_1$ , which can be solved to produce  $\mathbf{x}_1$ .

The equation  $A_{21}\mathbf{x}_1 + A_{22}\mathbf{x}_2 = \mathbf{b}_2$  yields

$A_{22}\mathbf{x}_2 = \mathbf{b}_2 - A_{21}\mathbf{x}_1$ , which can be solved for  $\mathbf{x}_2$  by row reducing the matrix  $[A_{22} \quad \mathbf{c}]$ , where  $\mathbf{c} = \mathbf{b}_2 - A_{21}\mathbf{x}_1$ .

### Section 2.5, page 129

$$1. Ly = \mathbf{b} \Rightarrow y = \begin{bmatrix} -7 \\ -2 \\ 6 \end{bmatrix}, Ux = y \Rightarrow x = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$$

$$3. y = \begin{bmatrix} 6 \\ 12 \\ 0 \end{bmatrix}, x = \begin{bmatrix} -5 \\ -4 \\ 0 \end{bmatrix} \quad 5. y = \begin{bmatrix} 1 \\ 3 \\ 1 \\ -4 \end{bmatrix}, x = \begin{bmatrix} 38 \\ 16 \\ 8.5 \\ -4 \end{bmatrix}$$

$$7. LU = \begin{bmatrix} 1 & 0 \\ -3/2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 0 & 7/2 \end{bmatrix}$$

$$9. \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$11. \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 7 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$13. \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 4 & 5 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$15. \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 5 & 2 \\ 0 & 3 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$17. U^{-1} = \begin{bmatrix} 1/2 & -3/4 & -2 \\ 0 & -1/4 & -1 \\ 0 & 0 & -1/2 \end{bmatrix}, \\ L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix}, \\ A^{-1} = \begin{bmatrix} 3 & 5/4 & -2 \\ 3/2 & 3/4 & -1 \\ 1 & 1/2 & -1/2 \end{bmatrix}$$

19. Hint: Think about row reducing  $[A \quad I]$ .

21. Hint: Represent the row operations by a sequence of elementary matrices.

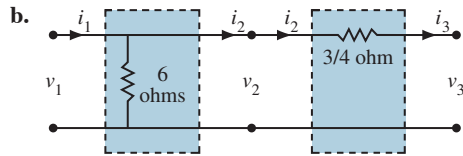
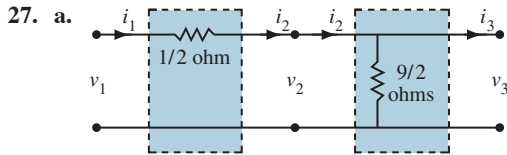
23. a. Denote the rows of  $D$  as transposes of column vectors. Then partitioned matrix multiplication yields

$$A = CD = \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_4 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_4^T \end{bmatrix} \\ = \mathbf{c}_1 \mathbf{v}_1^T + \cdots + \mathbf{c}_4 \mathbf{v}_4^T$$

- b.  $A$  has 40,000 entries. Since  $C$  has 1600 entries and  $D$  has 400 entries, together they occupy only 5% of the memory needed to store  $A$ .

## A28 Answers to Odd-Numbered Exercises

25. Explain why  $U$ ,  $D$ , and  $V^T$  are invertible. Then use a theorem on the inverse of a product of invertible matrices.



29. a. 
$$\begin{bmatrix} 1 + R_3/R_2 & -R_1 - R_3 - (R_1 R_3)/R_2 \\ -1/R_2 & 1 + R_1/R_2 \end{bmatrix}$$

b. 
$$A = \begin{bmatrix} 3 & -12 \\ -1/3 & 5/3 \end{bmatrix}$$
  

$$= \begin{bmatrix} 1 & -6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

Set  $R_1 = 2$  ohms,  $R_2 = 3$  ohms, and  $R_3 = 6$  ohms

31. [M]

a. 
$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -.25 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -.25 & -.0667 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -.2667 & -.2857 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -.2679 & -.0833 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -.2917 & -.2921 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -.2697 & -.0861 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -.2948 & -.2931 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.75 & -.25 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.7333 & -1.0667 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.4286 & -.2857 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.7083 & -1.0833 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.3919 & -.2921 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3.7052 & -1.0861 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.3868 \end{bmatrix}$$

b.  $\mathbf{x} =$

(3.9569, 6.5885, 4.2392, 7.3971, 5.6029, 8.7608, 9.4115, 12.0431)

c. 
$$A^{-1} = \begin{bmatrix} .2953 & .0866 & .0945 & .0509 & .0318 & .0227 & .0100 & .0082 \\ .0866 & .2953 & .0509 & .0945 & .0227 & .0318 & .0082 & .0100 \\ .0945 & .0509 & .3271 & .1093 & .1045 & .0591 & .0318 & .0227 \\ .0509 & .0945 & .1093 & .3271 & .0591 & .1045 & .0227 & .0318 \\ .0318 & .0227 & .1045 & .0591 & .3271 & .1093 & .0945 & .0509 \\ .0227 & .0318 & .0591 & .1045 & .1093 & .3271 & .0509 & .0945 \\ .0100 & .0082 & .0318 & .0227 & .0945 & .0509 & .2953 & .0866 \\ .0082 & .0100 & .0227 & .0318 & .0509 & .0945 & .0866 & .2953 \end{bmatrix}$$

Obtain  $A^{-1}$  directly and then compute  $A^{-1} - U^{-1}L^{-1}$  to compare the two methods for inverting a matrix.

## Section 2.6, page 136

1. 
$$C = \begin{bmatrix} .10 & .60 & .60 \\ .30 & .20 & 0 \\ .30 & .10 & .10 \end{bmatrix}, \begin{Bmatrix} \text{intermediate} \\ \text{demand} \end{Bmatrix} = \begin{bmatrix} 60 \\ 20 \\ 10 \end{bmatrix}$$

3. 
$$\mathbf{x} = \begin{bmatrix} 44.44 \\ 16.67 \\ 16.67 \end{bmatrix}$$

5. 
$$\mathbf{x} = \begin{bmatrix} 110 \\ 120 \end{bmatrix}$$

7. a. 
$$\begin{bmatrix} 1.6 \\ 1.2 \end{bmatrix}$$

b. 
$$\begin{bmatrix} 111.6 \\ 121.2 \end{bmatrix}$$

c. Hint: Find a formula involving  $(I - C)^{-1}$ . See the Study Guide.

9. 
$$\mathbf{x} = \begin{bmatrix} 82.8 \\ 131.0 \\ 110.3 \end{bmatrix}$$

11. Hint: Use properties of transposes to obtain

$$\mathbf{p}^T = \mathbf{p}^T C + \mathbf{v}^T, \text{ so that}$$

$$\mathbf{p}^T \mathbf{x} = (\mathbf{p}^T C + \mathbf{v}^T) \mathbf{x} = \mathbf{p}^T C \mathbf{x} + \mathbf{v}^T \mathbf{x}. \text{ Now compute } \mathbf{p}^T \mathbf{x}$$

from the production equation.

13. [M]  $\mathbf{x} =$

(99576, 97703, 51231, 131570, 49488, 329554, 13835).

The entries in  $\mathbf{x}$  suggest more precision in the answer than is warranted by the entries in  $\mathbf{d}$ , which appear to be accurate only to perhaps the nearest thousand. So a more realistic answer for  $\mathbf{x}$  might be

$$\mathbf{x} = 1000 \times (100, 98, 51, 132, 49, 330, 14).$$

15. [M]  $\mathbf{x}^{(12)}$  is the first vector whose entries are accurate to the nearest thousand. The calculation of  $\mathbf{x}^{(12)}$  takes about 1260 flops, while row reduction of  $[(I - C) \mathbf{d}]$  takes only about 550 flops. If  $C$  is larger than  $20 \times 20$ , then fewer flops are needed to compute  $\mathbf{x}^{(12)}$  by iteration than to compute the equilibrium vector  $\mathbf{x}$  by row reduction. As the size of  $C$  grows, the advantage of the iterative method increases. Also, because  $C$  becomes more sparse for larger models of the economy, fewer iterations are needed for reasonable accuracy.

## Section 2.7, page 144

1. 
$$\begin{bmatrix} 1 & .25 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. 
$$\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

5. 
$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

7. 
$$\begin{bmatrix} 1/2 & -\sqrt{3}/2 & 3 + 4\sqrt{3} \\ \sqrt{3}/2 & 1/2 & 4 - 3\sqrt{3} \\ 0 & 0 & 1 \end{bmatrix}$$

See the Practice Problem.

9.  $A(BD)$  requires 800 multiplications.  $(AB)D$  requires 408 multiplications. The first method uses about twice as many multiplications. If  $D$  had 10,000 columns, the counts would be 80,000 and 40,008, respectively.

11. Use the fact that

$$\sec \varphi - \tan \varphi \sin \varphi = \frac{1}{\cos \varphi} - \frac{\sin^2 \varphi}{\cos \varphi} = \cos \varphi$$

13. 
$$\begin{bmatrix} A & \mathbf{p} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} I & \mathbf{p} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}. \text{ First apply the linear transformation } A, \text{ and then translate by } \mathbf{p}.$$

$$15. (12, -6, -3) \quad 17. \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 & 0 \\ 0 & \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

19. The triangle with vertices at  $(7, 2, 0)$ ,  $(7.5, 5, 0)$ , and  $(5, 5, 0)$

$$21. [M] \begin{bmatrix} 2.2586 & -1.0395 & -.3473 \\ -1.3495 & 2.3441 & .0696 \\ .0910 & -.3046 & 1.2777 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} R \\ G \\ B \end{bmatrix}$$

### Section 2.8, page 151

- The set is closed under sums but not under multiplication by a negative scalar. (Sketch an example.)
- The set is not closed under sums or scalar multiples.
- Yes. The system corresponding to  $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{w}]$  is consistent.
- The three vectors  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ , and  $\mathbf{v}_3$
  - Infinitely many vectors
  - Yes, because  $A\mathbf{x} = \mathbf{p}$  has a solution.
- No, because  $A\mathbf{p} \neq \mathbf{0}$ .
- $p = 4$  and  $q = 3$ .  $\text{Nul } A$  is a subspace of  $\mathbb{R}^4$  because solutions of  $A\mathbf{x} = \mathbf{0}$  must have four entries, to match the columns of  $A$ .  $\text{Col } A$  is a subspace of  $\mathbb{R}^3$  because each column vector has three entries.
- For  $\text{Nul } A$ , choose  $(1, -2, 1, 0)$  or  $(-1, 4, 0, 1)$ , for example. For  $\text{Col } A$ , select any column of  $A$ .
- Let  $A$  be the matrix whose columns are the vectors given. Then  $A$  is invertible because its determinant is nonzero, and so its columns form a basis for  $\mathbb{R}^2$ , by the IMT (or by Example 5). (Other reasons for the invertibility of  $A$  could be given.)
- Let  $A$  be the matrix whose columns are the vectors given. Row reduction shows three pivots, so  $A$  is invertible. By the IMT, the columns of  $A$  form a basis for  $\mathbb{R}^3$ .
- Let  $A$  be the  $3 \times 2$  matrix whose columns are the vectors given. The columns of  $A$  cannot possibly span  $\mathbb{R}^3$  because  $A$  cannot have a pivot in every row. So the columns are not a basis for  $\mathbb{R}^3$ . (They are a basis for a plane in  $\mathbb{R}^3$ .)
- Read the section carefully, and write your answers before checking the *Study Guide*. This section has terms and key concepts that you must learn now before going on.

$$23. \text{Basis for Col } A: \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}$$

$$\text{Basis for Nul } A: \begin{bmatrix} 4 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

$$25. \text{Basis for Col } A: \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 5 \\ -5 \end{bmatrix}$$

$$\text{Basis for Nul } A: \begin{bmatrix} 2 \\ -2.5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ .5 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

- Construct a nonzero  $3 \times 3$  matrix  $A$ , and construct  $\mathbf{b}$  to be almost any convenient linear combination of the columns of  $A$ .
- Hint:* You need a nonzero matrix whose columns are linearly dependent.
- If  $\text{Col } F \neq \mathbb{R}^5$ , then the columns of  $F$  do not span  $\mathbb{R}^5$ . Since  $F$  is square, the IMT shows that  $F$  is not invertible and the equation  $F\mathbf{x} = \mathbf{0}$  has a nontrivial solution. That is,  $\text{Nul } F$  contains a nonzero vector. Another way to describe this is to write  $\text{Nul } F \neq \{\mathbf{0}\}$ .
- If  $\text{Nul } C = \{\mathbf{0}\}$ , then the equation  $C\mathbf{x} = \mathbf{0}$  has only the trivial solution. Since  $C$  is square, the IMT shows that  $C$  is invertible and the equation  $C\mathbf{x} = \mathbf{b}$  has a solution for each  $\mathbf{b}$  in  $\mathbb{R}^6$ . Also, each solution is unique, by Theorem 5 in Section 2.2.
- If  $\text{Nul } B$  contains nonzero vectors, then the equation  $B\mathbf{x} = \mathbf{0}$  has nontrivial solutions. Since  $B$  is square, the IMT shows that  $B$  is not invertible and the columns of  $B$  do not span  $\mathbb{R}^5$ . So  $\text{Col } B$  is a subspace of  $\mathbb{R}^5$ , but  $\text{Col } B \neq \mathbb{R}^5$ .
- [M] Display the reduced echelon form of  $A$ , and select the pivot columns of  $A$  as a basis for  $\text{Col } A$ . For  $\text{Nul } A$ , write the solution of  $A\mathbf{x} = \mathbf{0}$  in parametric vector form.

$$\text{Basis for Col } A: \begin{bmatrix} 3 \\ -7 \\ -5 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ 9 \\ 7 \\ -7 \end{bmatrix}$$

$$\text{Basis for Nul } A: \begin{bmatrix} -2.5 \\ -1.5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4.5 \\ 2.5 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3.5 \\ -1.5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

### Section 2.9, page 157

- $\mathbf{x} = 3\mathbf{b}_1 + 2\mathbf{b}_2 = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$
- $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$     5.  $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$     7.  $[\mathbf{w}]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ ,  $[\mathbf{x}]_B = \begin{bmatrix} 1.5 \\ .5 \end{bmatrix}$
- Basis for  $\text{Col } A: \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 9 \\ 14 \end{bmatrix}$ ;  $\dim \text{Col } A = 3$

### A30 Answers to Odd-Numbered Exercises

$$\text{Basis for Nul } A: \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}; \dim \text{Nul } A = 1$$

$$11. \text{ Basis for Col } A: \begin{bmatrix} 2 \\ 3 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -5 \\ -8 \\ 9 \\ -7 \end{bmatrix}; \dim \text{Col } A = 2$$

$$\text{Basis for Nul } A: \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}; \dim \text{Nul } A = 3$$

13. The vectors  $\mathbf{v}_1$ ,  $\mathbf{v}_3$ , and  $\mathbf{v}_4$  form a basis for the given subspace,  $H$ . So, dimension  $H = 3$ .
15.  $\text{Col } A = \mathbb{R}^4$ , because  $A$  has a pivot in each row, and so the columns of  $A$  span  $\mathbb{R}^4$ .  $\text{Nul } A$  cannot equal  $\mathbb{R}^2$ , because  $\text{Nul } A$  is a subspace of  $\mathbb{R}^6$ . It is true, however, that  $\text{Nul } A$  is two-dimensional. Reason: The equation  $A\mathbf{x} = \mathbf{0}$  has two free variables, because  $A$  has six columns and only four of them are pivot columns.
17. See the *Study Guide* after you write your justifications.
19. The fact that the solution space of  $A\mathbf{x} = \mathbf{0}$  has a basis of three vectors means that  $\dim \text{Nul } A = 3$ . Since a  $5 \times 7$  matrix  $A$  has seven columns, the Rank Theorem shows that  $\text{rank } A = 7 - \dim \text{Nul } A = 4$ . See the *Study Guide* for a justification that does not explicitly mention the Rank Theorem.
21. A  $9 \times 8$  matrix has eight columns. By the Rank Theorem,  $\dim \text{Nul } A = 8 - \text{rank } A$ . Since the rank is 7,  $\dim \text{Nul } A = 1$ . That is, the dimension of the solution space of  $A\mathbf{x} = \mathbf{0}$  is 1.
23. Create a  $3 \times 5$  matrix  $A$  with two pivot columns. The remaining three columns will correspond to free variables in the equation  $A\mathbf{x} = \mathbf{0}$ . So the desired construction is possible.
25. The  $p$  columns of  $A$  span  $\text{Col } A$  by definition. If  $\dim \text{Col } A = p$ , then the spanning set of  $p$  columns is automatically a basis for  $\text{Col } A$ , by the Basis Theorem. In particular, the columns are linearly independent.
27. a. *Hint:* The columns of  $B$  span  $W$ , and each vector  $\mathbf{a}_j$  is in  $W$ . The vector  $\mathbf{c}_j$  is in  $\mathbb{R}^p$  because  $B$  has  $p$  columns.  
b. *Hint:* What is the size of  $C$ ?  
c. *Hint:* How are  $B$  and  $C$  related to  $A$ ?
29. [M] Your calculations should show that the matrix  $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{x} \end{bmatrix}$  corresponds to a consistent system. The  $B$ -coordinate vector of  $\mathbf{x}$  is  $(2, -1)$ .

### Chapter 2 Supplementary Exercises, page 160

1. a. T b. F c. T d. F e. F f. F  
g. T h. T i. T j. F k. T l. F

m. F n. T o. F p. T

3.  $I$

5.  $A^2 = 2A - I$ . Multiply by  $A$ :  $A^3 = 2A^2 - A$ .  
Substitute  $A^2 = 2A - I$ :  
 $A^3 = 2(2A - I) - A = 3A - 2I$ .  
Multiply by  $A$  again:  $A^4 = A(3A - 2I) = 3A^2 - 2A$ .  
Substitute the identity  $A^2 = 2A - I$  again:  
 $A^4 = 3(2A - I) - 2A = 4A - 3I$ .

$$7. \begin{bmatrix} 10 & -1 \\ 9 & 10 \\ -5 & -3 \end{bmatrix} \quad 9. \begin{bmatrix} -3 & 13 \\ -8 & 27 \end{bmatrix}$$

$$11. \text{ a. } p(x_i) = c_0 + c_1x_i + \cdots + c_{n-1}x_i^{n-1} \\ = \text{row}_i(V) \cdot \begin{bmatrix} c_0 \\ \vdots \\ c_{n-1} \end{bmatrix} = \text{row}_i(V\mathbf{c}) = y_i$$

- b. Suppose  $x_1, \dots, x_n$  are distinct, and suppose  $V\mathbf{c} = \mathbf{0}$  for some vector  $\mathbf{c}$ . Then the entries in  $\mathbf{c}$  are the coefficients of a polynomial whose value is zero at the distinct points  $x_1, \dots, x_n$ . However, a nonzero polynomial of degree  $n - 1$  cannot have  $n$  zeros, so the polynomial must be identically zero. That is, the entries in  $\mathbf{c}$  must all be zero. This shows that the columns of  $V$  are linearly independent.
- c. *Hint:* When  $x_1, \dots, x_n$  are distinct, there is a vector  $\mathbf{c}$  such that  $V\mathbf{c} = \mathbf{y}$ . Why?
13. a.  $P^2 = (\mathbf{u}\mathbf{u}^T)(\mathbf{u}\mathbf{u}^T) = \mathbf{u}(\mathbf{u}^T\mathbf{u})\mathbf{u}^T = \mathbf{u}(1)\mathbf{u}^T = P$   
b.  $P^T = (\mathbf{u}\mathbf{u}^T)^T = \mathbf{u}^{TT}\mathbf{u}^T = \mathbf{u}\mathbf{u}^T = P$   
c.  $Q^2 = (I - 2P)(I - 2P) \\ = I - I(2P) - 2PI + 2P(2P) \\ = I - 4P + 4P^2 = I$ , because of part (a).
15. Left-multiplication by an elementary matrix produces an elementary row operation:  
 $B \sim E_1B \sim E_2E_1B \sim E_3E_2E_1B = C$   
So  $B$  is row equivalent to  $C$ . Since row operations are reversible,  $C$  is row equivalent to  $B$ . (Alternatively, show  $C$  being changed into  $B$  by row operations using the inverses of the  $E_i$ .)
17. Since  $B$  is  $4 \times 6$  (with more columns than rows), its six columns are linearly dependent and there is a nonzero  $\mathbf{x}$  such that  $B\mathbf{x} = \mathbf{0}$ . Thus  $AB\mathbf{x} = A\mathbf{0} = \mathbf{0}$ , which shows that the matrix  $AB$  is not invertible, by the Invertible Matrix Theorem.

19. [M] To four decimal places, as  $k$  increases,

$$A^k \rightarrow \begin{bmatrix} .2857 & .2857 & .2857 \\ .4286 & .4286 & .4286 \\ .2857 & .2857 & .2857 \end{bmatrix} \quad \text{and} \\ B^k \rightarrow \begin{bmatrix} .2022 & .2022 & .2022 \\ .3708 & .3708 & .3708 \\ .4270 & .4270 & .4270 \end{bmatrix}$$

or, in rational format,

$$A^k \rightarrow \begin{bmatrix} 2/7 & 2/7 & 2/7 \\ 3/7 & 3/7 & 3/7 \\ 2/7 & 2/7 & 2/7 \end{bmatrix} \text{ and}$$

$$B^k \rightarrow \begin{bmatrix} 18/89 & 18/89 & 18/89 \\ 33/89 & 33/89 & 33/89 \\ 38/89 & 38/89 & 38/89 \end{bmatrix}$$

## Chapter 3

### Section 3.1, page 167

1. 1      3. -5      5. -23      7. 4
9. 10. Start with row 3.
11. -12. Start with column 1 or row 4.
13. 6. Start with row 2 or column 2.
15. 1      17. -5
19.  $ad - bc, cb - da$ . Interchanging two rows changes the sign of the determinant.
21. -2,  $(18 + 12k) - (20 + 12k) = -2$ . A row replacement does not change the value of a determinant.
23. -5,  $k(4) - k(2) + k(-7) = -5k$ . Scaling a row by a constant  $k$  multiplies the determinant by  $k$ .
25. 1      27.  $k$       29. -1
31. 1. The matrix is upper or lower triangular, with only 1's on the diagonal. The determinant is 1, the product of the diagonal entries.
33.  $\det EA = \det \begin{bmatrix} c & d \\ a & b \end{bmatrix} = cb - ad = (-1)(ad - bc) = (\det E)(\det A)$
35.  $\det EA = \det \begin{bmatrix} a + kc & b + kd \\ c & d \end{bmatrix} = (a + kc)d - (b + kd)c = ad + kcd - bc - kdc = (+1)(ad - bc) = (\det E)(\det A)$
37.  $5A = \begin{bmatrix} 15 & 5 \\ 20 & 10 \end{bmatrix}$ ; no
39. Hints are in the *Study Guide*.
41. The area of the parallelogram and the determinant of  $\begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}$  both equal 6. If  $\mathbf{v} = \begin{bmatrix} x \\ 2 \end{bmatrix}$  for any  $x$ , the area is still 6. In each case the base of the parallelogram is unchanged, and the altitude remains 2 because the second coordinate of  $\mathbf{v}$  is always 2.
43. [M] In general,  $\det(A + B)$  is not equal to  $\det A + \det B$ .
45. [M] You can check your conjectures when you get to Section 3.2.

### Section 3.2, page 175

1. Interchanging two rows reverses the sign of the determinant.
3. A row replacement operation does not change the determinant.
5. 3      7. 0      9. 3      11. 120
13. 6      15. 35      17. -7      19. 14
21. Invertible      23. Not invertible
25. Linearly independent      27. See the *Study Guide*.
29. -32
31. *Hint*: Show that  $(\det A)(\det A^{-1}) = 1$ .
33. *Hint*: Use Theorem 6.
35. *Hint*: Use Theorem 6 and another theorem.
37.  $\det AB = \det \begin{bmatrix} 6 & 0 \\ 17 & 4 \end{bmatrix} = 24$ ;  
 $(\det A)(\det B) = 3 \cdot 8 = 24$
39. a. -12      b. 500      c. -3      d.  $\frac{1}{4}$       e. 64
41.  $\det A = (a + e)d - (b + f)c = ad + ed - bc - fc = (ad - bc) + (ed - fc) = \det B + \det C$
43. *Hint*: Compute  $\det A$  by a cofactor expansion down column 3.
45. [M] See the *Study Guide* after you have made a conjecture about  $A^T A$  and  $AA^T$ .

### Section 3.3, page 184

1.  $\begin{bmatrix} 5/6 \\ -1/6 \end{bmatrix}$       3.  $\begin{bmatrix} 4 \\ 5/2 \end{bmatrix}$       5.  $\begin{bmatrix} 3/2 \\ 4 \\ -7/2 \end{bmatrix}$
7.  $s \neq \pm\sqrt{3}$ ;  $x_1 = \frac{5s + 4}{6(s^2 - 3)}$ ,  $x_2 = \frac{-4s - 15}{4(s^2 - 3)}$
9.  $s \neq 0, -1$ ;  $x_1 = \frac{1}{3(s + 1)}$ ,  $x_2 = \frac{4s + 3}{6s(s + 1)}$
11.  $\text{adj } A = \begin{bmatrix} 0 & 1 & 0 \\ -3 & -1 & -3 \\ 3 & 2 & 6 \end{bmatrix}$ ,  $A^{-1} = \frac{1}{3} \begin{bmatrix} 0 & 1 & 0 \\ -3 & -1 & -3 \\ 3 & 2 & 6 \end{bmatrix}$
13.  $\text{adj } A = \begin{bmatrix} -1 & -1 & 5 \\ 1 & -5 & 1 \\ 1 & 7 & -5 \end{bmatrix}$ ,  $A^{-1} = \frac{1}{6} \begin{bmatrix} -1 & -1 & 5 \\ 1 & -5 & 1 \\ 1 & 7 & -5 \end{bmatrix}$
15.  $\text{adj } A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 6 & 0 \\ -1 & -9 & 3 \end{bmatrix}$ ,  $A^{-1} = \frac{1}{6} \begin{bmatrix} 2 & 0 & 0 \\ 2 & 6 & 0 \\ -1 & -9 & 3 \end{bmatrix}$
17. If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , then  $C_{11} = d$ ,  $C_{12} = -c$ ,  $C_{21} = -b$ ,  $C_{22} = a$ . The adjugate matrix is the transpose of cofactors:  
 $\text{adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$



### A32 Answers to Odd-Numbered Exercises

Following Theorem 8, we divide by  $\det A$ ; this produces the formula from Section 2.2.

19. 8                      21. 14                      23. 22

25. A  $3 \times 3$  matrix  $A$  is not invertible if and only if its columns are linearly dependent (by the Invertible Matrix Theorem). This happens if and only if one of the columns is in the plane spanned by the other two columns, which is equivalent to the condition that the parallelepiped determined by these columns has zero volume, which in turn is equivalent to the condition that  $\det A = 0$ .

27. 24                      29.  $\frac{1}{2} |\det [\mathbf{v}_1 \quad \mathbf{v}_2]|$

31. a. See Example 5.      b.  $4\pi abc/3$

33. [M] In MATLAB, the entries in  $B - \text{inv}(A)$  are approximately  $10^{-15}$  or smaller. See the *Study Guide* for suggestions that may save you keystrokes as you work.

35. [M] MATLAB Student Version 4.0 uses 57,771 flops for  $\text{inv}(A)$ , and 14,269,045 flops for the inverse formula. The `inv(A)` command requires only about 0.4% of the operations for the inverse formula. The *Study Guide* shows how to use the `flops` command.

### Chapter 3 Supplementary Exercises, page 185

1. a. T    b. T    c. F    d. F    e. F    f. F  
g. T    h. T    i. F    j. F    k. T    l. F  
m. F    n. T    o. F    p. T

The solution for Exercise 3 is based on the fact that if a matrix contains two rows (or two columns) that are multiples of each other, then the determinant of the matrix is zero, by Theorem 4, because the matrix cannot be invertible.

3. Make two row replacement operations, and then factor out a common multiple in row 2 and a common multiple in row 3.

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & b+c \\ 0 & b-a & a-b \\ 0 & c-a & a-c \end{vmatrix} \\ = (b-a)(c-a) \begin{vmatrix} 1 & a & b+c \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{vmatrix} \\ = 0$$

5. -12

7. When the determinant is expanded by cofactors of the first row, the equation has the form  $ax + by + c = 0$ , where at least one of  $a$  and  $b$  is not zero. This is the equation of a line. It is clear that  $(x_1, y_1)$  and  $(x_2, y_2)$  are on the line, because when the coordinates of one of the points are substituted for  $x$  and  $y$ , two rows of the matrix are equal and so the determinant is zero.

9.  $T \sim \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix}$ . Thus, by Theorem 3,

$$\begin{aligned} \det T &= (b-a)(c-a) \det \begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{bmatrix} \\ &= (b-a)(c-a) \det \begin{bmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{bmatrix} \\ &= (b-a)(c-a)(c-b) \end{aligned}$$

11. Area = 12. If one vertex is subtracted from all four vertices, and if the new vertices are  $\mathbf{0}$ ,  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ , and  $\mathbf{v}_3$ , then the translated figure (and hence the original figure) will be a parallelogram if and only if one of  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ , and  $\mathbf{v}_3$  is the sum of the other two vectors.

13. By the Inverse Formula,  $(\text{adj } A) \cdot \frac{1}{\det A} A = A^{-1}A = I$ . By the Invertible Matrix Theorem,  $\text{adj } A$  is invertible and  $(\text{adj } A)^{-1} = \frac{1}{\det A} A$ .

15. a.  $X = CA^{-1}$ ,  $Y = D - CA^{-1}B$ . Now use Exercise 14(c).

- b. From part (a), and the multiplicative property of determinants,

$$\begin{aligned} \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \det [A(D - CA^{-1}B)] \\ &= \det [AD - ACA^{-1}B] \\ &= \det [AD - CAA^{-1}B] \\ &= \det [AD - CB] \end{aligned}$$

where the equality  $AC = CA$  was used in the third step.

17. First consider the case  $n = 2$ , and prove that the result holds by directly computing the determinants of  $B$  and  $C$ . Now assume that the formula holds for all  $(k-1) \times (k-1)$  matrices, and let  $A$ ,  $B$ , and  $C$  be  $k \times k$  matrices. Use a cofactor expansion along the first column and the inductive hypothesis to find  $\det B$ . Use row replacement operations on  $C$  to create zeros below the first pivot and produce a triangular matrix. Find the determinant of this matrix and add it to  $\det B$  to get the result.

19. [M] Compute:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = 1, \quad \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{vmatrix} = 1,$$

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{vmatrix} = 1$$

Conjecture:

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & & 2 \\ 1 & 2 & 3 & & 3 \\ \vdots & & & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{vmatrix} = 1$$

To confirm the conjecture, use row replacement operations to create zeros below the first pivot, then the second pivot, and so on. The resulting matrix is

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & & 1 \\ 0 & 0 & 1 & & 1 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

which is an upper triangular matrix with determinant 1.

## Chapter 4

### Section 4.1, page 195

1. **a.**  $\mathbf{u} + \mathbf{v}$  is in  $V$  because its entries will both be nonnegative.
- b.** *Example:* If  $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$  and  $c = -1$ , then  $\mathbf{u}$  is in  $V$ , but  $c\mathbf{u}$  is not in  $V$ .
3. *Example:* If  $\mathbf{u} = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$  and  $c = 4$ , then  $\mathbf{u}$  is in  $H$ , but  $c\mathbf{u}$  is not in  $H$ .
5. Yes, by Theorem 1, because the set is  $\text{Span}\{t^2\}$ .
7. No, the set is not closed under multiplication by scalars that are not integers.
9.  $H = \text{Span}\{\mathbf{v}\}$ , where  $\mathbf{v} = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$ . By Theorem 1,  $H$  is a subspace of  $\mathbb{R}^3$ .
11.  $W = \text{Span}\{\mathbf{u}, \mathbf{v}\}$ , where  $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ ,  $\mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$ . By Theorem 1,  $W$  is a subspace of  $\mathbb{R}^3$ .
13. **a.** There are only three vectors in  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ , and  $\mathbf{w}$  is not one of them.
- b.** There are infinitely many vectors in  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ .
- c.**  $\mathbf{w}$  is in  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  because  $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2$ .
15.  $W$  is not a vector space because the zero vector is not in  $W$ .
17.  $S = \left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \\ 0 \end{bmatrix} \right\}$
19. *Hint:* Use Theorem 1.

**Warning:** Although the *Study Guide* has complete solutions for every odd-numbered exercise whose answer here is only a “Hint,” you *must* really try to work the solution yourself. Otherwise, you will not benefit from the exercise.

21. Yes. The conditions for a subspace are obviously satisfied: The zero matrix is in  $H$ , the sum of two upper triangular matrices is upper triangular, and any scalar multiple of an upper triangular matrix is again upper triangular.
23. See the *Study Guide* after you have written your answers.
25. 4      27. **a.** 8    **b.** 3    **c.** 5    **d.** 4
29.  $\mathbf{u} + (-1)\mathbf{u} = 1\mathbf{u} + (-1)\mathbf{u}$     Axiom 10  
 $= [1 + (-1)]\mathbf{u}$     Axiom 8  
 $= 0\mathbf{u} = \mathbf{0}$     Exercise 27  
 From Exercise 26, it follows that  $(-1)\mathbf{u} = -\mathbf{u}$ .
31. Any subspace  $H$  that contains  $\mathbf{u}$  and  $\mathbf{v}$  must also contain all scalar multiples of  $\mathbf{u}$  and  $\mathbf{v}$  and hence must contain all sums of scalar multiples of  $\mathbf{u}$  and  $\mathbf{v}$ . Thus  $H$  must contain  $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ .
33. *Hint:* For part of the solution, consider  $\mathbf{w}_1$  and  $\mathbf{w}_2$  in  $H + K$ , and write  $\mathbf{w}_1$  and  $\mathbf{w}_2$  in the form  $\mathbf{w}_1 = \mathbf{u}_1 + \mathbf{v}_1$  and  $\mathbf{w}_2 = \mathbf{u}_2 + \mathbf{v}_2$ , where  $\mathbf{u}_1$  and  $\mathbf{u}_2$  are in  $H$ , and  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are in  $K$ .
35. **[M]** The reduced echelon form of  $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}]$  shows that  $\mathbf{w} = \mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3$ . Hence  $\mathbf{w}$  is in the subspace spanned by  $\mathbf{v}_1, \mathbf{v}_2$ , and  $\mathbf{v}_3$ .
37. **[M]** The functions are  $\cos 4t$  and  $\cos 6t$ . See Exercise 34 in Section 4.5.

### Section 4.2, page 205

1.  $\begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ,  
 so  $\mathbf{w}$  is in  $\text{Nul } A$ .
3.  $\begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix}$     5.  $\begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}$
7.  $W$  is not a subspace of  $\mathbb{R}^3$  because the zero vector  $(0, 0, 0)$  is not in  $W$ .
9.  $W$  is a subspace of  $\mathbb{R}^4$  because  $W$  is the set of solutions of the system  
 $p - 3q - 4s = 0$   
 $2p - s - 5r = 0$
11.  $W$  is not a subspace because  $\mathbf{0}$  is not in  $W$ . *Justification:* If a typical element  $(s - 2t, 3 + 3s, 3s + t, 2s)$  were zero, then  $3 + 3s = 0$  and  $2s = 0$ , which is impossible.

### A34 Answers to Odd-Numbered Exercises

13.  $W = \text{Col } A$  for  $A = \begin{bmatrix} 1 & -6 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ , so  $W$  is a vector space by

Theorem 3.

15.  $\begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \\ 2 & -1 & -1 \end{bmatrix}$

17. a. 2      b. 4      19. a. 5      b. 2

21. The vector  $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$  is in  $\text{Nul } A$  and the vector  $\begin{bmatrix} 6 \\ -3 \\ -9 \\ 9 \end{bmatrix}$  is in

$\text{Col } A$ . Other answers are possible.

23.  $\mathbf{w}$  is in both  $\text{Nul } A$  and  $\text{Col } A$ .

25. See the *Study Guide*. By now you should know how to use it properly.

27. Let  $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$  and  $A = \begin{bmatrix} 1 & -3 & -3 \\ -2 & 4 & 2 \\ -1 & 5 & 7 \end{bmatrix}$ . Then  $\mathbf{x}$  is in

$\text{Nul } A$ . Since  $\text{Nul } A$  is a subspace of  $\mathbb{R}^3$ ,  $10\mathbf{x}$  is in  $\text{Nul } A$ .

29. a.  $A\mathbf{0} = \mathbf{0}$ , so the zero vector is in  $\text{Col } A$ .

- b. By a property of matrix multiplication,  $A\mathbf{x} + A\mathbf{w} = A(\mathbf{x} + \mathbf{w})$ , which shows that  $A\mathbf{x} + A\mathbf{w}$  is a linear combination of the columns of  $A$  and hence is in  $\text{Col } A$ .

- c.  $c(A\mathbf{x}) = A(c\mathbf{x})$ , which shows that  $c(A\mathbf{x})$  is in  $\text{Col } A$  for all scalars  $c$ .

31. a. For arbitrary polynomials  $\mathbf{p}, \mathbf{q}$  in  $\mathbb{P}_2$  and any scalar  $c$ ,

$$\begin{aligned} T(\mathbf{p} + \mathbf{q}) &= \begin{bmatrix} (\mathbf{p} + \mathbf{q})(0) \\ (\mathbf{p} + \mathbf{q})(1) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(0) + \mathbf{q}(0) \\ \mathbf{p}(1) + \mathbf{q}(1) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} + \begin{bmatrix} \mathbf{q}(0) \\ \mathbf{q}(1) \end{bmatrix} = T(\mathbf{p}) + T(\mathbf{q}) \\ T(c\mathbf{p}) &= \begin{bmatrix} c\mathbf{p}(0) \\ c\mathbf{p}(1) \end{bmatrix} = c \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} = cT(\mathbf{p}) \end{aligned}$$

So  $T$  is a linear transformation from  $\mathbb{P}_2$  into  $\mathbb{P}_2$ .

- b. Any quadratic polynomial that vanishes at 0 and 1 must be a multiple of  $\mathbf{p}(t) = t(t-1)$ . The range of  $T$  is  $\mathbb{R}^2$ .

33. a. For  $A, B$  in  $M_{2 \times 2}$  and any scalar  $c$ ,

$$\begin{aligned} T(A + B) &= (A + B) + (A + B)^T \\ &= A + B + A^T + B^T \quad \text{Transpose property} \\ &= (A + A^T) + (B + B^T) = T(A) + T(B) \\ T(cA) &= (cA) + (cA)^T = cA + cA^T \\ &= c(A + A^T) = cT(A) \end{aligned}$$

So  $T$  is a linear transformation from  $M_{2 \times 2}$  into  $M_{2 \times 2}$ .

- b. If  $B$  is any element in  $M_{2 \times 2}$  with the property that  $B^T = B$ , and if  $A = \frac{1}{2}B$ , then

$$T(A) = \frac{1}{2}B + \left(\frac{1}{2}B\right)^T = \frac{1}{2}B + \frac{1}{2}B = B$$

- c. Part (b) showed that the range of  $T$  contains all  $B$  such that  $B^T = B$ . So it suffices to show that any  $B$  in the range of  $T$  has this property. If  $B = T(A)$ , then by properties of transposes,

$$B^T = (A + A^T)^T = A^T + A^{TT} = A^T + A = B$$

- d. The kernel of  $T$  is  $\left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} : b \text{ real} \right\}$ .

35. *Hint:* Check the three conditions for a subspace. Typical elements of  $T(U)$  have the form  $T(\mathbf{u}_1)$  and  $T(\mathbf{u}_2)$ , where  $\mathbf{u}_1, \mathbf{u}_2$  are in  $U$ .

37. [M]  $\mathbf{w}$  is in  $\text{Col } A$  but not in  $\text{Nul } A$ . (Explain why.)

39. [M] The reduced echelon form of  $A$  is

$$\begin{bmatrix} 1 & 0 & 1/3 & 0 & 10/3 \\ 0 & 1 & 1/3 & 0 & -26/3 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

### Section 4.3, page 213

1. The  $3 \times 3$  matrix  $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$  has three pivot

positions. By the Invertible Matrix Theorem,  $A$  is invertible and its columns form a basis for  $\mathbb{R}^3$ . (See Example 3.)

3. This set does not form a basis for  $\mathbb{R}^3$ . The set is linearly dependent and does not span  $\mathbb{R}^3$ .

5. This set does not form a basis for  $\mathbb{R}^3$ . The set is linearly dependent because the zero vector is in the set. However,

$$\begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 7 & 0 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 4 & 0 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

The matrix has a pivot in each row and hence its columns span  $\mathbb{R}^3$ .

7. This set does not form a basis for  $\mathbb{R}^3$ . The set is linearly independent because one vector is not a multiple of the other. However, the vectors do not span  $\mathbb{R}^3$ . The matrix

$$\begin{bmatrix} -2 & 6 \\ 3 & -1 \\ 0 & 5 \end{bmatrix}$$

can have at most two pivots since it has only two columns. So there will not be a pivot in each row.

9.  $\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$       11.  $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

13. Basis for  $\text{Nul } A$ :  $\begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix}$

Basis for  $\text{Col } A$ :  $\begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix}$

15.  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_5\}$     17. [M]  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5\}$
19. The three simplest answers are  $\{\mathbf{v}_1, \mathbf{v}_2\}$ ,  $\{\mathbf{v}_1, \mathbf{v}_3\}$ , and  $\{\mathbf{v}_2, \mathbf{v}_3\}$ . Other answers are possible.
21. See the *Study Guide* for hints.
23. *Hint*: Use the Invertible Matrix Theorem.
25. No. (Why is the set not a basis for  $H$ ?)
27.  $\{\cos \omega t, \sin \omega t\}$
29. Let  $A$  be the  $n \times k$  matrix  $[\mathbf{v}_1 \ \cdots \ \mathbf{v}_k]$ . Since  $A$  has fewer columns than rows, there cannot be a pivot position in each row of  $A$ . By Theorem 4 in Section 1.4, the columns of  $A$  do not span  $\mathbb{R}^n$  and hence are not a basis for  $\mathbb{R}^n$ .
31. *Hint*: If  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is linearly dependent, then there exist  $c_1, \dots, c_p$ , not all zero, such that  $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$ . Use this equation.
33. Neither polynomial is a multiple of the other polynomial, so  $\{\mathbf{p}_1, \mathbf{p}_2\}$  is a linearly independent set in  $\mathbb{P}_3$ .
35. Let  $\{\mathbf{v}_1, \mathbf{v}_3\}$  be any linearly independent set in the vector space  $V$ , and let  $\mathbf{v}_2$  and  $\mathbf{v}_4$  be linear combinations of  $\mathbf{v}_1$  and  $\mathbf{v}_3$ . Then  $\{\mathbf{v}_1, \mathbf{v}_3\}$  is a basis for  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ .
37. [M] You could be clever and find special values of  $t$  that produce several zeros in (5), and thereby create a system of equations that can be solved easily by hand. Or, you could use values of  $t$  such as  $t = 0, .1, .2, \dots$  to create a system of equations that you can solve with a matrix program.

## Section 4.4, page 222

1.  $\begin{bmatrix} 3 \\ -7 \end{bmatrix}$     3.  $\begin{bmatrix} -7 \\ 4 \\ 3 \end{bmatrix}$     5.  $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$     7.  $\begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$
9.  $\begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix}$     11.  $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$     13.  $\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$
15. The *Study Guide* has hints.
17.  $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 5\mathbf{v}_1 - 2\mathbf{v}_2 = 10\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3 = -\mathbf{v}_2 - \mathbf{v}_3$   
(infinitely many answers)
19. *Hint*: By hypothesis, the zero vector has a unique representation as a linear combination of elements of  $S$ .
21.  $\begin{bmatrix} 9 & 2 \\ 4 & 1 \end{bmatrix}$
23. *Hint*: Suppose that  $[\mathbf{u}]_B = [\mathbf{w}]_B$  for some  $\mathbf{u}$  and  $\mathbf{w}$  in  $V$ , and denote the entries in  $[\mathbf{u}]_B$  by  $c_1, \dots, c_n$ . Use the definition of  $[\mathbf{u}]_B$ .
25. One possible approach: First, show that if  $\mathbf{u}_1, \dots, \mathbf{u}_p$  are linearly dependent, then  $[\mathbf{u}_1]_B, \dots, [\mathbf{u}_p]_B$  are linearly dependent. Second, show that if  $[\mathbf{u}_1]_B, \dots, [\mathbf{u}_p]_B$  are linearly dependent, then  $\mathbf{u}_1, \dots, \mathbf{u}_p$  are linearly dependent. Use the two equations displayed in the exercise. A slightly different proof is given in the *Study Guide*.

27. Linearly independent. (Justify answers to Exercises 27–34.)
29. Linearly dependent.

31. a. The coordinate vectors  $\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ -6 \end{bmatrix},$

$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$  do not span  $\mathbb{R}^3$ . Because of the isomorphism between  $\mathbb{R}^3$  and  $\mathbb{P}_2$ , the corresponding polynomials do not span  $\mathbb{P}_2$ .

b. The coordinate vectors  $\begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -8 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$  span  $\mathbb{R}^3$ . Because of the isomorphism between  $\mathbb{R}^3$  and  $\mathbb{P}_2$ , the corresponding polynomials span  $\mathbb{P}_2$ .

33. [M] The coordinate vectors  $\begin{bmatrix} 3 \\ 7 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix},$

$\begin{bmatrix} 1 \\ 16 \\ -6 \\ 2 \end{bmatrix}$  are a linearly dependent subset of  $\mathbb{R}^4$ . Because of the isomorphism between  $\mathbb{R}^4$  and  $\mathbb{P}_3$ , the corresponding polynomials form a linearly dependent subset of  $\mathbb{P}_3$ , and thus cannot be a basis for  $\mathbb{P}_3$ .

35. [M]  $[\mathbf{x}]_B = \begin{bmatrix} -5/3 \\ 8/3 \end{bmatrix}$     37. [M]  $\begin{bmatrix} 1.3 \\ 0 \\ 0.8 \end{bmatrix}$

## Section 4.5, page 229

1.  $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$ ; dim is 2
3.  $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}$ ; dim is 3
5.  $\begin{bmatrix} 1 \\ 2 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 2 \\ 6 \end{bmatrix}$ ; dim is 3
7. No basis; dim is 0    9. 2    11. 3    13. 2, 3
15. 2, 3    17. 0, 3
19. See the *Study Guide*.
21. *Hint*: You need only show that the first four Hermite polynomials are linearly independent. Why?
23.  $[\mathbf{p}]_B = (3, 6, 2, 1)$
25. *Hint*: Suppose  $S$  does span  $V$ , and use the Spanning Set Theorem. This leads to a contradiction, which shows that the spanning hypothesis is false.

## A36 Answers to Odd-Numbered Exercises

27. *Hint:* Use the fact that each  $\mathbb{P}_n$  is a subspace of  $\mathbb{P}$ .

29. Justify each answer.

- a. True      b. True      c. True

31. *Hint:* Since  $H$  is a nonzero subspace of a finite-dimensional space,  $H$  is finite-dimensional and has a basis, say,  $\mathbf{v}_1, \dots, \mathbf{v}_p$ . First show that  $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)\}$  spans  $T(H)$ .

33. [M] a. One basis is  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{e}_2, \mathbf{e}_3\}$ . In fact, any two of the vectors  $\mathbf{e}_2, \dots, \mathbf{e}_5$  will extend  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  to a basis of  $\mathbb{R}^5$ .

### Section 4.6, page 236

1.  $\text{rank } A = 2$ ;  $\dim \text{Nul } A = 2$ ;

$$\text{Basis for Col } A: \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix}$$

Basis for Row  $A$ :  $(1, 0, -1, 5), (0, -2, 5, -6)$

$$\text{Basis for Nul } A: \begin{bmatrix} 1 \\ 5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

3.  $\text{rank } A = 3$ ;  $\dim \text{Nul } A = 3$ ;

$$\text{Basis for Col } A: \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ 3 \end{bmatrix}$$

Basis for Row  $A$ :  $(2, 6, -6, 6, 3, 6), (0, 3, 0, 3, 3, 0), (0, 0, 0, 0, 3, 0)$

$$\text{Basis for Nul } A: \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

5. 4, 3, 3

7. Yes; no. Since  $\text{Col } A$  is a four-dimensional subspace of  $\mathbb{R}^4$ , it coincides with  $\mathbb{R}^4$ . The null space cannot be  $\mathbb{R}^3$ , because the vectors in  $\text{Nul } A$  have 7 entries.  $\text{Nul } A$  is a three-dimensional subspace of  $\mathbb{R}^7$ , by the Rank Theorem.

9. 3, no. Notice that the columns of a  $4 \times 6$  matrix are in  $\mathbb{R}^4$ , rather than  $\mathbb{R}^3$ .  $\text{Col } A$  is a three-dimensional subspace of  $\mathbb{R}^4$ .

11. 2

13. 5, 5. In both cases, the number of pivots cannot exceed the number of columns or the number of rows.

15. 4      17. See the *Study Guide*.

19. Yes. Try to write an explanation before you consult the *Study Guide*.

21. No. Explain why.

23. Yes. Only six homogeneous linear equations are necessary.

25. No. Explain why.

27. Row  $A$  and  $\text{Nul } A$  are in  $\mathbb{R}^n$ ;  $\text{Col } A$  and  $\text{Nul } A^T$  are in  $\mathbb{R}^m$ . There are only four distinct subspaces because  $\text{Row } A^T = \text{Col } A$  and  $\text{Col } A^T = \text{Row } A$ .

29. Recall that  $\dim \text{Col } A = m$  precisely when  $\text{Col } A = \mathbb{R}^m$ , or equivalently, when the equation  $A\mathbf{x} = \mathbf{b}$  is consistent for all  $\mathbf{b}$ . By Exercise 28(b),  $\dim \text{Col } A = m$  precisely when  $\dim \text{Nul } A^T = 0$ , or equivalently, when the equation  $A^T\mathbf{x} = \mathbf{0}$  has only the trivial solution.

31.  $\mathbf{u}\mathbf{v}^T = \begin{bmatrix} 2a & 2b & 2c \\ -3a & -3b & -3c \\ 5a & 5b & 5c \end{bmatrix}$ . The columns are all multiples of  $\mathbf{u}$ , so  $\text{Col } \mathbf{u}\mathbf{v}^T$  is one-dimensional, unless  $a = b = c = 0$ .

33. *Hint:* Let  $A = [\mathbf{u} \ \mathbf{u}_2 \ \mathbf{u}_3]$ . If  $\mathbf{u} \neq \mathbf{0}$ , then  $\mathbf{u}$  is a basis for  $\text{Col } A$ . Why?

35. [M] *Hint:* See Exercise 28 and the remarks before Example 4.

37. [M] The matrices  $C$  and  $R$  given for Exercise 35 work here, and  $A = CR$ .

### Section 4.7, page 242

1. a.  $\begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}$       b.  $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$

3. (ii)

5. a.  $\begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$       b.  $\begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}$

7.  ${}_{C \leftarrow B} P = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$ ,  ${}_{B \leftarrow C} P = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$

9.  ${}_{C \leftarrow B} P = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$ ,  ${}_{B \leftarrow C} P = \frac{1}{2} \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$

11. See the *Study Guide*.

13.  ${}_{C \leftarrow B} P = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}$ ,  $[-1 + 2t]_B = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$

15. a.  $\mathcal{B}$  is a basis for  $V$ .

b. The coordinate mapping is a linear transformation.

c. The product of a matrix and a vector

d. The coordinate vector of  $\mathbf{v}$  relative to  $\mathcal{B}$

17. a. [M]

$$P^{-1} = \frac{1}{32} \begin{bmatrix} 32 & 0 & 16 & 0 & 12 & 0 & 10 \\ & 32 & 0 & 24 & 0 & 20 & 0 \\ & & 16 & 0 & 16 & 0 & 15 \\ & & & 8 & 0 & 10 & 0 \\ & & & & 4 & 0 & 6 \\ & & & & & 2 & 0 \\ & & & & & & 1 \end{bmatrix}$$

$$\begin{aligned}\text{b. } \cos^2 t &= (1/2)[1 + \cos 2t] \\ \cos^3 t &= (1/4)[3 \cos t + \cos 3t] \\ \cos^4 t &= (1/8)[3 + 4 \cos 2t + \cos 4t] \\ \cos^5 t &= (1/16)[10 \cos t + 5 \cos 3t + \cos 5t] \\ \cos^6 t &= (1/32)[10 + 15 \cos 2t + 6 \cos 4t + \cos 6t]\end{aligned}$$

19. [M] *Hint:* Let  $\mathcal{C}$  be the basis  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ . Then the columns of  $P$  are  $[\mathbf{u}_1]_{\mathcal{C}}$ ,  $[\mathbf{u}_2]_{\mathcal{C}}$ , and  $[\mathbf{u}_3]_{\mathcal{C}}$ . Use the definition of  $\mathcal{C}$ -coordinate vectors and matrix algebra to compute  $\mathbf{u}_1$ ,  $\mathbf{u}_2$ ,  $\mathbf{u}_3$ . The solution method is discussed in the *Study Guide*. Here are the numerical answers:

$$\begin{aligned}\text{a. } \mathbf{u}_1 &= \begin{bmatrix} -6 \\ -5 \\ 21 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -6 \\ -9 \\ 32 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix} \\ \text{b. } \mathbf{w}_1 &= \begin{bmatrix} 28 \\ -9 \\ -3 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 38 \\ -13 \\ 2 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 21 \\ -7 \\ 3 \end{bmatrix}\end{aligned}$$

### Section 4.8, page 251

1. If  $y_k = 2^k$ , then  $y_{k+1} = 2^{k+1}$  and  $y_{k+2} = 2^{k+2}$ . Substituting these formulas into the left side of the equation gives

$$\begin{aligned}y_{k+2} + 2y_{k+1} - 8y_k &= 2^{k+2} + 2 \cdot 2^{k+1} - 8 \cdot 2^k \\ &= 2^k(2^2 + 2 \cdot 2 - 8) \\ &= 2^k(0) = 0 \quad \text{for all } k\end{aligned}$$

Since the difference equation holds for all  $k$ ,  $2^k$  is a solution. A similar calculation works for  $y_k = (-4)^k$ .

3. The signals  $2^k$  and  $(-4)^k$  are linearly independent because neither is a multiple of the other. For instance, there is no scalar  $c$  such that  $2^k = c(-4)^k$  for all  $k$ . By Theorem 17, the solution set  $H$  of the difference equation in Exercise 1 is two-dimensional. By the Basis Theorem in Section 4.5, the two linearly independent signals  $2^k$  and  $(-4)^k$  form a basis for  $H$ .
5. If  $y_k = (-2)^k$ , then

$$\begin{aligned}y_{k+2} + 4y_{k+1} + 4y_k &= (-2)^{k+2} + 4(-2)^{k+1} + 4(-2)^k \\ &= (-2)^k[(-2)^2 + 4(-2) + 4] \\ &= (-2)^k(0) = 0 \quad \text{for all } k\end{aligned}$$

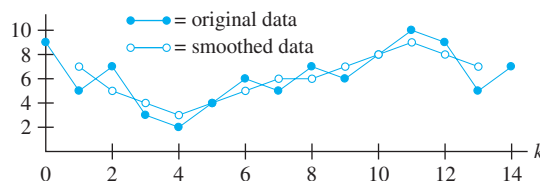
Similarly, if  $y_k = k(-2)^k$ , then

$$\begin{aligned}y_{k+2} + 4y_{k+1} + 4y_k &= (k+2)(-2)^{k+2} + 4(k+1)(-2)^{k+1} + 4k(-2)^k \\ &= (-2)^k[(k+2)(-2)^2 + 4(k+1)(-2) + 4k] \\ &= (-2)^k[4k + 8 - 8k - 8 + 4k] \\ &= (-2)^k(0) \quad \text{for all } k\end{aligned}$$

Thus both  $(-2)^k$  and  $k(-2)^k$  are in the solution space  $H$  of the difference equation. Also, there is no scalar  $c$  such that  $k(-2)^k = c(-2)^k$  for all  $k$ , because  $c$  must be chosen independently of  $k$ . Likewise, there is no scalar  $c$  such that  $(-2)^k = ck(-2)^k$  for all  $k$ . So the two signals are linearly

independent. Since  $\dim H = 2$ , the signals form a basis for  $H$ , by the Basis Theorem.

7. Yes                      9. Yes
11. No, two signals cannot span the three-dimensional solution space.
13.  $(\frac{1}{3})^k, (\frac{2}{3})^k$                       15.  $(\frac{1}{2})^k, (-\frac{2}{3})^k$
17.  $Y_k = c_1(.8)^k + c_2(.5)^k + 10 \rightarrow 10$  as  $k \rightarrow \infty$
19.  $y_k = c_1(-2 + \sqrt{3})^k + c_2(-2 - \sqrt{3})^k$
21. 7, 5, 4, 3, 4, 5, 6, 6, 7, 8, 9, 8, 7; see figure:



23. a.  $y_{k+1} - 1.01y_k = -450$ ,  $y_0 = 10,000$
25.  $k^2 + c_1 \cdot (-4)^k + c_2$                       27.  $-2 + k + c_1 \cdot 2^k + c_2 \cdot (-2)^k$
29.  $\mathbf{x}_{k+1} = A\mathbf{x}_k$ , where
- $$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & -6 & 8 & -3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \\ y_{k+3} \end{bmatrix}$$
31. The equation holds for all  $k$ , so it holds with  $k$  replaced by  $k-1$ , which transforms the equation into
- $$y_{k+2} + 5y_{k+1} + 6y_k = 0 \quad \text{for all } k$$
- The equation is of order 2.
33. For all  $k$ , the Casorati matrix  $C(k)$  is not invertible. In this case, the Casorati matrix gives no information about the linear independence/dependence of the set of signals. In fact, neither signal is a multiple of the other, so they are linearly independent.
35. *Hint:* Verify the two properties that define a linear transformation. For  $\{y_k\}$  and  $\{z_k\}$  in  $\mathbb{S}$ , study  $T(\{y_k\} + \{z_k\})$ . Note that if  $r$  is any scalar, then the  $k$ th term of  $r\{y_k\}$  is  $ry_k$ ; so  $T(r\{y_k\})$  is the sequence  $\{w_k\}$  given by
- $$w_k = ry_{k+2} + a(ry_{k+1}) + b(ry_k)$$
37. *Hint:* Find  $TD(y_0, y_1, y_2, \dots)$  and  $DT(y_0, y_1, y_2, \dots)$ .

### Section 4.9, page 260

1. a. From: 

N	M	To:
.7	.6	News
.3	.4	Music
- b.  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$                       c. 33%
3. a. From: 

H	I	To:
.95	.45	Healthy
.05	.55	Ill
- b. 15%, 12.5%

### A38 Answers to Odd-Numbered Exercises

- c. .925; use  $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .
5.  $\begin{bmatrix} 5/14 \\ 9/14 \end{bmatrix}$       7.  $\begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}$
9. Yes, because  $P^2$  has all positive entries.
11. a.  $\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$       b.  $2/3$
13. a.  $\begin{bmatrix} .9 \\ .1 \end{bmatrix}$       b. .10, no
15. [M] About 13.9% of the United States population
17. a. The entries in a column of  $P$  sum to 1. A column in the matrix  $P - I$  has the same entries as in  $P$  except that one of the entries is decreased by 1. Hence each column sum is 0.
- b. By part (a), the bottom row of  $P - I$  is the negative of the sum of the other rows.
- c. By part (b) and the Spanning Set Theorem, the bottom row of  $P - I$  can be removed and the remaining  $(n - 1)$  rows will still span the row space. Alternatively, use part (a) and the fact that row operations do not change the row space. Let  $A$  be the matrix obtained from  $P - I$  by adding to the bottom row all the other rows. By part (a), the row space is spanned by the first  $(n - 1)$  rows of  $A$ .
- d. By the Rank Theorem and part (c), the dimension of the column space of  $P - I$  is less than  $n$ , and hence the null space is nontrivial. Instead of the Rank Theorem, you may use the Invertible Matrix Theorem, since  $P - I$  is a square matrix.
19. a. The product  $S\mathbf{x}$  equals the sum of the entries in  $\mathbf{x}$ . For a probability vector, this sum must be 1.
- b.  $P = [\mathbf{p}_1 \ \mathbf{p}_2 \ \cdots \ \mathbf{p}_n]$ , where the  $\mathbf{p}_i$  are probability vectors. By matrix multiplication and part (a),  
 $SP = [S\mathbf{p}_1 \ S\mathbf{p}_2 \ \cdots \ S\mathbf{p}_n] = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} = S$
- c. By part (b),  $S(P\mathbf{x}) = (SP)\mathbf{x} = S\mathbf{x} = 1$ . Also, the entries in  $P\mathbf{x}$  are nonnegative (because  $P$  and  $\mathbf{x}$  have nonnegative entries). Hence, by (a),  $P\mathbf{x}$  is a probability vector.

### Chapter 4 Supplementary Exercises, page 262

1. a. T    b. T    c. F    d. F    e. T    f. T  
 g. F    h. F    i. T    j. F    k. F    l. F  
 m. T    n. F    o. T    p. T    q. F    r. T  
 s. T    t. F
3. The set of all  $(b_1, b_2, b_3)$  satisfying  $b_1 + 2b_2 + b_3 = 0$ .
5. The vector  $\mathbf{p}_1$  is not zero and  $\mathbf{p}_2$  is not a multiple of  $\mathbf{p}_1$ , so keep both of these vectors. Since  $\mathbf{p}_3 = 2\mathbf{p}_1 + 2\mathbf{p}_2$ , discard  $\mathbf{p}_3$ . Since  $\mathbf{p}_4$  has a  $t^2$  term, it cannot be a linear combination of  $\mathbf{p}_1$  and  $\mathbf{p}_2$ , so keep  $\mathbf{p}_4$ . Finally,  $\mathbf{p}_5 = \mathbf{p}_1 + \mathbf{p}_4$ , so discard  $\mathbf{p}_5$ . The resulting basis is  $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_4\}$ .

7. You would have to know that the solution set of the homogeneous system is spanned by two solutions. In this case, the null space of the  $18 \times 20$  coefficient matrix  $A$  is at most two-dimensional. By the Rank Theorem,  $\dim \text{Col } A \geq 20 - 2 = 18$ , which means that  $\text{Col } A = \mathbb{R}^{18}$ , because  $A$  has 18 rows, and every equation  $A\mathbf{x} = \mathbf{b}$  is consistent.
9. Let  $A$  be the standard  $m \times n$  matrix of the transformation  $T$ .
- a. If  $T$  is one-to-one, then the columns of  $A$  are linearly independent (Theorem 12 in Section 1.9), so  $\dim \text{Nul } A = 0$ . By the Rank Theorem,  $\dim \text{Col } A = \text{rank } A = n$ . Since the range of  $T$  is  $\text{Col } A$ , the dimension of the range of  $T$  is  $n$ .
- b. If  $T$  is onto, then the columns of  $A$  span  $\mathbb{R}^m$  (Theorem 12 in Section 1.9), so  $\dim \text{Col } A = m$ . By the Rank Theorem,  $\dim \text{Nul } A = n - \dim \text{Col } A = n - m$ . Since the kernel of  $T$  is  $\text{Nul } A$ , the dimension of the kernel of  $T$  is  $n - m$ .
11. If  $S$  is a finite spanning set for  $V$ , then a subset of  $S$ —say  $S'$ —is a basis for  $V$ . Since  $S'$  must span  $V$ ,  $S'$  cannot be a proper subset of  $S$  because of the minimality of  $S$ . Thus  $S' = S$ , which proves that  $S$  is a basis for  $V$ .
12. a. *Hint:* Any  $\mathbf{y}$  in  $\text{Col } AB$  has the form  $\mathbf{y} = AB\mathbf{x}$  for some  $\mathbf{x}$ .
13. By Exercise 9,  $\text{rank } PA \leq \text{rank } A$ , and  $\text{rank } A = \text{rank } P^{-1}PA \leq \text{rank } PA$ . Thus  $\text{rank } PA = \text{rank } A$ .
15. The equation  $AB = 0$  shows that each column of  $B$  is in  $\text{Nul } A$ . Since  $\text{Nul } A$  is a subspace, all linear combinations of the columns of  $B$  are in  $\text{Nul } A$ , so  $\text{Col } B$  is a subspace of  $\text{Nul } A$ . By Theorem 11 in Section 4.5,  $\dim \text{Col } B \leq \dim \text{Nul } A$ . Applying the Rank Theorem, we find that  
 $n = \text{rank } A + \dim \text{Nul } A \geq \text{rank } A + \text{rank } B$
17. a. Let  $A_1$  consist of the  $r$  pivot columns in  $A$ . The columns of  $A_1$  are linearly independent. So  $A_1$  is an  $m \times r$  submatrix with rank  $r$ .
- b. By the Rank Theorem applied to  $A_1$ , the dimension of  $\text{Row } A$  is  $r$ , so  $A_1$  has  $r$  linearly independent rows. Use them to form  $A_2$ . Then  $A_2$  is  $r \times r$  with linearly independent rows. By the Invertible Matrix Theorem,  $A_2$  is invertible.

$$19. \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -.9 & .81 \\ 1 & .5 & .25 \end{bmatrix} \\ \sim \begin{bmatrix} 1 & -.9 & .81 \\ 0 & 1 & 0 \\ 0 & 0 & -.56 \end{bmatrix}$$

This matrix has rank 3, so the pair  $(A, B)$  is controllable.

21. [M]  $\text{rank} \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = 3$ . The pair  $(A, B)$  is not controllable.



## Chapter 5

## Section 5.1, page 271

1. Yes      3. Yes,  $\lambda = -2$       5. Yes,  $\lambda = -5$

7. Yes,  $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

9.  $\lambda = 1: \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}; \quad \lambda = 3: \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

11.  $\lambda = -1: \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}; \quad \lambda = 7: \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

13.  $\lambda = 1: \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad \lambda = 2: \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}; \quad \lambda = 3: \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

15.  $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$       17. 0, 3, -2

19. 0. Justify your answer.

21. See the *Study Guide*, after you have written your answers.

23. *Hint:* Use Theorem 2.

25. *Hint:* Use the equation  $A\mathbf{x} = \lambda\mathbf{x}$  to find an equation involving  $A^{-1}$ .

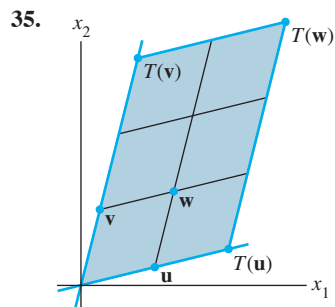
27. *Hint:* For any  $\lambda$ ,  $(A - \lambda I)^T = A^T - \lambda I$ . By a theorem (which one?),  $A^T - \lambda I$  is invertible if and only if  $A - \lambda I$  is invertible.

29. Let  $\mathbf{v}$  be the vector in  $\mathbb{R}^n$  whose entries are all 1's. Then  $A\mathbf{v} = s\mathbf{v}$ .

31. *Hint:* If  $A$  is the standard matrix of  $T$ , look for a nonzero vector  $\mathbf{v}$  (a point in the plane) such that  $A\mathbf{v} = \mathbf{v}$ .

33. a.  $\mathbf{x}_{k+1} = c_1\lambda^{k+1}\mathbf{u} + c_2\mu^{k+1}\mathbf{v}$

b.  $A\mathbf{x}_k = A(c_1\lambda^k\mathbf{u} + c_2\mu^k\mathbf{v})$   
 $= c_1\lambda^k A\mathbf{u} + c_2\mu^k A\mathbf{v}$       *Linearity*  
 $= c_1\lambda^k\lambda\mathbf{u} + c_2\mu^k\mu\mathbf{v}$        *$\mathbf{u}$  and  $\mathbf{v}$  are eigenvectors.*  
 $= \mathbf{x}_{k+1}$



37. [M]  $\lambda = 5: \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}; \quad \lambda = 10: \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}; \quad \lambda = 15: \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

39. [M]  $\lambda = -4: \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}; \quad \lambda = -8: \begin{bmatrix} 6 \\ 3 \\ 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix};$   
 $\lambda = 12: \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

## Section 5.2, page 279

1.  $\lambda^2 - 4\lambda - 45; 9, -5$       3.  $\lambda^2 - 3\lambda - 40; -5, 8$

5.  $\lambda^2 - 16\lambda + 48; 4, 12$

7.  $\lambda^2 - 9\lambda + 32$ ; no real eigenvalues

9.  $-\lambda^3 + 10\lambda^2 - 33\lambda + 36$       11.  $-\lambda^3 + 8\lambda^2 - 19\lambda + 12$

13.  $-\lambda^3 + 18\lambda^2 - 95\lambda + 150$       15. 2, 3, 5, 5

17. 3, 3, 1, 1, 0

19. *Hint:* The equation given holds for all  $\lambda$ .

21. The *Study Guide* has hints.

23. *Hint:* Find an invertible matrix  $P$  such that  $RQ = P^{-1}AP$ .

25. a.  $\{\mathbf{v}_1, \mathbf{v}_2\}$ , where  $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$  is an eigenvector for  $\lambda = .3$

b.  $\mathbf{x}_0 = \mathbf{v}_1 - \frac{1}{14}\mathbf{v}_2$

c.  $\mathbf{x}_1 = \mathbf{v}_1 - \frac{1}{14}(.3)\mathbf{v}_2$ ,  $\mathbf{x}_2 = \mathbf{v}_1 - \frac{1}{14}(.3)^2\mathbf{v}_2$ , and  
 $\mathbf{x}_k = \mathbf{v}_1 - \frac{1}{14}(.3)^k\mathbf{v}_2$ . As  $k \rightarrow \infty$ ,  $(.3)^k \rightarrow 0$  and  
 $\mathbf{x}_k \rightarrow \mathbf{v}_1$ .

27. a.  $A\mathbf{v}_1 = \mathbf{v}_1$ ,  $A\mathbf{v}_2 = .5\mathbf{v}_2$ ,  $A\mathbf{v}_3 = .2\mathbf{v}_3$ . (This also shows that the eigenvalues of  $A$  are 1, .5, and .2.)

b.  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly independent because the eigenvectors correspond to distinct eigenvalues (Theorem 2). Since there are 3 vectors in the set, the set is a basis for  $\mathbb{R}^3$ . So there exist (unique) constants such that

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

Then

$$\mathbf{w}\mathbf{x}_0 = c_1\mathbf{w}^T\mathbf{v}_1 + c_2\mathbf{w}^T\mathbf{v}_2 + c_3\mathbf{w}^T\mathbf{v}_3 \quad (*)$$

Since  $\mathbf{x}_0$  and  $\mathbf{v}_1$  are probability vectors and since the entries in  $\mathbf{v}_2$  and in  $\mathbf{v}_3$  each sum to 0, (\*) shows that  $1 = c_1$ .

c. By part (b),

$$\mathbf{x}_0 = \mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

Using part (a),

$$\begin{aligned} \mathbf{x}_k &= A^k\mathbf{x}_0 = A^k\mathbf{v}_1 + c_2A^k\mathbf{v}_2 + c_3A^k\mathbf{v}_3 \\ &= \mathbf{v}_1 + c_2(.5)^k\mathbf{v}_2 + c_3(.2)^k\mathbf{v}_3 \\ &\rightarrow \mathbf{v}_1 \text{ as } k \rightarrow \infty \end{aligned}$$

## A40 Answers to Odd-Numbered Exercises

29. [M] Report your results and conclusions. You can avoid tedious calculations if you use the program `gauss` discussed in the *Study Guide*.

### Section 5.3, page 286

$$1. \begin{bmatrix} 226 & -525 \\ 90 & -209 \end{bmatrix} \quad 3. \begin{bmatrix} a^k & 0 \\ 2(a^k - b^k) & b^k \end{bmatrix}$$

$$5. \lambda = 2: \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}; \quad \lambda = 3: \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

When an answer involves a diagonalization,  $A = PDP^{-1}$ , the factors  $P$  and  $D$  are not unique, so your answer may differ from that given here.

$$7. P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

9. Not diagonalizable

$$11. P = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$13. P = \begin{bmatrix} -1 & 2 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$15. P = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

17. Not diagonalizable

$$19. P = \begin{bmatrix} 1 & 3 & -1 & -1 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

21. See the *Study Guide*. 23. Yes. (Explain why.)

25. No,  $A$  must be diagonalizable. (Explain why.)

27. *Hint:* Write  $A = PDP^{-1}$ . Since  $A$  is invertible, 0 is not an eigenvalue of  $A$ , so  $D$  has nonzero entries on its diagonal.

29. One answer is  $P_1 = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$ , whose columns are eigenvectors corresponding to the eigenvalues in  $D_1$ .

31. *Hint:* Construct a suitable  $2 \times 2$  triangular matrix.

$$33. [M] P = \begin{bmatrix} 2 & 0 & -1 & 1 \\ 7 & -1 & 0 & -4 \\ 7 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 \end{bmatrix},$$

$$D = \begin{bmatrix} -12 & 0 & 0 & 0 \\ 0 & -12 & 0 & 0 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

$$35. [M] P = \begin{bmatrix} 2 & 1 & -3 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 7 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & -14 & 0 \\ 0 & 0 & 0 & 0 & -14 \end{bmatrix}$$

### Section 5.4, page 293

$$1. \begin{bmatrix} 3 & -1 & 0 \\ -5 & 6 & 4 \end{bmatrix}$$

3. a.  $T(\mathbf{e}_1) = \mathbf{b}_3$ ,  $T(\mathbf{e}_2) = -\mathbf{b}_1 - 2\mathbf{b}_2$ ,  $T(\mathbf{e}_3) = 2\mathbf{b}_1 + 3\mathbf{b}_3$

$$b. [T(\mathbf{e}_1)]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, [T(\mathbf{e}_2)]_B = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix},$$

$$[T(\mathbf{e}_3)]_B = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

$$c. \begin{bmatrix} 0 & -1 & 2 \\ 0 & -2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

5. a.  $9 - 3t + t^2 + t^3$

b. For any  $\mathbf{p}, \mathbf{q}$  in  $\mathbb{P}_2$  and any scalar  $c$ ,

$$\begin{aligned} T[\mathbf{p}(t) + \mathbf{q}(t)] &= (t+3)[\mathbf{p}(t) + \mathbf{q}(t)] \\ &= (t+3)\mathbf{p}(t) + (t+3)\mathbf{q}(t) \\ &= T[\mathbf{p}(t)] + T[\mathbf{q}(t)] \\ T[c \cdot \mathbf{p}(t)] &= (t+3)[c \cdot \mathbf{p}(t)] = c \cdot (t+3)\mathbf{p}(t) \\ &= c \cdot T[\mathbf{p}(t)] \end{aligned}$$

$$c. \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$7. \begin{bmatrix} 3 & 0 & 0 \\ 5 & -2 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

$$9. a. \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

b. *Hint:* Compute  $T(\mathbf{p} + \mathbf{q})$  and  $T(c \cdot \mathbf{p})$  for arbitrary  $\mathbf{p}, \mathbf{q}$  in  $\mathbb{P}_2$  and an arbitrary scalar  $c$ .

$$c. \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$11. \begin{bmatrix} -2 & -2 \\ 0 & -1 \end{bmatrix} \quad 13. \mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$15. \mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

17. a.  $A\mathbf{b}_1 = 3\mathbf{b}_1$ , so  $\mathbf{b}_1$  is an eigenvector of  $A$ . However,  $A$  has only one eigenvalue,  $\lambda = 3$ , and the eigenspace is only one-dimensional, so  $A$  is not diagonalizable.

b.  $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

19. By definition, if  $A$  is similar to  $B$ , there exists an invertible matrix  $P$  such that  $P^{-1}AP = B$ . (See Section 5.2.) Then  $B$  is invertible because it is the product of invertible matrices. To show that  $A^{-1}$  is similar to  $B^{-1}$ , use the equation  $P^{-1}AP = B$ . See the *Study Guide*.

21. *Hint*: Review Practice Problem 2.

23. *Hint*: Compute  $B(P^{-1}\mathbf{x})$ .

25. *Hint*: Write  $A = PBP^{-1} = (PB)P^{-1}$ , and use the trace property.

27. For each  $j$ ,  $I(\mathbf{b}_j) = \mathbf{b}_j$ . Since the standard coordinate vector of any vector in  $\mathbb{R}^n$  is just the vector itself,  $[I(\mathbf{b}_j)]_{\mathcal{E}} = \mathbf{b}_j$ . Thus the matrix for  $I$  relative to  $\mathcal{B}$  and the standard basis  $\mathcal{E}$  is simply  $[\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_n]$ . This matrix is precisely the *change-of-coordinates* matrix  $P_{\mathcal{B}}$  defined in Section 4.4.

29. The  $\mathcal{B}$ -matrix for the identity transformation is  $I_n$ , because the  $\mathcal{B}$ -coordinate vector of the  $j$ th basis vector  $\mathbf{b}_j$  is the  $j$ th column of  $I_n$ .

31.  $[\mathbf{M}] \begin{bmatrix} -7 & -2 & -6 \\ 0 & -4 & -6 \\ 0 & 0 & -1 \end{bmatrix}$

### Section 5.5, page 300

1.  $\lambda = 2 + i$ ,  $\begin{bmatrix} -1+i \\ 1 \end{bmatrix}$ ;  $\lambda = 2 - i$ ,  $\begin{bmatrix} -1-i \\ 1 \end{bmatrix}$
3.  $\lambda = 3 + 2i$ ,  $\begin{bmatrix} -1-i \\ 4 \end{bmatrix}$ ;  $\lambda = 3 - 2i$ ,  $\begin{bmatrix} -1+i \\ 4 \end{bmatrix}$
5.  $\lambda = 4 + i$ ,  $\begin{bmatrix} 1-i \\ 2 \end{bmatrix}$ ;  $\lambda = 4 - i$ ,  $\begin{bmatrix} 1+i \\ 2 \end{bmatrix}$
7.  $\lambda = \sqrt{3} \pm i$ ,  $\varphi = \pi/6$  radian,  $r = 2$
9.  $\lambda = \pm 2i$ ,  $\varphi = -\pi/2$  radians,  $r = 2$
11.  $\lambda = -\sqrt{3} \pm i$ ,  $\varphi = -5\pi/6$  radian,  $r = 2$

In Exercises 13–20, other answers are possible. Any  $P$  that makes  $P^{-1}AP$  equal to the given  $C$  or to  $C^T$  is a satisfactory answer. First find  $P$ ; then compute  $P^{-1}AP$ .

13.  $P = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$ ,  $C = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

15.  $P = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ ,  $C = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$

17.  $P = \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix}$ ,  $C = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix}$

19.  $P = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$ ,  $C = \begin{bmatrix} .96 & -.28 \\ .28 & .96 \end{bmatrix}$

21.  $\mathbf{y} = \begin{bmatrix} 2 \\ -1+2i \end{bmatrix} = \frac{-1+2i}{5} \begin{bmatrix} -2-4i \\ 5 \end{bmatrix}$

23. a. Properties of conjugates and the fact that  $\overline{\mathbf{x}^T} = \overline{\mathbf{x}}^T$ ;  
b.  $\overline{A\mathbf{x}} = A\overline{\mathbf{x}}$  and  $A$  is real; (c) because  $\mathbf{x}^T A\overline{\mathbf{x}}$  is a scalar and hence may be viewed as a  $1 \times 1$  matrix; (d) properties of transposes; (e)  $A^T = A$ , definition of  $q$

25. *Hint*: First write  $\mathbf{x} = \text{Re } \mathbf{x} + i(\text{Im } \mathbf{x})$ .

27.  $[\mathbf{M}] P = \begin{bmatrix} -1 & 1 & -1 & -1 \\ 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$ ,

$$C = \begin{bmatrix} -2 & 5 & 0 & 0 \\ -5 & -2 & 0 & 0 \\ 0 & 0 & -4 & 10 \\ 0 & 0 & -10 & -4 \end{bmatrix}$$

Other choices are possible, but  $C$  must equal  $P^{-1}AP$ .

### Section 5.6, page 309

1. a. *Hint*: Find  $c_1, c_2$  such that  $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ . Use this representation and the fact that  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are eigenvectors of  $A$  to compute  $\mathbf{x}_1 = \begin{bmatrix} 49/3 \\ 41/3 \end{bmatrix}$ .

- b. In general,  $\mathbf{x}_k = 5(3)^k\mathbf{v}_1 - 4(\frac{1}{3})^k\mathbf{v}_2$  for  $k \geq 0$ .

3. When  $p = .2$ , the eigenvalues of  $A$  are .9 and .7, and

$$\mathbf{x}_k = c_1(.9)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2(.7)^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow \mathbf{0} \quad \text{as } k \rightarrow \infty$$

The higher predation rate cuts down the owls' food supply, and eventually both predator and prey populations perish.

5. If  $p = .325$ , the eigenvalues are 1.05 and .55. Since  $1.05 > 1$ , both populations will grow at 5% per year. An eigenvector for 1.05 is (6, 13), so eventually there will be approximately 6 spotted owls to every 13 (thousand) flying squirrels.

7. a. The origin is a saddle point because  $A$  has one eigenvalue larger than 1 and one smaller than 1 (in absolute value).

- b. The direction of greatest attraction is given by the eigenvector corresponding to the eigenvalue  $1/3$ , namely,  $\mathbf{v}_2$ . All vectors that are multiples of  $\mathbf{v}_2$  are attracted to the origin. The direction of greatest repulsion is given by the eigenvector  $\mathbf{v}_1$ . All multiples of  $\mathbf{v}_1$  are repelled.

- c. See the *Study Guide*.

9. Saddle point; eigenvalues: 2, .5; direction of greatest repulsion: the line through (0, 0) and (-1, 1); direction of greatest attraction: the line through (0, 0) and (1, 4)

11. Attractor; eigenvalues: .9, .8; greatest attraction: line through (0, 0) and (5, 4)

13. Repeller; eigenvalues: 1.2, 1.1; greatest repulsion: line through (0, 0) and (3, 4)

## A42 Answers to Odd-Numbered Exercises

15.  $\mathbf{x}_k = \mathbf{v}_1 + .1(.5)^k \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + .3(.2)^k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \rightarrow \mathbf{v}_1$  as  $k \rightarrow \infty$

17. a.  $A = \begin{bmatrix} 0 & 1.6 \\ .3 & .8 \end{bmatrix}$

b. The population is growing because the largest eigenvalue of  $A$  is 1.2, which is larger than 1 in magnitude. The eventual growth rate is 1.2, which is 20% per year. The eigenvector  $(4, 3)$  for  $\lambda_1 = 1.2$  shows that there will be 4 juveniles for every 3 adults.

c. [M] The juvenile–adult ratio seems to stabilize after about 5 or 6 years. The *Study Guide* describes how to construct a matrix program to generate a data matrix whose columns list the numbers of juveniles and adults each year. Graphing the data is also discussed.

### Section 5.7, page 317

1.  $\mathbf{x}(t) = \frac{5}{2} \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{4t} - \frac{3}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t}$

3.  $-\frac{5}{2} \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^t + \frac{9}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$ . The origin is a saddle point.

The direction of greatest attraction is the line through  $(-1, 1)$  and the origin. The direction of greatest repulsion is the line through  $(-3, 1)$  and the origin.

5.  $-\frac{1}{2} \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{4t} + \frac{7}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}$ . The origin is a repeller. The direction of greatest repulsion is the line through  $(1, 1)$  and the origin.

7. Set  $P = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$  and  $D = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}$ . Then  $A = PDP^{-1}$ . Substituting  $\mathbf{x} = P\mathbf{y}$  into  $\mathbf{x}' = A\mathbf{x}$ , we have

$$\frac{d}{dt}(P\mathbf{y}) = A(P\mathbf{y})$$

$$P\mathbf{y}' = PDP^{-1}(P\mathbf{y}) = P D \mathbf{y}$$

Left-multiplying by  $P^{-1}$  gives

$$\mathbf{y}' = D\mathbf{y}, \quad \text{or} \quad \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

9. (complex solution):

$$c_1 \begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{(-2+i)t} + c_2 \begin{bmatrix} 1+i \\ 1 \end{bmatrix} e^{(-2-i)t}$$

(real solution):

$$c_1 \begin{bmatrix} \cos t + \sin t \\ \cos t \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} \sin t - \cos t \\ \sin t \end{bmatrix} e^{-2t}$$

The trajectories spiral in toward the origin.

11. (complex):  $c_1 \begin{bmatrix} -3+3i \\ 2 \end{bmatrix} e^{3it} + c_2 \begin{bmatrix} -3-3i \\ 2 \end{bmatrix} e^{-3it}$

(real):

$$c_1 \begin{bmatrix} -3 \cos 3t - 3 \sin 3t \\ 2 \cos 3t \end{bmatrix} + c_2 \begin{bmatrix} -3 \sin 3t + 3 \cos 3t \\ 2 \sin 3t \end{bmatrix}$$

The trajectories are ellipses about the origin.

13. (complex):  $c_1 \begin{bmatrix} 1+i \\ 2 \end{bmatrix} e^{(1+3i)t} + c_2 \begin{bmatrix} 1-i \\ 2 \end{bmatrix} e^{(1-3i)t}$

(real):  $c_1 \begin{bmatrix} \cos 3t - \sin 3t \\ 2 \cos 3t \end{bmatrix} e^t + c_2 \begin{bmatrix} \sin 3t + \cos 3t \\ 2 \sin 3t \end{bmatrix} e^t$

The trajectories spiral out, away from the origin.

15. [M]  $\mathbf{x}(t) = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -6 \\ 1 \\ 5 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} -4 \\ 1 \\ 4 \end{bmatrix} e^t$

The origin is a saddle point. A solution with  $c_3 = 0$  is attracted to the origin. A solution with  $c_1 = c_2 = 0$  is repelled.

17. [M] (complex):

$$c_1 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 23-34i \\ -9+14i \\ 3 \end{bmatrix} e^{(5+2i)t} +$$

$$c_3 \begin{bmatrix} 23+34i \\ -9-14i \\ 3 \end{bmatrix} e^{(5-2i)t}$$

(real):  $c_1 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 23 \cos 2t + 34 \sin 2t \\ -9 \cos 2t - 14 \sin 2t \\ 3 \cos 2t \end{bmatrix} e^{5t} +$

$$c_3 \begin{bmatrix} 23 \sin 2t - 34 \cos 2t \\ -9 \sin 2t + 14 \cos 2t \\ 3 \sin 2t \end{bmatrix} e^{5t}$$

The origin is a repeller. The trajectories spiral outward, away from the origin.

19. [M]  $A = \begin{bmatrix} -2 & 3/4 \\ 1 & -1 \end{bmatrix}$ ,

$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \frac{5}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-.5t} - \frac{1}{2} \begin{bmatrix} -3 \\ 2 \end{bmatrix} e^{-2.5t}$$

21. [M]  $A = \begin{bmatrix} -1 & -8 \\ 5 & -5 \end{bmatrix}$ ,

$$\begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} = \begin{bmatrix} -20 \sin 6t \\ 15 \cos 6t - 5 \sin 6t \end{bmatrix} e^{-3t}$$

### Section 5.8, page 324

1. Eigenvector:  $\mathbf{x}_4 = \begin{bmatrix} 1 \\ .3326 \end{bmatrix}$ , or  $A\mathbf{x}_4 = \begin{bmatrix} 4.9978 \\ 1.6652 \end{bmatrix}$ ;  
 $\lambda \approx 4.9978$

3. Eigenvector:  $\mathbf{x}_4 = \begin{bmatrix} .5188 \\ 1 \end{bmatrix}$ , or  $A\mathbf{x}_4 = \begin{bmatrix} .4594 \\ .9075 \end{bmatrix}$ ;  
 $\lambda \approx .9075$

5.  $\mathbf{x} = \begin{bmatrix} -.7999 \\ 1 \end{bmatrix}$ ,  $A\mathbf{x} = \begin{bmatrix} 4.0015 \\ -5.0020 \end{bmatrix}$ ;  
estimated  $\lambda = -5.0020$

7. [M]

$$\mathbf{x}_k: \begin{bmatrix} .75 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ .9565 \end{bmatrix}, \begin{bmatrix} .9932 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ .9990 \end{bmatrix}, \begin{bmatrix} .9998 \\ 1 \end{bmatrix}$$

$$\mu_k: 11.5, 12.78, 12.96, 12.9948, 12.9990$$

9. [M]  $\mu_5 = 8.4233$ ,  $\mu_6 = 8.4246$ ; actual value: 8.42443 (accurate to 5 places)

11.  $\mu_k$ : 5.8000, 5.9655, 5.9942, 5.9990 ( $k = 1, 2, 3, 4$ );  $R(\mathbf{x}_k)$ : 5.9655, 5.9990, 5.9997, 5.999993
13. Yes, but the sequences may converge very slowly.
15. *Hint*: Write  $A\mathbf{x} - \alpha\mathbf{x} = (A - \alpha I)\mathbf{x}$ , and use the fact that  $(A - \alpha I)$  is invertible when  $\alpha$  is *not* an eigenvalue of  $A$ .
17. [M]  $v_0 = 3.3384$ ,  $v_1 = 3.32119$  (accurate to 4 places with rounding),  $v_2 = 3.3212209$ . Actual value: 3.3212201 (accurate to 7 places)
19. [M] a.  $\mu_6 = 30.2887 = \mu_7$  to four decimal places. To six places, the largest eigenvalue is 30.288685, with eigenvector (.957629, .688937, 1, .943782).
- b. The inverse power method (with  $\alpha = 0$ ) produces  $\mu_1^{-1} = .010141$ ,  $\mu_2^{-1} = .010150$ . To seven places, the smallest eigenvalue is .0101500, with eigenvector  $(-.603972, 1, -.251135, .148953)$ . The reason for the rapid convergence is that the next-to-smallest eigenvalue is near .85.
21. a. If the eigenvalues of  $A$  are all less than 1 in magnitude, and if  $\mathbf{x} \neq \mathbf{0}$ , then  $A^k\mathbf{x}$  is approximately an eigenvector for large  $k$ .
- b. If the strictly dominant eigenvalue is 1, and if  $\mathbf{x}$  has a component in the direction of the corresponding eigenvector, then  $\{A^k\mathbf{x}\}$  will converge to a multiple of that eigenvector.
- c. If the eigenvalues of  $A$  are all greater than 1 in magnitude, and if  $\mathbf{x}$  is not an eigenvector, then the distance from  $A^k\mathbf{x}$  to the nearest eigenvector will increase as  $k \rightarrow \infty$ .

## Chapter 5 Supplementary Exercises, page 326

1. a. T b. F c. T d. F e. T f. T  
g. F h. T i. F j. T k. F l. F  
m. F n. T o. F p. T q. F r. T  
s. F t. T u. T v. T w. F x. T

3. a. Suppose  $A\mathbf{x} = \lambda\mathbf{x}$ , with  $\mathbf{x} \neq \mathbf{0}$ . Then  
 $(5I - A)\mathbf{x} = 5\mathbf{x} - A\mathbf{x} = 5\mathbf{x} - \lambda\mathbf{x} = (5 - \lambda)\mathbf{x}$ .

The eigenvalue is  $5 - \lambda$ .

- b.  $(5I - 3A + A^2)\mathbf{x} = 5\mathbf{x} - 3A\mathbf{x} + A(A\mathbf{x})$   
 $= 5\mathbf{x} - 3\lambda\mathbf{x} + \lambda^2\mathbf{x}$   
 $= (5 - 3\lambda + \lambda^2)\mathbf{x}$ .

The eigenvalue is  $5 - 3\lambda + \lambda^2$ .

5. Suppose  $A\mathbf{x} = \lambda\mathbf{x}$ , with  $\mathbf{x} \neq \mathbf{0}$ . Then

$$\begin{aligned} p(A)\mathbf{x} &= (c_0I + c_1A + c_2A^2 + \cdots + c_nA^n)\mathbf{x} \\ &= c_0\mathbf{x} + c_1A\mathbf{x} + c_2A^2\mathbf{x} + \cdots + c_nA^n\mathbf{x} \\ &= c_0\mathbf{x} + c_1\lambda\mathbf{x} + c_2\lambda^2\mathbf{x} + \cdots + c_n\lambda^n\mathbf{x} = p(\lambda)\mathbf{x} \end{aligned}$$

So  $p(\lambda)$  is an eigenvalue of the matrix  $p(A)$ .

7. If  $A = PDP^{-1}$ , then  $p(A) = Pp(D)P^{-1}$ , as shown in Exercise 6. If the  $(j, j)$  entry in  $D$  is  $\lambda$ , then the  $(j, j)$  entry in  $D^k$  is  $\lambda^k$ , and so the  $(j, j)$  entry in  $p(D)$  is  $p(\lambda)$ . If  $p$  is the characteristic polynomial of  $A$ , then  $p(\lambda) = 0$

for each diagonal entry of  $D$ , because these entries in  $D$  are the eigenvalues of  $A$ . Thus  $p(D)$  is the zero matrix. Thus  $p(A) = P \cdot 0 \cdot P^{-1} = 0$ .

9. If  $I - A$  were not invertible, then the equation  $(I - A)\mathbf{x} = \mathbf{0}$  would have a nontrivial solution  $\mathbf{x}$ . Then  $\mathbf{x} - A\mathbf{x} = \mathbf{0}$  and  $A\mathbf{x} = 1 \cdot \mathbf{x}$ , which shows that  $A$  would have 1 as an eigenvalue. This cannot happen if all the eigenvalues are less than 1 in magnitude. So  $I - A$  must be invertible.
11. a. Take  $\mathbf{x}$  in  $H$ . Then  $\mathbf{x} = c\mathbf{u}$  for some scalar  $c$ . So  $A\mathbf{x} = A(c\mathbf{u}) = c(A\mathbf{u}) = c(\lambda\mathbf{u}) = (c\lambda)\mathbf{u}$ , which shows that  $A\mathbf{x}$  is in  $H$ .
- b. Let  $\mathbf{x}$  be a nonzero vector in  $K$ . Since  $K$  is one-dimensional,  $K$  must be the set of all scalar multiples of  $\mathbf{x}$ . If  $K$  is invariant under  $A$ , then  $A\mathbf{x}$  is in  $K$  and hence  $A\mathbf{x}$  is a multiple of  $\mathbf{x}$ . Thus  $\mathbf{x}$  is an eigenvector of  $A$ .

13. 1, 3, 7

15. Replace  $a$  by  $a - \lambda$  in the determinant formula from Exercise 16 in Chapter 3 Supplementary Exercises:

$$\det(A - \lambda I) = (a - b - \lambda)^{n-1}[a - \lambda + (n-1)b]$$

This determinant is zero only if  $a - b - \lambda = 0$  or  $a - \lambda + (n-1)b = 0$ . Thus  $\lambda$  is an eigenvalue of  $A$  if and only if  $\lambda = a - b$  or  $\lambda = a + (n-1)b$ . From the formula for  $\det(A - \lambda I)$  above, the algebraic multiplicity is  $n-1$  for  $a - b$  and 1 for  $a + (n-1)b$ .

17.  $\det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = \lambda^2 - (\text{tr } A)\lambda + \det A$ . Use the quadratic formula to solve the characteristic equation:

$$\lambda = \frac{\text{tr } A \pm \sqrt{(\text{tr } A)^2 - 4 \det A}}{2}$$

The eigenvalues are both real if and only if the discriminant is nonnegative, that is,  $(\text{tr } A)^2 - 4 \det A \geq 0$ . This inequality

simplifies to  $(\text{tr } A)^2 \geq 4 \det A$  and  $\left(\frac{\text{tr } A}{2}\right)^2 \geq \det A$ .

19.  $C_p = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$ ;  $\det(C_p - \lambda I) = 6 - 5\lambda + \lambda^2 = p(\lambda)$

21. If  $p$  is a polynomial of order 2, then a calculation such as in Exercise 19 shows that the characteristic polynomial of  $C_p$  is  $p(\lambda) = (-1)^2 p(\lambda)$ , so the result is true for  $n = 2$ . Suppose the result is true for  $n = k$  for some  $k \geq 2$ , and consider a polynomial  $p$  of degree  $k + 1$ . Then, expanding  $\det(C_p - \lambda I)$  by cofactors down the first column, the determinant of  $C_p - \lambda I$  equals

$$(-\lambda) \det \begin{bmatrix} -\lambda & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & & 1 \\ -a_1 & -a_2 & \cdots & -a_k - \lambda \end{bmatrix} + (-1)^{k+1} a_0$$

## A44 Answers to Odd-Numbered Exercises

The  $k \times k$  matrix shown is  $C_q - \lambda I$ , where  $q(t) = a_1 + a_2 t + \cdots + a_k t^{k-1} + t^k$ . By the induction assumption, the determinant of  $C_q - \lambda I$  is  $(-1)^k q(\lambda)$ . Thus

$$\begin{aligned}\det(C_p - \lambda I) &= (-1)^{k+1} a_0 + (-\lambda)(-1)^k q(\lambda) \\ &= (-1)^{k+1} [a_0 + \lambda(a_1 + \cdots + a_k \lambda^{k-1} + \lambda^k)] \\ &= (-1)^{k+1} p(\lambda)\end{aligned}$$

So the formula holds for  $n = k + 1$  when it holds for  $n = k$ . By the principle of induction, the formula for  $\det(C_p - \lambda I)$  is true for all  $n \geq 2$ .

23. From Exercise 22, the columns of the Vandermonde matrix  $V$  are eigenvectors of  $C_p$ , corresponding to the eigenvalues  $\lambda_1, \lambda_2, \lambda_3$  (the roots of the polynomial  $p$ ). Since these eigenvalues are distinct, the eigenvectors form a linearly independent set, by Theorem 2 in Section 5.1. Thus  $V$  has linearly independent columns and hence is invertible, by the Invertible Matrix Theorem. Finally, since the columns of  $V$  are eigenvectors of  $C_p$ , the Diagonalization Theorem (Theorem 5 in Section 5.3) shows that  $V^{-1}C_p V$  is diagonal.
25. [M] If your matrix program computes eigenvalues and eigenvectors by iterative methods rather than symbolic calculations, you may have some difficulties. You should find that  $AP - PD$  has extremely small entries and  $PDP^{-1}$  is close to  $A$ . (This was true just a few years ago, but the situation could change as matrix programs continue to improve.) If you constructed  $P$  from the program's eigenvectors, check the condition number of  $P$ . This may indicate that you do not really have three linearly independent eigenvectors.

## Chapter 6

### Section 6.1, page 336

1.  $5, 8, \frac{8}{5}$     3.  $\begin{bmatrix} 3/35 \\ -1/35 \\ -1/7 \end{bmatrix}$     5.  $\begin{bmatrix} 8/13 \\ 12/13 \end{bmatrix}$
7.  $\sqrt{35}$     9.  $\begin{bmatrix} -.6 \\ .8 \end{bmatrix}$     11.  $\begin{bmatrix} 7/\sqrt{69} \\ 2/\sqrt{69} \\ 4/\sqrt{69} \end{bmatrix}$
13.  $5\sqrt{5}$     15. Not orthogonal    17. Orthogonal
19. Refer to the *Study Guide* after you have written your answers.
21. *Hint:* Use Theorems 3 and 2 from Section 2.1.
23.  $\mathbf{u} \cdot \mathbf{v} = 0$ ,  $\|\mathbf{u}\|^2 = 30$ ,  $\|\mathbf{v}\|^2 = 101$ ,  
 $\|\mathbf{u} + \mathbf{v}\|^2 = (-5)^2 + (-9)^2 + 5^2 = 131 = 30 + 101$
25. The set of all multiples of  $\begin{bmatrix} -b \\ a \end{bmatrix}$  (when  $\mathbf{v} \neq \mathbf{0}$ )
27. *Hint:* Use the definition of orthogonality.
29. *Hint:* Consider a typical vector  $\mathbf{w} = c_1 \mathbf{v}_1 + \cdots + c_p \mathbf{v}_p$  in  $W$ .

31. *Hint:* If  $\mathbf{x}$  is in  $W^\perp$ , then  $\mathbf{x}$  is orthogonal to every vector in  $W$ .

33. [M] State your conjecture and verify it algebraically.

### Section 6.2, page 344

1. Not orthogonal    3. Not orthogonal    5. Orthogonal
7. Show  $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$ , mention Theorem 4, and observe that two linearly independent vectors in  $\mathbb{R}^2$  form a basis. Then obtain
- $$\mathbf{x} = \frac{39}{13} \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \frac{26}{52} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$
9. Show  $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$ ,  $\mathbf{u}_1 \cdot \mathbf{u}_3 = 0$ , and  $\mathbf{u}_2 \cdot \mathbf{u}_3 = 0$ . Mention Theorem 4, and observe that three linearly independent vectors in  $\mathbb{R}^3$  form a basis. Then obtain
- $$\mathbf{x} = \frac{5}{2} \mathbf{u}_1 - \frac{27}{18} \mathbf{u}_2 + \frac{18}{9} \mathbf{u}_3 = \frac{5}{2} \mathbf{u}_1 - \frac{3}{2} \mathbf{u}_2 + 2 \mathbf{u}_3$$
11.  $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$     13.  $\mathbf{y} = \begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix} + \begin{bmatrix} 14/5 \\ 8/5 \end{bmatrix}$
15.  $\mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} .6 \\ -.8 \end{bmatrix}$ , distance is 1
17.  $\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$
19. Orthonormal    21. Orthonormal
23. See the *Study Guide*.
25. *Hint:*  $\|U\mathbf{x}\|^2 = (U\mathbf{x})^T (U\mathbf{x})$ . Also, parts (a) and (c) follow from (b).
27. *Hint:* You need two theorems, one of which applies only to square matrices.
29. *Hint:* If you have a candidate for an inverse, you can check to see whether the candidate works.
31. Suppose  $\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}$ . Replace  $\mathbf{u}$  by  $c\mathbf{u}$  with  $c \neq 0$ ; then
- $$\frac{\mathbf{y} \cdot (c\mathbf{u})}{(c\mathbf{u}) \cdot (c\mathbf{u})} (c\mathbf{u}) = \frac{c(\mathbf{y} \cdot \mathbf{u})}{c^2 \mathbf{u} \cdot \mathbf{u}} (c\mathbf{u}) = \hat{\mathbf{y}}$$
33. Let  $L = \text{Span}\{\mathbf{u}\}$ , where  $\mathbf{u}$  is nonzero, and let  $T(\mathbf{x}) = \text{proj}_L \mathbf{x}$ . By definition,
- $$T(\mathbf{x}) = \frac{\mathbf{x} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} = (\mathbf{x} \cdot \mathbf{u})(\mathbf{u} \cdot \mathbf{u})^{-1} \mathbf{u}$$
- For  $\mathbf{x}$  and  $\mathbf{y}$  in  $\mathbb{R}^n$  and any scalars  $c$  and  $d$ , properties of the inner product (Theorem 1) show that
- $$\begin{aligned}T(c\mathbf{x} + d\mathbf{y}) &= [(c\mathbf{x} + d\mathbf{y}) \cdot \mathbf{u}](\mathbf{u} \cdot \mathbf{u})^{-1} \mathbf{u} \\ &= [c(\mathbf{x} \cdot \mathbf{u}) + d(\mathbf{y} \cdot \mathbf{u})](\mathbf{u} \cdot \mathbf{u})^{-1} \mathbf{u} \\ &= c(\mathbf{x} \cdot \mathbf{u})(\mathbf{u} \cdot \mathbf{u})^{-1} \mathbf{u} + d(\mathbf{y} \cdot \mathbf{u})(\mathbf{u} \cdot \mathbf{u})^{-1} \mathbf{u} \\ &= cT(\mathbf{x}) + dT(\mathbf{y})\end{aligned}$$

Thus  $T$  is linear.

## Section 6.3, page 352

$$1. \mathbf{x} = -\frac{8}{9}\mathbf{u}_1 - \frac{2}{9}\mathbf{u}_2 + \frac{2}{3}\mathbf{u}_3 + 2\mathbf{u}_4; \quad \mathbf{x} = \begin{bmatrix} 0 \\ -2 \\ 4 \\ -2 \end{bmatrix} + \begin{bmatrix} 10 \\ -6 \\ -2 \\ 2 \end{bmatrix}$$

$$3. \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} \quad 5. \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix} = \mathbf{y}$$

$$7. \mathbf{y} = \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix} + \begin{bmatrix} -7/3 \\ 7/3 \\ 7/3 \end{bmatrix} \quad 9. \mathbf{y} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

$$11. \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad 13. \begin{bmatrix} -1 \\ -3 \\ -2 \\ 3 \end{bmatrix} \quad 15. \sqrt{40}$$

$$17. \text{ a. } U^T U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, U U^T = \begin{bmatrix} 8/9 & -2/9 & 2/9 \\ -2/9 & 5/9 & 4/9 \\ 2/9 & 4/9 & 5/9 \end{bmatrix}$$

$$\text{ b. } \text{proj}_W \mathbf{y} = 6\mathbf{u}_1 + 3\mathbf{u}_2 = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}, (U U^T) \mathbf{y} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

$$19. \text{ Any multiple of } \begin{bmatrix} 0 \\ 2/5 \\ 1/5 \end{bmatrix}, \text{ such as } \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

21. Write your answers before checking the *Study Guide*.

23. *Hint:* Use Theorem 3 and the Orthogonal Decomposition Theorem. For the uniqueness, suppose  $A\mathbf{p} = \mathbf{b}$  and  $A\mathbf{p}_1 = \mathbf{b}$ , and consider the equations  $\mathbf{p} = \mathbf{p}_1 + (\mathbf{p} - \mathbf{p}_1)$  and  $\mathbf{p} = \mathbf{p} + \mathbf{0}$ .

## Section 6.4, page 358

$$1. \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} \quad 3. \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix}$$

$$5. \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ -4 \\ -1 \end{bmatrix} \quad 7. \begin{bmatrix} 2/\sqrt{30} \\ -5/\sqrt{30} \\ 1/\sqrt{30} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$9. \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix} \quad 11. \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix}$$

$$13. R = \begin{bmatrix} 6 & 12 \\ 0 & 6 \end{bmatrix}$$

$$15. Q = \begin{bmatrix} 1/\sqrt{5} & 1/2 & 1/2 \\ -1/\sqrt{5} & 0 & 0 \\ -1/\sqrt{5} & 1/2 & 1/2 \\ 1/\sqrt{5} & -1/2 & 1/2 \\ 1/\sqrt{5} & 1/2 & -1/2 \end{bmatrix},$$

$$R = \begin{bmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

17. See the *Study Guide*.

19. Suppose  $\mathbf{x}$  satisfies  $R\mathbf{x} = \mathbf{0}$ ; then  $QR\mathbf{x} = Q\mathbf{0} = \mathbf{0}$ , and  $A\mathbf{x} = \mathbf{0}$ . Since the columns of  $A$  are linearly independent,  $\mathbf{x}$  must be zero. This fact, in turn, shows that the columns of  $R$  are linearly independent. Since  $R$  is square, it is invertible, by the Invertible Matrix Theorem.

21. Denote the columns of  $Q$  by  $\mathbf{q}_1, \dots, \mathbf{q}_n$ . Note that  $n \leq m$ , because  $A$  is  $m \times n$  and has linearly independent columns. Use the fact that the columns of  $Q$  can be extended to an orthonormal basis for  $\mathbb{R}^m$ , say,  $\{\mathbf{q}_1, \dots, \mathbf{q}_m\}$ . (The *Study Guide* describes one method.) Let  $Q_0 = [\mathbf{q}_{n+1} \cdots \mathbf{q}_m]$  and  $Q_1 = [Q \quad Q_0]$ . Then, using partitioned matrix multiplication,  $Q_1 \begin{bmatrix} R \\ 0 \end{bmatrix} = QR = A$ .

23. *Hint:* Partition  $R$  as a  $2 \times 2$  block matrix.

25. [M] The diagonal entries of  $R$  are 20, 6, 10.3923, and 7.0711, to four decimal places.

## Section 6.5, page 366

$$1. \text{ a. } \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix} \quad \text{ b. } \hat{\mathbf{x}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$3. \text{ a. } \begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix} \quad \text{ b. } \hat{\mathbf{x}} = \begin{bmatrix} 4/3 \\ -1/3 \end{bmatrix}$$

$$5. \hat{\mathbf{x}} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad 7. 2\sqrt{5}$$

$$9. \text{ a. } \hat{\mathbf{b}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{ b. } \hat{\mathbf{x}} = \begin{bmatrix} 2/7 \\ 1/7 \end{bmatrix}$$

$$11. \text{ a. } \hat{\mathbf{b}} = \begin{bmatrix} 3 \\ 1 \\ 4 \\ -1 \end{bmatrix} \quad \text{ b. } \hat{\mathbf{x}} = \begin{bmatrix} 2/3 \\ 0 \\ 1/3 \end{bmatrix}$$

$$13. A\mathbf{u} = \begin{bmatrix} 11 \\ -11 \\ 11 \end{bmatrix}, \quad A\mathbf{v} = \begin{bmatrix} 7 \\ -12 \\ 7 \end{bmatrix},$$

$$\mathbf{b} - A\mathbf{u} = \begin{bmatrix} 0 \\ 2 \\ -6 \end{bmatrix}, \quad \mathbf{b} - A\mathbf{v} = \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}. \text{ No, } \mathbf{u} \text{ could not possibly be a least-squares solution of } A\mathbf{x} = \mathbf{b}. \text{ Why?}$$

$$15. \hat{\mathbf{x}} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \quad 17. \text{ See the } \textit{Study Guide}.$$



## A46 Answers to Odd-Numbered Exercises

19. a. If  $A\mathbf{x} = \mathbf{0}$ , then  $A^T A\mathbf{x} = A^T \mathbf{0} = \mathbf{0}$ . This shows that  $\text{Nul } A$  is contained in  $\text{Nul } A^T A$ .  
 b. If  $A^T A\mathbf{x} = \mathbf{0}$ , then  $\mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T \mathbf{0} = 0$ . So  $(A\mathbf{x})^T (A\mathbf{x}) = 0$  (which means that  $\|A\mathbf{x}\|^2 = 0$ ), and hence  $A\mathbf{x} = \mathbf{0}$ . This shows that  $\text{Nul } A^T A$  is contained in  $\text{Nul } A$ .
21. *Hint:* For part (a), use an important theorem from Chapter 2.
23. By Theorem 14,  $\hat{\mathbf{b}} = A\hat{\mathbf{x}} = A(A^T A)^{-1} A^T \mathbf{b}$ . The matrix  $A(A^T A)^{-1} A^T$  occurs frequently in statistics, where it is sometimes called the *hat-matrix*.
25. The normal equations are  $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$ , whose solution is the set of  $(x, y)$  such that  $x + y = 3$ . The solutions correspond to points on the line midway between the lines  $x + y = 2$  and  $x + y = 4$ .

### Section 6.6, page 374

1.  $y = .9 + .4x$       3.  $y = 1.1 + 1.3x$
5. If two data points have different  $x$ -coordinates, then the two columns of the design matrix  $X$  cannot be multiples of each other and hence are linearly independent. By Theorem 14 in Section 6.5, the normal equations have a unique solution.
7. a.  $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$ , where  $\mathbf{y} = \begin{bmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{bmatrix}$ ,  $X = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{bmatrix}$ ,
- $$\boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix}$$
- b. [M]  $y = 1.76x - .20x^2$
9.  $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$ , where  $\mathbf{y} = \begin{bmatrix} 7.9 \\ 5.4 \\ -9 \end{bmatrix}$ ,  $X = \begin{bmatrix} \cos 1 & \sin 1 \\ \cos 2 & \sin 2 \\ \cos 3 & \sin 3 \end{bmatrix}$ ,
- $$\boldsymbol{\beta} = \begin{bmatrix} A \\ B \end{bmatrix}, \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}$$
11. [M]  $\beta = 1.45$  and  $e = .811$ ; the orbit is an ellipse. The equation  $r = \beta/(1 - e \cdot \cos \vartheta)$  produces  $r = 1.33$  when  $\vartheta = 4.6$ .
13. [M] a.  $y = -.8558 + 4.7025t + 5.5554t^2 - .0274t^3$   
 b. The velocity function is  $v(t) = 4.7025 + 11.1108t - .0822t^2$ , and  $v(4.5) = 53.0$  ft/sec.
15. *Hint:* Write  $X$  and  $\mathbf{y}$  as in equation (1), and compute  $X^T X$  and  $X^T \mathbf{y}$ .
17. a. The mean of the  $x$ -data is  $\bar{x} = 5.5$ . The data in mean-deviation form are  $(-3.5, 1)$ ,  $(-.5, 2)$ ,  $(1.5, 3)$ ,

and  $(2.5, 3)$ . The columns of  $X$  are orthogonal because the entries in the second column sum to 0.

b.  $\begin{bmatrix} 4 & 0 \\ 0 & 21 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 9 \\ 7.5 \end{bmatrix}$ ,  
 $y = \frac{9}{4} + \frac{5}{14}x^* = \frac{9}{4} + \frac{5}{14}(x - 5.5)$

19. *Hint:* The equation has a nice geometric interpretation.

### Section 6.7, page 382

1. a.  $3, \sqrt{105}, 225$       b. All multiples of  $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$
3. 28      5.  $5\sqrt{2}, 3\sqrt{3}$       7.  $\frac{56}{25} + \frac{14}{25}t$
9. a. Constant polynomial,  $p(t) = 5$   
 b.  $t^2 - 5$  is orthogonal to  $p_0$  and  $p_1$ ; values:  $(4, -4, -4, 4)$ ; answer:  $q(t) = \frac{1}{4}(t^2 - 5)$
11.  $\frac{17}{5}t$
13. Verify each of the four axioms. For instance:
1.  $\langle \mathbf{u}, \mathbf{v} \rangle = (A\mathbf{u}) \cdot (A\mathbf{v})$  *Definition*  
 $= (A\mathbf{v}) \cdot (A\mathbf{u})$  *Property of the dot product*  
 $= \langle \mathbf{v}, \mathbf{u} \rangle$  *Definition*
15.  $\langle \mathbf{u}, c\mathbf{v} \rangle = \langle c\mathbf{v}, \mathbf{u} \rangle$  *Axiom 1*  
 $= c\langle \mathbf{v}, \mathbf{u} \rangle$  *Axiom 3*  
 $= c\langle \mathbf{u}, \mathbf{v} \rangle$  *Axiom 1*
17. *Hint:* Compute 4 times the right-hand side.
19.  $\langle \mathbf{u}, \mathbf{v} \rangle = \sqrt{a}\sqrt{b} + \sqrt{b}\sqrt{a} = 2\sqrt{ab}$ ,  
 $\|\mathbf{u}\|^2 = (\sqrt{a})^2 + (\sqrt{b})^2 = a + b$ . Since  $a$  and  $b$  are nonnegative,  $\|\mathbf{u}\| = \sqrt{a + b}$ . Similarly,  $\|\mathbf{v}\| = \sqrt{b + a}$ . By Cauchy-Schwarz,  $2\sqrt{ab} \leq \sqrt{a + b}\sqrt{b + a} = a + b$ . Hence,  $\sqrt{ab} \leq \frac{a + b}{2}$ .
21. 0      23.  $2/\sqrt{5}$       25.  $1, t, 3t^2 - 1$
27. [M] The new orthogonal polynomials are multiples of  $-17t + 5t^3$  and  $72 - 155t^2 + 35t^4$ . Scale these polynomials so their values at  $-2, -1, 0, 1$ , and  $2$  are small integers.

### Section 6.8, page 389

1.  $y = 2 + \frac{3}{2}t$
3.  $p(t) = 4p_0 - .1p_1 - .5p_2 + .2p_3$   
 $= 4 - .1t - .5(t^2 - 2) + .2(\frac{5}{6}t^3 - \frac{17}{6}t)$   
 (This polynomial happens to fit the data exactly.)
5. Use the identity  $\sin mt \sin nt = \frac{1}{2}[\cos(mt - nt) - \cos(mt + nt)]$
7. Use the identity  $\cos^2 kt = \frac{1 + \cos 2kt}{2}$ .
9.  $\pi + 2\sin t + \sin 2t + \frac{2}{3}\sin 3t$  [*Hint:* Save time by using the results from Example 4.]
11.  $\frac{1}{2} - \frac{1}{2}\cos 2t$  (Why?)

13. *Hint:* Take functions  $f$  and  $g$  in  $C[0, 2\pi]$ , and fix an integer  $m \geq 0$ . Write the Fourier coefficient of  $f + g$  that involves  $\cos mt$ , and write the Fourier coefficient that involves  $\sin mt$  ( $m > 0$ ).
15. [M] The cubic curve is the graph of  $g(t) = -.2685 + 3.6095t + 5.8576t^2 - .0477t^3$ . The velocity at  $t = 4.5$  seconds is  $g'(4.5) = 53.4$  ft/sec. This is about .7% faster than the estimate obtained in Exercise 13 in Section 6.6.

## Chapter 6 Supplementary Exercises, page 390

1. a. F   b. T   c. T   d. F   e. F   f. T  
g. T   h. T   i. F   j. T   k. T   l. F  
m. T   n. F   o. F   p. T   q. T   r. F  
s. F
2. *Hint:* If  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is an orthonormal set and  $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ , then the vectors  $c_1\mathbf{v}_1$  and  $c_2\mathbf{v}_2$  are orthogonal, and
- $$\|\mathbf{x}\|^2 = \|c_1\mathbf{v}_1 + c_2\mathbf{v}_2\|^2 = \|c_1\mathbf{v}_1\|^2 + \|c_2\mathbf{v}_2\|^2 = (|c_1|\|\mathbf{v}_1\|)^2 + (|c_2|\|\mathbf{v}_2\|)^2 = |c_1|^2 + |c_2|^2$$
- (Explain why.) So the stated equality holds for  $p = 2$ . Suppose that the equality holds for  $p = k$ , with  $k \geq 2$ , let  $\{\mathbf{v}_1, \dots, \mathbf{v}_{k+1}\}$  be an orthonormal set, and consider  $\mathbf{x} = c_1\mathbf{v}_1 + \dots + c_k\mathbf{v}_k + c_{k+1}\mathbf{v}_{k+1} = \mathbf{u}_k + c_{k+1}\mathbf{v}_{k+1}$ , where  $\mathbf{u}_k = c_1\mathbf{v}_1 + \dots + c_k\mathbf{v}_k$ .
3. Given  $\mathbf{x}$  and an orthonormal set  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  in  $\mathbb{R}^n$ , let  $\hat{\mathbf{x}}$  be the orthogonal projection of  $\mathbf{x}$  onto the subspace spanned by  $\mathbf{v}_1, \dots, \mathbf{v}_p$ . By Theorem 10 in Section 6.3,
- $$\hat{\mathbf{x}} = (\mathbf{x} \cdot \mathbf{v}_1)\mathbf{v}_1 + \dots + (\mathbf{x} \cdot \mathbf{v}_p)\mathbf{v}_p$$
- By Exercise 2,  $\|\hat{\mathbf{x}}\|^2 = |\mathbf{x} \cdot \mathbf{v}_1|^2 + \dots + |\mathbf{x} \cdot \mathbf{v}_p|^2$ . Bessel's inequality follows from the fact that  $\|\hat{\mathbf{x}}\|^2 \leq \|\mathbf{x}\|^2$ , noted before the statement of the Cauchy–Schwarz inequality, in Section 6.7.
5. Suppose  $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$  for all  $\mathbf{x}, \mathbf{y}$  in  $\mathbb{R}^n$ , and let  $\mathbf{e}_1, \dots, \mathbf{e}_n$  be the standard basis for  $\mathbb{R}^n$ . For  $j = 1, \dots, n$ ,  $U\mathbf{e}_j$  is the  $j$ th column of  $U$ . Since  $\|U\mathbf{e}_j\|^2 = (U\mathbf{e}_j) \cdot (U\mathbf{e}_j) = \mathbf{e}_j \cdot \mathbf{e}_j = 1$ , the columns of  $U$  are unit vectors; since  $(U\mathbf{e}_j) \cdot (U\mathbf{e}_k) = \mathbf{e}_j \cdot \mathbf{e}_k = 0$  for  $j \neq k$ , the columns are pairwise orthogonal.
7. *Hint:* Compute  $Q^T Q$ , using the fact that  $(\mathbf{u}\mathbf{u}^T)^T = \mathbf{u}^T \mathbf{u}^T = \mathbf{u}\mathbf{u}^T$ .
9. Let  $W = \text{Span}\{\mathbf{u}, \mathbf{v}\}$ . Given  $\mathbf{z}$  in  $\mathbb{R}^n$ , let  $\hat{\mathbf{z}} = \text{proj}_W \mathbf{z}$ . Then  $\hat{\mathbf{z}}$  is in  $\text{Col } A$ , where  $A = [\mathbf{u} \ \mathbf{v}]$ , say,  $\hat{\mathbf{z}} = A\hat{\mathbf{x}}$  for some  $\hat{\mathbf{x}}$  in  $\mathbb{R}^2$ . So  $\hat{\mathbf{x}}$  is a least-squares solution of  $A\mathbf{x} = \mathbf{z}$ . The normal equations can be solved to produce  $\hat{\mathbf{x}}$ , and then  $\hat{\mathbf{z}}$  is found by computing  $A\hat{\mathbf{x}}$ .

11. *Hint:* Let  $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ ,  $\mathbf{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ ,  $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$ , and

$$A = \begin{bmatrix} \mathbf{v}^T \\ \mathbf{v}^T \\ \mathbf{v}^T \end{bmatrix} = \begin{bmatrix} 1 & -2 & 5 \\ 1 & -2 & 5 \\ 1 & -2 & 5 \end{bmatrix}. \text{ The given set of}$$

equations is  $A\mathbf{x} = \mathbf{b}$ , and the set of all least-squares solutions coincides with the set of solutions of  $A^T A\mathbf{x} = A^T \mathbf{b}$  (Theorem 13 in Section 6.5). Study this equation, and use the fact that  $(\mathbf{v}\mathbf{v}^T)\mathbf{x} = \mathbf{v}(\mathbf{v}^T \mathbf{x}) = (\mathbf{v}^T \mathbf{x})\mathbf{v}$ , because  $\mathbf{v}^T \mathbf{x}$  is a scalar.

13. a. The row–column calculation of  $A\mathbf{u}$  shows that each row of  $A$  is orthogonal to every  $\mathbf{u}$  in  $\text{Nul } A$ . So each row of  $A$  is in  $(\text{Nul } A)^\perp$ . Since  $(\text{Nul } A)^\perp$  is a subspace, it must contain all linear combinations of the rows of  $A$ ; hence  $(\text{Nul } A)^\perp$  contains  $\text{Row } A$ .
- b. If  $\text{rank } A = r$ , then  $\dim \text{Nul } A = n - r$ , by the Rank Theorem. By Exercise 24(c) in Section 6.3,
- $$\dim \text{Nul } A + \dim(\text{Nul } A)^\perp = n$$
- So  $\dim(\text{Nul } A)^\perp$  must be  $r$ . But  $\text{Row } A$  is an  $r$ -dimensional subspace of  $(\text{Nul } A)^\perp$ , by the Rank Theorem and part (a). Therefore,  $\text{Row } A$  must coincide with  $(\text{Nul } A)^\perp$ .
- c. Replace  $A$  by  $A^T$  in part (b) and conclude that  $\text{Row } A^T$  coincides with  $(\text{Nul } A^T)^\perp$ . Since  $\text{Row } A^T = \text{Col } A$ , this proves (c).

15. If  $A = URU^T$  with  $U$  orthogonal, then  $A$  is similar to  $R$  (because  $U$  is invertible and  $U^T = U^{-1}$ ) and so  $A$  has the same eigenvalues as  $R$  (by Theorem 4 in Section 5.2), namely, the  $n$  real numbers on the diagonal of  $R$ .

17. [M]  $\frac{\|\Delta \mathbf{x}\|}{\|\mathbf{x}\|} = .4618,$

$$\text{cond}(A) \times \frac{\|\Delta \mathbf{b}\|}{\|\mathbf{b}\|} = 3363 \times (1.548 \times 10^{-4}) = .5206.$$

Observe that  $\|\Delta \mathbf{x}\|/\|\mathbf{x}\|$  almost equals  $\text{cond}(A)$  times  $\|\Delta \mathbf{b}\|/\|\mathbf{b}\|$ .

19. [M]  $\frac{\|\Delta \mathbf{x}\|}{\|\mathbf{x}\|} = 7.178 \times 10^{-8}, \frac{\|\Delta \mathbf{b}\|}{\|\mathbf{b}\|} = 2.832 \times 10^{-4}.$

Observe that the relative change in  $\mathbf{x}$  is *much* smaller than the relative change in  $\mathbf{b}$ . In fact, since

$$\text{cond}(A) \times \frac{\|\Delta \mathbf{b}\|}{\|\mathbf{b}\|} = 23,683 \times (2.832 \times 10^{-4}) = 6.707$$

the theoretical bound on the relative change in  $\mathbf{x}$  is 6.707 (to four significant figures). This exercise shows that even when a condition number is large, the relative error in a solution need not be as large as you might expect.

## Chapter 7

### Section 7.1, page 399

1. Symmetric   3. Not symmetric   5. Not symmetric

7. Orthogonal,  $\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$    9. Not orthogonal

11. Orthogonal,  $\begin{bmatrix} 2/3 & 0 & \sqrt{5}/3 \\ 2/3 & 1/\sqrt{5} & -4/\sqrt{45} \\ 1/3 & -2/\sqrt{5} & -2/\sqrt{45} \end{bmatrix}$

**A48** Answers to Odd-Numbered Exercises

$$13. P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, D = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

$$15. P = \begin{bmatrix} -4/\sqrt{17} & 1/\sqrt{17} \\ 1/\sqrt{17} & 4/\sqrt{17} \end{bmatrix}, D = \begin{bmatrix} 17 & 0 \\ 0 & 0 \end{bmatrix}$$

$$17. P = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix},$$

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$19. P = \begin{bmatrix} -1/\sqrt{5} & 4/\sqrt{45} & -2/3 \\ 2/\sqrt{5} & 2/\sqrt{45} & -1/3 \\ 0 & 5/\sqrt{45} & 2/3 \end{bmatrix},$$

$$D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$21. P = \begin{bmatrix} .5 & -.5 & -1/\sqrt{2} & 0 \\ .5 & .5 & 0 & -1/\sqrt{2} \\ .5 & -.5 & 1/\sqrt{2} & 0 \\ .5 & .5 & 0 & 1/\sqrt{2} \end{bmatrix},$$

$$D = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$23. P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix},$$

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

25. See the *Study Guide*.

$$27. (B^T A B)^T = B^T A^T B^{TT} \quad \text{Product of transposes in reverse order}$$

$$= B^T A B \quad \text{Because } A \text{ is symmetric}$$

The result about  $B^T B$  is a special case when  $A = I$ .  
 $(BB^T)^T = B^{TT} B^T = BB^T$ , so  $BB^T$  is symmetric.

29. *Hint*: Use an orthogonal diagonalization of  $A$ , or appeal to Theorem 2.

31. The Diagonalization Theorem in Section 5.3 says that the columns of  $P$  are (linearly independent) eigenvectors corresponding to the eigenvalues of  $A$  listed on the diagonal of  $D$ . So  $P$  has exactly  $k$  columns of eigenvectors corresponding to  $\lambda$ . These  $k$  columns form a basis for the eigenspace.

$$33. A = 8\mathbf{u}_1\mathbf{u}_1^T + 6\mathbf{u}_2\mathbf{u}_2^T + 3\mathbf{u}_3\mathbf{u}_3^T$$

$$= 8 \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ 6 \begin{bmatrix} 1/6 & 1/6 & -2/6 \\ 1/6 & 1/6 & -2/6 \\ -2/6 & -2/6 & 4/6 \end{bmatrix}$$

$$+ 3 \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

35. *Hint*:  $(\mathbf{u}\mathbf{u}^T)\mathbf{x} = \mathbf{u}(\mathbf{u}^T\mathbf{x}) = (\mathbf{u}^T\mathbf{x})\mathbf{u}$ , because  $\mathbf{u}^T\mathbf{x}$  is a scalar.

**Section 7.2, page 406**

1. a.  $5x_1^2 + \frac{2}{3}x_1x_2 + x_2^2$       b. 185      c. 16

3. a.  $\begin{bmatrix} 10 & -3 \\ -3 & -3 \end{bmatrix}$       b.  $\begin{bmatrix} 5 & 3/2 \\ 3/2 & 0 \end{bmatrix}$

5. a.  $\begin{bmatrix} 8 & -3 & 2 \\ -3 & 7 & -1 \\ 2 & -1 & -3 \end{bmatrix}$       b.  $\begin{bmatrix} 0 & 2 & 3 \\ 2 & 0 & -4 \\ 3 & -4 & 0 \end{bmatrix}$

7.  $\mathbf{x} = P\mathbf{y}$ , where  $P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ ,  $\mathbf{y}^T D \mathbf{y} = 6y_1^2 - 4y_2^2$

In Exercises 9–14, other answers (change of variables and new quadratic form) are possible.

9. Positive definite; eigenvalues are 7 and 2

Change of variable:  $\mathbf{x} = P\mathbf{y}$ , with  $P = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$

New quadratic form:  $7y_1^2 + 2y_2^2$

11. Indefinite; eigenvalues are 7 and -3

Change of variable:  $\mathbf{x} = P\mathbf{y}$ , with  $P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

New quadratic form:  $7y_1^2 - 3y_2^2$

13. Positive semidefinite; eigenvalues are 10 and 0

Change of variable:  $\mathbf{x} = P\mathbf{y}$ , with  $P = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$

New quadratic form:  $10y_1^2$

15. [M] Negative semidefinite; eigenvalues are 0, -6, -8, -12

Change of variable:  $\mathbf{x} = P\mathbf{y}$ ;

$$P = \begin{bmatrix} 3/\sqrt{12} & 0 & -1/2 & 0 \\ 1/\sqrt{12} & -2/\sqrt{6} & 1/2 & 0 \\ 1/\sqrt{12} & 1/\sqrt{6} & 1/2 & -1/\sqrt{2} \\ 1/\sqrt{12} & 1/\sqrt{6} & 1/2 & 1/\sqrt{2} \end{bmatrix}$$

New quadratic form:  $-6y_2^2 - 8y_3^2 - 12y_4^2$

17. [M] Indefinite; eigenvalues are 8.5 and -6.5

Change of variable:  $\mathbf{x} = P\mathbf{y}$ ;

$$P = \frac{1}{\sqrt{50}} \begin{bmatrix} 3 & -4 & 3 & 4 \\ 5 & 0 & -5 & 0 \\ 4 & 3 & 4 & -3 \\ 0 & 5 & 0 & 5 \end{bmatrix}$$

New quadratic form:  $8.5y_1^2 + 8.5y_2^2 - 6.5y_3^2 - 6.5y_4^2$

19. 8      21. See the *Study Guide*.

23. Write the characteristic polynomial in two ways:

$$\det(A - \lambda I) = \det \begin{bmatrix} a - \lambda & b \\ b & d - \lambda \end{bmatrix} \\ = \lambda^2 - (a + d)\lambda + ad - b^2$$

and

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

Equate coefficients to obtain  $\lambda_1 + \lambda_2 = a + d$  and  $\lambda_1\lambda_2 = ad - b^2 = \det A$ .

25. Exercise 27 in Section 7.1 showed that  $B^TB$  is symmetric. Also,  $\mathbf{x}^TB^TB\mathbf{x} = (B\mathbf{x})^TB\mathbf{x} = \|B\mathbf{x}\|^2 \geq 0$ , so the quadratic form is positive semidefinite, and we say that the matrix  $B^TB$  is positive semidefinite. *Hint:* To show that  $B^TB$  is positive definite when  $B$  is square and invertible, suppose that  $\mathbf{x}^TB^TB\mathbf{x} = 0$  and deduce that  $\mathbf{x} = \mathbf{0}$ .

27. *Hint:* Show that  $A + B$  is symmetric and the quadratic form  $\mathbf{x}^T(A + B)\mathbf{x}$  is positive definite.

### Section 7.3, page 413

1.  $\mathbf{x} = P\mathbf{y}$ , where  $P = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$

3. a. 9      b.  $\pm \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$       c. 6

5. a. 7      b.  $\pm \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$       c. 3

7.  $\pm \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$       9.  $5 + \sqrt{5}$       11. 3

13. *Hint:* If  $m = M$ , take  $\alpha = 0$  in the formula for  $\mathbf{x}$ . That is, let  $\mathbf{x} = \mathbf{u}_n$ , and verify that  $\mathbf{x}^TA\mathbf{x} = m$ . If  $m < M$  and if  $t$  is a number between  $m$  and  $M$ , then  $0 \leq t - m \leq M - m$  and  $0 \leq (t - m)/(M - m) \leq 1$ . So let  $\alpha = (t - m)/(M - m)$ . Solve the expression for  $\alpha$  to see that  $t = (1 - \alpha)m + \alpha M$ . As  $\alpha$  goes from 0 to 1,  $t$  goes from  $m$  to  $M$ . Construct  $\mathbf{x}$  as in the statement of the exercise, and verify its properties.

15. [M] a. 7.5      b.  $\begin{bmatrix} .5 \\ .5 \\ .5 \\ .5 \end{bmatrix}$       c.  $-.5$

17. [M] a.  $-4$       b.  $\begin{bmatrix} -3/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \end{bmatrix}$       c.  $-10$

### Section 7.4, page 423

1. 3, 1      3. 3, 2

The answers in Exercises 5–13 are not the only possibilities.

5.  $\begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

7.  $\begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$

9.  $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix} \\ \times \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$

11.  $\begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \times \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$

13.  $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \\ = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \\ \times \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{18} & 1/\sqrt{18} & -4/\sqrt{18} \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$

15. a. rank  $A = 2$

b. Basis for Col  $A$ :  $\begin{bmatrix} .40 \\ .37 \\ -.84 \end{bmatrix}, \begin{bmatrix} -.78 \\ -.33 \\ -.52 \end{bmatrix}$

Basis for Nul  $A$ :  $\begin{bmatrix} .58 \\ -.58 \\ .58 \end{bmatrix}$

(Remember that  $V^T$  appears in the SVD.)

17. Let  $A = U\Sigma V^T = U\Sigma V^{-1}$ . Since  $A$  is square and invertible, rank  $A = n$ , and all the entries on the diagonal of  $\Sigma$  must be nonzero. So  $A^{-1} = (U\Sigma V^{-1})^{-1} = V\Sigma^{-1}U^{-1} = V\Sigma^{-1}U^T$ .

19. *Hint:* Since  $U$  and  $V$  are orthogonal,

$$A^TA = (U\Sigma V^T)^T U\Sigma V^T = V\Sigma^T U^T U\Sigma V^T \\ = V(\Sigma^T \Sigma)V^{-1}$$

Thus  $V$  diagonalizes  $A^TA$ . What does this tell you about  $V$ ?

21. Let  $A = U\Sigma V^T$ . The matrix  $PU$  is orthogonal, because  $P$  and  $U$  are both orthogonal. (See Exercise 29 in Section 6.2.) So the equation  $PA = (PU)\Sigma V^T$  has the form required for a singular value decomposition. By Exercise 19, the diagonal entries in  $\Sigma$  are the singular values of  $PA$ .

23. *Hint:* Use a column–row expansion of  $(U\Sigma)V^T$ .

## A50 Answers to Odd-Numbered Exercises

25. *Hint:* Consider the SVD for the standard matrix of  $T$ —say,  $A = U\Sigma V^T = U\Sigma V^{-1}$ . Let  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  and  $\mathcal{C} = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$  be bases constructed from the columns of  $V$  and  $U$ , respectively. Compute the matrix for  $T$  relative to  $\mathcal{B}$  and  $\mathcal{C}$ , as in Section 5.4. To do this, you must show that  $V^{-1}\mathbf{v}_j = \mathbf{e}_j$ , the  $j$ th column of  $I_n$ .

$$27. [\mathbf{M}] \begin{bmatrix} -.57 & -.65 & -.42 & .27 \\ .63 & -.24 & -.68 & -.29 \\ .07 & -.63 & .53 & -.56 \\ -.51 & .34 & -.29 & -.73 \end{bmatrix} \\ \times \begin{bmatrix} 16.46 & 0 & 0 & 0 & 0 \\ 0 & 12.16 & 0 & 0 & 0 \\ 0 & 0 & 4.87 & 0 & 0 \\ 0 & 0 & 0 & 4.31 & 0 \end{bmatrix} \\ \times \begin{bmatrix} -.10 & .61 & -.21 & -.52 & .55 \\ -.39 & .29 & .84 & -.14 & -.19 \\ -.74 & -.27 & -.07 & .38 & .49 \\ .41 & -.50 & .45 & -.23 & .58 \\ -.36 & -.48 & -.19 & -.72 & -.29 \end{bmatrix}$$

29.  $[\mathbf{M}]$  25.9343, 16.7554, 11.2917, 1.0785, .00037793;  
 $\sigma_1/\sigma_5 = 68,622$

### Section 7.5, page 430

1.  $M = \begin{bmatrix} 12 \\ 10 \end{bmatrix}; B = \begin{bmatrix} 7 & 10 & -6 & -9 & -10 & 8 \\ 2 & -4 & -1 & 5 & 3 & -5 \end{bmatrix};$   
 $S = \begin{bmatrix} 86 & -27 \\ -27 & 16 \end{bmatrix}$
3.  $\begin{bmatrix} .95 \\ -.32 \end{bmatrix}$  for  $\lambda = 95.2$ ,  $\begin{bmatrix} .32 \\ .95 \end{bmatrix}$  for  $\lambda = 6.8$
5.  $[\mathbf{M}]$  (.130, .874, .468), 75.9% of the variance
7.  $y_1 = .95x_1 - .32x_2$ ;  $y_1$  explains 93.3% of the variance.
9.  $c_1 = 1/3$ ,  $c_2 = 2/3$ ,  $c_3 = 2/3$ ; the variance of  $y$  is 9.
11. a. If  $\mathbf{w}$  is the vector in  $\mathbb{R}^N$  with a 1 in each position, then

$$[\mathbf{X}_1 \ \cdots \ \mathbf{X}_N] \mathbf{w} = \mathbf{X}_1 + \cdots + \mathbf{X}_N = \mathbf{0}$$

because the  $\mathbf{X}_k$  are in mean-deviation form. Then

$$\begin{aligned} [\mathbf{Y}_1 \ \cdots \ \mathbf{Y}_N] \mathbf{w} \\ &= [P^T \mathbf{X}_1 \ \cdots \ P^T \mathbf{X}_N] \mathbf{w} \quad \text{By definition} \\ &= P^T [\mathbf{X}_1 \ \cdots \ \mathbf{X}_N] \mathbf{w} = P^T \mathbf{0} = \mathbf{0} \end{aligned}$$

That is,  $\mathbf{Y}_1 + \cdots + \mathbf{Y}_N = \mathbf{0}$ , so the  $\mathbf{Y}_k$  are in mean-deviation form.

- b. *Hint:* Because the  $\mathbf{X}_j$  are in mean-deviation form, the covariance matrix of the  $\mathbf{X}_j$  is

$$1/(N-1)[\mathbf{X}_1 \ \cdots \ \mathbf{X}_N][\mathbf{X}_1 \ \cdots \ \mathbf{X}_N]^T$$

Compute the covariance matrix of the  $\mathbf{Y}_j$ , using part (a).

13. If  $B = [\hat{\mathbf{X}}_1 \ \cdots \ \hat{\mathbf{X}}_N]$ , then

$$\begin{aligned} S &= \frac{1}{N-1} BB^T = \frac{1}{N-1} [\hat{\mathbf{X}}_1 \ \cdots \ \hat{\mathbf{X}}_N] \begin{bmatrix} \hat{\mathbf{X}}_1^T \\ \vdots \\ \hat{\mathbf{X}}_N^T \end{bmatrix} \\ &= \frac{1}{N-1} \sum_1^N \hat{\mathbf{X}}_k \hat{\mathbf{X}}_k^T = \frac{1}{N-1} \sum_1^N (\mathbf{X}_k - \mathbf{M})(\mathbf{X}_k - \mathbf{M})^T \end{aligned}$$

### Chapter 7 Supplementary Exercises, page 432

1. a. T b. F c. T d. F e. F f. F  
g. F h. T i. F j. F k. F l. F  
m. T n. F o. T p. T q. F
3. If  $\text{rank } A = r$ , then  $\dim \text{Nul } A = n - r$ , by the Rank Theorem. So 0 is an eigenvalue of multiplicity  $n - r$ . Hence, of the  $n$  terms in the spectral decomposition of  $A$ , exactly  $n - r$  are zero. The remaining  $r$  terms (corresponding to the nonzero eigenvalues) are all rank 1 matrices, as mentioned in the discussion of the spectral decomposition.
5. If  $A\mathbf{v} = \lambda\mathbf{v}$  for some nonzero  $\lambda$ , then  $\mathbf{v} = \lambda^{-1}A\mathbf{v} = A(\lambda^{-1}\mathbf{v})$ , which shows that  $\mathbf{v}$  is a linear combination of the columns of  $A$ .
7. *Hint:* If  $A = R^T R$ , where  $R$  is invertible, then  $A$  is positive definite, by Exercise 25 in Section 7.2. Conversely, suppose that  $A$  is positive definite. Then by Exercise 26 in Section 7.2,  $A = B^T B$  for some positive definite matrix  $B$ . Explain why  $B$  admits a QR factorization, and use it to create the Cholesky factorization of  $A$ .
9. If  $A$  is  $m \times n$  and  $\mathbf{x}$  is in  $\mathbb{R}^n$ , then  $\mathbf{x}^T A^T A \mathbf{x} = (A\mathbf{x})^T (A\mathbf{x}) = \|A\mathbf{x}\|^2 \geq 0$ . Thus  $A^T A$  is positive semidefinite. By Exercise 22 in Section 6.5,  $\text{rank } A^T A = \text{rank } A$ .
11. *Hint:* Write an SVD of  $A$  in the form  $A = U\Sigma V^T = PQ$ , where  $P = U\Sigma U^T$  and  $Q = UV^T$ . Show that  $P$  is symmetric and has the same eigenvalues as  $\Sigma$ . Explain why  $Q$  is an orthogonal matrix.
13. a. If  $\mathbf{b} = A\mathbf{x}$ , then  $\mathbf{x}^+ = A^+ \mathbf{b} = A^+ A \mathbf{x}$ . By Exercise 12(a),  $\mathbf{x}^+$  is the orthogonal projection of  $\mathbf{x}$  onto  $\text{Row } A$ .  
b. From (a) and then Exercise 12(c),  $A\mathbf{x}^+ = A(A^+ A \mathbf{x}) = (AA^+) \mathbf{x} = A\mathbf{x} = \mathbf{b}$ .  
c. Since  $\mathbf{x}^+$  is the orthogonal projection onto  $\text{Row } A$ , the Pythagorean Theorem shows that  $\|\mathbf{u}\|^2 = \|\mathbf{x}^+\|^2 + \|\mathbf{u} - \mathbf{x}^+\|^2$ . Part (c) follows immediately.
15.  $[\mathbf{M}] A^+ = \frac{1}{40} \cdot \begin{bmatrix} -2 & -14 & 13 & 13 \\ -2 & -14 & 13 & 13 \\ -2 & 6 & -7 & -7 \\ 2 & -6 & 7 & 7 \\ 4 & -12 & -6 & -6 \end{bmatrix}, \hat{\mathbf{x}} = \begin{bmatrix} .7 \\ .7 \\ -.8 \\ .8 \\ .6 \end{bmatrix}$

The reduced echelon form of  $\begin{bmatrix} A \\ \mathbf{x}^T \end{bmatrix}$  is the same as the reduced echelon form of  $A$ , except for an extra row of

zeros. So adding scalar multiples of the rows of  $A$  to  $\mathbf{x}^T$  can produce the zero vector, which shows that  $\mathbf{x}^T$  is in Row  $A$ .

$$\text{Basis for Nul } A: \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

## Chapter 8

### Section 8.1, page 442

- Some possible answers:  $\mathbf{y} = 2\mathbf{v}_1 - 1.5\mathbf{v}_2 + .5\mathbf{v}_3$ ,  
 $\mathbf{y} = 2\mathbf{v}_1 - 2\mathbf{v}_3 + \mathbf{v}_4$ ,  $\mathbf{y} = 2\mathbf{v}_1 + 3\mathbf{v}_2 - 7\mathbf{v}_3 + 3\mathbf{v}_4$
- $\mathbf{y} = -3\mathbf{v}_1 + 2\mathbf{v}_2 + 2\mathbf{v}_3$ . The weights sum to 1, so this is an affine sum.
- $\mathbf{p}_1 = 3\mathbf{b}_1 - \mathbf{b}_2 - \mathbf{b}_3 \in \text{aff } S$  since the coefficients sum to 1.
  - $\mathbf{p}_2 = 2\mathbf{b}_1 + 0\mathbf{b}_2 + \mathbf{b}_3 \notin \text{aff } S$  since the coefficients do not sum to 1.
  - $\mathbf{p}_3 = -\mathbf{b}_1 + 2\mathbf{b}_2 + 0\mathbf{b}_3 \in \text{aff } S$  since the coefficients sum to 1.
- $\mathbf{p}_1 \in \text{Span } S$ , but  $\mathbf{p}_1 \notin \text{aff } S$
  - $\mathbf{p}_2 \in \text{Span } S$ , and  $\mathbf{p}_2 \in \text{aff } S$
  - $\mathbf{p}_3 \notin \text{Span } S$ , so  $\mathbf{p}_3 \notin \text{aff } S$
- $\mathbf{v}_1 = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$  and  $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ . Other answers are possible.
- See the *Study Guide*.
- Span  $\{\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1\}$  is a plane if and only if  $\{\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1\}$  is linearly independent. Suppose  $c_2$  and  $c_3$  satisfy  $c_2(\mathbf{v}_2 - \mathbf{v}_1) + c_3(\mathbf{v}_3 - \mathbf{v}_1) = \mathbf{0}$ . Show that this implies  $c_2 = c_3 = 0$ .
- Let  $S = \{\mathbf{x} : A\mathbf{x} = \mathbf{b}\}$ . To show that  $S$  is affine, it suffices to show that  $S$  is a flat, by Theorem 3. Let  $W = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$ . Then  $W$  is a subspace of  $\mathbb{R}^n$ , by Theorem 2 in Section 4.2 (or Theorem 12 in Section 2.8). Since  $S = W + \mathbf{p}$ , where  $\mathbf{p}$  satisfies  $A\mathbf{p} = \mathbf{b}$ , by Theorem 6 in Section 1.5,  $S$  is a translate of  $W$ , and hence  $S$  is a flat.
- A suitable set consists of any three vectors that are not collinear and have 5 as their third entry. If 5 is their third entry, they lie in the plane  $z = 5$ . If the vectors are not collinear, their affine hull cannot be a line, so it must be the plane.
- If  $\mathbf{p}, \mathbf{q} \in f(S)$ , then there exist  $\mathbf{r}, \mathbf{s} \in S$  such that  $f(\mathbf{r}) = \mathbf{p}$  and  $f(\mathbf{s}) = \mathbf{q}$ . Given any  $t \in \mathbb{R}$ , we must show that  $\mathbf{z} = (1-t)\mathbf{p} + t\mathbf{q}$  is in  $f(S)$ . Now use definitions of  $\mathbf{p}$  and  $\mathbf{q}$ , and the fact that  $f$  is linear. The complete proof is presented in the *Study Guide*.
- Since  $B$  is affine, Theorem 1 implies that  $B$  contains all affine combinations of points of  $B$ . Hence  $B$  contains all affine combinations of points of  $A$ . That is,  $\text{aff } A \subset B$ .
- Since  $A \subset (A \cup B)$ , it follows from Exercise 22 that  $\text{aff } A \subset \text{aff } (A \cup B)$ . Similarly,  $\text{aff } B \subset \text{aff } (A \cup B)$ , so  $[\text{aff } A \cup \text{aff } B] \subset \text{aff } (A \cup B)$ .

- To show that  $D \subset E \cap F$ , show that  $D \subset E$  and  $D \subset F$ . The complete proof is presented in the *Study Guide*.

### Section 8.2, page 452

- Affinely dependent and  $2\mathbf{v}_1 + \mathbf{v}_2 - 3\mathbf{v}_3 = \mathbf{0}$
- The set is affinely independent. If the points are called  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ , and  $\mathbf{v}_4$ , then  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is a basis for  $\mathbb{R}^3$  and  $\mathbf{v}_4 = 16\mathbf{v}_1 + 5\mathbf{v}_2 - 3\mathbf{v}_3$ , but the weights in the linear combination do not sum to 1.
- $-4\mathbf{v}_1 + 5\mathbf{v}_2 - 4\mathbf{v}_3 + 3\mathbf{v}_4 = \mathbf{0}$
- The barycentric coordinates are  $(-2, 4, -1)$ .
- See the *Study Guide*.
- When a set of five points is translated by subtracting, say, the first point, the new set of four points must be linearly dependent, by Theorem 8 in Section 1.7, because the four points are in  $\mathbb{R}^3$ . By Theorem 5, the original set of five points is affinely dependent.
- If  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is affinely dependent, then there exist  $c_1$  and  $c_2$ , not both zero, such that  $c_1 + c_2 = 0$  and  $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$ . Show that this implies  $\mathbf{v}_1 = \mathbf{v}_2$ . For the converse, suppose  $\mathbf{v}_1 = \mathbf{v}_2$  and select specific  $c_1$  and  $c_2$  that show their affine dependence. The details are in the *Study Guide*.
- The vectors  $\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$  are not multiples and hence are linearly independent. By Theorem 5,  $S$  is affinely independent.
  - $\mathbf{p}_1 \leftrightarrow (-\frac{6}{8}, \frac{9}{8}, \frac{5}{8})$ ,  $\mathbf{p}_2 \leftrightarrow (0, \frac{1}{2}, \frac{1}{2})$ ,  $\mathbf{p}_3 \leftrightarrow (\frac{14}{8}, -\frac{5}{8}, -\frac{1}{8})$ ,  
 $\mathbf{p}_4 \leftrightarrow (\frac{6}{8}, -\frac{5}{8}, \frac{7}{8})$ ,  $\mathbf{p}_5 \leftrightarrow (\frac{1}{4}, \frac{1}{8}, \frac{5}{8})$
  - $\mathbf{p}_6$  is  $(-, -, +)$ ,  $\mathbf{p}_7$  is  $(0, +, -)$ , and  $\mathbf{p}_8$  is  $(+, +, -)$ .
- Suppose  $S = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$  is an affinely independent set. Then equation (7) has a solution, because  $\mathbf{p}$  is in  $\text{aff } S$ . Hence equation (8) has a solution. By Theorem 5, the homogeneous forms of the points in  $S$  are linearly independent. Thus (8) has a unique solution. Then (7) also has a unique solution, because (8) encodes both equations that appear in (7).

The following argument mimics the proof of Theorem 7 in Section 4.4. If  $S = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$  is an affinely independent set, then scalars  $c_1, \dots, c_k$  exist that satisfy (7), by definition of  $\text{aff } S$ . Suppose  $\mathbf{x}$  also has the representation

$$\mathbf{x} = d_1\mathbf{b}_1 + \dots + d_k\mathbf{b}_k \quad \text{and} \quad d_1 + \dots + d_k = 1 \quad (7a)$$

for scalars  $d_1, \dots, d_k$ . Then subtraction produces the equation

$$\mathbf{0} = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\mathbf{b}_1 + \dots + (c_k - d_k)\mathbf{b}_k \quad (7b)$$

The weights in (7b) sum to 0 because the  $c$ 's and the  $d$ 's separately sum to 1. This is impossible, unless each weight in (8) is 0, because  $S$  is an affinely independent set. This proves that  $c_i = d_i$  for  $i = 1, \dots, k$ .

## A52 Answers to Odd-Numbered Exercises

19. If  $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$  is an affinely dependent set, then there exist scalars  $c_1, c_2$ , and  $c_3$ , not all zero, such that  $c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$  and  $c_1 + c_2 + c_3 = 0$ . Now use the linearity of  $f$ .

21. Let  $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ ,  $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ , and  $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ . Then

$$\det [\tilde{\mathbf{a}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}] = \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix} =$$

$$\det \begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{bmatrix}, \text{ by the transpose property of the}$$

determinant (Theorem 5 in Section 3.2). By Exercise 30 in Section 3.3, this determinant equals 2 times the area of the triangle with vertices at  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ .

23. If  $[\tilde{\mathbf{a}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}] \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \tilde{\mathbf{p}}$ , then Cramer's rule gives

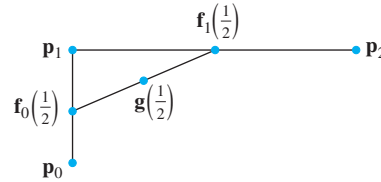
$r = \det [\tilde{\mathbf{p}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}] / \det [\tilde{\mathbf{a}} \quad \tilde{\mathbf{b}} \quad \tilde{\mathbf{c}}]$ . By Exercise 21, the numerator of this quotient is twice the area of  $\triangle \mathbf{pbc}$ , and the denominator is twice the area of  $\triangle \mathbf{abc}$ . This proves the formula for  $r$ . The other formulas are proved using Cramer's rule for  $s$  and  $t$ .

### Section 8.3, page 459

- See the *Study Guide*.
- None are in  $\text{conv } S$ .
- $\mathbf{p}_1 = -\frac{1}{6}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 + \frac{2}{3}\mathbf{v}_3 + \frac{1}{6}\mathbf{v}_4$ , so  $\mathbf{p}_1 \notin \text{conv } S$ .  
 $\mathbf{p}_2 = \frac{1}{3}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 + \frac{1}{6}\mathbf{v}_3 + \frac{1}{6}\mathbf{v}_4$ , so  $\mathbf{p}_2 \in \text{conv } S$ .
- a. The barycentric coordinates of  $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ , and  $\mathbf{p}_4$  are, respectively,  $(\frac{1}{3}, \frac{1}{6}, \frac{1}{2})$ ,  $(0, \frac{1}{2}, \frac{1}{2})$ ,  $(\frac{1}{2}, -\frac{1}{4}, \frac{3}{4})$ , and  $(\frac{1}{2}, \frac{3}{4}, -\frac{1}{4})$ .  
b.  $\mathbf{p}_3$  and  $\mathbf{p}_4$  are outside  $\text{conv } T$ .  $\mathbf{p}_1$  is inside  $\text{conv } T$ .  $\mathbf{p}_2$  is on the edge  $\overline{\mathbf{v}_2\mathbf{v}_3}$  of  $\text{conv } T$ .
- $\mathbf{p}_1$  and  $\mathbf{p}_3$  are outside the tetrahedron  $\text{conv } S$ .  $\mathbf{p}_2$  is on the face containing the vertices  $\mathbf{v}_2, \mathbf{v}_3$ , and  $\mathbf{v}_4$ .  $\mathbf{p}_4$  is inside  $\text{conv } S$ .  $\mathbf{p}_5$  is on the edge between  $\mathbf{v}_1$  and  $\mathbf{v}_3$ .
- See the *Study Guide*.
- If  $\mathbf{p}, \mathbf{q} \in f(S)$ , then there exist  $\mathbf{r}, \mathbf{s} \in S$  such that  $f(\mathbf{r}) = \mathbf{p}$  and  $f(\mathbf{s}) = \mathbf{q}$ . The goal is to show that the line segment  $\mathbf{y} = (1-t)\mathbf{p} + t\mathbf{q}$ , for  $0 \leq t \leq 1$ , is in  $f(S)$ . Use the linearity of  $f$  and the convexity of  $S$  to show that  $\mathbf{y} = f(\mathbf{w})$  for some  $\mathbf{w}$  in  $S$ . This will show that  $\mathbf{y}$  is in  $f(S)$  and that  $f(S)$  is convex.
- $\mathbf{p} = \frac{1}{6}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_4$  and  $\mathbf{p} = \frac{1}{2}\mathbf{v}_1 + \frac{1}{6}\mathbf{v}_2 + \frac{1}{3}\mathbf{v}_3$ .
- Suppose  $A \subset B$ , where  $B$  is convex. Then, since  $B$  is convex, Theorem 7 implies that  $B$  contains all convex combinations of points of  $B$ . Hence  $B$  contains all convex combinations of points of  $A$ . That is,  $\text{conv } A \subset B$ .
- a. Use Exercise 18 to show that  $\text{conv } A$  and  $\text{conv } B$  are both subsets of  $\text{conv } (A \cup B)$ . This will imply that their union is also a subset of  $\text{conv } (A \cup B)$ .

- b. One possibility is to let  $A$  be two adjacent corners of a square and let  $B$  be the other two corners. Then what is  $(\text{conv } A) \cup (\text{conv } B)$ , and what is  $\text{conv } (A \cup B)$ ?

21.



23.  $\mathbf{g}(t) = (1-t)\mathbf{f}_0(t) + t\mathbf{f}_1(t)$   
 $= (1-t)[(1-t)\mathbf{p}_0 + t\mathbf{p}_1] + t[(1-t)\mathbf{p}_1 + t\mathbf{p}_2]$   
 $= (1-t)^2\mathbf{p}_0 + 2t(1-t)\mathbf{p}_1 + t^2\mathbf{p}_2.$

The sum of the weights in the linear combination for  $\mathbf{g}$  is  $(1-t)^2 + 2t(1-t) + t^2$ , which equals  $(1-2t+t^2) + (2t-2t^2) + t^2 = 1$ . The weights are each between 0 and 1 when  $0 \leq t \leq 1$ , so  $\mathbf{g}(t)$  is in  $\text{conv } \{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$ .

### Section 8.4, page 467

- $f(x_1, x_2) = 3x_1 + 4x_2$  and  $d = 13$
- a. Open      b. Closed      c. Neither  
d. Closed      e. Closed
- a. Not compact, convex  
b. Compact, convex  
c. Not compact, convex  
d. Not compact, not convex  
e. Not compact, convex
- a.  $\mathbf{n} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$  or a multiple  
b.  $f(\mathbf{x}) = 2x_2 + 3x_3$ ,  $d = 11$
- a.  $\mathbf{n} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}$  or a multiple  
b.  $f(\mathbf{x}) = 3x_1 - x_2 + 2x_3 + x_4$ ,  $d = 5$
- $\mathbf{v}_2$  is on the same side as  $\mathbf{0}$ ,  $\mathbf{v}_1$  is on the other side, and  $\mathbf{v}_3$  is in  $H$ .
- One possibility is  $\mathbf{p} = \begin{bmatrix} 32 \\ -14 \\ 0 \\ 0 \end{bmatrix}$ ,  $\mathbf{v}_1 = \begin{bmatrix} 10 \\ -7 \\ 1 \\ 0 \end{bmatrix}$ ,  
 $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$
- $f(x_1, x_2, x_3) = x_1 - 3x_2 + 4x_3 - 2x_4$ , and  $d = 5$
- $f(x_1, x_2, x_3) = x_1 - 2x_2 + x_3$ , and  $d = 0$
- $f(x_1, x_2, x_3) = -5x_1 + 3x_2 + x_3$ , and  $d = 0$



21. See the *Study Guide*.

23.  $f(x_1, x_2) = 3x_1 - 2x_2$  with  $d$  satisfying  $9 < d < 10$  is one possibility.

25.  $f(x, y) = 4x + 1$ . A natural choice for  $d$  is 12.75, which equals  $f(3, .75)$ . The point  $(3, .75)$  is three-fourths of the distance between the center of  $B(\mathbf{0}, 3)$  and the center of  $B(\mathbf{p}, 1)$ .

27. Exercise 2(a) in Section 8.3 gives one possibility. Or let  $S = \{(x, y) : x^2 + y^2 = 1 \text{ and } y > 0\}$ . Then  $\text{conv } S$  is the upper (open) half-plane.

29. Let  $\mathbf{x}, \mathbf{y} \in B(\mathbf{p}, \delta)$  and suppose  $\mathbf{z} = (1 - t)\mathbf{x} + t\mathbf{y}$ , where  $0 \leq t \leq 1$ . Then show that

$$\begin{aligned}\|\mathbf{z} - \mathbf{p}\| &= \|(1 - t)\mathbf{x} + t\mathbf{y} - \mathbf{p}\| \\ &= \|(1 - t)(\mathbf{x} - \mathbf{p}) + t(\mathbf{y} - \mathbf{p})\| < \delta.\end{aligned}$$

### Section 8.5, page 479

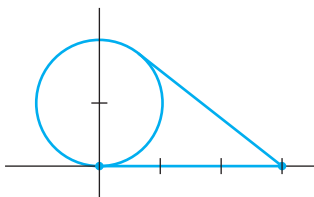
1. a.  $m = 1$  at the point  $\mathbf{p}_1$       b.  $m = 5$  at the point  $\mathbf{p}_2$   
c.  $m = 5$  at the point  $\mathbf{p}_3$

3. a.  $m = -3$  at the point  $\mathbf{p}_3$   
b.  $m = 1$  on the set  $\text{conv}\{\mathbf{p}_1, \mathbf{p}_3\}$   
c.  $m = -3$  on the set  $\text{conv}\{\mathbf{p}_1, \mathbf{p}_2\}$

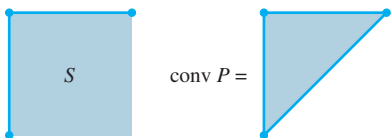
5.  $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \end{bmatrix} \right\}$

7.  $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \end{bmatrix} \right\}$

9. The origin is an extreme point, but it is not a vertex. Explain why.



11. One possibility is to let  $S$  be a square that includes part of the boundary but not all of it. For example, include just two adjacent edges. The convex hull of the profile  $P$  is a triangular region.



13. a.  $f_0(C^5) = 32$ ,  $f_1(C^5) = 80$ ,  $f_2(C^5) = 80$ ,  
 $f_3(C^5) = 40$ ,  $f_4(C^5) = 10$ , and  
 $32 - 80 + 80 - 40 + 10 = 2$ .

b.

	$f_0$	$f_1$	$f_2$	$f_3$	$f_4$
$S^1$	2				
$S^2$	4	4			
$S^3$	8	12	6		
$S^4$	16	32	24	8	
$S^5$	32	80	80	40	10

For a general formula, see the *Study Guide*.

15. a.  $f_0(P^n) = f_0(Q) + 1$   
b.  $f_k(P^n) = f_k(Q) + f_{k-1}(Q)$   
c.  $f_{n-1}(P^n) = f_{n-2}(Q) + 1$

17. See the *Study Guide*.

19. Let  $S$  be convex and let  $\mathbf{x} \in cS + dS$ , where  $c > 0$  and  $d > 0$ . Then there exist  $\mathbf{s}_1$  and  $\mathbf{s}_2$  in  $S$  such that  $\mathbf{x} = c\mathbf{s}_1 + d\mathbf{s}_2$ . But then

$$\mathbf{x} = c\mathbf{s}_1 + d\mathbf{s}_2 = (c + d) \left( \frac{c}{c + d} \mathbf{s}_1 + \frac{d}{c + d} \mathbf{s}_2 \right).$$

Now show that the expression on the right side is a member of  $(c + d)S$ .

For the converse, pick a typical point in  $(c + d)S$  and show it is in  $cS + dS$ .

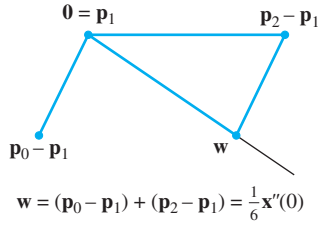
21. *Hint:* Suppose  $A$  and  $B$  are convex. Let  $\mathbf{x}, \mathbf{y} \in A + B$ . Then there exist  $\mathbf{a}, \mathbf{c} \in A$  and  $\mathbf{b}, \mathbf{d} \in B$  such that  $\mathbf{x} = \mathbf{a} + \mathbf{b}$  and  $\mathbf{y} = \mathbf{c} + \mathbf{d}$ . For any  $t$  such that  $0 \leq t \leq 1$ , show that  $\mathbf{w} = (1 - t)\mathbf{x} + t\mathbf{y} = (1 - t)(\mathbf{a} + \mathbf{b}) + t(\mathbf{c} + \mathbf{d})$  represents a point in  $A + B$ .

### Section 8.6, page 490

1. The control points for  $\mathbf{x}(t) + \mathbf{b}$  should be  $\mathbf{p}_0 + \mathbf{b}$ ,  $\mathbf{p}_1 + \mathbf{b}$ , and  $\mathbf{p}_3 + \mathbf{b}$ . Write the Bézier curve through these points, and show algebraically that this curve is  $\mathbf{x}(t) + \mathbf{b}$ . See the *Study Guide*.
3. a.  $\mathbf{x}'(t) = (-3 + 6t - 3t^2)\mathbf{p}_0 + (3 - 12t + 9t^2)\mathbf{p}_1 + (6t - 9t^2)\mathbf{p}_2 + 3t^2\mathbf{p}_3$ , so  $\mathbf{x}'(0) = -3\mathbf{p}_0 + 3\mathbf{p}_1 = 3(\mathbf{p}_1 - \mathbf{p}_0)$ , and  $\mathbf{x}'(1) = -3\mathbf{p}_2 + 3\mathbf{p}_3 = 3(\mathbf{p}_3 - \mathbf{p}_2)$ . This shows that the tangent vector  $\mathbf{x}'(0)$  points in the direction from  $\mathbf{p}_0$  to  $\mathbf{p}_1$  and is three times the length of  $\mathbf{p}_1 - \mathbf{p}_0$ . Likewise,  $\mathbf{x}'(1)$  points in the direction from  $\mathbf{p}_2$  to  $\mathbf{p}_3$  and is three times the length of  $\mathbf{p}_3 - \mathbf{p}_2$ . In particular,  $\mathbf{x}'(1) = \mathbf{0}$  if and only if  $\mathbf{p}_3 = \mathbf{p}_2$ .
- b.  $\mathbf{x}''(t) = (6 - 6t)\mathbf{p}_0 + (-12 + 18t)\mathbf{p}_1 + (6 - 18t)\mathbf{p}_2 + 6t\mathbf{p}_3$ , so that  $\mathbf{x}''(0) = 6\mathbf{p}_0 - 12\mathbf{p}_1 + 6\mathbf{p}_2 = 6(\mathbf{p}_0 - \mathbf{p}_1) + 6(\mathbf{p}_2 - \mathbf{p}_1)$  and  $\mathbf{x}''(1) = 6\mathbf{p}_1 - 12\mathbf{p}_2 + 6\mathbf{p}_3 = 6(\mathbf{p}_1 - \mathbf{p}_2) + 6(\mathbf{p}_3 - \mathbf{p}_2)$ . For a picture of  $\mathbf{x}''(0)$ , construct a coordinate system with the origin at  $\mathbf{p}_1$ , temporarily, label  $\mathbf{p}_0$  as  $\mathbf{p}_0 - \mathbf{p}_1$ , and label  $\mathbf{p}_2$  as  $\mathbf{p}_2 - \mathbf{p}_1$ . Finally, construct a line from

**A54** Answers to Odd-Numbered Exercises

this new origin through the sum of  $\mathbf{p}_0 - \mathbf{p}_1$  and  $\mathbf{p}_2 - \mathbf{p}_1$ , extended out a bit. That line points in the direction of  $\mathbf{x}''(0)$ .



5. a. From Exercise 3(a) or equation (9) in the text,

$$\mathbf{x}'(1) = 3(\mathbf{p}_3 - \mathbf{p}_2)$$

Use the formula for  $\mathbf{x}'(0)$ , with the control points from  $\mathbf{y}(t)$ , and obtain

$$\mathbf{y}'(0) = 3\mathbf{p}_3 + 3\mathbf{p}_4 = 3(\mathbf{p}_4 - \mathbf{p}_3)$$

For  $C^1$  continuity,  $3(\mathbf{p}_3 - \mathbf{p}_2) = 3(\mathbf{p}_4 - \mathbf{p}_3)$ , so  $\mathbf{p}_3 = (\mathbf{p}_4 + \mathbf{p}_2)/2$ , and  $\mathbf{p}_3$  is the midpoint of the line segment from  $\mathbf{p}_2$  to  $\mathbf{p}_4$ .

- b. If  $\mathbf{x}'(1) = \mathbf{y}'(0) = \mathbf{0}$ , then  $\mathbf{p}_2 = \mathbf{p}_3$  and  $\mathbf{p}_3 = \mathbf{p}_4$ . Thus, the “line segment” from  $\mathbf{p}_2$  to  $\mathbf{p}_4$  is just the point  $\mathbf{p}_3$ . [Note: In this case, the combined curve is still  $C^1$  continuous, by definition. However, some choices of the other “control” points,  $\mathbf{p}_0$ ,  $\mathbf{p}_1$ ,  $\mathbf{p}_5$ , and  $\mathbf{p}_6$ , can produce a curve with a visible corner at  $\mathbf{p}_3$ , in which case the curve is not  $G^1$  continuous at  $\mathbf{p}_3$ .]

7. Hint: Use  $\mathbf{x}''(t)$  from Exercise 3 and adapt this for the second curve to see that

$$\mathbf{y}''(t) = 6(1-t)\mathbf{p}_3 + 6(-2+3t)\mathbf{p}_4 + 6(1-3t)\mathbf{p}_5 + 6t\mathbf{p}_6$$

Then set  $\mathbf{x}''(1) = \mathbf{y}''(0)$ . Since the curve is  $C^1$  continuous at  $\mathbf{p}_3$ , Exercise 5(a) says that the point  $\mathbf{p}_3$  is the midpoint of the segment from  $\mathbf{p}_2$  to  $\mathbf{p}_4$ . This implies that

$\mathbf{p}_4 - \mathbf{p}_3 = \mathbf{p}_3 - \mathbf{p}_2$ . Use this substitution to show that  $\mathbf{p}_4$  and  $\mathbf{p}_5$  are uniquely determined by  $\mathbf{p}_1$ ,  $\mathbf{p}_2$ , and  $\mathbf{p}_3$ . Only  $\mathbf{p}_6$  can be chosen arbitrarily.

9. Write a vector of the polynomial weights for  $\mathbf{x}(t)$ , expand the polynomial weights, and factor the vector as  $M_B \mathbf{u}(t)$ :

$$\begin{bmatrix} 1 - 4t + 6t^2 - 4t^3 + t^4 \\ 4t - 12t^2 + 12t^3 - 4t^4 \\ 6t^2 - 12t^3 + 6t^4 \\ 4t^3 - 4t^4 \\ t^4 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ 0 & 4 & -12 & 12 & -4 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \end{bmatrix},$$

$$M_B = \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ 0 & 4 & -12 & 12 & -4 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

11. See the *Study Guide*.

13. a. Hint: Use the fact that  $\mathbf{q}_0 = \mathbf{p}_0$ .

- b. Multiply the first and last parts of equation (13) by  $\frac{8}{3}$  and solve for  $8\mathbf{q}_2$ .  
c. Use equation (8) to substitute for  $8\mathbf{q}_3$  and then apply part (a).

15. a. From equation (11),  $\mathbf{y}'(1) = .5\mathbf{x}'(.5) = \mathbf{z}'(0)$ .

- b. Observe that  $\mathbf{y}'(1) = 3(\mathbf{q}_3 - \mathbf{q}_2)$ . This follows from equation (9), with  $\mathbf{y}(t)$  and its control points in place of  $\mathbf{x}(t)$  and its control points. Similarly, for  $\mathbf{z}(t)$  and its control points,  $\mathbf{z}'(0) = 3(\mathbf{r}_1 - \mathbf{r}_0)$ . By part (a),  $3(\mathbf{q}_3 - \mathbf{q}_2) = 3(\mathbf{r}_1 - \mathbf{r}_0)$ . Replace  $\mathbf{r}_0$  by  $\mathbf{q}_3$ , and obtain  $\mathbf{q}_3 - \mathbf{q}_2 = \mathbf{r}_1 - \mathbf{q}_3$ , and hence  $\mathbf{q}_3 = (\mathbf{q}_2 + \mathbf{r}_1)/2$ .

- c. Set  $\mathbf{q}_0 = \mathbf{p}_0$  and  $\mathbf{r}_3 = \mathbf{p}_3$ . Compute  $\mathbf{q}_1 = (\mathbf{p}_0 + \mathbf{p}_1)/2$  and  $\mathbf{r}_2 = (\mathbf{p}_2 + \mathbf{p}_3)/2$ . Compute  $\mathbf{m} = (\mathbf{p}_1 + \mathbf{p}_2)/2$ . Compute  $\mathbf{q}_2 = (\mathbf{q}_1 + \mathbf{m})/2$  and  $\mathbf{r}_1 = (\mathbf{m} + \mathbf{r}_2)/2$ . Compute  $\mathbf{q}_3 = (\mathbf{q}_2 + \mathbf{r}_1)/2$  and set  $\mathbf{r}_0 = \mathbf{q}_3$ .

17. a.  $\mathbf{r}_0 = \mathbf{p}_0$ ,  $\mathbf{r}_1 = \frac{\mathbf{p}_0 + 2\mathbf{p}_1}{3}$ ,  $\mathbf{r}_2 = \frac{2\mathbf{p}_1 + \mathbf{p}_2}{3}$ ,  $\mathbf{r}_3 = \mathbf{p}_2$

- b. Hint: Write the standard formula (7) in this section, with  $\mathbf{r}_i$  in place of  $\mathbf{p}_i$  for  $i = 0, \dots, 3$ , and then replace  $\mathbf{r}_0$  and  $\mathbf{r}_3$  by  $\mathbf{p}_0$  and  $\mathbf{p}_2$ , respectively:

$$\begin{aligned} \mathbf{x}(t) = & (1 - 3t + 3t^2 - t^3)\mathbf{p}_0 \\ & + (3t - 6t^2 + 3t^3)\mathbf{r}_1 \\ & + (3t^2 - 3t^3)\mathbf{r}_2 + t^3\mathbf{p}_2 \end{aligned} \quad (iii)$$

Use the formulas for  $\mathbf{r}_1$  and  $\mathbf{r}_2$  from part (a) to examine the second and third terms in this expression for  $\mathbf{x}(t)$ .

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