15. Dynamic Games

15.1 One-sided incomplete information

At the conceptual level, situations of incomplete information involving dynamic (or extensive-form) games are essentially the same as situations involving static games: the only difference in the representation is that one would associate with every state a dynamic game instead of a static game.[1]

As in the case of static games, we will distinguish between one-sided and multi-sided incomplete information. In this section we will go through two examples of the former, while the latter will be discussed in Section 15.2.

Recall that a situation of incomplete information is said to be one-sided if only one of the players has some uncertainty about what game is being played. In Chapter 14 we restricted attention to the case where the uncertainty is about the payoffs and we will continue to do so in this chapter. The player who is uncertain about the game is often referred to as the "uninformed" player, whereas the other players are referred to as the "informed" players. It is common knowledge among all the players that the informed players know what game they are playing; furthermore, the beliefs of the uninformed player are common knowledge among all the players.

[1]The reader might have noticed that in Exercise 14.2 (Chapter 14) we "sneaked in" a dynamic game (where Bill moved first and decided whether or not to offer a gift and Ann – if offered a gift – decided whether or not to accept it). However, the game can also be written as a simultaneous game, where Ann decides whether or not to accept before knowing whether Bill will offer a gift (without knowing Ann's decision).

Consider the perfect-information frame shown in Figure 15.1, where the question marks in place of Player 2's payoffs indicate that her payoffs are not common knowledge between the two players.

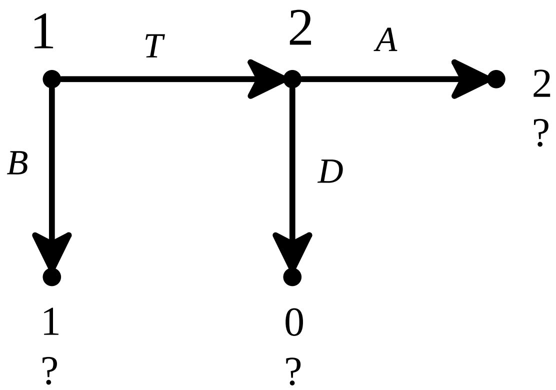

Figure 15.1: A perfect-information game-frame where the question marks indicate uncertainty about Player 2's payoffs.

Suppose that Player 1's payoffs are common knowledge between the two players. Furthermore, suppose that Player 1 is uncertain between the two possibilities shown in Figure 15.2 and assigns probability $\frac{1}{3}$ to the one on the left and probability $\frac{2}{3}$ to the one on the right. The thick edges represent the backward-induction solutions of the two games. Thus, if Player 1 knew that he was playing the game on the left, he would choose B, while if he knew that he was playing the game on the right, he would choose T.

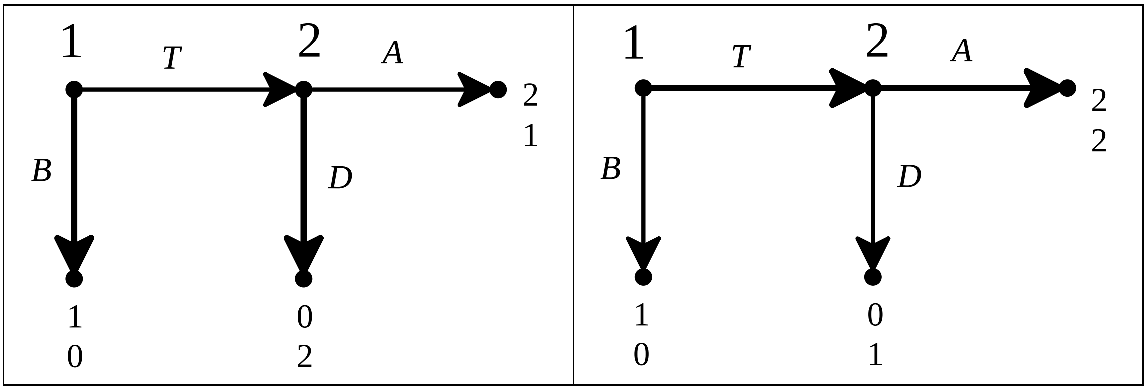

Figure 15.2: The two possibilities in the mind of Player 1.

If we assume that Player 1's beliefs are common knowledge between the players, then we have the situation of one-sided incomplete information shown in Figure 15.3. Let us take state α to be the true state.
Using the Harsanyi transformation we can convert the situation illustrated in Figure 15.3 into the extensive-form game shown in Figure 15.4.

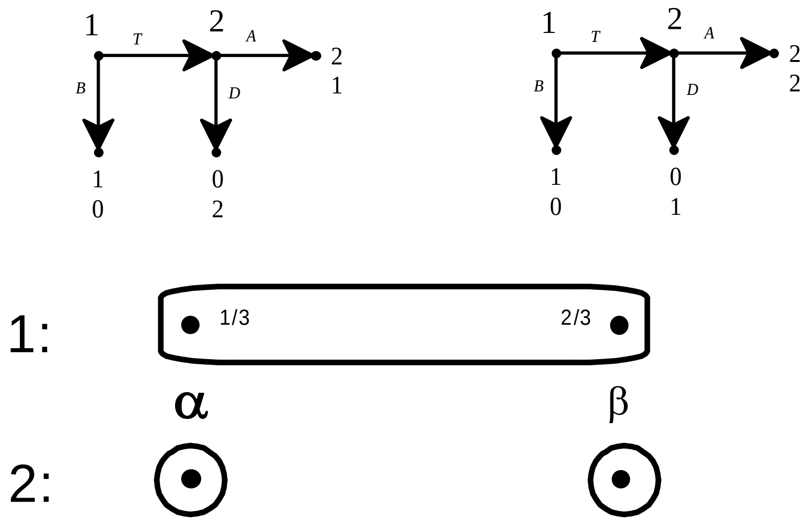

Figure 15.3: A one-sided situation of incomplete information involving the two games of Figure 15.2.

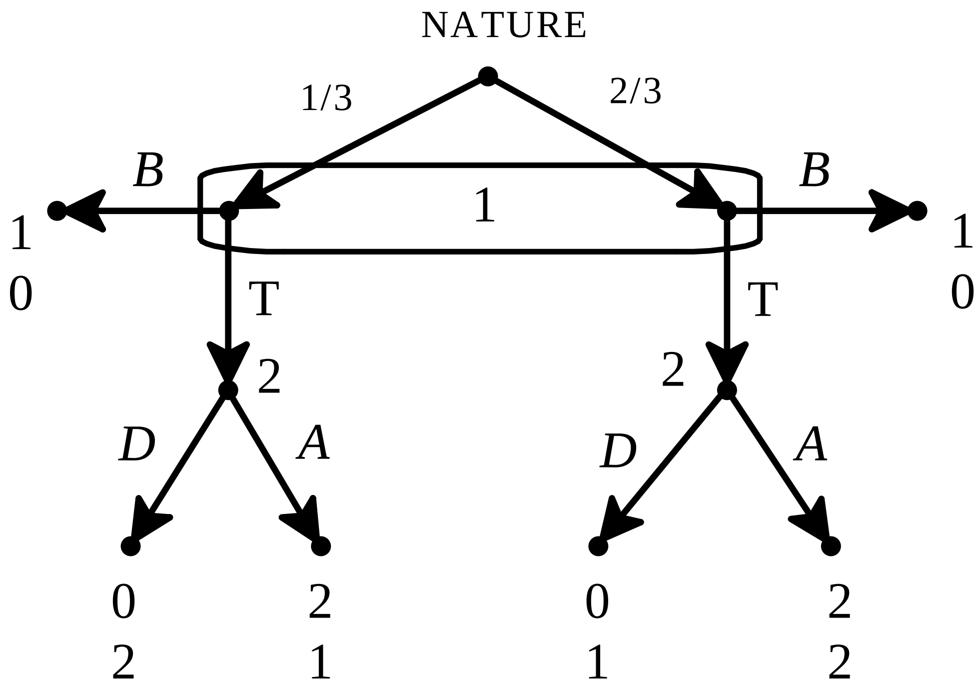

Figure 15.4: The game obtained by applying the Harsanyi transformation to the incomplete-information situation of Figure 15.3.

The solution concept used for the case of static games was Bayesian Nash equilibrium (that is, Nash equilibrium of the imperfect-information game obtained by applying the Harsanyi transformation to the situation of incomplete information).

In the case of dynamic games this is no longer an appropriate solution concept, since – as we know from Chapter 4 – it allows a player to "choose" a strictly dominated action at an unreached information set. To see this, consider the strategic-form game associated with the game of Figure 15.4, shown in Figure 15.5.

		Player 2			
		DD	*DA*	*AD*	*AA*
Player 1	*B*	1 0	1 0	1 0	1 0
	T	0 4/3	4/3 2	2/3 1	2 5/3

Figure 15.5: The strategic form of the game of Figure 15.4.

The Nash equilibria of the strategic-form game shown in Figure 15.5 (and thus the Bayesian Nash equilibria of the game shown in Figure 15.4) are:

$$(B, DD), \qquad (B, AD) \qquad \text{and} \qquad (T, DA).$$

Of these, only (T, DA) is a subgame-perfect equilibrium. From now on we shall use either the notion of subgame-perfect equilibrium or the notion of weak sequential equilibrium (in the game of Figure 15.4 the two notions coincide). Thus we will take (T, DA) to be the solution of the game of Figure 15.4. Since we postulated that the true game was the one associated with state α, as a matter of fact the players end up playing (T, D) which is not the backward induction solution of the true game. As pointed out in Chapter 14, this is not surprising given the uncertainty in the mind of Player 1 as to which game she is playing.

Next we consider a more complex example, built on Selten's Chain-Store game analyzed in Chapter 3 (Section 4). Recall that the story is as follows (we will consider the special case where m, the number of towns and thus of potential entrants, is 2). A chain store is a monopolist in an industry. It owns stores in two different towns. In each town the chain store makes \$5 million if left to enjoy its privileged position undisturbed. In each town there is a businesswoman who could enter the industry in that town, but earns \$1 million in an alternative investment if she chooses not to enter; if she decides to enter, then the monopolist can either fight the entrant, leading to zero profits for both the chain store and the entrant in that town, or it can accommodate entry and share the market with the entrant, in which case both players make \$1.5 million in that town. Thus, in each town the interaction between the incumbent monopolist and the potential entrant is as illustrated in Figure 15.6.

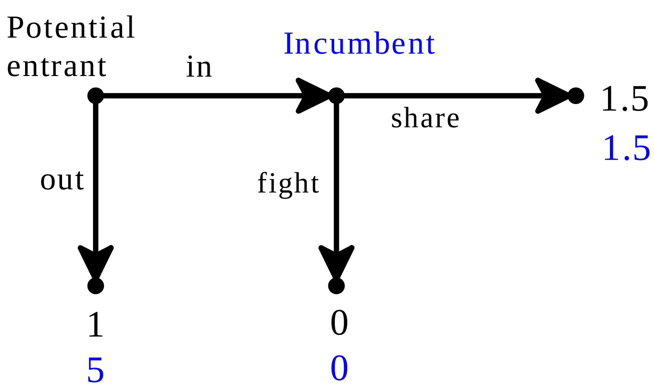

Figure 15.6: The game with only one potential entrant.

Decisions are made sequentially, as follows. At date t ($t = 1,2$) the businesswoman in town t decides whether or not to enter and, if she enters, then the chain store decides whether or not to fight in that town. What happens in town t (at date t) becomes known to everybody. Thus the businesswoman in town 2 at date 2 knows what happened in town 1 at date 1 (either that there was no entry or that entry was met with a fight or that entry was accommodated).

Intuition suggests that in this game the threat by the Incumbent to fight early entrants might be credible, for the following reason. The Incumbent could tell Businesswoman 1 the following:

> "It is true that, if you enter and I fight you, I will make zero profits, while by accommodating your entry I would make \$1.5 million and thus it would seem that it cannot be in my interest to fight you. However, somebody else is watching us, namely Businesswoman 2. If she sees that I have fought your entry then she might fear that I would do the same with her and decide to stay out, in which case, in town 2, I would make \$5 million, so that my total profits in towns 1 and 2 would be $\$(0+5) = \$5$ million. On the other hand, if I accommodate your entry, then she will be encouraged to enter herself and I will make \$1.5 million in each town, for a total profit of \$3 million. Hence, as you can see, it is indeed in my interest to fight you and thus you should stay out."

We showed in Chapter 3 that the notion of backward induction does not capture this intuition. In the game depicting the entire interaction (Figure 3.11, Chapter 3) there was a unique backward-induction solution whose corresponding outcome was that both businesswomen entered and the Incumbent accommodated entry in both towns. The reason why the backward-induction solution did not capture the "reputation" argument outlined above was explained in Chapter 3. We remarked there that, in order to capture the reputation effect, one would need to allow for some uncertainty in the mind of some of the players. This is what we now show.

Suppose that, in principle, there are two types of incumbent monopolists: the rational type and the hotheaded type. The payoffs of a rational type are shown in Figure 15.7 (a rational incumbent prefers 'share' to 'fight' if there is entry), while the payoffs of a hotheaded type are shown in Figure 15.8 (a hotheaded type prefers 'fight' to 'share' if there is entry). On the other hand, there is only one type of potential entrant.

		potential entrant	
		in	out
Incumbent	fight	0 , 0	5 , 1
	share	1.5 , 1.5	5 , 1

Figure 15.7: The payoffs of a **rational incumbent**.

		potential entrant	
		in	out
Incumbent	fight	2 , 0	5 , 1
	share	1.5 , 1.5	5 , 1

Figure 15.8: The payoffs of a **hotheaded incumbent**.

Consider the following situation of one-sided incomplete information.

- As a matter of fact, the Incumbent is rational and this fact is common knowledge between the Incumbent and Potential Entrant 1 (from now on denoted by PE-1).
- Potential Entrant 2 (PE-2) is uncertain whether the Incumbent is rational or hotheaded and attaches probability p to the latter case. The beliefs of PE-2 are common knowledge as are the payoffs of PE-1 and PE-2.

This situation can be illustrated with the knowledge-belief structure of Figure 15.9.

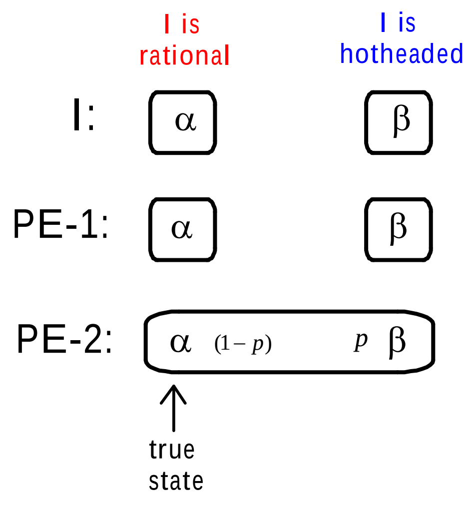

Figure 15.9: The one-sided incomplete-information situation.

Applying the Harsanyi transformation to the situation depicted in Figure 15.9 yields the extensive-form game shown in Figure 15.10. We want to see if this situation of one-sided incomplete information can indeed capture the reputation effect discussed above; in particular, we want to check if there is a weak sequential equilibrium of the game of Figure 15.10 where the Incumbent's strategy would be to fight *PE*-1's entry in order to scare off *PE*-2 and, as a consequence, *PE*-1 decides to stay out and so does *PE*-2.

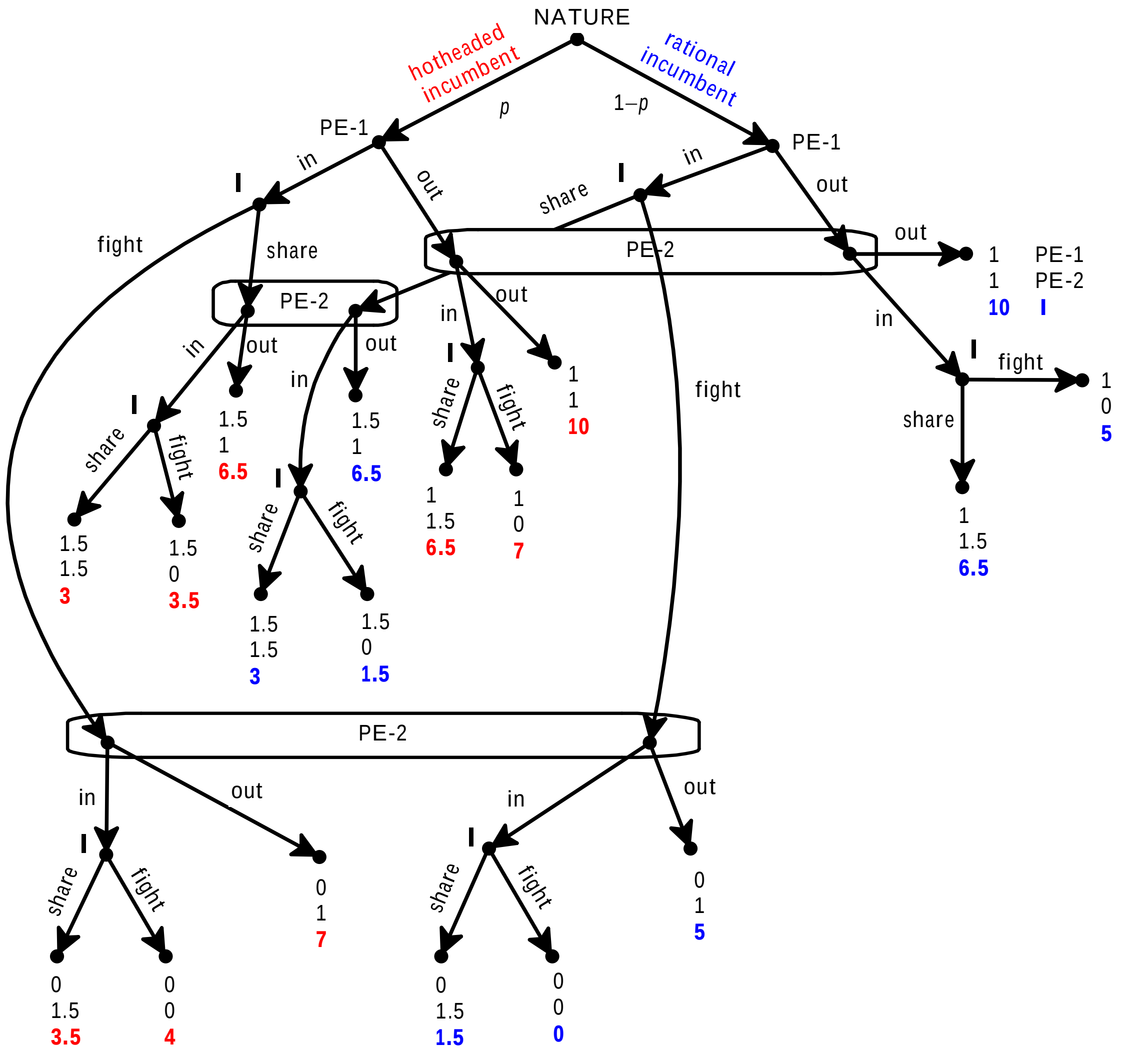

Figure 15.10: The game obtained by applying the Harsanyi transformation to the one-sided situation of incomplete information of Figure 15.9.

The notion of weak sequential equilibrium allows us to simplify the game, by selecting the strictly dominant choice for the Incumbent at each of his singleton nodes followed only by terminal nodes. It is straightforward to check that at such nodes a hotheaded Incumbent would choose "fight" while a rational Incumbent would choose "share". Thus we can delete those decision nodes and replace them with the payoff vectors associated with the optimal choice.

The simplified game is shown in Figure 15.11.

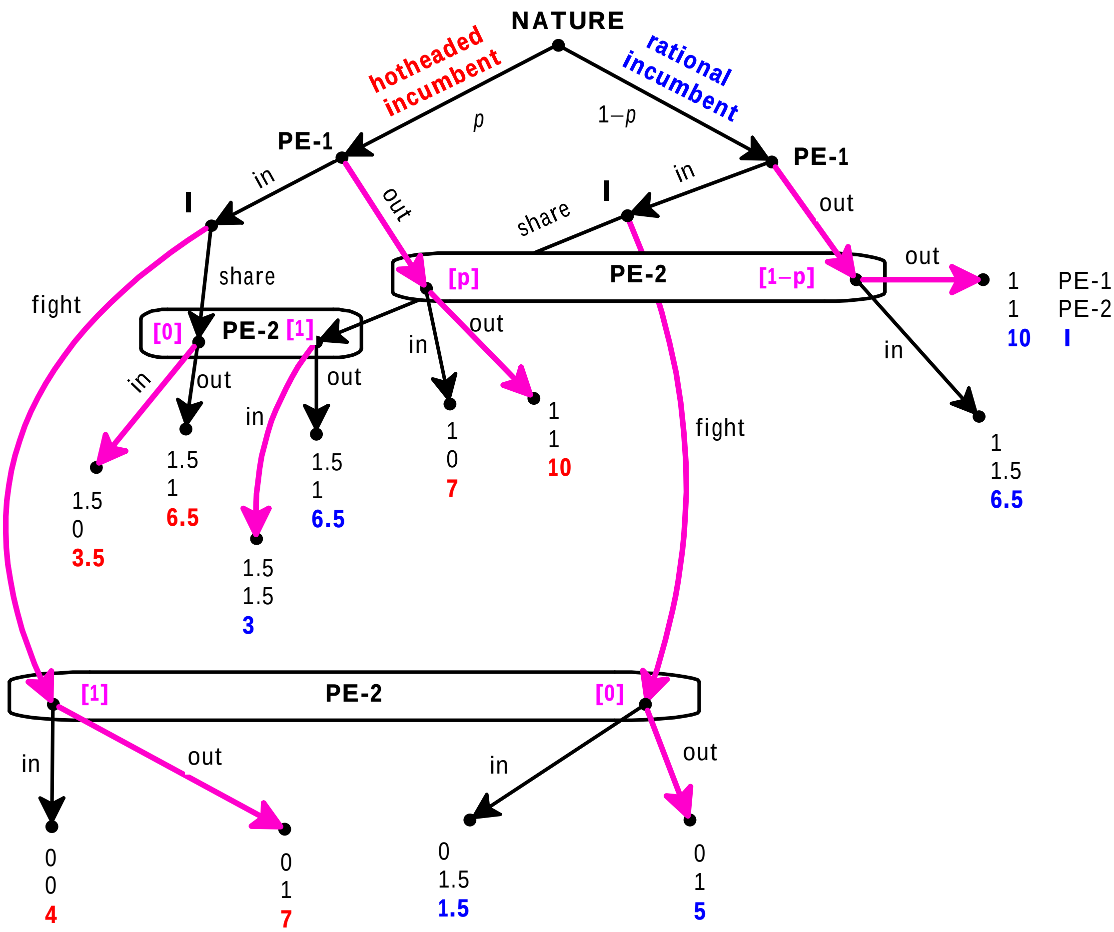

Figure 15.11: The reduced game obtained from Figure 15.10 by eliminating strictly dominated choices.

Consider the following pure-strategy profile, call it σ, for the simplified game of Figure 15.11 (it is highlighted by thick arrows in Figure 15.11):

1. *PE*-1's strategy is "out" at both nodes,
2. *PE*-2's strategy is
   - "out" at the top information set (after having observed that *PE*-1 stayed out),
   - "in" at the middle information set (after having observed that *PE*-1's entry was followed by the Incumbent sharing the market with *PE*-1)
   - "out" at the bottom information set (after having observed that *PE*-1's entry was followed by the Incumbent fighting against *PE*-1),
3. the Incumbent's strategy is to fight entry of *PE*-1 in any case (that is, whether the Incumbent himself is hotheaded or rational).

We want to show that σ, together with the system of beliefs μ shown in square brackets in Figure 15.11 (at her top information PE-2 assigns probability p to the Incumbent being hotheaded, at her middle information set she assigns probability 1 to the Incumbent being rational and at her bottom information set she assigns probability 1 to the Incumbent being hotheaded) constitutes a weak sequential equilibrium for any value of $p \geq \frac{1}{3}$.

Bayesian updating is satisfied at PE-2's top information set (the only information set reached by σ), and at the other two information sets of PE-2 any beliefs are allowed by the notion of weak sequential equilibrium.

Sequential rationality also holds:

- For PE-1, at the left node "in" yields 0 and "out" yields 1, so that "out" is sequentially rational; the same is true at the right node.
- For the Incumbent,
  - at the left node "fight" yields 7 and "share" yields 3.5, so that "fight" is sequentially rational;
  - at the right node "fight" yields 5 and "share" yields 3, so that "fight" is sequentially rational.
- For PE-2,
  - at the top information set "in" yields an expected payoff of $p(0)+(1-p)(1.5)$ and "out" yields 1, so that "out" is sequentially rational as long as $p \geq \frac{1}{3}$;
  - at the middle information set, given her belief that I is rational, "in" yields 1.5 and "out" yields 1, so that "in" is sequentially rational;
  - at the bottom information set, given her belief that I is hotheaded, "in" yields 0 and "out" yields 1, so that "out" is sequentially rational.

Thus the equilibrium described above captures the intuition suggested in Chapter 3, namely that – even though it is common knowledge between the Incumbent and PE-1 that the Incumbent is rational and thus would suffer a loss of 1.5 by fighting PE-1's entry – it is still credible for the Incumbent to threaten to fight PE-1's entry because it would influence the beliefs of PE-2 and induce her to stay out; understanding the credibility of this threat, it is optimal for PE-1 to stay out.

The above argument exploits the fact that the notion of weak sequential equilibrium allows for any beliefs whatsoever at unreached information sets. However, the beliefs postulated for PE-2 at her middle and bottom information sets seem highly reasonable. Indeed, as shown in Exercise 15.2, the assessment described above is also a sequential equilibrium.

We now turn to an example that deals with the issue of labor-management negotiations and the inefficiency of strikes. It is not uncommon to observe a firm and a union engaging in unsuccessful negotiations leading to a strike by the workers, followed by renewed negotiations and a final agreement. Strikes are costly for the workers, in terms of lost wages, and for the firm, in terms of lost production. Why, then, don't the parties reach the agreement at the very beginning, thus avoiding the inefficiency of a strike? The answer in many cases has to do with the fact that there is incomplete information on the side of the labor union and enduring a strike is the only credible way for the firm to convince the union to reduce its demands. We shall illustrate this in a simple example of one-sided incomplete information.

Consider the following game between a new firm and a labor union. The union requests a wage (either high, w_H, or low, w_L) and the firm can either accept or reject. If the union's request is accepted, then a contract is signed and production starts immediately. If the request is rejected, then the union goes on strike for one period and at the end of that period makes a second, and last, request to the firm, which the firm can accept or reject. If the firm rejects, then it cannot enter the industry. When no agreement is signed, both parties get a payoff of 0.
Both the firm and the union have a discount factor of δ with $0 < \delta < 1$, which means that \$1 accrued one period into the future is considered to be equivalent to \$$\delta$ at the present time (this captures the cost of waiting and thus the desirability of avoiding a strike).

The extensive-form game is shown in Figure 15.12, where π denotes the firm's profit (gross of labor costs).

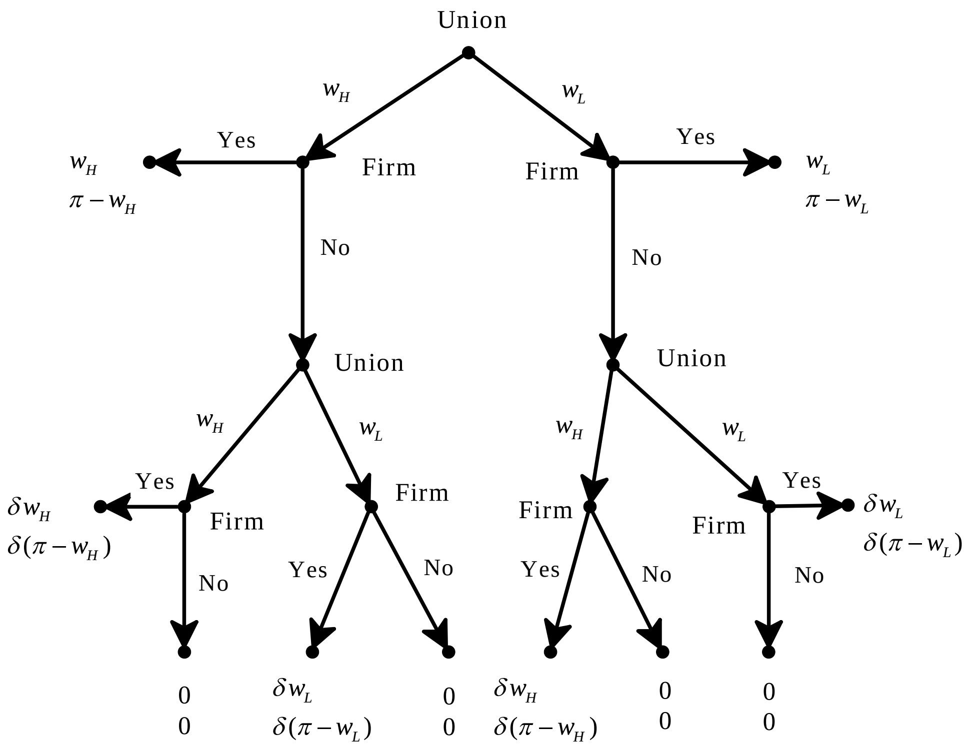

Figure 15.12: The structure of the wage bargaining game.

Suppose, however, that the union does not know the firm's expected profit π (gross of labor costs). It is not unlikely that the management will know what profits to expect, while the union will not (because, for example, the union does not have enough information on the intensity of competition, the firm's non-labor costs, etc.).

Suppose that we have a one-sided incomplete information situation where the union believes that π can have two values: π_H (high) and π_L (low) and assigns probability α to π_H and (1-α) to π_L.

This situation of one-sided incomplete information is illustrated in Figure 15.13.

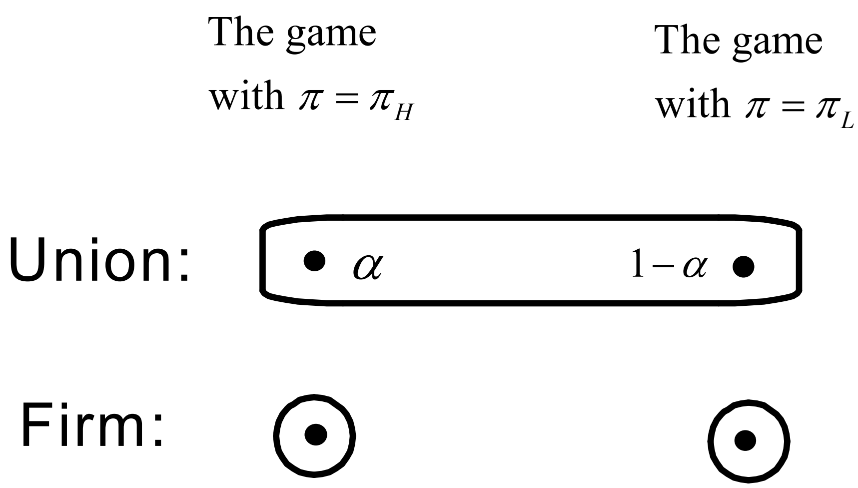

Figure 15.13: The one-sided situation of incomplete information involving the wage bargaining game of Figure 15.12.

Let $\pi_H > \pi_L > 0$ and $w_H > w_L > 0$ (so that H means "high" and L means "low").

We also assume that

- $\pi_H - w_H > 0$ (so that the high-profit firm could in fact afford to pay a high wage),
- $\pi_L - w_L > 0$ (that is, the low wage is sufficiently low for the low-profit firm to be able to pay it), and
- $\pi_L - w_H < 0$ (that is, the low-profit firm cannot afford to pay a high wage).

Finally, we assume that the true state is the one where $\pi = \pi_L$, that is, the firm's potential profits are in fact low. These assumptions imply that it is in the interest of both the firm and the union to sign a contract (even if the firm's profits are low).

Using the Harsanyi transformation we can convert the situation shown in Figure 15.13 into the extensive-form game shown in Figure 15.14.

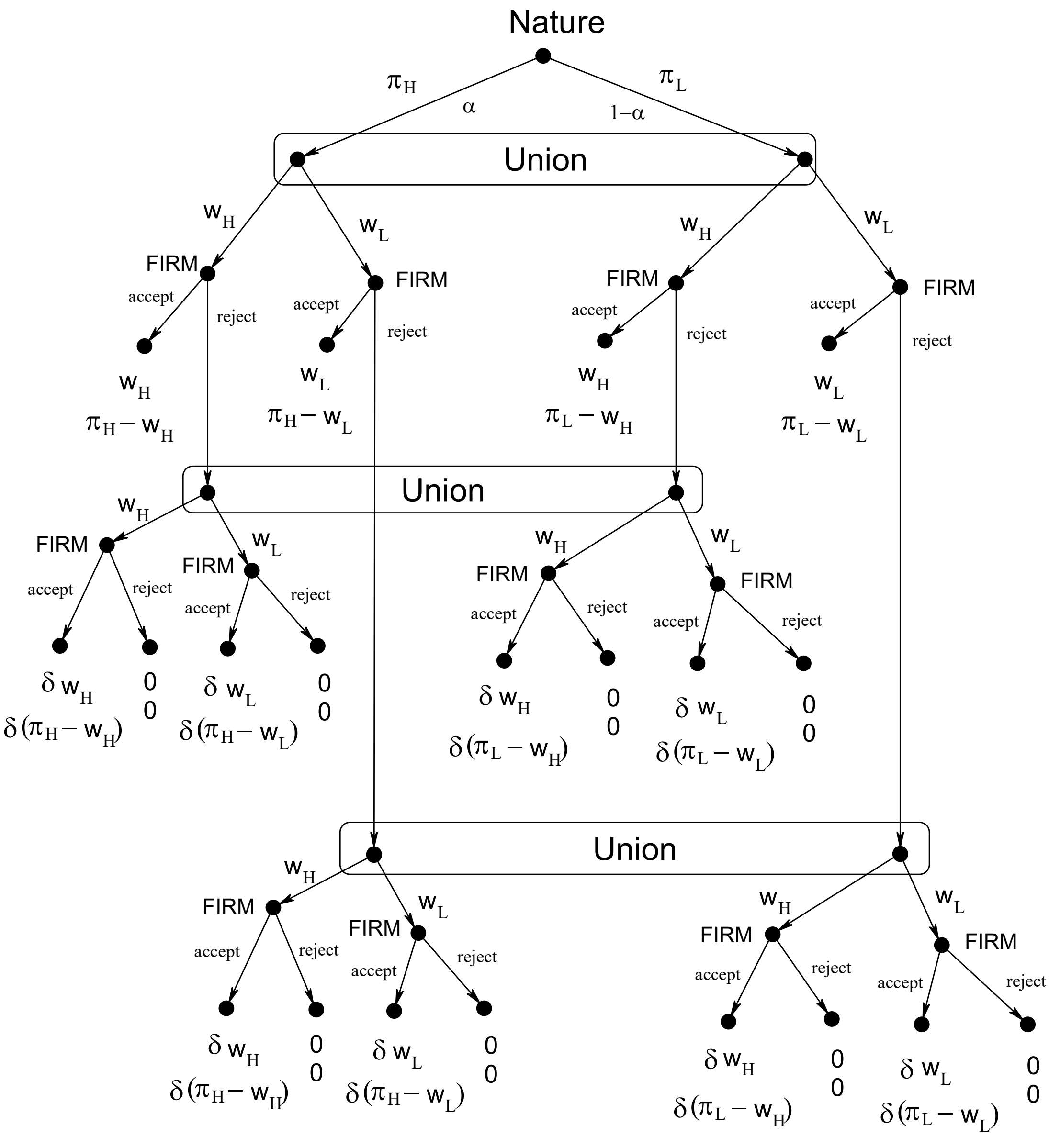

Figure 15.14: The game obtained by applying the Harsanyi transformation to the situation of incomplete information of Figure 15.13.

Let us find conditions under which the following strategy profile is part of a separating weak sequential equilibrium:

1. The union requests a high wage in the first period.

2. The high-profit firm accepts while the low-profit firm rejects.

3. After rejection of the first-period high-wage offer the union requests a low wage and the firm accepts.

That is, we want an equilibrium where the low-profit firm endures a strike to signal to the union that its expected profits are low and cannot afford to pay a high wage. The union reacts to the signal by lowering its demand.

Of course, we need to worry about the fact that the high-profit firm might want to masquerade as a low-profit firm by rejecting the first-period offer (that is, by sending the same signal as the low-profit firm).

First of all, we can simplify the game by replacing the second-period choice of the firm with the outcome that follows the best choice for the firm (which is "accept w_L" for both the π_H and the π_L firm, "accept w_H" for the π_H firm and "reject w_H" for the π_L firm).

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The simplified game is shown in Figure 15.15.

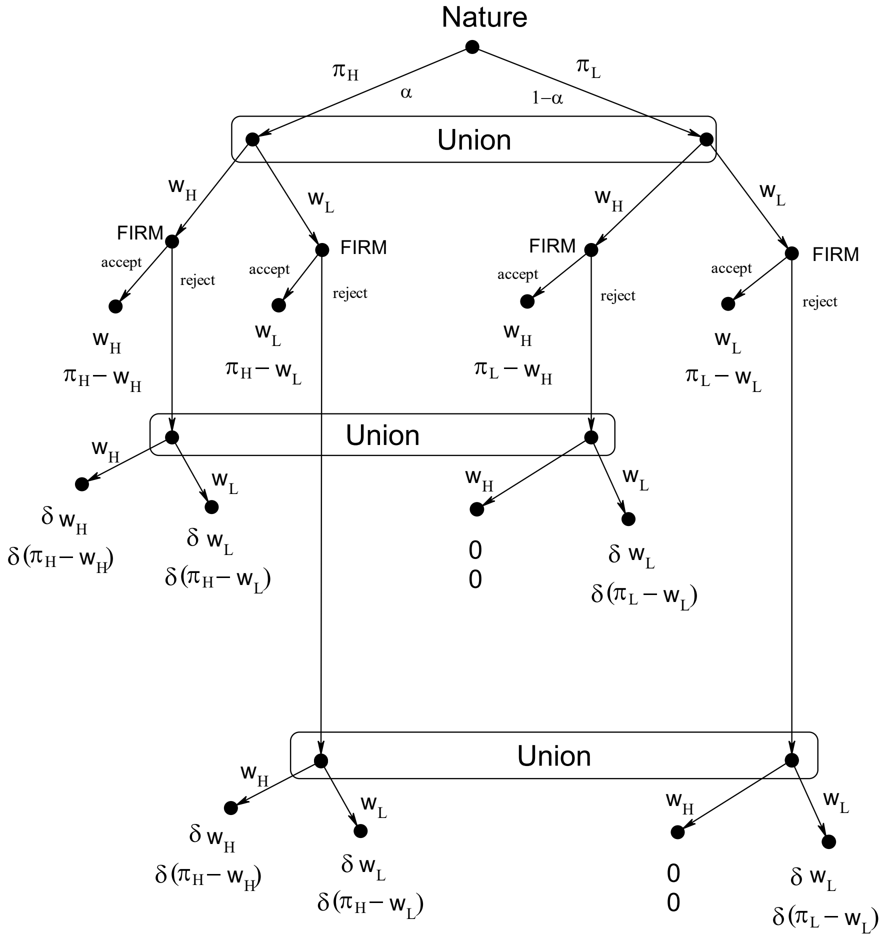


Figure 15.15: The simplified game after eliminating the second-period choice of the firm.

In order for a low-wage offer to be optimal for the union in Period 2 it is necessary for the union to assign sufficiently high probability to the firm being a low-profit one: if p is this probability, then we need the following (for example, $p = 1$ would be fine):

$$\delta w_L \geq (1-p)\delta w_H, \qquad \text{that is,} \qquad p \geq 1 - \frac{w_L}{w_H}$$

For the high-profit firm not to have an incentive to send the same signal as the low-profit firm it is necessary that

$$\pi_H - w_H \geq \delta\left(\pi_H - w_L\right), \qquad \text{that is,} \qquad \delta \leq \frac{\pi_H - w_H}{\pi_H - w_L}.$$

If this condition is satisfied, then the high-profit firm will accept w_H in period 1, the low-profit firm will reject and Bayesian updating requires the union to assign probability 1 to the right node of its middle information set (that is, to the firm being a low-profit type), in which case adjusting its demand to w_L is sequentially rational.

On the other hand, requesting a low wage in period 1 will lead to both types of the firm accepting immediately. So in order for a high-wage offer to be optimal for the union in period 1 it is necessary that $w_L \leq \alpha w_H + (1-\alpha)\delta w_L$, which will be true if α is sufficiently large, that is, if

$$\alpha \geq \frac{(1-\delta)w_L}{w_H - \delta w_L}.$$

Finally, given the above equilibrium choices, the notion of weak sequential equilibrium imposes no restrictions on the beliefs (and hence choice) of the union at the bottom information set.

For example, all the above inequalities are satisfied if:

$$\pi_H = 100, \quad \pi_L = 55, \quad w_H = 60, \quad w_L = 50, \quad \alpha = 0.7, \quad \delta = 0.6.$$

Given the above values, $\pi_H - w_H = 40 > \delta(\pi_H - w_L) = 0.6(50) = 30$
and $w_L = 50 < \alpha w_H + (1-\alpha)\delta w_L = (0.7)60 + 0.3(0.6)50 = 51$.

We conclude this section with one more example that has to do with the effectiveness of truth-in-advertising laws.

Consider the case of a seller and a buyer. The seller knows the quality x of his product, while the buyer does not, although she knows that he knows. The buyer can thus ask the seller to reveal the information he has. Suppose that there is a truth-in-advertising law which imposes harsh penalties for false claims. This is not quite enough, because the seller can tell the truth without necessarily revealing all the information.

For example, if x is the octane content of gasoline and x = 89, then the following are all true statements:

"the octane content of this gasoline is at least 70"

"the octane content of this gasoline is at least 85"

"the octane content of this gasoline is at most 89"

"the octane content of this gasoline is exactly 89", etc.

Other examples are:

- the fat content of food (the label on a package of ground meat might read "not more than 30% fat"),
- fuel consumption for cars (a car might be advertised as yielding "at least 35 miles per gallon"),
- the label on a mixed-nut package might read "not more than 40% peanuts", etc.

An interesting question is: Would the seller reveal all the information he has or would he try to be as vague as possible (while being truthful)? In the latter case, will the buyer be in a worse position than she would be in the case of complete information?

Milgrom and Roberts (1986) consider the following game (resulting from applying the Harsanyi transformation to the situation of one-sided incomplete information described above).

1. First Nature selects the value of x, representing the seller's information, from a finite set X; as usual, the probabilities with which Nature chooses reflect the buyer's beliefs.
2. The seller "observes" Nature's choice and makes an assertion A to the buyer; A is a subset of X. The seller is restricted to make true assertions, that is, we require that $x \in A$.
3. The buyer observes the seller's claim A and then selects a quantity $q \geq 0$ to buy.

Milgrom and Roberts show that (under reasonable hypotheses), if the buyer adopts a skeptical view concerning the seller's claim, that is, she always interprets the seller's claim in a way which is least favorable to the seller (for example "not more than 30% fat" is interpreted as "exactly 30% fat"), then there is an equilibrium where the outcome is the same as it would be in the case of complete information. We shall illustrate this phenomenon by means of a simple example.

Suppose that there are three possible quality levels for the good under consideration: low (l), medium (m) and high (h); thus $X = \{l,m,h\}$. The buyer believes that the three quality levels are equally likely. The buyer has to choose whether to buy one unit or two units. The seller can only make truthful claims. For example, if the quality is l, the seller's possible claims are $\{l\}$ (full revelation) or vague claims such as $\{l,m\}$, $\{l,h\}$, $\{l,m,h\}$. The extensive-form game is shown in Figure 15.16.

It is straightforward to check that the following is a weak sequential equilibrium:

- The seller claims {1} if Nature chooses l, $\{m\}$ if Nature chooses m and $\{h\}$ if Nature chooses h (that is, the seller reveals the whole truth by choosing not to make vague claims).
- The buyer buys one unit if told $\{l\}$, two units if told $\{m\}$ or if told $\{h\}$; furthermore, the buyer adopts beliefs which are least favorable to the seller:
  - if told $\{l,m\}$ or $\{l,h\}$ or $\{l,m,h\}$ she will believe l with probability one and buy one unit,
  - if told $\{m,h\}$ she will believe m with probability one and buy two units.

  All these beliefs are admissible because they concern information sets that are not reached in equilibrium and therefore Bayesian updating does not apply.

On the other hand, as shown in Exercise 15.4, if the buyer is naïve then there is an equilibrium where the seller chooses to make vague claims.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 15.3.1 at the end of this chapter.

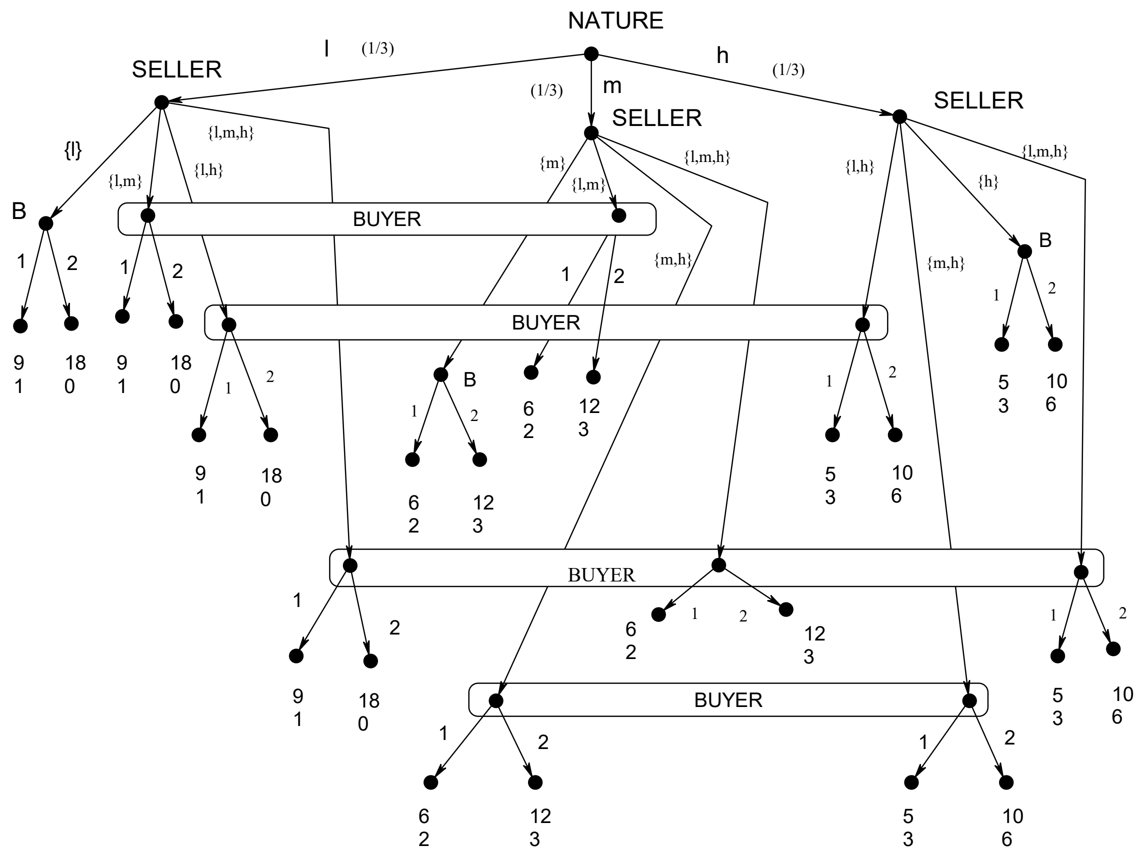


Figure 15.16: The buyer-seller game representing a situation of one-sided incomplete information.

15.2 Multi-sided incomplete information

The case of situations of multi-sided incomplete information involving dynamic games is conceptually the same as the case of multi-sided situations of incomplete information involving static games. In this section we shall go through one example.

Consider the following situation of two-sided incomplete information. A seller (player S) owns an item that a buyer (player B) would like to purchase. The seller's reservation price is s (that is, she is willing to sell if and only if the price paid by the buyer is at least s) and the buyer's reservation price is b (that is, he is willing to buy if and only if the price is less than or equal to b). It is common knowledge between the two that

- both b and s belong to the set $\{1,2,\ldots,n\}$,
- the buyer knows the value of b and the seller knows the value of s,
- both the buyer and the seller attach equal probability to all the possibilities among which they are uncertain.

Buyer and Seller play the following game. First the buyer makes an offer of a price $p \in \{1,\ldots,n\}$ to the seller. If $p = n$ the game ends and the object is exchanged for \$$p$.
If $p < n$ then the seller either accepts (in which case the game ends and the object is exchanged for p) or makes a counter-offer of $p' > p$, in which case either the buyer accepts (and the game ends and the object is exchanged for p') or the buyer rejects, in which case the game ends without an exchange.
Payoffs are as follows:

$$\pi_{seller} = \begin{cases} 0 & \text{if there is no exchange} \\ x - s & \text{if exchange takes place at price } \$x \end{cases}$$

$$\pi_{buyer} = \begin{cases} 0 & \text{if there is no exchange} \\ b - p & \text{if exchange takes place at price } \ \$p \ \ \text{(the initial offer)} \\ b - p' - \varepsilon & \text{if exchange takes place at price } \ \$p' \ \ \text{(the counter-offer)} \end{cases}$$

where $\varepsilon > 0$ is a measure of the buyer's "hurt feelings" for seeing his initial offer rejected. These are von Neumann-Morgenstern payoffs.

Let us start by focusing on the case $n = 2$. First we represent the situation described above by means of an interactive knowledge-belief structure. A possible state can be written as a pair (b,s), where b is the reservation price of the buyer and s that of the seller.

Thus when $n = 2$ the possible states are (1,1), (1,2), (2,1) and (2,2). Figure 15.17 represents this two-sided situation of incomplete information.

For each state there is a corresponding extensive-form game. The four possible games are shown in Figure 15.18 (the top number is the Buyer's payoff and the bottom number the Seller's payoff).

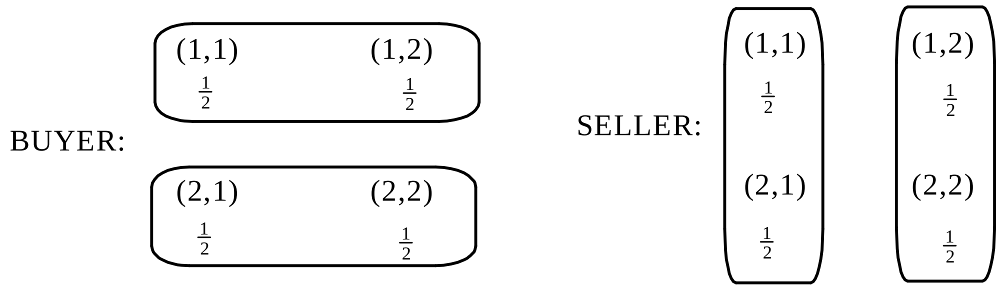


Figure 15.17: The two-sided situation of incomplete information between a buyer and a seller.

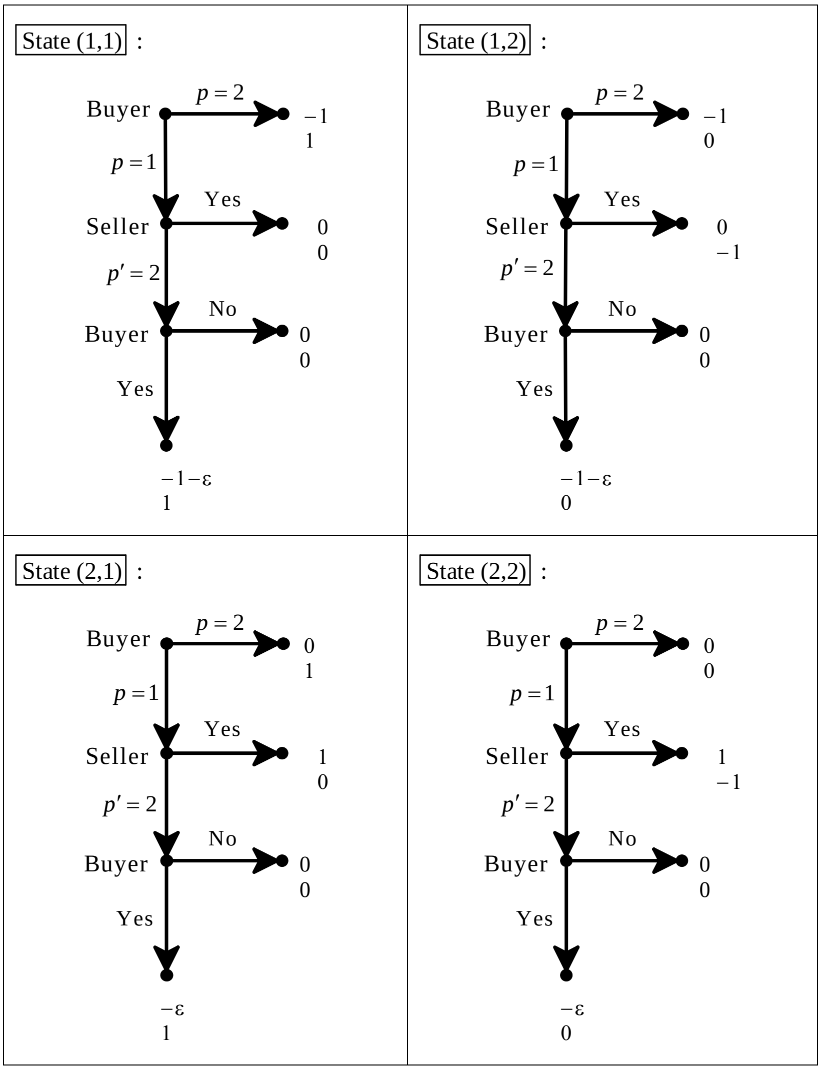


Figure 15.18: The four games corresponding to the four states of Figure 15.17.

The extensive-form game that results from applying the Harsanyi transformation to the situation illustrated in Figure 15.17 is shown in Figure 15.19.

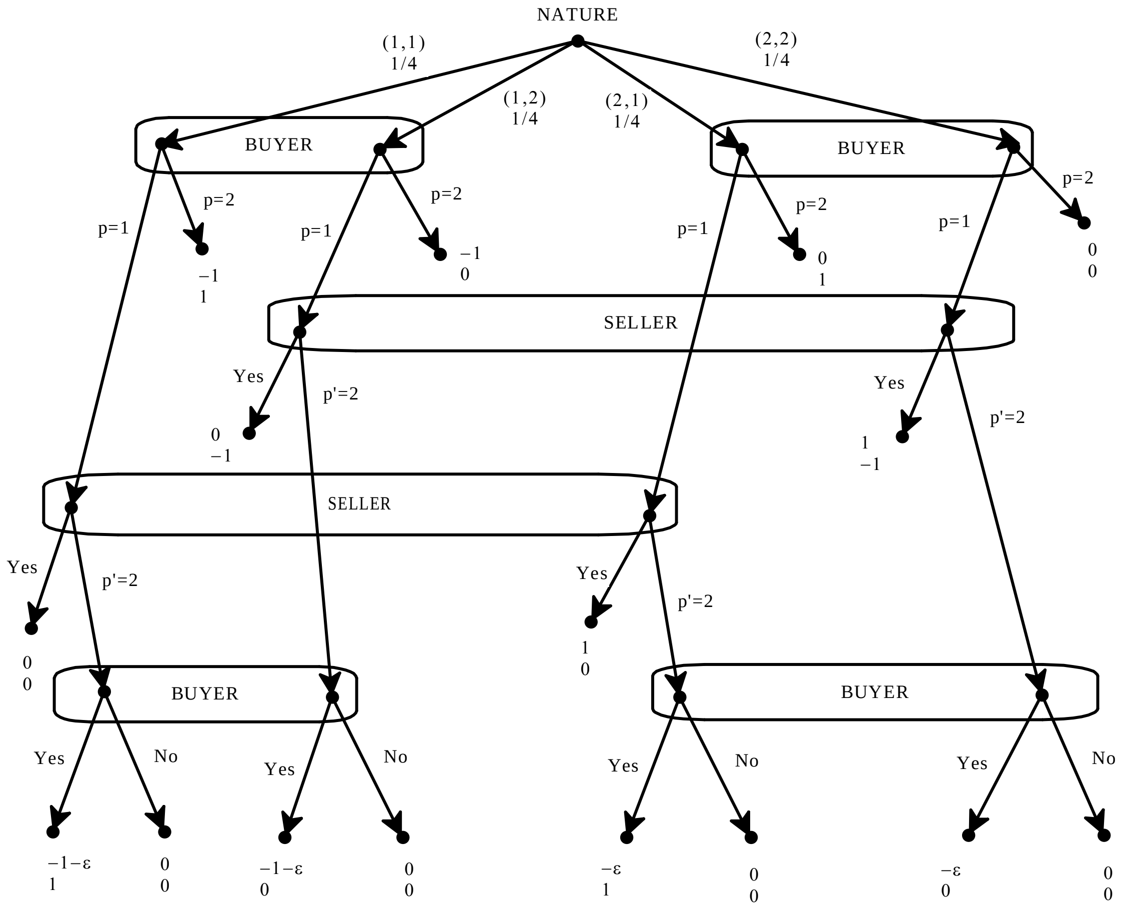


Figure 15.19: The extensive-form game that results from applying the Harsanyi transformation to the situation illustrated in Figure 15.17.

Let us find all the pure-strategy weak sequential equilibria of the game of Figure 15.19. First of all, note that at the bottom information sets of the Buyer, “Yes” is strictly dominated by “No” and thus a weak sequential equilibrium must select “No”.

Hence the game simplifies to the one shown in Figure 15.20.

In the game of Figure 15.20, at the middle information of the Seller, making a counteroffer of $p' = 2$ strictly dominates "Yes".

Thus the game can be further simplified as shown in Figure 15.21.

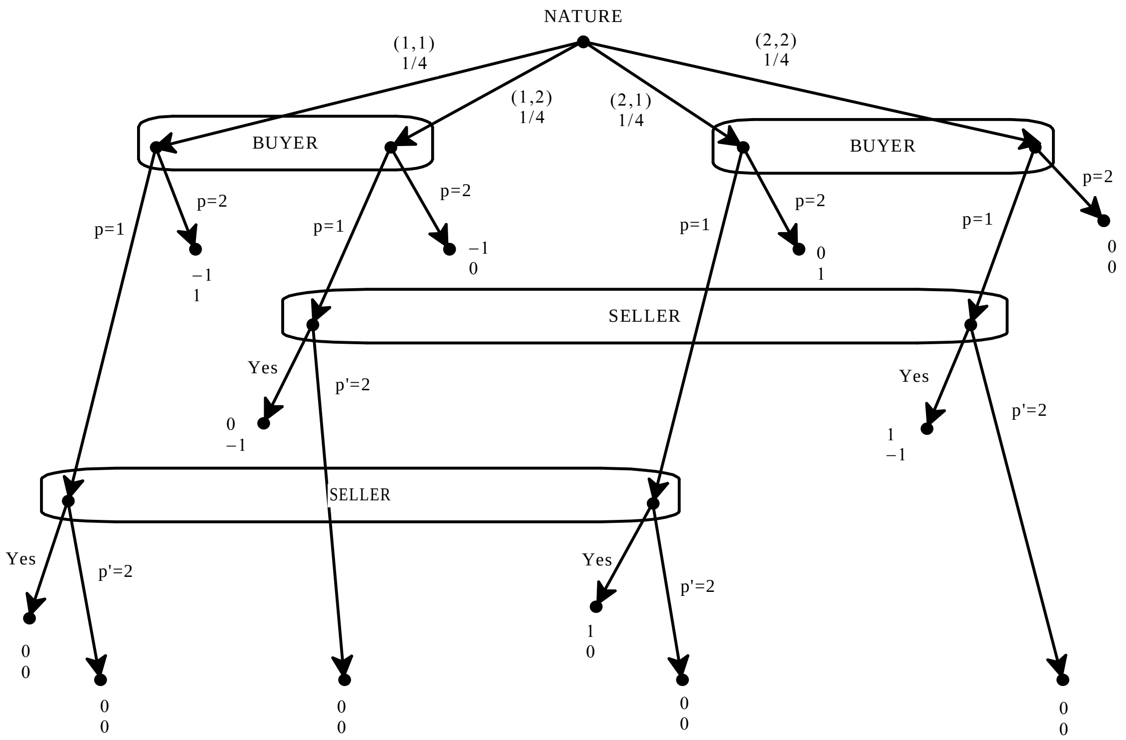


Figure 15.20: The reduced game after eliminating the strictly dominated choices.

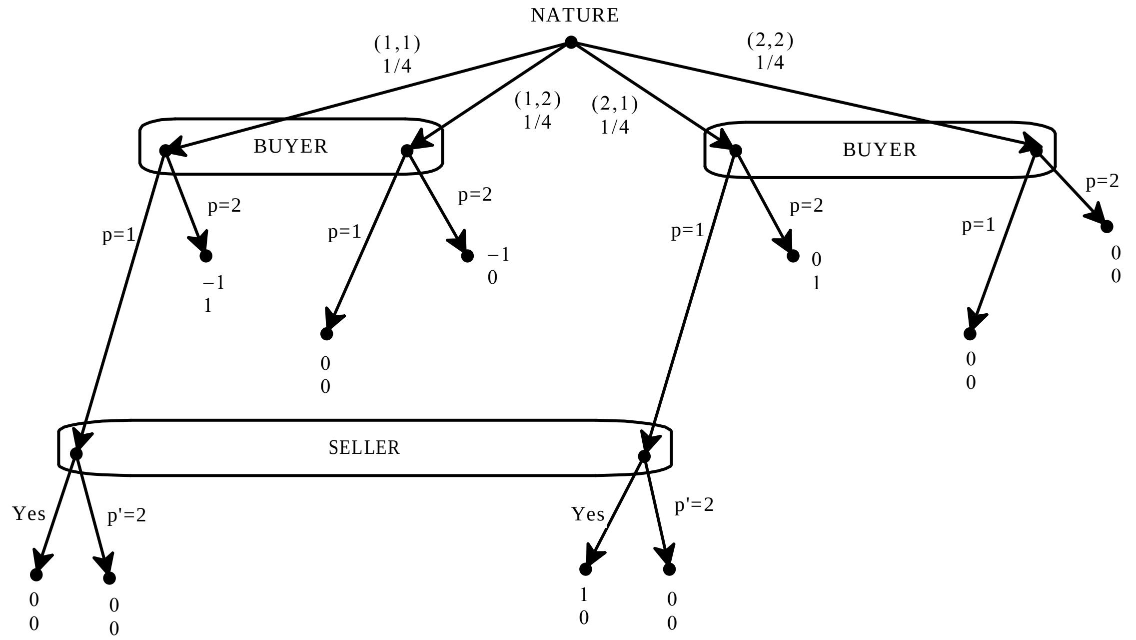


Figure 15.21: The further reduced game after eliminating strictly dominated choices from the game of Figure 15.20.

In the game of Figure 15.21 at the left information set of the Buyer $p = 1$ strictly dominates $p = 2$. At the right information set the Buyer's beliefs must be $\frac{1}{2}$ on each node so that if the Seller's strategy is to say "Yes", then $p = 1$ is the only sequentially rational choice there, otherwise both $p = 1$ and $p = 2$ are sequentially rational.
Thus the pure-strategy weak sequential equilibria of the reduced game shown in Figure 15.21 are as follows:

1. $\big((p=1, p=1)\,,\,Yes\big)$ with beliefs given by probability $\frac{1}{2}$ on each node at every information set.
2. $\big((p=1, p=1)\,,\,p'=2\big)$ with beliefs given by probability $\frac{1}{2}$ on each node at every information set.
3. $\big((p=1, p=2)\,,\,p'=2\big)$ with beliefs given by **(i)** probability $\frac{1}{2}$ on each node at both information sets of the Buyer and **(ii)** probability 1 on the left node at the information set of the Seller.

These equilibria can be extended to the original game of Figure 15.19 as follows:

1. $\big((p=1, p=1, No, No)\,,\,(p'=2, Yes)\big)$ with beliefs given by (i) probability $\frac{1}{2}$ on each node at both information sets of the Buyer at the top and at both information sets of the Seller and (ii) probability 1 on the right node at both information sets of Buyer at the bottom. The corresponding payoffs are: $\frac{1}{4}$ for the Buyer and 0 for the Seller.

2. $\big((p=1, p=1, No, No)\,,\,(p'=2, p'=2)\big)$ with beliefs given by probability $\frac{1}{2}$ on each node at every information set. The corresponding payoffs are 0 for both Buyer and Seller.

3. $\big((p=1, p=2, No, No)\,,\,(p'=2, p'=2)\big)$ with beliefs given by
(i) probability $\frac{1}{2}$ on each node at both information sets of the Buyer at the top and at the lower left information set of the Buyer,
(ii) any beliefs at the lower right information set of the Buyer and
(iii) probability 1 on the left node at each information set of the Seller.
The corresponding payoffs are: 0 for the Buyer and $\frac{1}{4}$ for the Seller.

Now let us consider the case $n = 100$. Drawing the interactive knowledge-belief structure and the corresponding extensive-form game (obtained by applying the Harsanyi transformation) is clearly not practical. However, one can still reason about what could be a pure-strategy Bayesian Nash equilibrium of that game. As a matter of fact, there are many Bayesian Nash equilibria.

One of them is the following:

- each type of the Buyer offers a price equal to his reservation price,
- the Seller accepts if, and only if, that price is greater than or equal to her reservation price,
- at information sets of the Seller that are not reached, the Seller rejects and counteroffers \$100,
- at information sets of the Buyer that are not reached the Buyer says "No".

The reader should convince himself/herself that this is indeed a Nash equilibrium.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 15.3.2 at the end of this chapter.

15.3 Exercises

15.3.1 Exercises for Section 15.1: One-sided incomplete information

The answers to the following exercises are in Section 15.4 at the end of this chapter.

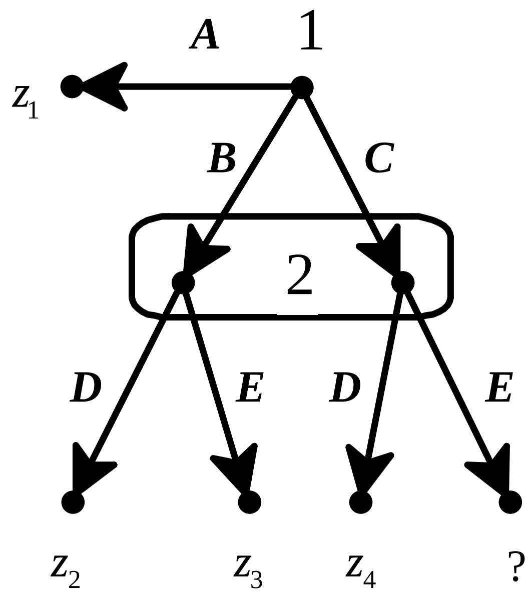


Figure 15.22: The extensive form for Exercise 15.1.

Exercise 15.1

Consider the following situation of one-sided incomplete information. Players 1 and 2 are playing the extensive-form game shown in Figure 15.22 (where z_1, z_2, etc. are outcomes and the question mark stands for either outcome z_5 or outcome z_6). The outcome that is behind the question mark is actually outcome z_5 and Player 1 knows this, but Player 2 does not know. Player 2 thinks that the outcome behind the question mark is either z_5 or z_6 and assigns probability 25% to it being z_5 and probability 75% to it being z_6. Player 2 also thinks that whatever the outcome is, Player 1 knows (that is, if it is z_5, then Player 1 knows that it is z_5, and if it is z_6 then Player 1 knows that it is z_6).

The beliefs of Player 2 are common knowledge between the two players.

(a) Represent this situation of incomplete information using an interactive knowledge-belief structure.

(b) Apply the Harsanyi transformation to transform the situation represented in part **(a)** into an extensive-form frame. [Don't worry about payoffs for the moment.]

From now on assume that the following is common knowledge:

1. Both players have von Neumann-Morgenstern preferences,
2. The ranking of Player 1 is

$$\begin{pmatrix} \mathit{best} & \text{second} & \mathit{worst} \\ z_4, z_6 & z_1 & z_2, z_3, z_5 \end{pmatrix}$$

and he is indifferent between z_1 for sure and the lottery $\begin{pmatrix} z_6 & z_5 \\ 0.5 & 0.5 \end{pmatrix}$.

3. The ranking of player 2 is

$$\begin{pmatrix} \mathit{best} & \text{second} & \text{third} & \mathit{worst} \\ z_6 & z_4 & z_2, z_5 & z_1, z_3 \end{pmatrix}$$

and she is indifferent between z_4 for sure and the lottery $\begin{pmatrix} z_6 & z_3 \\ 0.5 & 0.5 \end{pmatrix}$

and she is also indifferent between z_2 for sure and the lottery $\begin{pmatrix} z_6 & z_3 \\ 0.25 & 0.75 \end{pmatrix}$

(c) Calculate the von Neumann-Morgenstern normalized utility functions for the two players.

(d) Is there a weak sequential equilibrium of the game of part **(b)** where Player 1 always plays A (thus a pooling equilibrium)?

(e) Is there a weak sequential equilibrium of the game of part **(b)** where Player 1 always plays C (thus a pooling equilibrium)?

(f) Is there a pure-strategy weak sequential equilibrium of the game of part **(b)** where Player 1 does not always choose the same action (thus a separating equilibrium)? ■

Exercise 15.2

Show that the assessment highlighted in the Figure 15.23 (which reproduces Figure 15.11) is a sequential equilibrium as long as $p \geq \frac{1}{3}$. [The assessment is fully described on page 528.]

Exercise 15.3

Consider again the game of Figure 15.23 (which reproduces Figure 15.11).

(a) For the case where $p \leq \frac{1}{3}$, find a pure-strategy weak sequential equilibrium where *PE*-1 stays out but *PE*-2 enters; furthermore, the Incumbent would fight *PE*-1 if she entered. Prove that what you propose is a weak sequential equilibrium.

(b) Either prove that the weak sequential equilibrium of part **(a)** is a sequential equilibrium or prove that it is not a sequential equilibrium.

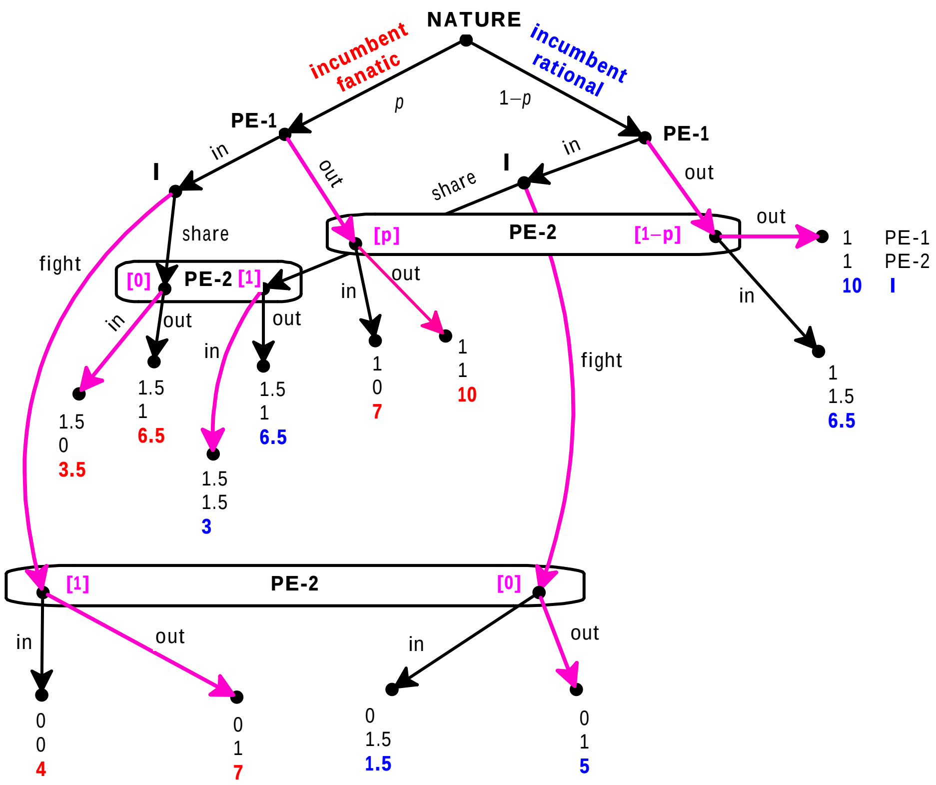
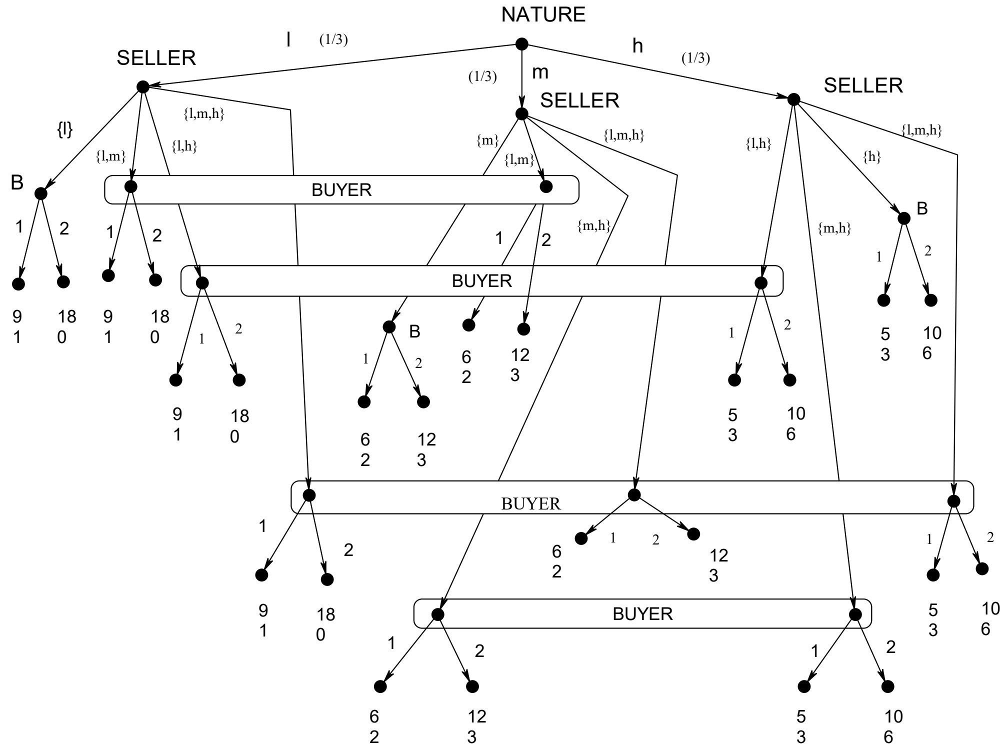


Figure 15.23: Copy of Figure 15.11

Exercise 15.4

Show that in the "truth-in-advertising" game of Figure 15.16, (which is reproduced on the next page), there is a weak sequential equilibrium where the seller makes vague claims (that is, does not reveal the whole truth).



Exercise 15.5

Consider a simpler version of the "truth-in-advertising" game, where there are only two quality levels: L and H. The payoffs are as follows:

- If the quality is L and the buyer buys one unit, the seller's payoff is 9 and the buyer's payoff is 1,
- If the quality is L and the buyer buys two units, the seller's payoff is 18 and the buyer's payoff is 0,
- If the quality is H and the buyer buys one unit, the seller's payoff is 6 and the buyer's payoff is 2,
- If the quality is H and the buyer buys two units, the seller's payoff is 12 and the buyer's payoff is 3,

Let the buyer's initial beliefs be as follows: the good is of quality L with probability $p \in (0,1)$ (and of quality H with probability $1-p$).

(a) Draw the extensive-form game that results from applying the Harsanyi transformation to this one-sided situation of incomplete information.

(b) Find the pure-strategy subgame-perfect equilibria of the game of part **(a)** for every possible value of $p \in (0,1)$.

15.3.2 Exercises for Section 15.2: Multi-sided incomplete information

The answers to the following exercises are in Section 15.4 at the end of this chapter.

Exercise 15.6

Consider the following situation of two-sided incomplete information. Players 1 and 2 are having dinner with friends and during dinner Player 2 insinuates that Player 1 is guilty of unethical behavior.

- Player 1 can either demand an apology (D) or ignore (I) Player 2's remark;
- if Player 1 demands an apology, then Player 2 can either apologize (A) or refuse to apologize (not-A);
- if Player 2 refuses to apologize, Player 1 can either concede (C) or start a fight (F).

Thus the sequence of moves is as shown in Figure 15.24 (where $z_1, \ldots, z_4$ are the possible outcomes).

Let U_i be the von Neumann-Morgenstern utility function of Player i $(i = 1, 2)$. The following is common knowledge between Players 1 and 2:

- $U_1(z_1) = 4, \qquad U_1(z_2) = 2, \qquad U_1(z_3) = 6$
- $U_2(z_1) = 6, \qquad U_2(z_2) = 8, \qquad U_2(z_3) = 2, \qquad U_2(z_4) = 0.$
- The only uncertainty concerns the value of $U_1(z_4)$. As a matter of fact, $U_1(z_4) = 0$; Player 1 knows this, but Player 2 does not. It is common knowledge between Players 1 and 2 that Player 2 thinks that either $U_1(z_4) = 0$ or $U_1(z_4) = 3$.

Furthermore,

- Player 2 assigns probability $\frac{1}{4}$ to $U_1(z_4) = 0$ and probability $\frac{3}{4}$ to $U_1(z_4) = 3$;
- Player 1 is uncertain as to whether Player 2's beliefs are

$$\begin{pmatrix} U_1(z_4) = 0 & U_1(z_4) = 3 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} U_1(z_4) = 0 \\ 1 \end{pmatrix}$$

  and he attaches probability $\frac{1}{2}$ to each.

(a) Construct a three-state interactive knowledge-belief structure that captures all of the above.

(b) Draw the extensive-form game that results from applying the Harsanyi transformation to the incomplete-information situation of part **(a)**.

(c) Find the pure-strategy weak sequential equilibria of the game of part **(b)**. ■

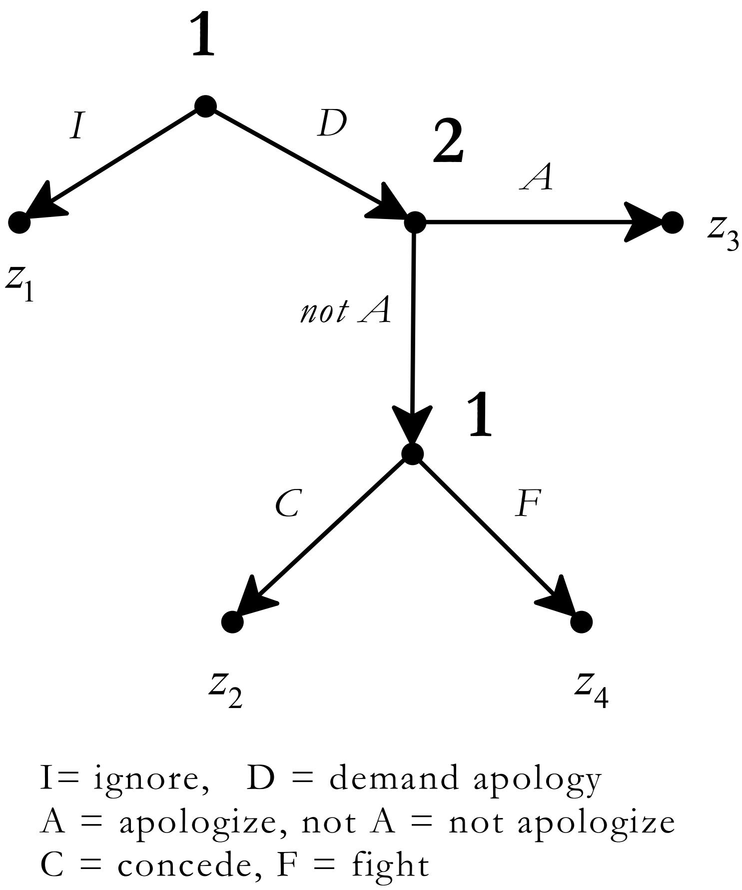


Figure 15.24: The extensive form for Exercise 15.6.

Exercise 15.7 — ⋆⋆⋆Challenging Question ⋆⋆⋆.

There are two parties to a potential lawsuit: the owner of a chemical plant and a supplier of safety equipment. The chemical plant owner, from now on called the plaintiff, alleges that the supplier, from now on called the defendant, was negligent in providing the safety equipment. The defendant knows whether or not he was negligent, while the plaintiff does not know; the plaintiff believes that there was negligence with probability q. These beliefs are common knowledge between the parties.

The plaintiff has to decide whether or not to sue. If she does not sue then nothing happens and both parties get a payoff of 0. If the plaintiff sues then the defendant can either offer an out-of-court settlement of \$$S$ or resist. If the defendant offers a settlement, the plaintiff can either accept (in which case her payoff is S and the defendant's payoff is –S) or go to trial. If the defendant resists then the plaintiff can either drop the case (in which case both parties get a payoff of 0) or go to trial. If the case goes to trial then legal costs are created in the amount of \$$P$ for the plaintiff and \$$D$ for the defendant.

Furthermore (if the case goes to trial), the judge is able to determine if there was negligence and, if there was, requires the defendant to pay \$$W$ to the plaintiff (and each party has to pay its own legal costs), while if there was no negligence the judge will drop the case without imposing any payments to either party (but each party still has to pay its own legal costs). It is common knowledge that each party is "selfish and greedy" (that is, only cares about its own wealth and prefers more money to less) and is risk neutral.

Assume the following about the parameters:

$$0 < q < 1, \qquad 0 < D < S, \qquad 0 < P < S < W - P.$$

(a) Represent this situation of incomplete information by means of an interactive knowledge-belief structure (the only two players are the plaintiff and the defendant).

(b) Apply the Harsanyi transformation to represent the situation in part **(a)** as an extensive-form game. [Don't forget to subtract the legal expenses from each party's payoff if the case goes to trial.]

(c) Write all the pure strategies of the plaintiff.

(d) Prove that there is no pure-strategy weak sequential equilibrium which (1) is a separating equilibrium and (2) involves suing.

(e) For what values of the parameters (q, S, P, W, D) are there pure-strategy weak sequential equilibria which (1) are pooling equilibria and (2) involve suing? Consider all types of pooling equilibria and prove your claim.

(f) Now drop the assumption that $S < W - P$. For the case where $q = \frac{1}{12}$, $P = 70$, $S = 80$ and $W = 100$ find all the pure-strategy weak sequential equilibria which (1) are pooling equilibria and (2) involve suing.

■

15.4 Solutions to Exercises

Solutions to Exercise 15.1

(a) Let G_1 be the game with outcome z_5 and G_2 the game with outcome z_6. Then the structure is as shown in Figure 15.25, with $q = \frac{1}{4}$. The true state is α.

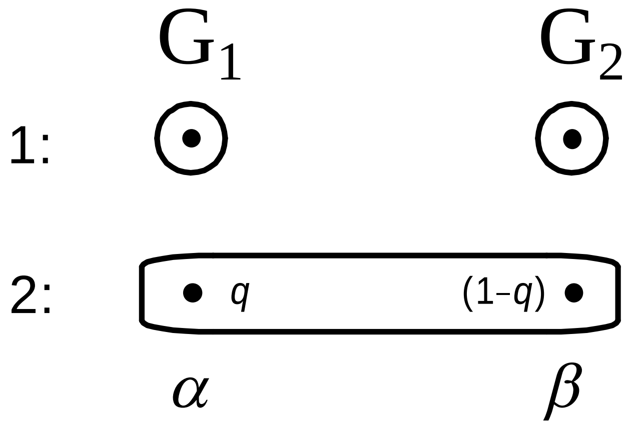


Figure 15.25: The one-sided incomplete-information situation of Exercise 15.1.

(b) The extensive form is shown in Figure 15.26.

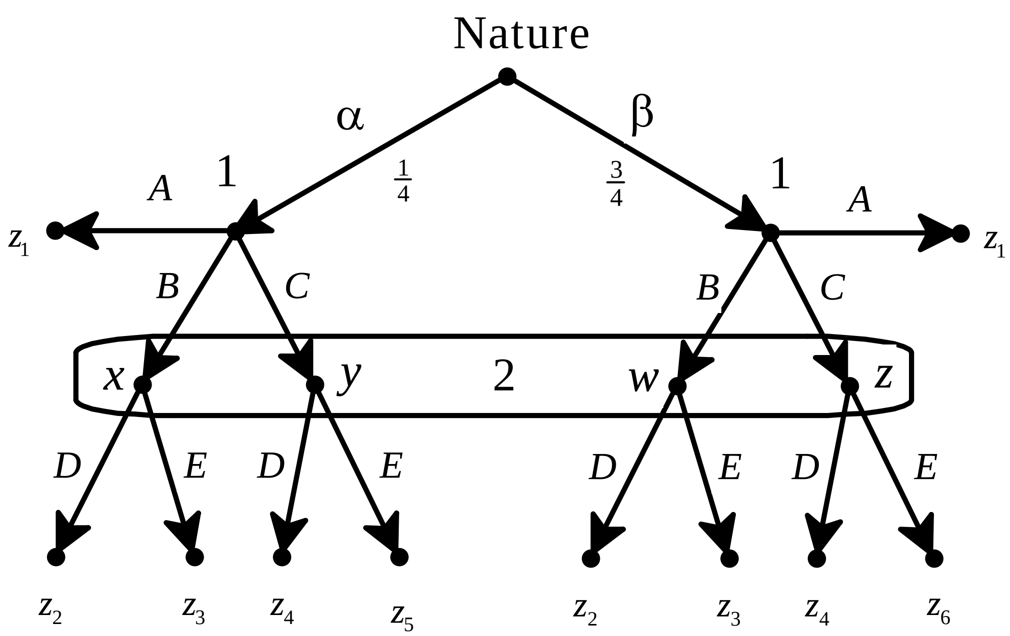


Figure 15.26: The extensive form obtained by applying the Harsanyi transformation to the incomplete-information situation shown in Figure 15.25.

(c) The von Neumann-Morgenstern utility functions are as follows:

$$\left(\begin{array}{lcccccc} & z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \\ U_1: & 0.5 & 0 & 0 & 1 & 0 & 1 \\ U_2: & 0 & 0.25 & 0 & 0.5 & 0.25 & 1 \end{array}\right).$$

Adding these payoffs to the extensive form we obtain the game shown in Figure 15.27.

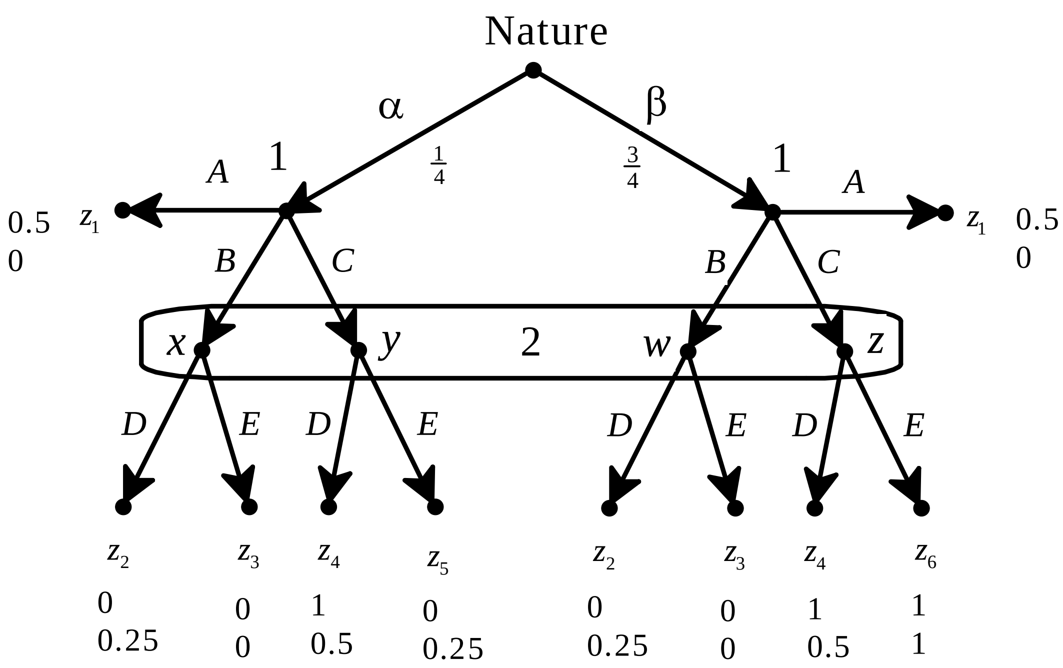


Figure 15.27: The game obtained by adding the payoffs to the game-frame of Figure 15.26.

(d) No, because at the right node of Player 1, C gives a payoff of 1 no matter what Player 2 does and thus Player 1 would choose C rather than A (which only gives him a payoff of 0.5).

(e) If Player 1 always plays C (that is, his strategy is CC), then – by Bayesian updating – Player 2 should assign probability $\frac{1}{4}$ to node y and probability $\frac{3}{4}$ to node z,
in which case D gives her a payoff of 0.5
while E gives her a payoff of $\frac{1}{4}(0.25)+\frac{3}{4}(1)=0.8125$; hence she must choose E.
But then at his left node Player 1 with C gets 0 while with A he gets 0.5.
Hence choosing C at the left node is not sequentially rational.

(f) Yes, the following is a weak sequential equilibrium:

$\sigma=(AC,E)$ with beliefs $\mu=\begin{pmatrix} x & y & w & z \\ 0 & 0 & 0 & 1 \end{pmatrix}$. It is straightforward to verify that sequential rationality and Bayesian updating are satisfied. □

Solutions to Exercise 15.2 Sequential rationality was verified in Section 15.1. Thus we only need to show that the highlighted pure-strategy profile together with the following system of beliefs $\mu = \left(\begin{array}{cc|cc|cc} s & t & u & v & x & w \\ p & 1-p & 0 & 1 & 1 & 0 \end{array}\right)$ (for the names of the nodes refer to Figure 15.28) constitutes a consistent assessment (Definition 12.1.1, Chapter 12).

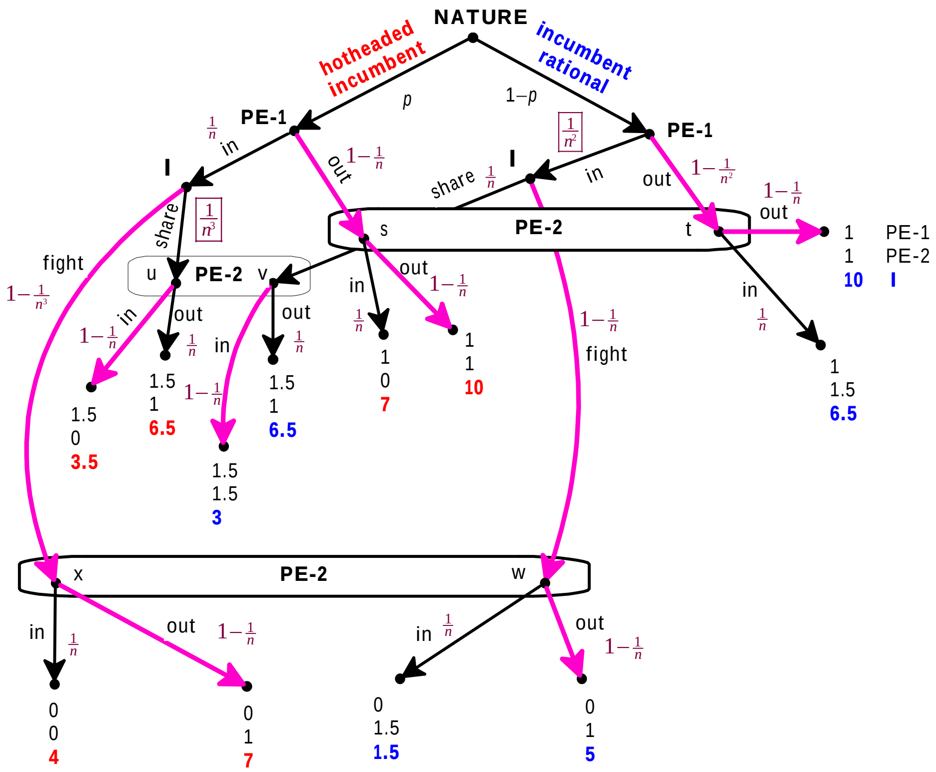


Figure 15.28: The game for Exercise 15.2.

Let $\langle \sigma_n \rangle_{n=1,2,...}$ be the sequence of completely mixed strategies shown in Figure 15.28. It is clear that $\lim_{n\to\infty} \sigma_n = \sigma$. Let us calculate the system of beliefs μ_n obtained from σ_n by using Bayesian updating:

$$\mu_n(s) = \frac{p\left(1-\frac{1}{n}\right)}{p\left(1-\frac{1}{n}\right)+(1-p)\left(1-\frac{1}{n^2}\right)} \quad \text{thus} \quad \lim_{n\to\infty}\mu_n(s) = \frac{p}{p+(1-p)} = p;$$

$$\mu_n(u) = \frac{p\left(\frac{1}{n}\right)\left(\frac{1}{n^3}\right)}{p\left(\frac{1}{n}\right)\left(\frac{1}{n^3}\right)+(1-p)\left(\frac{1}{n^2}\right)\left(\frac{1}{n}\right)} = \frac{p}{p+n(1-p)} \quad \text{thus} \quad \lim_{n\to\infty}\mu_n(u) = 0;$$

$$\mu_n(x) = \frac{p\left(\frac{1}{n}\right)\left(1-\frac{1}{n^3}\right)}{p\left(\frac{1}{n}\right)\left(1-\frac{1}{n^3}\right)+(1-p)\left(\frac{1}{n^2}\right)\left(1-\frac{1}{n}\right)} = \frac{p\left(1-\frac{1}{n^3}\right)}{p\left(1-\frac{1}{n^3}\right)+(1-p)\left(\frac{1}{n}\right)\left(1-\frac{1}{n}\right)}.$$

Thus, $\lim_{n\to\infty}\mu_n(x) = 1$ (the numerator tends to $p(1) = p$ and the denominator tends to $p(1)+(1-p)(0)(1) = p$). Hence (σ,μ) is consistent and sequentially rational and therefore it is a sequential equilibrium. $\square$

Solutions to Exercise 15.3

(a) Consider the followin pure-strategy profile, call it σ, which is highlighted by thick arrows in Figure 15.29:

- PE-1's strategy is "out" at both nodes,
- PE-2's strategy is
  - "in" at the top information set (after having observed that PE-1 stayed out),
  - "in" at the middle information set (after having observed that PE-1's entry was followed by the Incumbent sharing the market with PE-1),
  - "out" at the bottom information set (after having observed that PE-1's entry was followed by the Incumbent fighting against PE-1).
- The Incumbent's strategy is to fight entry of PE-1 in any case (that is, whether the Incumbent himself is hotheaded or rational).

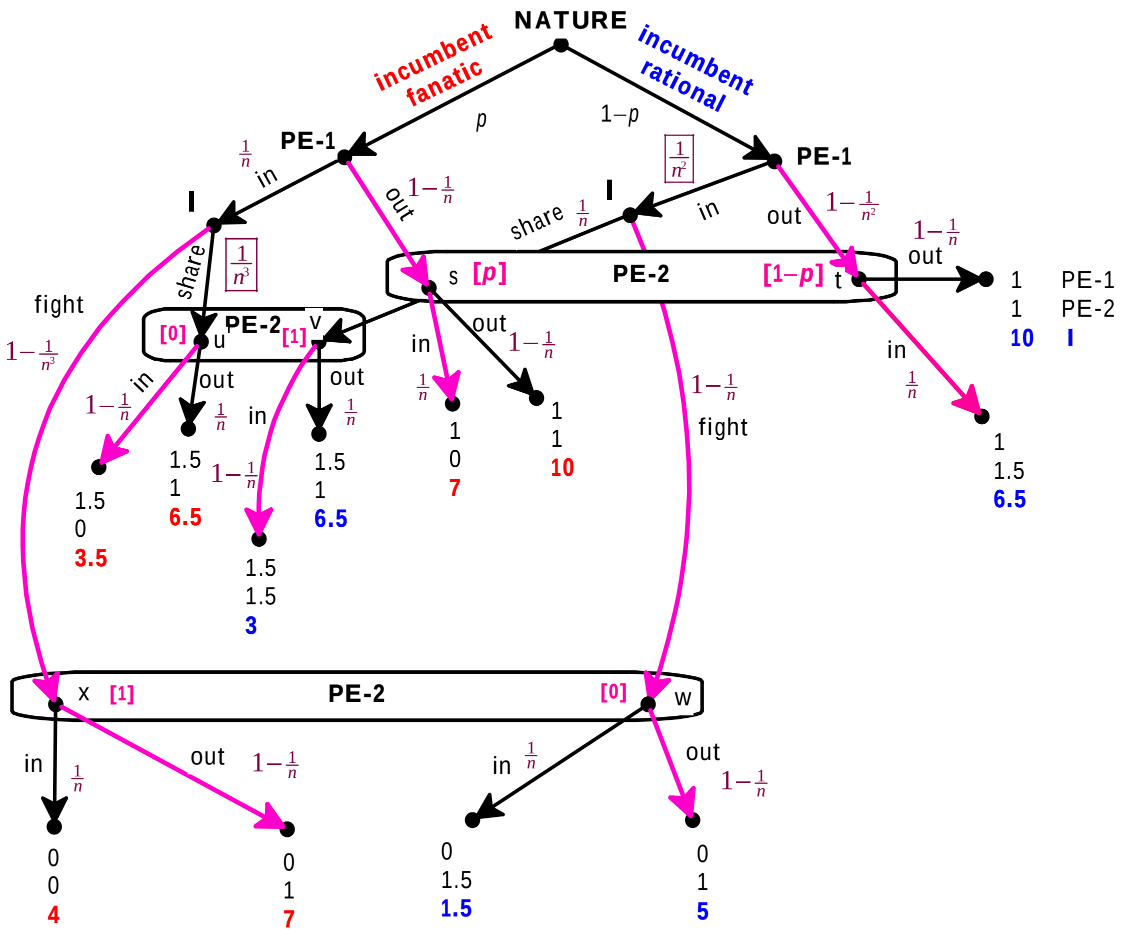


Figure 15.29: The game for Exercise 15.3.

We want to show that σ, together with the system of beliefs μ shown in Figure 15.29 (where at her top information PE-2 assigns probability p to the Incumbent being hotheaded, at her middle information set she assigns probability 1 to the Incumbent being rational and at her bottom information set she assigns probability 1 to the Incumbent being hotheaded) constitutes a weak sequential equilibrium for any value of $p \leq \frac{1}{3}$.

Bayesian updating is satisfied at PE-2's top information set (the only non-singleton information set reached by σ), and at the other two information sets of PE-2 any beliefs are allowed by the notion of weak sequential equilibrium.

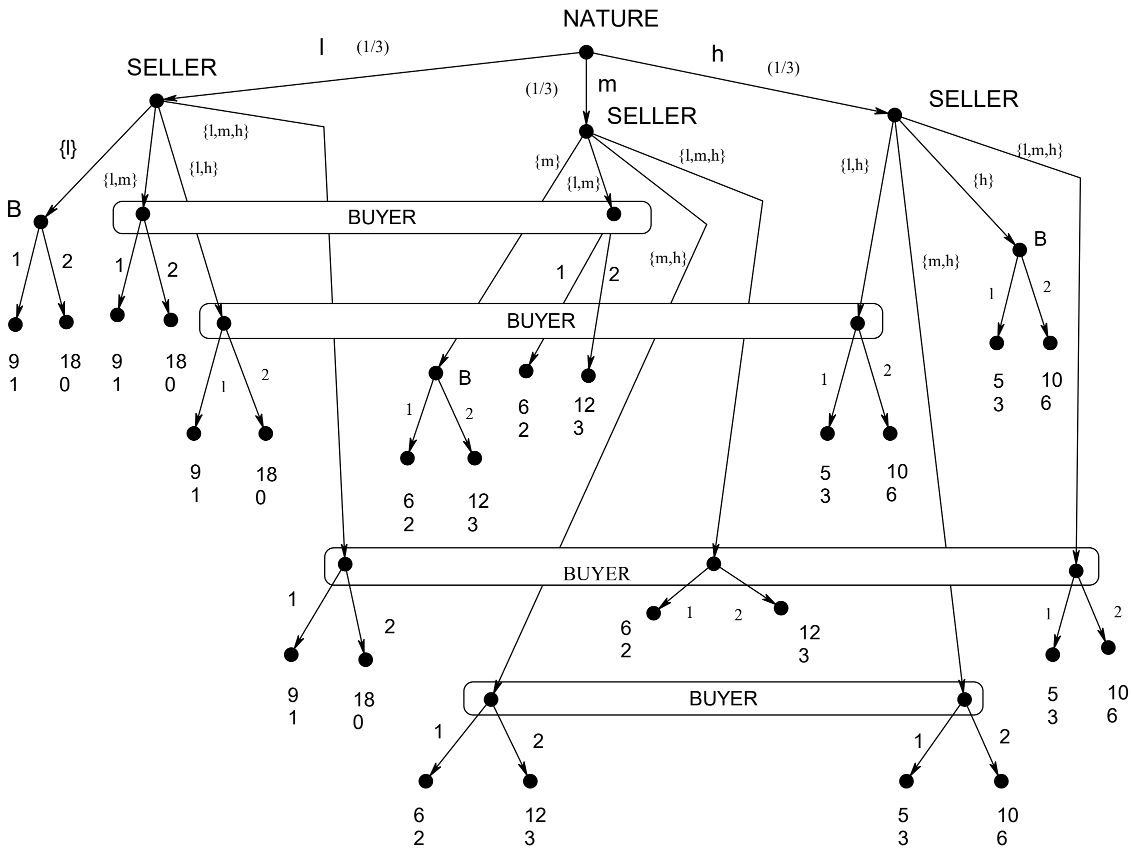
Sequential rationality also holds:

- For PE-1, at the left node "in" yields 0 and "out" yields 1, so that "out" is sequentially rational; the same is true at the right node.
- For the Incumbent,
  - at the left node "fight" yields 7 and "share" yields 3.5, so that "fight" is sequentially rational;
  - at the right node "fight" yields 5 and "share" yields 3, so that "fight" is sequentially rational.
- For PE-2,
  - at the top information set "in" yields an expected payoff of $p(0)+(1-p)(1.5)=(1-p)(1.5)$ and "out' yields 1, so that "in" is sequentially rational as long as $p \leq \frac{1}{3}$;
  - at the middle information set "in" yields 1.5 and "out" yields 1, so that "in" is sequentially rational;
  - at the bottom information set "in" yields 0 and "out" yields 1, so that "out" is sequentially rational.

(b) The assessment described in part **(a)** is in fact a sequential equilibrium. This can be shown using the sequence of completely mixed strategies marked in Figure 15.29, which coincides with the sequence of mixed strategies considered in Exercise 15.2; thus the calculations to show consistency are identical to the ones carried out in Exercise 15.2.

□

Solutions to Exercise 15.4 The game under consideration is the following (which reproduces Figure 15.16):



Let the buyer be naïve, in the sense that at every unreached information set she assigns equal probability to each node; furthermore, let the buyer's strategy be as follows: if told {l}, buy one unit, in every other case buy two units.

Let the seller's strategy be as follows: if Nature chooses l or m, claim {l,m} and if Nature chooses h, then claim {h}.

Let us verify that this assessment constitutes a weak sequential equilibrium. The only non-singleton information set that is reached by the strategy profile is the top information set (where the buyer hears the claim $\{l,m\}$) and Bayesian updating requires the buyer to assign equal probability to each node. Every other non-singleton information set is not reached and thus the notion of weak sequential equilibrium allows any beliefs: in particular beliefs that assign probability $\frac{1}{2}$ to each node.

Now we check sequential rationality.

Let us start with the buyer.

- At the singleton node following claim $\{l\}$, buying one unit is optimal and at the singleton nodes following claims $\{m\}$ and $\{h\}$ buying two units is optimal.

- At the top information set (after the claim $\{l,m\}$) choosing one unit gives an expected payoff of $\frac{1}{2}(1)+\frac{1}{2}(2)=1.5$ and choosing two units yields $\frac{1}{2}(0)+\frac{1}{2}(3)=1.5$, thus buying two units is sequentially rational.

- At the information set following claim $\{l,h\}$ choosing one unit gives an expected payoff of $\frac{1}{2}(1)+\frac{1}{2}(3)=2$ while choosing two units yields $\frac{1}{2}(0)+\frac{1}{2}(6)=3$, thus buying two units is sequentially rational.

- At the information set following claim $\{l,m,h\}$ choosing one unit gives an expected payoff of $\frac{1}{3}(1)+\frac{1}{3}(2)+\frac{1}{3}(3)=2$ while choosing two units yields $\frac{1}{3}(0)+\frac{1}{3}(3)+\frac{1}{3}(6)=3$, thus buying two units is sequentially rational.

- Finally, at the information set following claim $\{m,h\}$ choosing one unit gives an expected payoff of $\frac{1}{2}(2)+\frac{1}{2}(3)=2.5$ while choosing two units yields $\frac{1}{2}(3)+\frac{1}{2}(6)=4.5$, thus buying two units is sequentially rational.

Now let us check sequential rationality for the seller's strategy.

- ∘ At the left node (after Nature chooses l) claiming $\{l\}$ yields a payoff of 9, while every other claim yields a payoff of 18. Thus $\{l,m\}$ is sequentially rational.

- ∘ At the other two nodes, the seller is indifferent among all the claims, because they yield the same payoff (12 at the node after Nature chooses m and 10 at the node after Nature chooses h) thus the postulated choices are sequentially rational. $\square$

Solutions to Exercise 15.5

(a) The extensive form is shown in Figure 15.30.

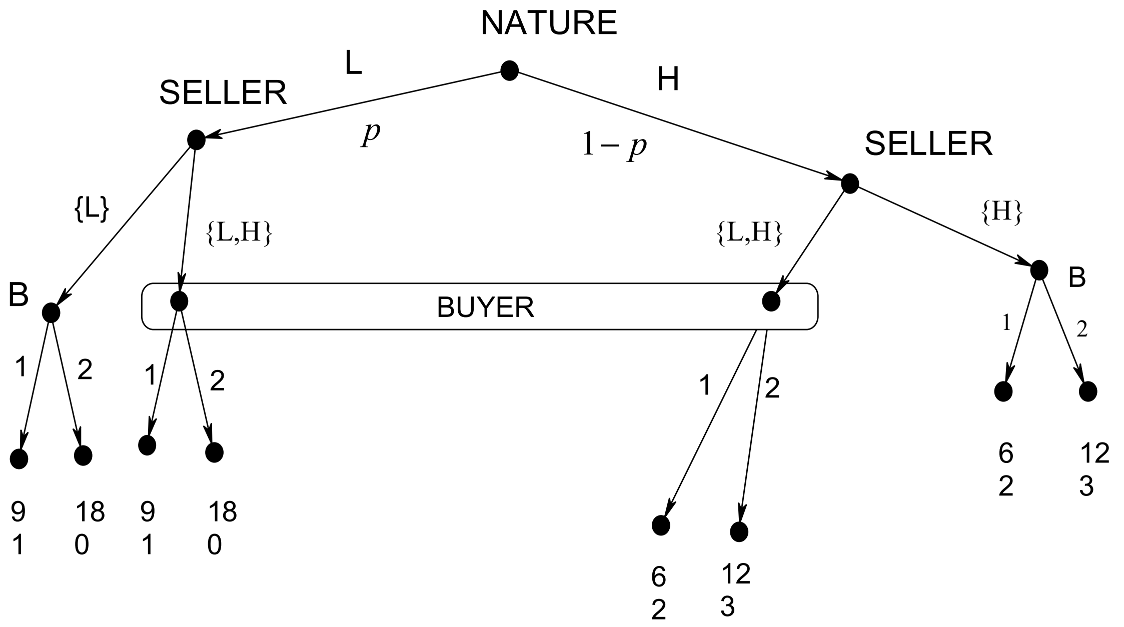


Figure 15.30: The game for Exercise 15.5.

(b) At any subgame-perfect equilibrium, the buyer buys one unit at the singleton information set on the left and two units at the singleton information set on the right. Thus we can simplify the game by replacing the buyer's node on the left with the payoff vector (9,1) and the buyer's node on the right with the payoff vector (12,3).

The strategic form corresponding to the simplified game is shown in Figure 15.31. The buyer's strategy refers to the buyer's only information set that has remained, where he is told $\{L, H\}$.

Buyer

		Buy 1 unit		Buy 2 units	
	L, LH	9p + 6(1–p),	p + 2(1–p)	18p + 12(1–p),	3(1–p)
Seller	L, H	9p + 12(1–p),	p + 3(1–p)	9p + 12(1–p),	p + 3(1–p)
	LH, LH	9p + 6(1–p),	p + 2(1-p)	18p + 12(1-p),	3(1–p)
	LH,H	9p + 12(1-p),	p + 3(1-p)	18p + 12(1-p),	3(1–p)

simplifying:

		Buy 1 unit		Buy 2 units	
	L, LH	6 + 3p,	2 – p	12 + 6p,	3 – 3p
Seller	L, H	12 – 3p,	3 – 2p	12 – 3p,	3 – 2p
	LH, LH	6 + 3p,	2 – p	12 + 6p,	3 – 3p
	LH,H	12 – 3p,	3 – 2p	12 + 6p,	3 – 3p

Figure 15.31: The strategic form of the simplified game.

Since $p < 1$, $12 - 3p > 6 + 3p$. Thus $\big((LH,H),1\big)$ and $\big((L,H),1\big)$ are always Nash equilibria for every value or p. Note that $2 - p > 3 - 3p$ if and only if $p > \frac{1}{2}$; thus if $p > \frac{1}{2}$ then there are no other pure-strategy Nash equilibria.

On the other hand, if $p \leq \frac{1}{2}$ then the following are also Nash equilibria: $\big((L,LH),2\big)$ and $\big((LH,LH),2\big)$.

To sum up, the pure-strategy Nash equilibria of this normal form, *which correspond to the pure-strategy subgame-perfect equilibria of the reduced extensive form*, are as follows:

- If $p > \frac{1}{2}$, $\big((LH,H),1\big)$ and $\big((L,H),1\big)$;
- If $p \leq \frac{1}{2}$, $\big((LH,H),1\big)$, $\big((L,H),1\big)$, $\big((L,LH),2\big)$ and $\big((LH,LH),2\big)$.

□

Solutions to Exercise 15.6

(a) Let G_1 and G_2 be the perfect-information games (whose backward induction solution has been highlighted by means of thick arrows) shown in Figure 15.32.

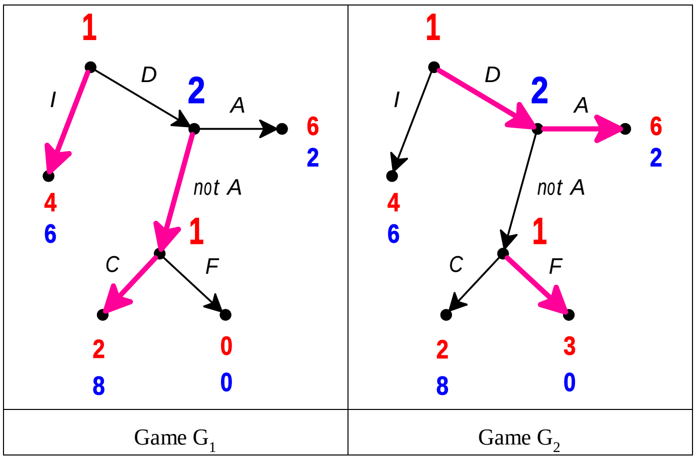


Figure 15.32: The possible perfect information games for Exercise 15.6.

The interactive knowledge-belief structure is shown in Figure 15.33

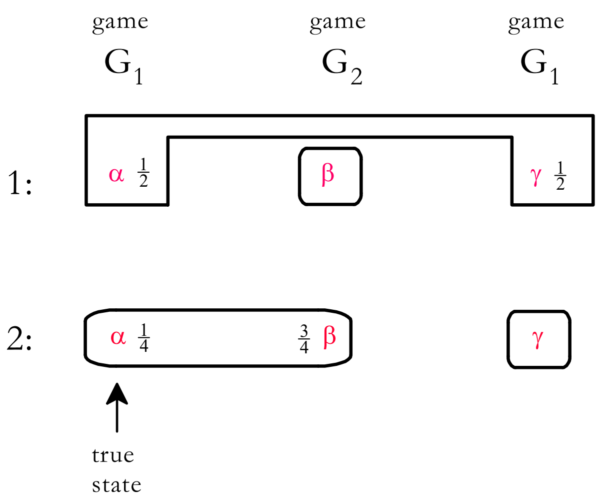


Figure 15.33: The two-sided incomplete-information situation.

(b) The common prior is given by $\begin{pmatrix} \alpha & \beta & \gamma \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{pmatrix}$.

The Harsanyi transformation yields the game shown in Figure 15.34.

(c)
- At the bottom information set of Player 1, C strictly dominates F and thus we can replace the two nodes in that information set with the payoff vector (2,8).
- At the bottom singleton node of Player 1, F strictly dominates C and thus we can replace that node with the payoff vector (3,0).

Thus – by appealing to sequential rationality – the game can be reduced to the game shown in Figure 15.35.

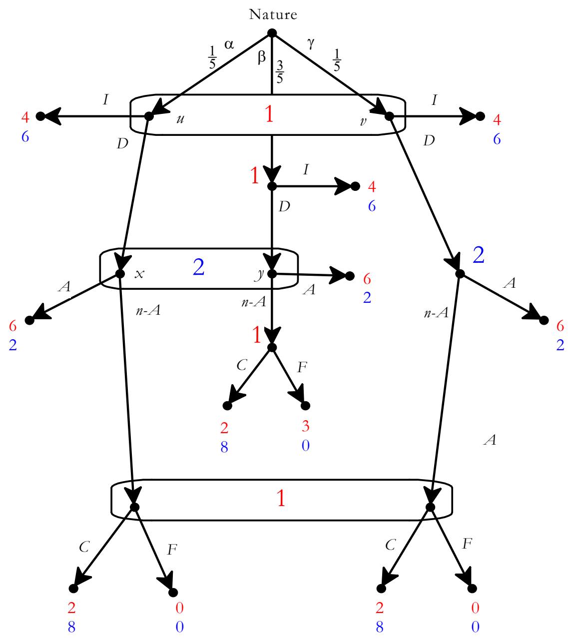


Figure 15.34: The game obtained by applying the Harsanyi transformation to Figure 15.33.

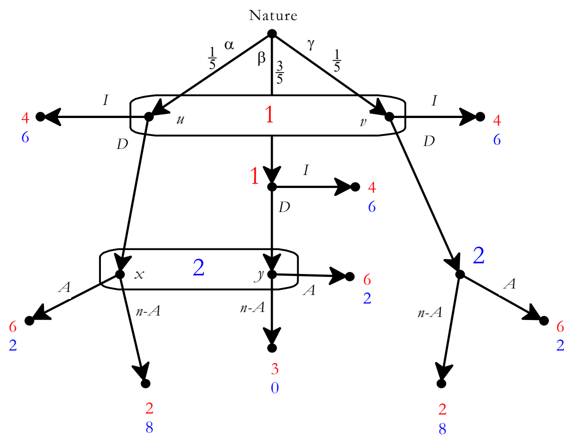


Figure 15.35: The reduced game.

Now at the bottom-right node of Player 2, *n-A* strictly dominates *A* and thus we can replace that node with the payoff vector (2,8) and further simplify the game as shown in Figure 15.36.

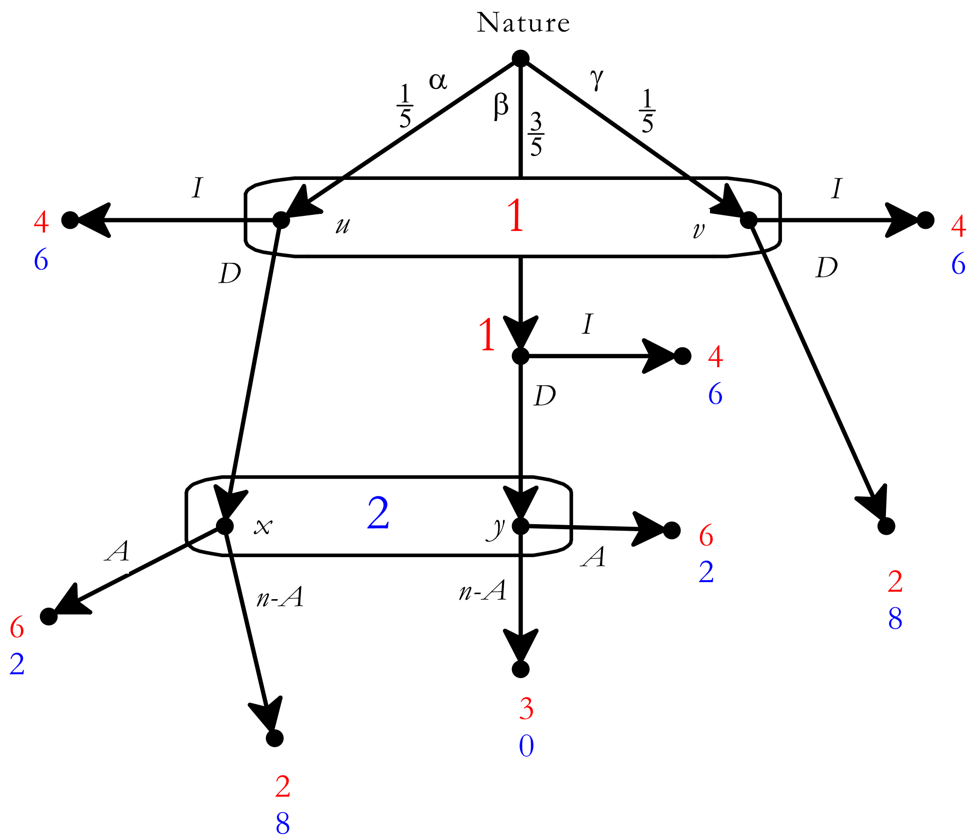


Figure 15.36: The further reduced game.

The following are the pure-strategy weak sequential equilibria of the reduced game (which can be extended into weak sequential equilibria of the original game by adding the choices that were selected during the simplification process):

- $((I,I),\, n\text{-}A)$ with beliefs $\mu = \left(\begin{array}{cc|cc} u & v & x & y \\ \frac{1}{2} & \frac{1}{2} & p & 1-p \end{array}\right)$ for any $p \geq \frac{1}{4}$.
- $((I,D),A)$ with beliefs $\mu = \left(\begin{array}{cc|cc} u & v & x & y \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 \end{array}\right)$.
- $((D,D),A)$ with beliefs $\mu = \left(\begin{array}{cc|cc} u & v & x & y \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \end{array}\right)$. $\square$

Solutions to Exercise 15.7

(a) Let G_1 and G_2 be the games shown in Figure 15.37 (in G_1 the defendant is negligent and in G_2 he is not).

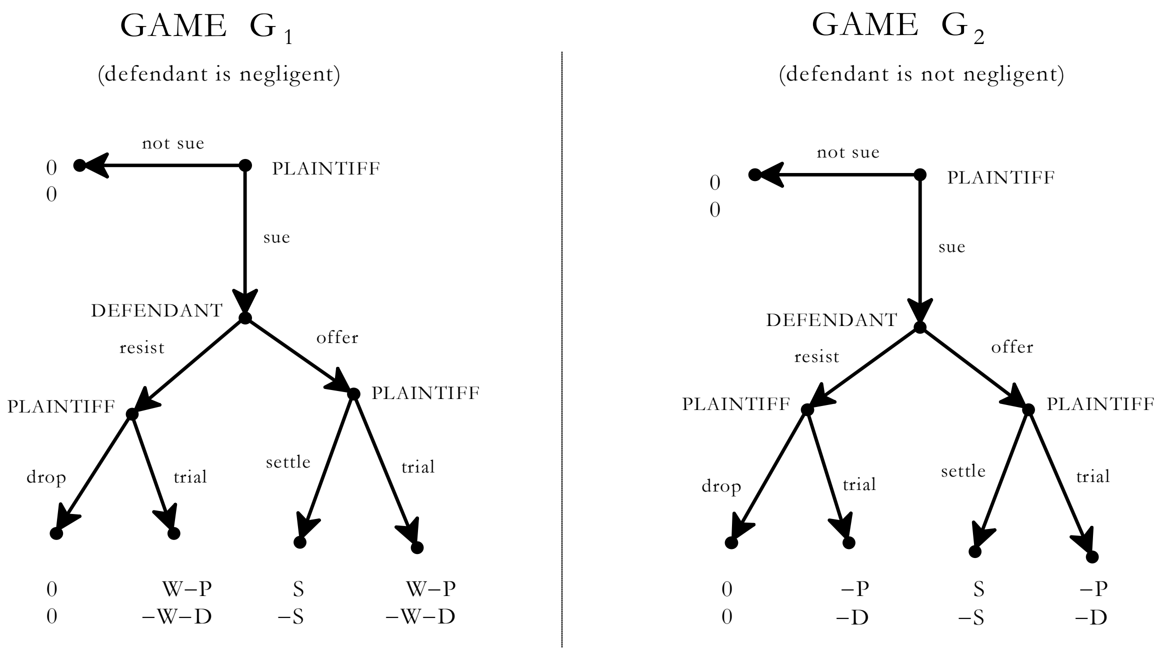


Figure 15.37: The two possible games.

Then the situation can be represented as shown in Figure 15.38 (at state α is the defendant is negligent and at state β the defendant is not negligent).

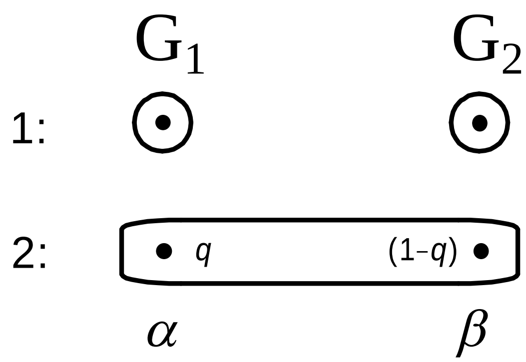


Figure 15.38: The one-sided incomplete-information situation.

(b) The extensive-form game is shown in Figure 15.39.

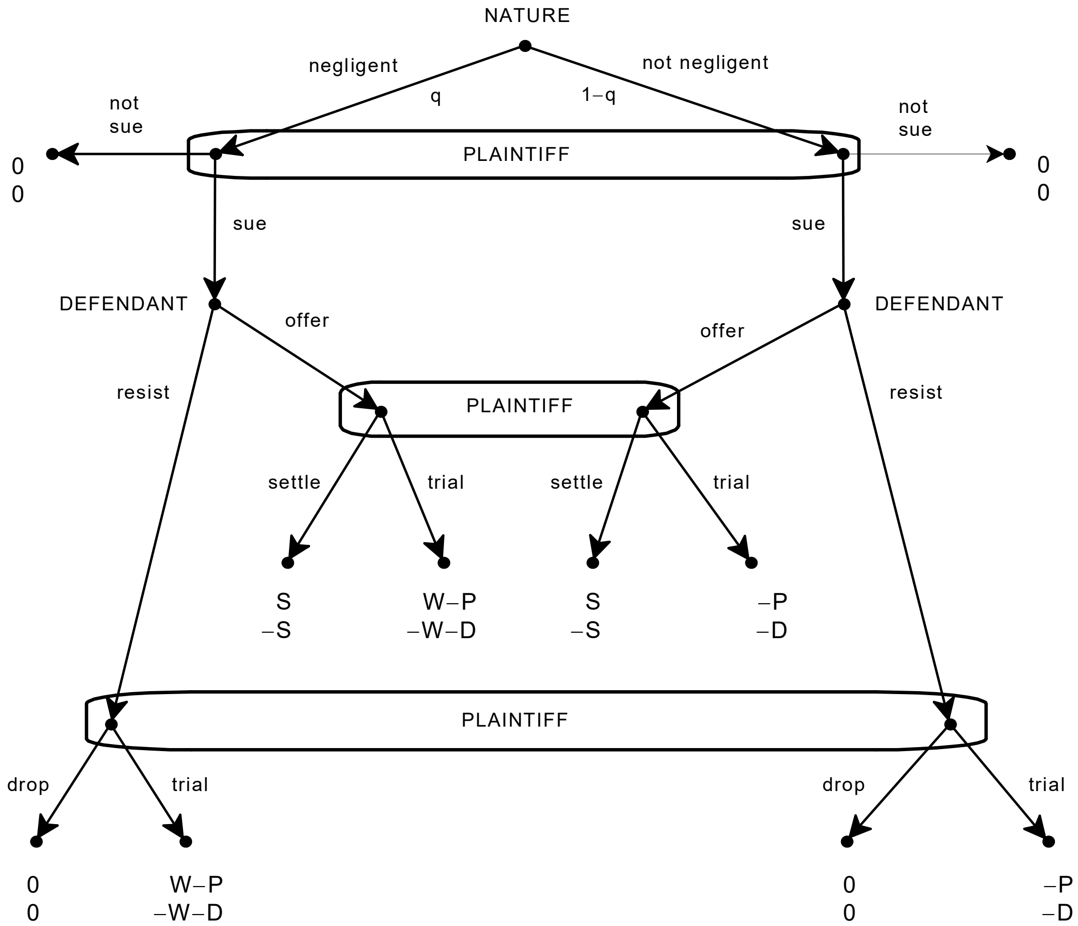


Figure 15.39:

(c) Plaintiff's strategies:

1. (sue; if offer settle; if resist drop),
2. (sue; if offer settle; if resist go to trial),
3. (sue; if offer go to trial; if resist drop),
4. (sue; if offer go to trial; if resist go to trial),
5. (not sue; if offer settle; if resist drop),
6. (not sue; if offer settle; if resist go to trial),
7. (not sue; if offer go to trial; if resist drop),
8. (not sue; if offer go to trial; if resist go to trial).

(d) There are two possibilities for a separating equilibrium:
Case S1: the defendant's strategy is "resist if negligent and offer if not negligent",
Case S2: the defendant's strategy is "offer if negligent and resist if not negligent".
In both cases we assume that the plaintiff's strategy involves suing with probability 1.

Consider Case S1 first. By Bayesian updating, at the bottom information set the plaintiff must attach probability 1 to the negligent type and thus, by sequential rationality, must choose "trial" (because $W - P > 0$).

Similarly, by Bayesian updating, at the middle information set the plaintiff must attach probability 1 to the non-negligent type and thus by sequential rationality must choose "settle". But then the negligent type of the defendant gets $-(W+D)$ by resisting and would get $-S$ by offering.

Since, by assumption, $S < W (< W + D)$, the choice of resisting is not sequentially rational.

Now consider Case S2. By Bayesian updating, at the bottom information set the plaintiff must attach probability 1 to the non-negligent type and thus by sequential rationality must choose "drop".

But then the negligent type of the defendant gets a negative payoff by offering, while he would get 0 by resisting. Hence the choice of offering is not sequentially rational.

(e) There are two candidates for a pure-strategy pooling equilibrium:
Case P1: both types of the defendant choose "offer" and
Case P2: both types of the defendant choose "resist".

Consider Case P1 first. (both types of the defendant choose "offer"). In order for "offer" to be sequentially rational for the non-negligent type, it cannot be that the plaintiff's strategy involves "settle" at the middle information set (the non-negligent type would get either 0 or $-D$ by resisting and both payoffs are greater than $-S$) and/or "drop" at the bottom information set. That is, it must be that the plaintiff chooses "trial" at both information sets.

By Bayesian updating, at the *middle* information set the plaintiff must attach probability q to the negligent type and probability $(1-q)$ to the non-negligent type.

Hence at the middle information set "trial" is sequentially rational if and only if $qW - P \geq S$, that is, $q \geq \frac{S+P}{W}$.

In order for "trial" to be sequentially rational at the *bottom* information set, the plaintiff must attach sufficiently high probability (namely $p \geq \frac{P}{W}$) to the negligent type. This is allowed by weak sequential equilibrium because the bottom information set is not reached.

Finally, in order for "sue" to be sequentially rational it must be that $qW - P \geq 0$, that is, $q \geq \frac{P}{W}$, which is implied by $q \geq \frac{S+P}{W}$.

Thus there is a pooling equilibrium with $\big((\text{sue,trial,trial}),(\text{offer,offer})\big)$

if and only if $q \geq \frac{S+P}{W}$.

Now consider Case P2. (both types of the defendant choose "resist"). [Note: since in part **(f)** below the restriction $S < W - P$ does not hold, we will carry out the analysis at first without imposing the restriction.]

If the plaintiff's strategy involves "drop" at the bottom information set, then it is indeed sequentially rational for both types of the defendant to choose "resist".

Furthermore, "drop" is sequentially rational in this case if, and only if, $qW - P \leq 0$ that is, $q \leq \frac{P}{W}$.

Then "sue" is also sequentially rational, since the Plaintiff's payoff is 0 no matter whether he sues or does not sue.

Thus there is a pooling equilibrium with $\big($(sue, x, drop),(resist,resist)$\big)$ if and only if $q \leq \frac{P}{W}$

and appropriate beliefs as follows (p is the probability on the left node of the unreached middle information set):

- x = settle and either any p if $W \leq S + P$ or $p \leq \frac{S+P}{W}$ if $W > S + P$, or
- x = trial and $p \geq \frac{S+P}{W}$, which requires $W \geq S + P$ (since $p \leq 1$).

Since it is assumed that $W > S + P$, we can conclude that

$\big($(sue,settle,drop),(resist,resist)$\big)$ is an equilibrium if and only if $q \leq \frac{P}{W}$ with $p \leq \frac{S+P}{W}$

$\big($(sue,trial,drop),(resist,resist)$\big)$ is an equilibrium if and only if $q \leq \frac{P}{W}$ with $p \geq \frac{S+P}{W}$

If, on he other hand, $q \geq \frac{P}{W}$, then "trial" is sequentially rational at the bottom information set. Then, in order for the non-negligent type of the defendant to choose "resist" it must be that the plaintiff's strategy involves "trial" also at the middle information set, for which we need him to assign probability $p \geq \frac{S+P}{W}$ to the negligent type (which is possible, since the middle information set is not reached); of course, this requires $W \geq S + P$. Thus,

$\big($(sue,trial,trial),(resist,resist)$\big)$ is an equilibrium if and only if $q \geq \frac{P}{W}$ with $p \geq \frac{S+P}{W}$.

(f) Note that here the restriction $W - P > S$ does not hold.

In this case, $q < \frac{S+P}{W} = \frac{80+70}{100} = \frac{3}{2}$ and thus, by the previous analysis, there is no pooling equilibrium of type **P1**, that is, $\big($(sue,trial,trial),(offer,offer)$\big)$ is not an equilibrium.

As for pooling equilibria of type **P2**, $\big($(sue,trial,trial),(resist,resist)$\big)$ is not an equilibrium because $S + P > W$; $\big($(sue,trial,drop),(resist,resist)$\big)$ is not an equilibrium either. However, there is a pooling equilibrium of type **P2** with ((sue,settle,drop),(resist,resist)) with any beliefs at the middle information set, since "settle" strictly dominates "trial" there (and, of course, belief $q = \frac{1}{12}$ on the left node of the bottom information set). □