Appendix C

Denavit–Hartenberg Parameters

The basic idea underlying the Denavit–Hartenberg approach to forward kinematics is to attach reference frames to each link of the open chain and then to derive the forward kinematics from the knowledge of the relative displacements between adjacent link frames. Assume that a fixed reference frame has been established and that a reference frame (the end-effector frame) has been attached to some point on the last link of the open chain. For a chain consisting of n one-degree-of-freedom joints, the links are numbered sequentially from 0 to n: the ground link is labeled 0, and the end-effector frame is attached to link n. Reference frames attached to the links are also correspondingly labeled from $\{0\}$ (the fixed frame) to $\{n\}$ (the end-effector frame). The joint variable corresponding to the ith joint is denoted θ_i. The forward kinematics of the n-link open chain can then be expressed as

$$T_{0n}(\theta_1, \ldots, \theta_n) = T_{01}(\theta_1)T_{12}(\theta_2) \cdots T_{n-1,n}(\theta_n), \tag{C.1}$$

where $T_{i,i-1} \in SE(3)$ denotes the relative displacement between link frames $\{i-1\}$ and $\{i\}$. Depending on how the link reference frames have been chosen, each $T_{i-1,i}$ can be obtained in a straightforward fashion.

C.1 Assigning Link Frames

Rather than attaching reference frames to each link in an arbitrary fashion, in the Denavit–Hartenberg convention a set of rules for assigning link frames

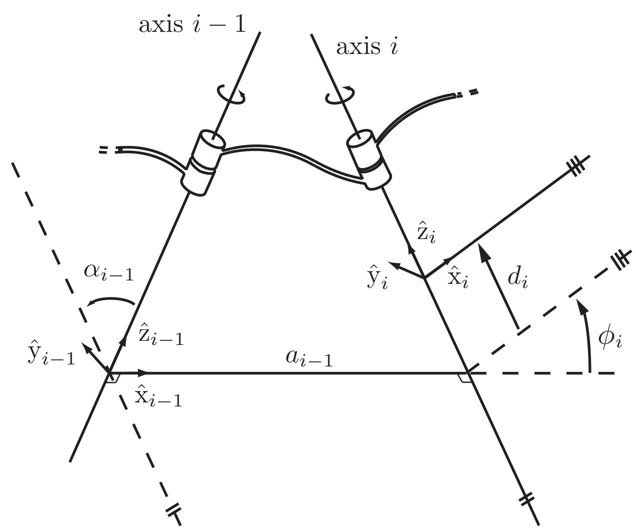


Figure C.1: Illustration of the Denavit–Hartenberg parameters.

is observed. Figure C.1 illustrates the frame-assignment convention for two adjacent revolute joints $i-1$ and i that are connected by link $i-1$.

The first rule is that the $\hat{z}_i$-axis coincides with joint axis i and the $\hat{z}_{i-1}$-axis coincides with joint axis $i-1$. The direction of positive rotation about each link's $\hat{z}$-axis is determined by the right-hand rule.

Once the $\hat{z}$-axis direction has been assigned, the next rule determines the origin of the link reference frame. First, find the line segment that orthogonally intersects both the joint axes $\hat{z}_{i-1}$ and $\hat{z}_i$. For now let us assume that this line segment is unique; the case where it is not unique (i.e., when the two joint axes are parallel), or fails to exist (i.e., when the two joint axes intersect), is addressed later. Connecting joint axes $i-1$ and i by a mutually perpendicular line, the origin of frame $\{i-1\}$ is then located at the point where this line intersects joint axis $i-1$.

Determining the remaining $\hat{x}$- and $\hat{y}$-axes of each link reference frame is now straightforward: the $\hat{x}$-axis is chosen to be in the direction of the mutually perpendicular line pointing from the $(i-1)$-axis to the i-axis. The $\hat{y}$-axis is then uniquely determined from the cross product $\hat{x} \times \hat{y} = \hat{z}$. Figure C.1 depicts the link frames $\{i\}$ and $\{i-1\}$ chosen according to this convention.

Having assigned reference frames in this fashion for links i and $i-1$, we now define four parameters that exactly specify $T_{i-1,i}$:

- The length of the mutually perpendicular line, denoted by the scalar a_{i-1}, is called the **link length** of link $i-1$. Despite its name, this link length does not necessarily correspond to the actual length of the physical link.
- The **link twist** α_{i-1} is the angle from $\hat{z}_{i-1}$ to $\hat{z}_i$, measured about $\hat{x}_{i-1}$.
- The **link offset** d_i is the distance from the intersection of $\hat{x}_{i-1}$ and $\hat{z}_i$ to the origin of the link-i frame (the positive direction is defined to be along the $\hat{z}_i$-axis).
- The **joint angle** ϕ_i is the angle from $\hat{x}_{i-1}$ to $\hat{x}_i$, measured about the $\hat{z}_i$-axis.

These parameters constitute the Denavit–Hartenberg (D–H) parameters. For an open chain with n one-degree-of-freedom joints, the $4n$ D–H parameters are sufficient to completely describe the forward kinematics. In the case of an open chain with all joints revolute, the link lengths a_{i-1}, twists α_{i-1}, and offset parameters d_i are all constant, while the joint angle parameters ϕ_i act as the joint variables.

We now consider the cases where the mutually perpendicular line is undefined or fails to be unique, or where some of the joints are prismatic; finally, we consider how to choose the ground and end-effector frames.

When Adjacent Revolute Joint Axes Intersect

If two adjacent revolute joint axes intersect each other then a mutually perpendicular line between the joint axes fails to exist. In this case the link length is set to zero, and we choose $\hat{x}_{i-1}$ to be perpendicular to the plane spanned by $\hat{z}_{i-1}$ and $\hat{z}_i$. There are two possibilities, both of which are acceptable: one leads to a positive value of the twist angle α_{i-1} while the other leads to a negative value.

When Adjacent Revolute Joint Axes Are Parallel

The second special case occurs when two adjacent revolute joint axes are parallel. In this case there exist many possibilities for a mutually perpendicular line, all of which are valid (more precisely, a one-dimensional family of mutual perpendicular lines is said to exist). A useful guide is to try to choose the mutually perpendicular line that is the most physically intuitive and that results in as many zero parameters as possible.

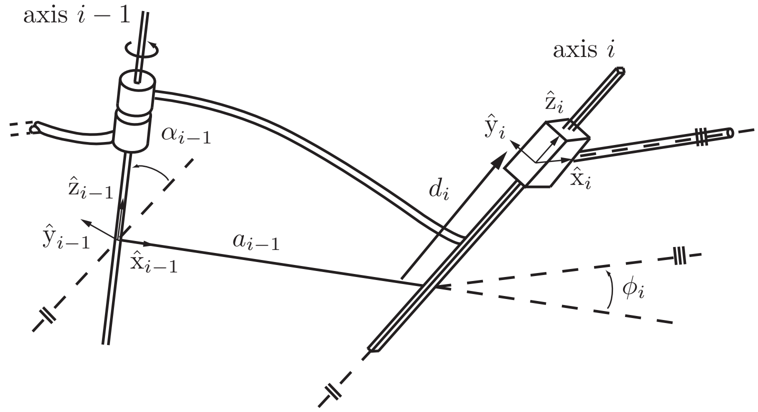


Figure C.2: Link frame assignment convention for prismatic joints. Joint $i-1$ is a revolute joint, while joint i is a prismatic joint.

Prismatic Joints

For prismatic joints, the $\hat{z}$-direction of the link reference frame is chosen to be along the positive direction of translation. This convention is consistent with that for revolute joints, in which the $\hat{z}$-axis indicates the positive axis of rotation. With this choice the link offset d_i is the joint variable and the joint angle ϕ_i is constant (see Figure C.2). The procedure for choosing the link-frame origin, as well as the remaining $\hat{x}$- and $\hat{y}$-axes, remains the same as for revolute joints.

Assigning the Ground and End-Effector Frames

Our frame-assignment procedure described thus far does not specify how to choose the ground and final link frames. Here, as before, a useful guideline is to choose initial and final frames that are the most physically intuitive and that simplify as many D–H parameters as possible. This usually implies that the ground frame is chosen to coincide with the link-1 frame in its zero (rest) position; in the event that the joint is revolute this choice forces $a_0 = \alpha_0 = d_1 = 0$, while for a prismatic joint we have $a_0 = \alpha_0 = \phi_1 = 0$. The end-effector frame is attached to some reference point on the end-effector, usually at a location that makes the description of the task intuitive and natural and also simplifies as many of the D–H parameters as possible (e.g., their values become zero).

It is important to realize that arbitrary choices of the ground and end-effector frames may not always be possible, since there may not exist a valid set of D–H parameters to describe the relative transformation. We elaborate on this point

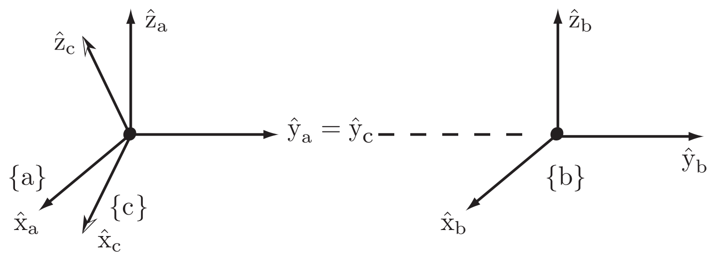


Figure C.3: An example of three frames {a}, {b}, and {c}, for which the transformations T_{ab} and T_{ac} cannot be described by any set of D–H parameters.

below.

C.2 Why Four Parameters are Sufficient

In our earlier study of spatial displacements we argued that a minimum of six independent parameters were required to describe the relative displacement between two frames in space, three for the orientation and three for the position. On the basis of this result, it would seem that, for an n-link arm, a total of $6n$ parameters would be required to completely describe the forward kinematics (each $T_{i-1,i}$ in the above equation would require six parameters). Surprisingly, in the D–H parameter representation only four parameters are required for each transformation $T_{i-1,i}$. Although this may at first appear to contradict our earlier results, the reduction in the number of parameters is accomplished by the carefully stipulated rules for assigning link reference frames. If the link reference frames are assigned in arbitrary fashion, then more parameters are required.

Consider, for example, the link frames shown in Figure C.3. The transformation from frame {a} to frame {b} is a pure translation along the ŷ-axis of frame {a}. If one were to try to express the transformation T_{ab} in terms of the D–H parameters (α, a, d, θ) as prescribed above, it should become apparent that no such set of parameter values exists. Similarly, the transformation T_{ac} also does not admit a description in terms of D–H parameters, as only rotations about the x̂- and ẑ-axes are permissible. Under our D–H convention, only rotations and translations along the x̂- and ẑ-axes are allowed, and no combination of such motions can achieve the transformation shown in Figure C.3.

Given that the D–H convention uses exactly four parameters to describe the transformation between link frames, one might naturally wonder whether the number of parameters can be reduced even further, by an even more clever set of link-frame assignment rules. Denavit and Hartenberg showed that this is not possible and that four is the minimum number of parameters [34].

We end this section with a reminder that there are alternative conventions for assigning link frames. Whereas we chose the $\hat{z}$-axis to coincide with the joint axis, some authors choose the $\hat{x}$-axis and reserve the $\hat{z}$-axis to be the direction of the mutually perpendicular line. To avoid ambiguities in the interpretation of the D–H parameters, it is essential to include a concise description of the link frames together with the parameter values.

C.3 Manipulator Forward Kinematics

Once all the transformations $T_{i-1,i}$ between adjacent link frames are known in terms of their D–H parameters, the forward kinematics is obtained by sequentially multiplying these link transformations. Each link frame transformation is of the form

$$\begin{aligned} T_{i-1,i} &= \mathrm{Rot}(\hat{\mathrm{x}}, \alpha_{i-1})\mathrm{Trans}(\hat{\mathrm{x}}, a_{i-1})\mathrm{Trans}(\hat{\mathrm{z}}, d_i)\mathrm{Rot}(\hat{\mathrm{z}}, \phi_i) \\ &= \begin{bmatrix} \cos\phi_i & -\sin\phi_i & 0 & a_{i-1} \\ \sin\phi_i\cos\alpha_{i-1} & \cos\phi_i\cos\alpha_{i-1} & -\sin\alpha_{i-1} & -d_i\sin\alpha_{i-1} \\ \sin\phi_i\sin\alpha_{i-1} & \cos\phi_i\sin\alpha_{i-1} & \cos\alpha_{i-1} & d_i\cos\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

where

$$\mathrm{Rot}(\hat{\mathrm{x}}, \alpha_{i-1}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_{i-1} & -\sin\alpha_{i-1} & 0 \\ 0 & \sin\alpha_{i-1} & \cos\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{C.2}$$

$$\mathrm{Trans}(\hat{\mathrm{x}}, a_{i-1}) = \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{C.3}$$

$$\mathrm{Trans}(\hat{\mathrm{z}}, d_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{C.4}$$

$$\mathrm{Rot}(\hat{\mathrm{z}}, \phi_i) = \begin{bmatrix} \cos\phi_{i-1} & -\sin\phi_{i-1} & 0 & 0 \\ \sin\phi_{i-1} & \cos\phi_{i-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{C.5}$$

A useful way to visualize $T_{i,i-1}$ is that it transports frame $\{i-1\}$ to frame $\{i\}$ via the following sequence of four transformations:

(a) A rotation of frame $\{i-1\}$ about its $\hat{\mathrm{x}}$-axis by an angle α_{i-1}.

(b) A translation of this new frame along its $\hat{\mathrm{x}}$-axis by a distance a_{i-1}.

(c) A translation of the new frame formed by (b) along its $\hat{\mathrm{z}}$-axis by a distance d_i.

(d) A rotation of the new frame formed by (c) about its $\hat{\mathrm{z}}$-axis by an angle ϕ_i.

Note that switching the order of the first and second steps will not change the final form of $T_{i-1,i}$. Similarly, the order of the third and fourth steps can also be switched without affecting $T_{i-1,i}$.

C.4 Examples

We now derive the D–H parameters for some common spatial open chain structures.

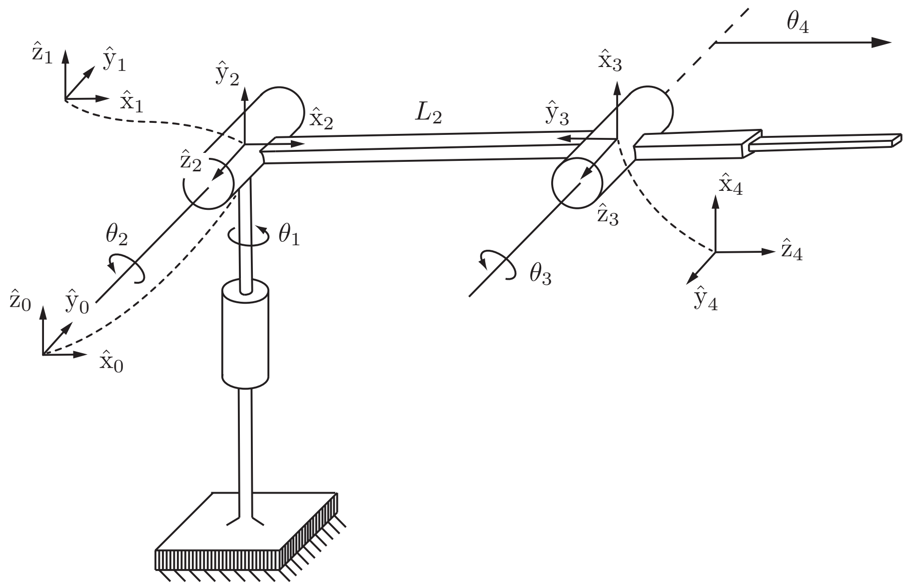
Example C.1 (A 3R spatial open chain)**.** Consider the 3R spatial open chain of Figure 4.3, shown in its zero position (i.e., with all its joint variables set to zero). The assigned link reference frames are shown in the figure, and the corresponding D–H parameters are listed in the following table:

i	α_{i-1}	a_{i-1}	d_i	ϕ_i
1	0	0	0	θ_1
2	$90°$	L_1	0	$\theta_2 - 90°$
3	$-90°$	L_2	0	θ_3

Note that frames $\{1\}$ and $\{2\}$ are uniquely specified from our frame assignment convention, but that we have some latitude in choosing frames $\{0\}$ and $\{3\}$. Here we choose the ground frame $\{0\}$ to coincide with frame $\{1\}$ (resulting in $\alpha_0 = a_0 = d_1 = 0$) and frame $\{3\}$ to be such that $\hat{\mathrm{x}}_3 = \hat{\mathrm{x}}_2$ (resulting in no offset to the joint angle θ_3).

Example C.2 (A spatial RRRP open chain)**.** The next example we consider is the four-dof RRRP spatial open chain of Figure C.4, here shown in its zero position. The link frame assignments are as shown, and the corresponding D–H parameters are listed in the figure.

The four joint variables are $(\theta_1, \theta_2, \theta_3, \theta_4)$, where θ_4 is the displacement of the prismatic joint. As in the previous example, the ground frame $\{0\}$ and final

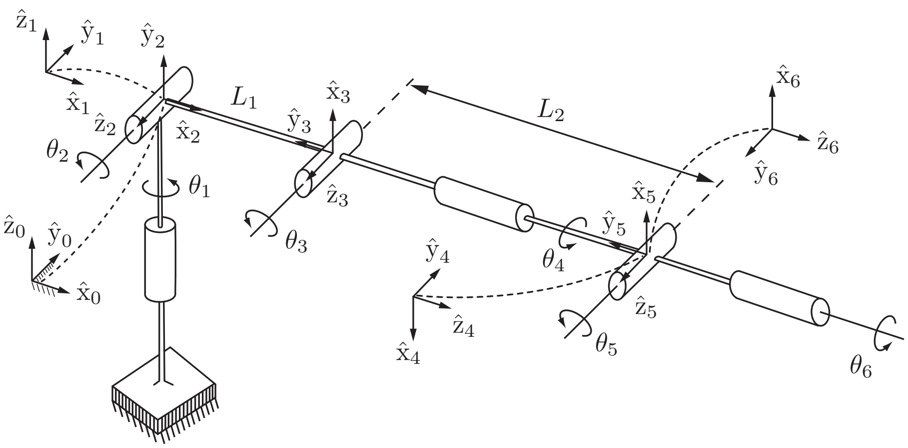


i	α_{i-1}	a_{i-1}	d_i	ϕ_i
1	0	0	0	θ_1
2	90°	0	0	θ_2
3	0	L_2	0	$\theta_3 + 90°$
4	90°	0	θ_4	0

Figure C.4: An RRRP spatial open chain.

link frame {4} have been chosen to make as many of the D–H parameters zero as possible.

Example C.3 (A spatial 6R open chain)**.** The final example we consider is the widely used 6R robot arm (Figure C.5). This open chain has six rotational joints: the first three joints function as a Cartesian positioning device, while the last three joints act as a ZYZ Euler angle-type wrist. The link frames are shown in the figure, and the corresponding D–H parameters are listed in the table accompanying the figure.



i	α_{i-1}	a_{i-1}	d_i	ϕ_i
1	0	0	0	θ_1
2	90°	0	0	θ_2
3	0	L_1	0	$\theta_3 + 90^\circ$
4	90°	0	L_2	$\theta_4 + 180^\circ$
5	90°	0	0	$\theta_5 + 180^\circ$
6	90°	0	0	θ_6

Figure C.5: A 6R spatial open chain.

C.5 Relation Between the PoE and D–H Representations

The product of exponentials formula can be derived directly from the D–H parameter-based representation of the forward kinematics. As before, denote the relative displacement between adjacent link frames by

$$T_{i-1,i} = \mathrm{Rot}(\hat{\mathrm{x}}, \alpha_{i-1})\mathrm{Trans}(\hat{\mathrm{x}}, a_{i-1})\mathrm{Trans}(\hat{\mathrm{z}}, d_i)\mathrm{Rot}(\hat{\mathrm{z}}, \phi_i).$$

If joint i is revolute, the first three matrices can be regarded as constant and ϕ_i becomes the revolute joint variable. Define $\theta_i = \phi_i$ and

$$M_i = \mathrm{Rot}(\hat{\mathrm{x}}, \alpha_{i-1})\mathrm{Trans}(\hat{\mathrm{x}}, a_{i-1})\mathrm{Trans}(\hat{\mathrm{z}}, d_i), \tag{C.6}$$

and write $\mathrm{Rot}(\hat{\mathrm{z}}, \theta_i)$ as the following matrix exponential:

$$\mathrm{Rot}(\hat{\mathrm{z}}, \theta_i) = e^{[\mathcal{A}_i]\theta_i}, \qquad [\mathcal{A}_i] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{C.7}$$

With the above definitions we can write $T_{i-1,i} = M_i e^{[\mathcal{A}_i]\theta_i}$.

If joint i is prismatic then d_i becomes the joint variable, ϕ_i is a constant parameter, and the order of $\mathrm{Trans}(\hat{\mathrm{z}}, d_i)$ and $\mathrm{Rot}(\hat{\mathrm{z}}, \phi_i)$ in $T_{i-1,i}$ can be reversed (recall that reversing translations and rotations taken along the same axis still results in the same motion). In this case we can still write $T_{i-1,i} = M_i e^{[\mathcal{A}_i]\theta_i}$, where $\theta_i = d_i$ and

$$\begin{aligned} M_i &= \mathrm{Rot}(\hat{\mathrm{x}}, \alpha_{i-1})\mathrm{Trans}(\hat{\mathrm{x}}, a_{i-1})\mathrm{Rot}(\hat{\mathrm{z}}, \phi_i), & \text{(C.8)} \\ [\mathcal{A}_i] &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. & \text{(C.9)} \end{aligned}$$

From the above, for an n-link open chain containing both revolute and prismatic joints, the forward kinematics can be written as

$$T_{0,n} = M_1 e^{[\mathcal{A}_1]\theta_1} M_2 e^{[\mathcal{A}_2]\theta_2} \cdots M_n e^{[\mathcal{A}_n]\theta_n} \tag{C.10}$$

where θ_i denotes joint variable i, and $[\mathcal{A}_i]$ is either of the form (C.7) or of the form (C.9), depending on whether joint i is revolute or prismatic.

We now make use of the matrix identity $Me^P M^{-1} = e^{MPM^{-1}}$, which holds for any nonsingular $M \in \mathbb{R}^{n\times n}$ and arbitrary $P \in \mathbb{R}^{n\times n}$. This identity can also be rearranged as $Me^P = e^{MPM^{-1}}M$. Beginning from the left of Equation (C.10), if we repeatedly apply the identity, after n iterations we obtain the product of exponentials formula as originally derived:

$$\begin{aligned} T_{0n} &= e^{M_1[\mathcal{A}_1]M_1^{-1}\theta_1}(M_1M_2)e^{[\mathcal{A}_2]\theta_2}\cdots e^{[\mathcal{A}_n]\theta_n} \\ &= e^{M_1[\mathcal{A}_1]M_1^{-1}\theta_1}e^{(M_1M_2)[\mathcal{A}_2](M_1M_2)^{-1}\theta_2}(M_1M_2M_3)e^{[\mathcal{A}_3]\theta_3}\cdots e^{[\mathcal{A}_n]\theta_n} \\ &= e^{[\mathcal{S}_1]\theta_1}\cdots e^{[\mathcal{S}_n]\theta_n}M, \end{aligned} \tag{C.11}$$

where

$$\begin{aligned} [\mathcal{S}_i] &= (M_1\cdots M_{i-1})[\mathcal{A}_i](M_1\cdots M_{i-1})^{-1}, \qquad i = 1,\ldots,n, & \text{(C.12)} \\ M &= M_1M_2\cdots M_n. & \text{(C.13)} \end{aligned}$$

We now re-examine the physical meaning of the $\mathcal{S}_i$ by recalling how a screw twist transforms under a change of reference frames. If $\mathcal{S}_a$ represents the screw twist for a given screw motion with respect to frame {a}, and $\mathcal{S}_b$ represents the screw twist for the same physical screw motion but this time with respect to frame {b}, then recall that $\mathcal{S}_a$ and $\mathcal{S}_b$ are related by

$$[\mathcal{S}_b] = T_{ba}[\mathcal{S}_a]T_{ba}^{-1} \tag{C.14}$$

or, using the adjoint notation $\mathrm{Ad}_{T_{ba}}$,

$$\mathcal{S}_b = \mathrm{Ad}_{T_{ba}}(\mathcal{S}_a). \tag{C.15}$$

Seen from the perspective of this transformation rule, Equation (C.12) suggests that $\mathcal{A}_i$ is the screw twist for joint axis i as seen from link frame $\{i\}$, while $\mathcal{S}_i$ is the screw twist for joint axis i as seen from the fixed frame $\{0\}$.

C.6 A Final Comparison

We now summarize the relative advantages and disadvantages of the PoE formula as compared with the D–H representation. Recall that the D–H parameters constitute a minimal parameter set, i.e., only four parameters are needed to describe the transformation between adjacent link frames. However, it is necessary to assign link frames in a way such that valid D–H parameters exist; they cannot be chosen arbitrarily. The same applies when choosing the base and end-effector frames. Moreover, there is more than one convention for assigning link frames; in some conventions the link frame is attached so that the joint axis is aligned in the $\hat{\mathrm{x}}$ rather than the $\hat{\mathrm{z}}$-direction as we have done. Further note that for revolute joints, the joint variable is taken to be θ whereas for prismatic joints the joint variable is d.

Another disadvantage of the D–H parameters is that they can become ill-conditioned. For example, when adjacent joint axes are nearly parallel, the common normal between the joint axes can vary wildly with small changes in the axes' orientation. This ill-conditioned behavior of the D–H parameters makes their accurate measurement and identification difficult, since robots typically have manufacturing and other errors so that, e.g., a collection of joint axes may indeed deviate from being exactly parallel or from intersecting at a single common point.

The requirements for identifying the D–H parameters of a robot can be contrasted with those for the PoE formula. Once a zero position for the robot has been specified, and a base frame and end-effector frame established (recall that, unlike the case with D–H parameters, the base and end-effector frames can be chosen arbitrarily with no restrictions), then the product of exponentials formula is completely defined. No link reference frames are necessary and no additional bookkeeping is needed to distinguish between revolute and prismatic joints. The interpretation of the parameters in the PoE formula, as the screws representing the joint axes, is natural and intuitive. Moreover the columns of the Jacobian can also be interpreted as the (configuration-dependent) screws of the joint axes.

The only disadvantage – that the PoE representation of the joint axes uses more parameters than the D–H representation – is more than offset by the many advantages. In short, there is little practical or other reason to use the D–H parameters in modeling the forward kinematics of open chains.