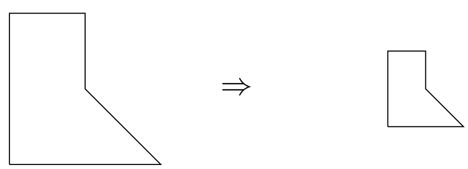
11 Wavelets

Given a vector space of functions, one would like an orthonormal set of basis functions that span the space. The Fourier transform provides a set of basis functions based on sines and cosines. Often we are dealing with functions that have local structure in which case we would like the basis vectors to have finite support. Also we would like to have an efficient algorithm for computing the coefficients of the expansion of a function in the basis.

11.1 Dilation

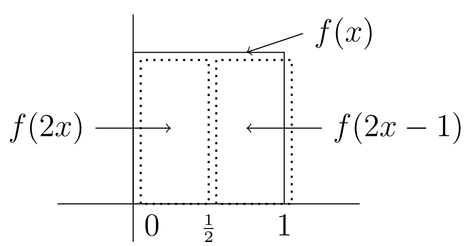
We begin our development of wavelets by first introducing dilation. A *dilation* is a mapping that scales all distances by the same factor.



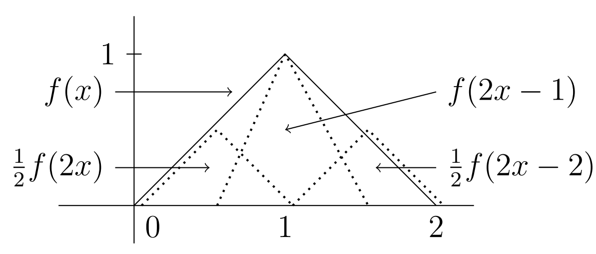
A dilation equation is an equation where a function is defined in terms of a linear combination of scaled, shifted versions of itself. For instance,

$$f(x) = \sum_{k=0}^{d-1} c_k f(2x-k).$$

An example of this is $f(x) = f(2x) + f(2x-1)$ which has a solution $f(x)$ equal to one for $0 \le x < 1$ and is zero elsewhere. The equation is illustrated in the figure below. The solid rectangle is $f(x)$ and the dotted rectangles are $f(2x)$ and $f(2x-1)$.



Another example is $f(x) = \frac{1}{2}f(2x) + f(2x-1) + \frac{1}{2}f(2x-2)$. A solution is illustrated in the figure below. The function $f(x)$ is indicated by solid lines. The functions $\frac{1}{2}f(2x)$, $f(2x+1)$, and $\frac{1}{2}f(2x-2)$ are indicated by dotted lines.



If a dilation equation is of the form $\sum_{k=1}^{d-1} c_k f(2x-k)$ then we say that all dilations in the equation are factor of two reductions.

Lemma 11.1 *If a dilation equation in which all the dilations are a factor of two reduction has a solution, then either the coefficients on the right hand side of the equation sum to two or the integral $\int_{-\infty}^{\infty} f(x)dx$ of the solution is zero.*

Proof: Integrate both sides of the dilation equation from $-\infty$ to $+\infty$.

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= \int_{-\infty}^{\infty} \sum_{k=0}^{d-1} c_k f(2x-k)dx = \sum_{k=0}^{d-1} c_k \int_{-\infty}^{\infty} f(2x-k)dx \\ &= \sum_{k=0}^{d-1} c_k \int_{-\infty}^{\infty} f(2x)dx = \frac{1}{2}\sum_{k=0}^{d-1} c_k \int_{-\infty}^{\infty} f(x)dx\end{aligned}$$

If $\int_{-\infty}^{\infty} f(x)dx \neq 0$, then dividing both sides by $\int\limits_{-\infty}^{\infty} f(x)dx$ gives $\sum\limits_{k=0}^{d-1} c_k = 2$ ∎

The above proof interchanged the order of the summation and the integral. This is valid provided the 1-norm of the function is finite. Also note that there are nonzero solutions to dilation equations in which all dilations are a factor of two reduction where the coefficients do not sum to two such as

$$f(x) = f(2x) + f(2x-1) + f(2x-2) + f(2x-3)$$

or

$$f(x) = f(2x) + 2f(2x-1) + 2f(2x-2) + 2f(2x-3) + f(2x-4).$$

In these examples $f(x)$ takes on both positive and negative values and $\int_{-\infty}^{\infty} f(x)dx = 0$.

11.2 The Haar Wavelet

Let $\phi(x)$ be a solution to the dilation equation $f(x) = f(2x) + f(2x-1)$. The function ϕ is called a *scale function* or *scale vector* and is used to generate the two dimensional family of functions, $\phi_{jk}(x) = \phi(2^j x - k)$, where j and k are non-negative integers. Other authors scale $\phi_{jk} = \phi(2^j x - k)$ by $2^{\frac{j}{2}}$ so that the 2-norm, $\int_{-\infty}^{\infty} \phi_{jk}^2(t)dt$, is 1. However, for educational purposes, simplifying the notation for ease of understanding was preferred.

For a given value of j, the shifted versions, $\{\phi_{jk}|k \geq 0\}$, span a space V_j. The spaces $V_0, V_1, V_2, \ldots$ are larger and larger spaces and allow better and better approximations to a function. The fact that $\phi(x)$ is the solution of a dilation equation implies that for any fixed j, ϕ_{jk} is a linear combination of the $\{\phi_{j+1,k'}|k' \geq 0\}$ and this ensures that $V_j \subseteq V_{j+1}$. It is for this reason that it is desirable in designing a wavelet system for the scale function to satisfy a dilation equation. For a given value of j, the shifted ϕ_{jk} are orthogonal in the

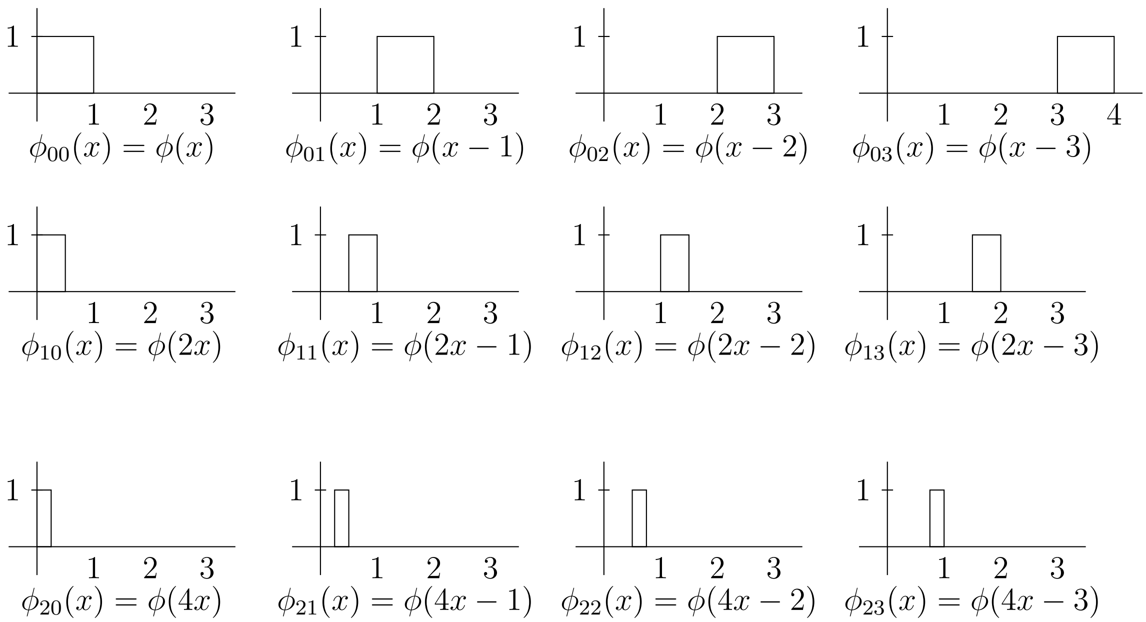


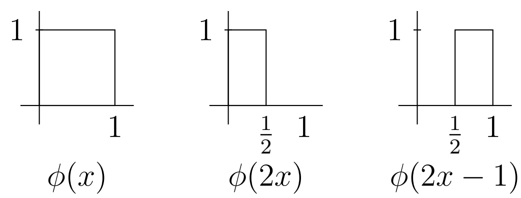
Figure 11.1: Set of scale functions associated with the Haar wavelet.

sense that $\int_x \phi_{jk}(x)\phi_{jl}(x)dx = 0$ for $k \neq l$.

Note that for each j, the set of functions ϕ_{jk}, $k = 0, 1, 2 \ldots$, form a basis for a vector space V_j and are orthogonal. The set of basis vectors ϕ_{jk}, for all j and k, form an over-complete basis and for different values of j are not orthogonal. Since ϕ_{jk}, $\phi_{j+1,2k}$, and $\phi_{j+1,2k+1}$ are linearly dependent, for each value of j delete $\phi_{j+1,k}$ for odd values of k to get a linearly independent set of basis vectors. To get an orthogonal set of basis vectors, define

$$\psi_{jk}(x) = \begin{cases} 1 & \frac{2k}{2^j} \leq x < \frac{2k+1}{2^j} \\ -1 & \frac{2k+1}{2^j} \leq x < \frac{2k+2}{2^j} \\ 0 & \text{otherwise} \end{cases}$$

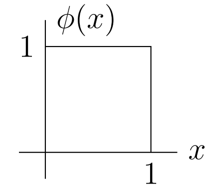
and replace $\phi_{j,2k}$ with $\psi_{j+1,2k}$. Basically, replace the three functions



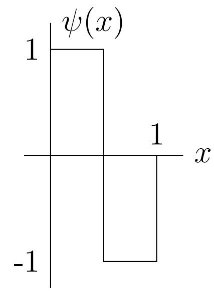
by the two functions

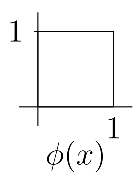
The Haar Wavelet

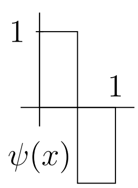
$$\phi(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$



$$\psi(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$







The basis set becomes

$$\begin{array}{llllllll} \phi_{00} & \psi_{10} & & & & & & \\ \psi_{20} & \psi_{22} & & & & & & \\ \psi_{30} & \psi_{32} & \psi_{34} & \psi_{36} & & & & \\ \psi_{40} & \psi_{42} & \psi_{44} & \psi_{46} & \psi_{48} & \psi_{4,10} & \psi_{4,12} & \psi_{4,14} \end{array}$$

To approximate a function that has only finite support, select a scale vector $\phi(x)$ whose scale is that of the support of the function to be represented. Next approximate the function by the set of scale functions $\phi(2^j x - k)$, $k = 0, 1, \ldots$, for some fixed value of j. The value of j is determined by the desired accuracy of the approximation. Basically the x axis has been divided into intervals of size 2^{-j} and in each interval the function is approximated by a fixed value. It is this approximation of the function that is expressed as a linear combination of the basis functions.

Once the value of j has been selected, the function is sampled at 2^j points, one in each interval of width 2^{-j}. Let the sample values be $s_0, s_1, \ldots$. The approximation to the function is $\sum_{k=0}^{2^j-1} s_k \phi(2^j x - k)$ and is represented by the vector $(s_0, s_1 \ldots, s_{2^j-1})$. The problem now is to represent the approximation to the function using the basis vectors rather than the nonorthogonal set of scale functions $\phi_{jk}(x)$. This is illustrated in the following example.

To represent the function corresponding to a vector such as $\begin{pmatrix} 3 & 1 & 4 & 8 & 3 & 5 & 7 & 9 \end{pmatrix}$, one needs to find the c_i such that

$$\begin{pmatrix} 3 \\ 1 \\ 4 \\ 8 \\ 3 \\ 5 \\ 7 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix}.$$

The first column represents the scale function $\phi(x)$ and subsequent columns the ψ's. The tree in Figure 11.2 illustrates an efficient way to find the coefficients representing the vector $\begin{pmatrix} 3 & 1 & 4 & 8 & 3 & 5 & 7 & 9 \end{pmatrix}$ in the basis. Each vertex in the tree contains the average of the quantities of its two children. The root gives the average of the elements in the vector, which is 5 in this example. This average is the coefficient of the basis vector in the first column of the above matrix. The second basis vector converts the average of the eight elements into the average of the first four elements, which is 4, and the last four elements, which is 6, with a coefficient of -1. Working up the tree determines the coefficients for each basis vector.

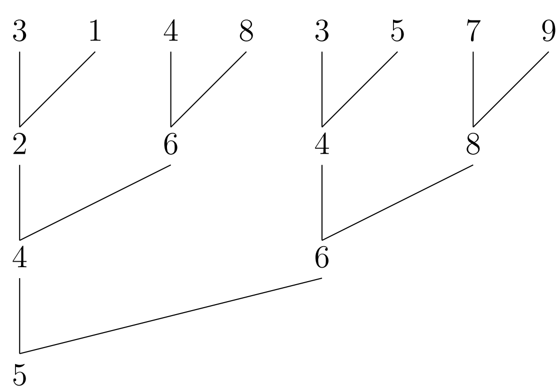


Figure 11.2: Tree of function averages

$$\begin{pmatrix} 3 \\ 1 \\ 4 \\ 8 \\ 3 \\ 5 \\ 7 \\ 9 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} -1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} -2 \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} -2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} +1 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} -2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} -1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} -1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

11.3 Wavelet Systems

So far we have explained wavelets using the simple-to-understand Haar wavelet. We now consider general wavelet systems. A wavelet system is built from a basic scaling function $\phi(x)$, which comes from a dilation equation. Scaling and shifting of the basic scaling function gives a two dimensional set of scaling functions ϕ_{jk} where

$$\phi_{jk}(x) = \phi(2^j x - k).$$

For a fixed value of j, the ϕ_{jk} span a space V_j. If $\phi(x)$ satisfies a dilation equation

$$\phi(x) = \sum_{k=0}^{d-1} c_k \phi(2x - k),$$

then ϕ_{jk} is a linear combination of the $\phi_{j+1,k}$'s and this implies that $V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \cdots$.

11.4 Solving the Dilation Equation

Consider solving a dilation equation

$$\phi(x) = \sum_{k=0}^{d-1} c_k \phi(2x - k)$$

to obtain the scale function for a wavelet system. Perhaps the easiest way is to assume a solution and then calculate the scale function by successive approximation as in the following program for the Daubechies scale function:

$$\phi(x) = \tfrac{1+\sqrt{3}}{4}\phi(2x) + \tfrac{3+\sqrt{3}}{4}\phi(2x-1) + \tfrac{3-\sqrt{3}}{4}\phi(2x-2) + \tfrac{1-\sqrt{3}}{4}\phi(2x-3),$$

The solution will actually be samples of $\phi(x)$ at some desired resolution.

Program Compute-Daubechies:

Set the initial approximation to $\phi(x)$ by generating a vector whose components approximate the samples of $\phi(x)$ at equally spaced values of x.

Begin with the coefficients of the dilation equation.

$$c_1 = \tfrac{1+\sqrt{3}}{4} \qquad c_2 = \tfrac{3+\sqrt{3}}{4} \qquad c_3 = \tfrac{3-\sqrt{3}}{4} \qquad c_4 = \tfrac{1-\sqrt{3}}{4}$$

Execute the following loop until the values for $\phi(x)$ converge.

begin

Calculate $\phi(2x)$ by averaging successive values of $\phi(x)$ together. Fill out the remaining half of the vector representing $\phi(2x)$ with zeros.

Calculate $\phi(2x-1)$, $\phi(2x-2)$, and $\phi(2x-3)$ by shifting the contents of $\phi(2x)$ the appropriate distance, discarding the zeros that move off the right end and adding zeros at the left end.

Calculate the new approximation for $\phi(x)$ using the above values for $\phi(2x-1)$, $\phi(2x-2)$, and $\phi(2x-3)$ in the dilation equation for $\phi(2x)$.

end

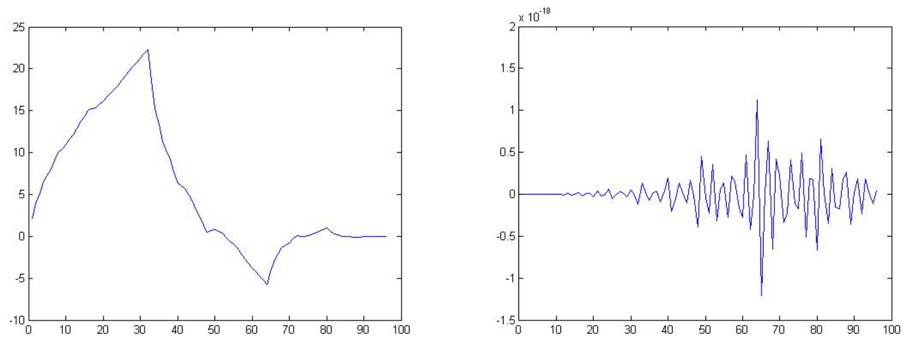


Figure 11.3: Daubechies scale function and associated wavelet

The convergence of the iterative procedure for computing is fast if the eigenvectors of a certain matrix are unity.

Another approach to solving the dilation equation

Consider the dilation equation $\phi(x) = \frac{1}{2}f(2x) + f(2x-1) + \frac{1}{2}f(2x-2)$ and consider continuous solutions with support in $0 \le x < 2$.

$$\begin{array}{ll} \phi(0) = \frac{1}{2}\phi(0) + \phi(-1) + \phi(-2) = \frac{1}{2}\phi(0) + 0 + 0 & \phi(0) = 0 \\ \phi(2) = \frac{1}{2}\phi(4) + \phi(3) + \phi(2) = \frac{1}{2}\phi(2) + 0 + 0 & \phi(2) = 0 \\ \phi(1) = \frac{1}{2}\phi(2) + \phi(1) + \phi(0) = 0 + \phi(1) + 0 & \phi(1) \quad \text{arbitrary} \end{array}$$

Set $\phi(1) = 1$. Then

$$\begin{aligned} \phi(\tfrac{1}{2}) &= \tfrac{1}{2}\phi(1) + \phi(0) + \tfrac{1}{2}\phi(-1) = \tfrac{1}{2} \\ \phi(\tfrac{3}{2}) &= \tfrac{1}{2}\phi(3) + \phi(2) + \tfrac{1}{2}\phi(1) = \tfrac{1}{2} \qquad . \\ \phi(\tfrac{1}{4}) &= \tfrac{1}{2}\phi(\tfrac{1}{2}) + \phi(-\tfrac{1}{2}) + \tfrac{1}{2}\phi(-\tfrac{3}{2}) = \tfrac{1}{4} \end{aligned}$$

One can continue this process and compute $\phi(\frac{i}{2^j})$ for larger values of j until $\phi(x)$ is approximated to a desired accuracy. If $\phi(x)$ is a simple equation as in this example, one could conjecture its form and verify that the form satisfies the dilation equation.

11.5 Conditions on the Dilation Equation

We would like a basis for a vector space of functions where each basis vector has finite support and the basis vectors are orthogonal. This is achieved by a wavelet system consisting of a shifted version of a scale function that satisfies a dilation equation along with a set of wavelets of various scales and shifts. For the scale function to have a nonzero integral, Lemma 11.1 requires that the coefficients of the dilation equation sum to two. Although the scale function $\phi(x)$ for the Haar system has the property that $\phi(x)$ and $\phi(x-k)$, $k>0$, are orthogonal, this is not true for the scale function for the dilation equation $\phi(x)=\frac{1}{2}\phi(2x)+\phi(2x-1)+\frac{1}{2}\phi(2x-2)$. The conditions that integer shifts of the scale function be orthogonal and that the scale function has finite support puts additional conditions on the coefficients of the dilation equation. These conditions are developed in the next two lemmas.

Lemma 11.2 *Let*

$$\phi(x)=\sum_{k=0}^{d-1} c_k\phi(2x-k).$$

If $\phi(x)$ and $\phi(x-k)$ are orthogonal for $k\neq 0$ and $\phi(x)$ has been normalized so that $\int_{-\infty}^{\infty}\phi(x)\phi(x-k)dx=\delta(k)$, then $\sum_{i=0}^{d-1}c_ic_{i-2k}=2\delta(k)$.

Proof: Assume $\phi(x)$ has been normalized so that $\int_{-\infty}^{\infty}\phi(x)\phi(x-k)dx=\delta(k)$. Then

$$\begin{aligned}\int_{x=-\infty}^{\infty}\phi(x)\phi(x-k)dx &= \int_{x=-\infty}^{\infty}\sum_{i=0}^{d-1}c_i\phi(2x-i)\sum_{j=0}^{d-1}c_j\phi(2x-2k-j)dx\\ &=\sum_{i=0}^{d-1}\sum_{j=0}^{d-1}c_ic_j\int_{x=-\infty}^{\infty}\phi(2x-i)\phi(2x-2k-j)dx\end{aligned}$$

Since

$$\begin{aligned}\int_{x=-\infty}^{\infty}\phi(2x-i)\phi(2x-2k-j)dx &= \frac{1}{2}\int_{x=-\infty}^{\infty}\phi(y-i)\phi(y-2k-j)dy\\ &=\frac{1}{2}\int_{x=-\infty}^{\infty}\phi(y)\phi(y+i-2k-j)dy\\ &=\frac{1}{2}\delta(2k+j-i),\end{aligned}$$

$\displaystyle\int_{x=-\infty}^{\infty}\phi(x)\phi(x-k)dx=\sum_{i=0}^{d-1}\sum_{j=0}^{d-1}c_ic_j\frac{1}{2}\delta(2k+j-i)=\frac{1}{2}\sum_{i=0}^{d-1}c_ic_{i-2k}$. Since $\phi(x)$ was normalized so that

$\displaystyle\int_{-\infty}^{\infty}\phi(x)\phi(x-k)dx=\delta(k)$, it follows that $\displaystyle\sum_{i=0}^{d-1}c_ic_{i-2k}=2\delta(k)$. ■

Scale and wavelet coefficients equations

$\phi(x) = \sum_{k=0}^{d-1} c_k \phi(2x-k)$

$\int\limits_{-\infty}^{\infty} \phi(x)\phi(x-k)dx = \delta(k)$

$\sum\limits_{j=0}^{d-1} c_j = 2$

$\sum\limits_{j=0}^{d-1} c_j c_{j-2k} = 2\delta(k)$

$c_k = 0$ unless $0 \le k \le d-1$

d even

$\sum\limits_{j=0}^{d-1} c_{2j} = \sum\limits_{j=0}^{d-1} c_{2j+1}$

$\psi(x) = \sum\limits_{k=0}^{d-1} b_k \phi(x-k)$

$\int\limits_{x=-\infty}^{\infty} \phi(x)\psi(x-k) = 0$

$\int\limits_{x=-\infty}^{\infty} \psi(x)dx = 0$

$\int\limits_{x=-\infty}^{\infty} \psi(x)\psi(x-k)dx = \delta(k)$

$\sum\limits_{i=0}^{d-1} (-1)^k b_i b_{i-2k} = 2\delta(k)$

$\sum\limits_{j=0}^{d-1} c_j b_{j-2k} = 0$

$\sum\limits_{j=0}^{d-1} b_j = 0$

$b_k = (-1)^k c_{d-1-k}$

One designs wavelet systems so the above conditions are satisfied.

Lemma 11.2 provides a necessary but not sufficient condition on the coefficients of the dilation equation for shifts of the scale function to be orthogonal. One should note that the conditions of Lemma 11.2 are not true for the triangular or piecewise quadratic solutions to

$$\phi(x) = \frac{1}{2}\phi(2x) + \phi(2x-1) + \frac{1}{2}\phi(2x-2)$$

and

$$\phi(x) = \frac{1}{4}\phi(2x) + \frac{3}{4}\phi(2x-1) + \frac{3}{4}\phi(2x-2) + \frac{1}{4}\phi(2x-3)$$

which overlap and are not orthogonal.

For $\phi(x)$ to have finite support the dilation equation can have only a finite number of terms. This is proved in the following lemma.

Lemma 11.3 *If $0 \le x < d$ is the support of $\phi(x)$, and the set of integer shifts, $\{\phi(x-k) | k \ge 0\}$, are linearly independent, then $c_k = 0$ unless $0 \le k \le d-1$.*

Proof: If the support of $\phi(x)$ is $0 \le x < d$, then the support of $\phi(2x)$ is $0 \le x < \frac{d}{2}$. If

$$\phi(x) = \sum_{k=-\infty}^{\infty} c_k \phi(2x-k)$$

the support of both sides of the equation must be the same. Since the $\phi(x-k)$ are linearly independent the limits of the summation are actually $k=0$ to $d-1$ and

$$\phi(x)=\sum_{k=0}^{d-1} c_k\phi(2x-k).$$

It follows that $c_k=0$ unless $0\leq k\leq d-1$.

The condition that the integer shifts are linearly independent is essential to the proof and the lemma is not true without this condition. ∎

One should also note that $\sum\limits_{i=0}^{d-1} c_i c_{i-2k}=0$ for $k\neq 0$ implies that d is even since for d odd and $k=\frac{d-1}{2}$

$$\sum_{i=0}^{d-1} c_i c_{i-2k}=\sum_{i=0}^{d-1} c_i c_{i-d+1}=c_{d-1}c_0.$$

For $c_{d-1}c_0$ to be zero either c_{d-1} or c_0 must be zero. Since either $c_0=0$ or $c_{d-1}=0$, there are only $d-1$ nonzero coefficients. From here on we assume that d is even. If the dilation equation has d terms and the coefficients satisfy the linear equation $\sum_{k=0}^{d-1} c_k=2$ and the $\frac{d}{2}$ quadratic equations $\sum_{i=0}^{d-1} c_i c_{i-2k}=2\delta(k)$ for $1\leq k\leq \frac{d-1}{2}$, then for $d>2$ there are $\frac{d}{2}-1$ coefficients that can be used to design the wavelet system to achieve desired properties.

11.6 Derivation of the Wavelets from the Scaling Function

In a wavelet system one develops a mother wavelet as a linear combination of integer shifts of a scaled version of the scale function $\phi(x)$. Let the mother wavelet $\psi(x)$ be given by $\psi(x)=\sum\limits_{k=0}^{d-1} b_k\phi(2x-k)$. One wants integer shifts of the mother wavelet $\psi(x-k)$ to be orthogonal and also for integer shifts of the mother wavelet to be orthogonal to the scaling function $\phi(x)$. These conditions place restrictions on the coefficients b_k which are the subject matter of the next two lemmas.

Lemma 11.4 *(Orthogonality of $\psi(x)$ and $\psi(x-k)$) Let $\psi(x)=\sum\limits_{k=0}^{d-1} b_k\phi(2x-k)$. If $\psi(x)$ and $\psi(x-k)$ are orthogonal for $k\neq 0$ and $\psi(x)$ has been normalized so that $\int_{-\infty}^{\infty}\psi(x)\psi(x-k)dx=\delta(k)$, then*

$$\sum_{i=0}^{d-1}(-1)^k b_i b_{i-2k}=2\delta(k).$$

Proof: Analogous to Lemma 11.2. ∎

Lemma 11.5 *(Orthogonality of $\phi(x)$ and $\psi(x-k)$) Let $\phi(x) = \sum\limits_{k=0}^{d-1} c_k \phi(2x-k)$ and $\psi(x) = \sum\limits_{k=0}^{d-1} b_k \phi(2x-k)$. If $\int\limits_{x=-\infty}^{\infty} \phi(x)\phi(x-k)dx = \delta(k)$ and $\int\limits_{x=-\infty}^{\infty} \phi(x)\psi(x-k)dx = 0$ for all k, then $\sum\limits_{i=0}^{d-1} c_i b_{i-2k} = 0$ for all k.*

Proof:

$$\int_{x=-\infty}^{\infty} \phi(x)\psi(x-k)dx = \int_{x=-\infty}^{\infty} \sum_{i=0}^{d-1} c_i \phi(2x-i) \sum_{j=1}^{d-1} b_j \phi(2x-2k-j)dx = 0.$$

Interchanging the order of integration and summation

$$\sum_{i=0}^{d-1} \sum_{j=0}^{d-1} c_i b_j \int_{x=-\infty}^{\infty} \phi(2x-i)\phi(2x-2k-j)dx = 0$$

Substituting $y = 2x - i$ yields

$$\frac{1}{2} \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} c_i b_j \int_{y=-\infty}^{\infty} \phi(y)\phi(y-2k-j+i)dy = 0$$

Thus,

$$\sum_{i=0}^{d-1} \sum_{j=0}^{d-1} c_i b_j \delta(2k+j-i) = 0$$

Summing over j gives

$$\sum_{i=0}^{d-1} c_i b_{i-2k} = 0$$

■

Lemma 11.5 gave a condition on the coefficients in the equations for $\phi(x)$ and $\psi(x)$ if integer shifts of the mother wavelet are to be orthogonal to the scale function. In addition, for integer shifts of the mother wavelet to be orthogonal to the scale function requires that $b_k = (-1)^k c_{d-1-k}$.

Lemma 11.6 *Let the scale function $\phi(x)$ equal $\sum\limits_{k=0}^{d-1} c_k \phi(2x-k)$ and let the wavelet function $\psi(x)$ equal $\sum\limits_{k=0}^{d-1} b_k \phi(2x-k)$. If the scale functions are orthogonal*

$$\int_{-\infty}^{\infty} \phi(x)\phi(x-k)dx = \delta(k)$$

and the wavelet functions are orthogonal with the scale function

$$\int_{x=-\infty}^{\infty} \phi(x)\psi(x-k)dx = 0$$

for all k, then $b_k = (-1)^k c_{d-1-k}$.

Proof: By Lemma 11.5, $\sum\limits_{j=0}^{d-1} c_j b_{j-2k} = 0$ for all k. Separating $\sum\limits_{j=0}^{d-1} c_j b_{j-2k} = 0$ into odd and even indices gives

$$\sum_{j=0}^{\frac{d}{2}-1} c_{2j} b_{2j-2k} + \sum_{j=0}^{\frac{d}{2}-1} c_{2j+1} b_{2j+1-2k} = 0 \tag{11.1}$$

for all k.

$$\begin{aligned} c_0b_0 + c_2b_2 + c_4b_4 + \cdots + c_1b_1 + c_3b_3 + c_5b_5 + \cdots &= 0 \qquad k=0 \\ c_2b_0 + c_4b_2 + \cdots \qquad + c_3b_1 + c_5b_3 + \cdots &= 0 \qquad k=1 \\ c_4b_0 + \cdots \qquad + c_5b_1 + \cdots &= 0 \qquad k=2 \end{aligned}$$

By Lemmas 11.2 and 11.4, $\sum\limits_{j=0}^{d-1} c_j c_{j-2k} = 2\delta(k)$ and $\sum\limits_{j=0}^{d-1} b_j b_{j-2k} = 2\delta(k)$ and for all k. Separating odd and even terms,

$$\sum_{j=0}^{\frac{d}{2}-1} c_{2j} c_{2j-2k} + \sum_{j=0}^{\frac{d}{2}-1} c_{2j+1} c_{2j+1-2k} = 2\delta(k) \tag{11.2}$$

and

$$\sum_{j=0}^{\frac{d}{2}-1} b_{2j} b_{2j-2k} + \sum_{j=0}^{\frac{d}{2}-1} (-1)^j b_{2j+1} b_{2j+1-2k} = 2\delta(k) \tag{11.3}$$

for all k.

$$\begin{aligned} c_0c_0 + c_2c_2 + c_4c_4 + \cdots + c_1c_1 + c_3c_3 + c_5c_5 + \cdots &= 2 \qquad k=0 \\ c_2c_0 + c_4c_2 + \cdots \qquad + c_3c_1 + c_5c_3 + \cdots &= 0 \qquad k=1 \\ c_4c_0 + \cdots \qquad + c_5c_1 + \cdots &= 0 \qquad k=2 \end{aligned}$$

$$\begin{aligned} b_0b_0 + b_2b_2 + b_4b_4 + \cdots + b_1b_1 - b_3b_3 + b_5b_5 - \cdots &= 2 \qquad k=0 \\ b_2b_0 + b_4b_2 + \cdots \qquad - b_3b_1 + b_5b_3 - \cdots &= 0 \qquad k=1 \\ b_4b_0 + \cdots \qquad + b_5b_1 - \cdots &= 0 \qquad k=2 \end{aligned}$$

Let $C_e = (c_0, c_2, \ldots, c_{d-2})$, $C_o = (c_1, c_3, \ldots, c_{d-1})$, $B_e = (b_0, b_2, \ldots, b_{d-2})$, and $B_o = (b_1, b_3, \ldots, b_{d-1})$. Equations 12.1, 12.2, and 11.3 can be expressed as convolutions[46] of these sequences. Equation 12.1 is $C_e * B_e^R + C_o * B_o^R = 0$, 12.2 is $C_e * C_e^R + C_o * C_o^R = \delta(k)$, and 11.3 is $B_e * B_e^R + B_o * B_o^R = \delta(k)$, where the superscript R stands for reversal of the sequence. These equations can be written in matrix format as

$$\begin{pmatrix} C_e & C_o \\ B_e & B_o \end{pmatrix} * \begin{pmatrix} C_e^R & B_e^R \\ C_o^R & B_o^R \end{pmatrix} = \begin{pmatrix} 2\delta & 0 \\ 0 & 2\delta \end{pmatrix}$$

Taking the Fourier or z-transform yields

$$\begin{pmatrix} F(C_e) & F(C_o) \\ F(B_e) & F(B_o) \end{pmatrix} \begin{pmatrix} F(C_e^R) & F(B_e^R) \\ F(C_o^R) & F(B_o^R) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

where F denotes the transform. Taking the determinant yields

$$\Big(F(C_e)F(B_o) - F(B_e)F(C_o)\Big)\Big(F(C_e)F(B_o) - F(C_o)F(B_e)\Big) = 4$$

Thus $F(C_e)F(B_o) - F(C_o)F(B_e) = 2$ and the inverse transform yields

$$C_e * B_o - C_o * B_e = 2\delta(k).$$

Convolution by C_e^R yields

$$C_e^R * C_e * B_o - C_e^R * B_e * C_o = C_e^R * 2\delta(k)$$

Now $\sum\limits_{j=0}^{d-1} c_j b_{j-2k} = 0$ so $-C_e^R * B_e = C_o^R * B_o$. Thus

$$\begin{aligned} C_e^R * C_e * B_o + C_o^R * B_o * C_o &= 2C_e^R * \delta(k) \\ (C_e^R * C_e + C_o^R * C_o) * B_o &= 2C_e^R * \delta(k) \\ 2\delta(k) * B_o &= 2C_e^R * \delta(k) \\ C_e &= B_o^R \end{aligned}$$

Thus, $c_i = 2b_{d-1-i}$ for even i. By a similar argument, convolution by C_0^R yields

$$C_0^R * C_e * B_0 - C_0^R * C_0 * B_e = 2C_0^R \delta(k)$$

Since $C_)^R * B_0 = -C_0^R * B_e$

$$\begin{aligned} -C_e^R * C_e^R * B_e - C_0^R * C_0 * B_e &= 2C_0^R \delta(k) \\ -(C_e * C_e^R + C_0^R * C_0) * B_e &= 2C_0^R \delta(k) \\ -2\delta(k) B_e &= 2C_0^R \delta(k) \\ -B_e &= C_0^R \end{aligned}$$

Thus, $c_i = -2b_{d-1-i}$ for all odd i and hence $c_i = (-1)^i 2b_{d-1-i}$ for all i. ∎

[46]The convolution of $(a_0, a_1, \ldots, a_{d-1})$ and $(b_0, b_1, \ldots, b_{d-1})$ denoted $(a_0, a_1, \ldots, a_{d-1}) * (b_0, b_1, \ldots, b_{d-1})$ is the sequence $(a_0 b_{d-1}, a_0 b_{d-2} + a_1 b_{d-1}, a_0 b_{d-3} + a_1 b_{d-2} + a_3 b_{d-1} \ldots, a_{d-1} b_0)$.

11.7 Sufficient Conditions for the Wavelets to be Orthogonal

Section 11.6 gave necessary conditions on the b_k and c_k in the definitions of the scale function and wavelets for certain orthogonality properties. In this section we show that these conditions are also sufficient for certain orthogonality conditions. One would like a wavelet system to satisfy certain conditions.

1. Wavelets, $\psi_j(2^j x - k)$, at all scales and shifts to be orthogonal to the scale function $\phi(x)$.
2. All wavelets to be orthogonal. That is
$$\int_{-\infty}^{\infty} \psi_j(2^j x - k)\psi_l(2^l x - m)dx = \delta(j-l)\delta(k-m)$$
3. $\phi(x)$ and ψ_{jk}, $j \leq l$ and all k, to span V_l, the space spanned by $\phi(2^l x - k)$ for all k.

These items are proved in the following lemmas. The first lemma gives sufficient conditions on the wavelet coefficients b_k in the definition
$$\psi(x) = \sum_k b_k \psi(2x - k)$$
for the mother wavelet so that the wavelets will be orthogonal to the scale function. The lemma shows that if the wavelet coefficients equal the scale coefficients in reverse order with alternating negative signs, then the wavelets will be orthogonal to the scale function.

Lemma 11.7 *If $b_k = (-1)^k c_{d-1-k}$, then $\int_{-\infty}^{\infty} \phi(x)\psi(2^j x - l)dx = 0$ for all j and l.*

Proof: Assume that $b_k = (-1)^k c_{d-1-k}$. We first show that $\phi(x)$ and $\psi(x-k)$ are orthogonal for all values of k. Then we modify the proof to show that $\phi(x)$ and $\psi(2^j x - k)$ are orthogonal for all j and k.

Assume $b_k = (-1)^k c_{d-1-k}$. Then

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(x)\psi(x-k) &= \int_{-\infty}^{\infty} \sum_{i=0}^{d-1} c_i \phi(2x-i) \sum_{j=0}^{d-1} b_j \phi(2x-2k-j)dx \\
&= \sum_{i=0}^{d-1}\sum_{j=0}^{d-1} c_i(-1)^j c_{d-1-j} \int_{-\infty}^{\infty} \phi(2x-i)\phi(2x-2k-j)dx \\
&= \sum_{i=0}^{d-1}\sum_{j=0}^{d-1} (-1)^j c_i c_{d-1-j}\delta(i-2k-j) \\
&= \sum_{j=0}^{d-1} (-1)^j c_{2k+j} c_{d-1-j} \\
&= c_{2k}c_{d-1} - c_{2k+1}c_{d-2} + \cdots + c_{d-2}c_{2k-1} - c_{d-1}c_{2k} \\
&= 0
\end{aligned}$$

The last step requires that d be even which we have assumed for all scale functions.

For the case where the wavelet is $\psi(2^j - l)$, first express $\phi(x)$ as a linear combination of $\phi(2^{j-1}x - n)$. Now for each these terms

$$\int_{-\infty}^{\infty} \phi(2^{j-1}x - m)\psi(2^j x - k)dx = 0$$

To see this, substitute $y = 2^{j-1}x$. Then

$$\int_{-\infty}^{\infty} \phi(2^j x - m)\psi(2^j x - k)dx = \frac{1}{2^{j-1}} \int_{-\infty}^{\infty} \phi(y - m)\psi(2y - k)dy$$

which by the previous argument is zero. ∎

The next lemma gives conditions on the coefficients b_k that are sufficient for the wavelets to be orthogonal.

Lemma 11.8 *If* $b_k = (-1)^k c_{d-1-k}$, *then*

$$\int_{-\infty}^{\infty} \frac{1}{2^j}\psi_j(2^j x - k)\frac{1}{2^k}\psi_l(2^l x - m)dx = \delta(j - l)\delta(k - m).$$

Proof: The first level wavelets are orthogonal.

$$\begin{aligned}
\int_{-\infty}^{\infty} \psi(x)\psi(x-k)dx &= \int_{-\infty}^{\infty} \sum_{i=0}^{d-1} b_i\phi(2x - i) \sum_{j=0}^{d-1} b_j\phi(2x - 2k - j)dx \\
&= \sum_{i=0}^{d-1} b_i \sum_{j=0}^{d-1} b_j \int_{-\infty}^{\infty} \phi(2x - i)\phi(2x - 2k - j)dx \\
&= \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} b_i b_j \delta(i - 2k - j) \\
&= \sum_{i=0}^{d-1} bib_{i-2k} \\
&= \sum_{i=0}^{d-1} (-1)^i c_{d-1-i}(-1)^{i-2k} c_{d-1-i+2k} \\
&= \sum_{i=0}^{d-1} (-1)^{2i-2k} c_{d-1-i} c_{d-1-i+2k}
\end{aligned}$$

Substituting j for $d - 1 - i$ yields

$$\sum_{j=0}^{d-1} c_j c_{j+2k} = 2\delta(k)$$

Example of orthogonality when wavelets are of different scale.

$$\int_{-\infty}^{\infty} \psi(2x)\psi(x-k)dx = \int_{-\infty}^{\infty} \sum_{i=0}^{d-1} b_i\phi(4x-i) \sum_{j=0}^{d-1} b_j\phi(2x-2k-j)dx$$
$$= \sum_{i=0}^{d-1}\sum_{i=0}^{d-1} b_i b_j \int_{-\infty}^{\infty} \phi(4x-i)\phi(2x-2k-j)dx$$

Since $\phi(2x-2k-j) = \sum\limits_{l=0}^{d-1} c_l\phi(4x-4k-2j-l)$

$$\int_{-\infty}^{\infty} \psi(2x)\psi(x-k)dx = \sum_{i=0}^{d-1}\sum_{j=0}^{d-1}\sum_{l=0}^{d-1} b_i b_j c_l \int_{-\infty}^{\infty} \psi(4x-i)\phi(4x-4k-2j-l)dx$$
$$= \sum_{i=0}^{d-1}\sum_{j=0}^{d-1}\sum_{l=0}^{d-1} b_i b_j c_l \delta(i-4k-2j-l)$$
$$= \sum_{i=0}^{d-1}\sum_{j=0}^{d-1} b_i b_j c_{i-4k-2j}$$

Since $\sum\limits_{j=0}^{d-1} c_j b_{j-2k} = 0$, $\sum\limits_{i=0}^{d-1} b_i c_{i-4k-2j} = \delta(j-2k)$ Thus

$$\int_{-\infty}^{\infty} \psi(2x)\psi(x-k)dx = \sum_{j=0}^{d-1} b_j\delta(j-2k) = 0.$$

Orthogonality of scale function with wavelet of different scale.

$$\int_{-\infty}^{\infty} \phi(x)\psi(2x-k)dx = \int_{-\infty}^{\infty} \sum_{j=0}^{d-1} c_j\phi(2x-j)\psi(2x-k)dx$$
$$= \sum_{j=0}^{d-1} c_j \int_{-\infty}^{\infty} \phi(2x-j)\psi(2x-k)dx$$
$$= \frac{1}{2}\sum_{j=0}^{d-1} c_j \int_{-\infty}^{\infty} \phi(y-j)\psi(y-k)dy$$
$$= 0$$

If ψ was of scale 2^j, ϕ would be expanded as a linear combination of ϕ of scale 2^j all of which would be orthogonal to ψ. ∎

11.8 Expressing a Function in Terms of Wavelets

Given a wavelet system with scale function ϕ and mother wavelet ψ we wish to express a function $f(x)$ in terms of an orthonormal basis of the wavelet system. First we will express $f(x)$ in terms of scale functions $\phi_{jk}(x) = \phi(2^j x - k)$. To do this we will build a tree similar to that in Figure 11.2 for the Haar system, except that computing the coefficients will be much more complex. Recall that the coefficients at a level in the tree are the coefficients to represent $f(x)$ using scale functions with the precision of the level.

Let $f(x) = \sum_{k=0}^{\infty} a_{jk}\phi_j(x-k)$ where the a_{jk} are the coefficients in the expansion of $f(x)$ using level j scale functions. Since the $\phi_j(x-k)$ are orthogonal

$$a_{jk} = \int_{x=-\infty}^{\infty} f(x)\phi_j(x-k)dx.$$

Expanding ϕ_j in terms of ϕ_{j+1} yields

$$\begin{aligned}
a_{jk} &= \int_{x=-\infty}^{\infty} f(x)\sum_{m=0}^{d-1} c_m\phi_{j+1}(2x-2k-m)dx \\
&= \sum_{m=0}^{d-1} c_m \int_{x=-\infty}^{\infty} f(x)\phi_{j+1}(2x-2k-m)dx \\
&= \sum_{m=0}^{d-1} c_m a_{j+1,2k+m}
\end{aligned}$$

Let $n = 2k+m$. Now $m = n-2k$. Then

$$a_{jk} = \sum_{n=2k}^{d-1} c_{n-2k} a_{j+1,n} \tag{11.4}$$

In construction the tree similar to that in Figure 11.2, the values at the leaves are the values of the function sampled in the intervals of size 2^{-j}. Equation 11.4 is used to compute values as one moves up the tree. The coefficients in the tree could be used if we wanted to represent $f(x)$ using scale functions. However, we want to represent $f(x)$ using one scale function whose scale is the support of $f(x)$ along with wavelets which gives us an orthogonal set of basis functions. To do this we need to calculate the coefficients for the wavelets. The value at the root of the tree is the coefficient for the scale function. We then move down the tree calculating the coefficients for the wavelets.

Finish by calculating wavelet coefficients
maybe add material on jpeg

Example: Add example using D_4.
Maybe example using sinc

11.9 Designing a Wavelet System

In designing a wavelet system there are a number of parameters in the dilation equation. If one uses d terms in the dilation equation, one degree of freedom can be used to satisfy

$$\sum_{i=0}^{d-1} c_i = 2$$

which insures the existence of a solution with a nonzero mean. Another $\frac{d}{2}$ degrees of freedom are used to satisfy

$$\sum_{i=0}^{d-1} c_i c_{i-2k} = \delta(k)$$

which insures the orthogonal properties. The remaining $\frac{d}{2} - 1$ degrees of freedom can be used to obtain some desirable properties such as smoothness. Smoothness appears to be related to vanishing moments of the scaling function. Material on the design of systems is beyond the scope of this book and can be found in the literature.

11.10 Applications

Wavelets are widely used in data compression for images and speech, as well as in computer vision for representing images. Unlike the sines and cosines of the Fourier transform, wavelets have spatial locality in addition to frequency information, which can be useful for better understanding the contents of an image and for relating pieces of different images to each other. Wavelets are also being used in power line communication protocols that send data over highly noisy channels.

11.11 Bibliographic Notes

In 1909 Alfred Haar presented an orthonormal basis for functions with finite support. Ingrid Daubechies