Mathematics III Assignment 1

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August 30, 2018

1. X is a random variable with PDF given by

$$f(n) = \begin{cases} cx^2 & \text{if } x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Constant c

   Summation of PDF over domain adds up to 1.
   Or,

$$\int_{-\infty}^{\infty} \mathrm{cx}^2 \, \mathrm{d}x = 1$$

   Since the function returns 0 everywhere except at $[-1, 1]$, we just calculate

$$\int_{-1}^{1} \mathrm{cx}^2 \, \mathrm{d}x = 1$$

$$c \times \left. \frac{x^3}{3} \right|_{-1}^{1} = 1$$

$$c \times \frac{2}{3} = 1$$

$$c = 1.5 \text{ (Answer)}$$

(b) $E\left[X\right]$ and $Var\left(X\right)$ $E[X]$ is

$$\begin{aligned}\int_{-1}^{1} \mathrm{xcx}^2 \, \mathrm{d}x &= 1.5 \times \int_{-1}^{1} \mathrm{x}^3 \, \mathrm{d}x \\ &= 1.5t \times \left.\frac{x^4}{4}\right|_{-1}^{1} \\ &= 0\end{aligned}$$

Now, $Var(X) = E[x^2] - (E[X])^2$, or

$$\begin{aligned}Var(X) &= 1.5 \times \int_{-1}^{1} \mathrm{x}^4 \, \mathrm{d}x - 0 \\ &= 1.5 \times \left.\frac{x^5}{5}\right|_{-1}^{1} \\ &= 1.5 \times 0.4 \\ &= 0.6\end{aligned}$$

(c) $P\left(X \geq \frac{1}{2}\right)$

Since the function given is a PDF, to get the $P(X \geq \frac{1}{2})$, all we need to do is integrate $f(x)$ from $\frac{1}{2}$ to 1
Or,

$$\begin{aligned}P\left(X \geq \frac{1}{2}\right) &= \int_{\frac{1}{2}}^{1} \mathrm{cx}^2 \, \mathrm{d}x \\ &= 1.5 \times \left.\frac{x^3}{3}\right|_{\frac{1}{2}}^{1} \\ &= 1.5 \times \frac{7}{24} \\ &= 0.4375\end{aligned}$$

2. Given, the CDF is:

$$F(x) = \frac{x^3 + k}{40} \quad x = 1, 2, 3$$

(a) Value of k

Since $F(x)$ is a CDF, value of $F(3) = 1$
or

$$\frac{27+k}{40} = 1$$
$$k = 13 \quad \text{(Q.E.D)}$$

(b) Find the probability distribution of X

This can be obtained by simple subtraction, answer is

$$\begin{aligned}
P(X=1) &= \frac{1+13}{40} \\
&= \frac{7}{20} \\
P(X=2) &= \frac{21-14}{40} \\
&= \frac{7}{40} \\
P(X=3) &= \frac{40-21}{40} \\
&= \frac{19}{40}
\end{aligned}$$

(c) Given $Var(X) = \frac{259}{320}$, calculate $Var(4X-5)$

$$\begin{aligned}
{\sigma_{ax+b}}^2 &= a^2 \times {\sigma_x}^2 \\
{\sigma_{4x+5}}^2 &= 16 \times \frac{259}{320} \\
&= \frac{259}{20} \\
&= 12.95
\end{aligned}$$

3.

$$
\begin{aligned}
P\,(\text{First 6 in 2nd throw} \mid \text{First 6 on even throw}) &= \frac{P\,(\text{2nd throw}) \cap P\,(\text{even throw})}{P\,(\text{even throw})} \\
&= \frac{\frac{5}{6} \times \frac{1}{6}}{\frac{1}{6} \times \left[\sum \frac{5}{6} + \left(\frac{5}{6}\right)^3 + \cdots\right]} \\
&= \frac{\frac{5}{36}}{\frac{1}{6} \times \frac{30}{11}} \\
&= \frac{\frac{5}{6}}{\frac{30}{11}} \\
&= \frac{5}{6} \times \frac{11}{30} \\
&= \frac{11}{36}\,\text{(Answer)}
\end{aligned}
$$

4.

5. (a) Sum of a PDF over its given range is 1
Given function:

$$
f(n) = \begin{cases} cx^2 & \text{if } 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}
$$

$$
\begin{aligned}
\int_0^3 \mathrm{cx}^2 \, \mathrm{d}x &= 1 \\
c \times \left.\frac{x^3}{3}\right|_0^3 &= 1 \\
c \times 9 &= 1 \\
c &= \frac{1}{9}
\end{aligned}
$$

(b) $P(1 < X < 2)$

$$
\begin{aligned}
P(1 < X < 2) &= \frac{1}{9} \times \int_1^2 \mathrm{x}^2 \, \mathrm{d}x \\
&= \frac{1}{9} \times \left.\frac{x^3}{3}\right|_1^2 \\
&= \frac{1}{9} \times \left[\frac{8}{3} - \frac{1}{3}\right] \\
&= \frac{7}{27}
\end{aligned}
$$

6. (a)

$$\begin{aligned} c \times \int_{-\infty}^{\infty} \frac{1}{\mathrm{x}^2+1}\,\mathrm{d}x &= 1 \\ c \times tan^{-1}(x)\Big|_{-\infty}^{\infty} &= 1 \\ c \times \Pi &= 1 \\ c &= \frac{1}{\Pi} \end{aligned}$$

(b)

$$x \in \left(-1, -\sqrt{\frac{1}{3}}\right) \cup \left(\sqrt{\frac{1}{3}}, 1\right)$$

$$\begin{aligned} P\left(\frac{1}{3} < x^2 < 1\right) &= \frac{1}{\Pi} \times \left[\int_{-1}^{-\sqrt{\frac{1}{3}}} \frac{1}{1+\mathrm{x}^2}\,\mathrm{d}x + \int_{\sqrt{\frac{1}{3}}}^{1} \frac{1}{1+\mathrm{x}^2}\,\mathrm{d}x\right] \\ &= \frac{1}{\Pi} \times \left[tan^{-1}(x)\Big|_{-1}^{-\sqrt{\frac{1}{3}}} + tan^{-1}(x)\Big|_{\sqrt{\frac{1}{3}}}^{1}\right] \\ &= \frac{1}{\Pi} \times \left[-\frac{\Pi}{6} + \frac{\Pi}{4} + \frac{\Pi}{4} - \frac{\Pi}{6}\right] \\ &= \frac{1}{\Pi} \times \frac{\Pi}{6} \\ &= \frac{1}{6} \end{aligned}$$

7. (a) For values of $x > 0$

$$\begin{aligned} \int_0^x \mathrm{f(x)}\,\mathrm{d}x = F(x), \quad F(x) = 1 - e^{-2x} \\ f(x) = \frac{dF(x)}{dx} \\ f(x) = 2e^{-2x} \end{aligned}$$

for values of $x < 0$, $f(x) = 0$
Answer:

$$\text{PDF } f(x) = \begin{cases} 2e^{-2x} & \text{if x} > 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) For $P(X > 2)$, we do $F(\infty) - F(2)$

$$\begin{aligned} F(\infty) - F(2) &= 1 - \left(1 - e^{-4}\right) \\ &= e^{-4} \quad \text{(Ans)} \end{aligned}$$

(c) $P(-3 < X \leq 4) = P(X \leq 4)$

$$\begin{aligned} &= F(4) \\ &= 1 - e^{-8} \quad \text{(Ans)} \end{aligned}$$

8. (a)

   (b)

9.

10.

11.

12.

13.

14.

15.

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17.