Mathematics III Assignment 1

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1. X is a random variable with PDF given by

$$f(n) = \begin{cases} cx^2 & \text{if } x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Constant c

   Summation of PDF over domain adds up to 1.
   Or,

$$\int_{-\infty}^{\infty} \mathrm{cx}^2 \,\mathrm{d}x = 1$$

   Since the function returns 0 everywhere except at $[-1, 1]$, we just calculate

$$\int_{-1}^{1} \mathrm{cx}^2 \,\mathrm{d}x = 1$$

$$c \times \left.\frac{x^3}{3}\right|_{-1}^{1} = 1$$

$$c \times \frac{2}{3} = 1$$

$$c = 1.5 \text{ (Answer)}$$

(b) $E[X]$ and $Var(X)$ $E[X]$ is

$$\int_{-1}^{1} \mathrm{xcx}^2 \, \mathrm{d}x = 1.5 \times \int_{-1}^{1} \mathrm{x}^3 \, \mathrm{d}x$$
$$= 1.5t \times \left.\frac{x^4}{4}\right|_{-1}^{1}$$
$$= 0$$

Now, $Var(X) = E[x^2] - (E[X])^2$, or

$$Var(X) = 1.5 \times \int_{-1}^{1} \mathrm{x}^4 \, \mathrm{d}x - 0$$
$$= 1.5 \times \left.\frac{x^5}{5}\right|_{-1}^{1}$$
$$= 1.5 \times 0.4$$
$$= 0.6$$

(c) $P\left(X \geq \frac{1}{2}\right)$

Since the function given is a PDF, to get the $P(X \geq \frac{1}{2})$, all we need to do is integrate $f(x)$ from $\frac{1}{2}$ to 1
Or,

$$P\left(X \geq \frac{1}{2}\right) = \int_{\frac{1}{2}}^{1} \mathrm{cx}^2 \, \mathrm{d}x$$
$$= 1.5 \times \left.\frac{x^3}{3}\right|_{\frac{1}{2}}^{1}$$
$$= 1.5 \times \frac{7}{24}$$
$$= 0.4375$$

2. Given, the CDF is:

$$F(x) = \frac{x^3 + k}{40} \quad x = 1, 2, 3$$

(a) Value of k

Since $F(x)$ is a CDF, value of $F(3) = 1$
or

$$\frac{27+k}{40} = 1$$
$$k = 13 \quad \text{(Q.E.D)}$$

(b) Find the probability distribution of X

This can be obtained by simple subtraction, answer is

$$\begin{aligned}
P(X=1) &= \frac{1+13}{40} \\
&= \frac{7}{20} \\
P(X=2) &= \frac{21-14}{40} \\
&= \frac{7}{40} \\
P(X=3) &= \frac{40-21}{40} \\
&= \frac{19}{40}
\end{aligned}$$

(c) Given $Var(X) = \frac{259}{320}$, calculate $Var(4X-5)$

$$\begin{aligned}
{\sigma_{ax+b}}^2 &= a^2 \times {\sigma_x}^2 \\
{\sigma_{4x+5}}^2 &= 16 \times \frac{259}{320} \\
&= \frac{259}{20} \\
&= 12.95
\end{aligned}$$

3.

$$\begin{aligned}
P\,(\text{First 6 in 2nd throw} \mid \text{First 6 on even throw}) &= \frac{P\,(\text{2nd throw}) \cap P\,(\text{even throw})}{P\,(\text{even throw})} \\
&= \frac{\frac{5}{6} \times \frac{1}{6}}{\frac{1}{6} \times \left[\sum \frac{5}{6} + \left(\frac{5}{6}\right)^3 + \cdots\right]} \\
&= \frac{\frac{5}{36}}{\frac{1}{6} \times \frac{30}{11}} \\
&= \frac{\frac{5}{6}}{\frac{30}{11}} \\
&= \frac{5}{6} \times \frac{11}{30} \\
&= \frac{11}{36} \text{ (Answer)}
\end{aligned}$$

4. By Bayes' Theorerm,

$$\begin{aligned}
P(0\text{ sent}|0\text{ received}) &= \frac{P(0\text{ received}|0\text{ sent}) \times P(0\text{ sent})}{P(0\text{ received})} \\
&= \frac{\frac{4}{5} \times \frac{2}{3}}{P(0\text{ received}|0\text{ sent}) \times P(0\text{ sent}) + P(0\text{ received}|1\text{ sent}) \times P(1\text{ sent})} \\
&= \frac{\frac{8}{15}}{\frac{8}{15} \times \frac{1}{18}} \\
&= \frac{\frac{8}{15}}{\frac{53}{90}} \\
&= \frac{48}{53} \quad \text{(Ans)}
\end{aligned}$$

5. (a) Sum of a PDF over its given range is 1
Given function:

$$f(n) = \begin{cases} cx^2 & \text{if } 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^3 \mathrm{cx}^2 \, dx = 1$$
$$c \times \left. \frac{x^3}{3} \right|_0^3 = 1$$
$$c \times 9 = 1$$
$$c = \frac{1}{9}$$

(b) $P(1 < X < 2)$

$$P(1 < X < 2) = \frac{1}{9} \times \int_1^2 \mathrm{x}^2 \, dx$$
$$= \frac{1}{9} \times \left. \frac{x^3}{3} \right|_1^2$$
$$= \frac{1}{9} \times \left[\frac{8}{3} - \frac{1}{3} \right]$$
$$= \frac{7}{27}$$

6. (a)

$$c \times \int_{-\infty}^{\infty} \frac{1}{\mathrm{x}^2 + 1} \, dx = 1$$
$$c \times tan^{-1}(x) \big|_{-\infty}^{\infty} = 1$$
$$c \times \Pi = 1$$
$$c = \frac{1}{\Pi}$$

(b)

$$x \in \left(-1, -\sqrt{\frac{1}{3}}\right) \cup \left(\sqrt{\frac{1}{3}}, 1\right)$$

$$\begin{aligned}
P\left(\frac{1}{3} < x^2 < 1\right) &= \frac{1}{\Pi} \times \left[\int_{-1}^{-\sqrt{\frac{1}{3}}} \frac{1}{1+\mathrm{x}^2}\,\mathrm{d}x + \int_{\sqrt{\frac{1}{3}}}^{1} \frac{1}{1+\mathrm{x}^2}\,\mathrm{d}x\right] \\
&= \frac{1}{\Pi} \times \left[tan^{-1}(x)\Big|_{-1}^{-\sqrt{\frac{1}{3}}} + tan^{-1}(x)\Big|_{\sqrt{\frac{1}{3}}}^{1}\right] \\
&= \frac{1}{\Pi} \times \left[-\frac{\Pi}{6} + \frac{\Pi}{4} + \frac{\Pi}{4} - \frac{\Pi}{6}\right] \\
&= \frac{1}{\Pi} \times \frac{\Pi}{6} \\
&= \frac{1}{6}
\end{aligned}$$

7. (a) For values of $x > 0$

$$\int_0^x \mathrm{f(x)}\,\mathrm{d}x = F(x), \quad F(x) = 1 - e^{-2x}$$

$$f(x) = \frac{dF(x)}{dx}$$

$$f(x) = 2e^{-2x}$$

for values of $x < 0$, $f(x) = 0$
Answer:

$$\text{PDF } f(x) = \begin{cases} 2e^{-2x} & \text{if x} > 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) For $P(X > 2)$, we do $F(\infty) - F(2)$

$$\begin{aligned}
F(\infty) - F(2) &= 1 - \left(1 - e^{-4}\right) \\
&= e^{-4} \quad \text{(Ans)}
\end{aligned}$$

(c) $P(-3 < X \leq 4) = P(X \leq 4)$

$$\begin{aligned}
&= F(4) \\
&= 1 - e^{-8} \quad \text{(Ans)}
\end{aligned}$$

8. (a)

$$\begin{aligned}F1(-\infty) &= 0\\ F1(\infty) &= 1\\ \frac{d}{dx}F1(x) &\geq 0 \forall x \subset domain\\ &\Rightarrow \text{the function is a CDF}\end{aligned}$$

(b)

$$\begin{aligned}F1(-\infty) &\neq 0\\ F1(\infty) &= 1\\ &\Rightarrow \text{the function is not a CDF}\end{aligned}$$

9. B = black, O = orange, W = white
Assumption: the balls are picked out one by one, so the following combinations are all possible and distinct
possible outcomes = BB(+4), WW(-2), OO(0), BW(1), WB(1), BO(2), OB(2), WO(-1), OW(-1)

Outcome is +4:

$$\begin{aligned}\text{BB} &= \frac{4}{14} \times \frac{3}{13}\\ &= \frac{6}{91}\end{aligned}$$

Outcome is -2:

$$\begin{aligned}\text{WW} &= \frac{8}{14} \times \frac{7}{13}\\ &= \frac{28}{91}\end{aligned}$$

Outcome is 0:

$$\begin{aligned}\text{OO} &= \frac{2}{14} \times \frac{1}{13}\\ &= \frac{1}{91}\end{aligned}$$

Outcome is +1:

$$\begin{aligned}\text{BW and WB} &= 2 \times \frac{4}{14} \times \frac{8}{13} \\ &= \frac{16}{91} \\ &= \frac{32}{91}\end{aligned}$$

Outcome is +2:

$$\begin{aligned}\text{BO and OB} &= 2 \times \frac{4}{14} \times \frac{2}{13} \\ &= \frac{4}{91} \\ &= \frac{8}{91}\end{aligned}$$

Outcome is -1:

$$\begin{aligned}\text{WO and OW} &= 2 \times \frac{8}{14} \times \frac{2}{13} \\ &= \frac{8}{91} \\ &= \frac{16}{91}\end{aligned}$$

10.

$$
\begin{aligned}
P(X=0) &= \frac{\binom{5}{2}\times 3!}{5!} \\
&= \frac{1}{2}
\end{aligned}
$$

similarly,

$$
\begin{aligned}
P(X=1) &= \frac{\binom{5}{3}\times 2!}{5!} \\
&= \frac{1}{6} \\
P(X=2) &= \frac{\binom{5}{4}\times 2!\times 1!}{5!} \\
&= \frac{1}{12} \\
P(X=3) &= \frac{\binom{5}{5}\times 3!}{5!} \\
&= \frac{1}{20} \\
P(X=4) &= \frac{4!}{5!} \\
&= \frac{1}{5}
\end{aligned}
$$

11. Assume X is the number of defective items in the sample
    or, possible values of $X = 0, 1, 2, 3$

$$
\begin{aligned}
E(X) &= \sum xf(x) \\
&= f(1) + 2\times f(2) + 3\times f(3) \\
&= \frac{4\times 16\times 15}{20\times 19\times 18} + 2\times\frac{4\times 3\times 16}{20\times 19\times 18} + 3\times\frac{4\times 3\times 2}{20\times 19\times 18} \\
&= \frac{1416}{6840} \\
&= 0.207 \quad \text{(Ans)}
\end{aligned}
$$

12. For errors, at least three bits should be flipped
    Therefore, probability

$$
\begin{aligned}
&= flip(3) + flip(4) + flip(5) \\
&= \binom{5}{3} \times 0.8^2 \times 0.2^3 + \binom{5}{4} \times 0.8 \times 0.2^4 + \binom{5}{5} \times 0.2^5 \\
&= 0.0512 + 0.0064 + 0.00032 \\
&= 0.05792 \quad \text{(Ans)}
\end{aligned}
$$

13.

$$
\begin{aligned}
P(X = 1) &= \frac{\binom{5}{1} \times 9!}{10!} \\
&= \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
P(X = 2) &= \frac{\binom{5}{1} \times \binom{5}{1} \times 8!}{10!} \\
&= \frac{5}{18}
\end{aligned}
$$

$$
\begin{aligned}
P(X = 3) &= \frac{(2! \times \binom{5}{2}) \times \binom{5}{1} \times 7!}{10!} \\
&= \frac{5}{36}
\end{aligned}
$$

$$
\begin{aligned}
P(X = 4) &= \frac{(3! \times \binom{5}{3}) \times \binom{5}{1} \times 6!}{10!} \\
&= \frac{5}{84}
\end{aligned}
$$

$$P(X=5) = \frac{(4! \times \binom{5}{4}) \times \binom{5}{1} \times 5!}{10!}$$
$$= \frac{5}{252}$$

$$P(X=6) = \frac{(5! \times \binom{5}{5}) \times \binom{5}{1} \times 4!}{10!}$$
$$= \frac{1}{252}$$

$$P(X >= 7) = 0$$

14.

$$\frac{\text{Number of ways to have 3 men in each team}}{\text{All possible combinations}}$$
$$= \frac{\binom{6}{3} \times \binom{6}{3}}{\binom{12}{6}}$$
$$= \frac{400}{924}$$
$$= 0.4329$$

15. E = probability that Elvis' twin was male
A = probability that Elvis' twin was identical = $\frac{2}{19}$
B = probability that Elvis' twin was fraternal = $1 - \frac{2}{19} = \frac{17}{19}$

Now,

$$P(M|A) = 1$$
$$P(M|B) = 0.5$$

$$P(A|M) = \frac{P(M|A) \times P(A)}{P(M|A) \times P(A) + P(B) \times P(M|B)}$$
$$= \frac{1 \times \frac{2}{19}}{1 \times \frac{2}{19} + \frac{1}{2} \times \frac{17}{19}}$$
$$= \frac{4}{21} \quad \text{(Ans)}$$

16.

$$f_{xy}(x,y) = \begin{cases} 2 & \text{if } x+y \leq 1, x, y > 0 \quad \Rightarrow 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\sigma_{x,y} = E(x,y) - \mu_x \mu_y$$

$$\begin{aligned} E(x,y) &= \int_0^1 \int_0^y xyf(x,y)dydx \\ &= 2\int_0^1 x\left[\frac{y^2}{2}\right]dx \\ &= \int_0^1 x(x^2-2x+1)dx \\ &= \int_0^1 (x^3-2x^2+x)dx \\ &= \frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\bigg|_0^1 \\ &= \frac{1}{4} - \frac{2}{3} + \frac{1}{2} \\ &= \frac{1}{12} \end{aligned}$$

$$\begin{aligned} \mu_x &= \int_x \text{xg(x)dx} \\ &= \int_x x \int_y \text{f(x,y) dy dx} \\ &= \int_0^1 x \int_0^{1-x} \text{2 dy dx} \\ &= 2 \times \int_0^1 \text{x(1-x)dx} \\ &= 2 \times \frac{x^2}{2} - \frac{x^3}{3}\bigg|_0^1 \\ &= 2 \times \frac{1}{6} \\ &= \frac{1}{3} \end{aligned}$$

$$
\begin{aligned}
\mu_y &= \int_y \text{yh(y)dy} \\
&= \int_y y \int_x \text{f(x,y) dx dy} \\
&= \int_0^1 y \int_0^{1-y} 2 \text{ dx dy} \\
&= 2 \times \int_0^1 \text{y(1-y) dy} \\
&= 2 \times \frac{1}{6} \\
&= \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{xy} &= E(x,y) - \mu_x\mu_y \\
&= \frac{1}{12} - \frac{1}{3} \times \frac{1}{3} \\
&= \frac{1}{12} - \frac{1}{9} \\
&= \frac{-3}{108} \\
&= \frac{-1}{36} \quad \text{(Ans)}
\end{aligned}
$$

17. X = Probability of getting head and getting tail in a coin is equal, $= \frac{1}{2}$

    Assumption of ideal conditions:
    Coin toss is unaffected by surrounding conditions and is perfectly unbiased.
    Ans: P(X) = 1

    Assumption of non ideal conditions:
    Coin toss is affected by surrounding conditions and there is bias. So probabilities are not equal
    Ans: P(X) = 0