LINEAR ALGEBRA

Matrix Multiplication .... $\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}\begin{bmatrix} e & f & g \\ h & i & j \end{bmatrix}_{2\times 3}=\begin{bmatrix} ae+bh & af+bi & ag+bj \\ ce+dh & cf+di & cg+dj \end{bmatrix}_{2\times 3}$

For matrices A_{mxn} and B_{nxp} to obtain AB, $m\cdot p\cdot n$ number of multiplications and $m\cdot p\cdot(n-1)$ number of Additions are required to be done.

Matrix multiplication is associative, if conformability is assured i.e. A(BC) = (AB)C

Matrix multiplication is distributive w.r.t. addition of matrices i.e. A·(B+C) = A·B + A·C

The matrix multiplication is not always commutative i.e. AB ≠ BA (AB is not always equal to BA)

But, when AB = BA, then the matrices are said to commute

$AB^n = B^nA$	$(AB)^n = A^nB^n$	$(A+B)(A-B) = A^2 - B^2$

If |A| = 0, then A is a singular matrix

The equation AB = O doesn't necessarily imply that atleast one of the matrices A and B is a zero matrix i.e. product of two non-zero matrices can also be a zero matrix. Eg. $A=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

If the product of two non-zero matrices A and B is a zero matrix, then both A and B are singular matrices i.e. |A| = 0, |B| = 0.

If the product of two non-zero matrices A and B is a zero matrix, and A is a non-singular matrix (|A| ≠ 0), then B has to be a zero matrix.

$$L=\begin{bmatrix} a & 0 & 0 & 0 \\ b & e & 0 & 0 \\ c & f & g & 0 \\ d & h & i & j \end{bmatrix}, U=\begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & g & h \\ 0 & 0 & 0 & i \end{bmatrix}, D=\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & j \end{bmatrix}, S=\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix}, I=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Principle diagonal

Upper Triangular Matrix: A square matrix is said to be upper triangular if all the elements below its principle diagonal are zeros. eg. U

Lower Triangular Matrix: A square matrix is said to be lower triangular if all the elements above its principle diagonal are zeros. eg. L

Diagonal matrix: A square matrix is said to be diagonal matrix if all the elements below and above the principle diagonal are zeros. eg. D

- Product of two diagonal matrices of the same order is a diagonal matrix and follows commutative law i.e. AB = BA
- Rank of a diagonal matrix is equal to the number of non-zero elements in the principle diagonal.

Scalar matrix: It is a **diagonal** matrix with same diagonal elements. Eg. S

If A is any square matrix of order n, then $A_nS_n = S_nA_n$….i.e. Scalar matrix is commutative with any matrix of same order

Unit or Identity matrix: A **scalar** matrix with diagonal elements being 1. eg. I

Trace: It is the sum of elements of principle diagonal. Eg. tr(U) = a+e+g+i

$tr(\lambda A) = \lambda\, tr(A)$	$tr(A+B) = tr(A) + tr(B)$	$tr(AB) = tr(BA)$

Transpose: Transpose of a matrix can be obtained by interchanging the rows and columns of that matrix

If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ then $A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$

If A is matrix of order m x n then A^T will be of order n x m

$(A^T)^T = A$	$(A+B)^T = A^T + B^T$
$(kA)^T = k(A)^T$	$(AB)^T = B^T A^T$

$|A| = |A^T|$

Symmetric matrix: a **square** matrix A is symmetric if $A^T = A$

Skew-symmetric matrix: a square matrix A is symmetric if $A^T = -A$……Rank of A ≠ 1

If A is a square matrix then,

$A + A^T$	Symmetric
$A - A^T$	Skew-symmetric
$A \cdot A^T$	Symmetric

Every square matrix can be represented by the sum of a symmetric and a skew-symmetric matrix

i.e. $A = 1/2(A + A^T) + 1/2(A - A^T)$

If A and B are symmetric then AB + BA is symmetric and AB – BA is Skew-symmetric

If A is symmetric then A^n is symmetric for n = 2,3,4,5,……

If A is Skew-symmetric then A^n is Skew-symmetric when n is odd

A^n is symmetric when n is even

Hermitian matrix: A square matrix A in which $(I,j)^{th}$ element is equal to the complex conjugate of the $(j,i)^{th}$ element i.e. $a_{ij} = \overline{a}_{ji}$ for all I and j. If $\overline{A}$ is the conjugate matrix of A (i.e. a matrix formed of complex conjugates of elements of A) and if $A = (\overline{A})^T$, then A is called Hermitian matrix.

Skew-hermitian matrix: $A = -(\overline{A})^T$, $\overline{A}$ is also written as A^e

Inverse of a matrix: If A is a square matrix of order n, its inverse exists is it is non-singular i.e. |A| ≠ 0.
Inverse of A can be written as A^{-1}.

$A \cdot A^{-1}$ = Identity matrix of order n, $(AB)^{-1} = B^{-1} A^{-1}$

$$A^{-1} = \frac{(\text{adj } A)}{|A|}$$

Adj A = Transpose of the cofactor matrix, Cofactor = $(-1)^{i+j} \cdot$ Minor
If Rank of A_{nxn} = (n – 2), then Rank of Adj(A) = zero
If Rank of A_{nxn} = (n – 1), then Rank of Adj(A) = 1
If Rank of A_{nxn} = n, then Rank of Adj(A) = n

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$	$(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$	$(A^{-1})^T = (A^T)^{-1}$

- If a non-singular matrix A is symmetric, then A^{-1} is also symmetric

Orthogonal matrix: $A^T = A^{-1}$

- If A is orthogonal then A^T & A^{-1} are also orthogonal.
- If A & B are orthogonal matrices of the same order then AB is also orthogonal.

Determinants:

If two parallel lines (rows or columns) of a determinant are interchanged, then the determinant retains its numerical value but changes in sign. In general, if any line of determinant is passed over 'm' parallel lines, the **resulting determinant = $(-1)^m$.** Δ, where Δ is the initial determinant value

A determinant vanishes if two parallel lines are identical.

If each element of a line be multiplied by the same factor, then the whole determinant can be multiplied by that factor. Note: if $R_i = k R_j$ or $C_i = k C_j$ then the value of the determinant is zero

$$\begin{vmatrix} ka_1 & kb_1 \\ a_2 & b_2 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 & lc_1 \\ a_2 & b_2 & lc_2 \\ a_3 & b_3 & lc_3 \end{vmatrix} = l\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ ka_1 & kb_1 & kc_1 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = k(0) = 0$$

$$\begin{vmatrix} a_1 & b_1+c_1 & d_1 \\ a_2 & b_2+c_2 & d_2 \\ a_3 & b_3+c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} + \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}$$

If to each elements of a line be added equi-multiples of the corresponding elements of one or more parallel lines the determinant remains unaltered.

The determinant of an upper/lower /diagonal/scalar matrix is equal to the product of leading elements of the matrix.

A, B are square matrices of the same order then $|AB| = |A|\,|B| = |BA|$

If A is a non-singular matrix i.e. $|A| \neq 0$, then $|A^{-1}| = 1/|A|$

If A is a square matrix of order n, then (i) $|\text{Adj } A| = |A|^{n-1}$ (ii) $|\text{Adj }(\text{Adj } A)| = |A|^{(n-1)^2}$

Determinant of a Skew-symmetric matrix of **odd order** is zero.

If A is an orthogonal matrix then $|A| = \pm 1$

If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta^1 = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$

Where A, B,C are co-factors of a, b, c, then $\Delta^1 = \Delta^2$, which is called Reciprocal/Adjucate determinant of Δ.

$|I_n| = 1 \quad V \quad n \in Z^+$

If 'A' is a square matrix of order 'n', then $|kA| = k^n\ |A\in|$

Rank of a matrix is equal to:

- Order of the largest non vanishing minor of the matrix.
- The number of linearly independent rows or columns in the matrix
- The number of non zero rows or columns in the row echelon or column echelon form of the matrix

  - ➢ If 'r' is the rank of matrix A, then A has 'r' linearly independent vectors (here vector means either row or column of the matrix) i.e. No. of linearly independent vectors = rank.
  - ➢ If $|A| \neq 0$ ……then all the rows and columns (vectors) of A are linearly independent.
  - ➢ If $|A| = 0$ ……then all the rows and columns (vectors) of A are linearly dependent

Observations in ranks:

- If 'r' is the rank of matrix A, then there exists at least one non zero minor of order r.
- Rank of an elementary matrix is equal to the order of the matrix
- If $|A_{nxn}| \neq 0$, then

$\rho(A) = \rho(A^T)$	$\rho(A) = \rho(A^{-1})$	$\rho(A) = \rho\ (I_n)$

(i). $\rho(A + B) \leq \rho(A) + \rho(B)$ (ii). $\rho(A - B) \geq \rho(A) - \rho(B)$ (iii). $\rho(AB) \leq \min \{ \rho(A) , \rho(B) \}$

Echelon form:

Consider equations $3x_1+2x_2+ x_3 = 3$; $2x_1+x_2+x_3 = 0$; $6x_1+2x_2+4x_3 = 6$ which can be written in matrix form as

$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$ its echelon form after elementary row transformation can be obtained as $\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 6 & 0 & 0 & 12 \end{array}\right]$

This echelon form can be used to find the unknown variables of the equation by back-substituting and solving $3x_1+2x_2+x_3 = 3$; $-x_1/3+x_2/3 = -2$; $0x_3 = 12$ simultaneously

A system of **non homogeneous** equation AX = B has a solution if and only if Rank of A = Rank of [A|B]

It has	If
unique solution	Rank (A) = Rank (A|B) = number of variables
Infinitely many solutions	Rank (A) = Rank (A|B) < number of variables
No solution	Rank (A) ≠ Rank (A|B) i.e. $\rho(A) < \rho(A\|B)$

A system of **homogeneous** equation AX = O has

- A zero (trivial) solution if $|A| \neq 0$ (zero solution means all roots equal to zero i.e. x = 0, y = 0, z = 0)
- A Non-zero solution if $|A| = 0$.
- (n - r) linearly independent solutions.
- Necessarily a non-zero solution if Number of variables > Number of equations

It has	If
unique solution (zero or trivial solution)	Rank (A) = number of variables
Infinitely many solutions (non-zero or non trivial solution)	Rank (A) = Rank (A|B) < number of variables

Infinitely many solutions is the case when rank is less than number of variables. Any of those solutions can be obtained by choosing values at pleasure for the unknowns x_{r+1}, x_{r+2},……, x_n and solving the r^{th} equation for x_r then $(r-1)^{th}$ equation for x_{r-1} and so on.

Eigen values:
Let A be a square matrix of order n and λ be a scalar, then **$|A - \lambda I| = 0$** is called the **characteristic equation** of A and the roots of the characteristic equation are called **Eigen values (Characteristic roots/ Latent roots)** of A.

Eigen vectors:
Corresponding to each eigen value, there exists a non-zero solution X such that $(A - \lambda I)X = 0$. X is called the Eigen vector (Characteristic vector or latent vector) of A

Properties of Eigen values and Eigen vectors

1. Sum of the Eigen values of a matrix is equal to the trace (sum of principle diagonal elements) of the matrix.
2. The product of the engine values of a matrix is equal to the determinant of the matrix.
3. The eigen values of A^T are same as the eigen values of A.
4. If λ is an eigen value of a non-singular matrix A then
    a) 1/ λ is an eigen value of A^{-1} ($AA^{-1} = I = A^{-1}A$)
    b) |A|/ λ is an eigen value of Adj A (A·Adj A = |A|I)
5. If A is a square matrix such that |A| = 0 (i.e. singular), eigen value of A is zero.
6. If λ is an eigen value of an Orthogonal matrix A (i.e. AT = A-1) then 1/ λ is also an eigen value of A.
7. If $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are eigen values of A, then
    a) The eigen values of kA are $k\lambda_1, k\lambda_2, \ldots, k\lambda_n$ (where k is a scalar)
    b) A^m has eigen values $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ (where $m \in z^+$)
    c) A + kI has eigen values $\lambda_1+k, \lambda_2+k, \ldots, \lambda_n+k$
    d) $(A - kI)^2$ has eigen values $(\lambda_1 - k)^2, (\lambda_2 - k)^2, \ldots, (\lambda_n - k)^2$
8. The eigen values of an orthogonal matrix have absolute value '1'
9. The eigen values of a symmetric matrix are purely real.
10. The eigen values of a Skew-symmetric matrix are either purely imaginary or zero.
11. The set all characteristics of a matrix is called ***Spectrum*** of the matrix
12. Zero is an eigen value of a matrix if and only if the matrix is singular.
13. λ is an eigen value of a non-singular matrix ↔ λ ≠ 0.
14. If λ is an eigen value of matrix A, then the corresponding eigen vector X is not unique i.e. we have infinite number of eigen vector corresponding to a single eigen value.
15. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct eigen values of an n x n matrix A, then the corresponding eigen vectors $X_1, X_2, \ldots X_n$ form a linearly independent set.
16. For a matrix A of order n x n, if some eigen values are repeated, then it may/may not be possible to get 'n' linearly independent eigen vectors for A.

Determinant of n x n matrix:

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

For n = 1, $|A| = a_{11}$

For $n \geq 2$, $|A| = a_{j1}C_{j1} + a_{j2}C_{j2} + \dots\dots + a_{jn}C_{jn}$ for (j = 1,2,…., or n)

(or)

$$|A| = a_{1k}C_{1k} + a_{2k}C_{2k} + \dots\dots + a_{nk}C_{nk}$$ for (k = 1,2,…., or n)

Where $C_{jk} = (-1)^{j+k} M_{jk}$ ……..i.e. C_{jk} is the cofactor of a_{jk}

Therefore we can write,

$$|A| = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk}$$ for (j = 1,2,…., or n)

(or)

$$|A| = \sum_{j=1}^{n} (-1)^{j+k} a_{jk} M_{jk}$$ for (k = 1,2,…., or n)

Cramers rule:

For linear system of three equations with three unknowns,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$ which can be written as $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$x_1 = \frac{D_1}{D}, x_1 = \frac{D_2}{D}, x_1 = \frac{D_3}{D}$$ $D \neq 0$

Where $D_1 = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$, $D_2 = \begin{bmatrix} a_{11} & d_1 & a_{13} \\ a_{21} & d_2 & a_{23} \\ a_{31} & d_3 & a_{33} \end{bmatrix}$, $D_3 = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$