Ex : 13.1

(1) Solve $\frac{d^2x}{dt^2} + 3a\frac{dx}{dt} - 4a^2x = 0$

Sol. In symbolic form, the given diff. eq. can be written as

$$(D^2 + 3aD - 4a^2)x = 0$$

Its A.E. is

$$D^2 + 3aD - 4a^2 = 0$$
$$D^2 + 4aD - aD - 4a^2 = 0$$
$$D(D+4a) - a(D+4a) = 0$$
$$(D+4a)(D-a) = 0$$
$$\Rightarrow \quad D - a = 0 \quad \text{or} \quad D + 4a = 0$$
$$D = a \quad \text{or} \quad D = -4a$$
$$\Rightarrow \quad D = a, -4a$$

$$C.S = c_1 e^{at} + c_2 e^{-4at}$$

Q No 2 $y'' - 2y' + 10y = 0$ or $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$, $y(0) = 4$, $y'(0) = 1$

In symbolic form, the given diff. eq. can be written as

$$(D^2 - 2D + 10)y = 0$$

Its A.E. is

$$D^2 - 2D + 10 = 0$$

~~$D = \frac{2}{5}$~~

$$D = \frac{2 \pm \sqrt{4 - 4\times1\times10}}{2}$$
$$= \frac{2 \pm \sqrt{4-40}}{2}$$
$$= \frac{2 \pm \sqrt{-36}}{2}$$
$$= \frac{2 \pm \sqrt{36\times -1}}{2}$$
$$= \frac{2 \pm \sqrt{36 i^2}}{2} \qquad i^2 = -1$$
$$= \frac{2 \pm 6i}{2} = 1 \pm 3i$$

$y(x) = C.S. = e^x[c_1\cos 3x + c_2 \sin 3x]$ (1)

Given that $y(0) = 0$ (2)

Put $x = 0$ in (1), we get

$$y(0) = e^0[c_1\cos 0 + c_2 \sin 0]$$
$$= 1[c_1 + c_2 \cdot 0]$$
$$= c_1$$

But $y(0) = 0$

$\therefore c_1 = 0$

Differentiating (1), w.r.t. x, we get

$y'(x) = \frac{dy}{dx} = e^x[c_1\cos 3x + c_2 \sin 3x] + e^x[-3c_1 \sin 3x + c_2 3\cos 3x]$

Put $x = 0$, we get

$y'(0) = e^0[c_1\cos 0 + c_2 \sin 0] + e^0[-3c_1 \sin 0 + 3c_2 \cos 0]$

But $y'(0) = 1$ (given)

$1 = 1[0 \cdot 1 + 0 \ldots$

$1 = 3c_2$ $\quad [\cos 0 = 1, \sin 0 = 0]$

$c_2 = \frac{1}{3}$

Putting the value of c_1 and c_2 in (1), we get

$y(x) = C.S. = e^x[0.\cos 3x + \frac{1}{3}\sin 3x] = \frac{e^x \sin 3x}{3}$

Q.No. $4\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$ or $4y''' + 4y'' + y' = 0$

Solve it. In symbolic form, the given diff. eq. can be written as

$(4D^3 + 4D^2 + D)y = 0$

Its A.E. is

$4D^3 + 4D^2 + D = 0$

$D(4D^2 + 4D + 1) = 0$

$\Rightarrow D = 0$ or $4D^2 + 4D + 1 = 0$

$D = 0$ or $(2D+1)^2 = 0$

$D = 0$ or $(2D+1)(2D+1) = 0$

$\Rightarrow D = 0, -\frac{1}{2}, -\frac{1}{2}$

$C.S. = c_1 e^{0x} + (c_2 + c_3 x)e^{-\frac{x}{2}}$

$= c_1 + (c_2 + c_3 x)e^{-x/2}$ $\quad | e^{0x} = e^0) = 1$

Q $\frac{d^3y}{dx^3} + y = 0$

In symbolic form

$(D^3 + 1)y = 0$

Its A.E.

$D^3 + 1 = 0$ $\quad | a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$(D+1)(D^2 - D + 1) = 0$

or $D^2 - D + 1 = 0$

$D = \frac{1 \pm \sqrt{1-4}}{2}$

$= \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3i^2}}{2}$

$= \frac{1 \pm i\sqrt{3}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

So

$C.S. = c_1 e^{-x}$ $e^{x/2}$ ...

Q $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$

In symbolic form, the given diff. can be written as

$(D^3 - 3D^2 + 3D - 1)y = 0$

Its A.E.

$D^3 - 3D^2 + 3D - 1 = 0$

$(D-1)^3 = 0$

$(D-1)(D-1)(D-1) = 0$

$\Rightarrow$

$\Rightarrow D = 1, 1, 1$

$C.S. = (c_1 + c_2x + c_3x^2)e^x$

Q $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0$

In symbolic form

$(D^4 + 8D^2 + 16)y = 0$

Its A.E. is

$D^4 + 8D^2 + 16 = 0$

$(D^2+4)^2 = 0$

$(D^2+4)(D^2+4) = 0$

$\Rightarrow D^2 = -4$ or $D^2 = -4$

$\Rightarrow D^2 = 4i^2$ or $D^2 = 4i^2$

$\Rightarrow D = \pm 2i$ or $D = \pm 2i$

$C.S. = (c_1 + c_2x)\cos 2x + (c_3 + c_4)\sin 2x$

Q $\frac{d^4x}{dt^4} = m^4x$, prove that

$x = c_1\cos mt + c_2\sin mt$

$x = c_1\cos mt + c_2\sin mt + c_4\sinh mt + c_3\cosh mt$

Sol $\frac{d^4x}{dt^4} - m^4x = 0$

In symbolic form

$(D^4 - m^4)x = 0$

Its A.E. is

$D^4 - m^4 = 0$

$(D^2 - m^2)(D^2 + m^2)$

$\Rightarrow D^2 - m^2 = 0$ or $D^2 + m^2 = 0$

$D^2 = m^2$ $D = \pm m$	$D^2 = -m^2$ $= m^2i^2$ $D = \pm im$

~~C.S. = c_1e^{mt} +~~

$C.S. = c_1\cos mt + c_2\sin mt + A_3e^{mt} + Be^{-mt}$ (1)

~~$= c_1\cos mt + c_2\sin mt$~~

Now ~~given~~

$x = c_1\cos mt + c_2\sin mt + c_3\cosh mt + c_4\sinh mt$

$= c_1\cos mt + c_2\sin mt + c_3\left(\frac{e^{mt} + e^{-mt}}{2}\right) + c_4\left[\frac{e^{mt} - e^{-mt}}{2}\right]$

$= c_1\cos mt + c_2\sin mt + \left(\frac{c_3+c_4}{2}\right)e^{mt} + \left(\frac{c_3 - c_4}{2}\right)e^{-mt}$

$= c_1\cos mt + c_2\sin mt + Ae^{mt} + Be^{-mt}$

$\left[A = \frac{c_3+c_4}{2}\right.$

$B = \frac{c_3 - c_4}{2}$

Which is same as (1)

Hence it is proved.

Q $\frac{d^4y}{dx^4} + a^4 y = 0$

In symbolic form

$(D^4 + a^4)\,y = 0$

Its A.E. is

$D^4 + a^4 = 0$

$a^2 - b^2 = (a+b)(a-b)$

$D^4 + 2a^2D^2 + a^4 - 2a^2D^2 = 0$

$(D^2 + a^2)^2 - (\sqrt{2}aD)^2 = 0$

$(D^2 + a^2 - \sqrt{2}aD)(D^2 + a^2 + \sqrt{2}aD) = 0$

$\Rightarrow \quad D^2 - \sqrt{2}aD + a^2 = 0$ or $D^2 + \sqrt{2}aD + a^2 = 0$

$D = \frac{\sqrt{2}a \pm \sqrt{2a^2 - 4a^2}}{2}$

$D = \frac{\sqrt{2}a \pm \sqrt{-2a^2}}{2}$

$= \frac{\sqrt{2}a \pm \sqrt{2a^2i^2}}{2}$

$= \frac{\sqrt{2}a \pm \sqrt{2}ai}{2}$

$= \frac{\sqrt{2}(a \pm ia)}{2}$

$= \frac{a \pm ia}{\sqrt{2}} = \frac{a}{\sqrt{2}} \pm \frac{ia}{\sqrt{2}}$

$D = \frac{-\sqrt{2}\cdot a \pm \sqrt{2a^2 - 4a^2}}{2}$

$D = \frac{-\sqrt{2}a \pm \sqrt{-2a^2}}{2}$

$= \frac{-\sqrt{2}a \pm \sqrt{2a^2i^2}}{2}$

$= \frac{-\sqrt{2}a \pm \sqrt{2}ia}{2}$

$= \sqrt{2}\left[\frac{-a \pm ia}{2}\right]$

$= \frac{-a \pm ia}{\sqrt{2}}$

$= \frac{-a}{\sqrt{2}} \pm \frac{ia}{\sqrt{2}}$

$C.S = e^{\frac{a}{\sqrt{2}}x}\left[c_1 \cos\frac{ax}{\sqrt{2}} + c_2 \sin\frac{ax}{\sqrt{2}}\right) + e^{-\frac{ax}{\sqrt{2}}}\left[c_3 \cos\frac{ax}{\sqrt{2}} + c_2 \sin\frac{ax}{\sqrt{2}}\right)$

Q $\frac{d^2y}{dx^2} - 4y = \cosh(2x-1) + 3^x$

$(D^2-4)y = \frac{e^{2x-1} + e^{-(2x-1)}}{2} + 3^x$

Its A.E.

$D^2 - 4 = 0$

$D = \pm 2$

$C.F = c_1 e^{2x} + c_2 e^{-2x}$

$P.I = \frac{1}{D^2-4}\left[\frac{e^{2x-1} - e^{-2x+1}}{2} + e^{\log 3^x}\right]$

$= \frac{1}{2}\cdot\frac{1}{D^2-4} e^{2x-1} - \frac{1}{2(D^2-4)} e^{-2x+1} + \frac{e^{x\log 3}}{D^2-4}$

$\cancel{\frac{1}{2(D^2-4)}}$

$= \frac{1}{2(4-4)} e^{2x-1} - \frac{1}{2}\frac{e^{-2x+1}}{(4-4)} + \frac{e^{x\log 3}}{(\log 3)^2 - 4}$

case of failure

$C.S = C.F + P.I$

$= \frac{x}{2\cdot 2D} e^{2x-1} - \frac{1}{2}x\frac{e^{-2x+1}}{2D} + \frac{3^x}{(\log 3)^2-4}$

$= \frac{x}{2\times2\times2} e^{2x-1} - \frac{1}{2}x\frac{e^{-2x+1}}{2\times -2} + \frac{3^x}{(\log 3)^2-4}$

$= \frac{x}{8}\left[e^{2x-1} - e^{-(2x-1)}\right] + \frac{3^x}{(\log 3)^2-4}$

$= \frac{x}{4}\cosh(2x-1) + \frac{3^x}{(\log 3)^2-4}$

Q $\frac{d^2y}{dx^2} + 4y = x^2 + \cos 2x$

$(D^2+4)y = x^2 + \cos 2x$

Its A.E.

$D^2 + 4 = 0$

$D^2 = -4 = 4i^2$

$D = \pm 2i$

$C.F = c_1\cos 2x + c_2 \sin 2x$

$P.I = \frac{1}{D^2+4}(x^2 + \cos 2x)$

$= \frac{1}{D^2+4}x^2 + \frac{1}{D^2+4}\cos 2x$

$= \frac{1}{4(1+\frac{D^2}{4})}x^2 + \frac{1}{-4+4}\cos 2x$

(case of failure)

$= \cancel{\frac{1}{4}\left[\left(1+\frac{D^2}{4}\right)^{-1}\right]x^2}$

$= \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}x^2 + x\frac{1}{2D}\cos 2x$

$= \frac{1}{4}\left[1 - \frac{D^2}{4} + \cdots\right]x^2 + \frac{x\sin 2x}{4}$

$= \frac{1}{4}\left[x^2 - \frac{D^2}{4}x^2\right] + \frac{x\sin 2x}{4}$

$= \frac{1}{4}\left[x^2 - \frac{2x}{4}\right] + \frac{x\sin 2x}{4}$

9 ts A.E

$$D^3+2D^2+D=0$$

$$D(D^2+2D+1)=0$$

$$D(D+1)^2=0$$

$$D(D+1)(D+1)=0$$

$$D=0,-1,-1$$

$$C.F=C_1e^{0\cdot x}+(C_2+C_3x)e^{-x}$$

$$=C_1+(C_2+C_3x)e^{-x}$$

$$C.S.=C.F+P.I$$

$$P.I=\frac{1}{D^3+2D^2+D}\,e^{-x}+\sin 2x$$

$$=\frac{1}{D^3+2D^2+D}\,e^{-x}+\frac{1}{D^3+2D^2+D}\sin 2x$$

$$=\frac{1}{(-1)^3+2(-1)^2+(-1)}\,e^{-x}$$

$$+\frac{1}{D^2\cdot D+D^2+D}\sin 2x$$

$$=\frac{1}{\underbrace{-1+2-1}_{0}}\,e^{-x}+\frac{1}{-\cancel{4}-1+\cancel{0}}\sin 2x$$

$$=\frac{x\cdot e^{-x}}{3D^2+4D+1}-\sin 2x$$

$$=x\,\frac{e^{-x}}{\underbrace{3-4+1}_{0}}-\sin 2x$$

$$=\frac{x^2e^{-x}}{6D+4}-\sin 2x$$

$$=\frac{x^2e^{-x}}{-6+4}-\sin 2x$$

$$=-\frac{x^2}{2}e^{-x}-\sin 2x$$

Q $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=e^{2x}-\cos^2x$

$$(D^2+2D+1)y=e^{2x}-\cos^2x$$

its A.E

$$D^2+2D+1=0$$

$$(D+1)^2=0$$

$$(D+1)(D+1)=0$$

$$\Rightarrow D=-1,-1 \text{ (equal roots)}$$

$$C.F=(C_1+C_2x)e^{-x}$$

$$P.I\ \frac{1}{D^2+2D+1}\,e^{2x}-\cos^2x=\frac{1}{D^2+2D+1}e^{2x}-\frac{\cos^2x}{D^2+2D+1}$$

$$=\frac{1}{(2)^2+2\times 2+1}e^{2x}-\frac{1}{D^2+2D+1}\,\frac{(1+\cos 2x)}{2}$$

$$=\frac{e^{2x}}{9}-\frac{1}{2(D^2+2D+1)}e^{0x}-\frac{1}{2(D^2+2D+1)}\cos 2x$$

$$=\frac{e^{2x}}{9}-\frac{1}{2}-\frac{1}{2(-4+2D+1)}\cos 2x$$

Putting the value of c_1 and c_2 in (1), we get

$$y(x) = \frac{e^{-2x}}{5}(3\cos x - 2\sin x) - \left(\frac{e^{x}}{10} + \frac{1}{2}e^{-x}\right)$$

Q. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4\cos^2 x$

$(D^2+3D+2)y = \frac{4(1+\cos 2x)}{2} = 2 + 2\cos 2x$

Its A.E.

$D^2+3D+2=0$

$(D+1)(D+2)=0$

$\Rightarrow D = -1, -2$

$C.F = c_1 e^{-x} + c_2 e^{-2x}$

$P.I = \frac{1}{D^2+3D+2}(2+2\cos 2x)$

$= \frac{1}{D^2+3D+2} 2e^{0\cdot x} + \frac{1}{D^2+3D+2} 2\cos 2x$

$= \frac{2e^{0x}}{0+3\cdot 0+2} + \frac{1}{-4+3D+2} 2\cos 2x$

$= \frac{2}{2} + \frac{1}{3D+1} 2\cos 2x$

$= 1 + \frac{(3D-1)}{(3D+1)(3D-1)} 2\cos 2x$

$= 1 + \frac{3D-1}{9D^2-1} 2\cos 2x$

$= 1 + \frac{(3D-1)2\cos 2x}{-9-1}$

$= 1 - \frac{1}{10}(3D\cos 2x - 2\cos 2x)$

$= 1 + \frac{12}{10}\sin 2x + \frac{2}{10}\cos 2x$

$= 1 + \frac{5}{6}\sin 2x + \frac{1}{5}\cos 2x$

$C.S = C.F + P.I$

$= c_1 e^{-x} + c_2 e^{-2x} + 1 + \frac{5}{6}\sin 2x + \frac{1}{5}\cos 2x$

Q. $(D^2 - 4D + 3)y = \sin 3x \cos 2x$

Its A.E. is

$D^2 - 4D + 3 = 0$

$D^2 - 3D - D + 3 = 0$

$D(D-3) - (D-3) = 0$

$\Rightarrow (D-3)(D-1) = 0$

$\Rightarrow D = 1, 3$

$C.F = c_1 e^{x} + c_2 e^{3x}$

$P.I = \frac{1}{D^2-4D+3}\sin 3x\cos 2x$

$= \frac{1}{D^2-4D+3} \times \frac{1}{2}\, 2\sin 3x\cos 2x$

$= \frac{1}{2}\frac{1}{D^2-4D+3}[\sin 5x + \sin x]$

$$= \frac{1}{2}\left[\frac{1}{D^2-4D+3}\sin 5x + \frac{1}{D^2-4D+3}\sin x\right]$$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$$= \frac{1}{2}\left[\frac{1}{-25-4D+3}\sin 5x + \frac{1}{-1-4D+3}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-22\;4D}\sin 5x + \frac{1}{2-4D}\sin x\right]$$

$$= \frac{1}{2}\left[-\frac{1}{4D+22}\sin 5x + \frac{1}{2-4D}\sin x\right]$$

$$= \frac{1}{2}\left[-\frac{(4D-22)}{(4D+22)(4D-22)}\sin 5x + \frac{1\,(2+4D)}{(2-4D)(2+4D)}\sin x\right]$$

$$= \frac{1}{2}\left[-\frac{(4D-22)}{16D^2-484}\sin 5x + \frac{1\,(2+4D)}{4-16D^2}\sin x\right]$$

$$= \frac{1}{2}\left[-\frac{(4D-22)}{-16\times 25-484}\sin 5x + \frac{1\,(2+4D)}{4-16\times -1}\sin x\right]$$

$$= \frac{1}{2}\left[+\frac{1}{884}(4D\sin 5x - 22\sin 5x) + \frac{1}{20}(2\sin x + 4D\sin x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{884}\,20\cos 5x - \frac{22\sin 5x}{884} + \frac{1}{20}(2\sin x + 4\cos x)\right]$$

$$= \frac{1}{2}\left[\frac{5\cos 5x}{221} - \frac{11\sin 5x}{442} + \frac{1}{10}\sin x + \frac{1}{5}\cos x\right]$$

$C.S = C.F + P.I$

Q $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{-x} + \sin 2x$

$(D^3 + 2D^2 + D)y = e^{-x} + \sin 2x$

A.E

$D^3 + 2D^2 + D = 0$

Ex · 13.2

Solve $(D^3 - 6D^2 + 11D - 6)y = e^{-2x} + e^{-3x}$

Its A.E. is

$D^3 - 6D^2 + 11D - 6 = 0$

$(D-1)(D^2 - 5D + 6) = 0$

$\Rightarrow$ $D - 1 = 0$ or $D^2 - 5D + 6 = 0$

$D = 1$

$D^2 - 3D - 2D + 6 = 0$

$D(D-3) - 2(D-3) = 0$

$(D-3)(D-2) = 0$

$\Rightarrow D = 2, 3$

C.F $= c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$

P.I $= \frac{1}{D^3 - 6D^2 + 11D - 6}(e^{-2x} + e^{-3x})$

$= \frac{1}{D^3 - 6D^2 + 11D - 6} e^{-2x} + \frac{1}{D^3 - 6D^2 + 11D - 6} e^{-3x}$

$= \frac{1}{(-2)^3 - 6(-2)^2 + 11(-2) - 6} e^{-2x} + \frac{1}{(-3)^3 - 6(-3)^2 + 11 \times -3 - 6} e^{-3x}$

$= \frac{1}{-8 - 24 - 22 - 6} e^{-2x} + \frac{1}{-27 - 54 - 33 - 6} e^{-3x}$

$= -\frac{e^{-2x}}{+60} - \frac{e^{-3x}}{120} = -\frac{1}{60}\left[e^{-2x} + \frac{e^{-3x}}{2}\right]$

C.S $=$ C.F + P.I

$= c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} - \frac{1}{60}\left(e^{-2x} + \frac{e^{-3x}}{2}\right)$

Q No 2 $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\cosh x$. Also find y when $y = 0$, $\frac{dy}{dx} = 1$ at $x = 0$

Sol)

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\cosh x$$

In symbolic form, the given diff. eq. can be written as

$$(D^2 + 4D + 5)y = -2\cosh x$$

Its A.E.

$$D^2 + 4D + 5 = 0$$

$$\cancel{(D+4)(D+1)} = 0$$

$$\Rightarrow \cancel{D+1 = 0} \text{ or } \cancel{D+4} = 0$$

$$D = \frac{-4 \pm \sqrt{16-20}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm \sqrt{4i^2}}{2}$$

$$D = \frac{-4 \pm 2i}{2} = -2 \pm i$$

$$C.F = e^{-2x}[c_1\cos x + c_2\sin x].$$

$$P.I = \frac{1}{D^2+4D+5} - \cancel{2}\left[\frac{e^x + e^{-x}}{\cancel{2}}\right)$$

$$= -\left[\frac{1}{D^2+4D+5}e^x + \frac{1}{D^2+4D+5}e^{-x}\right]$$

$$= -\left[\frac{e^x}{(1)^2+4\times 1+5} + \frac{1}{(-1)^2+4\times -1+5}e^{-x}\right]$$

$$+ \frac{1}{1-4+5}$$

$$= -\left[\frac{e^x}{10} + \frac{1}{2}e^{-x}\right]$$

$$y(x) = C.S = e^{-2x}[c_1\cos x + c_2\sin x] - \left[\frac{e^x}{10} + \frac{1}{2}e^{-x}\right] \quad (1)$$

when $x=0$, $y=0$

Put in (1), we get

$$0 = e^{-2\times 0}[c_1\cos 0 + c_2\sin 0] - \left(\frac{e^0}{10} + \frac{1}{2}e^{-0}\right)$$

$$0 = c_1 - \frac{1}{10} - \frac{1}{2}$$

$$c_1 = \frac{1}{2} + \frac{1}{10} = \frac{6}{10} = \frac{3}{5}$$

$e^0 = 1$

$e^{-0} = \frac{1}{e^0} = \frac{1}{1} = 1$

$\cos 0 = 1$

$\sin 0 = 0$

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = e^{-2x}[-c_1\sin x + c_2\cos x] - \left[\frac{e^x}{10} - \frac{1}{2}e^{-x}\right]$$

Put $x=0$, $\frac{dy}{dx} = 0$ (given)

$$0 = e^{-0}[-c_1\sin 0 + c_2\cos 0) - \left(\frac{e^0}{10} - \frac{1}{2}e^{-0}\right)$$

$$0 = c_2 - \frac{1}{10} + \frac{1}{2}$$

$$c_2 = -\frac{1}{2} + \frac{1}{10}$$

$$= \frac{-5+1}{10} = \frac{-4}{10}$$

$$= -\frac{2}{5}$$

Q. $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$ (2)

or

$$\frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y}$$

which is of the type $\frac{dx}{dy} + Px = Q$

Here $P = \frac{1}{1+y^2}$, $Q = \frac{\tan^{-1}y}{1+y^2}$

$$I.F = e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1}y}$$

Hence its sol. is

$$x.e^{\tan^{-1}y} = \int Q.I.F.dy + c$$

$$= \int \frac{\tan^{-1}y}{1+y^2} . e^{\tan^{-1}y} dy + c$$

Put $\tan^{-1}y = z$

$\frac{1}{1+y^2} = \frac{dz}{dy}$ or $\frac{dy}{1+y^2} = dz$

$$x\,e^{\tan^{-1}y} = \int z e^z dz + c \quad = z\int e^z - \int\left[\frac{d}{dz}z \int e^z dz\right]dz + c$$

$$= z e^z - \int e^z dz + c$$

$$= z e^z - e^z + c$$

$$= \tan^{-1}y\, e^{\tan^{-1}y} - e^{\tan^{-1}y} + c$$

$$= e^{\tan^{-1}y}[\tan^{-1}y - 1] + c$$

$$x = \frac{\cancel{e^{\tan^{-1}y}}(\tan^{-1}y - 1)}{\cancel{e^{\tan^{-1}y}}} + \frac{c}{e^{\tan^{-1}y}}$$

$$= \tan^{-1}y - 1 + c\,e^{-\tan^{-1}y}$$

Q. solve $(D^2+5D+6)y = e^x$

its A.E.

$D^2+5D+6=0$

$(D+3)(D+2)=0$

$\Rightarrow D=-2,-3$

$C.F = c_1e^{-2x}+c_2e^{-3x}$

$P.I = \frac{1}{D^2+5D+6}e^x$

$= \frac{e^x}{(1)^2+5\times1+6}$

$= \frac{e^x}{1+5+6} = \frac{e^x}{12}$

Hence sol^n is

$C.S = c_1e^{-2x}+c_2e^{-3x} + \frac{e^x}{12}$

<u>solve</u> $(D^4+4)y=0$

its A.E.

$D^4+4=0$

$D^4+4+4D^2-4D^2=0$

$(D^2+2)^2-(2D)^2=0$

$(D^2+2-2D)(D^2+2+2D)=0$

$\Rightarrow D^2+2D+2=0$ or $D^2-2D+2=0$

$D = \frac{-2\pm\sqrt{4-8}}{2}$

$= \frac{-2\pm\sqrt{-4}}{2}$

$= \frac{-2\pm\sqrt{4i^2}}{2}$

$= \frac{-2\pm2i}{2}$

$= -1\pm i$

$-1+i, -1-i$

$D = \frac{2\pm\sqrt{4-8}}{2} = \frac{2\pm\sqrt{-4}}{2}$

$= \frac{2\pm\sqrt{4i^2}}{2} = \frac{2\pm2i}{2}$

$= 1\pm i$

$D = 1+i, 1-i$

Q. Solve by the method of variation of parameters

$(D^2-2D+1)y = e^x \log x$

Its A.E. is

$D^2-2D+1=0$

$(D-1)^2=0$

$\Rightarrow D=1,1$

$C.F = (c_1+c_2x)e^x$

Here $y_1 = e^x$, $y_2 = xe^x$

$X = e^x \log x$

W = wronskian of y_1 and y_2

$$= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x+xe^x \end{vmatrix}$$

$= e^x(e^x+xe^x) - e^x \cdot xe^x$

$= e^{2x} + \cancel{xe^{2x}} - \cancel{xe^{2x}}$

$= e^{2x}$.

$$P.I = -y_1\int \frac{y_2 X}{W}dx + y_2\int \frac{y_1 X}{W}dx$$

$$= -e^x\int \frac{xe^x \cdot e^x \log x}{e^{2x}}dx + xe^x\int \frac{e^x \cdot e^x \log x}{e^{2x}}dx$$

$$= -e^x\int x\log x\,dx + xe^x\int \log x\,dx$$

$$= -e^x\left[\log x\int x\,dx - \int\left[\frac{d}{dx}\log x \cdot \int x\,dx\right]dx\right] + xe^x\left[\log x\int dx - \int\left[\frac{d}{dx}\log x\int dx\right]dx\right]$$

$$= -e^x\left(\log x \cdot \frac{x^2}{2} - \int \frac{1}{x}\cdot\frac{x^2}{2}dx\right) + xe^x\left[x\log x - \int \frac{1}{x}\cdot x\,dx\right]$$

$$= -e^x\left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right] + xe^x[x\log x - x]$$

$$= -\frac{e^x x^2\log x}{2} + \frac{e^x x^2}{4} + x^2e^x\log x - x^2e^x$$

$$= \frac{1}{2}\frac{e^x x^2\log x}{2} - \frac{3}{4}x^2e^x$$

$C.S = C.F + P.I$

(b)

$y = 2px + p^n$ (1) (3)

$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + np^{n-1}$.

$p = 2p + 2x\frac{dp}{dx} + np^{n-1}$

or $2x\frac{dp}{dx} = -p - np^{n-1}\frac{dp}{dx}$

$\cancel{\frac{dp}{dx} = \frac{-p \quad np^{n-1}}{2x}}$ $\qquad \frac{dp}{dx} + \frac{p}{2x} =$

$\cancel{\frac{dp}{p+np^{n-1}} = \frac{dx}{2x}}$

$\cancel{\frac{dp}{p(1+np^{n-1})} = -\frac{dx}{2x}}$

$(2x + np^{n-1})\frac{dp}{dx} = -p$

$\frac{dp}{dx} = \frac{-p}{2x+np^{n-1}}$ $\quad$ or $\frac{dx}{dp} = -\frac{2x+np^{n-1}}{p}$

$\frac{dx}{dp} = -\frac{2x}{p} - np^{n-2}$

$\frac{dx}{dp} + \frac{2x}{p} = -np^{n-2}$

$I.F = e^{\int\frac{2}{p}dp} = e^{2\log p} = e^{\log p^2} = p^2$

Its sol.

$xp^2 = \int -np^{n-2}.p^2 dp + c = -n\int p^n dp + c$

$= \frac{n p^{n+1}}{n+1} + c$

$x = \frac{n p^{n-1}}{n+1} + \frac{c}{p^2}$ (2)

$y = 2p\left(\frac{n p^{n-1}}{n+1} + \frac{c}{p^2}\right) + p^n$ (3)

(2) & (3) together gives the required solution

(1)

Q solve $y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0$ (1)

which is of the type $f_1(xy)y\,dx + f_2(xy)x\,dy = 0$

$\therefore$ P.I $= \frac{1}{Mx - Ny} = \frac{1}{xy(xy + 2x^2y^2) - xy(xy - x^2y^2)}$

$= \frac{1}{\cancel{x^2y^2} + 2x^3y^3 - \cancel{x^2y^2} + x^3y^3}$

$= \frac{1}{3x^3y^3}$

Multiply both sides of (1) by $\frac{1}{3x^3y^3}$, we get

$\frac{y(xy + 2x^2y^2)\,dx}{3x^3y^3} + \frac{x(xy - x^2y^2)\,dy}{3x^3y^3} = 0$

$\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0$

which is exact

Hence its sol is

$\int M\,dx + \int(\text{term of } N \text{ not containing } x)\,dy = c$

keeping y constant

$\int\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int -\frac{1}{3y}\,dy = c$

$-\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c$

Solve $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$ (solve for p)

$p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}$ $\left(\frac{dy}{dx} = p \quad \therefore \frac{dx}{dy} = \frac{1}{p}\right.$

$\frac{p^2 - 1}{p} = \frac{x^2 - y^2}{xy}$

$(p^2 - 1)xy = p(x^2 - y^2)$

$$p^2xy - p(x^2-y^2) - xy = 0$$

$$p = \frac{(x^2-y^2) \pm \sqrt{(x^2-y^2)^2 + 4x^2y^2}}{2xy}$$

$$p = \frac{x^2-y^2 \pm \sqrt{x^4+y^4-2x^2y^2+4x^2y^2}}{2xy}$$

$$= \frac{x^2-y^2 \pm \sqrt{x^4+y^4+2x^2y^2}}{2xy}$$

$$= \frac{x^2-y^2 \pm \sqrt{(x^2+y^2)^2}}{2xy}$$

$$= \frac{x^2-y^2 \pm x^2+y^2}{2xy}$$

Hence sol. is

$(y^2-x^2-c)(xy-c) = 0$

$$p = \frac{x^2-\cancel{y^2}+x^2+\cancel{y^2}}{2xy} \quad \frac{\cancel{2}x^2}{\cancel{2}xy} = \frac{x}{y}$$

$$\text{or } p = \frac{\cancel{x^2}-y^2-\cancel{x^2}-y^2}{2xy} = \frac{-2y^2}{2xy} = -\frac{y}{x}$$

$p = \frac{x}{y}$

$\frac{dy}{dx} = \frac{x}{y}$

$y\,dy = x\,dx$

Integral

$\frac{y^2}{2} = \frac{x^2}{2} + A$

$y^2 = x^2 + 2A$

$x+c$

or $y^2 - x^2 - c = 0$

$p = -\frac{y}{x}$

$\frac{dy}{dx} = -\frac{y}{x}$

$\frac{dy}{y} = -\frac{dx}{x}$

Integral

$\int \frac{dy}{y} = -\int \frac{dx}{x} + B$

$\log y = -\log x + \log C$

$\log y + \log x = \log C$

$\log xy = \log C$

$xy = C$

$xy - C = 0$

Q $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x}\sin 2x$

$(D^2+5D+6)y = e^{-2x}\sin 2x$

Its A.E. is

$D^2+5D+6=0$

$D^2+3D+2D+6=0$

$D(D+3)+2(D+3)=0$

$(D+3)(D+2)=0$

$\Rightarrow D+2=0$ or $D+3=0$

$\Rightarrow D=-2,-3$

$P.I = \frac{1}{D^2+5D+6} e^{-2x}\sin 2x$

$= e^{-2x}\frac{1}{(D-2)^2+5(D-2)+6}\sin 2x$

$= e^{-2x}\frac{1}{D^2+4-4D+5D-10+6}\sin 2x$

$= e^{-2x}\frac{1}{D^2+D}\sin 2x$

$= e^{-2x}\frac{1}{-4+D}\sin 2x$

$= e^{-2x}\frac{1}{D-4}\sin 2x$

$= e^{-2x}\frac{(D+4)}{(D-4)(D+4)}\sin 2x$

$= e^{-2x}\frac{(D+4)}{D^2-16}\sin 2x = e^{-2x}\frac{(D+4)\sin 2x}{-4-16}$

$= -\frac{e^{-2x}}{20}[D\sin 2x + 4\sin 2x]$

$= -\frac{e^{-2x}}{20}[2\cos 2x + 4\sin 2x] = -\frac{e^{-2x}}{10}[\cos 2x + 2\sin 2x]$

C.S. = C.F + P.I

Q $(D^4-1)y = e^x\cos x$

A.E. $D^4-1=0$

$(D^2+1)(D^2-1)=0$

$D^2+1=0$ or $D^2-1=0$

$D^2=-1=i^2$ | $D^2=1$

$=\pm i$ | $D=\pm 1$

C.F $= c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x$

$P.I = \frac{1}{D^4-1}e^x\cos x$

$= e^x\frac{1}{(D+1)^4-1}\cos x$

$= e^x\frac{1}{D^4+4D^3+6D^2+4D+1-1}\cos x$

$= e^x\frac{1}{D^4+4D^3+6D^2+4D}\cos x$

$= e^x\frac{1}{1-\cancel{4D}-6+\cancel{4D}}\cos x$

$= -\frac{e^x}{5}\cos x$

C.F + P.I

Q. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x$

$(D^2 - 3D + 2)y = xe^{3x} + \sin 2x$

Its A.E.

$D^2 - 3D + 2 = 0$

$(D-1)(D-2) = 0$

$\Rightarrow D = 1, 2$

$C.F = c_1 e^{x} + c_2 e^{2x}$

$P.I = \frac{1}{D^2-3D+2}(xe^{3x} + \sin 2x)$

$= \frac{1}{D^2-3D+2} xe^{3x} + \frac{1}{D^2-3D+2}\sin 2x$

$= e^{3x}\frac{1}{(D+3)^2 - 3(D+3)+2} x + \frac{1}{-4-3D+2}\sin 2x$

$= e^{3x}\frac{1}{D^2+6D+9-3D-9+2} x + \frac{1}{-2-3D}\sin 2x$

$= e^{3x}\frac{1}{D^2+3D+2} x - \frac{1}{(2+3D)}\sin 2x$

$= \frac{e^{3x}}{2\left(1+\frac{D^2+3D}{2}\right)} x - \frac{(2-3D)}{(2+3D)(2-3D)}\sin 2x$

$= \frac{e^{3x}}{2}\left[1+\frac{D^2+3D}{2}\right]^{-1} x - \frac{(2-3D)}{4-9D^2}\sin 2x$

$= \frac{e^{3x}}{2}\left[1 - \frac{(D^2+3D)}{2} + \cdots\right]x - \frac{(2-3D)\sin 2x}{4-9\times(-4)}$

$= \frac{e^{3x}}{2}\left[x - \frac{D^2}{2}x - \frac{3D}{2}x\right] - \frac{1}{4+36}(2\sin 2x - 3D\sin 2x)$

$= \frac{e^{3x}}{2}\left[x - \frac{0}{2} - \frac{3}{2}\right] - \frac{1}{40}[2\sin 2x - 3\times 2\cos 2x]$

$= \frac{e^{3x}}{2}(x - 3/2) - \frac{1}{20}(\sin 2x - 3\cos 2x)$

$= \frac{e^{3x}}{4}(2x-3) + \frac{3\cos 2x}{20} - \frac{1}{2}\sin 2x$

$G.S = C.F + P.I$

Q $(D^3 + 2D^2 + D)y = x^2e^{2x} + \sin^2 x$

Its A.E. is

$D^3 + 2D^2 + D = 0$

$D(D^2+2D+1) = 0$

$D(D+1)^2 = 0$

$\Rightarrow D = 0, (D+1)^2 = 0$

$\Rightarrow D = 0, -1, -1$

~~C.F = c_1e^{0x} + c_2 e^{-x}~~