

## Module - 9:

### Vector Integration

#### Line Integral

(1)

Any integral which is evaluated along a curve is called a line integral.

Let  $\bar{F}$  be a continuous vector point function, defined at each point of a curve  $C$ . Then the line integral  $\bar{F}$  along  $C$  is given by

$$\int_C \bar{F} \cdot d\bar{s} = \int_C F_1 dx + F_2 dy + F_3 dz,$$

$$\text{when } \bar{F} = F_1 \bar{i} + F_2 \bar{j} + F_3 \bar{k} \text{ and } d\bar{s} = \bar{i} dx + \bar{j} dy + \bar{k} dz$$

#### Circulation:

If  $\bar{V}$  represents the velocity of a fluid particle and  $C$  is a closed curve, then the integral  $\oint_C \bar{V} \cdot d\bar{s}$  is called the circulation of  $\bar{V}$  around  $C$ .

#### Work done by a force:

If  $\bar{F}$  represents the force vector acting on a particle moving along an arc  $AB$ , then work done during a small displacement  $d\bar{s}$  is  $\bar{F} \cdot d\bar{s}$ .

Hence, the total work done by  $\bar{F}$  during the displacement from  $A$  to  $B$  is given by the line integral  $\int_A^B \bar{F} \cdot d\bar{s}$ .

Problems:

1. If  $\vec{F} = (5xy - 6x^2)\vec{i} + (2y - 4x)\vec{j}$ , evaluate  $\int_C \vec{F} \cdot d\vec{s}$  along the curve  $C$  in XY-plane  $y = x^3$  from the point  $(1, 1)$  to  $(2, 8)$

Sol. Given curve  $C$  is  $y = x^3$

$$\begin{aligned}\vec{F} \cdot d\vec{s} &= [(5xy - 6x^2)\vec{i} + (2y - 4x)\vec{j}] \cdot (dx\vec{i} + dy\vec{j}) \\ &= (5xy - 6x^2)dx + (2y - 4x)dy\end{aligned}$$

Along  $y = x^3$ , we have  $dy = 3x^2 dx$

and  $x$  varied from 1 to 2.

$$\begin{aligned}\therefore \int_C \vec{F} \cdot d\vec{s} &= \int_{(1,1)}^{(2,8)} [(5xy - 6x^2)dx + (2y - 4x)dy] \\ &= \int_{x=1}^2 [(5x^4 - 6x^2)dx + (2x^3 - 4x)3x^2 dx] \\ &= \int_1^2 [5x^4 - 6x^2 + 6x^5 - 12x^3] dx \\ &= [x^5 - 2x^3 + x^6 - 3x^4]_1^2 \\ &= (32 - 16 + 64 - 48) - (1 - 2 + 1 - 3) \\ &= 35\end{aligned}$$

(3)

2. Find the work done in moving a particle  
in the force field  $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$   
along the straight line from  $(0, 0, 0)$  to  $(2, 1, 3)$

Sol. Given  $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$

$$\therefore \vec{F} \cdot d\vec{r} = 3x^2 dx + (2xz - y)dy + zdz$$

The equation of the straight line joining

$(0, 0, 0)$  to  $(2, 1, 3)$  is

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \text{ (say)}$$

Then  $x = 2t, y = t, z = 3t$ .  $t$  varies from 0 to 1.

$$\text{Workdone by } \vec{F} = \int_{(0,0,0)}^{(2,1,3)} \vec{F} \cdot d\vec{r}$$

$$= \int_{t=0}^1 [3(2t)^2 2dt + (2 \cdot 2t \cdot 3t - t)dt + 3t \cdot 3dt]$$

$$= \int_0^1 (36t^2 + 8t) dt$$

$$= (12t^3 + 4t^2) \Big|_0^1$$

$$= 12 + 4$$

$$= 16$$

$$\begin{aligned} &= \\ &= 16 \end{aligned}$$

(4)

3. If  $\bar{F} = (y - 2x)\bar{i} + (3x + 2y)\bar{j}$ , calculate the circulation of  $\bar{F}$  about the circle  $C$  in the plane  $x^2 + y^2 = 4$  oriented in the anticlockwise direction.

Sol. The curve is  $C: x^2 + y^2 = 4$  in XY plane

Then the parametric equations are

$$x = 2\cos t, y = 2\sin t, t \text{ varies from } 0 \text{ to } 2\pi$$

$$\Rightarrow dx = -2\sin t dt, dy = 2\cos t dt$$

$$\bar{F} \cdot d\bar{s} = (y - 2x)dx + (3x + 2y)dy$$

$$= (2\sin t - 4\cos t)(-2\sin t dt)$$

$$+ (6\cos t + 4\sin t)(2\cos t dt)$$

$$= (-4\sin^2 t + 8\sin t \cos t + 12\cos^2 t + 8\sin t \cos t)dt$$

$$= (16\sin t \cos t + 16\cos^2 t - 4)dt$$

$$\therefore \oint_C \bar{F} \cdot d\bar{s} = \int_0^{2\pi} (16\sin t \cos t + 16\cos^2 t - 4)dt$$

$$= \int_0^{2\pi} \left[ 8\sin 2t + 16\left(1 + \frac{\cos 2t}{2}\right) - 4 \right] dt$$

$$= \left[ 8\left(-\frac{\cos 2t}{2}\right) + 8\left(t + \frac{\sin 2t}{2}\right) - 4t \right]_0^{2\pi}$$

$$= -4 + 8(2\pi + 0) - 8\pi - (-4)$$

$$= 16\pi - 8\pi = 8\pi$$

(5)

## Surface integrals

Any integral which is to be evaluated over a surface is called a surface integral.

Let  $\bar{F}$  be a continuous vector point function.

Then the surface integral of  $\bar{F}$  over a surface  $S$  is given by  $\iint_S \bar{F} \cdot \bar{n} dS$ , where  $\bar{n}$  is the outward drawn unit normal vector at any point of  $S$ .

If  $\bar{F} = F_1 \bar{i} + F_2 \bar{j} + F_3 \bar{k}$ , then we have

$$\iint_S \bar{F} \cdot \bar{n} dS = \iint_{R_1} F_1 dy dz + \iint_{R_2} F_2 dx dy + \iint_{R_3} F_3 dx dy$$

Note: Let  $R_1, R_2, R_3$  be the projections of the surface  $S$  on  $XY$  plane,  $YZ$  plane,  $ZX$  planes respectively. Then

$$\iint_S \bar{F} \cdot \bar{n} dS = \iint_{R_1} \frac{\bar{F} \cdot \bar{n}}{|\bar{n} \cdot \bar{k}|} dx dy$$

$$\iint_S \bar{F} \cdot \bar{n} dS = \iint_{R_2} \frac{\bar{F} \cdot \bar{n}}{|\bar{n} \cdot \bar{i}|} dy dz$$

$$\iint_S \bar{F} \cdot \bar{n} dS = \iint_{R_3} \frac{\bar{F} \cdot \bar{n}}{|\bar{n} \cdot \bar{j}|} dx dz$$

(6)

Problems:

1. Evaluate  $\iint_S \bar{F} \cdot \bar{n} dS$  where  $\bar{F} = (x+y)\bar{i} - 2x\bar{j} + 2y\bar{k}$   
 and  $S$  is the surface of the plane  $2x+y+2z=6$   
 in the first octant.

Sol. The normal to the surface  $S$  is given by

$$\nabla(2x+y+2z) = 2\bar{i} + \bar{j} + 2\bar{k}$$

$$\therefore \text{unit normal is } \bar{n} = \frac{2\bar{i} + \bar{j} + 2\bar{k}}{\sqrt{4+1+4}} \\ = \frac{1}{3}(2\bar{i} + \bar{j} + 2\bar{k})$$

$$\Rightarrow \bar{n} \cdot \bar{k} = \frac{2}{3}.$$

$$\begin{aligned} \text{Also } \bar{F} \cdot \bar{n} &= [(x+y)\bar{i} - 2x\bar{j} + 2y\bar{k}] \cdot \frac{1}{3}(2\bar{i} + \bar{j} + 2\bar{k}) \\ &= \frac{2}{3}(x+y) - \frac{2}{3}x + \frac{4}{3}yz \\ &= \frac{2}{3}y + \frac{4}{3}y \left( \frac{6-2x-y}{2} \right) \\ &= \frac{4y}{3}(3-x) \quad (\because \text{since on } S, \\ &\qquad\qquad z = \frac{1}{2}(6-2x-y)) \end{aligned}$$

$$\therefore \iint_S \bar{F} \cdot \bar{n} dS = \iint_R \frac{\bar{F} \cdot \bar{n}}{|\bar{n} \cdot \bar{k}|} dx dy, \text{ where } R \text{ is projection of } S \text{ on } xy \text{ plane}$$

$$= \iint_R \frac{4y}{3}(3-x) \cdot \frac{3}{2} dx dy$$

$$\begin{aligned}
 &= \int_{x=0}^3 \int_{y=0}^{6-2x} 2y(3-x) dy dx \\
 &= \int_0^3 2(3-x) \left[ \frac{y^2}{2} \right]_0^{6-2x} dx \\
 &= 4 \int_0^3 (3-x)^3 dx \\
 &= 4 \left[ \frac{(3-x)^4}{4(-1)} \right]_0^3 \\
 &= -[0 - 81] = 81
 \end{aligned}$$

2. Evaluate  $\int_S \bar{F} \cdot \bar{n} ds$ , where  $\bar{F} = 2\bar{i} + x\bar{j} + 3y\bar{z}\bar{k}$   
 and  $S$  is the surface of the cylinder  $x^2 + y^2 = 16$   
 included in the first octant between  $z=0$  and  $z=5$

Sol. Given that  $S$  is the surface of  $x^2 + y^2 = 16$

normal to the surface =  $\nabla S$

$$\begin{aligned}
 &= \bar{i} \frac{\partial S}{\partial x} + \bar{j} \frac{\partial S}{\partial y} + \bar{k} \frac{\partial S}{\partial z} \\
 &= 2x\bar{i} + 2y\bar{j} \\
 &= 2(x\bar{i} + y\bar{j})
 \end{aligned}$$

$$\therefore \text{unit normal } \bar{n} = \frac{2(x\bar{i} + y\bar{j})}{2\sqrt{x^2 + y^2}}$$

(8)

$$\text{i.e. } \bar{n} = \frac{2(x\bar{i} + y\bar{j})}{2\sqrt{16}} = \frac{1}{4}(x\bar{i} + y\bar{j})$$

Let R be the projection of S on XY plane

$$\bar{F} \cdot \bar{n} = (z\bar{i} + x\bar{j} + 3y\bar{k}) \cdot \frac{1}{4}(x\bar{i} + y\bar{j})$$

$$= \frac{1}{4}(xz + xy) = \frac{1}{4}x(y+z)$$

$$\text{and } \bar{n} \cdot \bar{i} = \frac{1}{4}x$$

$$\begin{aligned} \therefore \iint_S \bar{F} \cdot \bar{n} dS &= \iint_R \frac{\bar{F} \cdot \bar{n}}{|\bar{n} \cdot \bar{i}|} dy dz \\ &= \iint_{z=0}^{5} \iint_{y=0}^4 \frac{1}{4}x(y+z) \cdot \frac{4}{x} dy dz \end{aligned}$$

(since the region R is enclosed by  
 $y=0$  to  $y=4$  and  $z=0$  to  $z=5$ )

$$= \int_{z=0}^5 \left( \frac{y}{2} + yz \right)^4 dz$$

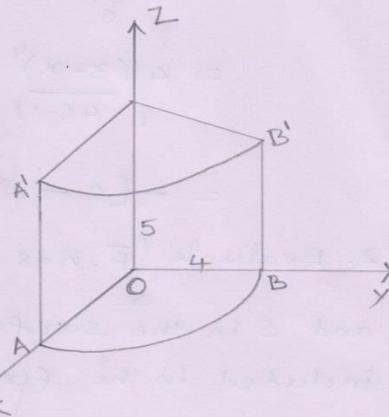
$$= \int_0^5 (8+4z) dz$$

$$= (8z + 2z^2) \Big|_0^5$$

$$= 40 + 50$$

$$= 90$$

=



## Volume Integrals

Any integral which is evaluated over a volume is called a volume integral.

If  $V$  is the volume bounded by a surface  $S$ , then  $\iiint_V \bar{F} dV$  is called the volume integral of  $\bar{F}$ .

### Problems

1. If  $\bar{F} = 2xz\bar{i} - x\bar{j} + y\bar{k}$ , evaluate  $\int \bar{F} dV$ , where  $V$  is the volume of the region bounded by the surfaces  $x=0, x=1, y=0, y=6, z=x^2, z=4$ .

Sol.  $\int \bar{F} dV = \int_0^1 \int_{x=0}^6 \int_{z=x^2}^4 (2xz\bar{i} - x\bar{j} + y\bar{k}) dy dz dx$

$$= \bar{i} \int_0^1 \int_{x^2}^6 \int_0^4 2xz dx dy dz - \bar{j} \int_0^1 \int_{x^2}^6 \int_0^4 x dx dy dz + \bar{k} \int_0^1 \int_{x^2}^6 \int_0^4 y dx dy dz$$

Consider,  $\int_0^1 \int_{x^2}^6 \int_0^4 2xz dx dy dz$

$$= \int_0^1 \int_0^6 x (z^4)_{x^2}^4 dy dx$$

(10)

$$= \int_{x=0}^1 \int_{y=0}^6 x(16-x^4) dy dx$$

$$= \int_0^1 x(16-x^4) [y]_0^6 dx$$

$$= \int_0^1 6x(16-x^4) dx$$

$$= 6 \int_0^1 (16x-x^5) dx$$

$$= 6 \left[ 8x^2 - \frac{x^6}{6} \right]_0^1$$

$$= 6 \left[ 8 - \frac{1}{6} \right] = 47.$$

$$\text{Now } \int_{x=0}^1 \int_{y=0}^6 \int_{z=x}^4 x dx dy dz = \int_0^1 \int_0^6 x(z^4) dy dx$$

$$= \int_0^1 \int_0^6 x(4-x^2) dy dx$$

$$= \int_0^1 (4x-x^3) [y]_0^6 dx$$

$$= 6 \int_0^1 (4x-x^3) dx$$

$$= 6 \left[ 2x^2 - \frac{x^4}{4} \right]_0^1$$

$$= 6 \left[ 2 - \frac{1}{4} \right]$$

$$= 2\frac{1}{2}.$$

(11)

Finally, consider  $\int_{x=0}^1 \int_{y=0}^6 \int_{z=x}^4 y^2 dx dy dz$

$$= \int_0^1 \int_0^6 y^2 (z)^4_x dy dz$$

$$= \int_0^1 \left(\frac{y^3}{3}\right)_0^6 (4-x) dx$$

$$= \frac{6^3}{3} \int_0^1 (4-x) dx$$

$$= 72 \left[ 4x - \frac{x^2}{3} \right]_0^1$$

$$= 72(4 - \frac{1}{3})$$

$$= 72(\frac{11}{3})$$

$$= 264$$

Substitute these values in eq(1), we get

$$\int_V \bar{F} dV = 47\bar{i} - \frac{21}{2}\bar{j} + 264\bar{k}$$

2. If  $\bar{F} = (2x^2 - 3z)\bar{i} - 2xy\bar{j} - 4x\bar{k}$ , evaluate

$\iiint_V \operatorname{curl} \bar{F} dV$ , where V is the closed region

bounded by the planes  $x=0, y=0, z=0$  and  
 $2x+2y+z=4$ .

(12)

$$\text{Sol. } \text{curl } \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-3z & -2xy & -4x \end{vmatrix}$$

$$= \bar{i}(0) - \bar{j}(-4+3) + \bar{k}(-2y-0)$$

$$= \bar{j} - 2y\bar{k}$$

The limits are  $z=0$  to  $z=4-2x-2y$

$$y=0 \text{ to } y=2-x$$

$$x=0 \text{ to } x=2$$

$$\therefore \iiint \nabla \times \bar{F} dv = \int_{x=0}^2 \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} (\bar{j} - 2y\bar{k}) dx dy dz$$

$$= \int_0^2 \int_0^{2-x} (\bar{j} - 2y\bar{k}) [z]_0^{4-2x-2y} dx dy$$

$$= \int_0^2 \int_0^{2-x} (\bar{j} - 2y\bar{k})(4-2x-2y) dy dx$$

$$= \int_0^2 \int_0^{2-x} [\bar{j}(4-2x-2y) - 2y\bar{k}(4-2x-2y)] dy dx$$

$$= \int_0^2 \bar{j} [4y - 2xy - y^2]_{y=0}^{2-x} - 2\bar{k} [2y^2 - xy - \frac{y^3}{3}]_{y=0}^{2-x} dx$$

$$= \int_0^2 \bar{j} [4(2-x) - 2x(2-x) - (2-x)^2] dx$$

$$\begin{aligned}
 & -\frac{2}{3}\bar{K} \left[ 6(2-x)^2 - 3x(2-x) - 2(2-x)^3 \right] \} dx \\
 &= \int_{x=0}^2 \left[ (2-x)^2 \bar{j} - \frac{2}{3}(2-x)^3 \bar{K} \right] dx \\
 &= \left[ \frac{(2-x)^3}{-3} \bar{j} + \frac{2}{3} \frac{(2-x)^4}{4} \bar{K} \right] \Big|_{x=0}^2 \\
 &= \frac{8}{3} \bar{j} - \frac{8}{3} \bar{K} = \frac{8}{3} (\bar{j} - \bar{K})
 \end{aligned}$$

### Vector integral Theorems

#### Gauss Divergence theorem

If  $\bar{F}$  is a vector point function having continuous first order partial derivatives over a closed surface  $S$  enclosing a volume  $V$ , then

$$\iiint_V \nabla \cdot \bar{F} dV = \iint_S \bar{F} \cdot \bar{n} ds,$$

where  $\bar{n}$  is the outward drawn unit normal vector to the surface  $S$ .

(14)

Problems

1. Verify Gauss divergence theorem for

$\vec{F} = (x^3 - yz)\vec{i} - 2xy\vec{j} + z\vec{k}$  taken over the surface of the cube bounded by the planes  $x=y=z=2$  and the coordinate planes.

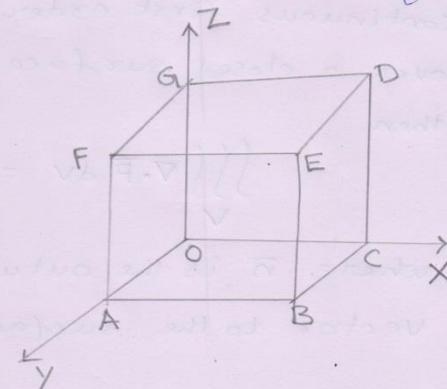
Sol. By Gauss divergence theorem,

$$\int_S \vec{F} \cdot \vec{n} dS = \int_V \nabla \cdot \vec{F} dV$$

consider the surface integral

$$\int_S \vec{F} \cdot \vec{n} dS = \int_{S_1} \vec{F} \cdot \vec{n} dS + \int_{S_2} \vec{F} \cdot \vec{n} dS + \int_{S_3} \vec{F} \cdot \vec{n} dS + \int_{S_4} \vec{F} \cdot \vec{n} dS + \int_{S_5} \vec{F} \cdot \vec{n} dS + \int_{S_6} \vec{F} \cdot \vec{n} dS \rightarrow 1)$$

where  $S_1, S_2, S_3$  are the six faces of the cube



(15)

Over the face  $S_1$ , i.e. BCDE

$$\underline{x=2}$$

$$\underline{n = \hat{i}} \text{ and } ds = dy dz$$

$$0 \leq y \leq 2 \text{ and } 0 \leq z \leq 2$$

$$\underline{\underline{F} \cdot \underline{n} = ((x^3 - yz)\hat{i} - 2xz^2y\hat{j} + zk\hat{k}) \cdot \hat{i}}$$

$$= x^3 - yz$$

$$= 8 - yz \quad (\because x=2)$$

$$\therefore \int_{S_1} \underline{\underline{F} \cdot \underline{n}} ds = \iint_{\substack{z=0 \\ z=0 \\ y=0}}^{2 \ 2} (8 - yz) dy dz$$

$$= \int_{z=0}^2 \left( 8y - \frac{y}{2}z^2 \right)_0^2 dz$$

$$= \int_0^2 (16 - 2z) dz$$

$$= (16z - z^2)_0^2$$

$$= 32 - 4 = 28$$

over the face  $S_2$ , i.e. AOGF

$$\text{we have } \underline{x=0}$$

$$\underline{n = -\hat{i}} \text{ and } ds = dy dz$$

$$0 \leq y \leq 2, \quad 0 \leq z \leq 2$$

$$\underline{\underline{F} \cdot \underline{n} = -(x^3 - yz)} = yz \quad (\because x=0)$$

(16)

$$\begin{aligned}\therefore \int_{S_2} \bar{F} \cdot \bar{n} dS &= \iint_{z=0, y=0}^{2, 2} yz dy dz \\ &= \int_0^2 z dz \int_0^2 y dy \\ &= \left(\frac{z^2}{2}\right)_0^2 \left(\frac{y^2}{2}\right)_0^2 \\ &= (2)(2) = 4\end{aligned}$$

over the face  $S_3$  i.e FEDG

$$z=2 \Rightarrow dz=0$$

$$\bar{n} = \bar{k} \text{ and } dS = dx dy$$

$$0 \leq x \leq 2 \text{ and } 0 \leq y \leq 2$$

$$\begin{aligned}\bar{F} \cdot \bar{n} &= z = 2 \\ \therefore \int_{S_3} \bar{F} \cdot \bar{n} dS &= \iint_{y=0, x=0}^{2, 2} 2 dx dy \\ &= 2 \left(x\right)_0^2 \left(y\right)_0^2 \\ &= 2(2)(2) = 8\end{aligned}$$

over the face  $S_4$  i.e OABC

$$z=0$$

$$\bar{n} = -\bar{k} \text{ and } dS = dx dy$$

$$0 \leq x \leq 2, 0 \leq y \leq 2$$

$$\bar{F} \cdot \bar{n} = -z = 0 \quad (\because z=0)$$

(17)

$$\therefore \int_{S_4} \bar{F} \cdot \bar{n} dS = \iint_{y=0, x=0}^2 0 \, dx \, dy = 0$$

over the face  $S_5$  i.e. ABEF

we have  $y = 2$

$$\bar{n} = \bar{j} \text{ and } dS = dx \, dz$$

$$0 \leq x \leq 2 \text{ and } 0 \leq z \leq 2.$$

$$\int_{S_5} \bar{F} \cdot \bar{n} dS = \iint_{x=0, z=0}^{2, 2} (-2xy) \, dx \, dz$$

$$= -4 \int_{x=0}^2 \int_{z=0}^2 x^2 \, dx \, dz$$

$$= -4 \left( \frac{x^3}{3} \right)_0^2 (z)^2_0$$

$$= -4 \left( \frac{8}{3} \right) (2)$$

$$= -\frac{64}{3}$$

over the face  $S_6$  i.e. OEDG

$$y = 0$$

$$\bar{n} = -\bar{j} \text{ and } dS = dx \, dz$$

$$\int_{S_6} \bar{F} \cdot \bar{n} dS = \iint_{x=0, z=0}^{2, 2} 2x^2y \, dx \, dz$$

$$= 0 \quad (\because y = 0)$$

(18)

$\therefore$  from 1), we get

$$\int_S \vec{F} \cdot \vec{n} dS = 28 + 4 + 8 + 0 - \frac{64}{3} + 0 = \frac{56}{3} \rightarrow 2)$$

Consider the volume integral

$$\begin{aligned} \int_V \nabla \cdot \vec{F} dV &= \int_V (3x^2 - 2x^2 + 1) dV \\ &= \int_0^2 \int_0^2 \int_{z=0}^2 (x^2 + 1) dx dy dz \\ &= \int_0^2 \int_0^2 \left( \frac{x^3}{3} + x \right)_0^2 dy dz \\ &= \int_0^2 \int_0^2 \left( \frac{8}{3} + 2 \right) dy dz \\ &= \frac{14}{3} \int_0^2 dy \int_0^2 dz \\ &= \frac{14}{3} (2) (2) \\ &= \frac{56}{3} \rightarrow 3) \end{aligned}$$

$\therefore$  from 2) and 3), we have

$$\int_S \vec{F} \cdot \vec{n} dS = \int_V \nabla \cdot \vec{F} dV$$

Hence Gauss divergence theorem is ~~not~~ verified

(19)

2. Evaluate  $\int_S \vec{F} \cdot \vec{n} dS$  by Gauss divergence theorem

where  $\vec{F} = 2x^2y\hat{i} - \hat{j} + 4x^2z\hat{k}$  taken over  
the region  $y+z=9$ ,  $x=2$  in the first octant.

Sol. By Gauss divergence theorem,

$$\int_S \vec{F} \cdot \vec{n} dS = \int_V \nabla \cdot \vec{F} dv$$

$$\begin{aligned} \text{Now } \nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(2x^2y) + \frac{\partial}{\partial y}(-\hat{j}) + \frac{\partial}{\partial z}(4x^2z) \\ &= 4xy - 2y + 8xz \end{aligned}$$

The limits are  $z = 0$  to  $\sqrt{9-y^2}$   
 $y = 0$  to 3 ( $\because$  In first quadrant)  
 $x = 0$  to 2

$$\begin{aligned} \int_V \nabla \cdot \vec{F} dv &= \int_{y=0}^3 \int_{z=0}^{\sqrt{9-y^2}} \int_{x=0}^2 (4xy - 2y + 8xz) dx dy dz \\ &= \int_{y=0}^3 \int_{z=0}^{\sqrt{9-y^2}} [2x^2y - 2xy + 4x^2z]_0^2 dz dy \\ &= \int_{y=0}^3 \int_{z=0}^{\sqrt{9-y^2}} (8y - 4y + 16z) dz dy \\ &= \int_{y=0}^3 \int_{z=0}^{\sqrt{9-y^2}} (4y + 16z) dz dy \end{aligned}$$

$$\begin{aligned}
 &= \int_0^3 [4yz + 8z^2] \sqrt{9-y^2} dy \\
 &= 4 \int_0^3 [y\sqrt{9-y^2} + 2(9-y^2)] dy \\
 &= 4 \left[ -\frac{1}{2}(9-y^2)^{3/2} \cdot \frac{2}{3} + 2(9y - \frac{y^3}{3}) \right]_0^3 \\
 &= 4 [0 + 9 + 36 - 0] \\
 &= 180
 \end{aligned}$$

Green's theorem in a plane

If R is a closed region in xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

$$\oint_C M dx + N dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

where C is traversed in positive (anticlockwise) direction.

Problems

1. Verify Green's theorem for  $\int [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$   
 where C is the boundary of the region bounded by  
 $x=0, y=0$  and  $x+y=1$ .

Sol. Given  $\int [(3x^2 - 8y^2)dx + (4y - 6xy)dy] = \int_C \vec{F} \cdot d\vec{\sigma}$

where  $\vec{F} = (3x^2 - 8y^2)\vec{i} + (4y - 6xy)\vec{j}$

and  $d\vec{\sigma} = dx\vec{i} + dy\vec{j}$

By Green's theorem,

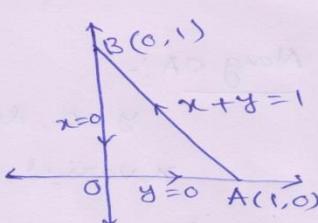
$$\int_C M dx + N dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

where R is the region bounded by  $x=0, y=0$   
 and  $x+y=1$

Here  $M = 3x^2 - 8y^2$

$N = 4y - 6xy$

$\frac{\partial M}{\partial y} = -16y, \quad \frac{\partial N}{\partial x} = -6y$



Now  $\iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \iint_R (16y + 6y) dx dy$

$$= 10 \int_{x=0}^1 \int_{y=0}^{1-x} y dy dx$$

(22)

$$\begin{aligned}
 &= 10 \int_0^1 \left(\frac{y^2}{2}\right)^{1-x} dx \\
 &= 5 \int_0^1 (1-x)^2 dx \\
 &= 5 \left[ \frac{(1-x)^3}{-3} \right]_0^1 \\
 &= -\frac{5}{3} (0-1) \\
 &= \frac{5}{3}.
 \end{aligned}$$

Now  $\int_C Mdx + Ndy = \int_{OA} Mdx + Ndy + \int_{AB} Mdx + Ndy + \int_{BO} Mdx + Ndy \rightarrow 2)$

Along OA:-

$$y=0, dy=0$$

$x$  varies from 0 to 1

$$\begin{aligned}
 \therefore \int_{OA} Mdx + Ndy &= \int_{x=0}^1 3x^2 dx \quad (\because y=0 \text{ and } dy=0) \\
 &= (x^3)_0^1 \\
 &= 1
 \end{aligned}$$

(23)

Along AB

$$\text{we have } x+y=1 \Rightarrow y=1-x$$

$$\Rightarrow dy = -dx$$

$x$  varies from 1 to 0

$$\int_{AB} M dx + N dy = \int_{OA} [(3x^2 - 8y^2) dx + (4y - 6xy) dy]$$

OA

$$= \int_{x=1}^0 [3x^2 - 8(1-x)^2] dx + (4(1-x) - 6x(1-x))(-dx)$$

$$= \int_1^0 [3x^2 - 8 + 16x - 8x^2 - 4 + 4x + 6x - 6x^2] dx$$

$$= \int_1^0 (-11x^2 + 26x - 12) dx$$

$$= \left[ -\frac{11x^3}{3} + \frac{26x^2}{2} - 12x \right]_1^0$$

$$= 0 - \left( -\frac{11}{3} + 13 - 12 \right)$$

$$= \frac{11}{3} - 1$$

$$= \frac{8}{3} \rightarrow 3)$$

Along BO,  
 $x=0, dx=0$   
and  $y$  varies from 1 to 0.

$$\therefore \int_{BO} (M dx + N dy) = \int_{y=1}^0 4y dy$$

$$= -2 \rightarrow (4)$$

Hence  
 $\int_C M dx + N dy = 1 + \frac{8}{3} - 2 = \frac{5}{3} \rightarrow (5)$

$\therefore$  from 1) and 5), we get

$$\int_C M dx + N dy = \iint_C \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \frac{5}{3} \rightarrow (5)$$

Hence, Green's theorem is verified.

(24)

2. Using Green's theorem, evaluate the integral

$\int_C (2xy - x^2) dx + (x^2 + y) dy$ , where  $C$  is the closed curve of the region bounded by  $y = \sqrt{x}$  and  $y = x$

$$\text{Sol: Given that } \int_C (2xy - x^2) dx + (x^2 + y) dy \\ = \int_C M dx + N dy$$

$$\text{where } M = 2xy - x^2, N = x^2 + y$$

$$\frac{\partial M}{\partial y} = 2x, \frac{\partial N}{\partial x} = 2x$$

By Green's theorem,

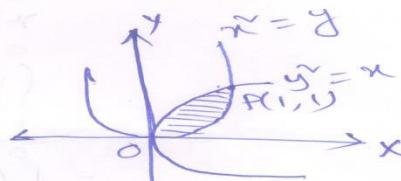
$$\int_C M dx + N dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dy dx$$

$$= \iint_R (2x - 2x) dy dx$$

$$= \int_{x=0}^1 \int_{y=x^2}^{\sqrt{x}} 0 dy dx$$

$$= 0$$

$$\text{i.e. } \int_C (2xy - x^2) dx + (x^2 + y) dy = 0$$



3. Evaluate by Green's theorem  $\int_C [(y - \sin x)dx + \cos y dy]$

where C is the triangle enclosed by the lines

$$y=0, x=\frac{\pi}{2}, y=\frac{2x}{\pi}$$

Sol: Given that  $\int_C [(y - \sin x)dx + \cos y dy] = \int_C M dx + N dy$

$$\text{where } M = y - \sin x, N = \cos x$$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = -\sin x$$

By Green's theorem,

$$\int_C M dx + N dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dy dx$$

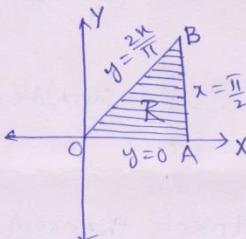
$$= \int_{x=0}^{\frac{\pi}{2}} \int_{y=0}^{\frac{2x}{\pi}} (-\sin x - 1) dy dx$$

$$= - \int_{x=0}^{\frac{\pi}{2}} \int_{y=0}^{\frac{2x}{\pi}} (1 + \sin x) dy dx$$

$$= - \int_0^{\frac{\pi}{2}} (1 + \sin x) \left[ y \right]_0^{\frac{2x}{\pi}} dx$$

$$= - \int_0^{\frac{\pi}{2}} (1 + \sin x) \left( \frac{2x}{\pi} \right) dx$$

$$= - \frac{2}{\pi} \int_0^{\frac{\pi}{2}} x (1 + \sin x) dx$$



(26)

$$= -\frac{2}{\pi} \left[ x(-\cos x + x) - 1 \cdot (-\sin x + \frac{x^2}{2}) \right]_0^{\frac{\pi}{2}}$$

$$= -\frac{2}{\pi} \left[ \frac{\pi}{2} (0 + \frac{\pi}{2}) + 1 - \frac{\pi^2}{8} \right]$$

$$= -\frac{2}{\pi} \left( \frac{\pi^2}{4} - \frac{\pi^2}{8} + 1 \right)$$

$$= -\frac{2}{\pi} \left( \frac{\pi^2}{8} + 1 \right)$$

$$= -\left( \frac{\pi}{4} + \frac{2}{\pi} \right)$$

$$\therefore \int_C [y - \sin x] dx + [\cos x] dy = -\left( \frac{\pi}{4} + \frac{2}{\pi} \right)$$

### Stoke's theorem

If  $S$  is an open surface bounded by a closed curve  $C$  and if  $\vec{F}$  is any continuously differentiable vector field, then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} dS$$

where  $\vec{n}$  is outward drawn unit normal at any point of  $S$ .

(27)

Problems:-

$$1. \text{ verify Stoke's theorem for } \vec{F} = (2x-y)\vec{i} + y\vec{z}\vec{j} - y\vec{z}\vec{k}$$

over the upper half of the sphere  $\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 = 1$   
bounded by the projection of the xy plane.

$$\text{Sol. Given } \vec{F} = (2x-y)\vec{i} - y\vec{z}\vec{j} - y\vec{z}\vec{k}$$

The bound C of S is a circle in xy plane i.e.  $z=0$

By Stoke's theorem

$$\int_C \vec{F} \cdot d\vec{s} = \iint_S \text{curl } \vec{F} \cdot \vec{n} dS$$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -y\tilde{z} & -y\tilde{z} \end{vmatrix}$$

$$= \vec{i}(-2y\tilde{z} + 2y\tilde{z}) - \vec{j}(0-0) + \vec{k}(0+1)$$

$$= \vec{k}$$

$$\begin{aligned} \iint_S \text{curl } \vec{F} \cdot \vec{n} dS &= \iint_S \vec{k} \cdot \vec{n} dS \\ &= \iint_S dx dy \\ &= \pi ( \text{ which is area of the circle } \\ &\quad \tilde{x}^2 + \tilde{y}^2 = 1 \text{ i.e. } \pi(1^2) = \pi ) \end{aligned}$$

(28)

Now, we have to evaluate the integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$   
 where  $C$  is the circle  $x^2 + y^2 = 1$

Then  $x = \cos\theta, y = \sin\theta$  ( $\because r=1$ )

$$\Rightarrow dx = -\sin\theta d\theta, dy = \cos\theta d\theta$$

$\theta$  varies from  $0$  to  $2\pi$  over the circle.

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_C [(2x-y)dx - yzdy - z^2dz] \\ &= \iint_C (2x-y)dx, \text{ since } z=0, dz=0 \\ &= - \int_0^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta d\theta) \\ &= \int_0^{2\pi} (\sin^2\theta - 2\sin\theta\cos\theta) d\theta \\ &= \int_0^{2\pi} \left[ \frac{1 - \cos 2\theta}{2} - \sin 2\theta \right] d\theta \\ &= \left[ \frac{1}{2} \left( \theta - \frac{\sin 2\theta}{2} \right) + \frac{\cos 2\theta}{2} \right]_0^{2\pi} \\ &= \frac{1}{2} (2\pi - 0) + \frac{1}{2} - 0 - \frac{1}{2} \quad (\because \cos 4\pi = 1 \\ &\quad \sin 4\pi = 0) \\ &= \pi. \rightarrow 2)\end{aligned}$$

from (1) & (2).  $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot \hat{n} ds.$

Hence, Green's theorem is verified.

(29)

2. Verify Stokes theorem for the function

$\vec{F} = (x^2 - y)\vec{i} + 2xy\vec{j}$ , integrated round the rectangle  
in the plane  $z=0$  and bounded by the lines  
 $x=0, y=0, x=a$  and  $y=b$ .

Sol: By Stoke's theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot \vec{n} dS$$

Here  $\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y & 2xy & 0 \end{vmatrix}$

$$= \begin{vmatrix} \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(2y+2y) \\ 4y\vec{k} \end{vmatrix}$$

$$\begin{aligned} \iint_S \operatorname{curl} \vec{F} \cdot \vec{n} dS &= \iint_S 4y\vec{k} \cdot \vec{n} dS \\ &= \iint_S 4y dx dy \\ &\quad \text{with } x=0, y=0 \\ &= \int_0^a \left( \int_0^b 4y dy \right) dx \\ &= \int_0^a (2y^2)_0^b dx \\ &= 2b^2 \int_0^a dx = 2b^2 (x)_0^a \\ &= 2ab^2 \rightarrow 1) \end{aligned}$$

(30)

Here C is the rectangle OABC

$$\int_C \vec{F} \cdot d\vec{r} = \int_{OA} \vec{F} \cdot d\vec{r} + \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CO} \vec{F} \cdot d\vec{r}$$

(2)

$$\text{Here } \vec{F} \cdot d\vec{r} = (x - y) dx + 2xy dy$$

Along OA

$$y = 0, dy = 0$$

x varies from 0 to a

$$\therefore \int_{OA} \vec{F} \cdot d\vec{r} = \int_0^a x^2 dx$$

$$= \left( \frac{x^3}{3} \right)_0^a = \frac{a^3}{3}$$

Along AB

$$x = a, dx = 0$$

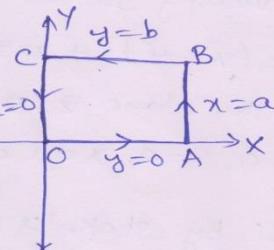
y varies from 0 to b

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_0^b 2ay dy$$

$$= 2a \left( \frac{y^2}{2} \right)_0^b$$

$$= 2a \left( \frac{b^2}{2} \right)$$

$$= ab^2$$



Along BC:

$$y = b, dy = 0$$

$x$  varies from  $a$  to  $0$

$$\begin{aligned} \int_{BC} \vec{F} \cdot d\vec{s} &= \int_a^0 (x\hat{i} - b\hat{j}) dx \\ &= \left( \frac{x^3}{3} - xb^2 \right) \Big|_a^0 \\ &= 0 - \left( \frac{a^3}{3} - ab^2 \right) \\ &= -\frac{a^3}{3} + ab^2 \end{aligned}$$

Along CO:

$$x = 0, dx = 0$$

$y$  varies from  $b$  to  $0$

$$\begin{aligned} \int_{CO} \vec{F} \cdot d\vec{s} &= \int_b^0 0 dy \\ &= 0 \end{aligned}$$

Substitute all these in eq2), we get

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{s} &= \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 + 0 \\ &= 2ab^2 \rightarrow 3) \end{aligned}$$

$$\therefore \text{from 1 and 3), } \oint_C \vec{F} \cdot d\vec{s} = \iint_S \operatorname{curl} \vec{F} \cdot \vec{n} ds$$

Hence, Stokes theorem is verified.