Module-4

(Applications of Multi Variable Calculus)

TAYLOR'S THEOREM FOR FUNCTIONS OF TWO VARIABLES

Let $f(x, y)$ be a function of two independent variables x and y. If the function $f(x, y)$ and its partial derivatives up to nth order are continuous throughout the domain centred at a point (x, y). Then

$$f(a+h, b+k) = f(a, b) + \left[h\frac{\partial f(a,b)}{\partial x} + k\frac{\partial f(a,b)}{\partial y}\right]$$

$$+ \frac{1}{\lfloor 2}\left[h^2\frac{\partial^2 f(a,b)}{\partial x^2} + 2hk\frac{\partial^2 f(a,b)}{\partial x \partial y} + k^2\frac{\partial^2 f(a,b)}{\partial y^2}\right]$$

$$+ \frac{1}{\lfloor 3}\left[h^3\frac{\partial^3 f(a,b)}{\partial x^3} + 3h^2k\frac{\partial^3 f(a,b)}{\partial x^2 \partial y} + 3hk^2\frac{\partial^3 f(a,b)}{\partial x \partial y^2} + k^3\frac{\partial^3 f(a,b)}{\partial y^3}\right] + \ldots$$

Or

$$f(a+h, b+k) = f(a, b) + \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right] f(a,b) + \frac{1}{\lfloor 2}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^2 f(a, b)$$

$$+ \frac{1}{\lfloor 3}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^3 f(a,b) + \ldots$$

Proof. Suppose $P(x, y)$ and $Q(x + h, y + k)$ be two neighbouring points. Then $f(x + h, y + k)$, the value of f at Q can be expressed in terms of f and its derivatives at P.

Here, we treat $f(x + h, y + k)$ as a function of single variable x and keeping y as a constant. Expanded as follows using Taylor's theorem for single variable.*

$$f(x+h, y+k) = f(x, y+k) + h\frac{\partial f(x,y+k)}{\partial x} + \frac{h^2}{\lfloor 2}\frac{\partial^2 f(x,y+k)}{\partial x^2} + \ldots \quad \ldots(i)$$

Now expanding all the terms on the R.H.S. of (i) as function of y, keeping x as constant.

$$f(x+h, y+k) = \left[f(x,y) + k\frac{\partial f(x,y)}{\partial y} + \frac{k^2}{\lfloor 2}\frac{\partial^2 f(x,y)}{\partial y^2} + \ldots\right]$$

$$+ h\frac{\partial}{\partial x}\left[f(x,y) + k\frac{\partial f(x,y)}{\partial y} + \frac{k^2}{\lfloor 2}\frac{\partial^2 f(x,y)}{\partial y^2} + \ldots\right]$$

$$+ \frac{h^2}{\lfloor 2}\frac{\partial^2}{\partial x^2}\left[f(x,y) + k\frac{\partial f(x,y)}{\partial y} + \frac{k^2}{\lfloor 2}\frac{\partial^2 f(x,y)}{\partial y^2} + \ldots\right] + \ldots$$

$$f(x+h, y+k) = f(x, y) + \left[h\frac{\partial f(x,y)}{\partial x} + k\frac{\partial f(x,y)}{\partial y}\right]$$

$$+ \frac{1}{\lfloor 2}\left[h^2\frac{\partial^2 f(x,y)}{\partial x^2} + 2hk\frac{\partial^2 f(x,y)}{\partial x \partial y} + k^2\frac{\partial^2 f(x,y)}{\partial y^2}\right] + \ldots$$

For any point (a, b) putting $x = a$, $y = b$ in above equation then, we get

$$f(a+h, b+k) = f(a, b) + \left[h\frac{\partial f(a,b)}{\partial x} + k\frac{\partial f(a,b)}{\partial y}\right]$$

$$+ \frac{1}{\lfloor 2}\left[h^2\frac{\partial^2 f(a,b)}{\partial x^2} + 2hk\frac{\partial^2 f(a,b)}{\partial x \partial y} + k^2\frac{\partial^2 f(a,b)}{\partial y^2}\right] + \ldots$$

Or

$$f(a+h, b+k) = f(a, b) + \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right] f(a,b) + \frac{1}{\lfloor 2}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^2 f(a, b)$$

$$+ \frac{1}{\lfloor 3}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^3 f(a,b) + \ldots$$

Hence proved.

Alternative form:

Putting $\quad a+h = x \Rightarrow h = x-a$

$b+k = y \Rightarrow k = y-b$

$$\text{then } f(x,y) = f(a,b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a,b) + \frac{1}{\lfloor 2} \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f(a,b) + \ldots \quad \ldots(ii)$$

Maclaurin's Series Expansion

It is a special case of Taylor's series when the expansion is about the origin (0, 0).

So, putting $a = 0$ and $b = 0$ in equation (2), we get

$$f(x, y) = f(0, 0) + \left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right] f(0,0) + \frac{1}{\underline{|2}}\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right]^2 f(0,0) + \ldots$$

Example

Expand $e^x \cos y$ about the point $\left(1, \frac{\pi}{4}\right)$

Sol. We have $f(x, y) = e^x \cos y$ ...(i)

and $a = 1, b = \frac{\pi}{4}, f\left(1, \frac{\pi}{4}\right) = e \cos \frac{\pi}{4} = \frac{e}{\sqrt{2}}$

$\therefore$ From (1)

$$\frac{\partial f}{\partial x} = e^x \cos y \Rightarrow \frac{\partial f\left(1, \frac{\pi}{4}\right)}{\partial x} = e \cos \frac{\pi}{4} = \frac{e}{\sqrt{2}}$$

$$\frac{\partial f}{\partial y} = -e^x \sin y \Rightarrow \frac{\partial f\left(1, \frac{\pi}{4}\right)}{\partial y} = -e \sin \frac{\pi}{4} = -\frac{e}{\sqrt{2}}$$

$$\frac{\partial^2 f}{\partial x^2} = e^x \cos y \Rightarrow \frac{\partial^2 f\left(1, \frac{\pi}{4}\right)}{\partial x^2} = \frac{e}{\sqrt{2}}, \frac{\partial^2 f}{\partial x \partial y} = -e^x \sin y$$

$$\frac{\partial^2 f\left(1, \frac{\pi}{4}\right)}{\partial x \partial y} = -\frac{e}{\sqrt{2}}, \frac{\partial^2 f}{\partial y^2} = -e^x \cos y = -\frac{e}{\sqrt{2}}.$$

By Taylor's theorem, we have

$$f\left(1+h,\frac{\pi}{4}+k\right) = f\left(1,\frac{\pi}{4}\right) + \left(h\frac{\partial f\left(1,\frac{\pi}{4}\right)}{\partial x}+k\frac{\partial f\left(1,\frac{\pi}{4}\right)}{\partial y}\right) + \ldots \qquad \ldots(ii)$$

Let $1 + h = x \Rightarrow h = x - 1$ and $\frac{\pi}{4} + k = y \Rightarrow k = y - \frac{\pi}{4}$, equation (2) reduce in the form

$$f((x, y) = e^x \cos y = \frac{e}{\sqrt{2}} + (x-1)\cdot\frac{e}{\sqrt{2}} + \left(y-\frac{\pi}{4}\right)\left(-\frac{e}{\sqrt{2}}\right) + \frac{1}{\lfloor 2}\left[(x-1)^2\cdot\frac{e}{\sqrt{2}}\right.$$

$$\left. + 2(x-1)\left(y-\frac{\pi}{4}\right)\left(-\frac{e}{\sqrt{2}}\right)+\left(y-\frac{\pi}{4}\right)^2\left(-\frac{e}{\sqrt{2}}\right)\right] + \ldots$$

$$\Rightarrow f(x, y) = \frac{e}{\sqrt{2}}\left[1+(x-1)-\left(y-\frac{\pi}{4}\right)+\frac{(x-1)^2}{2}-(x-1)\left(y-\frac{\pi}{4}\right)-\left(y-\frac{\pi}{4}\right)^2+\ldots\right].$$

Example

Expand $f(x, y) = e^y \log(1 + x)$ in powers of x and y about $(0, 0)$

Sol. We have $f(x, y) = e^y \log(1 + x)$

Here, $a = 0$ and $b = 0$, then $f(0, 0) = e^0 \log 1 = 0$

Now, $\dfrac{\partial f}{\partial x} = \dfrac{e^y}{1+x} \Rightarrow \dfrac{\partial f}{\partial x}(0,0) = 1$

$$\frac{\partial f}{\partial y} = e^y \log(1 + x) \Rightarrow \frac{\partial f(0,0)}{\partial x} = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{e^y}{1+x} \Rightarrow \frac{\partial^2 f(0,0)}{\partial x \partial y} = 1$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{e^y}{(1+x)^2} \Rightarrow \frac{\partial^2 f(0,0)}{\partial x^2} = -1$$

$$\frac{\partial^2 f}{\partial y^2} = e^y \log(1 + x) \Rightarrow \frac{\partial^2 f(0,0)}{\partial y^2} = 0$$

Now, applying Taylor's theorem, we get

$$f(0+h, 0+k) = f(h, k) = f(0, 0) + \left(h\frac{\partial f(0,0)}{\partial x} + k\frac{\partial f(0,0)}{\partial y}\right) + \frac{1}{\lfloor 2}\left(h^2\frac{\partial^2 f(0,0)}{\partial x^2}\right.$$

$$\left. + 2hk\,\frac{\partial^2 f(0,0)}{\partial x \partial y} + k^2\frac{\partial^2 f(0,0)}{\partial y^2}\right) + \ldots$$

Let $h = x$, $k = y$, then, we get

$$f(x, y) = e^y \log(1+x) = f(0,0) + \left(x\frac{\partial f(0,0)}{\partial x} + y\frac{\partial f(0,0)}{\partial y}\right) + \ldots$$

$$= 0 + (x \times 1 + y \times 0) + \frac{1}{\lfloor 2}[x^2(-1) + 2xy \times 1 + y^2 \times 0] + \ldots$$

$$\Rightarrow \qquad e^y \log(1+x) = x - \frac{x^2}{2} + xy + \ldots .$$

1 Taylor's theorem for Functions of Two Variables

Let $n \geq 1$. Suppose that $f(x,y)$ and its partial derivatives through order $n+1$ are continuous in a small neighborhood $\mathscr{R}(a,b)$, centred at the point (a,b). Then

$$
\begin{aligned}
f(a+h,b+k) &= \left|\left\{e^{h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}}\right\} f\right|_{(a,b)} \\
&= \left\{1+\tfrac{1}{1!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)+\tfrac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2+\cdots+\tfrac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n\right\} f+R_n \\
&= f+\tfrac{1}{1!}\left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right)+\tfrac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\partial y}+k^2\frac{\partial^2 f}{\partial y^2}\right) \\
&\qquad \cdots+\tfrac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f+R_n, \qquad (1.1)
\end{aligned}
$$

where f and all its partial derivatives are evaluated at the point (a,b), and

$$R_n = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(a+ch+b+ck),$$

for some $0 < c < 1$, and R_n is called the Taylor's remainder after n terms.

Formula (1.1) is useful in expanding ascending powers of h and k about the point (a, b). To expand f in ascending powers of $x-a$ and $y-b$, we use the following Taylor's formula:

$$
\begin{aligned}
f(x,y) = f(a,b) &+ \frac{1}{1!}\left\{(x-a)\frac{\partial f(a,b)}{\partial x} + (y-b)\frac{\partial f(a,b)}{\partial y}\right\} \\
&+ \frac{1}{2!}\left\{(x-a)^2\frac{\partial^2 f(a,b)}{\partial x^2} + 2(x-a)(y-b)\frac{\partial^2 f(a,b)}{\partial x\partial y} + (y-b)^2\frac{\partial^2 f(a,b)}{\partial y^2}\right\} \\
&+ \frac{1}{3!}\left\{(x-a)^3\frac{\partial^3 f(a,b)}{\partial x^3} + 3(x-a)^2(y-b)\frac{\partial^3 f(a,b)}{\partial y\partial x^2}\right. \\
&\left. + 3(x-a)(y-b)^2\frac{\partial^3 f(a,b)}{\partial x\partial y^2} + (y-b)^3\frac{\partial^3 f(a,b)}{\partial y^3}\right\} + \cdots \qquad (1.2)
\end{aligned}
$$

To expand f in ascending powers of x and y, we use (1.2) with $a = b = 0$.

2 Approximations using Taylor's theorem

For $n = 1$: the linear approximation of f about the origin $(0,0)$, is given by

$$f(x,y) \approx f(0,0) + \frac{1}{1!}\left(x\frac{\partial f(0,0)}{\partial x} + y\frac{\partial f(0,0)}{\partial y}\right), \tag{2.1}$$

and the error in the linear approximation is given by

$$\begin{aligned} E(x,y) &= \frac{1}{2!}\left\{x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2}\right\} \\ &= \frac{1}{2!}\left\{x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy}\right\}, \end{aligned} \tag{2.2}$$

where the second order partial derivatives are evaluated at (cx, cy) for some $0 < c < 1$.

For $n = 2$, the quadratic approximation of f is given by

$$f(x,y) \approx f(0,0) + \frac{1}{1!}\left(x\frac{\partial f(0,0)}{\partial x} + y\frac{\partial f(0,0)}{\partial y}\right)$$
$$+ \frac{1}{2!}\left(x^2\frac{\partial^2 f(0,0)}{\partial x^2} + 2xy\frac{\partial^2 f(0,0)}{\partial x \partial y} + y^2\frac{\partial^2 f(0,0)}{\partial y^2}\right). \quad (2.3)$$

and the error in the approximation is given by

$$E(x,y) = \frac{1}{3!}\left(x^3\frac{\partial^3 f}{\partial x^3} + 3x^2 y\frac{\partial^3 f}{\partial x^2 \partial y} + 3xy^2\frac{\partial^3 f}{\partial x \partial y^2} + y^2\frac{\partial^3 f}{\partial y^3}\right)$$
$$= \frac{1}{3!}\left\{x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{yyx} + y^3 f_{yyy}\right\}, \quad (2.4)$$

where the third order partial derivatives are evaluated at (cx, cy) for some $0 < c < 1$.

For $n = 3$, the cubic approximation of f is given by

$$f(x,y) \approx f(0,0) + \frac{1}{1!}\left(x\frac{\partial f(0,0)}{\partial x} + y\frac{\partial f(0,0)}{\partial y}\right) + \frac{1}{2!}\left(x^2\frac{\partial^2 f(0,0)}{\partial x^2} + 2xy\frac{\partial^2 f(0,0)}{\partial x\partial y} + y^2\frac{\partial^2 f(0,0)}{\partial y^2}\right) + \frac{1}{3!}\left(x^3\frac{\partial^3 f(0,0)}{\partial x^3} + 3x^2 y\frac{\partial^3 f(0,0)}{\partial x^2\partial y} + 3xy^2\frac{\partial^3 f(0,0)}{\partial x\partial y^2} + y^2\frac{\partial^3 f(0,0)}{\partial y^3}\right) \quad (2.5)$$

or

$$f(x,y) \approx f + \frac{1}{1!}\left(xf_x + yf_y\right) + \frac{1}{2!}\left(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy}\right) + \frac{1}{3!}\left(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{yyx} + y^3 f_{yyy}\right), \quad (2.6)$$

where f and all its partial derivatives are evaluated at $(0,0)$.

Example 2.1. Find quadratic approximation for $f(x,y) = \cos x \cos y$ at the origin, using Taylor's formula. Also, estimate the error in approximation, if $|x| \le 0.1$ and $|x| \le 0.1$.

Solution. We employ (2.3) for quadratic approximation, and (2.4) for the error in the approximation. In fact,

$$
\begin{aligned}
f(x,y) &= \cos x \cos y & &\Rightarrow & f(0,0) &= \cos 0 . \cos 0 = 1 \\
f_x(x,y) &= \frac{\partial f}{\partial x} = -\sin x \cos y & &\Rightarrow & f_x(0,0) &= -\sin 0 . \cos 0 = 0 \\
f_y(x,y) &= \frac{\partial f}{\partial y} = -\cos x \sin y & &\Rightarrow & f_y(0,0) &= -\cos 0 . \sin 0 = 0 \\
f_{xx}(x,y) &= \frac{\partial^2 f}{\partial x^2} = -\cos x \cos y & &\Rightarrow & f_{xx}(0,0) &= -\cos 0 . \cos 0 = -1 \\
f_{yy}(x,y) &= \frac{\partial^2 f}{\partial y^2} = -\cos x \cos y & &\Rightarrow & f_{yy}(0,0) &= -\cos 0 . \cos 0 = -1 \\
f_{xy}(x,y) &= \frac{\partial^2 f}{\partial y \partial x} = \sin x \sin y & &\Rightarrow & f_{xy}(0,0) &= \sin 0 . \sin 0 = 0.
\end{aligned}
$$

Substituting these in (2.3), we obtain that

$$f(x,y) \approx 1 + \frac{1}{1!}(x \cdot 0 + y \cdot 0) + \frac{1}{2!}\left[x^2(-1) + 2xy \cdot 0 + y^2(-1)\right] = 1 - \frac{x^2}{2} - \frac{y^2}{2},$$

which is the quadratic approximation, we need. Note that the partial derivatives of third order of f, being the products of sines and cosines, have the absolute values less than or equal to 1. Since $|x| \le 0.1$ and $|x| \le 0.1$, the error of approximation is estimated by

$$\begin{aligned} |E(x,y)| &\le \frac{1}{3!}\left\{\left|x^3 f_{xxx}\right| + 3\left|x^2 y f_{xxy}\right| + 3\left|xy^2 f_{yyx}\right| + \left|y^3 f_{yyy}\right|\right\} \\ &\le \frac{1}{6}\left\{|x|^3 + 3|x|^2|y| + 3|x||y|^2 + |y|^3\right\} \\ &\le \frac{1}{6}\left[(0.1)^3 + 3(0.1)^2(0.1) + 3(0.1)(0.1)^2 + (0.1)^3\right] = 0.00134. \end{aligned}$$

Example 2.2. Find cubic approximation for $f(x,y)=xe^y$ at the origin, using Taylor's formula.

Solution. We employ (2.6) for finding cubic approximation:

$$
\begin{aligned}
f(x,y) &= xe^y && \Rightarrow \quad f(0,0)=0.e^0=0;\\
f_x(x,y) &= \frac{\partial f}{\partial x}=e^y && \Rightarrow \quad f_x(0,0)=e^0=1,\\
f_y(x,y) &= \frac{\partial f}{\partial y}=xe^y && \Rightarrow \quad f_y(0,0)=0.e^0=0;\\
f_{xx}(x,y) &= \frac{\partial^2 f}{\partial x^2}=0 && \Rightarrow \quad f_{xx}(0,0)=0,\\
f_{yy}(x,y) &= \frac{\partial^2 f}{\partial y^2}=xe^y && \Rightarrow \quad f_{yy}(0,0)=0.e^0=0,\\
f_{xy}(x,y) &= \frac{\partial^2 f}{\partial y\partial x}=e^y && \Rightarrow \quad f_{xy}(0,0)=e^0=1;\\
f_{xxx}(x,y) &= \frac{\partial^3 f}{\partial x^3}=0 && \Rightarrow \quad f_{xxx}(0,0)=0,\\
f_{xxy}(x,y) &= \frac{\partial^3 f}{\partial y\partial x^2}=0 && \Rightarrow \quad f_{xxy}(0,0)=0,\\
f_{yyx}(x,y) &= \frac{\partial^3 f}{\partial x\partial y^2}=e^y && \Rightarrow \quad f_{yyx}(0,0)=e^0=1\\
f_{yyy}(x,y) &= \frac{\partial^3 f}{\partial y^3}=xe^y && \Rightarrow \quad f_{yyy}(0,0)=0.e^0=0.
\end{aligned}
$$

Substituting these in (2.6), we obtain that

$$\begin{aligned} f(x,y) &\approx 0 + \frac{1}{1!}(x\cdot 1 + y\cdot 0) + \frac{1}{2!}\left[x^2\cdot 0 + 2xy\cdot 1 + y^2\cdot 0\right] \\ &\quad + \frac{1}{3!}\left[x^3\cdot 0 + 3x^2y\cdot 0 + 3xy^2\cdot 1 + y^3.0\right] \\ &= x + xy + \frac{1}{2}\cdot xy^2 \end{aligned}$$

which is the required cubic approximation.

Exercise 2.1. Use Taylor's formula to find the quadratic and cubic approximations for each of the following functions $f(x,y)$ at the origin:

(a) $y \sin x$

(b) $\sin x \cos y$

(c) $e^x \cos y$

(d) $\sin(x^2 + y^2)$

(e) $\log(2x + y + 1)$

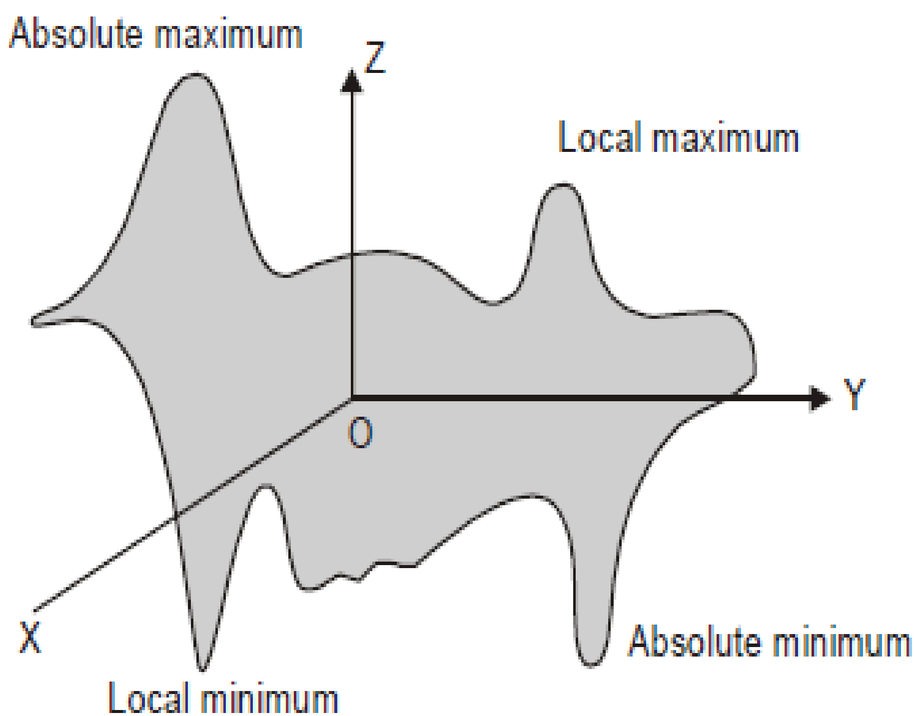
(f) $\frac{1}{1-x-y}$

(g) $\frac{1}{1-x-y+xy}$

EXTREMA OF FUNCTION OF SEVERAL VARIABLES

Introduction

In some practical and theoretical problems, it is required to find the largest and smallest values of a function of two variables where the variables are connected by some given relation or condition known as a constraint. For example, if we plot the function $z = f(x, y)$ to look like a mountain range, then the mountain tops or the high points are called local maxima of $f(x, y)$ and valley bottoms or the low points are called local minima of $f(x, y)$. The highest mountain and lowest valley in the entire range are said to be absolute maximum and absolute minimum. The graphical representation is as follows.



Definition

Let $f(x, y)$ be a function of two independent variables x, y such that it is continuous and finite for all values of x and y in the neighbourhood of their values a and b (say) respectively.

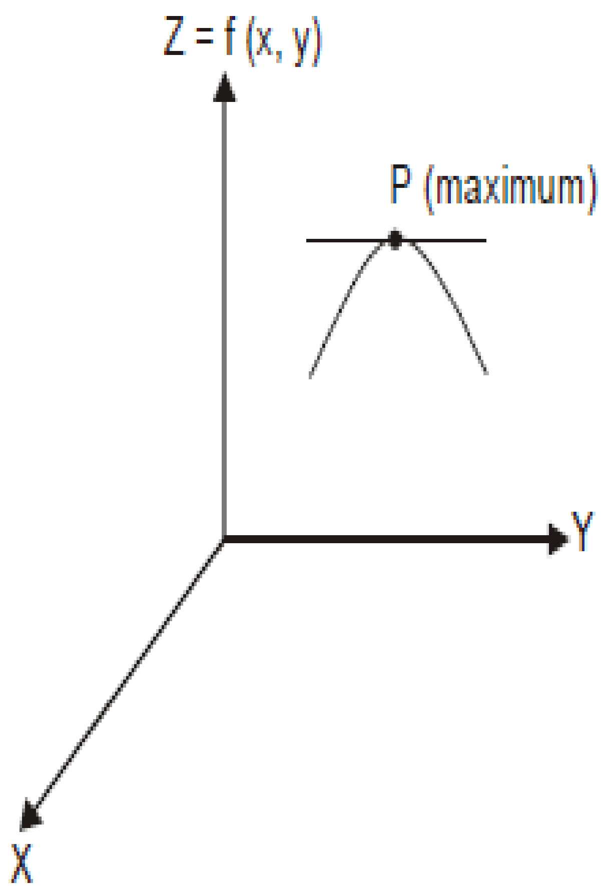
Maximum value: $f(a, b)$ is called maximum value of $f(x, y)$ if $f(a, b) > f(a + h, b + k)$. For small positive or negative values of h and k *i.e.*, $f(a, b)$ is greater than the value of function $f(x, y,)$ at all points in some small *nbd* of (a, b).

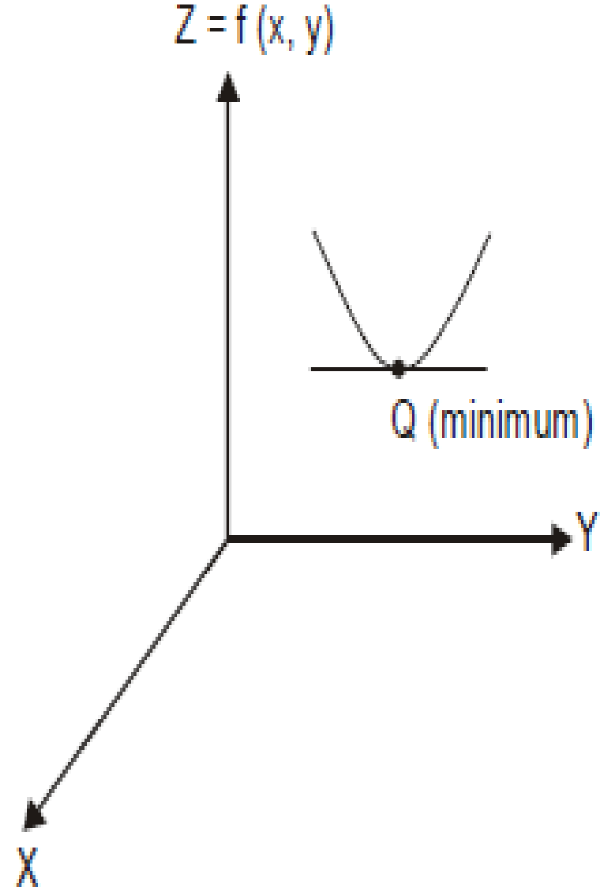
Minimum value: $f(a, b)$ is called minimum value of $f(x, y)$ if $f(a, b) < f(a + h, b + k)$.

Note: $f(a + h, b + k) - f(a, b)$ = positive, for Minimum value.
$f(a + h, b + k) - f(a, b)$ = negative, for Maximum value.

Extremum: The maximum or minimum value of the function $f(x, y)$ at any point $x = a$ and $y = b$ is called the extremum value and the point is called "extremum point".

Geometrical representation of maxima and minima: The function $f(x, y)$ represents a surface. The maximum is a point on the surface (hill top). The minimum is a point on the surface (bottom) from which the surface ascends (climbs up) in every direction.





Saddle point: It is a point where function is neither maximum nor minimum. At such point f is maximum in one direction while minimum in another direction.

Condition for the Existence of Maxima and Minima (Extrema)

By Taylor's theorem

$$f(a+h, b+k) = f(a, b) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{(a,b)} + \frac{1}{\underline{|2}}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)} + \ldots \quad \ldots(i)$$

Neglecting higher order terms of h^2, hk, k^2, etc. Since h, k are small, the above expansion reduce to

$$f(a+h, b+k) = f(a, b) + h\frac{\partial f(a,b)}{\partial x} + k\frac{\partial f(a,b)}{\partial y}$$

$$\Rightarrow f(a+h, b+k) - f(a, b) = h\frac{\partial f(a,b)}{\partial x} + k\frac{\partial f(a,b)}{\partial y} \quad \ldots(ii)$$

The necessary condition for a maximum or minimum value (L.H.S. of eqn (*ii*) negative or positive) is

$$h\,\frac{\partial f(a,b)}{\partial x} + k\,\frac{\partial f(a,b)}{\partial y} = 0$$

$$\Rightarrow \qquad \frac{\partial f(a,b)}{\partial x} = 0,\ \frac{\partial f(a,b)}{\partial y} = 0 \ \Big|\, h \text{ and } k \text{ can take both } +\text{ve and } -\text{ve value} \quad \ldots(iii)$$

The conditions (*iii*) are necessary conditions for a maxmium or a minimum value of $f(x, y)$.

Note: The conditions given by (*iii*) are not sufficient for existence of a maximum or a minimum value of $f(x, y)$.

Lagrange's Conditions for Maximum or Minimum (Extrema)

Using the conditions (*iii*) in equation (*i*) neglecting the higher order term $h^3, k^3, h^2 k$ etc. we get

$$f(a+h, b+k) - f(a, b) = \frac{1}{\underline{\lfloor 2}}\left[h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right]_{(a,b)}$$

Putting $\dfrac{\partial^2 f}{\partial x^2} = r, \; \dfrac{\partial^2 f}{\partial x \partial y} = s, \; \dfrac{\partial^2 f}{\partial y^2} = t$, then

$$f(a+h, b+k) - f(a, b) = \frac{1}{2}\left[h^2 r + 2hks + k^2 t\right]$$

$$= \frac{1}{2}\left[\frac{h^2 r^2 + 2hkrs + k^2 tr}{r}\right]$$

$$f(a+h, b+k) - f(a, b) = \frac{1}{2}\left[\frac{(hr+ks)^2 + k^2\left(rt - s^2\right)}{r}\right] \quad \ldots(iv)$$

If $rt - s^2 > 0$ then the numerator in R.H.S. of (iv) is positive. Here sign of L.H.S. = sign of r.

Thus, if $rt - s^2 > 0$ and $r < 0$, then $f(a + h, b + k) - f(a, b) < 0$

if $rt - s^2 > 0$ and $r > 0$, then $f(a + h, b + k) - f(a, b) > 0$.

Therefore, **the Lagrange's conditions for maximum or minimum are:**

1. If $rt - s^2 > 0$ and $r < 0$, then $f(x, y)$ has maximum value at (a, b).
2. If $rt - s^2 > 0$ and $r > 0$, then $f(x, y)$ has minimum value at (a, b).
3. If $rt - s^2 < 0$, then $f(x, y)$ has neither a maximum nor minimum *i.e.*, (a, b) is saddle point.
4. If $rt - s^2 = 0$, then case fail and here again investigate more for the nature of function.

Method of Finding Maxima or Minima

1. Solve $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, for the values of x and y. Let $x = a$, $y = b$.

   The point $P(a, b)$ is called critical or stationary point.

2. Find r, s and t at $x = a$, $y = b$.

3. Now check the following conditions:
   (*i*) If $rt - s^2 > 0$ and $r < 0$, $f(x, y)$ has maximum at $x = a$, $y = b$.
   (*ii*) If $rt - s^2 > 0$ and $r > 0$, $f(x, y)$ has minimum at $x = a$, $y = b$.
   (*iii*) If $rt - s^2 < 0$, $f(x, y)$ has neither maximum nor minimum.
   (*iv*) If $rt - s^2 = 0$, case fail.

Example

Discuss the maximum or minimum values of u when $u = x^3 + y^3 - 3axy$.

Sol. $\dfrac{\partial u}{\partial x} = 3x^2 - 3ay$; $\dfrac{\partial u}{\partial y} = 3y^2 - 3ax$; $r = \dfrac{\partial^2 u}{\partial x^2} = 6x$;

$$s = \frac{\partial^2 u}{\partial x \partial y} = -3a, \; t = \frac{\partial^2 u}{\partial y^2} = 6y.$$

Now for maximum or minimum, we must have $\dfrac{\partial u}{\partial x} = 0, \dfrac{\partial u}{\partial y} = 0$

So from $\dfrac{\partial u}{\partial x} = 0$, we get $x^2 - ay = 0$ ...(*i*)

and from $\dfrac{\partial u}{\partial y} = 0$, we get $y^2 - ax = 0$ ...(*ii*)

Solving (*i*) and (*ii*), we get $(y^2/a)^2 - ay = 0$

or $y^4 - a^3y = 0$ or $y(y^3 - a^3) = 0$ or $y = 0, a$.

Now from (*i*), we have when $y = 0$, $x = 0$, and when $y = a$, $x = \pm a$.

But $x = -a$, $y = a$, do not satisfy (*ii*), here are not solutions.

Hence the solutions are $x = 0$, $y = 0$; $x = a$, $y = a$;

At $x = 0$, $y = 0$, we have $r = 0$, $s = -3a$, $t = 0$.

$\therefore$ $rt - s^2 = 0 - (-3a)^2$ = negative and there is neither maximum nor minimum at $x = 0$, $y = 0$.

At $x = a$, $y = a$, we get $r = 6a$, $s = -3a$, $t = 6a$

$\therefore$ $rt - s^2 = (6a)(6a) - (-3a)^2 = 36a^2 - 9a^2 > 0$

Also $r = 6a > 0$ if $a > 0$ and $r < 0$ if $a < 0$.

Hence there is maximum or minimum according as $a < 0$ or $a > 0$. The maximum or minimum value of $u = -a^3$ according as $a < 0$ or $a > 0$.

Example

$$f(x, y) = 2 - x^2 - xy - y^2$$

For this function

$$f_x = -2x - y$$
$$f_y = -x - 2y$$
$$f_{xx} = -2$$
$$f_{yy} = -2$$
$$f_{xy} = -1$$

For stationary points, $-2x - y = 0$ and $-x - 2y = 0$ so again the only possibility is $(x, y) = (0, 0)$. We have

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - (-1)^2 = 3 > 0$$

so that $(0, 0)$ is either a max or a min. Since $f_{xx} < 0$ and $f_{yy} < 0$ it is a maximum.

Example

$$f(x,y) = 2x^3 + 6xy^2 - 3y^3 - 150x$$

The function in this example has four stationary points.

The first and second order partial derivatives of this function are

$$\begin{aligned} f_x &= 6x^2 + 6y^2 - 150 \\ f_y &= 12xy - 9y^2 \\ f_{xx} &= 12x \\ f_{yy} &= 12x - 18y \\ f_{xy} &= 12y \end{aligned}$$

For stationary points we need

$$6x^2 + 6y^2 - 150 = 0 \quad \text{and} \quad 12xy - 9y^2 = 0$$

i.e.

$$x^2 + y^2 = 25 \quad \text{and} \quad y(4x - 3y) = 0$$

The second of these equation s implies either that $y = 0$ or that $4x = 3y$ and both of these possibilities now need to be considered. If $y = 0$ then the first equation implies that $x^2 = 25$ so that $x = \pm 5$ giving $(5, 0)$ and $(-5, 0)$ as stationary points.
If $4x = 3y$ then $x = \frac{3}{4}y$ and so the first equation becomes

$$\frac{9}{16}y^2 + y^2 = 25$$

so that $y = \pm 4$. $y = 4$ gives $x = 3$ and $y = -4$ gives $x = -3$, so we have two further stationary points $(3, 4)$ and $(-3, -4)$.
Thus in total there are four stationary points $(5, 0)$, $(-5, 0)$, $(3, 4)$ and $(-3, -4)$.
Each of these must now be classified into max, min or saddle.

- Lets start with $(5,0)$. For this stationary point, $f_{xx}f_{yy} - f_{xy}^2 = 60^2 > 0$ so it is either a max or a min. But $f_{xx} = 60 > 0$ and $f_{yy} = 60 > 0$. Hence $(5,0)$ is a minimum.

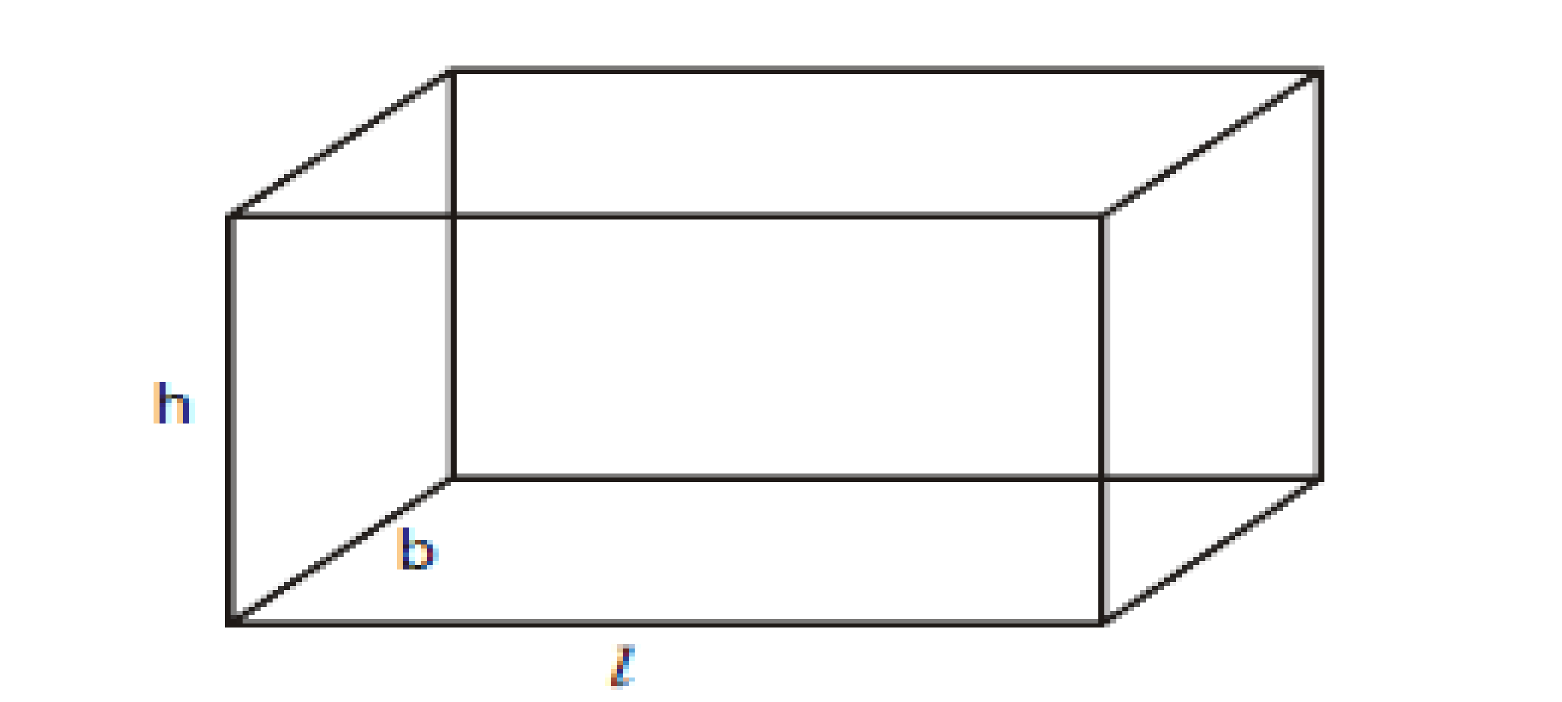
- Now deal with $(-5,0)$. For this stationary point, $f_{xx}f_{yy} - f_{xy}^2 = (-60)^2 > 0$ so it is either a max or a min. But $f_{xx} = -60 < 0$ and $f_{yy} = -60 < 0$. Hence $(-5,0)$ is a maximum.

- Now deal with $(3,4)$. For this stationary point, $f_{xx}f_{yy} - f_{xy}^2 = -3600 < 0$ so $(3,4)$ is a saddle.

- Now deal with $(-3,-4)$. For this stationary point, $f_{xx}f_{yy} - f_{xy}^2 = -3600 < 0$ so $(-3,-4)$ is a saddle.

Example

A rectangular box, open at the top, is to have a volume of 32 c.c. Find the dimensions of the box requiring least material for its construction.



Sol. $V = 32$ c.c.

Let length $= l$, breadth $= b$ and height $= h$

Total surface area $S = 2lh + 2bh + lb$ ...(*i*)

$$S = 2(l + b)h + lb$$

Now volume $V = lbh = 32 \Rightarrow b = \dfrac{32}{lh}$ ...(*ii*)

Putting the value of '*b*' in equation (*i*)

$$S = 2\left(l + \frac{32}{lh}\right)h + l\left(\frac{32}{lh}\right)$$

$$S = 2lh + \frac{64}{l} + \frac{32}{h}$$

$$\therefore \quad \frac{\partial S}{\partial l} = 2h - \frac{64}{l^2}, \quad \frac{\partial S}{\partial h} = 2l - \frac{32}{h^2}$$

For minimum S, we get

$$\frac{\partial S}{\partial l} = 0 \Rightarrow 2h - \frac{64}{l^2} = 0 \Rightarrow h = \frac{32}{l^2} \qquad ...(iv)$$

and $$\frac{\partial S}{\partial h} = 0 \Rightarrow 2l - \frac{32}{h^2} = 0 \Rightarrow l = \frac{16}{h^2} \qquad ...(v)$$

From (iv) and (v), we get

$$h = \frac{32 \times h^4}{256} \Rightarrow h^3 = 8 \Rightarrow h = 2$$

Putting $h = 2$, in equation (v), we get $l = \dfrac{16}{4} = 4$

From (ii) $$b = \frac{32}{4 \times 2} = 4$$

Now, $$\frac{\partial^2 S}{\partial l^2} = \frac{128}{l^3} = \frac{128}{64} = 2 \Rightarrow r = 2 > 0$$

and $\dfrac{\partial^2 S}{\partial l \partial h} = 2 \Rightarrow s = 2$ and $\dfrac{\partial^2 S}{\partial h^2} = \dfrac{64}{h^3} = \dfrac{64}{8} = 8 \Rightarrow t = 8$

$$\therefore \quad rt - s^2 = 2 \times 8 - 4 = 12 > 0$$

$\Rightarrow \ rt - s^2 > 0$ and $r > 0$

Hence, S is minimum, for least material

$$l = 4,\ b = 4,\ h = 2.$$

Example

Find the maximum and minimum values of the function

$$z = \sin x \sin y \sin (x + y)$$

Sol. Given $z = \sin x \sin y \sin (x + y)$

$$= \frac{1}{2} [2 \sin x \sin y] \sin (x + y)$$

$$= \frac{1}{2} [\cos (x - y) - \cos (x + y)] \sin (x + y)$$

$$= \frac{1}{4} [2 \sin (x + y) \cos (x - y) - 2 \sin (x + y) \cos (x + y)]$$

or $$z = \frac{1}{4} [\sin 2x + \sin 2y - \sin(2x + 2y)]$$

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{1}{2}[\cos 2x - \cos(2x + 2y)]$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}[\cos 2y - \cos(2x + 2y)]$$

$$r = \frac{\partial^2 z}{\partial x^2} = -\sin 2x + \sin(2x + 2y) \qquad \text{...}(A)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \sin(2x + 2y) \qquad \text{...}(B)$$

$$t = \frac{\partial^2 z}{\partial y^2} = -\sin 2y + \sin(2x + 2y) \qquad \text{...}(C)$$

For maximum or minimum, we must have $\frac{\partial z}{\partial x} = 0, \ \frac{\partial z}{\partial y} = 0$

From $\dfrac{\partial z}{\partial x} = 0$, we get $\cos 2x - \cos(2x + 2y) = 0$ ...(*i*)

From $\dfrac{\partial z}{\partial y} = 0$, we get $\cos 2y - \cos(2x + 2y) = 0$ ...(*ii*)

Solving (*i*) and (*ii*), we get $\cos 2x = \cos 2y$ which gives

$$2x = 2n\pi \pm 2y. \text{ In particular } 2x = 2y \text{ or } x = y$$

When $x = y$, from (*i*), we get $\cos 2x - \cos 4x = 0$

$$\cos 2x - (2\cos^2 2x - 1) = 0 \quad \because \cos 2\theta = 2\cos^2\theta - 1$$

$$2\cos^2 2x - \cos 2x - 1 = 0$$

or $\quad \cos 2x = \dfrac{1 \pm \sqrt{(1+8)}}{4} = \dfrac{1 \pm 3}{4} = 1, -\dfrac{1}{2}$

or $\quad 2x = 2n\pi \pm 0,\ 2m\pi \pm \dfrac{2\pi}{3}$, where m, n are zero or any integers

or $\quad x = n\pi,\ m\pi \pm \dfrac{\pi}{3}$

In particular $x = \dfrac{\pi}{3}$

When $x = \dfrac{\pi}{3}$, we have $y = x = \dfrac{\pi}{3}$

and then $r = -\sin \dfrac{2\pi}{3} + \sin \dfrac{4\pi}{3}$, from (A)

$$= -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}\ ;$$

$$s = \sin \frac{4\pi}{3}, \text{ from } (B)$$

or $$s = -\frac{\sqrt{3}}{2}$$

and $$t = -\sin \frac{2\pi}{3} + \sin \frac{4\pi}{3}, \text{ from (C)}$$

$$= -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$$

$$\therefore \qquad rt - s^2 = \left(-\sqrt{3}\right)\left(-\sqrt{3}\right) - \left(-\frac{\sqrt{3}}{2}\right)^2 = 3 - \frac{3}{4} = \frac{9}{4}$$

$$= \text{positive.}$$

Thus at $x = \dfrac{\pi}{3} = y$, $rt - s^2 > 0$, $r < 0$, so there is a maximum at $x = \dfrac{\pi}{3} = y$.

Hence, maximum value $= \sin \dfrac{\pi}{3} \cdot \sin \dfrac{\pi}{3} \cdot \sin \left(\dfrac{\pi}{3} + \dfrac{\pi}{3}\right)$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}.$$

If we take $\quad x = -\dfrac{\pi}{3}$, then $y = x = -\dfrac{\pi}{3}$

$$r = \sqrt{3}, \quad s = \frac{1}{2}\sqrt{3}, \; t = \sqrt{3}$$

$$\therefore \qquad rt - s^2 = \frac{9}{4} > 0 , \; r > 0$$

There is a minimum at $x = -\dfrac{\pi}{3} = y$.

Hence the minimum value $= \sin \left(-\dfrac{\pi}{3}\right) \sin \left(-\dfrac{\pi}{3}\right) \sin \left\{\left(-\dfrac{\pi}{3}\right) + \left(-\dfrac{\pi}{3}\right)\right\}$

$$= -\sin\frac{\pi}{3}\cdot\sin\frac{\pi}{3}\;\sin\frac{2\pi}{3}$$

$$= -\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{8}.$$

LAGRANGE'S* METHOD OF UNDETERMINED MULTIPLIERS

Let $\phi\,(x, y, z)$ is a function of three independent variables, where x, y, z are related by a known constraint $g(x, y, z) = 0$

Thus the problem is Extrema of

$$u = f(x, y, z) \qquad \text{...}(i)$$

Subject to $\qquad g\,(x, y, z) = 0 \qquad \text{...}(ii)$

For stationary point $\quad \dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = \dfrac{\partial f}{\partial z} = 0$

$$\therefore \qquad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \qquad \text{...}(iii)$$

From (ii)

$$dg = \frac{\partial g}{\partial x}\,dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}\,dz = 0 \qquad \text{...}(iv)$$

Multiplying eqn. (iv) by λ and adding to (iii), we obtain

$$\left(\frac{\partial f}{\partial x}+\lambda\frac{\partial g}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}+\lambda\frac{\partial g}{\partial y}\right)dy+\left(\frac{\partial f}{\partial z}+\lambda\frac{\partial g}{\partial z}\right)dz = 0 \qquad \text{...}(v)$$

Since x, y, z are independent variables

$$\therefore \qquad \frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \qquad \ldots(vi)$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \qquad \ldots(vii)$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial g}{\partial z} = 0 \qquad \ldots(viii)$$

On solving (*ii*), (*vi*), (*vii*) and (*viii*), we can find x, y, z and λ for which $f(x, y, z)$ has maximum or minimum.

Example

Determine the maxima and minima of $x^2 + y^2 + z^2$ when $ax^2 + by^2 + cz^2 = 1$.

Sol. Let $f(x, y, z) = x^2 + y^2 + z^2$ ...(*i*)

and $g(x, y, z) \equiv ax^2 + by^2 + cz^2 - 1 = 0$ ...(*ii*)

From (*i*) $$\frac{\partial f}{\partial x} = 2x,\ \frac{\partial f}{\partial y} = 2y,\ \frac{\partial f}{\partial z} = 2z$$

From (*ii*) $$\frac{\partial g}{\partial x} = 2ax,\ \frac{\partial g}{\partial y} = 2by,\ \frac{\partial g}{\partial z} = 2cz.$$

Now from Lagrange's equations, we get

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x} = 2x + \lambda \,.\, 2ax = 0 \Rightarrow 2x\,(1 + \lambda a) = 0 \Rightarrow x\,(1 + \lambda a) = 0 \quad ...(iii)$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y} = 0 \Rightarrow 2y + \lambda \cdot 2by = 0 \Rightarrow 2y\,(1 + \lambda b) = 0 \Rightarrow y\,(1 + \lambda b) = 0 \quad ...(iv)$$

and $$\frac{\partial f}{\partial z} + \lambda\frac{\partial g}{\partial z} = 0 \Rightarrow 2z + \lambda \cdot 2cz = 0 \Rightarrow 2z\,(1 + \lambda c) = 0 \Rightarrow z\,(1 + \lambda b) = 0 \quad ...(v)$$

Multiplying these equations by x, y, z respectively and adding, we get

$$x^2(1 + \lambda a) + y^2(1 + \lambda b) + z^2(1 + \lambda c) = 0$$

or $$(x^2 + y^2 + z^2) + \lambda(ax^2 + by^2 + cz^2) = 0 \quad ...(vi)$$

Using (i) and (ii) in above equation, we get

$$f + \lambda = 0 \Rightarrow \lambda = -f$$

Putting $\lambda = -f$ in equations (iii), (iv) and (v), we get

$$x(1 - fa) = 0, y(1 - fb) = 0, z(1 - fc) = 0$$

$\Rightarrow$ $$1 - fa = 0, 1 - fb = 0, 1 - fc = 0$$

i.e., $f = \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$. These give the max. and min. values of f.

Example

Find the extreme value of $x^2 + y^2 + z^2$, given that $ax + by + cz = p$.

Sol. Let $$u = x^2 + y^2 + z^2$$

Given $$ax + by + cz = p.$$

For max. or min. from (*i*), we have

$$du = 2x\,dx + 2y\,dy + 2z\,dz = 0. \qquad ...(iii)$$

Also from (*ii*), $$a\,dx + b\,dy + c\,dz = 0. \qquad ...(iv)$$

Multiplying (*iv*) by λ and adding in (*iii*), we get $(x\,dx + y\,dy + z\,dz) + \lambda\,(a\,dx + b\,dy + c\,dz) = 0$.

Equating the coefficients of dx, dy and dz to zero, we get

$$x + \lambda a = 0,\ y + \lambda b = 0,\ z + \lambda c = 0 \qquad ...(v)$$

These are Lagrange's equations.

Multiplying these by x, y, z respectively and adding, we get

$$x\,(x + \lambda a) + y\,(y + \lambda b) + z\,(z + \lambda c) = 0$$

or $$(x^2 + y^2 + z^2) + \lambda\,(ax + by + cz) = 0$$

or $$u + \lambda p = 0 \text{ or } \lambda = -u/p.$$

$\therefore$ From (*v*), we get

$$x - \left(\frac{au}{p}\right) = 0, \quad y - \left(\frac{bu}{p}\right) = 0, \quad z - \left(\frac{cu}{p}\right) = 0$$

or $$\frac{x}{a} = \frac{u}{p} = \frac{y}{b} = \frac{z}{c} \quad \text{or} \quad \frac{x}{a} = \frac{y}{b} = \frac{z}{c} \qquad \ldots(vi)$$

From (*ii*), we get $a^2\left(\frac{x}{a}\right) + b^2\left(\frac{y}{b}\right) + c^2\left(\frac{z}{c}\right) = p$

or $$a^2\left(\frac{x}{a}\right) + b^2\left(\frac{x}{a}\right) + c^2\left(\frac{x}{a}\right) = p, \text{ from } (vi)$$

or $$(a^2 + b^2 + c^2)\left(\frac{x}{a}\right) = p \quad \text{or} \quad x = \frac{ap}{a^2 + b^2 + c^2}$$

Similarly, $$y = \frac{bp}{a^2 + b^2 + c^2}, \quad z = \frac{cp}{a^2 + b^2 + c^2}$$

These give the minimum value of u.

Hence minimum value of u is

$$u = \frac{a^2p^2}{(a^2 + b^2 + c^2)^2} + \frac{b^2p^2}{(a^2 + b^2 + c^2)^2} + \frac{c^2p^2}{(a^2 + b^2 + c^2)^2}$$

$$= \frac{(a^2 + b^2 + c^2)p^2}{(a^2 + b^2 + c^2)^2} = \frac{p^2}{(a^2 + b^2 + c^2)}.$$

Example 1.1. Find two non-negative numbers whose sum is 9 and the product of one number and the square of the other number is a maximum.

Solution. Let x and y represent two non-negative numbers. We wish to maximize the product $f(x,y)=xy^2$, subject to the condition that

$$g(x,y)=0, \tag{1.3}$$

where $g(x,y)=x+y-9$. From the relations (1.2), we get $y^2=\lambda$, $2xy=\lambda$. These imply that $y^2=2xy$ or $y(y-2x)=0$ so that $y=2x$, since both x and y are not zero. Substituting $y=2x$ in (1.3), we get $x=3$, $y=6$. Thus the critical point is $P(3,6)$. The maximum value of f is $f(P)=3(6)^2=108$.

Exercise 1.1. Find the extreme values of $f(x,y) = xy$ on (a) the unit circle, and (b) the ellipse $x^2 + 2y^2 = 1$.

Exercise 1.2. Find the points on the curve $xy^2 = 54$, nearest to the origin.

Hint: Minimize the squared distance function $f(x,y) = x^2 + y^2$ subject to the side condition $\underbrace{xy^2 - 4}_{g(x,y)} = 0$

Example 1.2. A rectangular box with square base, open at the top, is to be made from 48 square feet of material. What dimensions will result in a box with the largest possible volume?

Solution. Let x be the side of the square base, and y be the height of the box. Since the box is open at the top, the surface area of the box equals the area of base plus four times the area of one side plane, that is $x^2 + 4xy$. We wish to maximize the volume $f(x, y) = x^2 y$, subject to the condition:

$$\underbrace{x^2 + 4xy - 48}_{g(x,y)} = 0, \tag{1.4}$$

The Lagrangian relations (1.2) reduce to $2xy = \lambda(2x + 4y)$, $x^2 = 4\lambda x$. That is

$$xy = \lambda(x + 2y) \tag{1.5}$$

$$x = 4\lambda. \tag{1.6}$$

Dividing (1.6) with (1.5), we get $\frac{xy}{x+2y} = \frac{x}{4}$ or $x = 2y$. Substituting this in (1.4), we get $(2y)^2 + 4(2y)(y) = 48$ or $y = 2$ so that $x = 4$. Thus the critical point is $P(4, 2)$. The maximum volume is $f(P) = (4)^2(2) = 32$ cubic feet.

Example 1.3. A right circular cylindrical container with open top has surface area 3π square feet. What height h and base radius r will maximize its volume?

Solution. We wish to maximize the volume $f(r,h) = \pi r^2 h$ of the container, subject to the condition:

$$\underbrace{\pi r^2 + 2\pi r h - 3\pi}_{g(r,h)} = 0, \tag{1.7}$$

The Lagrangian relations $\frac{\partial f}{\partial r} = \lambda \frac{\partial g}{\partial r}$, $\frac{\partial f}{\partial h} = \lambda \frac{\partial g}{\partial h}$ reduce to $2\pi r h = \lambda(2\pi r + 2\pi h)$ and $\pi r^2 = \lambda 2\pi r$ or

$$r = \lambda(r + h) \tag{1.8}$$

$$r = 2\lambda. \tag{1.9}$$

Solving (1.8) and (1.9), we get $r + h = 2$ or $h = 2 - r$. Substituting this in (1.7), we get $\pi r^2 + 2\pi r(2 - r) = 3\pi$ or $r^2 - 4r + 3 = 0$. The two roots are $r = 1, 3$. If $r = 3$, we see that $h = 2 - r = -1$, which is not possible. Therefore, we take $r = 1$, and hence $h = 1$. Thus the largest possible volume of the cylinder is $f(1,1) = \pi(1)^2(1) = \pi$ cubic feet.

Exercise 1.3. A closed right circular cylinder will have a volume of 1000 cubic feet. The top and the bottom of the cylinder are made of metal that costs 2 dollars per square foot. The lateral side is wrapped in metal costing 2.50 dollars per square foot. Find the minimum cost of construction.

Exercise 1.4. Find the dimensions of the closed right circular cylindrical can of the least surface area containing the volume 16π cubic cm.

Exercise 1.5. If a cylinder is formed by revolving a rectangle of perimeter 12 inches about one of its edges, what dimensions of the rectangle will result in the cylinder of maximum volume?

Example 1.4. Consider all triangles formed by lines passing through the point $(8/9, 3)$ and both the x and y axes. Find the dimensions of the triangle with the shortest hypotenuse.

Solution. Let a and b be the x and y-intercepts of the hypotenuse respectively. We maximize the square of the length $H(a,b) = a^2 + b^2$, when the hypotenuse passes through the point $(8/9, 3)$, that is when

$$\underbrace{\frac{8}{9a} + \frac{3}{b} - 1}_{g(a,b)} = 0. \tag{1.10}$$

Now, the Lagrangian relations $\frac{\partial f}{\partial a} = \lambda \frac{\partial g}{\partial a}$, $\frac{\partial f}{\partial b} = \lambda \frac{\partial g}{\partial b}$ reduce to

$$2a = -\frac{8\lambda}{9a^2} \text{ and } 2b = -\frac{3\lambda}{b^2}.$$

Eliminating λ from these, and simplifying, we get $a = 2b/3$. Substituting this in (1.10), we get $\frac{8}{9} \cdot \frac{3}{2b} + \frac{3}{b} = 1$ or $y = 13/3$ and hence $x = 26/9$. Thus the minimum length of the hypotenuse is $f(a,b) = f(13/3, 26/9) = \sqrt{a^2 + b^2} = 13\sqrt{13}/9$ units.

Exercise 1.6. Find the length of the shortest ladder that will reach over an 8 feet high fence to a large wall which is 3 feet behind the fence.

Exercise 1.7. Find the maximum area of a right angled triangle whose perimeter is 4 units.

Example 1.5. Find a rectangle of largest area, that can be inscribed in the closed region bounded by the x-axis, y-axis, and graph of $y = 8 - x^3$.

Solution. Let R be the largest rectangle, which can be best fit in the region A enclosed by the coordinate axes and the curve C: $y = 8 - x^3$. Then one corner (x, y) of R will lie on the curve C. The problem is to maximize the area function $f(x, y) = xy$ such that

$$\underbrace{x^3 + y - 8}_{g(r,h)} = 0. \tag{1.11}$$

The Lagrangian relations $\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$, $\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$ reduce to

$$y = \lambda(3x^2) \text{ and } x = \lambda\cdot$$

Eliminating λ from these, we obtain $y = 3x^3$. Substituting this in (1.11), we get $3x^3 = 8 - x^3$ or $x = 2^{1/3}$ and hence $y = 8 - 2 = 6$. Thus the critical point of f is $P(2^{1/3}, 6)$. The maximum volume of the cone is $f(P) = xy = 6 \cdot 2^{2/3}$ square units.

Example 1.6. Find the dimensions (radius r and height h) of the cone of maximum volume, which can be inscribed in a sphere of radius 2.

Solution. Let C be a cone of maximum volume, which can be inscribed in a sphere S of radius 2. Then the vertex of the cone lies on the surface of S, and the height of the cone is along the radius of S. If r is the radius of the cross-section of the cone, and h is the length of perpendicular from the centre O of S onto it, then from the geometry, we find that

$$r^2 + h^2 = 2^2 \text{ or } \underbrace{r^2 + h^2 - 4}_{g(r,h)} = 0. \tag{1.12}$$

We maximize the volume function $f(r,h) = \pi r^2(h+2)/3$ subject to the condition (1.12). Now, the Lagrangian relations $\frac{\partial f}{\partial r} = \lambda \frac{\partial g}{\partial r}$, $\frac{\partial f}{\partial h} = \lambda \frac{\partial g}{\partial h}$ reduce to

$$\frac{2\pi r(h+2)}{3} = \lambda(2r) \text{ and } \frac{\pi r^2}{3} = \lambda(2h).$$

Eliminating λ from these, simplifying, and using (1.12), we get $2h^2 + 4h = r^2 = 4 - h^2$ or $3h^2 + 4h - 4 = 0$. The two roots are $h = -2, 2/3$. Discarding the negative value, $h = 2/3$ so that $r^2 = 4 - 4/9 = 32/9$. Thus the critical point of f is $P(4\sqrt{2}/3, 2/3)$. The maximum volume of the cone is $f(P) = \pi r^2(h+2)/3 = \pi(32/9)(2+2/3) = 256/27$ cubic units.

Exercise 1.8. The sum of the length and girth of a container of square cross section is a inches. Find its maximum volume.