The Laplace Transform

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What do you mean by the Laplace Transform?

Let $f : [0, \infty) \longrightarrow \mathbb{R}$. The Laplace transform of $f(t)$ is given by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \hat{F}(s), \tag{1.1}$$

where s is a *parameter* (real or complex).

- For the improper integral (1.1) to have finite value, $s > 0$ if s is real, or the real part of s must be positive if s is complex

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1. The Laplace transform converts a function of t to a function of s
2. $\mathscr{L}\{f(t)\}$ exists, provided $f(t)$ is piecewise continuous over every finite interval, and is of exponential order as $t \longrightarrow \infty$, that is $\lim_{t\to\infty} f(t)e^{-kt} = 0$ for some $k > 0$
3. $f(t)$ is piecewise continuous on $(0, \infty)$, and is of exponential order as $t \longrightarrow \infty$, with $\mathscr{L}\{f(t)\} = \hat{F}(s)$ then $\lim_{s\to\infty} \hat{F}(s) = 0$

Laplace Transform of e^{at} where $a \in \mathbb{R}$

The Laplace transform of e^{at} is given by

$$\mathscr{L}\left\{e^{at}\right\} = \int_0^\infty e^{at}.e^{-st}dt = \int_0^\infty e^{-(s-a)t}dt = \frac{1}{s-a},$$

provided $s > a$

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1. For $s > a$, $e^{-(s-a)t} \longrightarrow 0$ as $t \longrightarrow \infty$ so that the integral converges to $\frac{1}{s-a}$
2. $\mathscr{L}\{1\} = \int_0^\infty 1.e^{-st}dt = \frac{1}{s}$
3. Does $f(t) = e^{t^2}$ has the exponential order?

The Laplace Transform of Power Function

Let p be a real number with $p > -1$, and $s > 0$. Then

$$\mathscr{L}\left\{t^p\right\} = \int_0^\infty t^p . e^{-st} dt = \frac{\Gamma(p+1)}{s^{p+1}}$$

In particular, if p is a positive integer, then $\Gamma(n+1) = n!$ for $n = 1, 2, 3, ...$
Therefore,

$$\mathscr{L}\left\{t^n\right\} = \int_0^\infty t^n e^{-st} dt = \frac{n!}{s^{n+1}} \text{ for } n = 1, 2, 3, \ldots$$

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1. The gamma function is defined by $\Gamma(r) = \int_0^\infty t^{r-1}.e^{-t} dt$ for $r > 0$, $\Gamma(1/2) = \sqrt{\pi}$
2. Can you give a discontinuous function for which the Laplace transform exists?

The Laplace Transform of Ramp Function $f(t) = t$

$$\mathscr{L}\left\{t\right\} = \int_0^\infty te^{-st}dt = \left|(t)\left\{-\frac{e^{-st}}{s}\right\} - (1)\left\{\frac{e^{-st}}{s^2}\right\}\right|_{t=0}^{\infty} = \frac{1}{s^2}$$

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1 What is the Matlab command to find $\mathscr{L}\left\{t\right\}$?

The Laplace Transform of Higher powers of t

1. $\mathscr{L}\{t\} = \int_0^\infty t e^{-st} dt = \frac{1}{s^2}$
2. $\mathscr{L}\left\{t^2\right\} = \int_0^\infty t^2 e^{-st} dt = \frac{2}{s^3}$
3. $\mathscr{L}\left\{t^{1/2}\right\} = \int_0^\infty t^{1/2} e^{-st} dt = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$
4. $\mathscr{L}\left\{t^{3/2}\right\} = \int_0^\infty t^{3/2} e^{-st} dt = \frac{\Gamma(5/2)}{s^{5/2}} = \frac{\frac{3}{2}\cdot\Gamma(3/2)}{s^{5/2}} = \frac{3\sqrt{\pi}}{4s^{5/2}}$

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1. $\Gamma(p+1) = p\Gamma(p)$: $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}/2$

Laplace Transform of $\sin at$ and $\cos at$

The Laplace transforms of $\sin at$ and $\cos at$ are given by

$$\mathscr{L}\{\sin at\} = \int_0^\infty (\sin at)e^{-st}\,dt = \frac{a}{s^2+a^2}$$

$$\mathscr{L}\{\cos at\} = \int_0^\infty (\cos at)e^{-st}\,dt = \frac{s}{s^2+a^2}$$

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1. $\int_0^\infty (\sin At)e^{Bt}\,dt = \frac{A}{A^2+B^2}$, $\int_0^\infty (\cos At)e^{Bt}\,dt = \frac{B}{A^2+B^2}$
2. What is the choice of s?

Linearity or Superposition of $\mathscr{L}$

Let $\mathscr{L}\{f(t)\} = \hat{F}(S)$, $\mathscr{L}\{g(t)\} = \hat{G}(S)$. If a and b are real numbers, not both zero, then

$$\mathscr{L}\{af(t) + bg(t)\} = a\hat{F}(s) + b\hat{G}(s). \tag{3.1}$$

That is, the Laplace transform of linear combination of $f(t)$ and $g(t)$ equals the linear combination of their transforms $\hat{F}(s)$ and $\hat{G}(s)$

Laplace Transform of $\sinh at$ and $\cosh at$

As an immediate consequence of the linearity of the operator $\mathscr{L}$, the Laplace transforms of $\sinh at$ and $\cosh at$ are given by

$$\mathscr{L}\{\sinh at\} = \mathscr{L}\left\{\frac{e^{at}-e^{-at}}{2}\right\} = \frac{1}{2}\left[\mathscr{L}\left\{e^{at}\right\} - \mathscr{L}\left\{e^{-at}\right\}\right] = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2-a^2},$$

$$\mathscr{L}\{\cosh at\} = \mathscr{L}\left\{\frac{e^{at}+e^{-at}}{2}\right\} = \frac{1}{2}\left[\mathscr{L}\left\{e^{at}\right\} + \mathscr{L}\left\{e^{-at}\right\}\right] = \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{s}{s^2-a^2}$$

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1. $\sinh\theta = \dfrac{e^{\theta}-e^{-\theta}}{2}$, $\cosh\theta = \dfrac{e^{\theta}+e^{-\theta}}{2}$
2. What is the choice of s in these formulae?

Laplace Transform of, $f(t)$ Multiplied by t

Let $\mathscr{L}\{f(t)\} = \hat{F}(s)$. Then

$$\mathscr{L}\{tf(t)\} = -\frac{d\hat{F}}{ds} \tag{4.1}$$

- The Laplace transform of t times $f(t)$, is the negative of the derivative of Laplace transform

In general,

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n\hat{F}}{ds^n}, \text{ for } n \geq 1 \tag{4.2}$$

Multiplication by t

Example 4.1

$$(a)\quad \mathscr{L}\left\{te^{at}\right\} = -\frac{d}{ds}\left(\mathscr{L}\left\{e^{at}\right\}\right) = -\frac{d}{ds}\left(\frac{1}{s-a}\right) = \frac{1}{(s-a)^2}$$

$$(b)\quad \mathscr{L}\{t\sin at\} = -\frac{d}{ds}\left(\mathscr{L}\{\sin at\}\right) = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \frac{2as}{(s^2+a^2)^2}$$

$$(c)\quad \mathscr{L}\{t\cos at\} = -\frac{d}{ds}\left(\mathscr{L}\{\cos at\}\right) = -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = \frac{s^2-a^2}{(s^2+a^2)^2}$$

$$(d)\quad \mathscr{L}\{\sin at + at\cos at\} = \frac{a}{s^2+a^2} + \frac{a(s^2-a^2)}{(s^2+a^2)^2} = \frac{2as^2}{(s^2+a^2)^2}$$

$$(e)\quad \mathscr{L}\{\sin at - at\cos at\} = \frac{a}{s^2+a^2} - \frac{a(s^2-a^2)}{(s^2+a^2)^2} = \frac{2a^3}{(s^2+a^2)^2}$$

Laplace Transform of, $f(t)$ divided by t

Let $\mathscr{L}\{f(t)\} = \hat{F}(s)$. Then

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_{u=s}^{\infty} \hat{F}(u)du \tag{4.3}$$

- The Laplace transform of $f(t)$ divided by t, is the integral of Laplace transform from s to ∞

First Shifting or Frequency Shifting

Let $\mathscr{L}\left\{f(t)\right\} = \hat{F}(s)$. Then

$$\mathscr{L}\left\{e^{at}f(t)\right\} = \hat{F}(s-a),\;\; s > a. \tag{5.1}$$

- The Laplace transform of e^{at} times $f(t)$ is the Laplace transform, shifted a units to the right

First Shifting

Example 5.1

$$(a)\quad \mathscr{L}\left\{e^{at}t\right\} = \left|\mathscr{L}\left\{t\right\}\right|_{s\to s-a} = \left|\frac{1}{s^2}\right|_{s\to s-a} = \frac{1}{(s-a)^2}$$

$$(b)\quad \mathscr{L}\left\{e^{at}\sin bt\right\} = \left|\mathscr{L}\left\{\sin bt\right\}\right|_{s\to s-a} = \left|\frac{b}{s^2+b^2}\right|_{s\to s-a} = \frac{b}{(s-a)^2+b^2}$$

$$(c)\quad \mathscr{L}\left\{e^{at}\cos bt\right\} = \left|\mathscr{L}\left\{\cos bt\right\}\right|_{s\to s-a} = \left|\frac{s}{s^2+b^2}\right|_{s\to s-a} = \frac{s-a}{(s-a)^2+b^2}$$

$$(d)\quad \mathscr{L}\left\{e^{at}\sin bt\right\} = \left|\mathscr{L}\left\{\sin bt\right\}\right|_{s\to s-a} = \left|\frac{b}{s^2+b^2}\right|_{s\to s-a} = \frac{b}{(s-a)^2+b^2}$$

$$(e)\quad \mathscr{L}\left\{e^{at}\sin bt\right\} = \left|\mathscr{L}\left\{\sin bt\right\}\right|_{s\to s-a} = \left|\frac{b}{s^2+b^2}\right|_{s\to s-a} = \frac{b}{(s-a)^2+b^2}$$

Well-known Discontinuous Functions

Mathematical models of mechanical or electrical systems often involve functions with discontinuities corresponding to external forces that are turned abruptly on or off.

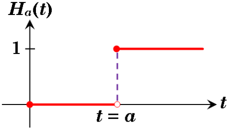
Such discontinuous functions are

1. Heaviside Unit Step Function
2. Dirac Delta Function

Heaviside Unit Step Function and its Transform

The Heaviside Unit Step Function with parameter $a \geq 0$, is defined by

$$H_a(t) = \begin{cases} 0, & \text{if } t < a \\ 1, & \text{if } t \geq a. \end{cases} \tag{7.1}$$



$$\mathscr{L}\{H_a(t)\} = \int_0^\infty H_a(t)e^{-st}\,dt = \int_a^\infty e^{-st}\,dt = \left| -\frac{e^{-st}}{s} \right|_a^\infty = \frac{e^{-as}}{s}, \quad s > 0 \tag{7.2}$$

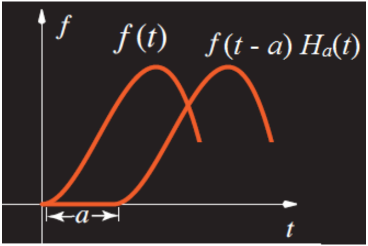
$H_a(t)$ has a finite discontinuity at $t = a$ with jump $H_a(a+0) - H_a(a-0) = 1$

Shifted Function

Let $f(t)$ be a real valued function defined on $[0, \infty)$. Then

$$G_a(t) = f(t-a)H_a(t) = \begin{cases} 0, & \text{if } t < a \\ f(t-a), & \text{if } t \geq a, \end{cases} \tag{8.1}$$

represents $f(t)$, shifted by a units to the right.



Second Shifting or Time Shifting

If $\hat{F}(s) = \mathscr{L}\{f(t)\}$, and $a > 0$. Then the Laplace transform of, $f(t)$ shifted by a units to the right, is given by

$$\mathscr{L}\{G_a(t)\} = \mathscr{L}\{f(t-a)H_a(t)\} = e^{-as}\hat{F}(s) = e^{-as}\mathscr{L}\{f(t)\}. \tag{8.2}$$

Second Shifting or Time Shifting

Example 1

Find the Laplace transform of

$$f(t) = \begin{cases} -1, & \text{if } t < 3 \\ 5, & \text{if } t \geq 3, \end{cases}$$

We write $f(t)$ in terms of $H_a(t)$ as follows:

$$f(t) = \begin{cases} -1+0, & \text{if } t < 3 \\ -1+6, & \text{if } t \geq 3 \end{cases} = -1 + \begin{cases} 0, & \text{if } t < 3 \\ 6, & \text{if } t \geq 3 \end{cases} = -1 + 6 \cdot H_3(t)$$

Hence, by the linearity and second shifting properties, we have

$$\mathscr{L}\{f(t)\} = -\mathscr{L}\{1\} + 6 \cdot \mathscr{L}\{H_3(t)\} = -\mathscr{L}\{1\} + 6e^{-3s}\mathscr{L}\{1\}$$

$$= (6e^{-3s} - 1)\mathscr{L}\{1\} = -\frac{6e^{-3s} - 1}{s}$$

Second Shifting or Time Shifting

Example 2

Find the Laplace transform of the rectangular pulse function

$$f(t) = \begin{cases} k, & \text{if } \alpha < t < \beta \\ 0, & \text{elsewhere.} \end{cases}$$

We write $f(t)$ in terms of $H_a(t)$ as follows:

$$\begin{aligned} f(t) &= k \begin{cases} 1, & \text{if } \alpha < t < \beta \\ 0, & \text{elsewhere.} \end{cases} = k \left[\begin{cases} 0, & \text{if } t < \alpha \\ 1, & \text{if } t \geq \alpha \end{cases} - \begin{cases} 0, & \text{if } t < \beta \\ 1, & \text{if } t \geq \beta \end{cases} \right] \\ &= k[H_\alpha(t) - H_\beta(t)] \end{aligned}$$

Hence, by the linearity, we have

$$\mathscr{L}\{f(t)\} = k\left[\mathscr{L}\{H_\alpha(t)\} - \mathscr{L}\left\{H_\beta(t)\right\}\right] = \frac{k(e^{-\alpha s} - e^{-\beta s})}{s}$$

ϵ-Impulse Function and its Transform

Let $a \geq 0$. Given $\epsilon > 0$, we define the ϵ-impulse function by

$$I_\epsilon(t) = \begin{cases} \frac{1}{\epsilon}, & \text{if } a \leq t < a + \epsilon \\ 0, & \text{elsewhere.} \end{cases}$$

Then

$$\mathscr{L}\left\{I_\epsilon(t)\right\} = \int_0^\infty I_\epsilon(t)e^{-st}dt = \int_a^{a+\epsilon} \frac{1}{\epsilon} \cdot e^{-st}dt = \frac{1}{\epsilon}\left|-\frac{e^{-st}}{s}\right|_a^{a+\epsilon} = \frac{e^{-as}}{s} \cdot \frac{1-e^{-s\epsilon}}{\epsilon}$$

Dirac Delta Function and its Transform

Let $a \geq 0$. The limit of $I_\epsilon(t)$ as $\epsilon \longrightarrow 0$ is called the Dirac Delta function $\mathscr{D}_a(t)$. Then

$$\begin{aligned}
\mathscr{L}\left\{\mathscr{D}_a(t)\right\} &= \mathscr{L}\left\{\lim_{\epsilon\longrightarrow 0} I_\epsilon(t)\right\} \\
&= \lim_{\epsilon\longrightarrow 0} \mathscr{L}\left\{I_\epsilon(t)\right\} \\
&= \lim_{\epsilon\longrightarrow 0} \frac{e^{-as}}{s} \cdot \frac{1-e^{-s\epsilon}}{\epsilon} \\
&= \frac{e^{-as}}{s} \cdot s = e^{-as}
\end{aligned}$$

Laplace Transform of Periodic Functions

Let $f(t)$ be a periodic function with period $T > 0$. Then its graph is repeated in regular intervals of length T.

The Laplace transform of f is given by Then

$$\mathscr{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T f(t) e^{-st} dt$$