VECTOR DIFFERENTIATION

❖ Introduction:

If vector r is a function of a scalar variable t, then we write

$$\vec{r} = \vec{r}(t)$$

If a particle is moving along a curved path then the position vector $\vec{r}$ of the particle is a function of t. If the component of $f(t)$ along $x - axis$, $y - axis$, $z - axis$ are $f_1(t)$, $f_2(t)$, $f_3(t)$ respectively.

Then,

$$\overrightarrow{f(t)} = f_1(t)\,\hat{\imath} + f_2(t)\,\hat{\jmath} + f_3(t)\,\hat{k}$$

1.1 Scalar and Vector Point Function :

Point function: A variable quantity whose value at any point in a region of space depends upon the position of the point, is called a point function.

There are two types of point functions.

1) **Scalar point function:**
   If to each point $P(x, y, z)$ of a region R in space there corresponds a unique scalar $f(P)$, then f is called a scalar point function.
   For example:
   The temperature distribution in a heated body, density of a body and potential due to gravity are the examples of a scalar point function.
2) **Vector point function:**
   If to each point $P(x, y, z)$ of a region R in space there corresponds a unique vector $f(P)$, then f is called a vector point function.
   For example:
   The velocities of a moving fluid, gravitational force are the examples of vector point function.

2.1 Vector Differential Operator Del *i.e.* ∇

The vector differential operator Del is denoted by ∇. It is defined as

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\imath}}\frac{\partial}{\partial x} + \hat{\boldsymbol{\jmath}}\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z}$$

Note: ∇ is read Del or nebla.

3.1 Gradient of a Scalar Function:

If $\emptyset(x, y, z)$ be a scalar function then $\hat{\imath}\frac{\partial\emptyset}{\partial x} + \hat{\jmath}\frac{\partial\emptyset}{\partial y} + \hat{k}\frac{\partial\emptyset}{\partial z}$ is called the gradient of the scalar function $\emptyset$.

And is denoted by grad $\emptyset$.

Thus,

$$\text{grad } \emptyset = \hat{\boldsymbol{\imath}}\frac{\partial\emptyset}{\partial x} + \hat{\boldsymbol{\jmath}}\frac{\partial\emptyset}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial\emptyset}{\partial z}$$

$$\text{grad } \emptyset = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \emptyset(x, y, z)$$

$$\text{grad } \emptyset = \boldsymbol{\nabla}\, \emptyset$$

4.1 Normal and Directional Derivative:

1) **Normal:**

If $\emptyset(x, y, z) = c$ represents a family of surfaces for different values of the constant c. On differentiating $\emptyset$, we get $d\emptyset = 0$

But $$d\emptyset = \nabla \emptyset \, . \, d\vec{r}$$

So $$\nabla\emptyset . dr = 0$$

The scalar product of two vectors $\nabla\emptyset$ and $d\vec{r}$ being zero, $\nabla \emptyset$ and $d\vec{r}$ are perpendicular to each other. $d\vec{r}$ is in the direction of tangent to the giving surface.

Thus $\nabla\emptyset$ is a vector normal to the surface $\emptyset(x, y, z) = c$.

2) **Directional derivative:**

The component of $\nabla \emptyset$ in the direction of a vector $\vec{d}$ is equal to $\nabla \emptyset . \vec{d}$ and is called the directional derivative of $\emptyset$ in the direction of $\vec{d}$.

$$\frac{\partial \emptyset}{\partial r} = \lim_{\delta r \to 0} \frac{\delta \emptyset}{\delta r} \qquad \text{Where, } \delta r = PQ$$

$\dfrac{\partial \emptyset}{\partial r}$ is called the directional derivative of $\emptyset$ at P in the direction of PQ.

Examples:

Example 1: If $\emptyset = 3x^2y - y^3z^2$; find grad $\emptyset$ at the point (1,-2, 1).

Solution:

$$\text{grad } \emptyset = \nabla\emptyset$$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3x^2y - y^3z^2)$$

$$= \hat{\imath}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{\jmath}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= \hat{\imath}(6xy) + \hat{\jmath}(3x^2 - 3y^2z^2) + \hat{k}(-2y^3z)$$

$$\text{grad } \emptyset \text{ at } (1, -2, 1)$$

$$= \hat{\imath}(6)(1)(-2) + \hat{\jmath}[(3)(1) - 3(4)(1)] + \hat{k}(-2)(-8)(-1)$$

$$= -12\hat{\imath} - 9\hat{\jmath} - 16\hat{k}$$

Example 2: Find the directional derivative of $x^2y^2z^2$ at the point $(1,-1,1)$ in the direction of the tangent to the curve

$x = e^t, y = \sin 2t + 1, z = 1 - cost\ at\ t = 0.$

Solution: Let $\emptyset = x^2y^2z^2$

Direction Derivative of $\emptyset\ = \nabla\emptyset$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y^2z^2)$$

$$\nabla\emptyset = 2xy^2z^2\hat{\imath} + 2yx^2z^2\hat{\jmath} + 2zx^2y^2\hat{k}$$

Directional Derivative of $\emptyset$ at (1,1,-1)

$= 2(1)(1)^2(-1)^2\hat{\imath} + 2(1)(1)^2(-1)^2\hat{\jmath} + 2(-1)(1)^2(1)^2\hat{k}$

$= 2\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$ ____________ (1)

$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = e^t\,\hat{\imath} + (sin2t + 1)\hat{\jmath} + (1 - cost)\hat{k}$

$\vec{T} = \frac{d\vec{r}}{dt} = e^t\,\hat{\imath} + 2\,cos2t\,\hat{\jmath} + sint\,\hat{k}$

Tangent vector,

Tangent $(at\ t = 0) = e^{0}\ \hat{\imath} + 2\ (\cos 0)\hat{\jmath} + (sin0)\hat{k}$

$$= \hat{\imath} + \ 2\hat{\jmath} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_ (2)}$$

Required directional derivative along tangent $= (2\ \hat{\imath} + 2\hat{\jmath} - 2\hat{k})\frac{(\hat{\imath}+2\hat{\jmath})}{\sqrt{1+4}}$

$[from\ (1) and\ (2)]$

$$= \frac{2+4+0}{\sqrt{5}}$$

$$= \frac{6}{\sqrt{5}}$$

Example 3: Verify $\overline{\nabla} \times [f(r)\vec{r}] = 0$

Solution:

$$\overline{\nabla} \times [f(r)\vec{r}] = \left[\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] \times [f(r)\vec{r}]$$

$$= \left[\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] \times f(r)(x\hat{\imath} + y\hat{\jmath} + z\hat{k})$$

$$= \left[\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] \times \left[f(r)x\hat{\imath} + f(r)y\hat{\jmath} + f(r)z\hat{k}\right]$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f(r)x & f(r)y & f(r)z \end{vmatrix}$$

$$= \left[z\,\frac{\partial f(r)}{\partial y} - y\frac{\partial}{\partial z}f(r)\right]\hat{\imath} - \left[z\frac{\partial}{\partial x}f(r) - x\frac{\partial}{\partial z}f(r)\right]\hat{\jmath} + \left[y\frac{\partial}{\partial x}f(r) - x\frac{\partial}{\partial y}f(r)\right]\hat{k}$$

$$= \left[z\frac{d}{dr}f(r)\frac{\partial r}{\partial y} - y\frac{d}{dr}f(r)\frac{\partial r}{\partial z}\right]\hat{\imath} - \left[z\frac{d}{dr}f(r)\frac{\partial r}{\partial x} - x\frac{d}{dr}f(r)\frac{\partial r}{\partial z}\right]\hat{\jmath} + \left[y\frac{d}{dr}f(r)\frac{\partial r}{\partial x} - x\frac{d}{dr}f(r)\frac{\partial r}{\partial y}\right]\hat{k}$$

$$= \frac{f'(r)}{r}\left[(yz - yz)\hat{\imath} - (xz - xz)\hat{\jmath} + (xy - xy)\hat{k}\right] = 0$$

Exercise:

1) Find a unit vector normal to the surface $x^2 + 3y^2 + 2z^2 = 6\ at\ P(2,0,1)$.
2) Find the direction derivative of $\frac{1}{r}$ in the direction $\vec{r}$ where $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.
3) If $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, show that
   a) grad $r = \frac{\vec{r}}{r}$
   b) grad $\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$
4) Find the rate of change of $\emptyset = xyz$ in the direction normal to the surface $x^2y + y^2x + yz^2 = 3$ at the point (1, 1, 1).

DIVERGENCE OF A VECTOR FUNCTION

The divergence of a vector point function $\vec{F}$ is denoted by div F and is defined as below.

Let $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$

$$\text{div } \vec{F} = \vec{\nabla}.\vec{F} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\hat{\imath}F_1 + \hat{\jmath}F_2 + \hat{k}F_3\right)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

It is evident that div F is scalar function.

Note: If the fluid is compressible, there can be no gain or loss in the volume element. Hence

$$\textbf{div } \overline{\boldsymbol{V}} = \mathbf{0}$$

Here, V is called a Solenoidal vector function. And this equation is called equation of continuity or conservation of mass.

Example 1: If $u = x^2 + y^2 + z^2, and\ \bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, then find div $(u\bar{r})$ in terms of u.

Solution: div $(u\bar{r}) = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[(x^2 + y^2 + z^2)\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)\right]$

$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left[(x^2 + y^2 + z^2)x\hat{\imath} + (x^2 + y^2 + z^2)y\hat{\jmath} + (x^2 + y^2 + z^2)z\hat{k}\right]$

$= \frac{\partial}{\partial x}(x^3 + xy^2 + xz^2) + \frac{\partial}{\partial y}(x^2y + y^3 + yz^2) + \frac{\partial}{\partial z}(x^2z + y^2z + z^3)$

$= (3x^2 + y^2 + z^2) + (x^2 + 3y^2 + z^2) + (x^2 + y^2 + 3z^2)$

$= 5(x^2 + y^2 + z^2)$

$= 5u$

Example 2: Find the directional derivative of the divergence of $\bar{f}(x,y,z) = xy\hat{\imath} + xy^2\hat{\jmath} + z^2\hat{k}$ at the point (2, 1, 2) in the direction of the outer normal to the sphere $x^2 + y^2 + z^2 = 9$.

Solution: $\bar{f}(x,y,z) = xy\hat{\imath} + xy^2\hat{\jmath} + z^2\hat{k}$

Divergence of $\bar{f}(x,y,z) = \bar{\nabla}.\bar{f} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(xy\hat{\imath} + xy^2\hat{\jmath} + z^2\hat{k}\right)$

$$= y + 2xy + 2z$$

Directional derivative of divergence of $(y + 2xy + 2z)$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(y + 2xy + 2z)$$

$$= 2y\hat{\imath} + (1 + 2x)\hat{\jmath} + 2\hat{k} \quad \text{\_\_\_\_\_\_\_\_(i)}$$

Directional derivative at the point $(2,1,2) = 2\hat{\imath} + 5\hat{\jmath} + 2\hat{k}$

Normal to the sphere $= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 9)$

$$= 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$$

Normal at the point $(2,1,2) = 4\hat{\imath} + 2\hat{\jmath} + 4\hat{k}$ ____________ (ii)

Directional derivative along normal at $(2,1,2) = \left(2\hat{\imath} + 5\hat{\jmath} + 2\hat{k}\right).\frac{(4\hat{\imath}+2\hat{\jmath}+4\hat{k})}{\sqrt{16+4+16}}$

$$= \frac{1}{6}(8 + 10 + 8)$$

$$= \frac{13}{3}$$

Exercise:

1) If $\bar{v} = \frac{x\hat{\imath}+y\hat{\jmath}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$, find the value of div $\bar{v}$.
2) Find the directional derivative of div $(\vec{u})$ at the point (1, 2, 2) in the direction of the outer normal of the sphere $x^2 + y^2 + z^2 = 9$ for $\vec{u} = x^4\hat{\imath} + y^4\hat{\jmath} + z^4\hat{k}$.

CURL

The curl of a vector point function F is defined as below

Curl $\bar{F} = \bar{\nabla} \times \bar{F}$ $\quad\quad (\bar{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k})$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \hat{\imath}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Curl $\bar{F}$ is a vector quantity.

Note: Curl $\boldsymbol{\bar{F} = 0}$, the field F is termed as irrotational.

Example 1: Find the divergence and curl of

$$\bar{v} = (xyz)\hat{\imath} + (3x^2y)\hat{\jmath} + (xz^2 - y^2z)\hat{k}\ at\ (2, -1, 1)$$

Solution: Here, we have

$$\bar{v} = (xyz)\hat{\imath} + (3x^2y)\hat{\jmath} + (xz^2 - y^2z)\hat{k}$$

$$\text{Div}\quad \bar{v} = \nabla \emptyset$$

$$\text{Div}\quad \bar{v} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$$

$$= yz + 3x^2 + 2xz - y^2$$

$$\bar{v}_{(2,-1,1)} = -1 + 12 + 4 - 1 = 14$$

$$\text{Curl } \bar{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$= -2yz\hat{\imath} - (z^2 - xy)\hat{\jmath} + (6xy - xz)\hat{k}$$

$$= -2yz\hat{\imath} + (xy - z^2)\hat{\jmath} + (6xy - xz)\hat{k}$$

Curl at (2, -1, 1)

$$= 2(-1)(1)\hat{\imath} + \{2(-1) - 1\}\hat{\jmath} + \{6(2)(-1) - 2(1)\}\hat{k}$$

$$= 2\hat{\imath} - 3\hat{\jmath} - 14\hat{k}$$

Example 2: Prove that

$(y^2 - z^2 + 3yz - 2x)\hat{\imath} + (3xy + 2xy)\hat{\jmath} + (3xy - 2xz + 2z)\hat{k}$ is both Solenoidal and irrotational.

Solution:

Let $\bar{F} = (y^2 - z^2 + 3yz - 2x)\hat{\imath} + (3xy + 2xy)\hat{\jmath} + (3xy - 2xz + 2z)\hat{k}$

For Solenoidal, we have to prove $\bar{\nabla}.\bar{F} = 0$.

$$\bar{\nabla}.\bar{F} = \left[\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right].\left[(y^2 - z^2 + 3yz - 2x)\hat{\imath} + (3xy + 2xy)\hat{\jmath} + (3xy - 2xz + 2z)\hat{k}\right]$$

$$= -2 + 2x - 2x + 2$$

$$= 0$$

Thus, $\bar{F}$ is Solenoidal.

For irrotational, we have to prove Curl $\bar{F} = 0$

Now, Curl $\bar{F} =$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2 - z^2 + 3yz - 2x) & (3xy + 2xy) & (3xy - 2xz + 2z) \end{vmatrix}$$

$$= (3z + 2y - 2y + 3z)\hat{\imath} - (-2z + 3y - 3y + 2z)\hat{\jmath} + (3z + 2y - 2y - 3z)\hat{k}$$

$$= 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$$

$$= 0$$

Thus, $\bar{F}$ is irrotational.

Example 3: Find the scalar potential (velocity potential) function f for $\vec{A} = y^2\hat{\imath} + 2xy\hat{\jmath} - z^2\hat{k}$.

Solution: We have, $\vec{A} = y^2\hat{\imath} + 2xy\hat{\jmath} - z^2\hat{k}$.

Curl $\vec{A} = \nabla \times \vec{A}$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(y^2\hat{\imath} + 2xy\hat{\jmath} - z^2\hat{k}\right).$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy & -z^2 \end{vmatrix}$$

$$= \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(2y - 2y)$$

$$= 0$$

Hence, $\vec{A}$ is irrotational.

To find the scalar potential function f.

$$\vec{A} = \nabla f$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz)$$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f.d\vec{r}$$

$$= \nabla f.d\vec{r}$$

$$= \vec{A}.d\vec{r} \qquad (A = \nabla f)$$

$$= (y^2\hat{\imath} + 2xy\hat{\jmath} - z^2\hat{k}).(\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz)$$

$$= y^2dx + 2xy\,dy - z^2\,dz$$

$$= d(xy^2) - z^2dz$$

$$f = \int d(xy^2) - \int z^2dz \qquad \therefore\ f = xy^2 - \frac{z^3}{3} + c$$

Exercise

1) If a vector field is given by $\vec{F} = (x^2 - y^2 + x)\hat{\imath} - (2xy + y)\hat{\jmath}$. Is this field irrotational? If so, find its scalar potential.

2) Determine the constants a and b such that the curl of vector $\bar{A} = (2xy + 3yz)\hat{\imath} + (x^2 + axz - 4z^2)\hat{\jmath} - (3xy + byz)\hat{k}$ is zero.

3) A fluid motion is given by $\bar{v} = (y + z)\hat{\imath} + (z + x)\hat{\jmath} + (x + y)\hat{k}$. Show that the motion is irrotational and hence find the velocity potential.

4) Given that vector field $\bar{V} = (x^2 - y^2 + 2xz)\hat{\imath} + (xz - xy + yz)\hat{\jmath} + (z^2 + x^2)\hat{k}$ find curl V. Show that the vectors given by curl V at $P_0(1,2,-3)$ and $P_1(2,3,12)$ are orthogonal.

5) Suppose that $\vec{U}, \vec{V}$ and f are continuously differentiable fields then Prove that, $\operatorname{div}(\vec{U} \times \vec{V}) = \vec{V}.curl\, \vec{U} - \vec{U}.curl\, \vec{V}$.