Triple Integrals

The volume integral of $f(x,y,z)$ over the region R is denoted by $\iiint\limits_R f(x,y,z)\,dx\,dy\,dz$ (or, $\int\limits_V f(x,y,z)\,dV$; here V stands for the volume of R).

Thus,
$$\int\limits_V f(x,y,z)\,dV = \iiint\limits_R f(x,y,z)\,dx\,dy\,dz$$
$$= \int_{x=a}^{b} \int_{y=y_1(x)}^{y_2(x)} \int_{z=z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

<u>Note:</u> A volume integral is also called as a Triple Integral.

Problems:

1) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$

Sol:
$$\int_{x=0}^{1} \left(\int_{y=0}^{\sqrt{1-x^2}} \left(\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\,dz \right) dy \right) dx$$
$$= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} xy \left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$= \frac{1}{2} \int_{x=0}^{1} \left(\int_{y=0}^{\sqrt{1-x^2}} xy(1-x^2-y^2)\,dy \right) dx$$

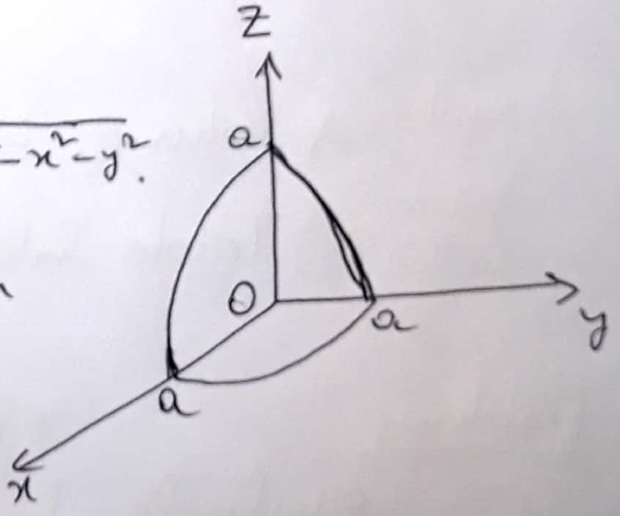
$$= \frac{1}{2} \int_{x=0}^{1} x\left\{(1-x^2)\frac{y^2}{2} - \frac{y^4}{4}\right\}_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{8} \int_{x=0}^{1} x(1-2x^2+x^4)\,dx$$

$$= \frac{1}{8}\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) = \frac{1}{48}$$

(2) Evaluate $\iiint_R xy\,dx\,dy\,dz$, where R is the positive octant of the sphere $x^2+y^2+z^2=a^2$

Sol: In the given region,



z varies from 0 to $\sqrt{a^2-x^2-y^2}$.

For $z=0$, y varies from 0 to $\sqrt{a^2-x^2}$.

For $z=0, y=0$,

x varies from 0 to a.

Therefore, $\iiint_R xy\,dx\,dy\,dz = \int_{x=0}^{a}\left(\int_{y=0}^{\sqrt{a^2-x^2}}\left(\int_{z=0}^{\sqrt{a^2-x^2-y^2}} xy\,dz\right)dy\right)dx$

$$= \int_{x=0}^{a} x \left(\int_{y=0}^{\sqrt{a^2-x^2}} y \, [z]_0^{\sqrt{a^2-x^2-y^2}} \, dy \right) dx$$

$$= \int_{x=0}^{a} x \left(\int_{y=0}^{\sqrt{a^2-x^2}} y \sqrt{a^2-x^2-y^2} \, dy \right) dx$$

$$= \int_{x=0}^{a} x \left(\int_{t=a^2-x^2}^{0} \cancel{y} \left(\frac{-1}{2}\right) \sqrt{t} \, dt \right) dx$$

Taking $t = (a^2-x^2) - y^2$

then $dt = -2y\,dy$

and $y \to 0,\ t \to a^2-x^2$

$y \to \sqrt{a^2-x^2},\ t \to 0$

$$= \frac{1}{2} \int_{x=0}^{a} x \left(\int_{0}^{a^2-x^2} \sqrt{t} \, dt \right) dx$$

$$= \frac{1}{2} \cdot \frac{2}{3} \int_{x=0}^{a} x \left[t^{3/2} \right]_0^{a^2-x^2} dx$$

$$= \frac{1}{3} \int_{x=0}^{a} x \left(a^2-x^2\right)^{3/2} dx$$

Taking $a^2 - x^2 = u$

then $-2x\,dx = du$

$x\,dx = -\frac{1}{2} du$

and $x \to 0,\ u \to a^2$

$x \to a,\ u \to 0$

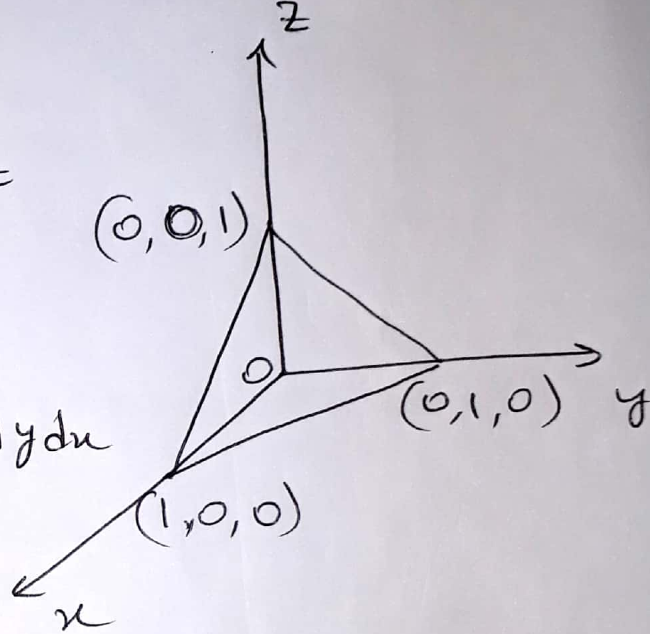
$$= \frac{1}{3} \int_{u=a^2}^{0} u^{3/2} \cdot \left(\frac{-1}{2}\right) du$$

$$= \frac{1}{6} \int_{u=0}^{a^2} u^{3/2} \, du$$

$$= \frac{1}{6} \left[\frac{u^{5/2}}{\left(\frac{5}{2}\right)} \right]_0^{a^2} = \frac{1}{15} \cdot a^5$$

③ Evaluate $\iiint_R (x+y+z)\,dx\,dy\,dz$, where R is the region bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$.

Sol:



$$\iiint_R (x+y+z)\,dx\,dy\,dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} (x+y+z)\,dz\,dy\,dx$$

$$= \int_{x=0}^{1}\left(\int_{y=0}^{1-x} \left\{(x+y)\left[z\right]_0^{1-x-y} + \left[\frac{z^2}{2}\right]_0^{1-x-y}\right\} dy\right) dx$$

$$= \frac{1}{2}\int_{x=0}^{1}\left(\int_{y=0}^{1-x} \left\{1-(x+y)^2\right\} dy\right) dx$$

$$= \frac{1}{2}\int_{x=0}^{1} \left[y - \frac{1}{3}(x+y)^3\right]_0^{1-x} dx$$

$$= \frac{1}{6}\cdot\int_{x=0}^{1} (2-3x+x^3)\,dx = \frac{1}{8}.$$

Problems

① Evaluate the following

(i) $$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx$$

(ii) $$\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dz\, dx\, dy$$

(iii) $$\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x+y+z)\, dy\, dx\, dz$$

(iv) $$\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^{x^2+y^2} z^2\, dz\, dy\, dx$$

② Evaluate $\iiint_R dx\,dy\,dz$, where R is the finite region of space formed by the planes $x=0, y=0, z=0$ and $2x+3y+4z=12$

③ Evaluate (i) $$\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$$

④ Evaluate $$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\, dy\, dx$$