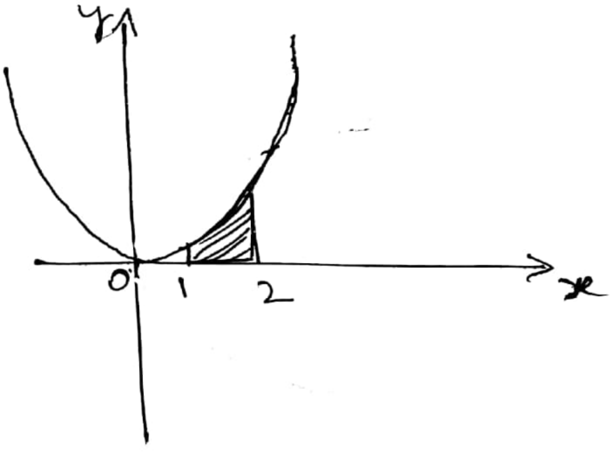
Areas :

The area which is bounded by the curve $y = f(x)$, lines $x = a$ and $x = b$ is denoted by

$$\int_{x=a}^{x=b} f(x)\,dx$$

Ex ① Obtain the area bounded by the curve $y = f(x) = x^2$ and the lines $x = 1$, $x = 2$

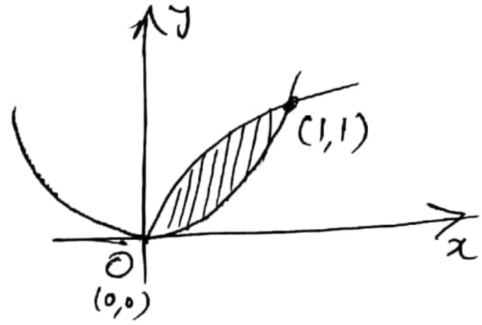


Sol. $\text{Area} = \int_1^2 x^2\,dx$

$$= \left(\frac{x^3}{3}\right)_1^2$$

$$= \frac{8}{3} - \frac{1}{3} = \frac{7}{3} \text{ Sq. units.}$$

(2) Find the area of the region enclosed by $y = x^2$ and $y = \sqrt{x}$.

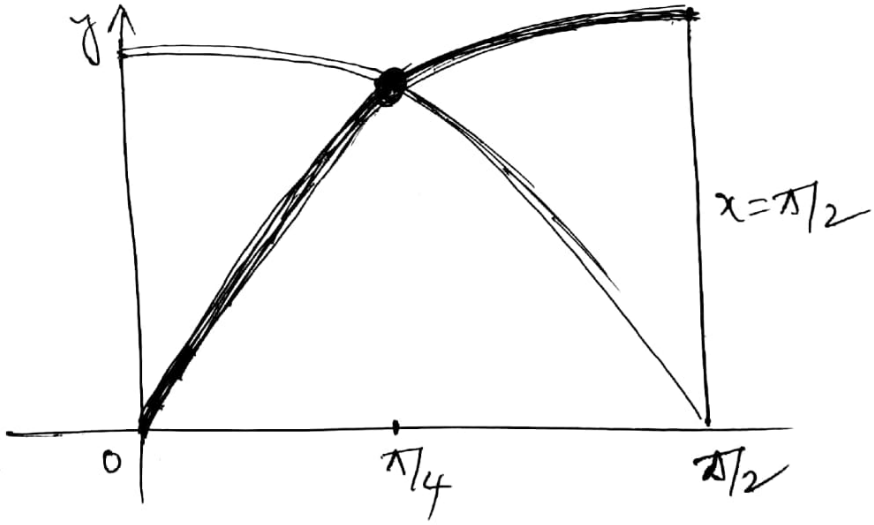


Sol. Area $= \int_0^1 (\sqrt{x} - x^2)\, dx$

$= \left[\frac{2}{3} x^{3/2} - \frac{1}{3} x^3 \right]_0^1$

$= \frac{1}{3}$ sq. units

(3) Determine the area of the region enclosed by $y = \sin x$, $y = \cos x$, $x = \frac{\pi}{2}$ and the y-axis



Area $= \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx$

$= \left(\sin x + \cos x \right)_0^{\pi/4} + \left(-\cos x - \sin x \right)_{\pi/4}^{\pi/2}$

$= (\sqrt{2} - 1) + (\sqrt{2} - 1) = 2\sqrt{2} - 2$

$= 0.828427$ sq. units

☆ Determine the area of the region enclosed by $x = \frac{1}{2}y^2 - 3$ and $y = x - 1$ ——— 18

☆ Determine the area of the region bounded by $x = -y^2 + 10$ and $x = (y-2)^2$. ——— 64/3

☆ Determine the area of the region bounded by $y = 2x^2 + 10$, $y = 4x + 16$, $x = -2$ and $x = 5$ —— 142/3

Volumes using Cross-Sections

Definition: The volume of a solid of integrable cross-section area $A(x)$ from $x=a$ to $x=b$ is

$$V = \int_a^b A(x)\,dx$$

Definition: The volume of a solid of revolution of integrable cross-section area $A(y)$ from $y=c$ to $y=d$ is

$$V = \int_c^d A(y)\,dy$$

To get the cross sectional area, it is required to cut the object $\perp^r$ to the axis of rotation. Doing this, the cross section will be either a solid disk if the object is solid (or) a ring if we have hollowed out a portion of the solid.

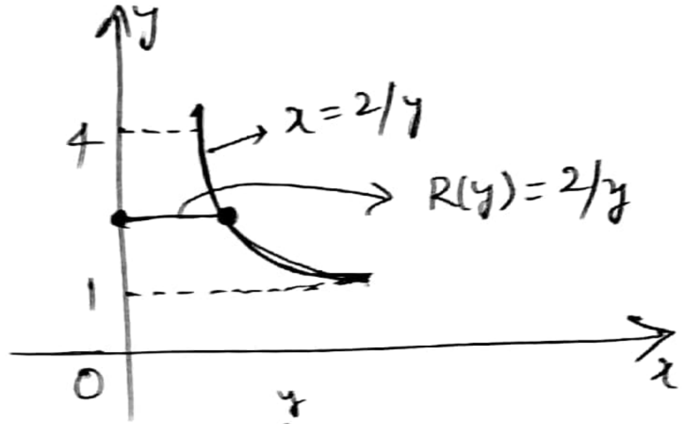
In the case of solid disk, area is $\pi (r)^2$
$\pi (\text{radius})^2$

where the radius depends on the function and the axis of rotation.

In the case of ring, the area is $\pi\left[(\text{Outer radius})^2 - (\text{Inner radius})^2\right]$ where again both the radii depends on the functions given and axis of rotation.

Examples.

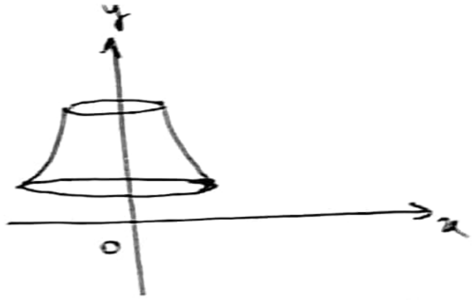
① Find the Volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$; $1 \leq y \leq 4$ about the y-axis



Sol. Volume (V)

$$= \int_c^d A(y)\, dy$$

$$= \int_1^4 \pi \left[\frac{2}{y}\right]^2 dy = 3\pi$$



② Find the Volume of the solid generated by revolving the region between the parabola $x = y^2+1$ and the line $x = 3$ about the line $x = 3$

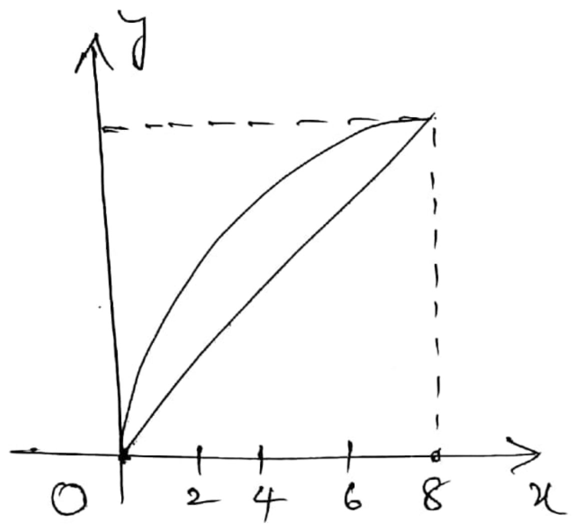
Sol. Volume (V) $= \int_{-\sqrt{2}}^{\sqrt{2}} \pi \left[R(y)\right]^2 dy$

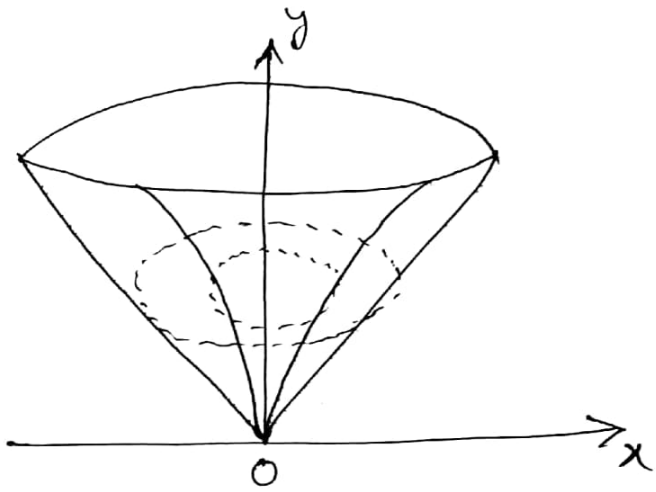
$$= \int_{-\sqrt{2}}^{\sqrt{2}} \pi \left[2 - y^2\right]^2 dy$$

$$= (64\pi\sqrt{2})/15.$$

③ Determine the volume of the solid generated by rotating the portion of the region bounded by $y = \sqrt[3]{x}$ and $y = \frac{x}{4}$, that lies in the first quadrant about the y-axis.

Sol:





Cross sectional area $(A(y))$: $\pi\left[[R_1(y)]^2 - [R_2(y)]^2\right]$

$$= \pi\left[(4y)^2 - (y^3)^2\right] = \pi\left[16y^2 - y^6\right]$$

$$\text{Volume } (V) = \int_0^2 \pi\left[16y^2 - y^6\right] dy$$

$$= \frac{512\pi}{21}$$

☆ The region between the Curve $y=\sqrt{x}$; $0\le x\le 4$, and the x-axis is revolved about the x-axis to generate a Solid. Find it's volume $\longrightarrow$ 8π

☆ Find the Volume of the Solid generated by revolving the region between the parabola $x=y^2+1$ and the line $x=3$ about the line $x=3$ $\longrightarrow$ $\boxed{\frac{64\pi\sqrt{2}}{15}}$

☆ The Circle $x^2+y^2=a^2$ is rotated about the x-axis to generate a Sphere. Find it's Volume $\longrightarrow$ $\boxed{\frac{4}{3}\pi a^3}$

☆ Determine the Volume of the Solid obtained by rotating the region bounded by $y=x^2-2x$ and $y=x$ about the line $y=4$ $\longrightarrow$ $\boxed{\frac{153\pi}{5}}$

☆ Find the Volume of the Solid generated by revolving the region between the y-axis and the Curve $x=2/y$; $1\le y\le 4$ about the y-axis $\longrightarrow$ 3π