APPLIED LINEAR ALGEBRA (MAT3004)

Why we study Linear Algebra?

- One of the primary objective of the linear algebra course is to study a new necessary and sufficient conditions to solve the system of linear equations .

Consider the following system of linear equations,

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_n \end{gathered}$$

Why we study Linear Algebra?

- The system we have seen in the last slide can be written in matrix form $AX = B$ Where $A = (a_{ij})$ is the coefficient $m \times n$ matrix, X is the $n \times 1$ matrix consisting of all variables and B is the $m \times 1$ matrix consists the right hand side of the system.
- We know if $m = n$. The following are equivalent

(i) Determinant of $A \neq 0$.

(ii) System $AX = B$ has a unique solution.

Why we study Linear Algebra?

- If $m \neq n$ then the system may or may not have solution. The following question arise naturally,

Question : **Can we derive some equivalent conditions for the system has either no solution or have infinitely many solution or have unique solution based on the character of** A?

Why we study Linear Algebra?

We answer to the above question by studying this linear algebra course.

Why we study Linear Algebra?

The necessity of studying linear algebra also includes the application of real world problems. Let's go through the image compression technique using linear algebra tools.

Image Compression and Linear Algebra

What is a pixel?

A digital photo is a dividable thing. If we zoom in far enough we shall see that our image is like a mosaic formed by small tiles, which in photography are called pixels.

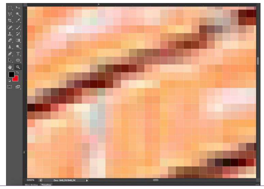


Image Compression and Linear Algebra

Pixel count

In order to calculate this resolution you just use the same formula you would use for the area of any rectangle multiply the length by the height.

For example, if you have a photo that has 4,500 pixels on the horizontal side, and 3,000 on the vertical size it gives you a total of 13,500,000.

Because this number is very unpractical to use, you can just divide it by a million to convert it into megapixels. So 13,500,000 / 1,000000 = 13.5 Megapixels.

Image Compression and Linear Algebra

- An mxn pixels image can be represented by mxn matrix representation.
- Suppose we have an 9 megapixel, gray-scale image, which is 3000 × 3000 pixels (a 3000 × 3000 matrix).
- For each pixel, we have some level of black and white color, given by some integer between 0 and 255.0 representing black color and 255 representing white color.

Image Compression and Linear Algebra

The basic idea here is each image can be represented as a matrix and we apply linear algebra (Singular Value Decomposition) on this matrix and get a reduced matrix out of this original matrix and the image corresponding to this reduced matrix requires much lesser storage space as compared to the original image.

Image Compression and Linear Algebra

- Any real matrix A can be factored as $A = U\Sigma V^T$ where U and V are orthogonal and Σ is diagonal with diagonal elements $\sigma_i \geq 0$ (called singular values of A).

Image Compression and Linear Algebra

Any real $m \times n$ matrix A can be expressed as a finite sum of rank 1 matrices in normalized form, that is $A = \sigma_1 R_1 + \sigma_2 R_2 + \cdots + \sigma_k R_k$, where $k = \min(m, n)$ and

1) $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0 = \sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_n, rank(A) = r \leq k$.
2) $R_i = p_i q_i^T$ where p_i is the i^{th} column of P and a unit eigenvector of AA^T an q_i is the i^{th} column of Q and a unit eigenvector of $A^T A$.
3) Each R_i has the sum of squares of its elements equal to 1.

Image Compression and Linear Algebra

- Let $A = U\Sigma V^T$ be the singular value decomposition of A. If $k < r = rank(A)$ and $A_k = \sum_{i=1}^{k} \sigma_i p_i q_i^T$ then

$$\min_{rank(B)=k} \|A - B\|_2 = \|A - A_k\|_2 = \sigma_{k+1}.$$

Image Compression and Linear Algebra

- Consider the following image

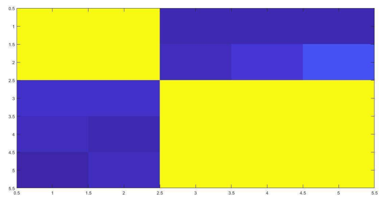


Image Compression and Linear Algebra

- Matrix corresponds to the above image is
- $$\begin{pmatrix} 255 & 255 & 2 & 2 & 2 \\ 255 & 255 & 3 & 5 & 10 \\ 4 & 4 & 255 & 253 & 255 \\ 3 & 2 & 255 & 255 & 255 \\ 1 & 3 & 255 & 255 & 254 \end{pmatrix}$$

Image Compression and Linear Algebra

- Singular value decomposition is

U =

$$\begin{bmatrix} -0.0195 & 0.7069 & 0.6800 & -0.1915 & 0.0298 \\ -0.0286 & 0.7065 & -0.6786 & 0.1959 & -0.0333 \\ -0.5763 & -0.0164 & -0.1637 & -0.4836 & 0.6379 \\ -0.5777 & -0.0206 & -0.0455 & -0.2964 & -0.7589 \\ -0.5770 & -0.0220 & 0.2196 & 0.7767 & 0.1233 \end{bmatrix}$$

Image Compression and Linear Algebra

S =

$$\begin{bmatrix} 764.2936 & 0 & 0 & 0 & 0 \\ 0 & 509.7433 & 0 & 0 & 0 \\ 0 & 0 & 3.7256 & 0 & 0 \\ 0 & 0 & 0 & 1.6245 & 0 \\ 0 & 0 & 0 & 0 & 1.2135 \end{bmatrix}$$

Image Compression and Linear Algebra

V =

$$\begin{matrix} -0.0221 & 0.7068 & -0.0601 & -0.5699 & -0.4142 \\ -0.0228 & 0.7067 & 0.0700 & 0.5687 & 0.4144 \\ -0.5777 & -0.0225 & 0.5366 & -0.4008 & 0.4660 \\ -0.5763 & -0.0197 & 0.2602 & 0.4358 & -0.6402 \\ -0.5772 & -0.0128 & -0.7974 & -0.0346 & 0.1723 \end{matrix}$$

Image Compression and Linear Algebra

One can check that

$$A = USV^T.$$

Image Compression and Linear Algebra

- Consider the following new matrix

S1 =

764.2936	0	0	0	0
0	509.7433	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

Image Compression and Linear Algebra

- Matrix of US_1V^T is

254.9898	254.9846	0.4989	1.4995	4.0030
255.0128	255.0128	4.5032	5.4933	8.0020
3.8363	4.1687	254.6516	253.9966	254.3532
2.3339	2.6673	255.3270	254.6643	255.0069
1.8302	2.1632	254.9968	254.3331	254.6703

Image Compression and Linear Algebra

- Image of US_1V^T is

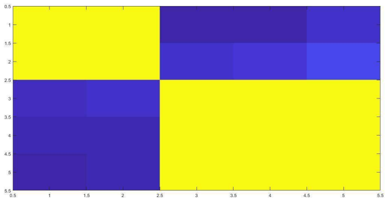


Image Compression and Linear Algebra

The image USV^T and US_1V^T looks almost same.

Image Compression and Linear Algebra

- Consider the following 340×280 pixel image.

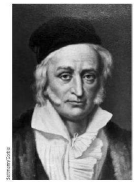


Image Compression and Linear Algebra

In order to store the above image, we need to store 95200 numbers. In this case transmission and manipulation of this image with 95200 numbers is very expensive.

So we apply singular value decomposition for compress the image resolution and without losing the important information which is available in the image.

Image Compression and Linear Algebra

If $A = \sigma_1 R_1 + \sigma_2 R_2 + \cdots + \sigma_r R_r$ is the singular value decomposition of A and

$$A_k = \sigma_1 R_1 + \sigma_2 R_2 + \cdots + \sigma_k R_k$$

Be the $m \times n$ matrix derived from A with $k \le r$.

We see the following diagram for various A_k.

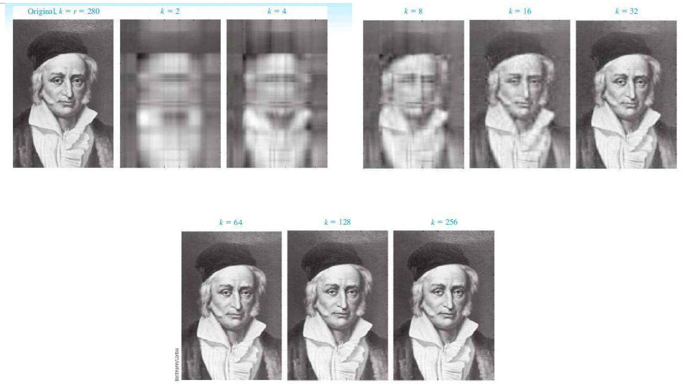


Image Compression and Linear Algebra

It is clear that for $k = 32$ we almost get the same image. Here A_{32} needs the only

$$32 + (32 \times 340) + (32 \times 260) = 19232$$

Numbers. So, storage is much lesser than original image.

Internal Assessments

Exam	Duration	Marks
CAT - 1	30 MINUTES	30
CAT - 2	30 MINUTES	30
Digital Assignment - 1	Due date will announced later	10
Digital Assignment - 2	Due date will announced later	10
Quiz -1	Date will be announced later	10
Additional learning	Not applicable	No marks

Module -1

System of linear equations

Gauss Elimination method :-

Consider the System of linear equations

$2x + y = 1 \longrightarrow (1)$

$x + 2y = 2 \longrightarrow (2)$

Method : 1 Eliminating variables :-

(1) $\Rightarrow$ $2x + y = 1$

(2) $\times 2$ $\Rightarrow$ $2x + 4y = 4$ (3 multiplication)

(1) $-$ (2) $\Rightarrow$ $-3y = -3$ (1 subtraction)

Thus $y = 1$ (1 divison)

Put $y=1$ in (1)

$$2x+y=1$$
$$2x+1=1$$
$$\Rightarrow 2x = 1-1 \quad (1 \text{ subtraction})$$
$$2x=0 \quad (1 \text{ division})$$
$$x=0$$

Totally we used 7 operations to get the values of x and y.

Another method:-

The matrix form $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$ax+by=e$
$cx+dy=f$; Grammer's Rule.

$$\Delta_x = \begin{vmatrix} e & b \\ f & d \end{vmatrix} \quad \& \quad \Delta_y = \begin{vmatrix} a & e \\ c & f \end{vmatrix}$$

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$x = \frac{\Delta x}{\Delta} \; ; \; y = \frac{\Delta_y}{\Delta}.$$

$2x+y=1$
$x+2y=2$; $\Delta_x = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$; $\Delta y = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$

$$\Delta = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$$\therefore x = \frac{\Delta x}{\Delta} = 0 \quad (7 \text{ operations})$$
$$y = \Delta y/\Delta = 1 \quad (7 \text{ operations})$$

Totally 14 operations are required.

Hence elimination is better method for algorithms.

Gaussian elimination method:-

Consider the following system of equations,

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

A matrix representation is

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{B}$$

Homogeneous System:-

If $B=0$ (ie) $b_1=b_2=b_3=\cdots=0$ then we say the system is homogeneous.

Example:-

$$2x+3y+4z=0$$
$$-7x+5y+1.73z=0$$

Non homogeneous System:-

A linear system of equation is said to be non homogeneous if atleast one b_i is non zero.

Example:-

$$x+3y+z=5$$
$$4x-7y+12z=0$$
$$5y-8z=12$$

Solution:-

Consider the system $Ax=B$.

Here $X = \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix}$. Suppose there exists $S = \begin{bmatrix} y_1 \\ y_2 \\ y_n \end{bmatrix}$ such that $AS=B$ then we say S is the solution of the given system.

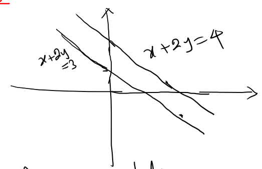
Q:- Does a system $AX=B$ has a solution?

Ans:- No.

See the example $x+2y=3$

$x+2y=4$.

Geometrically:-



This System has no solution.

Inconsistent System:-

Consider the system $AX=B$. We say the system is inconsistent if it does not have any solution

Example:- $x+y=1$

$x+y=3$

clearly this system does not have any solution and so it is inconsistent.

Q:- Suppose the system $AX=B$ has a solution. Does the solution unique?

Ans:- No.

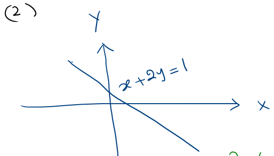
Example:- Consider the example

$x+2y=1 \rightarrow (1)$

$2x+4y=2 \rightarrow (2)$

$(1)\times 2 \Rightarrow (2)$

Geometrically:-



Some solutions are $\{(1,0),(3,-1),\cdots\} = \{(1-2t,t) \mid t \text{ is a real number}\}$

Elementary Row operations:-

Consider the following system.

$x+y+z=3 \rightarrow (1)$

$2x+y+z=4 \rightarrow (2)$

$x+2y+2z=5 \rightarrow (3)$

It is evident that $x=1, y=1, z=1$ is a solution

Suppose $(1) \Leftrightarrow (3)$, we have

$x+2y+2z=5$

$2x+y+z=4$

$x+y+z=3$

Q:- Is still $x=1, y=1, z=1$ a solution?

Ans:- Yes

Eventhough we interchange the equation. Still we have same solution set.

Consistent System:-

Let $AX=B$ be the given system of linear equation.

The system is said to be consistent if it has a solution.

Example:- $x+2y=1$

$2x+y=2$

clearly this system is consistent because it has a solution.

$(1)\times 4 \Rightarrow 4x+4y+4z=12$

$(2) \Rightarrow 2x+y+z=4$

$(3)\times(-5) \Rightarrow -5x-10y-10z=-25$

Q:- Is still $x=1, y=1, z=1$ a solution for the above system?

Ans:- Yes.

multiplying the equation by a constant does not alter the solution nature.

(1) $\Rightarrow$ $x+y+z=3$

(2)+2(1) $\Rightarrow$ $4x+3y+3z=10$

(3)−(1) $\Rightarrow$ $y+z=2$

Q:- Is still $x=1, y=1, z=1$ a solution?

Ans:- Yes.

Addition of two equations by multiplying a constant with it does not alter the nature of the solution.

Augmented matrix:-

Consider the following system,

$$a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=b_1$$
$$a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=b_2$$
$$\vdots$$
$$a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n=b_m$$

the augmented matrix corresponds to the above system is

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

Elementary Row operations:-

1) Addition of a multiple of one row to another

$(j^{th}\text{ row}) \rightarrow (j^{th}\text{ row}) + \alpha(k^{th}\text{ row})$

2) Multiplication of a row by a non zero constant

$(j^{th}\text{ row}) \rightarrow \alpha(j^{th}\text{ row})$

3) Interchange of two rows

$(j^{th}\text{ row}) \leftrightarrow (k^{th}\text{ row})$

Equivalent System

Two linear systems are said equivalent if their solution sets are identical (same)

Example:-

$$\begin{aligned} x+y+z&=3 \\ x+2y+2z&=5 \\ 2x+y+z&=4 \end{aligned} \quad \text{and} \quad \begin{aligned} x+y+z&=3 \\ 3x+4y+4z&=11 \\ x&=1 \end{aligned}$$

are Equivalent.

Theorem:-

If one linear system is obtained from other by a finite number of elementary operations then the two systems are equivalent.

Row equivalent augmented matrices:-

We say two augmented matrices are row equivalent if one can be obtained from other by finitely many row elementary operations.

Remark:- Consider a system $AX=B$ where A is a $m\times n$ matrix. X is a $n\times 1$ matrix and B is a $m\times 1$ matrix.

(1) If $m<n$ then (no. of equation < no. of Variable) system may be consistent (or) inconsistent.

Example:-

$$\begin{aligned} x+y+z&=1 \\ x+y+z&=3 \end{aligned} \quad \text{and} \quad \begin{aligned} x+y+z&=1 \\ 2x+2y+2z&=2 \end{aligned}$$

If it is consistent it has more than one solution

Augmented matrix:-

Consider the following system.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

Corresponding augmented matrix is

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

Elementary Row operations:-

1) $(J^{th}\text{ row}) \rightarrow (J^{th}\text{ row}) + \alpha(k^{th}\text{ row})$

2) $(J^{th}\text{ row}) \rightarrow \alpha(J^{th}\text{ row})$

3) $(J^{th}\text{ row}) \longleftrightarrow (k^{th}\text{ row})$

Pivot element:-

Pivot element of an augmented matrix is an entry of that matrix such that we want to make the elements below to be zero.

Gauss elimination method Procedure:-

1) Write the augmented matrix to the given system
2) Look at the element a_{11} and fix it as a pivot element and do the pivoting in the first column.

(Remark:- If $a_{11} = 0$ then we must interchange that row with a row below to it)

3) Move down to the diagonal to the element a_{22} and do the pivoting about this entry.
4) Continue pivoting in the successive diagonal entries until further pivoting is not possible
5) Stop the process.

Solve the following system by Gauss elimination method

$$x - y - z = 1$$
$$3x + y + z = 9$$
$$x - y + 4z = 8$$

Soln:-

Augmented matrix is

$$\begin{bmatrix} \boxed{1} & -1 & -1 & 1 \\ 3 & 1 & 1 & 9 \\ 1 & -1 & 4 & 8 \end{bmatrix}$$

(a_{11} 1st pivot)

$$\left.\begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \end{array}\right\} \Rightarrow \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & \boxed{4} & 4 & 6 \\ 0 & 0 & \boxed{5} & 7 \end{bmatrix}$$

(2nd pivot, 3rd pivot)

Process is stopped

System corresponds to the row reduced matrix is

$$x - y - z = 1$$
$$4y + 4z = 6$$
$$5z = 7$$

Hence $\boxed{z = 7/5;\ y = 1/10;\ x = 5/2}$

Problem 2

Solve the following system by Gauss elimination method.

$$x + 2y + 3z = 4$$
$$5x + 6y + 7z = 8$$
$$9x + 10y + 11z = 12$$

Solution:- Augmented matrix

$$\begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 - 9R_1 \end{array} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & \boxed{-4} & -8 & -12 \\ 0 & -8 & -16 & -24 \end{bmatrix}$$

(1st pivot, 2nd pivot)

$$R_3 \to R_3 - 2R_2 \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & \boxed{0} & 0 \end{bmatrix}$$

(3rd pivot)

Process stopped

Corresponding system to the row reduced matrix

$$x + 2y + 3z = 4 \rightarrow (4)$$
$$-4y - 8z = -12 \rightarrow (5)$$

Fix $\boxed{y = t}$ then (5) $\Rightarrow -4t - 8z = -12 \Rightarrow -8z = -12 + 4t$

$$\boxed{z = \frac{3 - t}{2}}$$

From 4, $x + 2t + 3\left(\frac{3-t}{2}\right) = 4$

$$\Rightarrow x = 4 - 2t + \frac{3(t-3)}{2}$$

$$= \frac{8 - 4t + 3t - 9}{2}$$

$$\boxed{x = \frac{-t-1}{2}}$$

Solutions are

$$x = \frac{-t-1}{2}; \quad y = t \quad ; z = \frac{3-t}{2} \quad , t \in \mathbb{R}.$$

Gauss-Jordan Elimination Method

The following **row operations** on the augmented matrix of a system produce the augmented matrix of an equivalent system, i.e., a system with the same solution as the original one.

- Interchange any two rows.
- Multiply each element of a row by a nonzero constant.
- Replace a row by the sum of itself and a constant multiple of another row of the matrix.

For these row operations, we will use the following notations.

- $R_i \leftrightarrow R_j$ means: Interchange row i and row j.
- αR_i means: Replace row i with α times row i.
- $R_i + \alpha R_j$ means: Replace row i with the sum of row i and α times row j.

The Gauss-Jordan elimination method to solve a system of linear equations is described in the following steps.

1. Write the augmented matrix of the system.
2. Use row operations to transform the augmented matrix in the form described below, which is called the **reduced row echelon form** (RREF).
   (a) The rows (if any) consisting entirely of zeros are grouped together at the bottom of the matrix.
   (b) In each row that does not consist entirely of zeros, the leftmost nonzero element is a 1 (called a leading 1 or a pivot).
   (c) Each column that contains a leading 1 has zeros in all other entries.
   (d) The leading 1 in any row is to the left of any leading 1's in the rows below it.
3. Stop process in step 2 if you obtain a row whose elements are all zeros except the last one on the right. In that case, the system is inconsistent and has no solutions. Otherwise, finish step 2 and read the solutions of the system from the final matrix.

Note: When doing step 2, row operations can be performed in any order. Try to choose row operations so that as few fractions as possible are carried through the computation. This makes calculation easier when working by hand.

Example 1. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} x + y + z = 5 \\ 2x + 3y + 5z = 8 \\ 4x + 5z = 2 \end{cases}$$

Solution: The augmented matrix of the system is the following.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

We will now perform row operations until we obtain a matrix in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

$$\xrightarrow{R_3-4R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right]$$

$$\xrightarrow{R_3+4R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array}\right]$$

$$\xrightarrow{\frac{1}{13}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2-3R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1-R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

From this final matrix, we can read the solution of the system. It is

$$\boxed{x = 3, \quad y = 4, \quad z = -2.}$$

Example 2. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} x + 2y - 3z = 2 \\ 6x + 3y - 9z = 6 \\ 7x + 14y - 21z = 13 \end{cases}$$

Solution: The augmented matrix of the system is the following.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 6 & 3 & -9 & 6 \\ 7 & 14 & -21 & 13 \end{array}\right]$$

Let's now perform row operations on this augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 6 & 3 & -9 & 6 \\ 7 & 14 & -21 & 13 \end{array}\right] \xrightarrow{R_2-6R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 0 & -9 & 9 & -6 \\ 7 & 14 & -21 & 13 \end{array}\right]$$

$$\xrightarrow{R_3-7R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 0 & -9 & 9 & -6 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

We obtain a row whose elements are all zeros except the last one on the right. Therefore, we conclude that the system of equations is inconsistent, i.e., it has no solutions.

Example 3. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} 4y + z = 2 \\ 2x + 6y - 2z = 3 \\ 4x + 8y - 5z = 4 \end{cases}$$

Solution: The augmented matrix of the system is the following.

$$\left[\begin{array}{ccc|c} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 4 \end{array}\right]$$

We will now perform row operations until we obtain a matrix in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 4 & 8 & -5 & 4 \end{array}\right]$$

$$\xrightarrow{R_3-2R_1} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 0 & -4 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_2} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1-6R_2} \left[\begin{array}{ccc|c} 2 & 0 & -7/2 & 0 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 0 & -7/4 & 0 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This last matrix is in reduced row echelon form so we can stop. It corresponds to the augmented matrix of the following system.

$$\begin{cases} x - \frac{7}{4}z = 0 \\ y + \frac{1}{4}z = \frac{1}{2} \end{cases}$$

We can express the solutions of this system as

$$x = \tfrac{7}{4}z, \quad y = \tfrac{1}{2} - \tfrac{1}{4}z.$$

Since there is no specific value for z, it can be chosen arbitrarily. This means that there are **infinitely many** solutions for this system. We can represent all the solutions by using a parameter t as follows.

$$\boxed{x = \tfrac{7}{4}t, \quad y = \tfrac{1}{2} - \tfrac{1}{4}t, \quad z = t}$$

Any value of the parameter t gives us a solution of the system. For example,

$$t = 4 \quad \text{gives the solution} \quad (x, y, z) = (7, -\tfrac{1}{2}, 4)$$

$$t = -2 \quad \text{gives the solution} \quad (x, y, z) = (-\tfrac{7}{2}, 1, -2).$$

Example 4. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} A+B+2C=1 \\ 2A-B+D=-2 \\ A-B-C-2D=4 \\ 2A-B+2C-D=0 \end{cases}$$

Solution: We will perform row operations on the augmented matrix of the system until we obtain a matrix in reduced row echelon form.

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 2 & -1 & 0 & 1 & -2 \\ 1 & -1 & -1 & -2 & 4 \\ 2 & -1 & 2 & -1 & 0 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 1 & -1 & -1 & -2 & 4 \\ 2 & -1 & 2 & -1 & 0 \end{array}\right] \xrightarrow{R_3-R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & -2 & -3 & -2 & 3 \\ 2 & -1 & 2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_4-2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & -3 & -2 & -1 & -2 \end{array}\right] \xrightarrow{R_4-R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right]$$

$$\xrightarrow{R_2\leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right]$$

$$\xrightarrow{R_3+3R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1/2 & 4 & -17/2 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right] \xrightarrow{2R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right]$$

$$\xrightarrow{R_4-2R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 0 & -18 & 36 \end{array}\right] \xrightarrow{-\frac{1}{18}R_4} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_3-8R_4} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_2-R_4} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2-\frac{3}{2}R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_1-2R_3,\ R_1-R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

From this final matrix, we can read the solution of the system. It is

$$\boxed{A=1, \quad B=2, \quad C=-1, \quad D=-2.}$$

5

In 1st row the nonzero leading entry is 1 which is the coefficient of x. Hence x is a basic variable. In 2nd row the nonzero leading entry is 7 which corresponds to the variable y. Hence y is also a basic variable. z is the free variable.

Problem 2 Investigate the values of λ and μ so that the equations

$$2x+3y+5z=9$$
$$7x+3y-2z=8$$
$$2x+3y+\lambda z=\mu$$

have (a) No solution (b) Unique solution
(c) Infinite number of solution.

Basic and free variables:-

Consider the system $Ax=B$. The variables corresponds to the leading nonzero entries of the row reduced augmented matrix is known as the basic variable and other variables are known as free variables.

Problem 1 Solve the following system by Gauss elimination method and find the basic and free variable of the system

$$2x+3y-2z=4$$
$$x-2y+z=3$$
$$7x-z=2$$

Soln:-

$$\begin{bmatrix} \boxed{2} & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - \frac{7}{2}R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \Rightarrow \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & \frac{-15}{2} & \frac{-39}{2} & \frac{-47}{2} \\ 0 & 0 & \lambda-5 & \mu-9 \end{bmatrix}$$

1st pivot

System corresponds to the row reduced matrix is

$$2x+3y+5z=9$$
$$\left(\frac{-15}{2}\right)y+\left(\frac{-39}{2}\right)z=\frac{-47}{2}$$
$$(\lambda-5)z=(\mu-9)$$

No solution:- If $\lambda=5$ and $\mu\neq 9$ then system has no solution.

Soln:-

$$\begin{bmatrix} 2 & 3 & -2 & 4 \\ 1 & -2 & 1 & 3 \\ 7 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} \boxed{1} & -2 & 1 & 3 \\ 2 & 3 & -2 & 4 \\ 7 & 0 & -1 & 2 \end{bmatrix}$$

1st pivot

$$\begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 7R_1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & \boxed{7} & -4 & -2 \\ 0 & 14 & -8 & -19 \end{bmatrix}$$

2nd pivot

$$R_3 \to R_3 - 2R_2 \Rightarrow \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 7 & -4 & -2 \\ 0 & 0 & 0 & -15 \end{bmatrix}$$

System corresponds to the row reduced equivalent matrix is

$$\left.\begin{matrix} x-2y+z=3 \\ 7y-4z=-2 \\ 0=-15 \end{matrix}\right\} \text{This system is inconsistent. Hence no solution.}$$

Unique solution:- If $\lambda\neq 5$ and μ is any number then the system has unique solution

Infinite solution:- If $\lambda=5$ and $\mu=9$ then the system has infinitely many solution.

Elementary matrices:-

Solve the following system by Gauss elimination method

$$x+y+z=3$$
$$2x+y+z=4$$
$$x+2y+2z=5$$

Soln:- Augmented matrix

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 3 \\ 2 & 1 & 1 & 4 \\ 1 & 2 & 2 & 5 \end{array}\right]$$

1st pivot

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$

$$\left[\boxed{\begin{matrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 2 \end{matrix}} \begin{matrix} 3 \\ -2 \\ 5 \end{matrix}\right] \Rightarrow \left[\boxed{\begin{matrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{matrix}} \begin{matrix} 3 \\ -2 \\ 2 \end{matrix}\right]$$

$$\overset{E_1}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$E_1 A X = E_1 B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$E_2 \cdot \underbrace{E_1 \cdot A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

Elementary matrices:-

A matrix E obtained from the identity matrix (I_n) by executing only one elementary row operation is called elementary matrix.

Examples:-

(i) $\begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}$ multiplying second row of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ by -5 we get this matrix.

2) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Consider $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_1 \to R_1 + 3R_3$

This is also an elementary matrix.

1.7 Elementary matrices

We now return to the system of linear equations $A\mathbf{x} = \mathbf{b}$. If A has a right inverse B such that $AB = I_m$, then $\mathbf{x} = B\mathbf{b}$ is a solution of the system since

$$A\mathbf{x} = A(B\mathbf{b}) = (AB)\mathbf{b} = \mathbf{b}.$$

In particular, if A is an invertible square matrix, then it has only one inverse A^{-1} by Lemma 1.6, and $\mathbf{x} = A^{-1}\mathbf{b}$ is the only solution of the system. In this section, we discuss how to compute A^{-1} when A is invertible.

Recall that Gaussian elimination is a process in which the augmented matrix is transformed into its row-echelon form by a finite number of elementary row operations. In the following, we will show that each elementary row operation can be expressed as a nonsingular matrix, called an *elementary matrix*, and hence the process of Gaussian elimination is simply multiplying a finite sequence of corresponding elementary matrices to the augmented matrix.

Definition 1.9 A matrix E obtained from the identity matrix I_n by executing only one elementary row operation is called an **elementary matrix**.

For example, the following matrices are three elementary matrices corresponding to each type of the three elementary row operations.

(1) $\begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}$: multiply the second row of I_2 by -5;

(2) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$: interchange the second and the fourth rows of I_4;

(3) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$: add 3 times the third row to the first row of I_3.

It is an interesting fact that, if E is an elementary matrix obtained by executing a certain elementary row operation on the identity matrix I_m, then for any $m \times n$ matrix A, *the product EA is exactly the matrix that is obtained when the same elementary row operation in E is executed on A.* The following example illustrates this argument. (Note that AE is not what we want. For this, see Problem 1.17).

Example 1.9 For simplicity, we work on a 3×1 column matrix $\mathbf{b}$. Suppose that we want to do the operation "adding $(-2) \times$ the first row to the second row" on matrix $\mathbf{b}$. Then, we execute this operation on the identity matrix I first to get an elementary matrix E:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiplying the elementary matrix E to $\mathbf{b}$ on the left produces the desired result:

$$E\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 \end{bmatrix},$$

Similarly, the operation "interchanging the first and third rows" on the matrix $\mathbf{b}$ can be achieved by multiplying a *permutation matrix* P, which is an elementary matrix obtained from I_3 by interchanging two rows, to $\mathbf{b}$ on the left:

$$P\mathbf{b} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_3 \\ b_2 \\ b_1 \end{bmatrix}.$$

□

Recall that each elementary row operation has an inverse operation, which is also an elementary operation, that brings the matrix back to the original one. Thus, suppose that E denotes an elementary matrix corresponding to an elementary row operation, and let E' be the elementary matrix corresponding to its "inverse" elementary row operation in E. Then,

(1) if E multiplies a row by $c \neq 0$, then E' multiplies the same row by $\frac{1}{c}$;

(2) if E interchanges two rows, then E' interchanges them again;

(3) if E adds a multiple of one row to another, then E' subtracts it back from the same row.

Thus, for any $m \times n$ matrix A, $E'EA = A$, and $E'E = I = EE'$. That is, *every elementary matrix is invertible so that $E^{-1} = E'$, which is also an elementary matrix.*

For instance, if

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{then}$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/c & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, E_3^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Definition 1.10 **A permutation matrix** is a square matrix obtained from the identity matrix by permuting the rows.

Problem 1.16 Prove:

(1) A permutation matrix is the product of a finite number of elementary matrices each of which is corresponding to the "row-interchanging" elementary row operation.
(2) Any permutation matrix P is invertible and $P^{-1} = P^T$.
(3) The product of any two permutation matrices is a permutation matrix.
(4) The transpose of a permutation matrix is also a permutation matrix.

Problem 1.17 Define the **elementary column operations** for a matrix by just replacing "row" by "column" in the definition of the elementary row operations. Show that if A is an $m \times n$ matrix and if E is an elementary matrix obtained by executing an elementary column operation on I_n, then AE is exactly the matrix that is obtained from A when the same column operation is executed on A.

The next theorem establishes some fundamental relationships between $n \times n$ square matrices and systems of n linear equations in n unknowns.

Theorem 1.8 *Let A be an $n \times n$ matrix. The following are equivalent:*

(1) *A has a left inverse;*

(2) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$;*

(3) *A is row-equivalent to I_n;*

(4) *A is a product of elementary matrices;*

(5) *A is invertible;*

(6) *A has a right inverse.*

Proof: **(1) ⇒ (2)** : Let $\mathbf{x}$ be a solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$, and let B be a left inverse of A. Then

$$\mathbf{x} = I_n\mathbf{x} = (BA)\mathbf{x} = BA\mathbf{x} = B\mathbf{0} = \mathbf{0}.$$

(2) ⇒ (3) : Suppose that the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$:

$$\begin{cases} x_1 & = 0 \\ \quad x_2 & = 0 \\ \qquad \ddots & \\ \qquad\quad x_n & = 0. \end{cases}$$

This means that the augmented matrix $[A\ \mathbf{0}]$ of the system $A\mathbf{x} = \mathbf{0}$ is reduced to the system $[I_n\ \mathbf{0}]$ by Gauss-Jordan elimination. Hence, A is row-equivalent to I_n.

(3) ⇒ (4) : Assume A is row-equivalent to I_n, so that A can be reduced to I_n by a finite sequence of elementary row operations. Thus, we can find elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k \cdots E_2 E_1 A = I_n.$$

Since $E_1, E_2, \ldots, E_k$ are invertible, by multiplying both sides of this equation on the left successively by $E_k^{-1}, \ldots, E_2^{-1}, E_1^{-1}$, we obtain

$$A = E_1^{-1}E_2^{-1} \cdots E_k^{-1} I_n = E_1^{-1}E_2^{-1} \cdots E_k^{-1},$$

which expresses A as the product of elementary matrices.

(4) ⇒ (5) is trivial, because any elementary matrix is invertible. In fact, $A^{-1} = E_k \cdots E_2 E_1$.

(5) ⇒ (1) and **(5) ⇒ (6)** are trivial.

(6) ⇒ (5) : If B is a right inverse of A, then A is a left inverse of B and we can apply (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) to B and conclude that B is invertible, with A as its unique inverse. That is, B is the inverse of A and so A is invertible. □

This theorem shows that a square matrix is invertible if it has a one-side inverse. In particular, if a square matrix A is invertible, then $\mathbf{x} = A^{-1}\mathbf{b}$ is a unique solution to the system $A\mathbf{x} = \mathbf{b}$.

Problem 1.18 Find the inverse of the product

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Recall:- (i) Aim to solve the system $AX=B$. Here A is $m\times n$ matrix, X is $n\times 1$ matrix and B is $m\times 1$ matrix.

(ii) Elimination method gives results very quickly.

(iii) Introduced augmented matrix $[A|B]$

(iv) Elementary row operation:- $R_i \to \alpha R_i$, $R_i \leftrightarrow R_j$,
$R_i \to R_i + \alpha R_j$

(v) We deduce a new row equivalent augmented matrix

(vi) Using row equivalent matrix we solved the system.

(v) Pivot element:- It is an entry in augmented matrix we want to make the elements below are zero.

(vi) Basic & Free variables:- Variables corresponds to the nonzero leading entries are known as basic variables

Q:- $E^{-1} = ?$

$$E^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \quad (R_2 \to R_2 - 2R_1 \text{ in } I_2)$$

Property 3:- If E adds a multiple of one row to another in I_n then E^{-1} subtracts it back from the same row in I_n

Elementary matrices:- A matrix derived from the identity matrix by just applying one elementary row operation.

Eg:- $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};\ R_2 \to 2R_2\right)$

$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ R_2 \to R_2 + 2R_1\right)$

Inverse of Elementary matrices:-

Take $E = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ matrix. What is E^{-1}?

Ans:- $E^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$ $\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \boxed{R_2 \to \frac{1}{2}R_2}\right)$

$E = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \boxed{R_2 \to 2R_2}\right)$

Gauss Jordan elimination method

1) Write the augmented matrix $[A|B]$
2) Apply elementary row operations such that the matrix A will be like Identity matrix.
3) Solve the system using row reduced equivalent matrix.

Problem:-1 Solve the system

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 5 \\ 1 & 4 & 7 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

By Gauss Jordan elimination method.

Property:1 If E multiplies a row $(\alpha \neq 0)$ then E^{-1} multiplies the same row by $\left(\frac{1}{\alpha}\right)$.

Example:-

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$ It is an elementary matrix $(R_2 \leftrightarrow R_3 \text{ in } I_3)$

Q: What is E^{-1}?

Ans:- $E^{-1} = E$.

Property 2:- If E interchanges two rows in I_n then E^{-1} interchanges them again I_n.

Example:- $E = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ This is an elementary matrix $(R_2 \to R_2 + 2R_1 \text{ in } I_2)$

Soln:- 1st pivot

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \\ 1 & 4 & 5 & 3 \\ 1 & 4 & 7 & 5 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & \boxed{3} & 4 & 2 \\ 0 & 3 & 6 & 4 \end{array}\right] R_2 \to \frac{1}{3}R_2$$

2nd pivot

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & \boxed{1} & 4/3 & 2/3 \\ 0 & 3 & 6 & 4 \end{array}\right] \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - 3R_2 \end{array} \Rightarrow \begin{bmatrix} 1 & 0 & -1/3 & +1/3 \\ 0 & 1 & 4/3 & 2/3 \\ 0 & 0 & \boxed{2} & 2 \end{bmatrix} R_3 \to \frac{1}{2}R_3$$

3rd pivot

$$\begin{bmatrix} 1 & 0 & -1/3 & +1/3 \\ 0 & 1 & 4/3 & 2/3 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix} \begin{array}{l} R_1 \to R_1 + \frac{1}{3}R_3 \\ R_2 \to R_2 - 4/3 R_3 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & -2/3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Stop the process.

Solutions are $\boxed{x = 2/3;\ y = -2/3;\ z = 1}$

As an application of the preceding theorem, we give a practical method for finding the inverse A^{-1} of an invertible $n \times n$ matrix A. If A is invertible, there are elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k \cdots E_2 E_1 A = I_n.$$

Hence,

$$A^{-1} = E_k \cdots E_2 E_1 = E_k \cdots E_2 E_1 I_n.$$

It follows that *the sequence of row operations that reduces an invertible matrix A to I_n will resolve I_n to A^{-1}*. In other words, let $[A \mid I]$ be the augmented matrix with the columns of A on the left half, the columns of I on the right half. A Gaussian elimination, applied to both sides, by some elementary row operations reduces the augmented matrix $[A \mid I]$ to $[U \mid K]$, where U is a row-echelon form of A. Next, the back substitution process by another series of elementary row operations reduces $[U \mid K]$ to $[I \mid A^{-1}]$:

$$\begin{aligned} [A \mid I] &\rightarrow [E_\ell \cdots E_1 A \mid E_\ell \cdots E_1 I] = [U \mid K] \\ &\rightarrow [F_k \cdots F_1 U \mid F_k \cdots F_1 K] = [I \mid A^{-1}], \end{aligned}$$

where $E_\ell \cdots E_1$ represents a Gaussian elimination and $F_k \cdots F_1$ represents the back substitution. The following example illustrates the computation of an inverse matrix.

Example 1.10 Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 0 & 2 \end{bmatrix}.$$

We apply Gauss-Jordan elimination to

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 5 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} (-2)\text{row } 1 + \text{row } 2 \\ (-1)\text{row } 1 + \text{row } 3 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right] (-1)\text{row } 2$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right] (2)\text{row } 2 + \text{row } 3$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right].$$

This is $[U \mid K]$ obtained by Gaussian elimination. Now continue the back substitution to reduce $[U \mid K]$ to $[I \mid A^{-1}]$

$$[U \mid K] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] \begin{array}{l} (-1)\text{row } 3 + \text{row } 2 \\ (-3)\text{row } 3 + \text{row } 1 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -8 & 6 & -3 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] (-2)\text{row } 2 + \text{row } 1$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & 4 & -1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right] = [I \mid A^{-1}].$$

Thus, we get

$$A^{-1} = \begin{bmatrix} -6 & 4 & -1 \\ -1 & 1 & -1 \\ 3 & -2 & 1 \end{bmatrix}.$$

(The reader should verify that $AA^{-1} = I = A^{-1}A$.) □

Note that if A is not invertible, then, at some step in Gaussian elimination, a zero row will show up on the left side in $[U \mid K]$. For example, the matrix $A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$ is row-equivalent to $\begin{bmatrix} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 0 & 0 \end{bmatrix}$ which is a noninvertible matrix.

Problem 1.19 Write A^{-1} as a product of elementary matrices for A in Example 1.10.

of A by using Gaussian elimination.

Problem 1.20 Find the inverse of each of the following matrices:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & 2 \\ -6 & 4 & 11 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 4 & 0 \\ 1 & 2 & 4 & 8 \end{bmatrix}, C = \begin{bmatrix} k & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & k \end{bmatrix} (k \neq 0).$$

Gauss Jordan Elimination method

Recall:- Consider the system $Ax = B$.

1) Write down the augmented matrix $[A|B]$
2) Apply elementary row operations until we get a row reduced equivalent matrix of the form of identity matrix
3) Give solution to the given problem with the help of row equivalent matrix.

Problem:1 Solve the following system using Gauss Jordan elemination method

$$\begin{aligned} x + 2y - 3z + 2t &= 2 \\ 2x + 5y - 8z + 6t &= 5 \\ 3x + 4y - 5z + 2t &= 4 \end{aligned}$$

Solution:-

$$\left(\begin{array}{cccc|c} \boxed{1} & 2 & -3 & 2 & 2 \\ 2 & 5 & -8 & 6 & 5 \\ 3 & 4 & -5 & 2 & 4 \end{array}\right) \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array} \Bigg\} \Rightarrow \left(\begin{array}{cccc|c} 1 & 2 & -3 & 2 & 2 \\ 0 & \boxed{1} & -2 & 2 & 1 \\ 0 & -2 & 4 & -4 & -2 \end{array}\right)$$

1st pivot; 2nd pivot

$$\begin{array}{l} R_1 \to R_1 - 2R_2 \\ R_3 \to R_3 + 2R_2 \end{array} \Bigg\} \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Stop the process.

System corresponds to the row equivalent matrix is

$$x + z - 2t = 0 \longrightarrow (4)$$

$$y - 2z + 2t = 1 \longrightarrow (5)$$

Fix $z = \alpha$, $t = \beta$ then (4) $\Rightarrow$ $x = 2\beta - \alpha$ and

(5) $\Rightarrow$ $y = 1 + 2\alpha - 2\beta$

Thus

$x = 2\beta - \alpha$; $y = 1 + 2\alpha - 2\beta$; $z = \alpha$, $t = \beta$, Here α, β are real numbers.

Invertible matrices:-

Let A be a $m \times n$ matrix, B and C are $n \times m$ matrices.

B is said to be left inverse of A if $BA = I_n$ (identity matrix of order n)

C is said to be right inverse of A if $AC = I_m$ (identity matrix of order m)

Examples:-

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 1 & -3 \\ -1 & 5 \\ -2 & 7 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ B is the right inverse of A

A is the left inverse of B

$BA = \begin{bmatrix} -5 & 2 & -4 \\ 9 & -2 & 6 \\ 12 & -4 & 9 \end{bmatrix}$ B is not the left inverse of A

A is not the right inverse of A

Q:- Let A be a $\boxed{m \times n}$ matrix. Is the left (or) right inverse of A unique?

Ans:- More than one left (or) right inverse exists.

Example:-

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \end{bmatrix}$ α, β are real numbers

$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Hence left inverse is not unique.

$AB = \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 0 \end{bmatrix}$

Let A be a $m \times n$ matrix. Suppose A is right invertible then A^T is left invertible.

Proof:

Given that A is right invertible, there exists B of order $n \times m$ s.t

$$AB = I_m$$

$$(AB)^T = I_m^T$$

$$B^T A^T = I_m$$

$\Rightarrow A^T$ is left invertible.

Note:- A is Right invertible $\Longleftrightarrow$ A^T is left invertible.

Theorem:-

Let A be a $n \times n$ matrix. Suppose A has left inverse B and a right inverse C then $B = C$

Proof:-

$$B = B \cdot I_n = B(AC) = (BA)C = I_n \cdot C = C$$

Theorem:- Let A be a $\boxed{n \times n}$ matrix. The following are equivalent

(a) A has a left inverse

(b) System $Ax = 0$ has only trivial solution $x = 0$

(c) A is row equivalent to I_n

(d) A is the product of elementary matrices

(e) A is invertible

(f) A has right inverse

Gauss-Jordan Elimination Method

The following **row operations** on the augmented matrix of a system produce the augmented matrix of an equivalent system, i.e., a system with the same solution as the original one.

- Interchange any two rows.
- Multiply each element of a row by a nonzero constant.
- Replace a row by the sum of itself and a constant multiple of another row of the matrix.

For these row operations, we will use the following notations.

- $R_i \leftrightarrow R_j$ means: Interchange row i and row j.
- αR_i means: Replace row i with α times row i.
- $R_i + \alpha R_j$ means: Replace row i with the sum of row i and α times row j.

The Gauss-Jordan elimination method to solve a system of linear equations is described in the following steps.

1. Write the augmented matrix of the system.
2. Use row operations to transform the augmented matrix in the form described below, which is called the **reduced row echelon form** (RREF).
   (a) The rows (if any) consisting entirely of zeros are grouped together at the bottom of the matrix.
   (b) In each row that does not consist entirely of zeros, the leftmost nonzero element is a 1 (called a leading 1 or a pivot).
   (c) Each column that contains a leading 1 has zeros in all other entries.
   (d) The leading 1 in any row is to the left of any leading 1's in the rows below it.
3. Stop process in step 2 if you obtain a row whose elements are all zeros except the last one on the right. In that case, the system is inconsistent and has no solutions. Otherwise, finish step 2 and read the solutions of the system from the final matrix.

Note: When doing step 2, row operations can be performed in any order. Try to choose row operations so that as few fractions as possible are carried through the computation. This makes calculation easier when working by hand.

Example 1. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} x + y + z = 5 \\ 2x + 3y + 5z = 8 \\ 4x + 5z = 2 \end{cases}$$

Solution: The augmented matrix of the system is the following.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

We will now perform row operations until we obtain a matrix in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 - 4R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right]$$

$$\xrightarrow{R_3 + 4R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array}\right]$$

$$\xrightarrow{\frac{1}{13}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2 - 3R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

From this final matrix, we can read the solution of the system. It is

$$\boxed{x = 3, \quad y = 4, \quad z = -2.}$$

Example 2. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} x + 2y - 3z = 2 \\ 6x + 3y - 9z = 6 \\ 7x + 14y - 21z = 13 \end{cases}$$

Solution: The augmented matrix of the system is the following.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 6 & 3 & -9 & 6 \\ 7 & 14 & -21 & 13 \end{array}\right]$$

Let's now perform row operations on this augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 6 & 3 & -9 & 6 \\ 7 & 14 & -21 & 13 \end{array}\right] \xrightarrow{R_2 - 6R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 0 & -9 & 9 & -6 \\ 7 & 14 & -21 & 13 \end{array}\right]$$

$$\xrightarrow{R_3 - 7R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 0 & -9 & 9 & -6 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

We obtain a row whose elements are all zeros except the last one on the right. Therefore, we conclude that the system of equations is inconsistent, i.e., it has no solutions.

Example 3. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} 4y + z = 2 \\ 2x + 6y - 2z = 3 \\ 4x + 8y - 5z = 4 \end{cases}$$

Solution: The augmented matrix of the system is the following.

$$\left[\begin{array}{ccc|c} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 4 \end{array}\right]$$

We will now perform row operations until we obtain a matrix in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 4 & 8 & -5 & 4 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_1} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 0 & -4 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_2} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - 6R_2} \left[\begin{array}{ccc|c} 2 & 0 & -7/2 & 0 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 0 & -7/4 & 0 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This last matrix is in reduced row echelon form so we can stop. It corresponds to the augmented matrix of the following system.

$$\begin{cases} x - \frac{7}{4}z = 0 \\ y + \frac{1}{4}z = \frac{1}{2} \end{cases}$$

We can express the solutions of this system as

$$x = \tfrac{7}{4}z, \quad y = \tfrac{1}{2} - \tfrac{1}{4}z.$$

Since there is no specific value for z, it can be chosen arbitrarily. This means that there are **infinitely many** solutions for this system. We can represent all the solutions by using a parameter t as follows.

$$\boxed{x = \tfrac{7}{4}t, \quad y = \tfrac{1}{2} - \tfrac{1}{4}t, \quad z = t}$$

Any value of the parameter t gives us a solution of the system. For example,

$$t = 4 \quad \text{gives the solution} \quad (x, y, z) = (7, -\tfrac{1}{2}, 4)$$

$$t = -2 \quad \text{gives the solution} \quad (x, y, z) = (-\tfrac{7}{2}, 1, -2).$$

Example 4. Solve the following system by using the Gauss-Jordan elimination method.

$$\begin{cases} A + B + 2C = 1 \\ 2A - B + D = -2 \\ A - B - C - 2D = 4 \\ 2A - B + 2C - D = 0 \end{cases}$$

Solution: We will perform row operations on the augmented matrix of the system until we obtain a matrix in reduced row echelon form.

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 2 & -1 & 0 & 1 & -2 \\ 1 & -1 & -1 & -2 & 4 \\ 2 & -1 & 2 & -1 & 0 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 1 & -1 & -1 & -2 & 4 \\ 2 & -1 & 2 & -1 & 0 \end{array}\right] \xrightarrow{R_3-R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & -2 & -3 & -2 & 3 \\ 2 & -1 & 2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_4-2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & -3 & -2 & -1 & -2 \end{array}\right] \xrightarrow{R_4-R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right]$$

$$\xrightarrow{R_2\leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right]$$

$$\xrightarrow{R_3+3R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1/2 & 4 & -17/2 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right] \xrightarrow{2R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right]$$

$$\xrightarrow{R_4-2R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 0 & -18 & 36 \end{array}\right] \xrightarrow{-\frac{1}{18}R_4} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_3-8R_4} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 1 & -3/2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_2-R_4} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2-\frac{3}{2}R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_1-2R_3,\ R_1-R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

From this final matrix, we can read the solution of the system. It is

$$\boxed{A = 1, \quad B = 2, \quad C = -1, \quad D = -2.}$$

Recall:- Aim:- Solve the system $AX = B$

1) Gauss elemination method → write $[A|B]$ → Apply elementary row operations $\left(\begin{array}{l} R_j \leftrightarrow R_i \\ R_j \to R_j + \alpha R_i \\ R_j \to \alpha R_j \end{array} \right)$ → Try to get triangular like form.

2) Elementary row operations mathematically known as Elementary matrices (Just applying one elementary row operation)

3) Gauss Jordan elimination method → we try to reduce the given system into identity like form

4) Invertible matrices → left & Right inverse of a $m \times n$ matrix
→ if $m = n$, (or) A is a square matrix then we have only one inverse (Right inverse = left inverse)

Calculating inverse by Gauss Jordan elimination method

Let A be a $n \times n$ invertible matrix (A^{-1} exists that is nothing but $\det A \neq 0$). We proceed by the following steps to find A^{-1}

1) Write the augmented matrix $[A | I_n]$ (I_n is $n \times n$ identity matrix)

2) Apply elementary operations successively untill we make A into I_n.

3) Suppose the row reduced augmented matrix is $[I_n | B]$ then $B = A^{-1}$.

Problem:1 Find inverse of the following matrix by Gauss Jordan elemination method

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 0 & 2 \end{bmatrix}$$

Soln:-

1st pivot

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 5 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array} \Big\} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right] R_2 \to (-1)R_2$$

2nd pivot

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 2 & -1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1 - 2R_2 \\ R_3 \to R_3 + 2R_2 \end{array} \Big\} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -3 & 2 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & \boxed{1} & 3 & -2 & 1 \end{array}\right]$$

3rd pivot

$$\begin{array}{l} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - R_3 \end{array} \Big\} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & 4 & -1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 3 & -2 & 1 \end{array}\right]$$

Now

$$A^{-1} = \begin{bmatrix} -6 & 4 & -1 \\ -1 & 1 & -1 \\ 3 & -2 & 1 \end{bmatrix}$$

Pb 2 Find the inverse of the following matrix by Gauss Jordan method

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

Soln:-

1st pivot

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right] R_1 \to R_2 - R_1 \Rightarrow \left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 0 & -1 & 1 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array} \Big\} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 3 & -2 & 0 \\ 0 & -1 & 2 & 2 & -2 & 1 \end{array}\right] R_2 \to (-1)R_2$$

2nd pivot

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & \boxed{+1} & -1 & -3 & 2 & 0 \\ 0 & -1 & 2 & 2 & -2 & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + R_2 \end{array}$$

3rd pivot

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & \boxed{1} & -1 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1 - R_3 \\ R_2 \to R_2 + R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

Hence

$$A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

LU Decomposition:-

$L \to$ Lower triangular matrix

$U \to$ Upper triangular matrix.

Example:-

$$\begin{aligned} x + y + z &= 3 \\ 2x + y + z &= 4 \\ x + 2y + 2z &= 5 \end{aligned}$$

Soln:-

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}}$$

$$\underset{E_1}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}} = \underset{U_1}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 2 \end{bmatrix}}$$

$$\underset{E_2}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}} \underset{U_1}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 2 \end{bmatrix}} = \underset{U_2}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}}$$

$$\underset{E_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}} \underset{U_2}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}} = \underset{U_3}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}$$

$$E_3 E_2 E_1 A = U_3$$

$$A = \underbrace{(E_1^{-1} E_2^{-1} E_3^{-1})}_{\text{Lower triangular}} U_3 \to \text{Upper triangul.}$$

Problem 1.12 Compute AB using block multiplication, where

$$A=\left[\begin{array}{rr|rr} 1 & 2 & 1 & 0 \\ -3 & 4 & 0 & 1 \\ \hline 0 & 0 & 2 & -1 \end{array}\right], \quad B=\left[\begin{array}{rr|r} 1 & 0 & 2 \\ 0 & 1 & 3 \\ \hline 2 & 3 & 4 \\ 3 & -2 & 1 \end{array}\right].$$

1.6 Inverse matrices

As we saw in Section 1.4, a system of linear equations can be written as $A\mathbf{x} = \mathbf{b}$ in matrix form. This form resembles one of the simplest linear equation in one variable $ax = b$ whose solution is simply $x = a^{-1}b$ when $a \neq 0$. Thus it is tempting to write the solution of the system as $\mathbf{x} = A^{-1}\mathbf{b}$. However, in the case of matrices we first have to have a precise meaning of A^{-1}. To discuss this we begin with the following definition.

Definition 1.7 For an $m \times n$ matrix A, an $n \times m$ matrix B is called a **left inverse** of A if $BA = I_n$, and an $n \times m$ matrix C is called a **right inverse** of A if $AC = I_m$.

Example 1.7 From a direct calculation for two matrices

$$A=\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix} \text{ and } B=\begin{bmatrix} 1 & -3 \\ -1 & 5 \\ -2 & 7 \end{bmatrix},$$

we have $AB = I_2$, and $BA = \begin{bmatrix} -5 & 2 & -4 \\ 9 & -2 & 6 \\ 12 & -4 & 9 \end{bmatrix} \neq I_3$.

Thus, the matrix B is a right inverse but not a left inverse of A, while A is a left inverse but not a right inverse of B. Since $(AB)^T = B^TA^T$ and $I^T = I$, a matrix A has a right inverse if and only if A^T has a left inverse. □

However, if A is a square matrix and has a left inverse, then we prove later (Theorem 1.8) that it has also a right inverse, and vice versa. Moreover, the following lemma shows that the left inverses and the right inverses of a square matrix are all equal. (This is not true for nonsquare matrices, of course).

Lemma 1.6 *If an $n \times n$ square matrix A has a left inverse B and a right inverse C, then B and C are equal, i.e., $B = C$.*

Proof: A direct calculation shows that

$$B = BI = B(AC) = (BA)C = IC = C.$$

Now any two left inverses must be both equal to a right inverse C, and hence to each other, and any two right inverses must be both equal to a left inverse B, and hence to each other. So there exist only one left and only one right inverse for a square matrix A ifit is known that A has both left and right inverses. Furthermore, the left and right inverses are equal. □

This theorem says that if a matrix A has both a right inverse and a left inverse, then they must be the same. However, we shall see in Chapter 3 that any $m\times n$ matrix A with $m \neq n$ cannot have both a right inverse and a left inverse: that is, a nonsquare matrix may have only a left inverse or only a right inverse. In this case, the matrix may have many left inverses or many right inverses.

Example 1.8 A nonsquare matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ can have more than one left inverse. In fact, for any $x, y \in \mathbb{R}$, one can easily check that the matrix $B = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \end{bmatrix}$ is a left inverse of A. □

Definition 1.8 An $n \times n$ square matrix A is said to be **invertible** (or **nonsingular**) if there exists a square matrix B of the same size such that

$$AB = I = BA.$$

Such a matrix B is called the **inverse** of A, and is denoted by A^{-1}. A matrix A is said to be **singular** if it is not invertible.

Note that Lemma 1.6 shows that if a square matrix A has both left and right inverses, then it must be unique. That is why we call B "the" inverse of A. For instance, consider a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then it is easy to verify that

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix},$$

since $AA^{-1} = I_2 = A^{-1}A$. (Check this product of matrices for practice!) Note that any zero matrix is singular.

Problem 1.13 Let A be an invertible matrix and k any nonzero scalar. Show that
(1) A^{-1} is invertible and $(A^{-1})^{-1} = A$;
(2) the matrix kA is invertible and $(kA)^{-1} = \frac{1}{k}A^{-1}$;
(3) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

Theorem 1.7 *The product of invertible matrices is also invertible, whose inverse is the product of the individual inverses in reverse order:*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof: Suppose that A and B are invertible matrices of the same size. Then $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$, and similarly $(B^{-1}A^{-1})(AB) = I$. Thus AB has the inverse $B^{-1}A^{-1}$. □

We have written the inverse of A as "A to the power -1", so we can give the meaning of A^k for any integer k: Let A be a square matrix. Define $A^0 = I$. Then, for any positive integer k, we define the power A^k of A inductively as

$$A^k = A(A^{k-1}).$$

Moreover, if A is invertible, then the negative integer power is defined as

$$A^{-k} = (A^{-1})^k \quad \text{for } k > 0.$$

It is easy to check that with these rules we have $A^{k+\ell} = A^kA^\ell$ whenever the right hand side is defined. (If A is not invertible, $A^{3+(-1)}$ is defined but A^{-1} is not.)

Problem 1.14 Prove:
(1) If A has a zero row, so does AB.
(2) If B has a zero column, so does AB.
(3) Any matrix with a zero row or a zero column cannot be invertible.

Problem 1.15 Let A be an invertible matrix. Is it true that $(A^k)^T = (A^T)^k$ for any integer k? Justify your answer.

LU decomposition

Consider a matrix A of order $m \times n$

$$E_k \cdots E_2 E_1 A = U$$

$$A = \underbrace{E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_k^{-1}}_{\text{Lower triangular matrices}} U \rightarrow \text{like upper triangular matrix}$$

$$A = LU$$

Calculating LU decomposition:-

(a) Apply elementary row operations to the matrix A to get the upper triangular like matrix U

(b) The element l_{ij} $(i>j)$ in the matrix L is the multiplier coefficients because of which the respective positions becomes zero and $l_{ii} = 1$

Solving the system of linear equation by LU decomposition method:-

(a) Find LU decomposition for A.

(b) Take $UX = Y$

(c) Solve the system $LY = B$ for Y

(d) Solve the system $UX = Y$ for X.

Problem:1 Solve the following system by LU factorization method

$$x - y = 1$$
$$-x + 2y - z = 1$$
$$-y + 2z = 1$$

Soln:- The matrix of the given system is

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Step:1 LU decomposition for given matrix A

$$\begin{bmatrix} \boxed{1}^{\text{1st pivot}} & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - (-1 R_1) \\ R_3 \to R_3 - (0 R_1) \end{matrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & \boxed{1}^{\text{2nd pivot}} & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$R_3 \to R_3 - (-1 R_2) \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Step 2:- We have $LU = A$, $\to$ (i)

we need to solve $AX = B$

$L\boxed{UX} = B$

Take $UX = Y$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

First we solve $LY = B$ for Y.

Step 3:-

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{y_1 = 1; \quad y_2 = 2 \quad y_3 = 3}$$

Step 4 Solve the system $UX = Y$ for X.

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \boxed{z = 3; \quad y = 5 \quad x = 6}$$

Problem 2 Solve the following system $AX = B$ by LU factorization method

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 1 & 0 & 1 \\ -2 & 2 & 1 & 1 \end{bmatrix}; \quad X = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}$$

Soln:- 1st pivot

$$\begin{bmatrix} \boxed{2} & 1 & 1 & 0 \\ 4 & 1 & 0 & 1 \\ -2 & 2 & 1 & 1 \end{bmatrix} \quad \left.\begin{matrix} R_2 \to R_2 - (2R_1) \\ R_3 \to R_3 - (-1R_1) \end{matrix}\right\} \Rightarrow \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & \boxed{-1} & -2 & 1 \\ 0 & 3 & 2 & 1 \end{bmatrix}$$

2nd pivot

$$R_3 \to R_3 - (-3R_2) \Rightarrow \underbrace{\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -4 & 4 \end{bmatrix}}_{U} \quad \text{(like upper triangular)}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix}$$

$$A = L \quad U$$
$$(3\times4) = (3\times3) \quad (3\times4)$$

Step 2: we need to solve $AX = B$

$LUX = B$

Take $UX = Y$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

So, we have $LY = B$

Step 3:- we solve the system $LY = B$ for Y.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}$$

$$\boxed{y_1 = 1; \quad y_2 = -4 \quad y_3 = -4}$$

Step 4: we solve the system $UX = Y$ for X

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -4 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ -4 \end{bmatrix}$$

$2x + y + z = 1 \to (5)$

$-y - 2z + t = -4 \to (6)$

$-4z + 4t = -4 \to (6)$

$-\alpha) + (1+\alpha)$

Take $\boxed{t = \alpha}$ then $\boxed{z = 1+\alpha}$

$(5) \Rightarrow -y - 2 - 2\alpha + \alpha = -4$

$-y - 2 - \alpha = -4$

$-y = -2 + \alpha$

$\boxed{y = 2 - \alpha}$

From (6) $\Rightarrow 2x + 2 - \alpha + 1 + \alpha = 1$

$2x + 3 = 1$

$2x = -2$

$\boxed{x = -1}$

Triangular Factorization

If an $n \times n$ matrix A can be reduced to strict upper triangular form using only row operation III, then it is possible to represent the reduction process in terms of a matrix factorization. We illustrate how this is done in the next example.

EXAMPLE 6 Let

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}$$

and let us use only row operation III to carry out the reduction process. At the first step, we subtract $\frac{1}{2}$ times the first row from the second and then we subtract twice the first row from the third.

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix}$$

To keep track of the multiples of the first row that were subtracted, we set $l_{21} = \frac{1}{2}$ and $l_{31} = 2$. We complete the elimination process by eliminating the -9 in the (3,2) position.

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix}$$

Let $l_{32} = -3$, the multiple of the second row subtracted from the third row. If we call the resulting matrix U and set

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$$

then it is easily verified that

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} = A$$ ■

The matrix L in the previous example is lower triangular with 1's on the diagonal. We say that L is *unit lower triangular*. The factorization of the matrix A into a product of a unit lower triangular matrix L times a strictly upper triangular matrix U is often referred to as an *LU factorization*.

To see why the factorization in Example 6 works, let us view the reduction process in terms of elementary matrices. The three row operations that were applied to the matrix A can be represented in terms of multiplications by elementary matrices

$$E_3E_2E_1A = U \tag{3}$$

where

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

correspond to the row operations in the reduction process. Since each of the elementary matrices is nonsingular, we can multiply equation (3) by their inverses.

$$A = E_1^{-1}E_2^{-1}E_3^{-1}U$$

[We multiply in reverse order because $(E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}$.] However, when the inverses are multiplied in this order, the multipliers l_{21}, l_{31}, l_{32} fill in below the diagonal in the product:

$$E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} = L$$

In general, if an $n \times n$ matrix A can be reduced to strict upper triangular form using only row operation III, then A has an LU factorization. The matrix L is unit lower triangular, and if $i > j$, then l_{ij} is the multiple of the jth row subtracted from the ith row during the reduction process.

The LU factorization is a very useful way of viewing the elimination process. We will find it particularly useful in Chapter 7 when we study computer methods for solving linear systems. Many of the major topics in linear algebra can be viewed in terms of matrix factorizations. We will study other interesting and important factorizations in Chapters 5 through 7.

SECTION 1.5 EXERCISES

1. Which of the matrices that follow are elementary matrices? Classify each elementary matrix by type.

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. Find the inverse of each matrix in Exercise 1. For each elementary matrix, verify that its inverse is an elementary matrix of the same type.

3. For each of the following pairs of matrices, find an elementary matrix E such that $EA = B$.

(a) $A = \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 2 \\ 5 & 3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 4 & 5 \\ 3 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ -2 & 4 & 5 \end{bmatrix}$

(c) $A = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ -2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ 0 & 3 & 5 \end{bmatrix}$

4. For each of the following pairs of matrices, find an elementary matrix E such that $AE = B$.

(a) $A = \begin{bmatrix} 4 & 1 & 3 \\ 2 & 1 & 4 \\ 1 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 & 4 \\ 4 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}$

(c) $A = \begin{bmatrix} 4 & -2 & 3 \\ -2 & 4 & 2 \\ 6 & 1 & -2 \end{bmatrix}$,

$B = \begin{bmatrix} 2 & -2 & 3 \\ -1 & 4 & 2 \\ 3 & 1 & -2 \end{bmatrix}$

5. Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 2 & 2 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 2 & 2 & 6 \end{bmatrix}$$

(a) Find an elementary matrix E such that $EA = B$.

(b) Find an elementary matrix F such that $FB = C$.

(c) Is C row equivalent to A? Explain.

6. Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{bmatrix}$$

(a) Find elementary matrices E_1, E_2, E_3 such that
$$E_3E_2E_1A = U$$
where U is an upper triangular matrix.

(b) Determine the inverses of E_1, E_2, E_3 and set $L = E_1^{-1}E_2^{-1}E_3^{-1}$. What type of matrix is L? Verify that $A = LU$.

7. Let

$$A = \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix}$$

(a) Express A^{-1} as a product of elementary matrices.

(b) Express A as a product of elementary matrices.

8. Compute the LU factorization of each of the following matrices.

(a) $\begin{bmatrix} 3 & 1 \\ 9 & 5 \end{bmatrix}$ **(b)** $\begin{bmatrix} 2 & 4 \\ -2 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ -2 & 2 & 7 \end{bmatrix}$ **(d)** $\begin{bmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{bmatrix}$

9. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{bmatrix}$$

(a) Verify that

$$A^{-1} = \begin{bmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{bmatrix}$$

(b) Use A^{-1} to solve $A\mathbf{x} = \mathbf{b}$ for the following choices of $\mathbf{b}$.

(i) $\mathbf{b} = (1, 1, 1)^T$ **(ii)** $\mathbf{b} = (1, 2, 3)^T$

(iii) $\mathbf{b} = (-2, 1, 0)^T$

10. Find the inverse of each of the following matrices.

(a) $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 6 \\ 3 & 8 \end{bmatrix}$ **(d)** $\begin{bmatrix} 3 & 0 \\ 9 & 3 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ **(f)** $\begin{bmatrix} 2 & 0 & 5 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

(g) $\begin{bmatrix} -1 & -3 & -3 \\ 2 & 6 & 1 \\ 3 & 8 & 3 \end{bmatrix}$ **(h)** $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -2 & -3 \end{bmatrix}$

11. Given

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

compute A^{-1} and use it to:

(a) Find a 2×2 matrix X such that $AX = B$.

(b) Find a 2×2 matrix Y such that $YA = B$.

12. Let

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$$

Solve each of the following matrix equations.

(a) $AX + B = C$ **(b)** $XA + B = C$

(c) $AX + B = X$ **(d)** $XA + C = X$

13. Is the transpose of an elementary matrix an elementary matrix of the same type? Is the product of two elementary matrices an elementary matrix?

14. Let U and R be $n \times n$ upper triangular matrices and set $T = UR$. Show that T is also upper triangular and that $t_{jj} = u_{jj}r_{jj}$ for $j = 1, \ldots, n$.

15. Let A be a 3×3 matrix and suppose that

$$2\mathbf{a}_1 + \mathbf{a}_2 - 4\mathbf{a}_3 = \mathbf{0}$$

How many solutions will the system $A\mathbf{x} = \mathbf{0}$ have? Explain. Is A nonsingular? Explain.

16. Let A be a 3×3 matrix and suppose that

$$\mathbf{a}_1 = 3\mathbf{a}_2 - 2\mathbf{a}_3$$

Will the system $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution? Is A nonsingular? Explain your answers.

17. Let A and B be $n \times n$ matrices and let $C = A - B$. Show that if $A\mathbf{x}_0 = B\mathbf{x}_0$ and $\mathbf{x}_0 \neq \mathbf{0}$, then C must be singular.

18. Let A and B be $n \times n$ matrices and let $C = AB$. Prove that if B is singular then C must be singular. *Hint*: Use Theorem 1.5.2.

19. Let U be an $n \times n$ upper triangular matrix with nonzero diagonal entries.

(a) Explain why U must be nonsingular.

(b) Explain why U^{-1} must be upper triangular.

20. Let A be a nonsingular $n \times n$ matrix and let B be an $n \times r$ matrix. Show that the reduced row echelon form of $(A|B)$ is $(I|C)$, where $C = A^{-1}B$.

21. In general, matrix multiplication is not commutative (i.e., $AB \neq BA$). However, in certain special cases the commutative property does hold. Show that

(a) if D_1 and D_2 are $n \times n$ diagonal matrices, then $D_1D_2 = D_2D_1$.

(b) if A is an $n \times n$ matrix and

$$B = a_0I + a_1A + a_2A^2 + \cdots + a_kA^k$$

where $a_0, a_1, \ldots, a_k$ are scalars, then $AB = BA$.

22. Show that if A is a symmetric nonsingular matrix then A^{-1} is also symmetric.

23. Prove that if A is row equivalent to B then B is row equivalent to A.

24. (a) Prove that if A is row equivalent to B and B is row equivalent to C, then A is row equivalent to C.

(b) Prove that any two nonsingular $n \times n$ matrices are row equivalent.

25. Let A and B be an $m \times n$ matrices. Prove that if B is row equivalent to A and U is any row echelon form of A, then B is row equivalent to U.

26. Prove that B is row equivalent to A if and only if there exists a nonsingular matrix M such that $B = MA$.

27. Is it possible for a singular matrix B to be row equivalent to a nonsingular matrix A? Explain.

28. Given a vector $\mathbf{x} \in \mathbb{R}^{n+1}$, the $(n+1)\times(n+1)$ matrix V defined by

$$v_{ij} = \begin{cases} 1 & \text{if } j = 1 \\ x_i^{j-1} & \text{for } j = 2, \ldots, n+1 \end{cases}$$

is called the Vandermonde matrix.

(a) Show that if

$$V\mathbf{c} = \mathbf{y}$$

and

$$p(x) = c_1 + c_2x + \cdots + c_{n+1}x^n$$

then

$$p(x_i) = y_i, \qquad i = 1, 2, \ldots, n+1$$

(b) Suppose that $x_1, x_2, \ldots, x_{n+1}$ are all distinct. Show that if $\mathbf{c}$ is a solution of $V\mathbf{x} = \mathbf{0}$ then the coefficients $c_1, c_2, \ldots, c_n$ must all be zero, and hence V must be nonsingular.

For each of following, answer true *if the statement is always true and answer* false *otherwise. In the case of a true statement, explain or prove your answer. In the case of a false statement, give an example to show that the statement is not always true.*

29. If A is row equivalent to I and $AB = AC$, then B must equal C.

30. If E and F are elementary matrices and $G = EF$, then G is nonsingular.

31. If A is a 4×4 matrix and $\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{a}_3 + 2\mathbf{a}_4$, then A must be singular.

32. If A is row equivalent to both B and C, then A is row equivalent to $B + C$.

1.6 Partitioned Matrices

Often it is useful to think of a matrix as being composed of a number of submatrices. A matrix C can be partitioned into smaller matrices by drawing horizontal lines between the rows and vertical lines between the columns. The smaller matrices are often referred to as *blocks*. For example, let

$$C = \begin{bmatrix} 1 & -2 & 4 & 1 & 3 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 3 & 2 & -1 & 2 \\ 4 & 6 & 2 & 2 & 4 \end{bmatrix}$$

If lines are drawn between the second and third rows and between the third and fourth columns, then C will be divided into four submatrices, C_{11}, C_{12}, C_{21}, and C_{22}.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \left[\begin{array}{ccc|cc} 1 & -2 & 4 & 1 & 3 \\ 2 & 1 & 1 & 1 & 1 \\ \hline 3 & 3 & 2 & -1 & 2 \\ 4 & 6 & 2 & 2 & 4 \end{array}\right]$$

One useful way of partitioning a matrix is to partition it into columns. For example, if

$$B = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 4 & 1 \end{bmatrix}$$

we can partition B into three column submatrices:

$$B = (\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3) = \left[\begin{array}{c|c|c} -1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 4 & 1 \end{array}\right]$$

Suppose that we are given a matrix A with three columns; then the product AB can be viewed as a block multiplication. Each block of B is multiplied by A and the result is a matrix with three blocks: $A\mathbf{b}_1$, $A\mathbf{b}_2$, and $A\mathbf{b}_3$; that is,

$$AB = A(\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3) = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \end{bmatrix}$$

For example, if

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

then

$$A\mathbf{b}_1 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}, \quad A\mathbf{b}_2 = \begin{bmatrix} 15 \\ -1 \end{bmatrix}, \quad A\mathbf{b}_3 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

and hence

$$A(\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3) = \left[\begin{array}{c|c|c} 6 & 15 & 5 \\ -2 & -1 & 1 \end{array}\right]$$

In general, if A is an $m \times n$ matrix and B is an $n \times r$ matrix that has been partitioned into columns $\begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_r \end{bmatrix}$, then the block multiplication of A times B is given by

$$\boxed{AB = (A\mathbf{b}_1\ A\mathbf{b}_2\ \cdots\ A\mathbf{b}_r)}$$

In particular,

$$(\mathbf{a}_1\ \cdots\ \mathbf{a}_n) = A = AI = (A\mathbf{e}_1\ \cdots\ A\mathbf{e}_n)$$

Permutation matrix:-

A matrix A which is derived from the identity matrix by permuting its rows (Rearrangement) is known as the permutation matrix.

Example:-

The matrix $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is a permutation matrix

The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is not a permutation matrix

Module-2

Vector space

Why do we need vector space concept:-

$$x + y + z = 1 \quad \to (1)$$
$$2x + y + z = 2 \quad \to (2)$$
$$2x + 2y + 2z = 4 \quad \to (3)$$

(1) & (3) $\Rightarrow$ $\left.\begin{array}{l} x + y + z = 1 \\ x + y + z = 2 \end{array}\right\} \Rightarrow 1 = 2$

This system has no solution.

In real life situation we have many systems they do not have solution.

Aim:- Try to find an "approximate" solution to the system

Notations:- $\mathbb{R} \to$ set of real numbers.

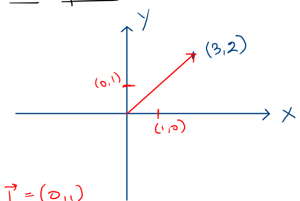
$\mathbb{R}^2 \to$ 2d-space (X-Y plane)

$\mathbb{R}^3 \to$ 3d-space (X-Y-Z-plane)

$\vdots$

$\mathbb{R}^m \to$ md-space

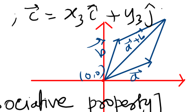
Facts about 2d-space:-



$\vec{i} = (1,0)$; $\vec{j} = (0,1)$

$(3,2) = 3(1,0) + 2(0,1) \sim 3\vec{i} + 2\vec{j}$

$\vec{a} = x_1\hat{i} + y_1\hat{j}$; $\vec{b} = x_2\hat{i} + y_2\hat{j}$; $\vec{c} = x_3\hat{i} + y_3\hat{j}$



$\vec{a} + \vec{b} = (x_1 + x_2)\hat{i} + (y_1 + y_2)\hat{j}$

$= \vec{b} + \vec{a}$

$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ [Associative property]

There exists a point $\vec{0} = 0\hat{i} + 0\hat{j}$ such that

$\vec{a} + \vec{0} = \vec{a}$

for every $\vec{a}$ there exists $\vec{d}$ such that

$\vec{a} + \vec{d} = \vec{0}$. $\quad \vec{d} = -x_1\hat{i} + (-y_1)\hat{j}$

$= -\vec{a}$

If $5\hat{i} + 2\hat{j}$ is in 2d space. Is $3(5\hat{i} + 2\hat{j})$ in 2d space?

$3(5\hat{i} + 2\hat{j}) = 15\hat{i} + 6\hat{j}$

$k.(\vec{a})$ in 2d space

Let k and l are two numbers

then $(k+l)\vec{a} = k\vec{a} + l\vec{a}$,

$k(\vec{a}+\vec{b}) = k\vec{a} + k\vec{b}$

$(kl)(\vec{a}) = k(l\vec{a})$

$1\cdot\vec{a} = \vec{a}$

Vector space is nothing but generalization of 2d-space.

Vector Spaces:- Let V be a non empty set. We call V is a vector space over $\mathbb{R}$ if it possess the following properties

(a) $x+y \in V$, $\forall\, x,y \in V$

(b) $x+(y+z) = (x+y)+z$, $\forall\, x,y,z \in V$

(c) There exists a zero vector $0_V \in V$ such that $x+0_V = x$, $\forall\, x \in V$

(d) For every $x \in V$, there exists $-x \in V$ such that $x+(-x) = 0_V$

(e) $x+y = y+x$, $\forall\, x,y \in V$

(f) $kx \in V$, $\forall\, k \in \mathbb{R}$, $x \in V$

(g) $k(x+y) = kx+ky$, $\forall\, k \in \mathbb{R}$, $x,y \in V$

(h) $(k+l)x = kx+lx$, $\forall\, k,l \in \mathbb{R}$, $x \in V$

(i) $(kl)x = k(lx)$, $\forall\, k,l \in \mathbb{R}$, $x \in V$

(j) $1.x = x$ $\forall\, x \in V$

Vector space:-

Let V be a non empty set [collection of well defined object]. V is said to be a vector space over $\mathbb{R}$, if it satisfies the following properties

(a) $x+y \in V$ for all $x,y \in V$

(b) $(x+y)+z = x+(y+z)$ for all $x,y,z \in V$

(c) There exists an element $0_V \in V$ such that $x + 0_V = x$ for all $x \in V$

(d) For every $x \in V$ there exists $-x \in V$ such that $x+(-x) = 0_V$

(e) $x+y = y+x$ for all $x,y \in V$

(f) $k(x+y) = kx+ky$ for all $k \in \mathbb{R}, x,y \in V$

(g) $(k+l)x = kx+lx$ for all $x \in V, k,l \in \mathbb{R}$

(h) $(kl)x = k(lx)$ for all $k,l \in \mathbb{R}, x \in V$

(i) $1.x = x$ for all $x \in V$

Example:1 Consider the following set

$\mathbb{R}^3 = \{(x,y,z) : x,y,z \in \mathbb{R}\}$

Define addition on $\mathbb{R}^3$ as follows

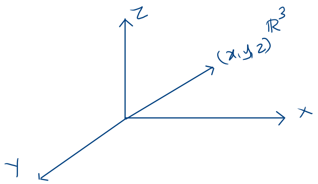
$(x_1,y_1,z_1)+(x_2,y_2,z_2) = (x_1+x_2, y_1+y_2, z_1+z_2)$

Define scalar multiplication on $\mathbb{R}^3$ as follows

$\alpha(x,y,z) = (\alpha x, \alpha y, \alpha z)$

We prove $\mathbb{R}^3$ is a vector space over $\mathbb{R}$.

Soln:-



Note:- Elements of a vector space is called as vectors.

Q:- Consider

$\mathbb{R}^2 = \{(x,y) : x,y \in \mathbb{R}\}$

Define addition of two elements in $\mathbb{R}^2$ by

$(x_1,y_1)+(x_2,y_2) = (x_1+x_2, y_1+y_2)$

and scalar multiplication by

$k(x,y) = (kx, ky)$ where $k \in \mathbb{R}$

Does $\mathbb{R}^2$ forms a vector space over $\mathbb{R}$?

Ans:-

Let $X = (x_1,y_1)$; $Y = (x_2,y_2)$

$X+Y = (x_1,y_1)+(x_2,y_2) = (x_1+x_2, y_1+y_2) \in \mathbb{R}^2$

$\therefore X+Y \in \mathbb{R}^2$ ((a) is verified)

Solution:- Take $X = (x_1,y_1,z_1)$; $Y = (x_2,y_2,z_2)$

$Z = (x_3,y_3,z_3)$

clearly $X+Y \in \mathbb{R}^3$ ((a) is verified)

$(X+Y)+Z = X+(Y+Z)$ (check!)

Zero vector in $\mathbb{R}^3$ is a point $0_V = (0,0,0)$, then

$X + 0_V = X$.

For every $X = (x,y,z) \in \mathbb{R}^3$ we have

$-X = (-x,-y,-z)$ such that

$X+(-X) = 0_V$

It is easy to see that $X+Y = Y+X$

for all $X,Y \in \mathbb{R}^3$.

Let α be a number $(x,y,z) \in \mathbb{R}^3$

then $\alpha(x,y,z) = (\alpha x, \alpha y, \alpha z) \in \mathbb{R}^3$

check!

$$(\alpha+\beta)(x,y,z) = \alpha(x,y,z) + \beta(x,y,z)$$

$$\alpha\big((x_1,y_1,z_1)+(x_2,y_2,z_2)\big) = \alpha(x_1,y_1,z_1)+\alpha(x_2,y_2,z_2)$$

$$(\alpha\beta)(x,y,z) = \alpha(\beta(x,y,z))$$

$$1(x,y,z) = (x,y,z)$$

What is the zero vector in $\mathbb{R}^2$ with respect to our new addition.

$\mathbb{R}^2$ with respect to the given addition does not have zero vector.

So, $\mathbb{R}^2$ is not a vector space over $\mathbb{R}$.

$$\Big(0_V = (\alpha,\beta)$$

$$(x,y) + (\alpha,\beta) = (x+\alpha, 0) \neq (x,y)\Big)$$

Example 2 Consider

$$\mathbb{R}^3_+ = \{(x,y,z) : x \neq 0\}$$

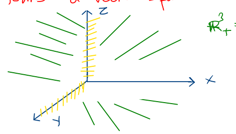
Define addition on $\mathbb{R}^3$ by

$$(x_1,y_1,z_1)+(x_2,y_2,z_2) = (x_1+x_2, y_1+y_2, z_1+z_2)$$

Scalar multiplication by

$$\alpha(x,y,z) = (\alpha x, \alpha y, \alpha z)$$

Q:- Does $\mathbb{R}^3_+$ forms a vector space over $\mathbb{R}$?



$\mathbb{R}^3_+ = \mathbb{R}^3$ except YZ plane.

Soln: Take $x=(-4,5,7)$; $y=(4,5,7)$

$$x+y = (0,10,14) \notin \mathbb{R}^3_+$$

Hence $\mathbb{R}^3_+$ is not a vector space over $\mathbb{R}$.

Example 3 Consider

$$Q = \left\{\frac{p}{q} \,\middle|\, p,q \text{ are integers, } q \neq 0, \text{ Gcd}(p,q)=1\right\}$$

(Set of rationals)

Define addition on Q by

$$\frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1q_2 + p_2q_1}{q_1q_2}$$

Define scalar multiplication on Q by

$$k\left(\frac{p}{q}\right) = \frac{kp}{q}$$

Q:- Does Q form a vector space over $\mathbb{R}$?

Take the irrational number $\sqrt{2}$ and rational number 1

$$\sqrt{2}\cdot 1 = \sqrt{2} \notin Q.$$

Scalar multiplication property does not hold thus Q is not a vector space over $\mathbb{R}$.

Example 2 Consider $\mathbb{R}^2 = \{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}\}$

Define addition on $\mathbb{R}^2$ by

$$(x,y)+(u,v) = (x+u, 0)$$

Define scalar multiplication on $\mathbb{R}^2$ by

$$k(x,y) = (kx, ky)$$

Q:- Does $\mathbb{R}^2$ forms a vector space with respect to the addition and scalar multiplication defined above

Ans:- Let $X=(x,y)$; $Y=(u,v)$, take $Z=(\alpha,\beta)$

then $X+Y = (x+u,0) \in \mathbb{R}^2$ ((a) is verified)

$$\begin{aligned} X+(Y+Z) &= X+((u,v)+(\alpha,\beta)) \\ &= X+(u+\alpha,0) \\ &= (x,y)+(u+\alpha,0) \\ &= (x+u+\alpha,0) \\ &= (X+Y)+Z. \end{aligned}$$

((b) is verified)

Vector space: (i) Consider $\mathbb{R}^2 = \{(x,y) : x,y \in \mathbb{R}\}$. We define addition on $\mathbb{R}^2$ by $(x_1,y_1)+(x_2,y_2) = (x_1+x_2, y_1+y_2)$ and scalar multiplication is defined as

$$k(x,y) = (kx, ky)$$

$\mathbb{R}^2$ is a vector space over $\mathbb{R}$

(ii) Consider (n- natural number)

$$\mathbb{R}^n = \{(x_1,x_2,\dots x_n) : x_1,x_2,\dots x_n \in \mathbb{R}\}$$

We can define addition on $\mathbb{R}^n$ by the following way.

$$(x_1,x_2,\dots x_n)+(y_1,y_2,\dots y_n) = (x_1+y_1, x_2+y_2, \dots x_n+y_n)$$

define scalar multiplication on $\mathbb{R}^n$ by

$$k(x_1,x_2,\dots x_n) = (kx_1, kx_2, \dots kx_n)$$

$\mathbb{R}^n$ is a vector space over $\mathbb{R}$.

Example 3:- polynomial of a variable x.

$$\underbrace{2x^2+3x+1}, \quad x^2+x+1, \quad x^3+1, \quad 1=x^0$$

degree of a polynomial highest power of x.

$$(2x^2+3x+1)+(x^2+x+1)=3x^2+4x+2$$

$$(x^3+1)+(x^2+x+1) =$$

$$(x^3+0x^2+0x+1)+(0\cdot x^3+x^2+x+1)=x^3+x^2+x+1$$

$$5(x^2+x+1)=5x^2+5x+1$$

$$(-7)(2x^2+3x+1)=-14x^2-21x-7$$

Vector Space of polynomials:- Let n be a natural number

$$P_n(\mathbb{R})=\{p(x) \mid \text{degree } p(x)\le n\}$$

$$=\{a_mx^m+a_{m-1}x^{m-1}+\dots+a_1x+a_0 \mid m\le n\}$$

Example:- Consider $f:[0,1]\to\mathbb{R}$ by $f(x)=\sin x$

$g:[0,1]\to\mathbb{R}$ by $g(x)=\cos x$.

What is the function $f+g$.

$f+g:[0,1]\to\mathbb{R}$ by $(f+g)(x)=f(x)+g(x)$
$=\sin x+\cos x$

What is $2f$?

$2f:[0,1]\to\mathbb{R}$ by $(2f)(x)=2f(x)$
$=2\sin x$

Vector Space of functions:-

$$C[0,1]=\{f \mid f:[0,1]\to\mathbb{R},\ f \text{ is continuous}\}$$

Define addition on $C[0,1]$ by

$$(f+g)(x)=f(x)+g(x)$$

scalar multiplication on $C[0,1]$ by

$$(kf)(x)=k\,f(x)$$

$C[0,1]$ is a vector space over $\mathbb{R}$.

Note:- The zero element is the zero function, $0:[0,1]\to\mathbb{R}$, $0(x)=0$

Let $f:[0,1]\to\mathbb{R}$ then its additive inverse is

$-f:[0,1]\to\mathbb{R}$ by $(-f)(x)=-f(x)$.

We define addition of two polynomials in the following way

$$p(x)=a_mx^m+a_{m-1}x^{m-1}+\dots+a_1x+a_0$$

$$q(x)=b_rx^r+b_{r-1}x^{r-1}+\dots+b_1x+b_0$$

Here $r>m$ then

$$p(x)+q(x)=b_rx^r+\dots+b_{m+1}x^{m+1}+(b_m+a_m)x^m+(b_{m-1}+a_{m-1})x^{m-1}+\dots+(b_1+a_1)x+(b_0+a_0)$$

We define the scalar multiplication by

$$k\,p(x)=k(a_mx^m+\dots+a_1x+a_0)$$
$$=(ka_m)x^m+\dots+(ka_1)x+(ka_0)$$

$P_n(\mathbb{R})$ is a vector space over $\mathbb{R}$.

Note:- The zero vector of $P_n(\mathbb{R})$ is the zero polynomial.

If $p(x)=a_mx^m+\dots+a_1x+a_0$ then its additive inverse is

$$-p(x)=(-a_m)x^m+\dots+(-a_1)x+(-a_0)$$

What is $P_2(\mathbb{R})$

$P_2(\mathbb{R})=$ Collection of all polynomials of degree atmost 2

$$=\{x,\ 2x+1,\ x+5,\ k,\ x^2,\}$$

$$=\{a_2x^2+a_1x+a_0 \mid a_0,a_1,a_2\in\mathbb{R}\}$$

$$P_4(\mathbb{R}) = \{a_4x^4+a_3x^3+a_2x^2+a_1x+a_0 \mid a_0,a_1,a_2,a_3,a_4\in\mathbb{R}\}$$

Consider

$$A=\begin{bmatrix}4 & 3 & 2\\ -7 & 1 & 6\end{bmatrix}_{(2\times 3)};\quad B=\begin{bmatrix}7 & 8 & 9\\ 1 & 2 & 3\end{bmatrix}_{(2\times 3)}$$

$$A+B=\begin{bmatrix}11 & 11 & 11\\ -6 & 3 & 9\end{bmatrix}$$

$$7A = 7\begin{bmatrix}4 & 3 & 2\\ -7 & 1 & 6\end{bmatrix}=\begin{bmatrix}28 & 21 & 14\\ -49 & 7 & 42\end{bmatrix}$$

Vector space of matrices:- Let m and n be two natural numbers

Let $M_{m\times n}(\mathbb{R})=$ collection of $m\times n$ matrices

$$=\{[a_{ij}]_{m\times n} : a_{ij}\in\mathbb{R}\}$$

We define addition on $M_{m\times n}(\mathbb{R})$ in the following way

$$[a_{ij}]+[b_{ij}]=[a_{ij}+b_{ij}]$$

we define scalar multiplication on $M_{m\times n}(\mathbb{R})$ by

$$k[a_{ij}] = [ka_{ij}]$$

$M_{m\times n}(\mathbb{R})$ forms a vector space over $\mathbb{R}$.

Note:- The zero element is the zero matrix

If $A=[a_{ij}]$ then its additive inverse is $-A=[-a_{ij}]$

Chapter 3

Vector Spaces

3.1 Vector spaces and subspaces

We discussed how to solve a system $A\mathbf{x} = \mathbf{b}$ of linear equations, and we saw that the basic questions of the existence or uniqueness of the solution were much easier to answer after Gaussian-elimination. In this chapter, we introduce the notion of a vector space, which is an abstraction of the usual algebraic structures of the 3-space $\mathbb{R}^3$ and then elaborate our study of a system of linear equations to this framework.

Usually, many physical quantities, such as length, area, mass, temperature are described by real numbers as magnitudes. Other physical quantities like force or velocity have directions as well as magnitudes. Such quantities with direction are called **vectors**, while the numbers are called **scalars**. For instance, an element (or a point) $\mathbf{x}$ in the 3-space $\mathbb{R}^3$ is usually represented as a triple of real numbers:

$$\mathbf{x} = (x_1,\ x_2,\ x_3),$$

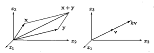
where $x_i \in \mathbb{R}$, $i = 1,\ 2,\ 3$, are called the **coordinates** of $\mathbf{x}$. This expression provides a rectangular coordinate system in a natural way. On the other hand, pictorially such a point in the 3-space $\mathbb{R}^3$ can also be represented by an arrow from the origin to $\mathbf{x}$. In this way, a point in the 3-space $\mathbb{R}^3$ can be understood as a vector. The direction of the arrow specifies the direction of the vector, and the length of the arrow describes its magnitude.

In order to have a more general definition of vectors, we extract the most basic properties of those arrows in $\mathbb{R}^3$. Note that for all vectors (or points) in $\mathbb{R}^3$, there are two algebraic operations: the addition of any two vectors

and scalar multiplication of a vector by a scalar. That is, for two vectors $\mathbf{x} = (x_1,\ x_2,\ x_3)$, $\mathbf{y} = (y_1,\ y_2,\ y_3)$ in $\mathbb{R}^3$ and k a scalar, we define

$$\begin{aligned} \mathbf{x}+\mathbf{y} &= (x_1+y_1,\ x_2+y_2,\ x_3+y_3), \\ k\mathbf{x} &= (kx_1,\ kx_2,\ kx_3). \end{aligned}$$

The addition of vectors and scalar multiplication of a vector in the 3-space $\mathbb{R}^3$ are illustrated as follows:



Even though our geometric visualization of vectors does not go beyond the 3-space $\mathbb{R}^3$, it is possible to extend the above algebraic operations of vectors in the 3-space $\mathbb{R}^3$ to the general **n-space** $\mathbb{R}^n$ for any positive integer n. It is defined to be the set of all ordered n-tuples $(a_1,\ a_2,\ \ldots,\ a_n)$ of real numbers, called *vectors*: *i.e.*,

$$\mathbb{R}^n = \{(a_1,\ a_2,\ \ldots,\ a_n)\ :\ a_i \in \mathbb{R},\ i = 1,\ 2,\ \ldots,\ n\}.$$

For any two vectors $\mathbf{x} = (x_1,\ x_2,\ \ldots,\ x_n)$ and $\mathbf{y} = (y_1,\ y_2,\ \ldots,\ y_n)$ in the n-space $\mathbb{R}^n$, and a scalar k, the *sum* $\mathbf{x}+\mathbf{y}$ and the *scalar multiplication* $k\mathbf{x}$ of them are vectors in $\mathbb{R}^n$ defined by

$$\begin{aligned} \mathbf{x}+\mathbf{y} &= (x_1+y_1,\ x_2+y_2,\ \ldots,\ x_n+y_n), \\ k\mathbf{x} &= (kx_1,\ kx_2,\ \ldots,\ kx_n). \end{aligned}$$

It is easy to verify the following list of arithmetical rules of the operations:

Theorem 3.1 *For any scalars k and ℓ, and vectors $\mathbf{x} = (x_1,\ x_2,\ \ldots,\ x_n)$, $\mathbf{y} = (y_1,\ y_2,\ \ldots,\ y_n)$, and $\mathbf{z} = (z_1,\ z_2,\ \ldots,\ z_n)$ in the n-space $\mathbb{R}^n$, the following rules hold:*

(1) $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$,

(2) $\mathbf{x}+(\mathbf{y}+\mathbf{z}) = (\mathbf{x}+\mathbf{y})+\mathbf{z}$,

(3) $\mathbf{x}+\mathbf{0} = \mathbf{x} = \mathbf{0}+\mathbf{x}$,

(4) $\mathbf{x}+(-1)\mathbf{x} = \mathbf{0}$,

(5) $k(\mathbf{x}+\mathbf{y}) = k\mathbf{x}+k\mathbf{y}$,

(6) $(k+\ell)\mathbf{x} = k\mathbf{x}+\ell\mathbf{x}$,

(7) $k(\ell\mathbf{x}) = (k\ell)\mathbf{x}$,

(8) $1\mathbf{x} = \mathbf{x}$,

where $\mathbf{0} = (0,\ 0,\ \ldots,\ 0)$ *is the* **zero** *vector.*

We usually write a vector $(a_1,\ a_2,\ \ldots,\ a_n)$ in the n-space $\mathbb{R}^n$ as an $n \times 1$ column matrix

$$(a_1,\ a_2,\ \ldots,\ a_n) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = [a_1\ a_2\ \cdots\ a_n]^T,$$

also called a *column vector*. Then the two operations of the matrix sum and the scalar multiplication of column matrices coincide with those of vectors in $\mathbb{R}^n$, and the above theorem is just Theorem 1.2.

These rules of arithmetic of vectors are the most important ones because they are the only rules that we need to manipulate vectors in the n-space $\mathbb{R}^n$. Hence, an (abstract) vector space can be defined with respect to these rules of operations of vectors in the n-space $\mathbb{R}^n$ so that $\mathbb{R}^n$ itself becomes a vector space. In general, a *vector space* is defined to be a set with two operations: an addition and a scalar multiplication which satisfy the above rules of operations in $\mathbb{R}^n$.

Definition 3.1 A (real) **vector space** is a nonempty set V of elements, called **vectors**, with two algebraic operations that satisfy the following rules.

(A) There is an operation called *vector addition* that associates to every pair $\mathbf{x}$ and $\mathbf{y}$ of vectors in V a unique vector $\mathbf{x}+\mathbf{y}$ in V, called the the **sum** of $\mathbf{x}$ and $\mathbf{y}$, so that the following rules hold for all vectors $\mathbf{x},\ \mathbf{y},\ \mathbf{z}$ in V:

(1) $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$ (commutativity in addition),

(2) $\mathbf{x}+(\mathbf{y}+\mathbf{z}) = (\mathbf{x}+\mathbf{y})+\mathbf{z}(= \mathbf{x}+\mathbf{y}+\mathbf{z})$ (associativity in addition),

(3) there is a unique vector $\mathbf{0}$ in V such that $\mathbf{x}+\mathbf{0} = \mathbf{x} = \mathbf{0}+\mathbf{x}$ for all $\mathbf{x} \in V$ (it is called the **zero vector**),

(4) for any $\mathbf{x} \in V$, there is a vector $-\mathbf{x} \in V$, called the **negative** of $\mathbf{x}$, such that $\mathbf{x}+(-\mathbf{x}) = (-\mathbf{x})+\mathbf{x} = \mathbf{0}$.

(B) There is an operation called **scalar multiplication** that associates to each vector $\mathbf{x}$ in V and each scalar k a unique vector $k\mathbf{x}$ in V called the **multiplication** of $\mathbf{x}$ by a (real) scalar k, so that the following rules hold for all vectors $\mathbf{x},\ \mathbf{y},\ \mathbf{z}$ in V and all scalars $k,\ \ell$:

(5) $k(\mathbf{x}+\mathbf{y}) = k\mathbf{x}+k\mathbf{y}$ (distributivity with respect to vector addition),

(6) $(k+\ell)\mathbf{x} = k\mathbf{x}+\ell\mathbf{x}$ (distributivity with respect to scalar addition),

(7) $k(\ell\mathbf{x}) = (k\ell)\mathbf{x}$ (associativity in scalar multiplication),

(8) $1\mathbf{x} = \mathbf{x}$.

Clearly, the n-space $\mathbb{R}^n$ is a vector space by Theorem 3.1. A **complex vector space** is obtained if, instead of real numbers, we take complex numbers for scalars. For example, the set $\mathbb{C}^n$ of all ordered n-tuples of complex numbers is a complex vector space. In Chapter 7 we shall discuss complex vector spaces, but until then we will discuss only real vector spaces unless otherwise stated.

Example 3.1 (1) For any positive integer m and n, the set $M_{m\times n}(\mathbb{R})$ of all $m \times n$ matrices forms a vector space under the matrix sum and scalar multiplication defined in Section 1.3. The zero vector in this space is the zero matrix $\mathbf{0}_{m\times n}$, and $-A$ is the negative of a matrix A.

(2) Let $C(\mathbb{R})$ denote the set of real-valued continuous functions defined on the real line $\mathbb{R}$. For two functions $f(x)$ and $g(x)$, and a real number k, the sum $f+g$ and the scalar multiplication kf of them are defined by

$$\begin{aligned} (f+g)(x) &= f(x)+g(x), \\ (kf)(x) &= kf(x). \end{aligned}$$

Then one can easily verify, as an exercise, that the set $C(\mathbb{R})$ is a vector space under these operations. The zero vector in this space is the constant function whose value at each point is zero.

(3) Let A be an $m \times n$ matrix. Then it is easy to show that the set of solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is a vector space (under the sum and scalar multiplication of matrices).

Theorem 3.2 *Let V be a vector space and let $\mathbf{x}$, $\mathbf{y}$ be vectors in V. Then*

(1) $\mathbf{x}+\mathbf{y} = \mathbf{y}$ *implies* $\mathbf{x} = \mathbf{0}$,

(2) $0\mathbf{x} = \mathbf{0}$,

(3) $k\mathbf{0} = \mathbf{0}$ *for any* $k \in \mathbb{R}$,

(4) $-\mathbf{x}$ *is unique and* $-\mathbf{x} = (-1)\mathbf{x}$,

(5) *if* $k\mathbf{x} = \mathbf{0}$, *then* $k = 0$ *or* $\mathbf{x} = \mathbf{0}$.

Proof: **(1)** By adding $-\mathbf{y}$ to both sides of $\mathbf{x}+\mathbf{y} = \mathbf{y}$, we have

$$\mathbf{x} = \mathbf{x}+\mathbf{0} = \mathbf{x}+\mathbf{y}+(-\mathbf{y}) = \mathbf{y}+(-\mathbf{y}) = \mathbf{0}.$$

(2) $0\mathbf{x} = (0+0)\mathbf{x} = 0\mathbf{x}+0\mathbf{x}$ implies $0\mathbf{x} = \mathbf{0}$ by (1).

(3) This is an easy exercise.

(4) The uniqueness of the negative $-\mathbf{x}$ of $\mathbf{x}$ can be shown by a simple modification of Lemma 1.6. In fact, if $\bar{\mathbf{x}}$ is another negative of $\mathbf{x}$ such that $\mathbf{x}+\bar{\mathbf{x}} = \mathbf{0}$, then

$$-\mathbf{x} = -\mathbf{x}+\mathbf{0} = -\mathbf{x}+(\mathbf{x}+\bar{\mathbf{x}}) = (-\mathbf{x}+\mathbf{x})+\bar{\mathbf{x}} = \mathbf{0}+\bar{\mathbf{x}} = \bar{\mathbf{x}}.$$

On the other hand, the equation

$$\mathbf{x}+(-1)\mathbf{x} = 1\mathbf{x}+(-1)\mathbf{x} = (1-1)\mathbf{x} = 0\mathbf{x} = \mathbf{0}$$

shows that $(-1)\mathbf{x}$ is another negative of $\mathbf{x}$, and hence $-\mathbf{x} = (-1)\mathbf{x}$ by the uniqueness of $-\mathbf{x}$.

(5) Suppose $k\mathbf{x} = \mathbf{0}$ and $k \neq 0$. Then $\mathbf{x} = 1\mathbf{x} = \frac{1}{k}(k\mathbf{x}) = \frac{1}{k}\mathbf{0} = \mathbf{0}$. □

Problem 3.1 Let V be the set of all pairs $(x,\ y)$ of real numbers. Suppose that an addition and scalar multiplication of pairs are defined by

$$(x,\ y)+(u,\ v) = (x+2u,\ y+2v), \quad k(x,\ y) = (kx,\ ky).$$

Is the set V a vector space under those operations? Justify your answer.

A subset W of a vector space V is called a **subspace** of V if W is itself a vector space under the addition and scalar multiplication defined in V. Usually, in order to show that a subset W is a subspace, it is not necessary to verify all the rules of the definition of a vector space, because certain rules satisfied in the larger space V are automatically satisfied in every subset, if vector addition and scalar multiplication are closed in subset.

Theorem 3.3 *A nonempty subset W of a vector space V is a subspace if and only if $\mathbf{x}+\mathbf{y}$ and $k\mathbf{x}$ are contained in W (or equivalently, $\mathbf{x}+k\mathbf{y} \in W$) for any vectors $\mathbf{x}$ and $\mathbf{y}$ in W and any scalar $k \in \mathbb{R}$.*

<u>Recall</u>:- $\mathbb{R}^n = \{(x_1, x_2, \cdots x_n) \mid x_1, x_2, \cdots x_n \in \mathbb{R}\}$

We define addition on $\mathbb{R}^n$ by

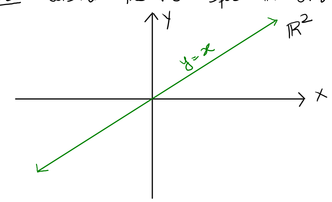
$$(x_1, x_2, \cdots x_n) + (y_1, y_2, \cdots y_n) = (x_1+y_1, x_2+y_2, \cdots, x_n+y_n) \rightarrow \text{standard addition}$$

and scalar multiplication on $\mathbb{R}^n$ by

$$k(x_1, x_2, \cdots x_n) = (kx_1, kx_2, \cdots kx_n) \rightarrow \text{standard scalar multiplication}$$

$\mathbb{R}^n$ is a vector space over $\mathbb{R}$.

<u>Vector</u> <u>Subspace</u>:- Consider the vector space $\mathbb{R}^2$ over $\mathbb{R}$



Consider $U=\{(x,y)\in\mathbb{R}^2 \mid y=x\}=\{(x,x) \mid x\in\mathbb{R}\}$

Q:- Does U forms a vector space over $\mathbb{R}$ with respect to the addition and scalar multiplication defined on $\mathbb{R}^2$?

Ans:- Take $X=(x,x)$, $Y=(y,y)$ and $Z=(z,z)$

$X+Y=(x,x)+(y,y)=(x+y,x+y)\in U$

$(X+Y)+Z = X+(Y+Z)$

Zero vector is $0_V=(0,0)\in U$

For every $X=(x,x)$ the vector $-X=(-x,x)\in U$ such that $X+(-X)=0_V$

$X+Y=Y+X \quad \forall X,Y\in U$

$kX=k(x,x)=(kx,kx)\in U$

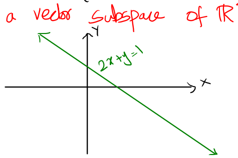
$(k+l)X=kX+lX$ and

$k(X+Y)=kX+kY$, $1.X=X$

$\therefore U$ forms a vector space over $\mathbb{R}$. ⇒ vector subspace of $\mathbb{R}^2$.

Example:- Consider $W=\{(x,y)\in\mathbb{R}^2 \mid 2x+y=1\}$

Q:- Does W a vector subspace of $\mathbb{R}^2$?



Soln:- Let $X=(x_1,y_1)\in W \Rightarrow 2x_1+y_1=1$

$Y=(x_2,y_2)\in W \Rightarrow 2x_2+y_2=1$

$X+Y=(x_1+x_2,\, y_1+y_2)$

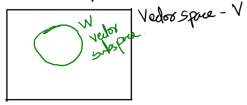
$2(x_1+x_2)+(y_1+y_2)=(2x_1+y_1)+(2x_2+y_2)$
$= 1+1 = 2$

$\therefore X+Y\notin W$

$\Rightarrow W$ is not a sub space of $\mathbb{R}^2$.

Definition:- Vector Subspace:-

A subset W of a vector space V is called a subspace if W is itself a vector space under addition and scalar multiplication defined in V



Theorem:- A non empty set W of a vector space V is a subspace if and only if

(i) $x+y\in W$ for every $x,y\in W$

(ii) $kx\in W$ for every $k\in\mathbb{R}$ and $x\in W$

Note:- Let $k_1,k_2,\ldots k_n,\beta$ be fixed real numbers.

Consider

$W=\{(x_1,x_2,\ldots x_n)\in\mathbb{R}^n \mid k_1x_1+k_2x_2+\cdots+k_nx_n=\beta\}$

W forms a vector subspace of $\mathbb{R}^n$ only if $\beta=0$.

Q:- Consider $U=\{(x,y,z,t)\in\mathbb{R}^4 \mid x+y+2z-t=1\}$

Does U forms a vector subspace of $\mathbb{R}^4$?

Ans:- No

Example: Consider the following subset → plane in $\mathbb{R}^3$

$W=\{(x,y,z)\in\mathbb{R}^3 \mid 2x+3y-4z=0\}$

Does W forms a vector subspace of $\mathbb{R}^3$?

Ans:- Let $X=(x_1,y_1,z_1)\in W \Rightarrow 2x_1+3y_1-4z_1=0$

$Y=(x_2,y_2,z_2)\in W \Rightarrow 2x_2+3y_2-4z_2=0$

$X+Y=(x_1+x_2,\, y_1+y_2,\, z_1+z_2)$

$2(x_1+x_2)+3(y_1+y_2)-4(z_1+z_2)=(2x_1+3y_1-4z_1)+(2x_2+3y_2-4z_2)$
$=0+0=0$

$\therefore X+Y\in W$

Let $k\in\mathbb{R}$, $X=(x_1,y_1,z_1)\in W$

$kX=(kx_1,ky_1,kz_1)$

$2(kx_1)+3(ky_1)-4(kz_1)=k(2x_1+3y_1-4z_1)$
$=0$

$\therefore kX\in W$

Thus W is a vector subspace of $\mathbb{R}^3$.

Example:- Consider the vector space of all $n\times n$ matrices (that is) $M_{n\times n}(\mathbb{R})$.

$W=\{A\in M_{n\times n}(\mathbb{R}) \mid A=A^T\}$ = Set of all symmetric matrices.

Does W forms a vector subspace of $M_{n\times n}(\mathbb{R})$?

Ans:- Let $A,B\in W \Rightarrow A=A^T$, $B=B^T$

$(A+B)^T=A^T+B^T=A+B$, $\therefore A+B\in W$

$(kA)^T=kA^T=kA \quad \therefore kA\in W$

$\therefore W$ is a vector subspace of $M_{n\times n}(\mathbb{R})$

Consider

$U=\{A\in M_{n\times n}(\mathbb{R}) \mid A^T=-A\}$

= Set of all skew symmetric matrices

U forms a vector subspace of $M_{n\times n}(\mathbb{R})$.

(4) $-\mathbf{x}$ *is unique and* $-\mathbf{x} = (-1)\mathbf{x}$,

(5) *if* $k\mathbf{x} = 0$, *then* $k = 0$ *or* $\mathbf{x} = 0$.

Proof: (1) By adding $-\mathbf{y}$ to both sides of $\mathbf{x} + \mathbf{y} = \mathbf{y}$, we have

$$\mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{x} + \mathbf{y} + (-\mathbf{y}) = \mathbf{y} + (-\mathbf{y}) = \mathbf{0}.$$

(2) $0\mathbf{x} = (0+0)\mathbf{x} = 0\mathbf{x} + 0\mathbf{x}$ implies $0\mathbf{x} = \mathbf{0}$ by (1).

(3) This is an easy exercise.

(4) The uniqueness of the negative $-\mathbf{x}$ of $\mathbf{x}$ can be shown by a simple modification of Lemma 1.6. In fact, if $\bar{\mathbf{x}}$ is another negative of $\mathbf{x}$ such that $\mathbf{x} + \bar{\mathbf{x}} = \mathbf{0}$, then

$$-\mathbf{x} = -\mathbf{x} + \mathbf{0} = -\mathbf{x} + (\mathbf{x} + \bar{\mathbf{x}}) = (-\mathbf{x} + \mathbf{x}) + \bar{\mathbf{x}} = \mathbf{0} + \bar{\mathbf{x}} = \bar{\mathbf{x}}.$$

On the other hand, the equation

$$\mathbf{x} + (-1)\mathbf{x} = 1\mathbf{x} + (-1)\mathbf{x} = (1-1)\mathbf{x} = 0\mathbf{x} = \mathbf{0}$$

shows that $(-1)\mathbf{x}$ is another negative of $\mathbf{x}$, and hence $-\mathbf{x} = (-1)\mathbf{x}$ by the uniqueness of $-\mathbf{x}$.

(5) Suppose $k\mathbf{x} = \mathbf{0}$ and $k \neq 0$. Then $\mathbf{x} = 1\mathbf{x} = \frac{1}{k}(k\mathbf{x}) = \frac{1}{k}\mathbf{0} = \mathbf{0}$. □

Problem 3.1 Let V be the set of all pairs (x, y) of real numbers. Suppose that an addition and scalar multiplication of pairs are defined by

$$(x, y) + (u, v) = (x + 2u, y + 2v), \quad k(x, y) = (kx, ky).$$

Is the set V a vector space under those operations? Justify your answer.

A subset W of a vector space V is called a **subspace** of V if W is itself a vector space under the addition and scalar multiplication defined in V. Usually, in order to show that a subset W is a subspace, it is not necessary to verify all the rules of the definition of a vector space, because certain rules satisfied in the larger space V are automatically satisfied in every subset, if vector addition and scalar multiplication are closed in subset.

Theorem 3.3 *A nonempty subset W of a vector space V is a subspace if and only if $\mathbf{x} + \mathbf{y}$ and $k\mathbf{x}$ are contained in W (or equivalently, $\mathbf{x} + k\mathbf{y} \in W$) for any vectors $\mathbf{x}$ and $\mathbf{y}$ in W and any scalar $k \in \mathbb{R}$.*

<u>Recall</u>:- <u>Subspace</u>:- Let V be a vector space over $\mathbb{R}$ and W be a nonempty subset of V. If W itself forms a vector space over $\mathbb{R}$ by the addition and scalar multiplication defined on V then we say W forms a subspace of V.

<u>Examples</u>:- (i) Let $\mathbb{R}^3$ be a vector space over $\mathbb{R}$ then

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid 2x + 3y - 4z = 0\}$$

is a vector subspace of $\mathbb{R}^3$.

(ii) Let $M_{n\times n}(\mathbb{R})$ be the vector space of all $n\times n$ matrices then the sets

$$W_1 = \{A \in M_{n\times n}(\mathbb{R}) \mid A^T = A\} \text{ and}$$
$$W_2 = \{A \in M_{n\times n}(\mathbb{R}) \mid A^T = -A\}$$

forms a vector subspace of $M_{n\times n}(\mathbb{R})$

(iii) Let $C[0,1]$ be the vector space of all continuous functions then

$$W = \{f \in C[0,1] \mid f(1) = 0\}$$

is a vector subspace of $C[0,1]$

<u>Theorem</u>:- Let V be a vector space.

W is a subspace $\iff$ $x + y \in W$ for all $x, y \in W$
$\alpha x \in W$ for all $\alpha \in \mathbb{R}$, $x \in W$

Proof: We need only to prove the sufficiency. Assume both conditions hold and let $\mathbf{x}$ be any vector in W. Since W is closed under scalar multiplication, $\mathbf{0} = 0\mathbf{x}$ and $-\mathbf{x} = (-1)\mathbf{x}$ are in W, so rules (3) and (4) for a vector space hold. All the other rules for a vector space are clear. □

A vector space V itself and the zero vector $\{\mathbf{0}\}$ are trivially subspaces. Some nontrivial subspaces are given in the following examples.

Example 3.2 Let $W = \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = 0\}$, where a, b, c are constants. If $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$ are points in W, then clearly $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$ is also a point in W, because it satisfies the equation in W. Similarly, $k\mathbf{x}$ also lies in W for any scalar k. Hence, W is a subspace of $\mathbb{R}^3$ and is a plane passing through the origin in $\mathbb{R}^3$.

Example 3.3 Let A be an $m \times n$ matrix. Then, as we have seen in Example 3.1 (3), the set

$$W = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$$

of solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is a vector space. Moreover, since the operations in W and in $\mathbb{R}^n$ coincide, W is a subspace of $\mathbb{R}^n$.

Example 3.4 For a nonnegative integer n, let $P_n(\mathbb{R})$ denote the set of all real polynomials in x with degree $\leq n$. Then $P_n(\mathbb{R})$ is a subspace of the vector space $C(\mathbb{R})$ of all continuous functions on $\mathbb{R}$.

Example 3.5 Let W be the set of all $n \times n$ real symmetric matrices. Then W is a subspace of the vector space $M_{n\times n}(\mathbb{R})$ of all $n \times n$ matrices, because the sum of two symmetric matrices is symmetric and a scalar multiplication of a symmetric matrix is also symmetric. Similarly, the set of all $n \times n$ skew-symmetric matrices is also a subspace of $M_{n\times n}(\mathbb{R})$.

Problem 3.2 Which of the following sets are subspaces of the 3-space $\mathbb{R}^3$? Justify your answer.

(1) $W = \{(x, y, z) \in \mathbb{R}^3 : xyz = 0\}$,
(2) $W = \{(2t, 3t, 4t) \in \mathbb{R}^3 : t \in \mathbb{R}\}$,
(3) $W = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 0\}$,
(4) $W = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x}^T\mathbf{u} = 0 = \mathbf{x}^T\mathbf{v}\}$, where $\mathbf{u}$ and $\mathbf{v}$ are any two fixed nonzero vectors in $\mathbb{R}^3$.

Can you describe all subspaces of the 3-space $\mathbb{R}^3$?

<u>Linear Combination</u>:- Consider the vector space $\mathbb{R}^3$ over $\mathbb{R}$ and vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. Then we know any $(x,y,z) \in \mathbb{R}^3$ can be written as

$$(x, y, z) = \underbrace{x(1,0,0) + y(0,1,0) + z(0,0,1)}_{\text{linear combination}}$$

Consider the vector space $\mathbb{R}^2$ over $\mathbb{R}$ and vectors $(1,0)$, $(0,1)$ then any $(x,y) \in \mathbb{R}^2$ can be written as

$$(x, y) = \underbrace{x(1,0) + y(0,1)}_{\text{linear combination.}}$$

Problem 3.3 Let $V = C(\mathbb{R})$ be the vector space of all continuous functions on $\mathbb{R}$. Which of the following sets W are subspaces of V? Justify your answer.

(1) W is the set of all differentiable functions on $\mathbb{R}$.
(2) W is the set of all bounded continuous functions on $\mathbb{R}$.
(3) W is the set of all continuous nonnegative-valued functions on $\mathbb{R}$, *i.e.*, $f(x) \geq 0$ for any $x \in \mathbb{R}$.
(4) W is the set of all continuous odd functions on $\mathbb{R}$, *i.e.*, $f(-x) = -f(x)$ for any $x \in \mathbb{R}$.
(5) W is the set of all polynomials with integer coefficients.

3.2 Bases

Recall that any vector in the 3-space $\mathbb{R}^3$ is of the form (x_1, x_2, x_3) which can alsobe written as

$$(x_1, x_2, x_3) = x_1(1, 0, 0) + x_2(0, 1, 0) + x_3(0, 0, 1).$$

That is, any vector in $\mathbb{R}^3$ can be expressed as the sum of scalar multiples of $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$ and $\mathbf{e}_3 = (0, 0, 1)$, which are also denoted by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, respectively. The following definition gives a name to such expressions.

Definition 3.2 Let V be a vector space, and let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ be a set of vectors in V. Then a vector $\mathbf{y}$ in V of the form

$$\mathbf{y} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_m\mathbf{x}_m,$$

where $a_1, \ldots, a_m$ are scalars, is called a **linear combination** of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$.

The next theorem shows that the set of all linear combinations of a finite set of vectors in a vector space forms a subspace.

Theorem 3.4 *Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ be vectors in a vector space V. Then the set $W = \{a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_m\mathbf{x}_m : a_i \in \mathbb{R}\}$ of all linear combinations of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ is a subspace of V called the* **subspace of V spanned by** *$\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$.*

<u>Definition</u>:- <u>Linear Combination</u>:-

Let V be a vector space $\{v_1, v_2, \cdots v_n\}$ be a set of vectors in V. A vector $y \in V$ is of the form

$$y = k_1v_1 + k_2v_2 + \cdots + k_nv_n$$

(where $k_1, k_2, \ldots k_n$ are real numbers) is called a linear combination of the vectors.

<u>Theorem</u>:- Let $v_1, v_2, \cdots v_n$ be vectors in a space V. The set

$$W = \{k_1v_1 + k_2v_2 + \cdots + k_nv_n \mid k_1, k_2, \ldots k_n \in \mathbb{R}\}$$

(All linear combinations of vectors $v_1, v_2, \cdots v_n$) is a subspace of V. We call W the subspace of V spanned by $v_1, v_2, \cdots v_n$.

Question1:- Express the vector $(2,5) \in \mathbb{R}^2$ as a linear combination of $(1,0)$, $(1,1)$.

Soln:-

$$(2,5) = k_1(1,0) + k_2(1,1) = (k_1, 0) + (k_2, k_2)$$
$$= (k_1 + k_2, k_2)$$
$$\Rightarrow k_2 = 5; \quad k_1 + k_2 = 2$$
$$k_1 = -3$$

Thus $(2,5) = (-3)(1,0) + 5(1,1)$

Question2:- Express the vector $(-1,5,7) \in \mathbb{R}^3$ as a combination of $(1,0,0)$, $(1,1,0)$, $(1,1,1)$

Ans:-

$$(-1,5,7) = k_1(1,0,0) + k_2(1,1,0) + k_3(1,1,1)$$
$$= (k_1 + k_2 + k_3, k_2 + k_3, k_3)$$
$$\Rightarrow k_3 = 7$$
$$k_2 + k_3 = 5 \Rightarrow k_2 = -2$$
$$k_1 + k_2 + k_3 = -1 \Rightarrow k_1 = -6$$
$$\therefore (-1,5,7) = (-6)(1,0,0) + (-2)(1,1,0) + 7(1,1,1)$$

Q:- Express the vector $(0,0,1) \in \mathbb{R}^3$ as a linear combination of vectors $\{(1,0,0), (0,1,0)\}$

Ans:-

$$(0,0,1) = k_1(1,0,0) + k_2(0,1,0)$$
$$\Rightarrow \boxed{(0,0,1) = (k_1, k_2, 0)}$$

This not possible as linear combination is not possible.

Q:- Is the polynomial $3t^2 - 3t + 1$ a linear combination of $p_1 = t^2 - t$; $p_2 = t^2 - 2t + 1$, $p_3 = -t^2 + 1$?

Ans:-

$$3t^2 - 3t + 1 = k_1(t^2 - t) + k_2(t^2 - 2t + 1) + k_3(-t^2 + 1)$$
$$= (k_1 + k_2 - k_3)t^2 + (-k_1 - 2k_2)t + (k_2 + k_3)$$

$$\Rightarrow \begin{aligned} k_1 + k_2 - k_3 &= 3 \\ -k_1 - 2k_2 &= -3 \\ k_2 + k_3 &= 1 \end{aligned} \quad \Rightarrow \begin{aligned} k_1 + k_2 - k_3 &= 3 \rightarrow (1) \\ k_1 + 2k_2 &= 3 \rightarrow (2) \\ k_2 + k_3 &= 1 \rightarrow (3) \end{aligned}$$

From (3) $\Rightarrow k_3 = 1 - k_2$ apply in (1),

$$k_1 + k_2 - (1 - k_2) = 3$$
$$\Rightarrow k_1 + 2k_2 = 4 \rightarrow (5)$$
$$(2) \Rightarrow k_1 + 2k_2 = 3$$

Thus we do not have solution

Hence linear combination is not possible.

Q:- Express the polynomial $t^2 + 4t - 3$ as a linear combination of $p_1 = t^2$; $p_2 = t$; $p_3 = 1 - t$.

Ans:-

$$(t^2 + 4t - 3) = k_1(t^2) + k_2(t) + k_3(1 - t)$$
$$= k_1 t^2 + (k_2 - k_3)t + k_3$$
$$\Rightarrow k_1 = 1; \quad k_2 - k_3 = 4; \quad k_3 = -3$$
$$\therefore k_2 = 1$$

Thus $(t^2 + 4t - 3) = t^2 + \quad t + (-3)(1 - t)$

Q:- Express every matrix of order 2×2 as a combination of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Ans:- Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Problem 3.3 Let $V = C(\mathbb{R})$ be the vector space of all continuous functions on $\mathbb{R}$. Which of the following sets W are subspaces of V? Justify your answer.

(1) W is the set of all differentiable functions on $\mathbb{R}$.
(2) W is the set of all bounded continuous functions on $\mathbb{R}$.
(3) W is the set of all continuous nonnegative-valued functions on $\mathbb{R}$, *i.e.*, $f(x) \geq 0$ for any $x \in \mathbb{R}$.
(4) W is the set of all continuous odd functions on $\mathbb{R}$, *i.e.*, $f(-x) = -f(x)$ for any $x \in \mathbb{R}$.
(5) W is the set of all polynomials with integer coefficients.

3.2 Bases

Recall that any vector in the 3-space $\mathbb{R}^3$ is of the form $(x_1,\ x_2,\ x_3)$ which can alsobe written as

$$(x_1,\ x_2,\ x_3) = x_1(1,\ 0,\ 0) + x_2(0,\ 1,\ 0) + x_3(0,\ 0,\ 1).$$

That is, any vector in $\mathbb{R}^3$ can be expressed as the sum of scalar multiples of $\mathbf{e}_1 = (1,\ 0,\ 0)$, $\mathbf{e}_2 = (0,\ 1,\ 0)$ and $\mathbf{e}_3 = (0,\ 0,\ 1)$, which are also denoted by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, respectively. The following definition gives a name to such expressions.

Definition 3.2 Let V be a vector space, and let $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m\}$ be a set of vectors in V. Then a vector $\mathbf{y}$ in V of the form

$$\mathbf{y} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_m\mathbf{x}_m,$$

where $a_1,\ \ldots,\ a_m$ are scalars, is called a **linear combination** of the vectors $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$.

The next theorem shows that the set of all linear combinations of a finite set of vectors in a vector space forms a subspace.

Theorem 3.4 *Let $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$ be vectors in a vector space V. Then the set $W = \{a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_m\mathbf{x}_m \ :\ a_i \in \mathbb{R}\}$ of all linear combinations of $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$ is a subspace of V called the* **subspace of V spanned** *by $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$.*

Proof: We want to show that W is closed under addition and scalar multiplication. Let $\mathbf{u}$ and $\mathbf{w}$ be any two vectors in W. Then

$$\begin{aligned}\mathbf{u} &= a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_m\mathbf{x}_m,\\ \mathbf{w} &= b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + \cdots + b_m\mathbf{x}_m\end{aligned}$$

for some scalars a_i's and b_i's. Therefore,

$$\mathbf{u} + \mathbf{w} = (a_1 + b_1)\mathbf{x}_1 + (a_2 + b_2)\mathbf{x}_2 + \cdots + (a_m + b_m)\mathbf{x}_m$$

and, for any scalar k,

$$k\mathbf{u} = (ka_1)\mathbf{x}_1 + (ka_2)\mathbf{x}_2 + \cdots + (ka_m)\mathbf{x}_m.$$

Thus, $\mathbf{u} + \mathbf{w}$ and $k\mathbf{u}$ are linear combinations of $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m$ and consequently contained in W. Therefore, W is a subspace of V. □

Suppose that $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m\}$ is any set of m vectors in a vector space V. If any vector in V can be written as a linear combination of these vector $\mathbf{x}_i$'s, we say that it is a **spanning set** of V.

Example 3.6 (1) For a nonzero vector $\mathbf{v}$ in a vector space V, linear combinations of $\mathbf{v}$ are simply scalar multiples of $\mathbf{v}$. Thus the subspace W of V spanned by $\mathbf{v}$ is $W = \{k\mathbf{v} : k \in \mathbb{R}\}$.

(2) Consider three vectors $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, -1, 1)$ and $\mathbf{v}_3 = (1, 0, 1)$ in $\mathbb{R}^3$. The subspace W_1 spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ is written as

$$W_1 = \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = (a_1 + a_2, a_1 - a_2, a_1 + a_2) : a_i \in \mathbb{R}\},$$

and the subspace W_2 spanned by $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ is written as

$$W_2 = \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = (a_1 + a_2 + a_3, a_1 - a_2, a_1 + a_2 + a_3) : a_i \in \mathbb{R}\}.$$

Then $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + 0\mathbf{v}_3$ implies $W_1 \subseteq W_2$. On the other hand, any vector in W_2 is of the form $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$. But, since $\mathbf{v}_3 = \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_2)$, this can be rewritten as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. This means that $W_2 \subseteq W_1$, thus $W_1 = W_2$ which is a plane in $\mathbb{R}^3$ containing the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. In general, a subspace in a vector space can have many different spanning sets. □

Example 3.7 Let

$$\begin{aligned}\mathbf{e}_1 &= (1,\ 0,\ 0,\ \ldots,\ 0),\\ \mathbf{e}_2 &= (0,\ 1,\ 0,\ \ldots,\ 0),\\ &\ \ \vdots\\ \mathbf{e}_n &= (0,\ 0,\ 0,\ \ldots,\ 1)\end{aligned}$$

be n vectors in the n-space $\mathbb{R}^n$ ($n \geq 3$). Then a linear combination of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is of the form

$$a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = (a_1,\ a_2,\ a_3,\ 0,\ \ldots,\ 0).$$

Hence, the set

$$W = \{(a_1,\ a_2,\ a_3,\ 0,\ \ldots,\ 0) \in \mathbb{R}^n : a_1,\ a_2,\ a_3 \in \mathbb{R}\}$$

is the subspace of the n-space $\mathbb{R}^n$ spanned by the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. Note that the subspace W can be identified with the 3-space $\mathbb{R}^3$ through the identification

$$(a_1,\ a_2,\ a_3,\ 0,\ \ldots,\ 0) \equiv (a_1,\ a_2,\ a_3)$$

with $a_i \in \mathbb{R}$. In general, for $m \leq n$, the m-space $\mathbb{R}^m$ can be identified as a subspace of the n-space $\mathbb{R}^n$. □

Example 3.8 Let $A = [\mathbf{c}_1\ \mathbf{c}_2\ \cdots\ \mathbf{c}_n]$ be an $m \times n$ matrix. Then the column vectors $\mathbf{c}_i$'s are in $\mathbb{R}^m$, and the matrix product $A\mathbf{x}$ represents the linear combination of the column vectors $\mathbf{c}_i$'s whose coefficients are the components of $\mathbf{x} \in \mathbb{R}^n$, *i.e.*, $A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n$. Therefore, the set

$$W = \{A\mathbf{x} \in \mathbb{R}^m : \mathbf{x} \in \mathbb{R}^n\}$$

of all linear combinations of the column vectors of A is a subspace of $\mathbb{R}^m$ called the **column space** of A. Therefore, $A\mathbf{x} = \mathbf{b}$ has a *solution* $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ *if and only if the vector* $\mathbf{b}$ *belongs to the subspace* W *spanned by the column vectors of* A. □

Problem 3.4 Show that $\mathbf{x}_1$, $\mathbf{x}_2$, …, $\mathbf{x}_m$ be vectors in a vector space V and let W be the subspace spanned by $\mathbf{x}_1$, $\mathbf{x}_2$, …, $\mathbf{x}_m$. Show that W is the smallest subspace of V containing $\mathbf{x}_1$, $\mathbf{x}_2$, …, $\mathbf{x}_m$, *i.e.*, if U is a subspace of V containing $\mathbf{x}_1$, $\mathbf{x}_2$, …, $\mathbf{x}_m$, then $W \subseteq U$.

Problem 3.5 Show that the set of all matrices of the form $AB - BA$ does not span the vector space $M_{n \times n}(\mathbb{R})$.

<u>Linear Combination</u>:- Let V be a vector space and $v_1, v_2, \ldots v_n$ be n vectors in V. The vector

$$y = k_1v_1 + k_2v_2 + \cdots + k_nv_n$$ is known as the linear combination of $\{v_1, v_2, \ldots v_n\}$.

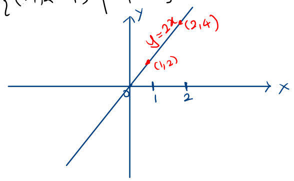
<u>Theorem</u>:-

$$W = \{k_1v_1 + k_2v_2 + \cdots + k_nv_n \mid k_1, k_2, \ldots k_n \in \mathbb{R}\}.$$

W forms a vector subspace of V. In this case we say W is the subspace of V spanned by $v_1, v_2, \ldots v_n$.

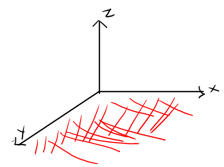
<u>Q.1</u> Find the subspace of $\mathbb{R}^2$ spanned by the vector $x_1 = (1, 2)$.

<u>Ans</u>:- $W = \{k_1x_1 \mid k_1 \in \mathbb{R}\} = \{k_1(1,2) \mid k_1 \in \mathbb{R}\}$
$= \{(k_1, 2k_1) \mid k_1 \in \mathbb{R}\}$



<u>Q.2</u> Find the subspace of $\mathbb{R}^3$ spanned by $x_1 = (1,0,0)$, $x_2 = (1,1,0)$.

<u>Ans</u>:- $W = \{k_1x_1 + k_2x_2 \mid k_1, k_2 \in \mathbb{R}\}$
$= \{k_1(1,0,0) + k_2(1,1,0) \mid k_1, k_2 \in \mathbb{R}\}$
$= \{(k_1+k_2, k_2, 0) \mid k_1, k_2 \in \mathbb{R}\}$ = XY plane.



Linearly independent and linearly dependent vectors:-

Q:- Is there a relationship (using linear combination) between the vectors $x_1 = (1,2)$ and $x_2 = (6,12)$ in $\mathbb{R}^2$?

Ans:- $x_2 = 6x_1$

$\Rightarrow \boxed{6}x_1 - x_2 = 0$

Q:- Is there any relationship (using linear combination) between the vectors $x_1 = (1,2,3)$, $x_2 = (2,4,6)$, $x_3 = (3,6,9)$ in $\mathbb{R}^3$

Ans:- $x_1 + x_2 = x_3$

$1x_1 + x_2 - x_3 = 0$

Q:- Is there a relationship (interms of linear combination) between the vectors $x_1 = (1,0,0)$, $x_2 = (0,1,0)$ and $x_3 = (0,0,1)$?

Ans:- Suppose $k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1) = 0$

$(k_1, k_2, k_3) = (0,0,0)$

$\Rightarrow k_1 = 0;\ k_2 = 0;\ k_3 = 0$

$0\cdot(1,0,0) + 0\cdot(0,1,0) + 0\cdot(0,0,1) = 0$

$0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 = 0$

Linearly independent vectors:- Let V be a vector space and $\{v_1, v_2, \dots v_n\}$ be a collection of vectors in V. We say $\{v_1, v_2, \dots v_n\}$ are linearly independent vectors if

$k_1v_1 + k_2v_2 + \dots + k_nv_n = 0$

implies $k_1 = k_2 = \dots = k_n = 0$.

Linearly dependent vectors:- We say $\{v_1, v_2, \dots v_n\}$ are linearly dependent if

$k_1v_1 + k_2v_2 + \dots + k_nv_n = 0$

implies atleast one k_i is non zero.

Pb1 Let $v_1 = (1,3,-2,2,3)$, $v_2 = (1,4,-3,4,2)$ and $v_3 = (1,3,0,2,3)$. Is $\{v_1, v_2, v_3\}$ linearly independent in $\mathbb{R}^5$?

Ans:- $k_1v_1 + k_2v_2 + k_3v_3 = 0$

$k_1(1,3,-2,2,3) + k_2(1,4,-3,4,2) + k_3(1,3,0,2,3) = (0,0,0,0,0)$

$(k_1 + k_2 + k_3,\ 3k_1 + 4k_2 + 3k_3,\ -2k_1 - 3k_2,\ 2k_1 + 4k_2 + 2k_3,\ 3k_1 + 2k_2 + 3k_3) = (0,0,0,0,0)$

$k_1 + k_2 + k_3 = 0 \to (1)$

$3k_1 + 4k_2 + 3k_3 = 0 \to (2)$

$-2k_1 - 3k_2 = 0 \to (3)$

$2k_1 + 4k_2 + 2k_3 = 0 \to (4)$

$3k_1 + 2k_2 + 3k_3 = 0 \to (5)$

$(2) - (5) \Rightarrow k_2 = 0$

$(3) \Rightarrow k_1 = 0$ from (1) $k_3 = 0$.

$\therefore \{v_1, v_2, v_3\}$ are linearly independent.

Q:- Let $V = P_3(\mathbb{R})$. Determine whether the following polynomials are linearly independent

$u = t^3 + 4t^2 - 2t + 3$

$v = t^3 + 6t^2 - t + 4$

$\omega = 3t^3 + 8t^2 - 8t + 7$

Ans:- $k_1u + k_2v + k_3\omega = 0$

$k_1(t^3 + 4t^2 - 2t + 3) + k_2(t^3 + 6t^2 - t + 4) + k_3(3t^3 + 8t^2 - 8t + 7) = 0$

$(k_1 + k_2 + 3k_3)t^3 + (4k_1 + 6k_2 + 8k_3)t^2 + (-2k_1 - k_2 - 8k_3)t + (3k_1 + 4k_2 + 7k_3) = 0$

$k_1 + k_2 + 3k_3 = 0 \to (1)$

$4k_1 + 6k_2 + 8k_3 = 0 \to (2)$

$-2k_1 - k_2 - 8k_3 = 0 \to (3)$

$3k_1 + 4k_2 + 7k_3 = 0 \to (4)$

1st pivot

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 3 & 0 \\ 4 & 6 & 8 & 0 \\ -2 & -1 & -8 & 0 \\ 3 & 4 & 7 & 0 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 + 2R_1 \\ R_4 \to R_4 - 3R_1 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & -4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] R_2 \to \tfrac{1}{2}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \begin{array}{l} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

reduced system is $k_1 + k_2 + 3k_3 = 0$

$k_2 - 2k_3 = 0$

Hence infinitely many non zero solutions are possible

Hence $\{u, v, \omega\}$ are linearly dependent.

As we saw above, any nonempty subset of a vector space V spans a subspace through the linear combinations of the vectors, and a subspace can have many spanning sets with a different number of vectors. This means that a vector can be written as linear combinations in various ways. If one can find a finite number of vectors in V such that any vector in V can be expressed in a unique way as a linear combination of them, then the study of the vector space V might be easier and the computations of vectors may be simplified. Thus, for a fixed set of vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m$ in a vector space V, we look at their linear combinations $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_m\mathbf{x}_m$ and see whether any vector in V can be written in this form in exactly one way. This problem can be rephrased as to whether or not a nontrivial linear combination produces the zero vector, while the trivial combination, with all scalars $c_i = 0$, obviously produces the zero vector.

Definition 3.3 A set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ in a vector space V is said to be **linearly independent** if the vector equation, called the **linear dependence** of $\mathbf{x}_i$'s,

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_m\mathbf{x}_m = \mathbf{0}$$

has only the trivial solution $c_1 = c_2 = \cdots = c_m = 0$. Otherwise, it is said to be **linearly dependent.**

Therefore, a set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ is linearly dependent if and only if there is a linear dependence

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_m\mathbf{x}_m = \mathbf{0}$$

with a nontrivial solution $(c_1, c_2, \dots, c_m)$. In this case, we may assume that $c_m \neq 0$. Then the equation can be rewritten as

$$\mathbf{x}_m = -\frac{c_1}{c_m}\mathbf{x}_1 - \frac{c_2}{c_m}\mathbf{x}_2 - \cdots - \frac{c_{m-1}}{c_m}\mathbf{x}_{m-1}.$$

That is, *a set of vectors is linearly dependent if and only if at least one of the vectors in the set can be written as a linear combination of the others.*

Example 3.9 Let $\mathbf{x} = (1,2,3)$ and $\mathbf{y} = (3,2,1)$ be two vectors in the 3-space $\mathbb{R}^3$. Then clearly $\mathbf{y} \neq \lambda\mathbf{x}$ for any $\lambda \in \mathbb{R}$ (or $a\mathbf{x} + b\mathbf{y} = \mathbf{0}$ only when $a = b = 0$). This means that $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent in $\mathbb{R}^3$. If $\mathbf{w} = (3,6,9)$, then $\{\mathbf{x}, \mathbf{w}\}$ is linearly dependent since $\mathbf{w} - 3\mathbf{x} = \mathbf{0}$. In general, if $\mathbf{x}, \mathbf{y}$ are noncollinear vectors in the 3-space $\mathbb{R}^3$, the set of all linear combinations of $\mathbf{x}$ and $\mathbf{y}$ determines a plane W through the origin in $\mathbb{R}^3$, *i.e.*, $W = \{a\mathbf{x} + b\mathbf{y} : a, b \in \mathbb{R}\}$. Let $\mathbf{z}$ be another nonzero vector in the 3-space $\mathbb{R}^3$. If $\mathbf{z} \in W$, then there are some numbers $a, b \in \mathbb{R}$, not all of them are zero, such that $\mathbf{z} = a\mathbf{x} + b\mathbf{y}$, that is, the set $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent. If $\mathbf{z} \notin W$, then $a\mathbf{x} + b\mathbf{y} + c\mathbf{z} = \mathbf{0}$ is possible only when $a = b = c = 0$ (prove it). Therefore, the set $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly independent if and only if $\mathbf{z}$ does not lie in W. □

By abuse of language, it is sometimes convenient to say that "the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ are linearly independent," although this is really a property of a set.

Example 3.10 The columns of the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 & 0 \\ 4 & 2 & 6 & 8 \\ 2 & -1 & 1 & 3 \end{bmatrix}$$

are linearly dependent in the 3-space $\mathbb{R}^3$, since the third column is the sum of the first and the second.

As this example shows, the concept of linear dependence can be applied to the row or column vectors of any matrix.

Example 3.11 Consider an upper triangular matrix

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 4 \end{bmatrix}.$$

The linear dependence of the column vectors of A may be written as

$$c_1 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which, in matrix notation, may be written as a homogeneous system:

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

From the third row, $c_3 = 0$, from the second row $c_2 = 0$, and substitution of them into the first row forces $c_1 = 0$, *i.e.*, the homogeneous system has only the trivial solution, so that the column vectors are linearly independent. □

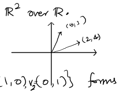
Basis of a vector space:-

Let V be a vector space over $\mathbb{R}$. A collection of vectors $\{v_1, v_2, \cdots v_m\}$ is said to form a basis for V if

(a) $\{v_1, v_2, \cdots v_m\}$ is linearly independent

(b) subspace spanned by $\{v_1, v_2, \cdots, v_m\}$ is V.

Examples:- Consider $\mathbb{R}^2$ over $\mathbb{R}$.



$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$

The collections $\{v_1 = (1,0), v_2 = (0,1)\}$ forms a basis for $\mathbb{R}^2$.

Is $\{v_1, v_2\}$ linearly independent?

$k_1 v_1 + k_2 v_2 = 0 \Rightarrow k_1(1,0) + k_2(0,1) = (0,0)$

$\Rightarrow (k_1, k_2) = (0,0) \therefore k_1 = 0; k_2 = 0$

Hence $\{v_1, v_2\}$ are linearly independent.

$W = \{k_1 v_1 + k_2 v_2 \mid k_1, k_2 \in \mathbb{R}\}$

$= \{k_1(1,0) + k_2(0,1) \mid k_1, k_2 \in \mathbb{R}\}$

$= \{(k_1, k_2) \mid k_1, k_2 \in \mathbb{R}\} = \mathbb{R}^2$

Standard basis for $\mathbb{R}^2$ is $\{(1,0), (0,1)\}$.

The following theorem can be proven by the same argument.

Theorem 3.5 *The nonzero rows of a matrix of a row-echelon form are linearly independent, and so are the columns that contain leading 1's.*

In particular, the rows of any triangular matrix with nonzero diagonals are linearly independent, and so are the columns.

In general, if $V = \mathbb{R}^m$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are n vectors in $\mathbb{R}^m$, then they form an $m \times n$ matrix $A = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n]$. On the other hand, Example 3.8 shows that the linear dependence $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ of $\mathbf{v}_i$'s is nothing but the homogeneous equation $A\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = (c_1, c_2, \cdots, c_n)$. Thus, *the column vectors $\mathbf{v}_i$'s of A are linearly independent in $\mathbb{R}^m$ if and only if the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution,* and *they are linearly dependent if and only if $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.*

If U is the reduced row-echelon form of A, then we know that $A\mathbf{x} = \mathbf{0}$ and $U\mathbf{x} = \mathbf{0}$ have the same set of solutions. Moreover, a homogeneous system $A\mathbf{x} = \mathbf{0}$ with unknowns more than the number of equations always has a nontrivial solution (see the remark on page 11). This proves the following lemma.

Lemma 3.6 **(1)** *Any set of n vectors in the m-space $\mathbb{R}^m$ is linearly dependent if $n > m$.*

(2) *If U is the reduced row-echelon form of A, then the columns of U are linearly independent if and only if the columns of A are linearly independent.*

Example 3.12 Consider the vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$ and $\mathbf{e}_3 = (0, 0, 1)$ in the 3-space $\mathbb{R}^3$. The vector equation $c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 = \mathbf{0}$ becomes

$$c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1) = (0, 0, 0)$$

or, equivalently, $(c_1, c_2, c_3) = (0, 0, 0)$. Thus, $c_1 = c_2 = c_3 = 0$, so the set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is linearly independent and also spans $\mathbb{R}^3$.

Example 3.13 In general, it is quite clear that the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ in $\mathbb{R}^n$ are linearly independent. Moreover, they span the n-space $\mathbb{R}^n$: In fact, when we write a vector $\mathbf{x} \in \mathbb{R}^n$ as $(x_1, \ldots, x_n)$, it means just the linear combination of the vector $\mathbf{e}_i$'s:

$$\mathbf{x} = (x_1, \ldots, x_n) = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n.$$

However, if any one of the $\mathbf{e}_i$'s is missed, then they cannot span $\mathbb{R}^n$. Thus, this kind of vector plays a special role in the vector space.

Example 2:- Consider $\mathbb{R}^3$ over $\mathbb{R}$. We prove that $\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ forms a basis for $\mathbb{R}^3$

Linearly independent:- $k_1 e_1 + k_2 e_2 + k_3 e_3 = 0$

$\Rightarrow k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1) = (0,0,0)$

$\Rightarrow (k_1, k_2, k_3) = (0,0,0)$

$\therefore k_1 = 0; \ k_2 = 0; \ k_3 = 0$

Spanning Subspace:- $W = \{k_1 e_1 + k_2 e_2 + k_3 e_3 \mid k_1, k_2, k_3 \in \mathbb{R}\}$

$= \{k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1) \mid k_1, k_2, k_3 \in \mathbb{R}\}$

$= \{(k_1, k_2, k_3) \mid k_1, k_2, k_3 \in \mathbb{R}\}$

$= \mathbb{R}^3$

Hence $\{(1,0,0), (0,1,0), (0,0,1)\}$ forms a basis for $\mathbb{R}^3$.

We call this as standard basis for $\mathbb{R}^3$

Recall:- 1) Let V be a vector space and $\{v_1, v_2, \cdots, v_n\}$ be a collection of vectors in V. The vector

$y = k_1 x_1 + k_2 x_2 + \cdots + k_n x_n$ where k_i's are scalars is known as the linear combination of $x_1, x_2, \cdots x_n$.

2) $W = \{k_1 x_1 + \cdots + k_n x_n \mid k_i\text{'s are scalars}\}$ forms a subspace of V. W is a subspace of V spanned by $x_1, x_2, \cdots x_n$

3) Linearly independent vectors:- The collection $\{v_1, v_2, \cdots v_n\}$ is said to be linearly independent if

$k_1 v_1 + k_2 v_2 + \cdots + k_n v_n = 0$ implies $k_1 = k_2 = \cdots = k_n = 0$.

4) Linearly dependent vectors:- The collection $\{v_1, v_2, \cdots v_n\}$ is said to be linearly dependent if

$k_1 v_1 + k_2 v_2 + \cdots + k_n v_n = 0$ then atleast one k_i is non zero.

Standard basis for $\mathbb{R}^n$:- Consider $\mathbb{R}^n$ over $\mathbb{R}$.

The standard basis of $\mathbb{R}^n$ is the collection of vectors $e_1, e_2, \cdots e_n$ where $e_i = \overbrace{(0, 0, \cdots, \underset{i^{th}\text{ place}}{\boxed{1}}, 0, \cdots 0)}^{n\text{ co-ordinates}}$

Basis for $P_2(\mathbb{R})$

$P_2(\mathbb{R}) = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\}$

= collection of all polynomials of degree atmost 2.

We prove that $\{1, x, x^2\}$ forms a basis for $P_2(\mathbb{R})$

Linearly independent $k_1 \cdot 1 + k_2 \cdot x + k_3 x^2 = 0 = 0 + 0x + 0x^2$

$k_1 = 0; k_2 = 0; k_3 = 0$

$\therefore \{1, x, x^2\}$ is linearly independent

Spanning subspace:- $W = \{k_1 \cdot 1 + k_2 x + k_3 x^2 \mid k_1, k_2, k_3 \in \mathbb{R}\}$

$= \{k_1 + k_2 x + k_3 x^2 \mid k_1, k_2, k_3 \in \mathbb{R}\}$

$= P_2(\mathbb{R})$

Hence standard basis for $P_2(\mathbb{R})$ is $\{1, x, x^2\}$

Standard basis for $P_n(\mathbb{R})$

$$P_n(\mathbb{R}) = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \mid a_1, a_2, \ldots a_n \in \mathbb{R}\}$$

is a vector space over $\mathbb{R}$. Standard basis for $P_n(\mathbb{R})$ is $\{1, x, x^2, x^3, \ldots, x^n\}$.

Basis for $M_{2\times 2}(\mathbb{R})$:-

$$M_{2\times2}(\mathbb{R}) = \left\{\begin{bmatrix} a & b \\ c & d\end{bmatrix} : a, b, c, d \in \mathbb{R}\right\}.$$

We prove that $\left\{E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}; E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix}\right\}$ forms a standard basis for $M_{2\times2}(\mathbb{R})$

Linearly independent set

$$k_1E_{11} + k_2E_{12} + k_3E_{21} + k_4E_{22} = 0$$

$$k_1\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix} + k_2\begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix} + k_3\begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix} + k_4\begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix}$$

$$\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4\end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix} \Rightarrow k_1 = k_2 = k_3 = k_4 = 0$$

Spanning Subspace:

$$w = \{k_1E_{11} + k_2E_{12} + k_3E_{21} + k_4E_{22} \mid k_1, k_2, k_3, k_4 \in \mathbb{R}\}$$

$$= \left\{k_1\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix} + k_2\begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix} + k_3\begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix} + k_4\begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix} \,\middle|\, k_1, k_2, k_3, k_4 \in \mathbb{R}\right\}$$

$$= \left\{\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4\end{bmatrix} \,\middle|\, k_1, k_2, k_3, k_4 \in \mathbb{R}\right\}$$

$$= M_{2\times2}(\mathbb{R})$$

Basis for $M_{m\times n}(\mathbb{R})$

A standard basis for $M_{m\times n}(\mathbb{R})$ is $\{E_{11}, E_{12}, \cdots E_{mn}\}$ where E_{ij} is the matrix of order $m\times n$ such that $(i,j)^{th}$ position of E_{ij} is 1 and other positions are zero.

Definition 3.4 Let V be a vector space. A **basis** for V is a set of linearly independent vectors that spans V.

For example, as we saw in Example 3.13, the set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ forms a basis, called the **standard basis** for the n-space $\mathbb{R}^n$. Of course, there are many other bases for $\mathbb{R}^n$.

Example 3.14 (1) The set of vectors $(1,1,0)$, $(0,-1,1)$, and $(1,0,1)$ is not a basis for the 3-space $\mathbb{R}^3$, since this set is linearly dependent (the third is the sum of the first two vectors), and cannot span $\mathbb{R}^3$. (The vector $(1,0,0)$ cannot be obtained as a linear combination of them (prove it).) This set does not have enough vectors spanning $\mathbb{R}^3$.

(2) The set of vectors $(1,0,0)$, $(0,1,1)$, $(1,0,1)$ and $(0,1,0)$ is not a basis either, since they are not linearly independent (the sum of the first two minus the third makes the fourth) even though they span $\mathbb{R}^3$. This set of vectors has some redundant vectors spanning $\mathbb{R}^3$.

(3) The set of vectors $(1,1,1)$, $(0,1,1)$, and $(0,0,1)$ is linearly independent and also spans $\mathbb{R}^3$. That is, it is a basis for $\mathbb{R}^3$, different from the standard basis. This set has the proper number of vectors spanning $\mathbb{R}^3$, since the set cannot be reduced to a smaller set nor does it need any additional vector spanning $\mathbb{R}^3$. □

By definition, in order to show that a set of vectors in a vector space is a basis, one needs to show two things: *it is linearly independent, and it spans the whole space.* The following theorem shows that a basis for a vector space represents a coordinate system just like the rectangular coordinate system by the standard basis for $\mathbb{R}^n$.

Theorem 3.7 *Let $\alpha = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V. Then each vector $\mathbf{x}$ in V can be uniquely expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, i.e., there are unique scalars a_i's, $i = 1, \ldots, n$, such that*

$$\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n.$$

In this case, the column vector $[a_1\ a_2\ \cdots\ a_n]^T$ is called the **coordinate vector** of $\mathbf{x}$ with respect to the basis α, and it is denoted $[\mathbf{x}]_\alpha$.

Proof: If $\mathbf{x}$ can be also expressed as $\mathbf{x} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n$, then we have $\mathbf{0} = (a_1 - b_1)\mathbf{v}_1 + (a_2 - b_2)\mathbf{v}_2 + \cdots + (a_n - b_n)\mathbf{v}_n$. By the linear independence of $\mathbf{x}_i$'s, $a_i = b_i$ for all $i = 1, \ldots, n$. □

Example 3.15 Let $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$, and let $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with $\mathbf{v}_1 = (1,1,1) = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, $\mathbf{v}_2 = (0,1,1) = \mathbf{e}_2 + \mathbf{e}_3$, $\mathbf{v}_3 = (0,0,1) = \mathbf{e}_3$. Then

$$[\mathbf{v}_1]_\alpha = \begin{bmatrix} 1 \\ 1 \\ 1\end{bmatrix},\ [\mathbf{v}_2]_\alpha = \begin{bmatrix} 0 \\ 1 \\ 1\end{bmatrix},\ [\mathbf{v}_3]_\alpha = \begin{bmatrix} 0 \\ 0 \\ 1\end{bmatrix},$$

while $[\mathbf{v}_1]_\beta = [1\ 0\ 0]^T$, $[\mathbf{v}_2]_\beta = [0\ 1\ 0]^T$, $[\mathbf{v}_3]_\beta = [0\ 0\ 1]^T$.

Problem 3.6 Show that the vectors $\mathbf{v}_1 = (1,\ 2,\ 1)$, $\mathbf{v}_2 = (2,\ 9,\ 0)$ and $\mathbf{v}_3 = (3,\ 3,\ 4)$ in the 3-space $\mathbb{R}^3$ form a basis.

Problem 3.7 Show that the set $\{1,\ x,\ x^2,\ \ldots,\ x^n\}$ is a basis for $P_n(\mathbb{R})$, the vector space of all polynomials of degree $\le n$ with real coefficients.

Problem 3.8 In the n-space $\mathbb{R}^n$, determine whether or not the set

$$\{\mathbf{e}_1 - \mathbf{e}_2,\ \mathbf{e}_2 - \mathbf{e}_3,\ \ldots,\ \mathbf{e}_{n-1} - \mathbf{e}_n,\ \mathbf{e}_n - \mathbf{e}_1\}$$

is linearly dependent.

Problem 3.9 Let $\mathbf{x}_k$ denote the vector in $\mathbb{R}^n$ whose first $k-1$ coordinates are zero and whose last $n-k+1$ coordinates are 1. Show that the set $\{\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_n\}$ is a basis for $\mathbb{R}^n$.

3.3 Dimensions

We often say that the line $\mathbb{R}^1$ is one-dimensional, the plane $\mathbb{R}^2$ is two-dimensional and the space $\mathbb{R}^3$ is three-dimensional, etc. This is mostly due to the fact that the freedom in choosing coordinates for each element in the space is 1, 2 or 3, respectively. This means that the concept of *dimension* is closely related to the concept of bases. Note that for a vector space in general there is no unique way to choose a basis. However, there is something common to all bases, and this is related to the notion of dimension. We first need the following important lemma from which one can define the dimension of a vector space.

Lemma 3.8 *Let V be a vector space and let $\alpha = \{\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m\}$ be a set of m-vectors in V.*

Recall:- Basis of a vector space:- Consider a vector space V and $\{v_1, v_2, \ldots v_n\}$ be a collection of vectors in V. We say $\{v_1, v_2, \ldots v_n\}$ forms a basis for V if

(a) $\{v_1, v_2, \ldots v_n\}$ is a linearly independent set

(b) subspace spanned by $\{v_1, v_2, \ldots v_n\}$ is V

1) Consider $\mathbb{R}^n$ over $\mathbb{R}$. The collection $\{e_1, e_2, \ldots e_n\}$ forms a basis for $\mathbb{R}^n$ where

$$e_i = \underbrace{(0, 0, 0, \ldots \overset{i^{th}\text{ position}}{\boxed{1}}, 0, 0, \ldots 0)}_{n \text{ coordinates}}$$

2) Consider $P_n(\mathbb{R}) = \{a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \mid a_n, a_{n-1}, \ldots a_0 \in \mathbb{R}\}$. The collection $\boxed{\{1, x, x^2, x^3, \ldots x^n\}}$ forms a basis for $P_n(\mathbb{R})$

3) Consider $M_{m\times n}(\mathbb{R})$ [Collection of all $m\times n$ matrices]. The collection $\{E_{11}, E_{12}, \ldots, E_{mn}\}$ forms a basis for $M_{m\times n}(\mathbb{R})$ where E_{ij} is a $m\times n$ matrix whose $(i,j)^{th}$ position is 1 and other positions are zero.

Consider $\mathbb{R}^2$ over $\mathbb{R}$ and the collection $\alpha = \{(1,0), (0,1), (1,1)\}$

For $(2,3) \in \mathbb{R}^2$ we have

$$(2,3) = 2(1,0) + 3(0,1) + 0\cdot(1,1) \quad \text{and}$$

$$(2,3) = (-1)(1,0) + 0\cdot(0,1) + 3(1,1) \quad \text{and}$$

$$(2,3) = 0\cdot(1,0) + 1\cdot(0,1) + 2(1,1)$$

Theorem:- Let $\alpha = \{v_1, v_2, \ldots v_n\}$ be a basis for a vector space V then each vector x in V can be uniquely expressed as a linear combination of vectors in α

Example:- Consider $\mathbb{R}^2$ over $\mathbb{R}$.

It is easy to see that

$$\alpha_1 = \{(1,0), (0,1)\}$$
$$\alpha_2 = \{(1,0), (1,1)\}$$
$$\alpha_3 = \{(1,1), (1,2)\}$$

all are different basis for $\mathbb{R}^2$. But they have same number of elements

Theorem:- If a basis for a vector space V consists of n vectors then so does every other basis.

Example:- Consider $\mathbb{R}^2$ over $\mathbb{R}$. We know $\{(1,0),(0,1)\}$ is a basis for $\mathbb{R}^2$. Spanning subspace of $\{(1,0),(0,1)\}$ is $\mathbb{R}^2$. What do you comment about the collection $\{(1,0),(0,1),(1,1)\}$?

Ans:- Collection $\{(1,0),(0,1),(1,1)\}$ is a linearly dependent set.

Theorem:- Let V be a vector space and $\alpha = \{v_1, v_2, \ldots v_m\}$ be a set of m-vectors then

(i) If α spans V then every set of vectors more than m-vectors are linearly dependent.

(ii) If α is linearly independent then any set of vectors with fewer than m-vectors cannot span V.

Dimension of a vector space:- Let V be a vector space. Dimension of a vector space is the number of elements in the basis.

Examples:- (i) A basis for $\mathbb{R}^2$ is $\{(1,0),(0,1)\}$

Dimension of $\mathbb{R}^2$ is 2.

(ii) A basis for $\mathbb{R}^3$ is $\{(1,0,0),(0,1,0),(0,0,1)\}$

Dimension of $\mathbb{R}^3$ is 3.

(iii) A basis for $\mathbb{R}^n$ is $\{e_1, e_2, \ldots e_n\}$

$\therefore$ Dimension is n.

(iv) A basis for $P_2(\mathbb{R})$ is $\{1, x, x^2\}$

$\therefore$ Dimension of $P_2(\mathbb{R})$ is 3.

(v) A basis for $P_n(\mathbb{R})$ is $\{1, x, x^2, \ldots x^n\}$

Dimension of $P_n(\mathbb{R})$ is $n+1$.

(vi) A basis for $M_{2\times 3}(\mathbb{R})$ is $\{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$

$E_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; $E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; $E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$E_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$; $E_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$; $E_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

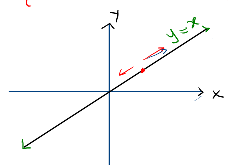
Dimension of $M_{2\times 3}(\mathbb{R})$ is 6

(vii) A basis for $M_{m\times n}(\mathbb{R})$ is $\{E_{11}, E_{12}, \ldots E_{mn}\}$

Dimension of $M_{m\times n}(\mathbb{R})$ is mn

Q:- What is the dimension of the subspace $W = \{(x,y) \in \mathbb{R}^2 \mid x - y = 0\}$?

Ans:-



Dimension of $W = 1$.

Recall:- Let V be a vector space. Dimension of V is the number of elements in the basis.

Examples:- dimension of $\mathbb{R}^n$ is n

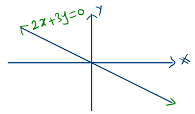
dimension of $P_n(\mathbb{R})$ is $n+1$

dimension of $M_{m\times n}(\mathbb{R})$ is mn

dimension of $\{(x,y) \in \mathbb{R}^2 \mid x - y = 0\}$ is 1

Q:- Find the dimension of the vector subspace $W = \{(x,y) \in \mathbb{R}^2 \mid 2x + 3y = 0\}$.

Ans:-



A basis is $\{(3,-2)\}$. Hence Dimension is 1.

Note:- Consider the vector space $\mathbb{R}^n$ and the subspace

$$W = \left\{ (x_1, x_2, \dots x_n) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0 \end{matrix} \right\}$$

dimension of W = no. of free variables in the associated system.

Example:- Consider the vector space $\mathbb{R}^3$ and the subspace

$$W = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \begin{matrix} x + y + z = 0 \\ x + 2y + 2z = 0 \\ 2x + y + z = 0 \end{matrix} \right\}.$$ Find the basis and dimension of W.

Ans:- We find the free of the given system.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right] \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$$R_3 \to R_3 + R_2 \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Reduced System is

$x + y + z = 0$

$y + z = 0$

$\therefore$ The Basis variables are x and y. The free variable is z $\therefore$ Dimension $(W) = 1$

Basis for $W = \{(0, 1, -1)\}$

Q:- Find a basis and dimension of the subspace

$$W = \{ (x, y, z, t) \in \mathbb{R}^4 \mid 5x + y + z + t = 0 \}.$$

Ans:- For the equation $5x + y + z + t = 0$ the variable x is basic and y, z, t are free variable.

Hence $\dim W = 3$

Basis $= \{ (0, 1, 0, -1), (0, 0, 1, -1), (\frac{-1}{5}, 1, 0, 0) \}$

Consider $M_{3\times3}(\mathbb{R})$. Find a basis and dimension of the vector subspace

$$W = \{ A \in M_{3\times3}(\mathbb{R}) \mid A = A^T \}$$

Soln:- $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ when do you say A is symmetric

Ans:- $a_{12} = a_{21}$; $a_{13} = a_{31}$; $a_{23} = a_{32}$

A basis is $\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$

Dimension of $W = 6$

Q:- Let us consider the vector space $M_{n\times n}(\mathbb{R})$

$$W = \{ A \in M_{n\times n}(\mathbb{R}) \mid A = A^T \}$$

What is the dimension of W?

Ans:- If it is symmetric

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$n + (n-1) + (n-2) + \dots + 1$

$$\text{Dim}(W) = \frac{n(n+1)}{2}.$$

H.W Find the dimension of the vector subspace

$$W = \{ A \in M_{n\times n}(\mathbb{R}) \mid A^T = -A \}$$

= Set of all skew symmetric matrices.

Ans:- $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $\quad A^T = -A$

$\dim(W) = 3$

$$A^T = \begin{bmatrix} \overset{0}{a_{11}} & a_{21} & a_{31} \\ a_{12} & \overset{0}{a_{22}} & a_{32} \\ a_{13} & a_{23} & \overset{0}{a_{33}} \end{bmatrix} = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \\ -a_{31} & -a_{32} & -a_{33} \end{bmatrix} = -A$$

In general $\dim W = \frac{(n-1)n}{2}$

Theorem:- Let V be a vector space of dimension n then

(i) Any set of n vectors that spans V is a basis for V and

(ii) Any set of n linearly independent vectors is a basis for V

Example:- Is the collection $\{(1,0,0), (1,1,0), (1,1,1)\}$ form a basis for $\mathbb{R}^3$?

Ans:- $k_1(1,0,0) + k_2(1,1,0) + k_3(1,1,1) = (0,0,0)$

$\Rightarrow (k_1+k_2+k_3,\ k_2+k_3,\ k_3) = (0,0,0)$

$k_1 = 0;\ k_2 = 0;\ k_3 = 0$

Hence the vectors are linearly independent so they form a basis.

Example 3.15 Let $\alpha = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$, and let $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with $\mathbf{v}_1 = (1,1,1) = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, $\mathbf{v}_2 = (0,1,1) = \mathbf{e}_2 + \mathbf{e}_3$, $\mathbf{v}_3 = (0,0,1) = \mathbf{e}_3$. Then

$$[\mathbf{v}_1]_\alpha = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ [\mathbf{v}_2]_\alpha = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},\ [\mathbf{v}_3]_\alpha = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};$$

while $[\mathbf{v}_1]_\beta = [1\ 0\ 0]^T$, $[\mathbf{v}_2]_\beta = [0\ 1\ 0]^T$, $[\mathbf{v}_3]_\beta = [0\ 0\ 1]^T$.

Problem 3.6 Show that the vectors $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (2, 9, 0)$ and $\mathbf{v}_3 = (3, 3, 4)$ in the 3-space $\mathbb{R}^3$ form a basis.

Problem 3.7 Show that the set $\{1, x, x^2, \ldots, x^n\}$ is a basis for $P_n(\mathbb{R})$, the vector space of all polynomials of degree $\leq n$ with real coefficients.

Problem 3.8 In the n-space $\mathbb{R}^n$, determine whether or not the set

$$\{\mathbf{e}_1 - \mathbf{e}_2,\ \mathbf{e}_2 - \mathbf{e}_3,\ \ldots,\ \mathbf{e}_{n-1} - \mathbf{e}_n,\ \mathbf{e}_n - \mathbf{e}_1\}$$

is linearly dependent.

Problem 3.9 Let $\mathbf{x}_k$ denote the vector in $\mathbb{R}^n$ whose first $k-1$ coordinates are zero and whose last $n-k+1$ coordinates are 1. Show that the set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is a basis for $\mathbb{R}^n$.

3.3 Dimensions

We often say that the line $\mathbb{R}^1$ is one-dimensional, the plane $\mathbb{R}^2$ is two-dimensional and the space $\mathbb{R}^3$ is three-dimensional, etc. This is mostly due to the fact that the freedom in choosing coordinates for each element in the space is 1, 2 or 3, respectively. This means that the concept of *dimension* is closely related to the concept of bases. Note that for a vector space in general there is no unique way to choose a basis. However, there is something common to all bases, and this is related to the notion of dimension. We first need the following important lemma from which one can define the dimension of a vector space.

Lemma 3.8 *Let V be a vector space and let $\alpha = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ be a set of m-vectors in V.*

(1) *If α spans V, then every set of vectors with more than m vectors cannot be linearly independent.*

(2) *If α is linearly independent, then any set of vectors with fewer than m vectors cannot span V.*

Proof: Since (2) follows from (1) directly, we prove only (1). Let $\beta = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ be a set of n-vectors in V with $n > m$. We will show that β is linearly dependent. Indeed, since each vector $\mathbf{y}_j$ is a linear combination of the vectors in the spanning set α, *i.e.*, for $j = 1, \ldots, n$,

$$\mathbf{y}_j = a_{1j}\mathbf{x}_1 + a_{2j}\mathbf{x}_2 + \cdots + a_{mj}\mathbf{x}_m = \sum_{i=1}^{m} a_{ij}\mathbf{x}_i,$$

we have

$$\begin{aligned} c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n &= c_1(a_{11}\mathbf{x}_1 + a_{21}\mathbf{x}_2 + \cdots + a_{m1}\mathbf{x}_m) \\ &\quad + c_2(a_{12}\mathbf{x}_1 + a_{22}\mathbf{x}_2 + \cdots + a_{m2}\mathbf{x}_m) \\ &\quad \vdots \\ &\quad + c_n(a_{1n}\mathbf{x}_1 + a_{2n}\mathbf{x}_2 + \cdots + a_{mn}\mathbf{x}_m) \\ &= (a_{11}c_1 + a_{12}c_2 + \cdots + a_{1n}c_n)\mathbf{x}_1 \\ &\quad + (a_{21}c_1 + a_{22}c_2 + \cdots + a_{2n}c_n)\mathbf{x}_2 \\ &\quad \vdots \\ &\quad + (a_{m1}c_1 + a_{m2}c_2 + \cdots + a_{mn}c_n)\mathbf{x}_m. \end{aligned}$$

Thus, β is linearly dependent if and only if the system of linear equations

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n = \mathbf{0}$$

has a nontrivial solution $(c_1, c_2, \ldots, c_n) \neq (0, 0, \cdots, 0)$. This is true if all the coefficients of $\mathbf{x}_i$'s are zero but not all of c_i's are zero. This means that the homogeneous system of linear equations in c_i's,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

has a nontrivial solution. This is guaranteed by Lemma 3.6, since A is an $m \times n$ matrix with $m < n$. □

It is clear by Lemma 3.8 that if a set $\alpha = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ of n vectors is a basis for a vector space V, then no other set $\beta = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_r\}$ of r vectors can be a basis for V if $r \neq n$. This means that all bases for a vector space V have the same number of vectors, even if there are many different bases for a vector space. Therefore, we obtain the following important result:

Theorem 3.9 *If a basis for a vector space V consists of n vectors, then so does every other basis.*

Definition 3.5 The **dimension** of a vector space V is the number, say n, of vectors in a basis for V, denoted by $\dim V = n$. When V has a basis of a finite number of vectors, V is said to be **finite dimensional**.

Example 3.16 The following can be easily verified:

(1) If V has only the zero vector: $V = \{\mathbf{0}\}$, then $\dim V = 0$.

(2) If $V = \mathbb{R}^n$, then $\dim \mathbb{R}^n = n$, since V has the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$.

(3) If $V = P_n(\mathbb{R})$ of all polynomials of degree less than or equal to n, then $\dim P_n(\mathbb{R}) = n + 1$ since $\{1, x, x^2, \ldots, x^n\}$ is a basis for V.

(4) If $V = M_{m\times n}(\mathbb{R})$ of all $m \times n$ matrices, then $\dim M_{m\times n}(\mathbb{R}) = mn$ since $\{E_{ij} : i = 1, \ldots, m,\ j = 1, \ldots, n\}$ is a basis for V, where E_{ij} is the $m \times n$ matrix whose (i,j)-th entry is 1 and all others are zero. □

If $V = C(\mathbb{R})$ of all real-valued continuous functions defined on the real line, then one can show that V is not finite dimensional. A vector space V is **infinite dimensional** if it is not finite dimensional. In this book, we are concerned only with finite dimensional vector spaces unless otherwise stated.

Theorem 3.10 *Let V be a finite dimensional vector space.*

(1) *Any linearly independent set in V can be extended to a basis by adding more vectors if necessary.*

(2) *Any set of vectors that spans V can be reduced to a basis by discarding vectors if necessary.*

Proof: We prove (1) only and leave (2) as an exercise. Let $\alpha = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ be a linearly independent set in V. If α spans V, then α is a basis. If α does not span V, then there exists a vector, say $\mathbf{x}_{k+1}$, in V that is not contained in the subspace spanned by the vectors in α. Now $\{\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1}\}$ is linearly independent (check why). If $\{\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1}\}$ spans V, then

this is a basis for V. If it does not span V, then the same procedure can be repeated, yielding a linearly independent set that spans V, *i.e.*, a basis for V. This procedure must stop in a finite step because of Lemma 3.8 for a finite dimensional vector space V. □

Theorem 3.10 shows that a basis for a vector space V is a set of vectors in V which is *maximally* independent and *minimally* spanning in the above sense. In particular, if W is a subspace of V, then any basis for W is linearly independent also in V, and can be extended to a basis for V. Thus $\dim W \leq \dim V$.

Corollary 3.11 *Let V be a vector space of dimension n. Then*

(1) *any set of n vectors that spans V is a basis for V, and*

(2) *any set of n linearly independent vectors is a basis for V.*

Proof: Again we prove (1) only. If a spanning set of n vectors were not linearly independent, then the set would be reduced to a basis that has a smaller number of vectors than n vectors. □

Corollary 3.11 means that if it is known that $\dim V = n$ and if a set of n vectors either is linearly independent or spans V, then it is already a basis for the space V.

Example 3.17 Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{x}_1 = (1,\ -2,\ 5,\ -3),\ \ \mathbf{x}_2 = (0,\ 1,\ 1,\ 4),\ \ \mathbf{x}_3 = (1,\ 0,\ 1,\ 0).$$

Find a basis for W and extend it to a basis for $\mathbb{R}^4$.

Solution: Note that $\dim W \leq 3$ since W is spanned by three vectors $\mathbf{x}_i$'s. Let A be the 3×4 matrix whose rows are $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$:

$$A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Reduce A to a row-echelon form:

$$U = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & \frac{5}{6} \end{bmatrix}.$$

The three nonzero row vectors of U are clearly linearly independent, and they also span W because the vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ can be expressed as a linear combination of these three nonzero row vectors of U. Hence, U provides a basis for W. (Note that this implies $\dim W = 3$ and hence $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ is also a basis for W by Corollary 3.11. The linear independence of $\mathbf{x}_i$'s is a by-product of this fact).

To extend this basis, just add any nonzero vector of the form $\mathbf{x}_4 = (0,\ 0,\ 0,\ t)$ to the rows of U to get a basis for the space $\mathbb{R}^4$. □

Problem 3.10 Let W be a subspace of a vector space V. Show that if $\dim W = \dim V$, then $W = V$.

Problem 3.11 Find a basis and the dimension of each of the following subspaces of $M_{n\times n}(\mathbb{R})$ of all $n \times n$ matrices:

(1) the space of all $n \times n$ diagonal matrices whose traces are zero;

(2) the space of all $n \times n$ symmetric matrices;

(3) the space of all $n \times n$ skew-symmetric matrices.

Now consider two subspaces U and W of a vector space V. The **sum** of these subspaces U and W is defined by

$$U + W = \{\mathbf{u} + \mathbf{w}\ :\ \mathbf{u} \in U,\ \mathbf{w} \in W\}.$$

It is not hard to see that this is a subspace of V.

Problem 3.12 Let U and W be subspaces of a vector space V.

(1) Show that $U + W$ is the smallest subspace of V containing U and W.

(2) Prove that $U \cap W$ is also a subspace of V. Is $U \cup W$ a subspace of V? Justify your answer.

Definition 3.6 A vector space V is called the **direct sum** of two subspaces U and W, written $V = U \oplus W$, if $V = U + W$ and $U \cap W = \{\mathbf{0}\}$.

For example, one can easily show that $\mathbb{R}^3 = \mathbb{R}^1 \oplus \mathbb{R}^2 = \mathbb{R}^1 \oplus \mathbb{R}^1 \oplus \mathbb{R}^1$.

Theorem 3.12 *A vector space V is the direct sum of subspaces U and W, i.e., $V = U \oplus W$, if and only if for any $\mathbf{v} \in V$ there exist unique $\mathbf{u} \in U$ and $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$.*

Construction of new subspaces:-

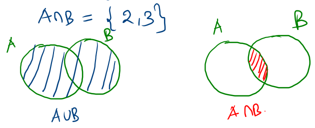
Recall:- Let A and B are two sets then

$$A \cup B = \{x \mid x \in A \text{ (or) } x \in B\}$$
$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Example:- $A = \{1, 2, 3, 4\}$ $B = \{2, 3, 8\}$

$A \cup B = \{1, 2, 3, 4, 8\}$ and

$A \cap B = \{2, 3\}$



Sum of subspaces:- Let V be a vector space over $\mathbb{R}$. U and W be subspaces of V. Then

$$U + W := \{x + y \mid x \in U, y \in W\}$$ is a subspace of V.

Examples:- Consider $\mathbb{R}^2$ over $\mathbb{R}$ and subspaces

$U = \{(x, 0) \mid x \in \mathbb{R}\}$ and $W = \{(0, y) \mid y \in \mathbb{R}\}$

$$U + W = \{(x, 0) + (0, y) \mid (x, 0) \in U, (0, y) \in W\}$$
$$= \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}\} = \underset{\substack{\downarrow \\ \text{It is a subspace}}}{\mathbb{R}^2}$$

Q:- Let V be a vector space over $\mathbb{R}$. U and W be two subspaces of V. Is it the set $U \cup W$ a subspace of V?

Ans- No

Example:- Consider $\mathbb{R}^2$ over $\mathbb{R}$ and subspaces

$U = \{(x, 0) \mid x \in \mathbb{R}\} \to X$ axis

$W = \{(0, y) \mid y \in \mathbb{R}\} \to Y$ axis

$U \cup W$ is not a subspace

Reason:- the elements $(x, 0), (0, y) \in U \cup W$

but $(x, 0) + (0, y) = (x, y) \notin U \cup W$

$\therefore$ $U \cup W$ is not a subspace

In general union of subspaces need not be a subspace

Suppose $U \cup W$ is a subspace if and only if either one of them is a subset of the other one.

[either $U \subseteq W$ (or) $W \subseteq U$]

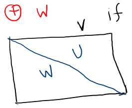
Direct Sum of Subspaces:-

A vector space V is called the direct sum of two subspaces U and W written as

$V = U \oplus W$ if

(i) $V = U + W$

(ii) $U \cap W = \{0\}$.



Example:- Consider $\mathbb{R}^2$ over $\mathbb{R}$ and subspaces

$U = \{(x, 0) \mid x \in \mathbb{R}\}$ and

$W = \{(0, y) \mid y \in \mathbb{R}\}$

We have seen that

$U + W = \mathbb{R}^2$ and $U \cap W = \{0\}$.

In this case, we can write $\boxed{\mathbb{R}^2 = U \oplus W}$

Note:- Let V be a vector space over $\mathbb{R}$. U and W be subspaces of V. Then

$U \cap W$ is always a subspace.

In general intersection of subspaces is a subspace.

Example:- Consider $\mathbb{R}^3$ over $\mathbb{R}$ and subspaces

$U = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\} = YZ$ plane

$W = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\} = XY$ plane

$U \cap W = Y$-axis $= \{(x, y, z) \mid x = z = 0\}$

clearly this is a subspace.

Exam Date: 9-8-2020 (Sunday)
Time : 9:50 – 10:45
Code : MAT 3004

Q:- Is $\{\underbrace{(1,0,0),(0,0,1)}_{\text{linearly independent set}}\}$ a basis a for $\mathbb{R}^3$?

Ans:- No,

1) Let V be a finite dimensional vector space. Then any linearly independent set in V can be extended to a basis by adding more vectors.

2) Any set of vectors that spans V can be reduced to a basis by discarding some vectors in the same spanning set.

this is a basis for V. If it does not span V, then the same procedure can be repeated, yielding a linearly independent set that spans V, *i.e.*, a basis for V. This procedure must stop in a finite step because of Lemma 3.8 for a finite dimensional vector space V. □

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Solution: Note that $\dim W \leq 3$ since W is spanned by three vectors $\mathbf{x}_i$'s. Let A be the 3×4 matrix whose rows are $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$:

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For example, one can easily show that $\mathbb{R}^3 = \mathbb{R}^1 \oplus \mathbb{R}^2 = \mathbb{R}^1 \oplus \mathbb{R}^1 \oplus \mathbb{R}^1$.

Theorem 3.12 *A vector space V is the direct sum of subspaces U and W, i.e., $V = U \oplus W$, if and only if for any $\mathbf{v} \in V$ there exist unique $\mathbf{u} \in U$ and $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$.*

Proof: Suppose that $V = U \oplus W$. Then, for any $\mathbf{v} \in V$, there exist vectors $\mathbf{u} \in U$ and $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$, since $V = U + W$. To show the uniqueness, suppose that $\mathbf{v}$ is also expressed as a sum $\mathbf{u}' + \mathbf{w}'$ for $\mathbf{u}' \in U$ and $\mathbf{w}' \in W$. Then $\mathbf{u} + \mathbf{w} = \mathbf{u}' + \mathbf{w}'$ implies

$$\mathbf{u} - \mathbf{u}' = \mathbf{w}' - \mathbf{w} \in U \cap W = \{\mathbf{0}\}.$$

Hence, $\mathbf{u} = \mathbf{u}'$ and $\mathbf{w} = \mathbf{w}'$.

Conversely, if there exists a nonzero vector $\mathbf{v}$ in $U \cap W$, then $\mathbf{v}$ can be written as sum of vectors in U and W in many different ways:

$$\mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{v} = \frac{1}{3}\mathbf{v} + \frac{2}{3}\mathbf{v} \in U + W. \quad \square$$

Example 3.18 Consider the three vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ in $\mathbb{R}^3$. Let $U = \{a_1\mathbf{e}_1 + b_3\mathbf{e}_3\ :\ a_1,\ b_3 \in \mathbb{R}\}$ be the subspace spanned by $\mathbf{e}_1$ and $\mathbf{e}_3$ (xz-plane), and let $W = \{a_2\mathbf{e}_2 + c_2\mathbf{e}_3\ :\ a_2, c_3 \in \mathbb{R}\}$ be the subspace of $\mathbb{R}^3$ spanned by $\mathbf{e}_2$ and $\mathbf{e}_3$ (yz-plane). Then a vector in $U + W$ is of the form

$$(a_1\mathbf{e}_1 + b_3\mathbf{e}_3) + (a_2\mathbf{e}_2 + c_2\mathbf{e}_3) = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + (b_3 + c_2)\mathbf{e}_3 = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$$

where $a_3 = b_3 + c_3$ and a_1, a_2, a_3 are arbitrary numbers. Thus $U + W = \mathbb{R}^3$. However, $\mathbb{R}^3 \neq U \oplus W$ since clearly $\mathbf{e}_3 \in U \cap W \neq \{\mathbf{0}\}$. In fact, the vector $\mathbf{e}_3 \in \mathbb{R}^3$ can be written as many linear combinations of vectors in U and W:

$$\mathbf{e}_3 = \frac{1}{2}\mathbf{e}_3 + \frac{1}{2}\mathbf{e}_3 = \frac{1}{3}\mathbf{e}_3 + \frac{2}{3}\mathbf{e}_3 \in U + W.$$

Note that if we had taken W to be the subspace spanned by $\mathbf{e}_2$ alone, then it would be easy to see that $\mathbb{R}^3 = U \oplus W$. Note also that there are many choices for W. □

As a direct consequence of Theorem 3.10 and the definition of the direct sum, one can show the following.

Corollary 3.13 *If U is a subspace of V, then there is a subspace W in V such that $V = U \oplus W$.*

Proof: Choose a basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ for U, and extend it to a basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ for V. Then the subspace W spanned by $\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ satisfies the requirement. □

Row and Column Space

Aim:- Solve the system $AX = B$ where A is (coefficient matrix) of order $m \times n$, X is the column vector of order $n \times 1$ and B is the column vector of order $m \times 1$

We have learnt the methods

(i) Gauss Elimination,

(ii) Gauss Jordan

(iii) LU decomposition to solve the system.

We need deep knowledge about the system $AX = B$

We introduced the notion vector spaces

(V together with two operations + and · satisfies the 9 axioms (vector space axioms) is known as a vector space)

Examples:- $\mathbb{R}^n$ (Euclidean n-dimensional space) over $\mathbb{R}$

$P_n(\mathbb{R})$ (polynomial of degree atmost n) over $\mathbb{R}$

$M_{m\times n}(\mathbb{R})$ (set of all $m \times n$ matrices) over $\mathbb{R}$

Subspaces:- A nonempty W of a vector space V is said to be a subspace if

(i) $x + y \in W$ for all $x, y \in W$

(ii) $kx \in W$ for all $k \in \mathbb{R}$, $x \in W$.

Linear combination:- Let V be a vector space $\{v_1, v_2, \ldots v_n\}$ be a collection of vectors in V the vector of the form

$k_1v_1 + k_2v_2 + \cdots + k_nv_n$

is known as linear combination of the vectors $v_1, v_2, \cdots v_n$

Spanning subset:-

$W = \{k_1 v_1 + k_2 v_2 + \cdots + k_n v_n \mid k_1, k_2, \ldots k_n \in \mathbb{R}\}$ is a subspace of V. It is known as subspace spanned by $\{v_1, v_2, \ldots v_n\}$

Linearly dependent & Independent sets:-

A collection of vectors $\{v_1, v_2, \ldots v_n\}$ is said to be linearly independent if

$$k_1 v_1 + k_2 v_2 + \cdots + k_n v_n = 0 \Rightarrow k_1 = k_2 = \cdots = k_n = 0.$$

If atleast one k_i is non zero then we say the collection is linearly dependent.

Consider the column vectors

$$w_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \ w_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \ w_3 = \begin{pmatrix} 3 \\ 6 \end{pmatrix} \text{ and } w_4 = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

$\therefore w_1, w_2, w_3, w_4 \in \mathbb{R}^2$

Subspace spanned by w_1, w_2, w_3, w_4 :-

$$W = \left\{k_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + k_2 \begin{pmatrix} 2 \\ 4 \end{pmatrix} + k_3 \begin{pmatrix} 3 \\ 6 \end{pmatrix} + k_4 \begin{pmatrix} 4 \\ 8 \end{pmatrix} \,\middle|\, k_1, k_2, k_3, k_4 \in \mathbb{R}\right\}$$

$$= \left\{k_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2k_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 3k_3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 4k_4 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \,\middle|\, k_1, k_2, k_3, k_4 \in \mathbb{R}\right\}$$

$$= \left\{(k_1 + 2k_2 + 3k_3 + 4k_4) \begin{pmatrix} 1 \\ 2 \end{pmatrix} \,\middle|\, k_1, k_2, k_3, k_4 \in \mathbb{R}\right\}$$

$$= \underbrace{\left\{k_5 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \,\middle|\, k_5 \in \mathbb{R}\right\}}_{\text{column Space of } A}$$

Basis and dimension:-

Let V be a vector space if the collection of vectors $\{v_1, v_2, \ldots v_n\}$ is said to form a basis if

(a) $\{v_1, v_2, \ldots v_n\}$ is a linearly independent set.

(b) The subspace spanned by $\{v_1, v_2, \ldots v_n\}$ is V.

Dimension:- No. of elements in the basis is known as the dimension of the vector space.

Let A be a $m \times n$ matrix given in the following form.

$$A = \begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{array}{l} \to \text{Row} 1 \\ \to \text{Row} 2 \\ \\ \to \text{Row } m \end{array} \\ \begin{array}{cccc} \downarrow & \downarrow & & \downarrow \\ \text{col } 1 & \text{col } 2 & \cdots & \text{col } n \end{array} \end{array} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix}$$

$$= [c_1 \quad c_2 \quad \cdots \quad c_n]$$

Here each r_i represents i^{th} row and $r_i \in \mathbb{R}^n$ and each c_i represents i^{th} column and $c_i \in \mathbb{R}^m$

Row space:-

The row space of A is the subspace of $\mathbb{R}^n$ spanned by the row vectors $\{r_1, r_2, \ldots r_m\}$

Row and column space

Example:- Consider the following matrix

$$A = \begin{array}{c} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix} \begin{array}{l} \to \text{Row } 1 \\ \to \text{Row } 2 \end{array} \\ \begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \\ \text{col } 1 & \text{col } 2 & \text{col } 3 & \text{col } 4 \end{array} \end{array}$$

look at rows

$v_1 = (1, 2, 3, 4)$, $v_2 = (2, 4, 6, 8)$. clearly $v_1, v_2 \in \mathbb{R}^4$

Subspace spanned by v_1, v_2.

$$W = \{k_1(1, 2, 3, 4) + k_2(2, 4, 6, 8) \mid k_1, k_2 \in \mathbb{R}\}$$

$$= \{k_1(1, 2, 3, 4) + 2k_2(1, 2, 3, 4) \mid k_1, k_2 \in \mathbb{R}\}$$

$$= \{\underbrace{(k_1 + 2k_2)}(1, 2, 3, 4) \mid k_1, k_2 \in \mathbb{R}\}$$

$$= \{k_3(1, 2, 3, 4) \mid k_3 \in \mathbb{R}\}$$

$\therefore$ W is known as the $\boxed{\text{row space of the matrix } A}$

Column space:- The column space of A is the subspace of $\mathbb{R}^m$ spanned by the column vectors $\{c_1, c_2, \ldots c_n\}$.

Null space:- The solution set of the homogeneous equation $AX = 0$ is called the null space of A.

Example:-

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

Take $X = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$

what is the solution of $AX = 0$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x + 2y + 3z + 4t = 0$$

$$\text{Nullspace} = \{X \mid AX = 0\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} : x + 2y + 3z + 4t = 0 \right\}$$

Notation:- Row space is denoted by $R(A)$
column space is denoted by $C(A)$
Null space is denoted by $N(A)$.

Row space, Column space and Null space:-

Consider

$$A = \begin{array}{c} \\ \end{array}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}\begin{array}{l} \to r_1 \text{ (Row 1)} \\ \to r_2 \text{ (Row 2)} \\ \vdots \\ \to r_m \text{ (Row m)} \end{array}$$

$$\begin{array}{cccc} \downarrow & \downarrow & & \downarrow \\ c_1 & c_2 & \cdots & c_n \\ \text{(col 1)} & \text{(col 2)} & & \text{(col n)} \end{array}$$

Each r_i represents i^{th} row and $r_i \in \mathbb{R}^n$
c_i represent i^{th} column and $c_i \in \mathbb{R}^m$

Row space:- It is a subspace of $\mathbb{R}^n$ spanned by $\{r_1, r_2, \dots r_m\}$ (Notation: $R(A)$)

Column space:- It is a subspace of $\mathbb{R}^m$ spanned by $\{c_1, c_2, \dots c_n\}$ (Notation: $C(A)$)

Null space:- It is a solution set of the homogeneous equation $Ax = 0$

(or) $N(A) = \{ X \in \mathbb{R}^n \mid AX = 0 \}$

Rank and Nullity:-

Dimension of $C(A)$ is known as rank of A.
Dimension of $N(A)$ is known as Nullity of A.

Theorem:- Let A be a $m \times n$ matrix and U be the row reduced equivalent matrix of A, then
$R(A) = R(U)$ and $N(A) = N(U)$

Remarks:-

(i) If U has r nonzero row vectors containing leading 1's then they form a basis for the row space of A.

(ii) The basis for the column space of A is the column vectors of A corresponds to the basic variable in the system $UX=0$.

Pb:1 Find a basis and dimension of row space, null space and column space of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Soln:-

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U$$

Basis for Row space $= \{(1, 2, 3)\}$ $\therefore \dim(R(A)) = 1$

Basis for column space:-

Form the system $UX = 0$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \underset{\substack{\downarrow \\ \text{Basis Variable}}}{x} + 2y + z = 0$$

$\therefore$ Basis for $C(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$ $\therefore \dim C(A) = 1$

Basis for Null space:-

$$N(A) = N(U) = \{X \in \mathbb{R}^3 \mid UX = 0\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + \underbrace{2y + 3z}_{\text{free variable}} = 0 \right\}$$

$\therefore$ dim $N(A) = 2$

Basis for $N(A) = \{(1, 1, -1), (-2, 1, 0)\}$

1) Let A be a $m \times n$ matrix and U be reduced row echelon form of A. Which of the following statement is true

a) $\mathcal{R}(A) = \mathcal{R}(U), \mathcal{C}(A) = \mathcal{C}(U)$ and $\mathcal{N}(A) = \mathcal{N}(U)$.

b) $\mathcal{R}(A)$ *need not be equal to* $\mathcal{R}(U)$ but $\mathcal{C}(A) = \mathcal{C}(U)$ and $\mathcal{N}(A) = \mathcal{N}(U)$.

c) $\mathcal{R}(A) = \mathcal{R}(U), \mathcal{N}(A) = \mathcal{N}(U)$ *but* $\mathcal{C}(A)$ *need not be equal to* $\mathcal{C}(U)$ ✓

d) $\mathcal{R}(A) = \mathcal{R}(U), \mathcal{C}(A) = \mathcal{C}(U)$ but $\mathcal{N}(A)$ *need not be equal to* $\mathcal{N}(U)$.

Soln:-

Theorem

2) Consider the matrix $A = \begin{pmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{pmatrix}$. Then

a) $\mathcal{R}(A) = \{(0, k, 4k, 0) : k \in \mathbb{R}\}$ and $\dim(\mathcal{R}(A)) = 1$. ✓

b) $\mathcal{R}(A) = \{(0,0,4k, 0) : k \in \mathbb{R}\}$ and $\dim(\mathcal{R}(A)) = 1$.

c) $\mathcal{R}(A) = \{(k, 2k) : k \in \mathbb{R}\}$ and $\dim(\mathcal{R}(A)) = 1$.

d) $\mathcal{R}(A) = \{(0,2k) : k \in \mathbb{R}\}$ and $\dim(\mathcal{R}(A)) = 1$.

Soln:-

$$\begin{pmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U$$

(Row reduced matrix)

Basis for $R(A) = \{(0, 1, 4, 0)\}$ $\therefore$ dim $R(A) = 1$

$R(A) = \{k(0, 1, 4, 0) : k \in \mathbb{R}\}$

$= \{(0, k, 4k, 0) : k \in \mathbb{R}\}$ $\therefore$

3) Consider the matrix $A = \begin{pmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{pmatrix}$. Then

a) $\mathcal{C}(A) = \{(0, k, 4k, 0) : k \in \mathbb{R}\}$ and $Rank(A) = 1$.

b) $\mathcal{C}(A) = \{(0,0,4k, 0) : k \in \mathbb{R}\}$ and $Rank(A) = 1$.

c) $\mathcal{C}(A) = \{(k, 2k) : k \in \mathbb{R}\}$ and $\dim(\mathcal{C}(A)) = 1$. ✓

d) $\mathcal{C}(A) = \{(0,2k) : k \in \mathbb{R}\}$ and $\dim(\mathcal{C}(A)) = 1$.

Soln:- The reduced from of the given matrix is

$U = \begin{pmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Form the equation

$$UX = 0 \Rightarrow \begin{pmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow y + 4z = 0$

y ↓ Basis variable

$\therefore$ Basis for $C(A) = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$ dim $C(A) = 1$

$\therefore C(A) = \left\{ k \begin{pmatrix} 1 \\ 2 \end{pmatrix} : k \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} k \\ 2k \end{pmatrix} : k \in \mathbb{R} \right\}$

4) Consider the matrix A = $\begin{pmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{pmatrix}$. Then

a) $\mathcal{N}(A) = \{(x, y, z, t) \in \mathbb{R}^4 : x + y + 2z = 0\}$ and nullity$(A) = 2$.
b) $\mathcal{N}(A) = \{(x, y, z, t) \in \mathbb{R}^4 : y + 4z = 0\}$ and nullity$(A) = 3$. ✓
c) $\mathcal{N}(A) = \{(x, y, z, t) \in \mathbb{R}^4 : x + y = 0\}$ and nullity$(A) = 4$.
d) $\mathcal{N}(A) = \{(x, y, z, t) \in \mathbb{R}^4 : z = 0\}$ and nullity$(A) = 3$.

Soln:- We know $N(A) = N(U)$ [U is the row reduced matrix]

$$N(U) = \{X \in \mathbb{R}^4 \mid UX = 0\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} : \begin{pmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = 0 \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} : y + 4z = 0 \right\}$$ (There are 3 free variables)

Hence nullity of $A = 3$

$$N(U) = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} : y + 4z = 0 \right\}$$

5) Consider the matrix A = $\begin{pmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{pmatrix}$. Then

a) $\mathcal{R}(A) \subseteq \mathbb{R}^4$, $\mathcal{C}(A) \subseteq \mathbb{R}^3$ and $\mathcal{N}(A) \subseteq \mathbb{R}^2$
b) $\mathcal{R}(A) \subseteq \mathbb{R}^4$, $\mathcal{C}(A) \subseteq \mathbb{R}^2$ and $\mathcal{N}(A) \subseteq \mathbb{R}^3$
c) $\mathcal{R}(A) \subseteq \mathbb{R}^4$, $\mathcal{C}(A) \subseteq \mathbb{R}^2$ and $\mathcal{N}(A) \subseteq \mathbb{R}^4$ ✓
d) $\mathcal{R}(A) \subseteq \mathbb{R}^3$, $\mathcal{C}(A) \subseteq \mathbb{R}^2$ and $\mathcal{N}(A) \subseteq \mathbb{R}^3$

Soln:- Refer problem 2, 3, 4.

6) Consider the matrix A = $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$. Then

a) $\mathcal{R}(A) = \{(a, 2a + b, 0, a) : a, b \in \mathbb{R}\}$ and $\dim(\mathcal{R}(A)) = 3$.
b) $\mathcal{R}(A) = \{(a, 2a + b, 0, 0) : a, b \in \mathbb{R}\}$ and $\dim(\mathcal{R}(A)) = 4$.
c) $\mathcal{R}(A) = \{(a, 2a + b, b, a) : a, b \in \mathbb{R}\}$ and $\dim(\mathcal{R}(A)) = 2$. ✓
d) $\mathcal{R}(A) = \{(a, 0, 0, 0) : a \in \mathbb{R}\}$ and $\dim(\mathcal{R}(A)) = 1$.

Soln:-

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U$$

Basis for $R(A) = \{(1, 2, 0, 1), (0, 1, 1, 0)\}$ $\therefore \dim R(A) = 2$

6) Consider the matrix A = $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$. Then

a) $\mathcal{R}(A) = \{(a, 2a + b, 0, a) : a, b \in \mathbb{R}\}$ and $\dim(\mathcal{R}(A)) = 3$.
b) $\mathcal{R}(A) = \{(a, 2a + b, 0, 0) : a, b \in \mathbb{R}\}$ and $\dim(\mathcal{R}(A)) = 4$.
c) $\mathcal{R}(A) = \{(a, 2a + b, b, a) : a, b \in \mathbb{R}\}$ and $\dim(\mathcal{R}(A)) = 2$. ✓
d) $\mathcal{R}(A) = \{(a, 0, 0, 0) : a \in \mathbb{R}\}$ and $\dim(\mathcal{R}(A)) = 1$.

Soln:-

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U$$

Basis for $R(A) = \{(1, 2, 0, 1), (0, 1, 1, 0)\}$ $\therefore \dim R(A) = 2$

$$= \{a(1, 2, 0, 1) + b(0, 1, 1, 0) : a, b \in \mathbb{R}\}$$

$$= \{(a, 2a + b, b, a) : b, a \in \mathbb{R}\}$$

Problems on Row Space and Column Space – 2.

7) Consider the matrix $A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$. Then

a) $\mathcal{C}(A) = \{(x, y, z) \in \mathbb{R}^3 : x = z\}$ and $\dim(\mathcal{C}(A)) = 3$.
b) $\mathcal{C}(A) = \{(x, y, z) \in \mathbb{R}^3 : x = y\}$ and $\dim(\mathcal{C}(A)) = 4$.
c) $\mathcal{C}(A) = \{(x, y, z) \in \mathbb{R}^3 : x + z = 0\}$ and $\dim(\mathcal{C}(A)) = 2$.
d) $\mathcal{C}(A) = \{(x, y, z) \in \mathbb{R}^3 : x = z\}$ and $\dim(\mathcal{C}(A)) = 2$. ✓

Soln:-

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U$$

Form the equation $UX = 0 \Rightarrow \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$x + 2y + t = 0$$
$$y + z = 0$$

Basic variables are x and y

$\therefore$ Basis for $C(A) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\}$ $\therefore \dim C(A) = 2$

$$C(A) = \left\{ k_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} : k_1, k_2 \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} k_1 + 2k_2 \\ k_2 \\ k_1 + 2k_2 \end{pmatrix} : k_1, k_2 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x = z \right\}$$

8) Consider the matrix $A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$. Then

a) A basis of $\mathcal{N}(A)$ *is* $\{(1,0,-1,0), (2,0,-2,0)\}$
b) A basis of $\mathcal{N}(A)$ *is* $\{\cancel{(1,0,-1,0), (0,-1,1,2)}\}$ $\{(1,0,0,-1), (0,-1,1,2)\}$ ✓
c) A basis of $\mathcal{N}(A)$ *is* $\{(0,-1,1,2), (0,-2,-2,0)\}$
d) A basis of $\mathcal{N}(A)$ *is* $\{(1,0,-1,0), (2,1,-2,0)\}$

Soln:- Consider the row reduced matrix U. We know

$$N(A) = N(U) = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} : \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} : \begin{array}{l} x + 2y + t = 0 \\ y + z = 0 \end{array} \right\} \quad \therefore \dim N(A) = 2$$

9) Consider the matrix $A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$. Then

a) $\mathcal{R}(A) \subseteq \mathbb{R}^4$, $\mathcal{C}(A) \subseteq \mathbb{R}^3$ and $\mathcal{N}(A) \subseteq \mathbb{R}^2$
b) $\mathcal{R}(A) \subseteq \mathbb{R}^4$, $\mathcal{C}(A) \subseteq \mathbb{R}^2$ and $\mathcal{N}(A) \subseteq \mathbb{R}^3$
c) $\mathcal{R}(A) \subseteq \mathbb{R}^4$, $\mathcal{C}(A) \subseteq \mathbb{R}^3$ and $\mathcal{N}(A) \subseteq \mathbb{R}^4$ ✓
d) $\mathcal{R}(A) \subseteq \mathbb{R}^3$, $\mathcal{C}(A) \subseteq \mathbb{R}^2$ and $\mathcal{N}(A) \subseteq \mathbb{R}^3$

Soln:-

Hint:- Refer pb 7, 8.

10) Conisder the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Which of the following is true

a) $\mathcal{R}(A) = \{(x, y, z) \in \mathbb{R}^3 : x = 0\}$, $\mathcal{C}(A) = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ *and* $\mathcal{N}(A) = \{(x, y, z) \in \mathbb{R}^3 : y = z = 0\}$
b) $\mathcal{R}(A) = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$, $\mathcal{C}(A) = \{(x, y, z) \in \mathbb{R}^3 : x = 0\}$ *and* $\mathcal{N}(A) = \{(x, y, z) \in \mathbb{R}^3 : y = z = 0\}$
c) $\mathcal{R}(A) = \{(x, y, z) \in \mathbb{R}^3 : x = 0\}$, $\mathcal{C}(A) = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ *and* $\mathcal{N}(A) = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$
d) $\mathcal{R}(A) = \{(x, y, z) \in \mathbb{R}^3 : x = 0\}$, $\mathcal{C}(A) = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ *and* $\mathcal{N}(A) = \{(x, y, z) \in \mathbb{R}^3 : y = 0\}$.

12) Let A be a $m \times n$ matrix. Which of the following is incorrect?
a) $\mathcal{R}(A) = \mathcal{C}(A^T)$.
b) $\mathcal{C}(A) = \mathcal{R}(A^T)$.
c) $\mathcal{N}(A) = \mathcal{N}(A^T)$. ✓
d) $\mathcal{N}(A^T) = \mathcal{C}(A)$. ✓

13) Let A be a $m \times n$ matrix. Which of the following is correct
a) If $m > n$ then the row space of A equals the column space of A.
b) $\mathcal{C}(A) = \{AX : X \in \mathbb{R}^n\}$. ✓
c) If $m = n$ then the row space of A equals the null space of A.
d) If $m < n$ then $\dim(\mathcal{R}(A)) + \dim(\mathcal{C}(A)) = 0$.

Soln:-

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$C(A) = \left\{ k_1 \begin{pmatrix} a_{11} \\ a_{21} \\ a_{m1} \end{pmatrix} + k_2 \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + k_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} : k_1, k_2 \ldots k_n \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} \middle| \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} \in \mathbb{R}^n \right\}$$

$$= \{ AX : X \in \mathbb{R}^n \}$$

Rank and Nullity:-
Rank is the dimension of $C(A)$
Nullity is the dimension of $N(A)$

Remarks:-
1) Let A be a $m \times n$ matrix then
$\text{Rank}(A) \le \min\{m, n\}$
2) Let A and B be two matrices and suppose AB and BA is possible then
$\text{Rank}(AB) \le \min\{\text{Rank}(A), \text{Rank}\,B\}$
$\text{Rank}(BA) \le \min\{\text{Rank}(A), \text{Rank}(B)\}$

3) $\text{Rank}(A) + \text{Nullity}(A) = n$ (no. of columns in A).

4) $\dim(R(A)) = \dim(C(A)) = \text{Rank}(A)$

Remark:- The system $AX = B$ has a solution if and only if $B \in C(A)$

9) Let A be a 12×11 matrix with Rank(A) = 5 then
Nullity(A)=-----------------------------

Soln:-
We know $\text{Rank}(A) + \text{Nullity}(A) = 11$
$5 + \text{''} = 11$
$\boxed{\therefore \text{Nullity } A = 6}$

PROBLEMS ON RANK-NULLITY

11) Let A be a 24×25 matrix with Nullity(A) $= 12$ then

$\dim C(A^T)$=-----------------------------

Soln:- Recall:- (i) $R(A) = C(A^T)$

$\therefore \dim(R(A)) = \dim(C(A^T))$?

(ii) $\dim(R(A)) = \dim(C(A)) = \text{Rank}(A)$

we know Rank A + Nullity $A = 25$

Rank A + $12 = 25$

$\therefore$ Rank $A = 13$

$\therefore \dim C(A^T) = 13$

Properties:- Let A be a $m \times n$ matrix then

(i) $\dim(R(A)) = \dim(C(A)) = \text{Rank}(A)$

(ii) $\text{Rank}(A) \leq \min\{m, n\}$.

(iii) $\dim(C(A)) + \dim(N(A)) = n$

$\text{Rank}(A) + \text{Nullity}(A) = n$

$\dim(R(A^T)) + \dim N(A) = n$

$\dim(C(A)) + \dim N(A^T) = n$

Let A and B are two matrices. Suppose AB and BA are possible then

$\text{Rank}(AB) \leq \min\{\text{Rank}(A), \text{Rank}(B)\}$.

Let A be a $m \times n$ matrix then

$\dim C(A)$ = Number of basic variables in the System $UX = 0$ (U = is the row reduced matrix of A)

= Maximal number of linearly independent column vector.

Since $\dim C(A)) = \dim(R(A))$

= Maximal number of linearly independent row vectors

= Maximal number of linearly independent column vectors.

Rank - Nullity Let A be a $m \times n$ matrix

dimension of the column space ($C(A)$) is known as the rank(A)

dimension of the null space ($N(A)$) is known as the nullity(A)

$\dim N(A)$ = number of free variables in the system $UX = 0$ (Here U is the row reduced matrix).

1) Let A be a $m \times n$ matrix. Which of the following is correct
a) If $m = n$ then the row space of A equals the column space of A.
b) For any value of m, n, we have $\dim(\mathcal{R}(A)) = \dim(\mathcal{C}(A))$.
c) If $m = n$ then the row space of A equals the null space of A.
d) If $m < n$ then $\dim(\mathcal{R}(A)) + \dim(\mathcal{C}(A)) = 0$.

Soln:-
(b) is correct.

2) Let A be a $m \times n$ matrix and U be reduced row echelon form of A. Which of the following is incorrect.
a) $\dim(\mathcal{R}(A))$=The number of row vectors in U.
b) $\dim(\mathcal{R}(A))$=The number of Basic variables in the system $Ux = 0$.
c) $\dim(\mathcal{R}(A))$=The maximal number of linearly independent column vectors in A.
d) $\dim(\mathcal{R}(A))$=The maximal number of linearly independent row vectors in A.

(a) is incorrect equality.

3) Let A be a $m \times n$ matrix and U be reduced row echelon form of A. Which of the following is correct.
a) $\dim(\mathcal{N}(A))$=The number of row vectors in U.
b) $\dim(\mathcal{N}(A))$=The number of free variables in the system $Ux = 0$.
c) $\dim(\mathcal{N}(A))$=The maximal number of linearly independent column vectors in A.
d) $\dim(\mathcal{N}(A))$=The maximal number of linearly independent row vectors in A.

(b) is the correct equality.

$= \{ \begin{pmatrix} \cdot \\ y \\ z \end{pmatrix}$

$=$ XZ plane in $\mathbb{R}^3$

4) Let A be a $m \times n$ matrix. Which of the following is not true?
A) Rank of A is the dimension of the row space of A.
B) Rank of A is the dimension of the column space of A.
C) Rank of A is the dimension of the null space of A.
D) Rank of A is the dimension of the row space of A^T.

5)Let A be a m $\times$ n matrix. Which of the following need not be true?
A) Rank(A)$\leq m$.
B) Rank(A)$\leq n$.
C) Rank(A)$\leq \min\{m, n\}$.
D) Rank(A)$\leq \min\{m-1, n-1\}$.

6) Let A be a $7\times$ 5 matrix. Which of the following need not be true?
A) Rank(A)$\leq$ 5.
B) Rank(A)$\leq$ 7.
C) Rank(A)$\leq$ 1.
D) Rank(A)$\leq$ 6.

Soln!-
Rank (A) $\leq \min\{5, 7\} = 5$
(C) need not be true.

7) Let A be a $m \times n$ matrix. Which of the following is not true?
A) $\text{Rank}(A) + \text{Nullity}(A) = n$.
B) $\text{Rank}(A) + \text{Nullity}(A) = m$.
C) $\text{Rank}(A^T) + \text{Nullity}(A) = n$.
D) $\dim(\mathcal{C}(A^T)) + \text{Nullity}(A^T) = n$.

(B) is not true.

8) Let A be a 20×50 matrix with nullity of $A = 32$ then Rank(A)=------------------------------

Soln:-

$\text{Rank}(A) + \text{Nullity } A = 50$

$\text{Rank}(A) + 32 = 50$

$\boxed{\text{Rank } A = 18}$

9) Let A be a 12×11 matrix with $\text{Rank}(A) = 5$ then

Nullity(A)=-------------6-------------------

Soln:-

$\text{Rank}(A) + \text{Nullity}(A) = 11$

$5 + \text{Nullity}(A) = 11$

$\boxed{\text{Nullity}(A) = 6}$

10) Let A be a 14×15 matrix with $\text{Rank}(A) = 8$ then

Nullity(A^T)=-------------7-------------------

Soln:-

Hint:- $\text{Nullity } A = \text{Nullity } A^T$

$\text{Rank } A + \text{Nullity } A = 15$

$8 + \text{''} = 15$

$\boxed{\text{Nullity } A^T = 7}$

11) Let A be a 24×25 matrix with $\text{Nullity}(A) = 12$ then

$\dim \mathcal{C}(A^T)$=-----------------------------

Soln:-

Hint:- $\text{Rank } A = \dim(\mathcal{C}(A)) = \dim(\mathcal{R}(A))$
$= \dim(\mathcal{C}(A^T))$

$\text{Nullity } A + \text{Rank } A = 25$

$\therefore \boxed{\dim(\mathcal{C}(A^T)) = 25 - 12 = 13}$

12) Let A and B be two matrices such that AB can be defined. Which of the following need not be true
a) $\mathcal{N}(AB) \supseteq \mathcal{N}(B)$.
b) $\mathcal{N}\left((AB)^T\right) \subseteq \mathcal{N}(A^T)$.
c) $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$.
d) $\mathcal{R}(AB) \subseteq \mathcal{R}(B)$.

13) Let A and B be two matrices such that AB can be defined. Which of the following need not be true

a) $rank(AB) \leq rank(B)$.
B) $rank(AB) \leq rank(A)$.
C) $rank(AB) \leq \min\{rank(A), rank(B)\}$.
D) $rank(AB) < \frac{rank(A)}{2}$. ✓

Soln:-

Hint:- $\text{Rank}(AB) \leq \min\{\text{Rank } A, \text{Rank } B\}$

14) Let A be an invertible matrix. Then for any matrix B , which of the following need not be true

a) $Rank(AB) = Rank(B)$.
B) $Rank(BA) = Rank(B)$.
C) $Rank(A) = Rank(B)$.
D) $Rank(AB)$=$Rank(BA)$.

15) Let A be a m $\times$ n matrix of rank r. Then which of the following need not be true

a) Every submatrix C of A has rank less than or equal to r.
b)There exists an invertible submatrix C of A of order r.
c)There exists an invertible submatrix C of A of order r+1.
d) There exists an invertible submatrix C of A of order 1.

16) What is the rank of the matrix $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$?

A) 2
B) 3
C) 4
D) 5 ✓

Soln:-

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 - R_1$
$R_5 \to R_5 - R_1$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 3 & 3 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}$$

$R_3 \to R_3 - R_2$
$R_4 \to R_4 - R_2$
$R_5 \to R_5 - R_2$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix}$$ (do this continuously)

$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

17) Let A be a 4×4 matrix. Suppose that the null space $\mathcal{N}(A)$ of A is

$$\{(x, y, z, w) \in \mathbb{R}^4 : x + y + z = 0, x + y + w = 0\}.$$

Then

a) dim(column space (A)) $= 1$.
b) dim(column space (A)) $= 2$.
c) rank$(A) = 1$.
d) $S = \{(1,1,1,0), (1,1,0,1)\}$ is a basis for$\mathcal{N}(A)$.

18) Let A be a $n \times m$ matrix with each entry equal to $1, -1$ or 0 such that every column has exactly one 1 and exactly one -1. We can conclude that

a) $Rank(A) \leq m - 1$.
b) $Rank(A) = m$.
C) $n \leq m$.
D) $n - 1 \leq m$.

Recall:- Let

$$\begin{array}{c|c} A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} & \begin{array}{l} \to r_1 \\ \to r_2 \\ \\ \to r_m \end{array} \\ \quad\ \ \downarrow c_1 \quad \downarrow c_2 \quad \cdots \quad \downarrow c_n & \end{array}$$

Row space is the subspace of $\mathbb{R}^n$ spanned by the vectors $r_1, r_2, \cdots r_m$ (Notation: $R(A)$)

Column space is the subspace of $\mathbb{R}^m$ spanned by the vectors $c_1, c_2, \ldots c_n$ (Notation $C(A)$)

Null space is the subspace of $\mathbb{R}^n$ which is the set of solutions of the homogeneous equation $AX=0$

Notation : $N(A)$

Rank:- dimension of $C(A)$

Nullity:- dimension of $N(A)$

Properties:-

Rank $A = \dim(C(A)) = \dim R(A) = \dim(C(A^T)) = \dim R(A^T)$

= the maximal number of linearly independent row vectors

= the maximal number of linearly independent column vectors

= Number of basic variables in the system $UX=0$ [U is the reduced form of A]

Nullity (A) = Number of free variables in the system $UX=0$.

Rank (A) + Nullity $(A) = n$

Rank$(A) \le \min\{m, n\}$.

Let A & B are two matrices suppose AB and BA is possible then

Rank $(AB) \le \min\{\text{Rank}(A), \text{Rank}(B)\}$.

Bases for Subspaces:- Consider the vector space $\mathbb{R}^n$ and subspace V and W. Suppose A basis for V is $\{v_1, v_2, \ldots v_k\}$ and a basis for W is $\{w_1, w_2, \ldots w_\ell\}$

Q:- What is the basis and dimension of the subspaces $V+W$ and $V \cap W$?

$V+W = \{x+y \mid x \in V \text{ and } y \in W\}$

clearly the vectors $\{v_1, v_2, \ldots v_k, w_1, w_2, \ldots w_\ell\}$ belongs to $V+W$

clearly $\{v_1, v_2, \ldots v_k, w_1, w_2, \ldots w_\ell\}$ is a spanning set for $V+W$

$\therefore$ If we remove some vectors from this collection we can get a basis for $V+W$.

We know $V \cap W \subseteq V$ and $V \cap W \subseteq W$

So $\{v_1, v_2, \ldots v_k\}$ and $\{w_1, w_2, \ldots w_\ell\}$ are spanning sets for $V \cap W$. If we remove some vectors either from the collection $\{v_1, v_2, \ldots v_k\}$ (or) from the collection $\{w_1, \ldots w_\ell\}$ then one can find basis and dimension for $V \cap W$.

Procedure to find the basis and dimension for $V+W$ and $V \cap W$:-

1) Form the matrix A of order $n \times (k+\ell)$ such that whose column vectors are the elements $\{v_1, v_2, \ldots v_k, w_1, w_2, \ldots w_\ell\}$ (that is)

$$A = [v_1 \ v_2 \ v_3 \cdots v_k \ w_1 \ w_2 \cdots w_\ell]$$

2) Find $C(A)$ and $N(A)$
3) Basis for $C(A)$ = Basis for $V+W$.
4) dimension of $N(A) = \dim(V \cap W)$
5) Suppose
$$\begin{aligned} X_1 &= (a_{11}, a_{12}, \ldots, a_{1k}, b_{11}, b_{12}, \ldots, b_{1\ell}) \\ X_2 &= (a_{21}, a_{22}, \ldots a_{2k}, b_{21}, b_{22}, \ldots b_{2\ell}) \\ &\vdots \\ X_s &= (a_{s1}, a_{s2}, \ldots a_{sk}, b_{s1}, b_{s2}, \ldots b_{s\ell}) \end{aligned}$$
is a basis for $N(A)$

then the vectors
$$\begin{aligned} y_1 &= a_{11}v_1 + a_{12}v_2 + \cdots + a_{1k}v_k \\ y_2 &= a_{21}v_1 + a_{22}v_2 + \cdots + a_{2k}v_k \\ &\vdots \\ y_s &= a_{s1}v_1 + a_{s2}v_2 + \cdots + a_{sk}v_k \end{aligned}$$
is a basis for $(V \cap W)$

Note:- Let V and W be subspaces of $\mathbb{R}^n$ then

$\dim(V+W) + \dim(V \cap W) = \dim V + \dim W$

Pb: 1 Find a basis and dimension of the subspaces $V+W$ and $V \cap W$ where

$$V = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\}$$
$$W = \{(x,y,z) \in \mathbb{R}^3 \mid x-y-z=0\}$$

Soln:-

Since $\dim V = 2$, Basis $V = \{\underbrace{(1,-1,0)}_{v_1}, \underbrace{(0,-1,1)}_{v_2}\}$

$\dim W = 2$, Basis $W = \{\underbrace{(2,1,1)}_{w_1}, \underbrace{(1,0,1)}_{w_2}\}$

Step: 1 Form the matrix $A = [v_1 \ v_2 \ w_1 \ w_2]$

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad R_2 \to R_2 + R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

Basis for $V+W$ = Basis for $C(A)$

= The column vectors in A corresponds to the basic variable in the system $Ux = 0$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} x + 2z + t &= 0 \\ -y + 3z + t &= 0 \\ 4z + 2t &= 0 \end{aligned}} \qquad \begin{aligned} x &= 0 \\ y &= 1 \\ z &= 1 \\ t &= -2 \end{aligned}$$

The basic variables are x, y, z.

So

Basis for $V+W$ = Basis for $C(A)$

$$= \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$\dim N(A)$ = No. of free variables in the system $Ux = 0$

= The free variable is t

$\therefore \dim N(U) = 1$

Basis for $N(U) = \{(\underbrace{0}_{a_{11}}, \underbrace{1}_{a_{12}}, \underbrace{1}_{b_{11}}, \underbrace{-2}_{b_{12}})\}$

Basis for $V \cap W = a_{11} \cdot v_1 + a_{12} v_2$

$= 0 \cdot (1,-1,0) + 1\,(1,0,-1)$

$= (1,0,-1)$

$\therefore \dim(V \cap W) = 1$.

Bases for Subspaces:- Consider the vector space $\mathbb{R}^n$ and subspace V and W. Suppose A basis for V is $\{v_1, v_2, \ldots v_k\}$ and a basis for W is $\{w_1, w_2, \ldots w_l\}$

Q:- What is the basis and dimension of the subspaces $V+W$ and $V \cap W$?

$V+W = \{x+y \mid x \in V \text{ and } y \in W\}$

clearly the vectors $\{v_1, v_2, \ldots v_k, w_1, w_2, \ldots w_l\}$ belongs to $V+W$.

clearly $\{v_1, v_2, \ldots v_k, w_1, w_2, \ldots w_l\}$ is a spanning set for $V+W$

$\therefore$ If we remove some vector from this collection we can get a basis for $V+W$.

We know $V \cap W \subseteq V$ and $V \cap W \subseteq W$

So $\{v_1, v_2, \ldots v_k\}$ and $\{w_1, w_2, \ldots w_l\}$ are spanning sets for $V \cap W$. If we remove some vectors either from the collection $\{v_1, v_2, \ldots v_k\}$ (or) from the collection $\{w_1, \ldots w_l\}$ then one can find basis and dimension for $V \cap W$.

Procedure to find the basis and dimension for $V+W$ and $V \cap W$:-

1) Form the matrix A of order $n \times (k+l)$ such that whose column vectors are the elements $\{v_1, v_2, \ldots v_k, w_1, w_2, \ldots w_l\}$ (that is)

$$A = [v_1 \ v_2 \ v_3 \cdots v_k \ w_1 \ w_2 \cdots w_l]$$

2) Find $C(A)$ and $N(A)$
3) Basis for $C(A)$ = Basis for $V+W$.
4) dimension of $N(A) = \dim(V \cap W)$
5) Suppose $x_1 = (a_{11}, a_{12}, \ldots, a_{1k}, b_{11}, b_{12}, \ldots, b_{1l})$

$x_2 = (a_{21}, a_{22}, \ldots a_{2k}, b_{21}, b_{22}, \ldots b_{2l})$

$\vdots$

$x_s = (a_{s1}, a_{s2}, \ldots a_{sk}, b_{s1}, b_{s2}, \ldots b_{sl})$

is a basis for $N(A)$ then the vectors

$y_1 = a_{11} v_1 + a_{12} v_2 + \cdots + a_{1k} v_k$

$y_2 = a_{21} v_1 + a_{22} v_2 + \cdots + a_{2k} v_k$

$\vdots$

$y_s = a_{s1} v_1 + a_{s2} v_2 + \cdots + a_{sk} v_k$

is a basis for $(V \cap W)$

Note:- Let V and W be subspaces of $\mathbb{R}^n$ then

$$\dim(V+W) + \dim(V \cap W) = \dim V + \dim W$$

Q:- Find a basis and dimension of the Subspaces $V \cap W$ and $V + W$ where

$V = \{(a,b,c,d) \in \mathbb{R}^4 \mid b+c+d=0\}$ and

$W = \{(a,b,c,d) \in \mathbb{R}^4 \mid a+b=0, \ c=2d\}$

Ans:- Since $\dim V = 3$, a basis for V is

$\{\underset{v_1}{(1,0,0,0)}, \underset{v_2}{(0,1,-1,0)}, \underset{v_3}{(0,1,0,-1)}\}$

Since $\dim W = 2$, a basis for W is

$\{\underset{w_1}{(1,-1,0,0)}, \underset{w_2}{(0,0,2,1)}\}$

Form the matrix $A = [v_1 \ v_2 \ v_3 \ w_1 \ w_2]$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \xrightarrow{R_4 \to R_4 + R_3}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix} = U$$

Form the System $UX = 0$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} x + t &= 0 \\ y + z - t &= 0 \\ z - t + 2w &= 0 \\ -t + 3w &= 0 \end{aligned}$$

The basic variables are x, y, z, t $\therefore$ First 4 columns of A is the basis for $V+W$.

Basis for $V+W = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right\}$

The system $UX = 0$ is

$$\begin{aligned} x + t &= 0 \\ y + z - t &= 0 \\ z - t + 2w &= 0 \\ -t + 3w &= 0. \end{aligned}$$

$x = -3$
$y = 2$
$z = 1$
$t = 3$
$w = 1$

Free variable is w. Hence $\dim(V \cap W) = \dim N(A) = 1$.

Basis for $N(A) = \{(\underset{a_{11}}{-3}, \underset{a_{12}}{2}, \underset{a_{13}}{1}, \underset{b_{11}}{3}, \underset{b_{12}}{1})\}$

Basis for $V \cap W = (-3) v_1 + 2 v_2 + 1 \cdot v_3$

$= (-3)(1,0,0,0) + 2(0,1,-1,0) + 1 \cdot (0,1,0,-1)$

Basis for $V \cap W = \{(-3, 3, -2, -1)\}$

Invertible matrices:-

Example:- consider the matrix

$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ order 2×3

What is the rank A? Form the equation

$AX = 0 \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow y = 0$
$z = 0$

$\therefore$ Basic variables are y, z

$\therefore$ Basis for $C(A) = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$\dim C(A) = 2.$

$\therefore \text{Rank}(A) = 2.$

$C(A) \subseteq \mathbb{R}^2$

Basis for $C(A) = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$C(A) = \mathbb{R}^2$

$\text{Rank } A = 2$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

does the equation

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

has any solution? Yes.

$x = 5 \quad y = 2, \quad z = 3$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$ b_1, b_2 are numbers.

$AX = B \Leftrightarrow \text{Rank}(A) = 2$

Theorem:- Let A be a $m\times n$ matrix. The following statements are equivalent

(a) For each $b \in \mathbb{R}^m$, $Ax=b$ has atleast one solution X in $\mathbb{R}^n$

(b) $C(A) = \mathbb{R}^m$

(c) $\text{Rank}(A) = m$

(d) There exists an $n\times m$ right inverse B of A exists such that $AB = I_m$.

Theorem:- Let A be a $m\times n$ matrix. The following statements are equivalent.

(a) For each $b \in \mathbb{R}^m$ the system $Ax=b$ has atmost one solution

(b) The column vectors of A are linearly independent.

(c) $R(A) = \mathbb{R}^n$

(d) $\text{Rank}(A) = n$ Hence $n \le m$.

(e) A has left inverse (which means there exists a $n\times m$ matrix B such that $BA = I_n$)

Invertibility:- Let A be a $m\times n$ matrix. The following statements are equivalent.

(a) For each $b \in \mathbb{R}^m$ the equation $Ax=b$ has a solution

(b) $C(A) = \mathbb{R}^m$

(c) $\text{Rank}(A) = m$ and hence $m \le n$.

(d) A has right inverse (which means, there exists a $n\times m$ matrix B such that $AB = I_m$)

Q:- Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 5 \\ 2 & 4 \end{bmatrix}$$

Does A has left inverse?

Soln:- Order A is 3×2.

$$\begin{bmatrix} 1 & 2 \\ 0 & 5 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & 2 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} = U$$

Form the equation $UX=0$

$$\begin{bmatrix} 1 & 2 \\ 0 & 5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + 2y = 0$
$5y = 0$

No. of basic variables are 2

$\therefore \text{Rank}(A) = 2$

$\therefore$ A has left inverse.

Does the equation

$$\begin{bmatrix} 1 & 2 \\ 0 & 5 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

has solution?

$x + 2y = 2 \to (1)$
$5y = 3 \to (2)$
$2x + 4y = 5 \to (3)$

Q:- Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Does A has right inverse?

Ans:- Order of A is 4×5

$$\begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = U \text{ (Reduced form)}$$

The equation $UX=0$ is

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + 3t = 0$
$y = 0$
$2z = 0$
$5w = 0$

$\therefore$ Basic variables are x, y, z, w

$\therefore \dim C(A) = 4$

$\text{Rank}(A) = 4$.

Yes, the matrix A has right inverse.

Theorem:- Let A be a $n\times n$ matrix. The following are equivalent.

(a) A is invertible

(b) $\det A \neq 0$

(c) A is row equivalent to I_n

(d) $Ax=b$ has a solution for every $b \in \mathbb{R}^n$

(e) $N(A) = \{0\}$

(f) The column space of A are linearly independent.

(g) $C(A) = \mathbb{R}^n$

(h) A has a left inverse

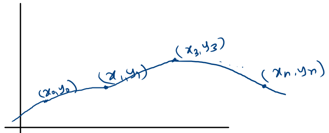
(i) $\text{Rank}(A) = n$

(j) Rows of A are linearly independent.

(k) $R(A) = \mathbb{R}^n$

(l) A has a right inverse.

Application of invertibility in polynomial interpolation:-



Given a set of n-data points.

x:	x_0	x_1	x_2	$\cdots$	x_n
y:	y_0	y_1	y_2		y_n

Is it possible to find a polynomial $p(x)$ such that $p(x_i)=y_i$ for all $i=0,1,2,\ldots n$?

Try! Find the equation of the circle passes through the three points $(2,-2)$, $(3,5)$ and $(-4,6)$

Hint:- Equation of the circle

$$(x-h)^2+(y-k)^2=r^2$$

where (h,k) is the centre and r is the radius.

We assume that the required polynomial is

$$p(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$$

we have, $p(x_0)=a_0+a_1x_0+a_2x_0^2+\cdots+a_nx_0^n=y_0$

$$p(x_1)=a_0+a_1x_1+a_2x_1^2+\cdots+a_nx_1^n=y_1$$

$$\vdots$$

$$p(x_n)=a_0+a_1x_n+a_2x_n^2+\cdots+a_nx_n^n=y_n$$

It's matrix form is

$$\underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}}_{V} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

We can calculate the required polynomial only if V is invertible (or) $\det V \neq 0$.

Module-4

Linear transformations

Aim: Find the relation between two different vector spaces over $\mathbb{R}$.

$A=\{1,2,3,4\}$ $B=\{2,4,6,8\}$

Define $f: A\to B$ such that $f(x)=2x$.

Function- Let X and Y be two non empty sets. A function is a relation from X to Y such that each element in X is related to a unique element in Y.

Examples:-(i) $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=x^2$

(ii) $g:\mathbb{R}\to\mathbb{R}$ by $g(x)=x^3$

$h:\mathbb{R}\to\mathbb{R}$ by $h(x)=\frac{1}{x}$. Is h a function?

h is not a function because $h(0)$ is not defined.

Pb: Find the polynomial $p(x)=a+bx+cx^2+dx^3$ that satisfies

$$p(0)=1,\ p'(0)=2,\ p(1)=4;\ p'(1)=4.$$

Ans:- If $x=0$; then $\boxed{a=1}$

$p'(x)=b+2cx+3dx^2$

$p'(0)=2 \Rightarrow \boxed{b=2}$

$p(1)=4 \Rightarrow a+b+c+d=4$

$\Rightarrow \boxed{c+d=1}$

$p'(1)=4 \Rightarrow b+2c+3d=4$

$\boxed{2c+3d=2}$

$\therefore \boxed{d=0 \text{ and } c=1}$

$T:\mathbb{R}^+\to\mathbb{R}$ by $T(x)=\sqrt{x}$ is T a function?

Ans:- No, $T(4)=\begin{cases}2\\-2\end{cases}$ Image is not unique.

So T is not a function.

Linear transformation:-

Let V and W be vector spaces. A function $T:V\to W$ is called a linear transformation from V to W if for all $x,y\in V$ and a scalar k the following conditions hold.

(a) $T(x+y)=T(x)+T(y)$

(b) $T(kx)=kT(x)$

(c) $T(0_V)=0_W$

Example:- $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x,y) = (2x, 2y)$.

check Is T represents a linear transformation?

Ans:- Take $X = (x_1, y_1)$; $Y = (x_2, y_2)$

$$T(X+Y) = T\big((x_1,y_1) + (x_2,y_2)\big) = T\big((x_1+x_2,\ y_1+y_2)\big)$$
$$= \big(2(x_1+x_2),\ 2(y_1+y_2)\big) = (2x_1, 2y_1) + (2x_2, 2y_2)$$
$$= T(X) + T(Y)$$

$$T(kX) = T(k(x_1,y_1)) = T(kx_1, ky_1) = (2kx_1, 2ky_1)$$
$$= k(2x_1, 2y_1) = kT(X)$$

$$T(0,0) = (2\cdot 0,\ 2\cdot 0) = (0,0).$$

$\therefore T$ is a linear transformation.

Zero linear transformation:- Let V and W be two vector spaces over $\mathbb{R}$. We call the transformation
$$T: V \longrightarrow W \text{ by } T(v) = 0_W \text{ for all } v \in V$$
is the Zero linear transformation.

Identity linear transformation:- Let V be a vector space over $\mathbb{R}$. We call the transformation
$$T: V \to V \text{ by } T(v) = v$$
is the identity linear transformation.

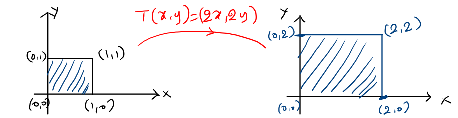
Example:- Let $T_1: \mathbb{R}^3 \to \mathbb{R}^3$ by $T_1(x,y,z) = (x,y,z)$

$T_2: \mathbb{R}^4 \to \mathbb{R}^2$ by $T_2(x,y,z,w) = (0,0)$

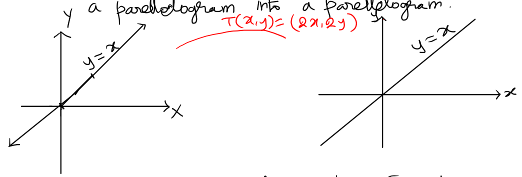
be linear transformations then

T_1 is the identity transformation.

T_2 is the zero transformation.



Note:- In general any linear transformation maps a parallelogram into a parallelogram.



Note:- In general a linear transformation maps a straight line into a straight line.

Linear transformation Let V and W be two vector spaces. A function $T: V \to W$ is a linear transformation if it satisfies the following conditions

(a) $T(x+y) = T(x) + T(y)$ for all $x, y \in V$

(b) $T(kx) = k\,T(x)$ for all $k \in \mathbb{R}$, $x \in V$

(c) $T(0_V) = 0_W$

Remarks:- Linear transformation maps a parallelogram into a parallelogram and a straight line into a straight line.

$T: V \to W$ by $T(x) = 0_W$ for all $x \in V$ is called as zero linear transformation and

$I_d: V \to V$ by $I_d(x) = x$ is called as identity linear transformation

Example:- Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x,y) = (x+2, y)$

Question:- Does T represent a linear transformation?

Ans:- No, Since $T(0,0) = (2,0) \neq (0,0)$

Hence T is not a linear transformation.

Example:- Define $T: \mathbb{R}^2 \to \mathbb{R}$ by
$$T(x,y) = x^2 + y^2$$

Question:- Does T represent a linear transformation?

Ans:- Let $X = (x_1, y_1)$; $Y = (x_2, y_2)$

$$T(X+Y) = T\big((x_1,y_1) + (x_2,y_2)\big) = T(x_1+x_2,\ y_1+y_2)$$
$$= (x_1+x_2)^2 + (y_1+y_2)^2$$
$$= x_1^2 + 2x_1x_2 + x_2^2 + y_1^2 + 2y_1y_2 + y_2^2$$
$$T(X) + T(Y) = x_1^2 + y_1^2 + x_2^2 + y_2^2$$

Hence $T(X+Y) \neq T(X) + T(Y)$

Thus T is not a linear transformation

Example:- Let A be a $m \times n$ matrix. Define a function $T: \mathbb{R}^n \to \mathbb{R}^m$ by $T(X) = AX$.

Then T represents a linear transformation.

$$T(X+Y) = A(X+Y) = AX + AY = T(X) + T(Y)$$
$$T(kX) = A(kX) = kAX = kT(X)$$
$$T(0) = A\cdot 0 = 0 =$$

Example 2 Consider the vector spaces $P_n(\mathbb{R})$, $P_{n-1}(\mathbb{R})$.

Define $D: P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ by $D(p(x)) = p'(x)$

$$D(p_1(x) + p_2(x)) = \big(p_1(x) + p_2(x)\big)' = p_1'(x) + p_2'(x)$$
$$= D(p_1(x)) + D(p_2(x))$$

$$D(kp(x)) = (kp(x))' = k\,p'(x) = k\,D(p(x))$$
$$D(0) = (0)' = 0$$

D is a linear transformation.

(Try!) 1) Define $I : P_{n+1}(\mathbb{R}) \to P_n(\mathbb{R})$ by

$$I(p(x)) = \int_0^x p(x)\,dx.$$

then I represents a linear transformation.

2) Define $tr : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ by

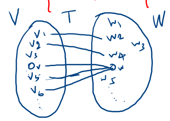
$$tr([a_{ij}]) = a_{11} + a_{22} + \cdots + a_{nn}$$

then tr is a linear transformation.

Definition:- Let V and W be two vector spaces, and let $T : V \to W$ be a linear transformation

(i) $Ker(T) = \{ v \in V \mid T(v) = 0 \}$ is called kernal of T.

(ii) $Im(T) = \{ T(v) \in W \mid v \in V \}$ is called Image of T. (Range of T)



Theorem:- $Ker(T)$ is a subspace of V

$Im(T)$ is a subspace of W.

Consider a linear transformation

$T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x,y) = (0,y)$

then

$$Ker(T) = \{ (x,y) \in \mathbb{R}^2 \mid T(x,y) = (0,0) \}$$

$$= \{ (x,y) \in \mathbb{R}^2 \mid (0,y) = (0,0) \}$$

$$= \{ (x,y) \in \mathbb{R}^2 \mid y = 0 \} = X\text{-axis}.$$

$Im(T) = Y$-axis

Theorem:- $T : V \to W$ be a linear transformation then

$$T(k_1x_1 + k_2x_2 + \cdots + k_nx_n) = k_1T(x_1) + k_2T(x_2) + \cdots + k_nT(x_n)$$

Example:- Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(1,0,0) = (8,6,1)$

$T(0,1,0) = (2,0,5)$

$T(0,0,1) = (1,2,0)$. Find $T(x,y,z)$ for any $(x,y,z) \in \mathbb{R}^3$

Soln:- $(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$

$$T(x,y,z) = T(\underbrace{x}_{k_1}\underbrace{(1,0,0)}_{x_1} + \underbrace{y}_{k_2}\underbrace{(0,1,0)}_{x_2} + \underbrace{z}_{k_3}\underbrace{(0,0,1)}_{x_3})$$

$$= xT(1,0,0) + yT(0,1,0) + zT(0,0,1)$$

$$= x(8,6,1) + y(2,0,5) + z(1,2,0)$$

$$\boxed{T(x,y,z) = (8x + 2y + z,\ 6x + 2z,\ x + 5z)}$$

Example 9:- Consider the vector space $\mathbb{R}^2$ with basis $\{(1,1), (1,-1)\}$. Given $T : \mathbb{R}^2 \to \mathbb{R}^3$ a linear transformation such that

$T(1,1) = (2,4,6)$

$T(1,-1) = (0,8,10)$

Find $T(x,y)$.

Soln:- $(x,y) = \alpha(1,1) + \beta(1,-1) \to ①$

we calculate α, β.

$① \Rightarrow \left.\begin{array}{l} \alpha + \beta = x \\ \alpha - \beta = y \end{array}\right\} \Rightarrow \begin{array}{l} \alpha = \frac{x+y}{2} \\ \beta = \frac{x-y}{2} \end{array}$

$$\boxed{(x,y) = \left(\frac{x+y}{2}\right)(1,1) + \left(\frac{x-y}{2}\right)(1,-1)}$$

$$T(x,y) = T\left(\frac{x+y}{2}(1,1) + \frac{x-y}{2}(1,-1)\right)$$

$$= \frac{x+y}{2}T(1,1) + \frac{x-y}{2}T(1,-1)$$

$$= \frac{x+y}{2}(2,4,6) + \frac{x-y}{2}(0,8,10)$$

$$= x+y(1,2,3) + x-y(0,4,5)$$

$$T(x,y) = (x+y,\ 6x-2y,\ 8x-2y)$$

Q:- When do we say two linear transformations are equal.

Ans:- Let V and W be two vector spaces and $\{v_1, v_2, \ldots v_n\}$ be a basis for V. Suppose $S : V \to W$, $T : V \to W$ be two linear transformation with

$$S(v_i) = T(v_i) \quad \text{for all } i$$

then $S = T$ (or) $S(x) = T(x)$. $\forall x \in V$

How to construct new linear transformations?

Example:- Consider the vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ over $\mathbb{R}$.

A basis for $\mathbb{R}^2$ is $\{(1,0),(0,1)\}$

2 elements in $\mathbb{R}^3$ $\{(1,0,0),(0,1,0)\}$

Consider a map $T: \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$T(1,0) = (1,0,0)$$
$$T(0,1) = (0,1,0)$$

Suppose T is a linear transformation then

$$\begin{aligned} T(x,y) &= T(x(1,0)+y(0,1)) \\ &= xT(1,0)+yT(0,1) \\ &= x(1,0,0)+y(0,1,0) \end{aligned}$$

$$T(x,y) = (x,y,0)$$

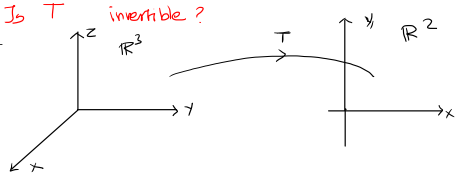
Hence $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x,y)=(x,y,0)$ is a linear transformation.

Example 2:- Consider the linear transformation

$$T: \mathbb{R}^3 \to \mathbb{R}^2 \text{ by } T(x,y,z) = (x,y)$$

Q:- Is T invertible?

Ans:-



Since $T(0,0,1) = (0,0)$

$T(0,0,2) = (0,0)$

So T is not 1-1. Hence T is not invertible.

Theorem:- Let V and W be vector spaces. Let $\{v_1, v_2, \dots v_n\}$ be a basis for V and let $\{w_1, w_2, \dots w_n\}$ be any vectors (possibly repeated) in W. Then there exists a unique linear transformation $T: V \to W$ such that $T(v_i) = w_i$.

Example:- Consider the vector spaces $\mathbb{R}^3$ and $P_2(\mathbb{R})$

A basis for $P_2(\mathbb{R})$ is $\{1, x, x^2\}$

3 elements in $\mathbb{R}^3$ $\{(1,0,0),(0,1,0),(0,0,1)\}$

By theorem there exists a linear transformation

$T: P_2(\mathbb{R}) \to \mathbb{R}^3$ such that

$$T(1) = (1,0,0)$$
$$T(x) = (0,1,0)$$
$$T(x^2) = (0,0,1)$$

we know $T(a_0+a_1x+a_2x^2) = a_0T(1)+a_1T(x)+a_3T(x^2)$

$$= a_0(1,0,0)+a_1(0,1,0)+a_2(0,0,1)$$

$$T(a_0+a_1x+a_2x^2) = (a_0, a_1, a_2)$$

Example:- consider the linear transformation

$$T: \mathbb{R}^2 \to \mathbb{R}^3 \text{ by } T(x,y) = (x,y,0)$$

Q:- Is T invertible?

Ans:- What is the preimage of $(0,0,1)$?

It does not exist

Hence T is not onto

$\therefore$ T is not invertible.

Theorem:- Let $T: V \to W$ be a linear transformation. Suppose T^{-1} exists then it is also a linear transformation.

Invertible linear transformations:-

Suppose $f: X \to Y$ be a function then when f^{-1} exists?

1-1 function:- f is a 1-1 function if

$$f(x) = f(y) \Rightarrow x = y$$

(or) $\quad x \neq y \Rightarrow f(x) \neq f(y)$

onto function:- Every element in the co-domain has pre image (or) Range(f) = Co-domain

f is 1-1 and onto then f^{-1} exists

Invertible linear transformation:- Let V and W be ~~two~~ vector spaces. $T: V \to W$ be a linear transformation. T is invertible if T is 1-1 and T is onto

Examples:- Identity linear transformation is always
1) invertible but zero linear transformation is not invertible.

Isomorphism:- 1 2 3 4 …

I II III IV

ONE TWO THREE FOUR

A linear transformation $T: V \to W$ from a vector space V to a vector space W is called an isomorphism if it is invertible. In this case we say V and W are isomorphic to each other.

Theorem:- Two vector spaces V and W are isomorphic to each other $\iff$ $\dim V = \dim W$.

Examples:- 1) Consider the vector space $P_2(\mathbb{R})$ and $\mathbb{R}^3$. Since $\dim(P_2(\mathbb{R})) = 3 = \dim(\mathbb{R}^3)$, the vector spaces $P_2(\mathbb{R})$ and $\mathbb{R}^3$ is isomorphic.

2) Consider the vector spaces $P_n(\mathbb{R})$ and $\mathbb{R}^{n+1}$. Since $\dim(P_n(\mathbb{R})) = n+1 = \dim(\mathbb{R}^{n+1})$, $P_n(\mathbb{R})$ and $\mathbb{R}^{n+1}$ are isomorphic.

3) Consider the vectorspace $M_{2\times 3}(\mathbb{R})$ and $\mathbb{R}^6$. Since $\dim(M_{2\times 3}(\mathbb{R})) = 6 = \dim(\mathbb{R}^6)$, the spaces $M_{2\times 3}(\mathbb{R})$ is isomorphic to $\mathbb{R}^6$.

4) Ingeneral the vector spaces $M_{m\times n}(\mathbb{R})$ is isomorphic to $\mathbb{R}^{mn}$

5) Any n-dimensional vector space is isomorphic to $\mathbb{R}^n$.

Pb:1 Find the matrix representation of the following linear transformations T on $\mathbb{R}^3$ with respect to the standard basis

(a) $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ by $T_1(x,y,z) = (2x-3y+4z,\ 5x-y+2z,\ 4x+7y)$

(b) $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ by $T_2(x,y,z) = (2y+z,\ x-4y,\ 3x)$

Solution:- Basis $\alpha = \{\underset{e_1}{(1,0,0)}, \underset{e_2}{(0,1,0)}, \underset{e_3}{(0,0,1)}\}$

$T_1(1,0,0) = (2,5,4) = 2(1,0,0) + 5(0,1,0) + 4(0,0,1)$

$T_1(0,1,0) = (-3,-1,7) = (-3)(1,0,0) + (-1)(0,1,0) + 7(0,0,1)$

$T_1(0,0,1) = (4,2,0) = 4(1,0,0) + 2(0,1,0) + 0\cdot(0,0,1)$

$$[T]_\alpha^\alpha = \begin{bmatrix} 2 & -3 & 4 \\ 5 & -1 & 2 \\ 4 & 7 & 0 \end{bmatrix}$$

Matrices for linear transformation:-

Consider the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Order of the matrix is 3×2. Then

$T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x,y) = A\begin{bmatrix} x \\ y \end{bmatrix}$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 2x+3y \\ x+y \\ y \end{bmatrix}$$

Hence $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x,y) = (2x+3y,\ x+y,\ y)$ is a linear transformation.

Q:- If T is a linear transformation then construct a matrix related to T.

(b) $T_2(1,0,0) = (0,1,3) = 0\cdot(1,0,0) + 1\cdot(0,1,0) + 3(0,0,1)$

$T_2(0,1,0) = (2,-4,0) = 2(1,0,0) + (-4)(0,1,0) + 0\cdot(0,0,1)$

$T_2(0,0,1) = (1,0,0) = 1\cdot(1,0,0) + 0\cdot(0,1,0) + 0\cdot(0,0,1)$

$$[T_2]_\alpha^\alpha = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

Process:- Let $\alpha = \{v_1, v_2, \dots v_n\}$ be a basis for vector space V. and $\beta = \{w_1, w_2, \dots w_m\}$ be a basis for vector space W. Suppose $T : V \to W$ be a linear transformation such that

$T(v_1) = a_{11}w_1 + a_{21}w_2 + \dots + a_{m1}w_m$

$T(v_2) = a_{12}w_1 + a_{22}w_2 + \dots + a_{m2}w_m$

$\vdots$

$T(v_n) = a_{1n}w_1 + a_{2n}w_2 + \dots + a_{mn}w_m$

then the matrix of T is

$$[T]_\alpha^\beta = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Process:- Let $\alpha = \{v_1, v_2, \dots v_n\}$ be a basis for vector space V. and $\beta = \{w_1, w_2, \dots w_m\}$ be a basis for vector space W. Suppose $T : V \to W$ be a linear transformation such that

$T(v_1) = a_{11}w_1 + a_{21}w_2 + \dots + a_{m1}w_m$

$T(v_2) = a_{12}w_1 + a_{22}w_2 + \dots + a_{m2}w_m$

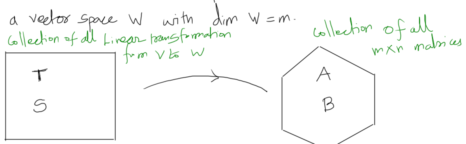
$\vdots$

$T(v_n) = a_{1n}w_1 + a_{2n}w_2 + \dots + a_{mn}w_m$

then the matrix of T is

$$[T]_\alpha^\beta = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Remarks:- consider a vector space V with $\dim V = n$ and a vector space W with $\dim W = m$.



If T is a linear transformation then one can construct a matrix A

If A is a $m \times n$ matrix then one can construct a linear transformation from V to W.

2) Suppose A is a $m \times n$ matrix then we can think of A as a linear transformation from $\mathbb{R}^n$ (with standard basis α) to $\mathbb{R}^m$ (with standard basis β) [which means] $[A]_\alpha^\beta = A$.

Pb:2 Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x,y,z) = (3x, x-y, 2x+y+z)$ be a linear transformation. Find

(a) The matrix of T with respect to the standard basis of $\mathbb{R}^3$

(b) Prove that T is invertible and $(T^2 - I)(T^2 - 3I) = 0$.

Soln:- Basis of $\mathbb{R}^3$ is $\alpha = \{(1,0,0),(0,1,0),(0,0,1)\}$

$T(1,0,0) = (3,1,2) = 3(1,0,0) + 1(0,1,0) + 2(0,0,1)$

$T(0,1,0) = (0,-1,1) = 0((1,0,0) + (-1)(0,1,0) + 1\cdot(0,0,1)$

$T(0,0,1) = (0,0,1) = 0\cdot(1,0,0) + 0(0,1,0) + 1(0,0,1)$

$$[T]_\alpha^\alpha = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} = A.$$

Since $\det A = -3 \neq 0$ Hence A is invertible so T is invertible.

To prove $(T^2 - I)(T^2 - 3I) = 0$

Try It is enough to check $(A^2 - I)(A^2 - 3I) = 0$

3) Let V and W be vector spaces with bases α and β. $T: V \to W$ be a linear transformation such that $[T]_\alpha^\beta = A$ then

(a) $\ker(T)$ is isomorphic to $N(A)$

(b) $\operatorname{Im}(T)$ is isomorphic to $C(A)$

(c) T is invertible iff A is invertible.

(d) If $V = \mathbb{R}^n$; $W = \mathbb{R}^m$ with standard basis on $\mathbb{R}^n$ and $\mathbb{R}^m$ then $\ker(T) = N(A)$, $\operatorname{Im}(T) = C(A)$

$\dim(\ker(T)) + \dim(\operatorname{Im}(T)) = \dim V$.

Pb:3 Consider a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ such that

$T(1,0,0) = (0,0,1) = 0\cdot(1,0,0) + 0(0,1,0) + 1\cdot(0,0,1)$

$T(0,1,0) = (0,1,0) = 0\cdot(1,0,0) + 1\cdot(0,1,0) + 0\cdot(0,0,1)$

$T(0,0,1) = (1,0,0) = 1\cdot(1,0,0) + 0\cdot(0,1,0) + 0(0,0,1)$

a) Find the matrix of T with respect to the standard basis of $\mathbb{R}^3$

b) Find $\dim(\ker(T))$ and $\dim(\operatorname{Im}(T))$.

Soln:-

The matrix is

$$[T]_\alpha^\beta = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = A.$$

$\dim \ker(T) = \dim(N(A)) = 0$; $\dim(\operatorname{Im}(T)) = \dim(C(A))$; $\dim(\operatorname{Im}(T)) = 3$

Pb:1 Let $T: P_3(\mathbb{R}) \to P_5(\mathbb{R})$ be the linear transformation defined by

$$T(p(x)) = (x^2 - 2)\, p(x)$$

Find the matrix representation of T relative to the standard basis of $P_3(\mathbb{R})$ and $P_5(\mathbb{R})$.

Soln:- Basis for $P_3(\mathbb{R})$ is $\alpha = \{1, x, x^2, x^3\}$ and Basis for $P_5(\mathbb{R})$ is $\beta = \{1, x, x^2, x^3, x^4, x^5\}$

$T(1) = (x^2-2)\cdot 1 = x^2 - 2 = (-2)\cdot 1 + 0\cdot x + 1\cdot x^2 + 0\cdot x^3 + 0\cdot x^4 + 0\cdot x^5$

$T(x) = (x^2-2)x = x^3 - 2x = 0\cdot 1 + (-2)x + 0\cdot x^2 + 1\cdot x^3 + 0\cdot x^4 + 0\cdot x^5$

$T(x^2) = (x^2-2)x^2 = x^4 - 2x^2 = 0\cdot 1 + 0\cdot x + (-2)x^2 + 0\cdot x^3 + 1\cdot x^4 + 0\cdot x^5$

$T(x^3) = (x^2-2)x^3 = x^5 - 2x^3 = 0\cdot 1 + 0\cdot x + 0\cdot x^2 + (-2)x^3 + 0\cdot x^4 + 1\cdot x^5$

$$[T]_\alpha^\beta = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Pb:1 Let $T: \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation defined by $T(x,y,z,t) = (x - y + z + t,\ 2x - 2y + 3z + 4t,\ 3x - 3y + 4z + 5t)$. Consider the standard basis on $\mathbb{R}^4$ and $\mathbb{R}^3$. Find the basis and dimension of $\ker(T)$, $\operatorname{Im}(T)$.

Soln:- standard basis of $\mathbb{R}^4$ is $\alpha = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

Standard basis of $\mathbb{R}^3$ is $\beta = \{(1,0,0), (0,1,0), (0,0,1)\}$.

$T(1,0,0,0) = (1,2,3)$

$T(0,1,0,0) = (-1,-2,-3)$; $[T]_\alpha^\beta = A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -2 & 3 & 4 \\ 3 & -3 & 4 & 5 \end{bmatrix}$

$T(0,0,1,0) = (1,3,4)$

$T(0,0,0,1) = (1,4,5)$

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -2 & 3 & 4 \\ 3 & -3 & 4 & 5 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

We form the equation $UX = 0$

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} x - y + z + t = 0 \\ z + 2t = 0 \end{array}$$

The basic variables are x and z. So the 1st and 3rd column of A are basis for $C(A)$

Basis for $Im(T)$ = Basis for $C(A)$

$= \{(1,2,3), (1,3,4)\}$

$\dim Im(T) = \dim(C(A)) = 2.$

System $UX = 0$ is $x - y + z + t = 0$

$z + 2t = 0.$

Free variables are y and t. So,

$\dim(Ker(T)) = \dim(N(A)) = \dim(N(U)) = 2.$

A basis for $(Ker(T))$ = A basis for $(N(U))$

$= \{(1,1,0,0), (1,0,-2,1)\}$

Pb:2 Let $\{(1,0,0), (0,2,1), (2,0,1)\}$ be a basis of $\mathbb{R}^3$. If $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$T(1,0,0) = (0,0,1)$

$T(0,2,1) = (1,2,0)$

$T(2,0,1) = (1,1,1)$

Find $Ker(T)$ and $Im(T)$.

Soln:-

$T(1,0,0) = (0,0,1) = (-2)(1,0,0) + 0(0,2,1) + 1\cdot(2,0,1)$

$T(0,2,1) = (1,2,0) = 3(1,0,0) + 1(0,2,1) + (-1)(2,0,1)$

$T(2,0,1) = (1,1,1) = 0(1,0,0) + (\frac{1}{2})(0,2,1) + (\frac{1}{2})(2,0,1)$

$$[T]_\alpha^\alpha = \begin{bmatrix} -2 & 3 & 0 \\ 0 & 1 & \frac{1}{2} \\ 1 & -1 & 1/2 \end{bmatrix} = A$$

Since $\det A \neq 0$, A is invertible.

$\Rightarrow T$ is also invertible

Note:- (a) T is one to one linear transformation then $Ker(T) = \{0\}$

(b) T is onto then $Im(T)$ = codomain.

T is invertible = T is 1-1 + T is onto

$\therefore Ker(T) = \{0\}$

$Im(T) = \mathbb{R}^3$

Pb:3 Consider the linear transformation

$T: P_2(\mathbb{R}) \to P_4(\mathbb{R})$ by $T(p(t)) = p(t) + 2t^2 p(t)$.

(a) Find the matrix of T relative to the basis $\{t, 1+t, t+t^2\}$ and $\{1, t, t^2, t^3, t^4\}$

(b) Is T invertible (or) not?

Soln:-

$T(t) = t + 2t^2 \cdot t = t + 2t^3 = 0\cdot 1 + 1\cdot t + 0\cdot t^2 + 2t^3 + 0\, t^4$

$T(1+t) = (1+t) + 2t^2(1+t) = 1 + t + 2t^2 + 2t^3 = 1\cdot 1 + 1\cdot t + 2\cdot t^2 + 2\cdot t^3 + 0\cdot t^4$

$T(t+t^2) = (t+t^2) + 2t^2(t+t^2) = t + t^2 + 2t^3 + 2t^4 = 0\cdot 1 + 1\cdot t + 1\cdot t^2 + 2\cdot t^3 + 2\cdot t^4$

$$[T]_\alpha^\beta = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

Co-ordinate matrix:- Let V be a vector space with basis $\alpha = \{v_1, v_2, \ldots v_n\}$. Consider $x \in V$ then

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

The column matrix $[x]_\alpha^\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$ is called as the co-ordinate matrix.

Example:- Consider the vector space $\mathbb{R}^3$ with basis $\alpha = \{(1,0,0), (1,1,0), (1,1,1)\}$. Find the co-ordinate matrix for (i) $(1,0,1)$ (ii) $(1,1,1)$ (iii) (x,y,z)

Soln:- (i) $(1,0,1) = \alpha(1,0,0) + \beta(1,1,0) + \gamma(1,1,1)$

$= 1(1,0,0) + (-1)(1,1,0) + 1(1,1,1)$

$$\therefore [(1,0,1)]_\alpha^\alpha = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

(ii) $(1,1,1) = 0\cdot(1,0,0) + 0\cdot(1,1,0) + 1\cdot(1,1,1)$

$$[(1,1,1)]_\alpha^\alpha = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(iii) $(x,y,z) = (x-y)(1,0,0) + (y-z)(1,1,0) + z(1,1,1)$

$$[(x,y,z)]_\alpha^\alpha = \begin{bmatrix} x-y \\ y-z \\ z \end{bmatrix}$$

Theorem:- Let V and W be two vector spaces and α is a basis for V, β is a basis for W. If $T: V \to W$ be a linear transformation then

$$[T(v)]_\beta^\beta = [T]_\alpha^\beta [v]_\alpha^\alpha$$

Pb:1 Consider a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x,y) = (3x-2y, 0, x+4y)$. Find the matrix of T with respect to the basis $\alpha = \{(1,1), (0,2)\}$ and $\beta = \{(1,1,0), (1,0,1), (0,1,1)\}$. Also prove that

$$[T(v)]^{\beta}_{\beta} = [T]^{\beta}_{\alpha} [v]^{\alpha}_{\alpha}$$

Soln:-

$T(1,1) = (1,0,5) = -2(1,1,0) + 3(1,0,1) + 2(0,1,1)$

$T(0,2) = (-4,0,8) = -6(1,1,0) + 2(1,0,1) + 6(0,1,1)$

$$[T]^{\beta}_{\alpha} = \begin{bmatrix} -2 & -6 \\ 3 & 2 \\ 2 & 6 \end{bmatrix}$$

Let $v \in (x,y) \in \mathbb{R}^2$;

$$(x,y) = x(1,1) + \left(\frac{y-x}{2}\right)(0,2)$$

$$[v]^{\alpha}_{\alpha} = [(x,y)]^{\alpha}_{\alpha} = \begin{bmatrix} x \\ \frac{y-x}{2} \end{bmatrix}$$

$T(v) = T(x,y) = (3x-2y, 0, x+4y) = \alpha(1,1,0) + \beta(1,0,1) + \gamma(0,1,1)$

$= (x-3y)(1,1,0) + (2x+y)(1,0,1) + (-x+3y)(0,1,1)$

$$[T(v)]^{\beta}_{\beta} = \begin{bmatrix} x-3y \\ 2x+y \\ -x+3y \end{bmatrix}$$

$$[T]^{\beta}_{\alpha} [v]^{\alpha}_{\alpha} = \begin{bmatrix} -2 & -6 \\ 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x \\ \frac{y-x}{2} \end{bmatrix} = \begin{bmatrix} x-3y \\ 2x+y \\ -x+3y \end{bmatrix}$$

$$[T]^{\beta}_{\alpha} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} = A$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_4}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_4 \to R_4 + R_3}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} = U$$

system $UX = 0 \Rightarrow$

$$\begin{aligned} x + z &= 0 \\ y - t &= 0 \\ -z + t &= 0 \\ 2t &= 0 \end{aligned}$$

dim $C(A)$ = no. of basic variables in the system $UX = 0$ $= 4$

$\dim Im(T) = \dim C(A) = 4$

$\dim(Im(T)) = 4 = \dim(M_{2\times 2}(\mathbb{R}))$

$\therefore Im(T) = M_{2\times 2}(\mathbb{R})$.

dim $N(A)$ = no. of free variables in $UX = 0$ $= 0$

$\dim(Ker(T)) = 0 \Rightarrow Ker(T) = \{0\}$

$\Rightarrow T$ is 1-1

Vector spaces of linear transformation:-

Let V and W be two vector spaces. If $T_1: V \to W$ be a linear transformation and $T_2: V \to W$ be a linear transformation then

(i) $T_1 + T_2: V \to W$ is also a linear transformation

$$(T_1 + T_2)(v) = T_1(v) + T_2(v)$$

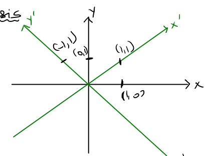
(ii) If $k \in \mathbb{R}$, then $kT_1: V \to W$ is also a linear transformation

$$(kT_1)(v) = kT_1(v)$$

Hence the collection of all linear transformation forms a vector space over $\mathbb{R}$. We denote it by $L(V,W)$.

Result:- If $V = \mathbb{R}^n$, $W = \mathbb{R}^m$ then $L(\mathbb{R}^n, \mathbb{R}^m)$ is isomorphic to $M_{m\times n}(\mathbb{R})$.

Change of Basis



Co-ordinate systems are XY plane with position vectors $\{(1,0), (0,1)\}$

new co-ordinate system is $X'Y'$ plane with position vectors $\alpha' = \{(1,1), (-1,1)\}$

What is the relationship?

$(1,0) = \frac{1}{2}(1,1) + \left(\frac{-1}{2}\right)(-1,1) \Rightarrow [(1,0)]^{\alpha'}_{\alpha'} = \begin{bmatrix} \frac{1}{2} \\ -1/2 \end{bmatrix}$

$(0,1) = \left(\frac{1}{2}\right)(1,1) + \left(\frac{1}{2}\right)(-1,1) \Rightarrow [(0,1)]^{\alpha'}_{\alpha'} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$

$(1,1) = 1\cdot(1,1) + 0\cdot(-1,1) = [(1,1)]^{\alpha'}_{\alpha} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$(-1,1) = 0\cdot(1,1) + 1\cdot(-1,1) = [(-1,1)]^{\alpha'}_{\alpha} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Pb:- Let $T: P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be a linear transformation defined by

$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = \begin{bmatrix} a_0 + a_2 & -a_0 + a_3 \\ a_1 - a_2 & a_1 - a_3 \end{bmatrix}$$

Find the matrix of T relative to the standard basis of $P_3(\mathbb{R})$ and $M_{2\times 2}(\mathbb{R})$. Further find a basis and dimension of $Im(T)$, $Ker(T)$. Prove that T is one to one.

Soln:- standard basis of $P_3(\mathbb{R})$ is $\alpha = \{1, x, x^2, x^3\}$

standard basis of $M_{2\times 2}(\mathbb{R})$ is

$$\beta = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$T(1) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = 1\cdot\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\cdot\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\cdot\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T(x) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = 0\cdot\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\cdot\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\cdot\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 1\cdot\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T(x^2) = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = 1\cdot\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\cdot\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\cdot\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T(x^3) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = 0\cdot\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1\cdot\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\cdot\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Transition matrix:- Let α and β be two bases for a vector space V. Consider the identity transformation $Id_V: V \to V$. The matrix representation of $[Id_V]^{\alpha}_{\beta}$ is called the transition matrix

(or) co-ordinate change matrix from β to α.

Vector space of linear transformations

Suppose $f_1 : \mathbb{R} \to \mathbb{R}$ by $f_1(x) = 5x^2$

$f_2 : \mathbb{R} \to \mathbb{R}$ by $f_2(x) = 2x$

What is $f_1 + f_2$?

$f_1 + f_2 : \mathbb{R} \to \mathbb{R}$ by $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
$= 5x^2 + 2x$

What is $5f_1$? $5f_1 : \mathbb{R} \to \mathbb{R}$ by $(5f_1)(x) = 5f_1(x)$
$= 25x^2$

What is $f_1 \circ f_2$?

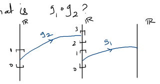
$f_1 \circ f_2 : \mathbb{R} \to \mathbb{R}$ by $(f_1 \circ f_2)(x) = f_1(f_2(x))$
$= f_1(2x)$
$= 5(2x)^2$
$= 20x^2$

$f_2 \circ f_1 : \mathbb{R} \to \mathbb{R}$ by $(f_2 \circ f_1)(x) = f_2(f_1(x)) = 10x^2$

$g_1 : [0,1] \to \mathbb{R}$ by $g_1(x) = x^2$

$g_2 : [0,1] \to [2,3]$ by $g_2(x) = x + 2$

What is $g_1 \circ g_2$?



$f \circ g$ is possible only if $\text{Rang}(g) \subseteq \text{domain}(f)$

Theorem:- Let V and W be vector spaces with basis α and β respectively and let $S, T : V \to W$ be linear transformations then

$$[S+T]_\alpha^\beta = [S]_\alpha^\beta + [T]_\alpha^\beta$$

$$[kS]_\alpha^\beta = k[S]_\alpha^\beta$$

If T is invertible

$$[T^{-1}]_\alpha^\beta = \left([T]_\alpha^\beta\right)^{-1}$$

Let V and W be two vector spaces. Let $L(V,W)$ denote set of all linear transformations from V to W.

$$L(V,W) = \{T \mid T \text{ is a linear transformation from } V \text{ to } W\}$$

For $S, T \in L(V,W)$ define $S+T$ and kS by

$$(S+T)(v) = S(v) + T(v)$$
$$kS(v) = k\,S(v)$$

clearly $S+T$ and kS belongs to $L(V,W)$. So $L(V,W)$ becomes a vector space.

Theorem:- Let V, W and Z be vector spaces with ordered basis α, β, and γ respectively. Suppose that $S : V \to W$ and $T : W \to Z$ are linear transformations then

$$[T \circ S]_\alpha^\gamma = [T]_\beta^\gamma [S]_\alpha^\beta$$

Note:- Let V be a vector space with $\dim m$, W be a vector space with $\dim n$ then $\dim L(V,W) = mn = \dim V \cdot \dim W$.

Result:- If $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ then $L(V,W)$ is isomorphic to $M_{m\times n}(\mathbb{R})$

Problem:1 Let $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$T_1(a,b,c) = (3a, b+c)$; $T_2(a,b,c) = (2a-3c, b)$

Compute $T_1 + T_2$, $5T_1$, $4T_1 - 5T_2$, $T_1 \circ T_2$ and $T_2 \circ T_1$.

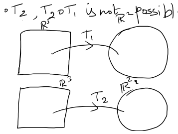
Soln:-

$(T_1+T_2)(a,b,c) = T_1(a,b,c) + T_2(a,b,c)$
$= (3a, b+c) + (2a-3c, b)$
$= (5a-3c, 2b+c)$

$5T_1(a,b,c) = (15a, 5b+5c)$

$(4T_1 - 5T_2)(a,b,c) = 4T_1(a,b,c) - 5T_2(a,b,c)$
$= (2a+15c, -b+4c)$

$T_1 \circ T_2$, $T_2 \circ T_1$ is not possible.



Pb: 2 Let α be the standard basis of $\mathbb{R}^3$ and let $S, T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$S(e_1) = (2,2,1)$ $\quad T(e_1) = (1,0,1)$

$S(e_2) = (0,1,2)$ $\quad T(e_2) = (0,1,1)$

$S(e_3) = (-1,2,1)$ $\quad T(e_3) = (1,1,2)$

Find $[S+T]_\alpha^\alpha$, $[2T-S]_\alpha^\alpha$; $[T\circ S]_\alpha^\alpha$

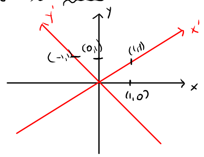
Soln:- $[S]_\alpha^\alpha = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, $[T]_\alpha^\alpha = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$$[S+T]_\alpha^\alpha = [S]_\alpha^\alpha + [T]_\alpha^\alpha = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix}$$

$$[2T-S]_\alpha^\alpha = 2[T]_\alpha^\alpha - [S]_\alpha^\alpha = \begin{bmatrix} 0 & 0 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$[T\circ S]_\alpha^\alpha = [T]_\alpha^\alpha [S]_\alpha^\alpha = \_\_\_\_\_$$

Change of Basis



Transition matrix:- Let α and β be two bases for a vector space V. Consider the identity transformation $Id_V : V \longrightarrow V$. The matrix representation $[Id_V]_\beta^\alpha$ is called the transition matrix (or) co-ordinate change matrix from β to α.

Pb1 Consider the following two bases of $\mathbb{R}^2$
$\alpha = \{(1,2), (3,5)\}$; $\beta = \{(1,-1), (1,-2)\}$
Find the transition matrix β to α.

Soln:- Define $Id : \mathbb{R}^2 \to \mathbb{R}^2$ by $Id(x,y) = (x,y)$

$Id(1,-1) = (1,-1) = (-8)(1,2) + 3(3,5)$

$Id(1,-2) = (1,-2) = (-11)(1,2) + 4(3,5)$

$$[Id]_\beta^\alpha = \begin{bmatrix} -8 & -11 \\ 3 & 4 \end{bmatrix}$$

Pb:2 Consider the following two bases of $\mathbb{R}^3$

$E = \{(1,0,0), (0,1,0), (0,0,1)\}$

$F = \{(1,0,1), (2,1,2), (1,2,2)\}$

Find the transition matrix from E to F.

Soln:- Define $Id : \mathbb{R}^3 \to \mathbb{R}^3$ by $Id(x,y,z) = (x,y,z)$

$Id(1,0,0) = (1,0,0) = (-2)(1,0,1) + 2\,(2,1,2) + (-1)\,(1,2,2)$

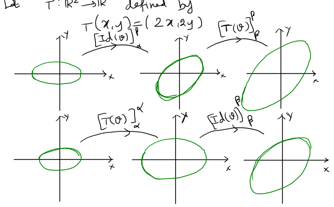
$Id(0,1,0) = (0,1,0) = (-2)(1,0,1) + 1\,(2,1,2) + 0\,(1,2,2)$

$Id(0,0,1) = (0,0,1) = c_1\,(1,0,1) + c_2(2,1,2) + c_3(1,2,2)$

$$[Id]_E^F = \begin{bmatrix} 2 & -2 & c_1 \\ 2 & 1 & c_2 \\ -1 & 0 & c_3 \end{bmatrix}$$

Similarity Transformations:-

Consider the vector space $\mathbb{R}^2$ with bases
$\alpha = \{(1,0), (0,1)\}$; $\beta = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}$.
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by
$T(x,y) = (2x, 2y)$



$$[T(\theta)]_\beta^\beta \circ [Id(\theta)]_\alpha^\beta = [Id(\theta)]_\alpha^\beta\, [T(\theta)]_\alpha^\alpha$$

$$\Rightarrow [T(\theta)]_\beta^\beta = [Id(\theta)]_\alpha^\beta\, [T(\theta)]_\alpha^\alpha \left([Id(\theta)]_\alpha^\beta\right)^{-1}$$

$$[T]_\beta^\beta = Q^{-1} [T]_\alpha^\alpha\, Q$$

Theorem:- Let $T : V \to V$ be a linear transformation. Consider the bases α and β on V. Let $Q = [Id]_\beta^\alpha$ be the transition matrix from β to α, then

(i) Q is invertible

(ii) For any $x \in V$, $[x]_\alpha = Q\,[x]_\beta$

(iii) $[T]_\beta^\beta = Q^{-1} [T]_\alpha^\alpha\, Q$

Definition:- For any square matrix A and B, A is said to be similar to B if there exists a non singular matrix Q such that
$$B = Q^{-1} A Q.$$

Theorem:- Let $T: V \to V$ be a linear transformation. Consider the bases α and β on V. Let $Q = [Id]^{\alpha}_{\beta}$ be the transition matrix from β to α then

(i) Q is invertible.

(ii) For any $x \in V$, $[x]_{\alpha} = Q[x]_{\beta}$

(iii) $[T]^{\beta}_{\beta} = Q^{-1}[T]^{\alpha}_{\alpha} Q$.

Definition:- For any square matrix A and B. A is said to be similar to B if there exist a non singular matrix Q such that

$$B = Q^{-1} A Q.$$

Calculation of Q^{-1}

$Id(1) = 1 = 1\cdot 1 + 0\cdot(2x) + 0\cdot(4x^2-2)$

$Id(x) = x = 0\cdot 1 + \frac{1}{2}(2x) + 0\cdot(4x^2-2)$

$Id(x^2) = x^2 = \frac{1}{2}\cdot 1 + 0\cdot(2x) + \frac{1}{4}(4x^2-2)$

$$Q^{-1} = [Id]^{\beta}_{\alpha} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

Pb.1 Let $D: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ such that

$$D(p(x)) = \frac{dp(x)}{dx}.$$

Find $[D]^{\alpha}_{\alpha}$ and $[D]^{\beta}_{\beta}$ for $\alpha = \{1, x, x^2\}$, $\beta = \{1, 2x, 4x^2-2\}$ by using similarity transformation.

Soln:- We know $[D]^{\beta}_{\beta} = Q^{-1}[D]^{\alpha}_{\alpha} Q$ where $Q = [Id]^{\alpha}_{\beta}$ and $Q^{-1} = [Id]^{\beta}_{\alpha}$.

Calculation of $[D]^{\alpha}_{\alpha}$:-

$D(1) = 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2$

$D(x) = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$

$D(x^2) = 2x = 0\cdot 1 + 2\cdot x + 0\cdot x^2$

$$[D]^{\alpha}_{\alpha} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Calculation of $[D]^{\beta}_{\beta}$

$$[D]^{\beta}_{\beta} = Q^{-1}[D]^{\alpha}_{\alpha} Q$$

$$= \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Calculation of Q:- Define $Id: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ by $Id(p(x)) = p(x)$

$Id(1) = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$

$Id(2x) = 2x = 0\cdot 1 + 2x + 0\cdot x^2$

$Id(4x^2-2) = 4x^2-2 = (-2)\cdot 1 + 0\cdot x + 4\cdot x^2$

$$Q = [Id]^{\alpha}_{\beta} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Pb.2 $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $T(x,y,z) = (x+2y+z, -y, x+4z)$. Find the matrix $[T]^{\alpha}_{\alpha}$ and $[T]^{\beta}_{\beta}$ using similarity where $\alpha = \{(1,0,0), (0,1,0), (0,0,1)\}$

$\beta = \{(1,0,0), (1,1,0), (1,1,1)\}$

Soln:- $[T]^{\beta}_{\beta} = Q^{-1}[T]^{\alpha}_{\alpha} Q$ where $Q = [Id]^{\alpha}_{\beta}$; $Q^{-1} = [Id]^{\beta}_{\alpha}$

Calculating $[T]^{\alpha}_{\alpha}$

$T(1,0,0) = (1,0,1) = 1\cdot(1,0,0) + 0\cdot(0,1,0) + 1\cdot(0,0,1)$

$T(0,1,0) = (2,-1,0) = (2)(1,0,0) + (-1)(0,1,0) + 0\cdot(0,0,1)$

$T(0,0,1) = (1,0,4) = 1(1,0,0) + 0\cdot(0,1,0) + 4(0,0,1)$

$$[T]^{\alpha}_{\alpha} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

Calculation of Q :-

Define $Id : \mathbb{R}^3 \to \mathbb{R}^3$ by $Id(x, y, z) = (x, y, z)$

$Id(1,0,0) = (1,0,0)$

$Id(1,1,0) = (1,1,0)$

$Id(1,1,1) = (1,1,1)$

$$\therefore Q = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = [Id]_\beta^\alpha$$

$Id(1,0,0) = (1,0,0) = 1\cdot(1,0,0) + 0\cdot(1,1,0) + 0\cdot(1,1,1)$

$Id(0,1,0) = (0,1,0) = (-1)(1,0,0) + 1(1,1,0) + 0\cdot(1,1,1)$

$Id(0,0,1) = (0,0,1) = 0\cdot(1,0,0) + (-1)(1,1,0) + 1(1,1,1)$

$$Q^{-1} = [Id]_\alpha^\beta = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T]_\beta^\beta = Q^{-1}[T]_\alpha^\alpha Q$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 5 \\ -1 & -2 & -6 \\ 1 & 1 & 5 \end{bmatrix} \quad //$$

(Try!) Let $G : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $G(x, y, z) = (2y + z, x - 4y, 3x)$

Let $\alpha = \{(1,0,0), (0,1,0), (0,0,1)\}$

$\beta = \{(1,0,0), (1,1,0), (1,1,1)\}$

Find $[G]_\alpha^\alpha$, $[G]_\beta^\beta$ using similarity transformation.

QUESTION BANK FOR MODULE 3

1) Consider the matrix $= \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}$.
   a) Determine whether $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in the column space of A?
   b) Determine whether $w = [4 \ \ 5]$ is in the row space of A?
   c) Find row space and column space of A?
2) Let u, v be two vectors in $\mathbb{R}^{10}$. If $A = uv^T$ then find the dimension of the row space of A?
3) Let A be a 4×3 matrix and suppose that the vectors $u = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ form a basis for null space of A. If $b = a_1 + 2a_2 + a_3$, find all solutions of the system $AX = b$. Here $a_1, a_2, a_3$are first, second and third columns of A.
4) Let A be a $m \times n$ matrix whose rank is n. If $Ac = Ad$, does this imply that $c = d$? What if the rank of A is less than n? Explain your answer.
5) Does the matrix $A = \begin{bmatrix} -2 & -2 & 1 & 3 \\ 3 & 3 & 0 & -1 \\ -1 & -1 & -2 & -5 \\ 2 & 2 & 1 & 1 \end{bmatrix}$ invertible?
6) Does the matrix $A = \begin{bmatrix} 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 \\ 10 & 9 & 8 & 7 & 6 & 5 & 4 \\ 11 & 10 & 9 & 8 & 7 & 6 & 5 \\ 12 & 11 & 10 & 9 & 8 & 7 & 6 \\ 13 & 12 & 11 & 10 & 9 & 8 & 7 \end{bmatrix}$ invertible?
7) Find the basis and dimension of the row space, column space and null space of the following matrix

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix}.$$

Answer : Basis for row space {(1,0,1,0,-1), (0,1,2,0,3),(0,0,0,1,4)} dimension of row space=3. Basis for column space {(1,2,-3,4),(1,-1,2,1),(1,1,-2,1)} dimension of column space=3. . Basis for null space {(1,-2,1,0,0),(1,-3,0,-4,1)} dimension of null space=2.

8) Find the basis and dimension of the row space, column space and null space of the following matrix

$$A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix}.$$

Answer : Basis for row space {(1,1,4,1,2), (0,1,2,1,1),(0,0,0,1,2)} dimension of row space=3. Basis for column space {(1,0,0,1,2),(1,1,0,-1,1),(1,1,1,0,0)} dimension of column space=3. .

9) Consider the vector space $\mathbb{R}^5$ and the subspaces $U = \{(x, y, z, t, w) \in \mathbb{R}^5 : 2x - y - z = 0, t - w = 0\}$ and $V == \{(x, y, z, t, w) \in \mathbb{R}^5 : z + t = 0\}$. Find a basis and dimension of the subspaces $U + V$ and $U \cap V$.

Answer : Basis for $U \cap V$ is {(1,2,0,0,0), (-3,0,-6,6,3)} dimension of $U \cap V$ =2. $U + V = \mathbb{R}^5$.

10) Consider the vector space $\mathbb{R}^4$ and the subsapces $U = \{(a, b, c, d) \in \mathbb{R}^4 : b - 2c + d = 0\}$ and $V = \{(a, b, c, d) \in \mathbb{R}^4 : a = d, b = 2c\}$. Find a basis and dimension of the subspaces $U + V$ and $U \cap V$.

QUESTION BANK MODULE 4

1) Which of the following is a linear transformation
   a) $T: \mathbb{R}^2 \to \mathbb{R}^2$ $defined\ by$ $T(x, y) = (x^2 + y^2, x^2 - y^2)$.
   b) $T: \mathbb{R}^3 \to \mathbb{R}^3$ $defined\ by$ $T(x, y, z) = (x + y, 0, 2x + 4z)$.
   c) $T: \mathbb{R}^3 \to \mathbb{R}^2$ $defined\ by$ $T(x, y, z) = (|x|, 0)$.
   d) $T: \mathbb{R}^2 \to \mathbb{R}^3$ $defined\ by$ $T(x, y) = (x + 1, 2y, x + y)$.
2) Let $T: \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ be a linear transformation such that $(1) = 1, T(x) = x^2$ and $T(x^3) = x^3 + x$. Find $T(ax^2 + bx + c)$.
3) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. If $T(1,2) = (-2,3), T(1,-1) = (5,2)$ then find $T(x, y)$.
4) Which of the following is incorrect.
   a) The map $T: \mathbb{R}^3 \to \mathbb{R}^2$ $such\ that$ $T(1,0,0) = (1,2)$, $T(0,1,0) = (3,4)$ and $T(0,0,1) = (5,6)$is a linear transformation.
   b) The map$T: \mathbb{R}^3 \to \mathbb{R}^2$ $such\ that$ $T(1,0,0) = (11,2)$, $T(0,1,0) = (-3 - ,4)$ and $T(0,0,1) = (5,16)$is a linear transformation.
   c) The map $T: \mathbb{R}^3 \to \mathbb{R}^2$ $such\ that$ $T(1,0,0) = (1,2)$, $T(2,0,0) = (3,4)$ and $T(0,0,1) = (5,6)$ is a linear transformation.
   d) The map$T: \mathbb{R}^3 \to \mathbb{R}^2$ $such\ that$$T(1,0,0) = (1,12)$, $T(0,1,0) = (3,14)$ and $T(0,0,1) = (15,6)$is a linear transformation.
5) Determine the kernel and image of the following linear transformations
   a) $T: \mathbb{R}^3 \to \mathbb{R}^3$ $defined\ by$ $T(x, y, z) = (z, y, x)$.
   b) $T: \mathbb{R}^3 \to \mathbb{R}^3$ $defined\ by$ $T(x, y, z) = (x, x, x)$.
   c) $T: \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ $defined\ by$ $T(p(x)) = p(0)x + p(1)$.
6) Let $T: \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ $defined\ by$ $T(p(x)) = p''(x) - 4p'(x) + p(x)$. Find the matrix $[T]_\alpha{}^\alpha$ where $\alpha = \{x, 1 + x, x + x^2, x^3\}$.
7) Find the matrix representation of the following transformations T on $\mathbb{R}^2$ with respect to the standard basis $\{e_1, e_2\}$.
   a) $T: \mathbb{R}^2 \to \mathbb{R}^2$ $defined\ by$ $T(x, y) = (2y, 3x - y)$.
   b) $T: \mathbb{R}^2 \to \mathbb{R}^2$ $defined\ by$ $T(x, y) = (3x - 4y, x + 5y)$.
8) Find the transition matrix of the $[Id]^\beta{}_\alpha$ from α to β when
   a) $\alpha = \{(2,3), (0,1)\}$$and$ $\beta = \{(6,4), (4,8)\}$
   b) $\alpha = \{(1,1,1), (1,1,0), (1,0,0)\}$$and$ $\beta = \{(2,0,3), (-1,4,1), (3,2,5)\}$
   c) $\alpha = \{1, t, t^2\}$$and$ $\beta = \{3 + 2t + t^2, t^2 - 4, 2 + t\}$
9) Let $T: \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ defined by $T(p(x)) = xp'(x) + p''(x)$.
   a) Find the matrix A of T relative to the standard basis of $\mathcal{P}_3(\mathbb{R})$.
   b) Find the matrix B of T relative to the basis $\{1, x, 1 + x^2\}$of $\mathcal{P}_3(\mathbb{R})$.
   c) Find the matrix S such that $B = S^{-1}AS$.
10) Let $= \begin{bmatrix} 4 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$. Find a unique linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ so that M is the matrix of T with respect to the basis $\alpha = \{(1,0,0), (1,1,0), (1,1,1)\}$ and $\beta = \{(1,0), (1,1)\}$. Find $T(x, y, z)$.

Module - 6

Inner product spaces

Consider the vector space $\mathbb{R}^2$ (2-dimensional plane)

Take $\vec{a} = x_1\hat{i} + y_1\hat{j} = (x_1, y_1) = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = X$

$\vec{b} = x_2\hat{i} + y_2\hat{j} = (x_2, y_2) = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = Y$

Dot product :- $\vec{a}\cdot\vec{b} = (x_1\hat{i} + y_1\hat{j})\cdot(x_2\hat{i} + y_2\hat{j})$

$= x_1x_2 + y_1y_2 = X^T Y$

1) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (Commutativity)

2) Let $\vec{c} = x_3\hat{i} + y_3\hat{j}$

$(\vec{a}+\vec{b})\cdot\vec{c} = \left[(x_1+x_2)\hat{i} + (y_1+y_2)\hat{j}\right](x_3\hat{i} + y_3\hat{j})$

$= (x_1+x_2)x_3 + (y_1+y_2)y_3$

$\vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c} = (x_1x_3 + y_1y_3) + (x_2x_3 + y_2y_3)$

Hence, $(\vec{a}+\vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$ (distribute law)

Perpendicular vectors

Consider $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^2$. We say $\vec{a}$ and $\vec{b}$ are perpendicular if

$\boxed{\vec{a}\cdot\vec{b} = 0}$ (or) $\theta = 90°$

Examples:- $\vec{a} = \hat{i}$; $\vec{b} = \hat{j}$

3) $\vec{a}\cdot\vec{a} = (x_1\hat{i} + y_1\hat{j})\cdot(x_1\hat{i} + y_1\hat{j}) = x_1^2 + y_1^2 \geq 0$

If $\vec{a}\cdot\vec{a} = 0$ then $\vec{a} = \vec{0}$

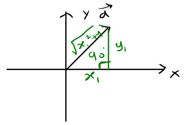
If $\vec{a} = \vec{0}$ then $\vec{a}\cdot\vec{a} = \vec{0}$

Length of a vector (or) magnitude of a vector.

Let $\vec{a}$ be a vector in $\mathbb{R}^2$. $\vec{a} = x_1\hat{i} + y_1\hat{j}$

length $= \sqrt{x_1^2 + y_1^2}$

$= \sqrt{\vec{a}\cdot\vec{a}}$



Suppose $X = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

length $= \sqrt{X^T X}$

Notation for length of $\vec{a}$ is $\|\vec{a}\|$

Inner product space definition:-

An inner product on a real vector space V is a function that associates a real number $\boxed{\langle x, y\rangle}$ to each pair of vectors x and y in V in such a way that the following rules are satisfied for all vectors x, y and z in V and all scalars k in $\mathbb{R}$

(i) $\langle x, y\rangle = \langle y, x\rangle$

(ii) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

(iii) $\langle kx, y\rangle = k\langle x, y\rangle$

(iv) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$

Angle between two vectors:-

Consider $\vec{a} = x_1\hat{i} + y_1\hat{j}$; $\vec{b} = x_2\hat{i} + y_2\hat{j}$

$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$ (Here θ is the angle between the vectors $\vec{a}$ and $\vec{b}$)

So, $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}$

$$\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}\right)$$

Inner product space :-

A pair $(V, \langle,\rangle)$ of a vector space V and an inner product $\langle,\rangle$ is called a inner product space.

Example:- Consider $\mathbb{R}^2$ define

$\langle X, Y\rangle = X^T Y = x_1y_1 + x_2y_2$

where $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

We already verified $\langle X, Y\rangle = \langle Y, X\rangle$

$\langle X+Y, Z\rangle = \langle X, Z\rangle + \langle Y, Z\rangle$

$\langle kX, Y\rangle = k\langle X, Y\rangle$

$\langle X, X\rangle \geq 0$ and, $\langle X, X\rangle = 0 \Leftrightarrow X = 0$

$\therefore \mathbb{R}^2$ is an inner product space with the above inner product.

Inner product space definition:-

An inner product on a real vector space V is a function that associates a real number $\boxed{\langle x,y\rangle}$ to each pair of vectors x and y in V in such a way that the following rules are satisfied for all vectors x, y and z in V and all scalars k in $\mathbb{R}$

(i) $\langle x,y\rangle = \langle y,x\rangle$

(ii) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

(iii) $\langle kx, y\rangle = k\langle x,y\rangle$

(iv) $\langle x, x\rangle \geq 0$ and $\langle x,x\rangle = 0$ if and only if $x=0$

Example : 1 For $\bar{x} = (x_1, x_2, x_3)$ and $\bar{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$ define

$$\langle \bar{x}, \bar{y}\rangle = x_1y_1 + 3x_2y_2 + 5x_3y_3$$

Prove $\langle\, ,\rangle$ is an inner product on $\mathbb{R}^3$.

Soln:- $\langle \bar{x}, \bar{y}\rangle = x_1y_1 + 3x_2y_2 + 5x_3y_3$

$$\langle \bar{y}, \bar{x}\rangle = \langle (y_1,y_2,y_3), (x_1,x_2,x_3)\rangle$$

$$= y_1x_1 + 3y_2x_2 + 5y_3x_3 = \langle \bar{x}, \bar{y}\rangle$$

$\Rightarrow \langle \bar{x}, \bar{y}\rangle = \langle \bar{y}, \bar{x}\rangle$ property ① is satisfied.

Take $\bar{z} = (z_1, z_2, z_3)$

$$\langle \bar{x}+\bar{y}, \bar{z}\rangle = \langle (x_1,x_2,x_3) + (y_1,y_2,y_3), (z_1,z_2,z_3)\rangle$$

$$= \langle (x_1+y_1, x_2+y_2, x_3+y_3), (z_1,z_2,z_3)\rangle$$

$$= (x_1+y_1)z_1 + 3(x_2+y_2)z_2 + 5(x_3+y_3)z_3$$

$$= (x_1z_1 + 3x_2z_2 + 5x_3z_3) + (y_1z_1 + 3y_2z_2 + 5y_3z_3)$$

$$= \langle \bar{x}, \bar{z}\rangle + \langle \bar{y}, \bar{z}\rangle$$

property ② is verified.

let $k \in \mathbb{R}$,

$$\langle k\bar{x}, \bar{y}\rangle = \langle k(x_1,x_2,x_3), (y_1,y_2,y_3)\rangle$$

$$= \langle (kx_1, kx_2, kx_3), (y_1,y_2,y_3)\rangle$$

$$= kx_1y_1 + 3kx_2y_2 + 5kx_3y_3$$

$$= k(x_1y_1 + 3x_2y_2 + 5x_3y_3)$$

$= k\langle \bar{x}, \bar{y}\rangle$ property ③ is verified.

$$\langle \bar{x}, \bar{x}\rangle = \langle (x_1,x_2,x_3), (x_1,x_2,x_3)\rangle$$

$$= x_1^2 + 3x_2^2 + 5x_3^2 \geq 0$$

$$\langle \bar{x}, \bar{x}\rangle = 0 \iff x_1^2 + 3x_2^2 + 5x_3^2 = 0$$

$$\iff x_1 = x_2 = x_3 = 0$$

$$\iff \bar{x} = (0,0,0)$$

Hence $\langle\, ,\rangle$ defines an inner product.

Example 2:- Let $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$

Define $\langle \bar{x}, \bar{y}\rangle = x_1y_2 + x_2y_1$

check $\langle\, ,\rangle$ defines an inner product on $\mathbb{R}^2$.

Soln:-

It is easy to see that $\langle \bar{x}, \bar{y}\rangle = \langle \bar{y}, \bar{x}\rangle$

$$\langle \bar{x}+\bar{y}, \bar{z}\rangle = \langle (x_1,x_2)+(y_1,y_2), (z_1,z_2)\rangle$$

$$= \langle (x_1+y_1, x_2+y_2), (z_1,z_2)\rangle$$

$$= (x_1+y_1)z_2 + (x_2+y_2)z_1$$

$$= (x_1z_2 + x_2z_1) + (y_1z_2 + y_2z_1)$$

$$= \langle \bar{x}, \bar{z}\rangle + \langle \bar{y}, \bar{z}\rangle$$

It is easy to see that $\langle k\bar{x}, \bar{y}\rangle = k\langle \bar{x}, \bar{y}\rangle$

$$\langle \bar{x}, \bar{x}\rangle = \langle (x_1,x_2), (x_1,x_2)\rangle$$

$$= x_1x_2 + x_1x_2 = 2x_1x_2$$

This need not be greater than or equal to zero

For example take $\bar{x} = (1, -2)$

$$\langle \bar{x}, \bar{x}\rangle = -4 < 0$$

Given product is not an inner product on $\mathbb{R}^2$.

Example 3:- For $\bar{x} = (x_1, x_2, x_3)$ and $\bar{y} = (y_1, y_2, y_3)$

define $\langle \bar{x}, \bar{y}\rangle = x_1y_3 - x_2y_1 - x_3y_2$

check $\langle\, ,\rangle$ is an inner product on $\mathbb{R}^3$.

Soln:- We prove that $\langle \bar{x}, \bar{y}\rangle \neq \langle \bar{y}, \bar{x}\rangle$.

Take $\bar{x} = (1,0,0)$; $\bar{y} = (0,1,0)$

$$\langle \bar{x}, \bar{y}\rangle = x_1y_3 - x_2y_1 - x_3y_1 = 0$$

$$\langle \bar{y}, \bar{x}\rangle = y_1x_3 - y_2x_1 - y_3x_2 = -1$$

$\therefore$ It is not an inner product.

Try! For $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$ in $\mathbb{R}^2$ define $\langle \bar{x}, \bar{y} \rangle = x_1 y_1 + 3 x_2 y_2$. check $\langle \, , \rangle$ defines an inner product on $\mathbb{R}^2$.

Inner product space definition:-

An inner product on a real vector space V is a function that associates a real number $\boxed{\langle x, y \rangle}$ to each pair of vectors x and y in V in such a way that the following rules are satisfied for all vectors x, y and z in V and all scalars k in $\mathbb{R}$

(i) $\langle x, y \rangle = \langle y, x \rangle$

(ii) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(iii) $\langle kx, y \rangle = k \langle x, y \rangle$

(iv) $\langle x, x \rangle \geqslant 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$

Example:- Consider the vector space $\mathbb{R}^n$.

For $\bar{x} = (x_1, x_2, x_3, \dots, x_n)$ and $\bar{y} = (y_1, y_2, \dots y_n)$ define $\langle \bar{x}, \bar{y} \rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$ on $\mathbb{R}^n$.

Then $\langle \, , \rangle$ is an inner product on $\mathbb{R}^n$.

Example 2:- Consider the vector space $P_n(\mathbb{R})$

[Vector space of all polynomials of degree atmost n]

For $p(x)$ and $q(x)$ in $P_n(\mathbb{R})$ define

$$\langle p(x), q(x) \rangle = \int_0^1 p(x) q(x) \, dx$$

then $\langle \, , \rangle$ is an innerproduct on $P_n(\mathbb{R})$.

Proof (i) $\langle p(x), q(x) \rangle = \int_0^1 p(x) q(x) \, dx$

$$\langle q(x), p(x) \rangle = \int_0^1 q(x) p(x) \, dx = \int_0^1 p(x) q(x) \, dx = \langle p(x), q(x) \rangle$$

property (1) is verified.

(ii) Take $r(x) \in P_n(\mathbb{R})$.

$$\langle p(x) + q(x), r(x) \rangle = \int_0^1 [p(x) + q(x)] \, r(x) \, dx$$

$$= \int_0^1 [p(x) r(x) + q(x) r(x)] \, dx$$

$$= \int_0^1 p(x) r(x) \, dx + \int_0^1 q(x) r(x) \, dx$$

$$= \langle p(x), r(x) \rangle + \langle q(x), r(x) \rangle$$

property (2) is verified.

3) Let $k \in \mathbb{R}$,

$$\langle k p(x), q(x) \rangle = \int_0^1 k p(x) q(x) \, dx = k \int_0^1 p(x) q(x) \, dx$$

$$= k \langle p(x), q(x) \rangle$$

[Recall:- Let f be a real valued function defined on $\mathbb{R}$, Suppose $f(x) \geqslant 0$ for all $x \in \mathbb{R}$ then $\int_0^1 f(x) \, dx \geqslant 0$]

$$\langle p(x), p(x) \rangle = \int_0^1 p(x) \cdot p(x) \, dx = \int_0^1 [p(x)]^2 \, dx \geqslant 0$$

$$\langle p(x), p(x) \rangle = 0 \iff \int_0^1 [p(x)]^2 \, dx = 0$$

$$\iff p(x) = 0$$

Hence property (4) is also verified.

Hence $\langle \, , \rangle$ is an inner product.

Properties:- Let V be an inner product space with inner product $\langle \, , \rangle$ then

(a) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$

(b) $\langle x, ky \rangle = k \langle x, y \rangle$

proof:-

(a) $\langle x, y + z \rangle = \langle y + z, x \rangle = \langle y, x \rangle + \langle z, x \rangle$

$= \langle x, y \rangle + \langle x, z \rangle$

(b) $\langle x, ky \rangle = \langle ky, x \rangle = k \langle y, x \rangle = k \langle x, y \rangle$.

Note:- Let x and y be ~~two~~ vectors in an inner product space V then

$$\langle \vec{0}, x \rangle = \langle y, \vec{0} \rangle = 0.$$

Magnitude (or) length of a vector:-

Let V be an inner product space. Then the magnitude (or) the length of a vector x denoted by $\|x\|$ is defined by

$$\|x\| = \sqrt{\langle x, x\rangle}$$

The distance between two vectors x and y denoted by

$$d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y\rangle}$$

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Example:- Consider the inner product space $\mathbb{R}^2$ with inner product

$$\langle \bar{x}, \bar{y}\rangle = x_1y_1 + x_2y_2$$

where $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$ on $\mathbb{R}^2$.
Find the length of vector $(3,5)$ and distance between vectors $(1,0)$, $(2,7)$.

Soln:- Take $\bar{x} = (3,5)$

$$\text{length} = \sqrt{\langle \bar{x}, \bar{x}\rangle} = \sqrt{\langle (3,5),(3,5)\rangle} = \sqrt{9+25}$$
$$= \sqrt{34} \text{ units.}$$

Take $\bar{x} = (1,0)$, $\bar{y} = (2,7)$.

$$\text{distance} = \|\bar{x}-\bar{y}\| = \sqrt{\langle \bar{x}-\bar{y}, \bar{x}-\bar{y}\rangle}$$
$$= \sqrt{\langle (1,0)-(2,7), (1,0)-(2,7)\rangle}$$
$$= \sqrt{\langle (-1,-7), (-1,-7)\rangle}$$
$$= \sqrt{1+49} = \sqrt{50} \text{ units.}$$

Angle between the given vectors:-

The real number θ in the interval $[0,\pi]$ that satisfies

$$\cos\theta = \frac{\langle x, y\rangle}{\|x\|\,\|y\|}$$

is the angle between x and y.

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$
$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

Inner product space definition:-

An inner product on a real vector space V is a function that associates a real number $\boxed{\langle x,y\rangle}$ to each pair of vectors x and y in V in such a way that the following rules are satisfied for all vectors x, y and z in V and all scalars k in $\mathbb{R}$

(i) $\langle x, y\rangle = \langle y, x\rangle$

(ii) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

(iii) $\langle kx, y\rangle = k\langle x, y\rangle$

(iv) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$

Pb:1 For $\bar{x} = (x_1, x_2, x_3)$ and $\bar{y} = (y_1, y_2, y_3)$ on $\mathbb{R}^3$ define $\langle \bar{x}, \bar{y}\rangle = x_1y_1 + 3x_2y_2 + 5x_3y_3$. Then $\langle , \rangle$ is an inner product on $\mathbb{R}^3$. Find the angle between $(1,2,3)$ and $(2,-1,3)$.

Soln:-

We know that the angle between the given vectors can be calculated by

$$\cos\theta = \frac{\langle x, y\rangle}{\|x\|\,\|y\|}$$

In our case $x = (1,2,3)$ and $y = (2,-1,3)$

$$\langle x, y\rangle = x_1y_1 + 3x_2y_2 + 5x_3y_3 = (1\cdot 2) + 3(2\cdot -1) + 5(3\cdot 3)$$
$$= 2 - 6 + 45 = 41$$

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{1\cdot 1 + 3(2\cdot 2) + 5(3\cdot 3)} = \sqrt{1+12+45} = \sqrt{58}$$

$$\|y\| = \sqrt{\langle y, y\rangle} = \sqrt{2\cdot 2 + 3(-1\cdot -1) + 5(3\cdot 3)} = \sqrt{52}$$

$$\cos\theta = \frac{41}{\sqrt{52}\,\sqrt{58}} = \frac{41}{2\sqrt{13}\,\sqrt{58}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{41}{2\sqrt{13}\,\sqrt{58}}\right)}$$

Pb 2 Consider the inner product space $P_n(\mathbb{R})$ with inner product
$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx.$$
Find the angle between the vectors $f(x) = x^m$ and $g(x) = x^n$ where m, n are positive integers.

Soln:-
$$\langle f(x), g(x)\rangle = \langle x^m, x^n\rangle = \int_0^1 x^m \cdot x^n\,dx = \int_0^1 x^{m+n}\,dx$$
$$= \left[\frac{x^{m+n+1}}{m+n+1}\right]_0^1$$
$$= \frac{1}{m+n+1}.$$

$$\|f(x)\| = \sqrt{\langle f(x), f(x)\rangle}$$
$$= \sqrt{\int_0^1 x^m \cdot x^m\,dx} = \sqrt{\int_0^1 x^{2m}\,dx} = \sqrt{\left[\frac{x^{2m+1}}{2m+1}\right]_0^1}$$
$$= \sqrt{\frac{1}{2m+1}} = \frac{1}{\sqrt{2m+1}}.$$
$$\|g(x)\| = \sqrt{\langle g(x), g(x)\rangle}$$
$$= \sqrt{\langle x^n, x^n\rangle} = \sqrt{\int_0^1 x^{2n}\,dx} = \sqrt{\left[\frac{x^{2n+1}}{2n+1}\right]_0^1}$$
$$= \frac{1}{\sqrt{2n+1}}.$$

We have
$$\cos\theta = \frac{\langle f(x), g(x)\rangle}{\|f(x)\|\,\|g(x)\|}$$
$$= \frac{\frac{1}{m+n+1}}{\frac{1}{\sqrt{2m+1}}\cdot\frac{1}{\sqrt{2n+1}}} = \frac{\sqrt{(2m+1)(2n+1)}}{m+n+1}$$
$$\theta = \cos^{-1}\left(\frac{\sqrt{(2m+1)(2n+1)}}{m+n+1}\right)$$

Pb 3 Let x and y be vectors in $\mathbb{R}^n$ such that $\|x\| = \|y\| = 1$ and their inner product $\langle x, y\rangle = \frac{-1}{2}$.
Determine $\|x - y\|$

Soln:-
$$\|x-y\| = \sqrt{\langle x-y, x-y\rangle} = \sqrt{\langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle}$$
$$= \sqrt{\|x\|^2 - 2\langle x,y\rangle + \|y\|^2} = \sqrt{1 - 2\left(\frac{-1}{2}\right) + 1}$$
$$= \sqrt{3}.$$

Orthogonal vectors:-

Let V be an inner product space. Two vectors x and y in an inner product space is said to be orthogonal if $\langle x, y\rangle = 0$

Example:- (i) Consider the inner product space $\mathbb{R}^2$ with inner product $\langle \bar{x}, \bar{y}\rangle = x_1y_1 + x_2y_2$ where $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$.

Then the vectors $(3, -1)$ and $(1, 3)$ are orthogonal.
$\langle (3,-1), (1,3)\rangle = 3\cdot 1 + (-1)\cdot 3 = 0$

The vectors $(-1, -2)$ and $(-2, 1)$ are orthogonal

Reason:- $\langle (-1,-2), (-2,1)\rangle = 2 - 2 = 0$

Example 2:- Consider the polynomial space $P_6(\mathbb{R})$ with the following inner product
$$\langle f(x), g(x)\rangle = \int_{-1}^1 f(x)\,g(x)\,dx.$$
Check the orthogonality of the following pairs.

(i) $f(x) = x$, $g(x) = x^2$

(ii) $f(x) = x^2$, $g(x) = x^4$.

Soln:-
$$\langle x, x^2\rangle = \int_{-1}^1 x\cdot x^2\,dx = \int_{-1}^1 x^3\,dx = 0$$
Thus x and x^2 are orthogonal
$$\langle x^2, x^4\rangle = \int_{-1}^1 x^2 x^4\,dx = \int_{-1}^1 x^6\,dx = \left[\frac{x^7}{7}\right]_{-1}^1 = \frac{2}{7} \neq 0$$
Thus x^2 and x^4 are not orthogonal.

Question:- Let V be a inner product space. Let x be a vector in V such that

$$\langle x, y\rangle = 0 \quad \text{for all } y \in V$$

[which means x is orthogonal to every vector y in V] then what is x?

Ans:- Given $\langle x, y\rangle = 0$ for all $y \in V$

in particular $\langle x, x\rangle = 0$

By the inner product property $x = 0_V$

Theorem:- Let V be an inner product space and let $\alpha = \{v_1, v_2, \dots v_n\}$ be a basis for V. A vector x in V is orthogonal to every basis vector v_i in α if and only if $x = 0_V$.

Cauchy - Schwartz inequality

If x and y are vectors in an inner product space V then

$$\langle x, y\rangle^2 \leq \langle x, x\rangle\langle y, y\rangle$$

proof:- If $x = 0_V$ (or) $y = 0_V$ then

$$\langle x, y\rangle^2 = 0^2 \text{ and } \langle x, x\rangle\langle y, y\rangle = 0$$

$\therefore$ inequality is true.

Assume $x \neq 0_V$ and $y \neq 0_V$ then for any scalar t we have

$$0 \leq \langle tx+y, tx+y\rangle$$
$$= \langle tx, tx\rangle + \langle tx, y\rangle + \langle y, tx\rangle + \langle y, y\rangle$$
$$= t^2\langle x, x\rangle + t\langle x, y\rangle + t\langle y, x\rangle + \langle y, y\rangle$$
$$0 \leq t^2\langle x, x\rangle + 2t\langle x, y\rangle + \langle y, y\rangle$$

The above inequality implies that the polynomial $\langle x, x\rangle t^2 + 2t\langle x, y\rangle + \langle y, y\rangle$ in t has either no real root or a repeated root. Its discriminant must be non positive.

$$4\langle x, y\rangle^2 - 4\langle x, x\rangle\langle y, y\rangle \leq 0$$
$$\Rightarrow 4\langle x, y\rangle^2 \leq 4\langle x, x\rangle\langle y, y\rangle$$
$$\Rightarrow \langle x, y\rangle^2 \leq \langle x, x\rangle\langle y, y\rangle$$

Cauchy - Schwartz inequality:- Let V be an inner product space. If x and y be ~~two~~ vectors in V then

$$\langle x, y\rangle^2 \leq \langle x, x\rangle\langle y, y\rangle$$

Orthogonal Vectors:- Let V be an inner product space and x and y are ~~two~~ vectors in V. We say x and y are orthogonal if

$$\langle x, y\rangle = 0$$

Theorem:- If $\{x_1, x_2, \cdots x_n\}$ be non zero vectors in an inner product space V are mutually orthogonal then they are linearly independant.

Proof:- Suppose

$$k_1 x_1 + k_2 x_2 + \dots + k_n x_n = 0_V \rightarrow (1)$$

we know that

$$\langle 0_V, x_i\rangle = 0$$

Substitute (1) in the above equation

$$\langle k_1 x_1 + k_2 x_2 + \dots + k_n x_n, x_i\rangle = 0$$
$$\langle k_1 x_1, x_i\rangle + \langle k_2 x_2, x_i\rangle + \dots + \langle k_n x_n, x_i\rangle = 0$$
$$k_1\langle x_1, x_i\rangle + k_2\langle x_2, x_i\rangle + \dots k_i\langle x_i, x_i\rangle + \dots + k_n\langle x_n, x_i\rangle = 0$$

We know $\langle x_1, x_i\rangle = 0, \langle x_2, x_i\rangle = 0, \dots, \langle x_n, x_i\rangle = 0$

but $\langle x_i, x_i\rangle \neq 0$.

Hence, we get $k_i\langle x_i, x_i\rangle = 0$

$$k_i \|x_i\|^2 = 0$$
$$\Rightarrow k_i = 0$$

Since i varies from 1 to k, we have

$$k_1 = k_2 = \dots = k_n = 0$$

Problem:- Let V be an innerproduct space over $\mathbb{R}$. If $u, v \in V$ then prove that

(a) $\langle u, v\rangle \leq \|u\|\,\|v\|$

(b) $\|u+v\| \leq \|u\| + \|v\|$

(c) $\big|\|u\| - \|v\|\big| \leq \|u - v\|$.

proof:-

(a) **Recall:-** $\langle u, v\rangle^2 \leq \langle u, u\rangle\langle v, v\rangle$

Now, $\langle u, v\rangle^2 \leq \langle u, u\rangle\langle v, v\rangle$

$$= \|u\|^2\,\|v\|^2$$

Taking square root on both sides, we have

$$\langle u, v\rangle \leq \|u\|\,\|v\|.$$

(a) is proved.

(b) $\|u+v\|^2 = \langle u+v, u+v\rangle$

$$= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle$$

$$= \|u\|^2 + \langle u,v\rangle + \langle u,v\rangle + \|v\|^2$$

$$= \|u\|^2 + 2\langle u,v\rangle + \|v\|^2 \longrightarrow (2)$$

In (a) we proved $\langle u,v\rangle \leq \|u\|\,\|v\|$

Thus (2) becomes

$$\|u+v\|^2 \leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$$

$$= (\|u\| + \|v\|)^2$$

Taking square root on both sides we have

$$\|u+v\| \leq \|u\| + \|v\|.$$

(c) $\big|\|u\| - \|v\|\big| \leq \|u-v\|$

proof:- $\|u\| = \|u - v + v\|$

$$\leq \|u-v\| + \|v\| \quad (\text{by (b)})$$

$$\Rightarrow \|u\| - \|v\| \leq \|u-v\| \longrightarrow (3)$$

Now, $\|v\| = \|v - u + u\|$

$$\leq \|v-u\| + \|u\| \quad (\text{by (b)})$$

$$\Rightarrow -\|v-u\| \leq \|u\| - \|v\|$$

$$\Rightarrow -\|u-v\| \leq \|u\| - \|v\| \longrightarrow (4)$$

From (3) & (4), we have

$$-\|u-v\| \leq \|u\| - \|v\| \leq \|u-v\|$$

$$\Rightarrow \big|\|u\| - \|v\|\big| \leq \|u-v\|.$$

Remark:- Let V be an inner product space $\alpha = \{v_1, v_2, \ldots, v_n\}$ be a basis for V. Let x and y be two vectors in V then

$$\langle x,y\rangle = [x]_\alpha^T A [y]_\alpha$$

where $[x]_\alpha$ is the co-ordinate matrix of x with respect to the basis α and $[y]_\alpha$ is the co-ordinate matrix of y with respect to the basis α.

The matrix $A = [a_{ij}]$ where $\boxed{a_{ij} = \langle v_i, v_j\rangle}$ is called the matrix representation of the inner product with respect to α

Note:- A is always symmetric.

Pb: 1 Consider the inner product space $\mathbb{R}^2$ with the inner product

$$\langle \bar{x}, \bar{y}\rangle = x_1y_1 + 3x_2y_2 + 5x_3y_3.$$

Find the matrix of the above inner product with respect to the standard basis of $\mathbb{R}^3$.

Solution:- Basis of $\mathbb{R}^3$ is

$$\{e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1)\}.$$

we need to find $A = [a_{ij}]$ such that $a_{ij} = \langle e_i, e_j\rangle$

$$A = \begin{bmatrix} \langle e_1,e_1\rangle & \langle e_1,e_2\rangle & \langle e_1,e_3\rangle \\ \langle e_2,e_1\rangle & \langle e_2,e_2\rangle & \langle e_2,e_3\rangle \\ \langle e_3,e_1\rangle & \langle e_3,e_2\rangle & \langle e_3,e_3\rangle \end{bmatrix}$$

$$\langle e_1,e_1\rangle = \langle (1,0,0), (1,0,0)\rangle = 1$$

$$\langle e_1,e_2\rangle = \langle (1,0,0), (0,1,0)\rangle = 0$$

$$\langle e_1,e_3\rangle = \langle (1,0,0), (0,0,1)\rangle = 0$$

$$\langle e_2,e_2\rangle = \langle (0,1,0), (0,1,0)\rangle = 3$$

$$\langle e_2,e_3\rangle = \langle (0,1,0), (0,0,1)\rangle = 0$$

$$\langle e_3,e_3\rangle = \langle (0,0,1), (0,0,1)\rangle = 5$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Matrix representations of inner product:-

Example:- consider the inner product space $\mathbb{R}^2$ with inner product $\langle \bar{x}, \bar{y}\rangle = x_1y_1 + x_2y_2$ where $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$. Let us consider the standard basis $\{e_1 = (1,0),\ e_2 = (0,1)\}$ on $\mathbb{R}^2$.

we know,

$$\bar{x} = (x_1, x_2) = x_1(1,0) + x_2(0,1) = x_1e_1 + x_2e_2$$

$$\bar{y} = (y_1, y_2) = y_1(1,0) + y_2(0,1) = y_1e_1 + y_2e_2$$

Now,

$$\langle \bar{x}, \bar{y}\rangle = \langle x_1e_1 + x_2e_2,\ y_1e_1 + y_2e_2\rangle$$

$$= \langle x_1e_1, y_1e_1\rangle + \langle x_1e_1, y_2e_2\rangle + \langle x_2e_2, y_1e_1\rangle + \langle x_2e_2, y_2e_2\rangle$$

$$= x_1y_1\langle e_1,e_1\rangle + x_1y_2\langle e_1,e_2\rangle + x_2y_1\langle e_2,e_1\rangle + x_2y_2\langle e_2,e_2\rangle$$

$$= x_1y_1\langle e_1,e_1\rangle + x_1y_2\langle e_1,e_2\rangle + x_2y_1\langle e_1,e_2\rangle + x_2y_2\langle e_2,e_2\rangle$$

$$= [x_1 \quad x_2] \begin{bmatrix} \langle e_1,e_1\rangle & \langle e_1,e_2\rangle \\ \langle e_1,e_2\rangle & \langle e_2,e_2\rangle \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= [\bar{x}]_\alpha^T \begin{bmatrix} \langle e_1,e_1\rangle & \langle e_1,e_2\rangle \\ \langle e_1,e_2\rangle & \langle e_2,e_2\rangle \end{bmatrix} [\bar{y}]_\alpha$$

where $\alpha = \{e_1, e_2\}$

Example:- Let $V = P_2(\mathbb{R})$ with an innerproduct

$$\langle f, g\rangle = \int_0^1 f(x)\,g(x)\,dx$$

Find the matrix representation of the above inner product with respect to the basis $\{1, x, x^2\}$.

Soln:- We need to calculate

$A = [a_{ij}]$ such that $a_{ij} = \langle f_i, f_j\rangle$

where $f_1 = 1$, $f_2 = x$; $f_3 = x^2$.

$$A = \begin{bmatrix} \langle f_1,f_1\rangle & \langle f_1,f_2\rangle & \langle f_1,f_3\rangle \\ \langle f_2,f_1\rangle & \langle f_2,f_2\rangle & \langle f_2,f_3\rangle \\ \langle f_3,f_1\rangle & \langle f_3,f_2\rangle & \langle f_3,f_3\rangle \end{bmatrix}$$

$$\langle f_1,f_1\rangle = \langle 1,1\rangle = \int_0^1 1.1\,dx = 1$$

$$\langle f_1,f_2\rangle = \langle 1,x\rangle = \int_0^1 x\,dx = 1/2$$

$$\langle f_1,f_3\rangle = \langle 1,x^2\rangle = \int_0^1 x^2\,dx = 1/3$$

$$\langle f_2,f_2\rangle = \langle x,x\rangle = \int_0^1 x^2\,dx = 1/3$$

$$\langle f_2,f_3\rangle = \langle x,x^2\rangle = \int_0^1 x^3\,dx = 1/4$$

$$\langle f_3,f_3\rangle = \langle x^2,x^2\rangle = \int_0^1 x^4\,dx = 1/5.$$

$\therefore$ The matrix

$$A = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

Unit Vector:- Let V be an inner product space. A vector x in V is said to be a unit vector if

$$\boxed{\|x\| = \sqrt{\langle x, x\rangle} = 1}$$

which means length of x is one.

Examples:- consider the inner product space $\mathbb{R}^3$ with inner product

$$\langle \bar{x}, \bar{y}\rangle = x_1 y_1 + x_2 y_2 + x_3 y_3.$$

The vector $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is a unit vector.

$$\left\|\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right\| = \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 1.$$

The vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ are all unit vectors.

Find the normalization of $(-1, -2, 5)$.

Ans:-

length of $(-1,-2,5) = \sqrt{30}$.

Normalized vector is $\left(\frac{-1}{\sqrt{30}}, \frac{-2}{\sqrt{30}}, \frac{5}{\sqrt{30}}\right)$.

Normalization:-

The process of obtaining a unit vector from a nonzero vector by multiplying the inverse of its length is called as normalization.

(That is) If x is a nonzero vector in an inner product space the vector $\frac{x}{\|x\|}$ is a unit vector.

Pb:- Consider the innerproduct space $P_2(\mathbb{R})$ with inner product

$$\langle f(x), g(x)\rangle = \int_0^1 f(x)\, g(x)\, dx.$$

Find the normalized vector of

(a) $f(x) = x^2 - 3$

(b) $g(x) = x + 5$.

Soln:-

length of $f(x) = \sqrt{\langle f(x), f(x)\rangle}$

$$= \sqrt{\int_0^1 (x^2-3)(x^2-3)\, dx}$$

$$= \sqrt{\int_0^1 (x^4 - 6x^2 + 9)\, dx}$$

$$= \sqrt{\frac{36}{5}} = \frac{6}{\sqrt{5}}.$$

$\therefore$ Normalized form $\dfrac{x^2-3}{6/\sqrt{5}} = \dfrac{\sqrt{5}(x^2-3)}{6}$

Examples:- consider the innerproduct space $\mathbb{R}^3$ with standard inner product. Find the normalized vector of $(-2, 3, 4)$

Soln:-

length of $(-2,3,4) = \sqrt{\langle(-2,3,4),(-2,3,4)\rangle}$

$$= \sqrt{4+9+16} = \sqrt{29}.$$

Normalized vector $\left(\frac{-2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}\right)$

(b) length of $g(x) = \sqrt{\langle x+5, x+5\rangle}$

$$= \sqrt{\int_0^1 (x+5)^2\, dx}$$

$$= \sqrt{\frac{6^3}{3}} = 6\sqrt{2}.$$

Normalized vector $= \dfrac{x+5}{6\sqrt{2}} =$

Orthonormal basis:– A set of vectors $\{x_1, x_2, \dots, x_k\}$ in an inner product space V is said to be $\boxed{\text{orthonormal}}$ if

$$\langle x_i, x_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

A set of vectors $\{x_1, x_2, \dots, x_n\}$ is called an $\boxed{\text{orthonormal basis}}$ for V if it is a $\boxed{\text{basis}}$ and $\boxed{\text{orthonormal}}$.

Pb:1 Consider the inner product space $\mathbb{R}^2$ with standard inner product. Which of the following is an orthonormal basis for $\mathbb{R}^2$

✓ (a) $\{\underset{x_1}{(1,0)}, \underset{x_2}{(0,1)}\}$

✗ (b) $\{\underbrace{(1,0), (1,1)}_{\text{they are not orthogonal}}\}$

✓ (c) $\left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)\right\}$

Projections:–

Example:– Consider the system
$$x+y=2$$
$$x+y=3$$

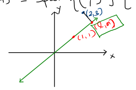
clearly this system does not have any solution.
The matrix form of the above equation is

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{B}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$

We find the column space of A.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Thus $C(A) = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} = \left\{k\begin{pmatrix} 1 \\ 1 \end{pmatrix} : k \in \mathbb{R}\right\}$



We find out the projection of $(2,3)$ onto the line $\left\{k\begin{pmatrix} 1 \\ 1 \end{pmatrix} : k \in \mathbb{R}\right\}$. Suppose the projection is (l, l) then

I solve the system
$$x+y=l$$
$$x+y=l$$

Projections:–

Recall:– **Linear transformation:**– Let V and W be vector spaces. A map $T: V \to W$ is called as a linear transformation if

(a) $T(x+y) = T(x) + T(y) \quad \forall x, y \in V$

(b) $T(kx) = kT(x) \quad$ for all $k \in \mathbb{R}, x \in V$

(c) $T(0_V) = 0_W$

Direct sum of subspaces:– Let V be a vector space U and W be subspaces of V. V is said to be direct sum of U and W if (a) $V = U + W$

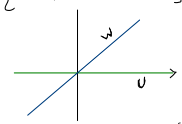
(b) $U \cap W = \{0_V\}$

In this case we write $\boxed{V = U \oplus W}$

Examples:– Consider the vector space $\mathbb{R}^2$ and subspaces

$U = \{k(1,0) : k \in \mathbb{R}\} = X\text{-axis}$

$W = \{k(1,1) : k \in \mathbb{R}\} =$ line passing through $(1,1)$.



It is clear that $U \cap W = \{(0,0)\}$

$(x,y) = (x-y)(1,0) + y(1,1)$

$\therefore$ every point in $\mathbb{R}^2$ belongs to $U+W$

$\therefore \mathbb{R}^2 = U + W$

Hence $\mathbb{R}^2 = U \oplus W$.

Definition:- Let U and W be subspaces of a vector space V. A linear transformation $T: V \to V$ is called the projection of V onto the subspace U along W if

(a) $V = U \oplus W$

(b) $T(x) = u$ for $x = u + w \in U \oplus W$.

Examples:- Consider $\mathbb{R}^2$ and subspaces

$U = \{k(1,0) : k \in \mathbb{R}\} = X-$axis

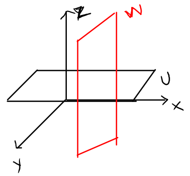
$W = \{k(0,1) : k \in \mathbb{R}\} = Y-$axis

It is clear that $\mathbb{R}^2 = U \oplus W$

Define $P: \mathbb{R}^2 \to \mathbb{R}^2$ by $P(x,y) = (x,0)$

then P is a projection

[Note:- $(x,y) = (x,0) + (0,y)$]



Definition let U and W be subspaces of an inner product space V. U and W is said to be orthogonal written by $U \perp W$ if

$\langle u, w \rangle = 0$ for each $u \in U$ and $w \in W$.

Example 2:- Consider the vector space $\mathbb{R}^3$ and the subspaces

$U = \{(x,y,0) : x,y \in \mathbb{R}\} = XY$-plane

$W = \{(0,0,z) : z \in \mathbb{R}\} = Z-$axis.

Then It is clear that

$\mathbb{R}^3 = U \oplus W$.

Define $P_2 : \mathbb{R}^3 \to \mathbb{R}^3$ by $P_2(x,y,z) = (0,0,z)$

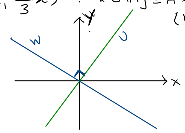
Then P_2 is a projection of $\mathbb{R}^3$ onto the subspace W along U

[Note:- $(x,y,z) = (x,y,0) + (0,0,z)$]

Example:- Consider the inner product space $\mathbb{R}^2$ and the subspaces

$U = \{(x, 3x) : x \in \mathbb{R}\}$ = A line passe through $(1,3)$

$W = \{(x, \frac{-1}{3}x) : x \in \mathbb{R}\}$ = A line passes through $(1, -1/3)$



Take a point $(k, 3k)$ in U and a vector $(l, -\frac{1}{3}l)$ in W then

$\langle (k,3k), (l, \frac{-1}{3}l) \rangle$

$= kl - kl = 0$

Thus U and W are orthogonal subspaces

Theorem:- A linear transformation $T: V \to V$ is a projection onto a subspace U if and only if

$T \circ T = T$

Theorem:- A linear transformation $T: V \to W$ is a projection onto U along W if and only if $Id_V - T$ is a projection of V onto a subspace W along U.

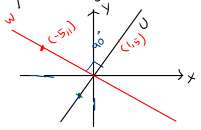
Orthogonal complement:- Let V be an inner product space, U be a subspace of V the set

(U perp)

$U^{\perp} = \{w \in V \mid \langle w, u \rangle = 0 \text{ for all } u \in U\}$

is called as the orthogonal complement of U. (which means) The set of all vectors in V that are orthogonal to every vector in U is called orthogonal complement of U.

Example:- Consider the inner product space $\mathbb{R}^2$ and the subspace $U = \{(x, 5x) : x \in \mathbb{R}\}$. Find the orthogonal complement of U.



$$U^\perp = \{w \in V \mid \langle w, u \rangle = 0 \text{ for all } u \in U\}$$

$$U^\perp = \{(x,y) \in \mathbb{R}^2 \mid \langle (x,y), k(1,5) \rangle = 0 \text{ for } k \in \mathbb{R}\}$$

$$= \{(x,y) \in \mathbb{R}^2 \mid x + 5y = 0\}$$

= line passes through $(-5,1)$.

Orthogonal projection:-

Let V be an inner product space and let U be a subspace of V so that $V = U \oplus U^\perp$. The projection of V onto U along $U^\perp$ is called orthogonal projection of V onto U denoted as Proj_U. For $x \in V$ the component vector $\mathrm{Proj}_U(x) \in U$ is called the orthogonal projection of x into U.

Example:- Consider the inner product space $\mathbb{R}^3$ with standard inner product. Let

$$U = \{(x,y,z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0\}.$$

Find $U^\perp$

Soln:- clearly, U is a plane and a subspace of $\mathbb{R}^3$.

A basis for U is $\{(2,-1,0), (3,0,-1)\}$.

Let $(k,l,m) \in U^\perp$. Then

$$\langle (k,l,m), (2,-1,0) \rangle = 0$$

$$\Rightarrow 2k - l = 0 \longrightarrow (1)$$

and

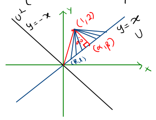
$$\langle (k,l,m), (3,0,-1) \rangle = 0$$

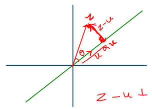
$$\Rightarrow 3k - m = 0 \longrightarrow (2)$$

$$\therefore U^\perp = \{(k,l,m) \in \mathbb{R}^2 : 2k - l = 0;\ 3k - m = 0\}$$

A basis for $U^\perp$ is $\{(1,2,3)\}$

$$\therefore U^\perp = \{t(1,2,3) \mid t \in \mathbb{R}\}.$$





$$z - u \perp \alpha u$$

$$\cos\theta = \frac{\alpha}{\|z\|}$$

$$\alpha = \|z\| \cos\theta$$

$$= \frac{\|u\|\,\|z\| \cos\theta}{\|u\|}$$

$$\boxed{\alpha = \frac{\langle u, z \rangle}{\|u\|}}$$

Orthogonal projection is $\dfrac{\langle u, z \rangle}{\|u\|} u$.

Theorem:- Let V be an inner product space U be a subspace of V then $U^\perp$ is also a subspace of V and

(a) $\dim U + \dim U^\perp = \dim V$

(b) $(U^\perp)^\perp = U$

(c) $V = U \oplus U^\perp$. This is called orthogonal decomposition of V by U.

Theorem:- Let U be a subspace of an inner product space V and let $\{u_1, u_2, \dots, u_m\}$ be an orthonormal basis for U. Then for any $x \in V$ the orthogonal projection $\mathrm{Proj}_U(x)$ of x into U is

$$\mathrm{Proj}_U(x) = \langle x, u_1 \rangle u_1 + \langle x, u_2 \rangle u_2 + \dots + \langle x, u_m \rangle u_m.$$

Theorem:- Let U be a subspace of an inner product space V and let $x \in V$. Then the orthogonal projection $\text{Proj}_U(x)$ of x satisfies

$$\|x - \text{Proj}_U(x)\| \le \|x - y\|$$

for all $y \in U$.

Orthogonal projection:-

Let V be an inner product space and let U be a subspace of V so that $V = U \oplus U^{\perp}$. The projection of V onto U along $U^{\perp}$ is called orthogonal projection of V onto U denoted as Proj_U. For $x \in V$ the component vector $\text{Proj}_U(x) \in U$ is called the orthogonal projection of x into U.

Theorem:- Let U be a subspace of an inner product space V and let $\{u_1, u_2, \dots, u_m\}$ be an orthonormal basis for U. Then for any $x \in V$ the orthogonal projection $\text{Proj}_U(x)$ of x into U is

$$\text{Proj}_U(x) = \langle x, u_1\rangle u_1 + \langle x, u_2\rangle u_2 + \dots + \langle x, u_m\rangle u_m.$$

Theorem:- Let U be a subspace of an inner product space V and let $x \in V$. Then the orthogonal projection $\text{Proj}_U(x)$ of x satisfies

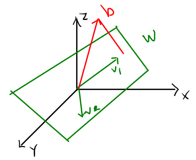
$$\|x - \text{Proj}_U(x)\| \le \|x - y\|$$

for all $y \in U$.

Pb:- Let W be the subspace of the inner product space $\mathbb{R}^3$ (with standard inner product) spanned by the vector $v_1 = (1,1,2)$ and $v_2 = (1,1,-1)$.

(a) Find the orthogonal projection of the vector $b = (1,3,-2)$ onto the subspace W.

(b) Find the shortest distance between b and W.



We first the orthonormal basis for W.

The given basis elements are $v_1 = (1,1,2)$ and $v_2 = (1,1,-1)$

$\langle v_1, v_2\rangle = \langle (1,1,2), (1,1,-1)\rangle = 1 + 1 - 2 = 0$

v_1, v_2 are orthogonal. Now,

$\|v_1\| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$;

$\|v_2\| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$

Construct $u_1 = \dfrac{v_1}{\|v_1\|} = \left(\dfrac{1}{\sqrt6}, \dfrac{1}{\sqrt6}, \dfrac{2}{\sqrt6}\right)$

and $u_2 = \dfrac{v_2}{\|v_2\|} = \left(\dfrac{1}{\sqrt3}, \dfrac{1}{\sqrt3}, \dfrac{-1}{\sqrt3}\right)$

Thus $\{u_1, u_2\}$ forms an orthonormal basis for W.

We know orthogonal projection of b is

$$\text{Proj}_W(b) = \langle b, u_1\rangle u_1 + \langle b, u_2\rangle u_2$$

$$\langle b, u_1\rangle = \langle (1,3,-2), \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)\rangle$$

$$= \frac{1}{\sqrt{6}} + \frac{3}{\sqrt{6}} - \frac{4}{\sqrt{6}} = 0$$

$$\langle b, u_2\rangle = \langle (1,3,-2), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)\rangle$$

$$= \frac{1}{\sqrt{3}} + \frac{3}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{6}{\sqrt{3}}$$

Hence

$$\text{Proj}_W(b) = 0\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right) + \frac{6}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

$$= 0 + \left(\frac{6}{3}, \frac{6}{3}, \frac{-6}{3}\right)$$

$$= (2, 2, -2)$$

Consider the inner product space $\mathbb{R}^2$ with standard inner product. Let $v_1 = (1,1)$ and $v_2 = (1,2)$. $\{v_1, v_2\}$ forms a basis for $\mathbb{R}^2$. Find an orthonormal basis using Gram-Schmidt orthogonalization process.

Soln:– $u_1 = v_1 = (1,1)$; $\|u_1\| = \sqrt{1^2+1^2} = \sqrt{2}$

$$w_1 = \frac{u_1}{\|u_1\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$u_2 = v_2 - \langle v_2, w_1\rangle w_1$$

$$= (1,2) - \underbrace{\langle (1,2), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\rangle}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$= (1,2) - \frac{3}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$= (1,2) - \left(\frac{3}{2}, \frac{3}{2}\right) = \left(\frac{-1}{2}, \frac{1}{2}\right)$$

$$\|u_2\| = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$$

$$\therefore w_2 = \frac{u_2}{\|u_2\|} = \left(\frac{-\frac{1}{2}}{\frac{1}{\sqrt{2}}}, \frac{\frac{1}{2}}{\frac{1}{\sqrt{2}}}\right)$$

$$= \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$\therefore$ The orthonormal basis for $\mathbb{R}^2$ is

$$\left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}$$

Shortest distance between the vector b and W is

$$\|b - \text{proj}_W(b)\| = \|(1,3,-2) - (2,2,-2)\|$$

$$= \|(-1, 1, 0)\|$$

$$= \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}.$$

Orthonormal vectors:– A set of vectors $\{v_1, v_2, \ldots, v_n\}$ in an inner product space V is said to be orthonormal if

$$\langle v_i, v_j\rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Orthonormal Basis:– A set of vectors $\{v_1, v_2, \ldots v_n\}$ is called an orthonormal basis if it is a basis and orthonormal

Gram-Schmidt Orthogonalization Process:–

Procedure:– Suppose $v_1, v_2, \ldots v_n$ is a basis for an inner product space V. Let

$u_1 = v_1$, $w_1 = \frac{u_1}{\|u_1\|}$.

$u_2 = v_2 - \langle v_2, w_1\rangle w_1$; $w_2 = \frac{u_2}{\|u_2\|}$.

$u_3 = v_3 - \langle v_3, w_1\rangle w_1 - \langle v_3, w_2\rangle w_2$; $w_3 = \frac{u_3}{\|u_3\|}$.

. . . .

$u_n = v_n - \langle v_n, w_1\rangle w_1 - \langle v_n, w_2\rangle w_2 - \cdots - \langle v_n, w_{n-1}\rangle w_{n-1}$

$w_n = \frac{u_n}{\|u_n\|}$

Then $\{w_1, w_2, \ldots w_n\}$ is an orthonormal basis for V.

Gram-Schmidt orthogonalization process:–

Suppose $v_1, v_2, \ldots v_n$ is a basis for an inner product space V. Let

$u_1 = v_1$, $w_1 = \frac{u_1}{\|u_1\|}$

$u_2 = v_2 - \langle v_2, w_1\rangle w_1$; $w_2 = \frac{u_2}{\|u_2\|}$

$u_3 = v_3 - \langle v_3, w_1\rangle w_1 - \langle v_3, w_2\rangle w_2$; $w_3 = \frac{u_3}{\|u_3\|}$

. . . .

$u_n = v_n - \langle v_n, w_1\rangle w_1 - \langle v_n, w_2\rangle w_2 - \cdots - \langle v_n, w_{n-1}\rangle w_{n-1}$

$w_n = \frac{u_n}{\|u_n\|}$.

Then $w_1, w_2, \ldots w_n$ is an orthonormal basis for V.

Pb:- Consider $P_2([-1,1]) = \{p(x) : p(x)$ is a polynomial of degree atmost $2\}$ is an inner product space with basis $\alpha = \{1, x, x^2\}$ with an inner product

$$\langle f(x), g(x)\rangle = \int_{-1}^{1} f(x)g(x)\,dx$$

Find an orthonormal basis for $P_2([-1,1])$.

Soln:- Take $v_1 = 1$; $v_2 = x$; $v_3 = x^2$

$u_1 = v_1 = 1$; $\omega_1 = \frac{u_1}{\|u_1\|}$. Here $\|u_1\| = \sqrt{\int_{-1}^{1} 1\cdot 1\,dx} = \sqrt{2}$

Hence $\omega_1 = \frac{1}{\sqrt{2}}$.

$$u_2 = v_2 - \langle v_2, \omega_1\rangle \omega_1 = x - \langle x, \tfrac{1}{\sqrt{2}}\rangle \tfrac{1}{\sqrt{2}}$$
$$= x - \left[\int_{-1}^{1} x\cdot\tfrac{1}{\sqrt{2}}\,dx\right]\tfrac{1}{\sqrt{2}} = x - 0\cdot\tfrac{1}{\sqrt{2}} = x$$

$$\|u_2\| = \sqrt{\int_{-1}^{1} x\cdot x\,dx} = \sqrt{\int_{-1}^{1} x^2\,dx} = \sqrt{2/3}.$$

Thus $\omega_2 = \frac{u_2}{\|u_2\|} = \frac{x}{\sqrt{2/3}} = \frac{\sqrt{3}x}{\sqrt{2}}$.

Next

$$u_3 = v_3 - \langle v_3, \omega_1\rangle\omega_1 - \langle v_3, \omega_2\rangle\omega_2$$
$$= x^2 - \langle x^2, \tfrac{1}{\sqrt{2}}\rangle\tfrac{1}{\sqrt{2}} - \langle x^2, \tfrac{\sqrt{3}}{\sqrt{2}}x\rangle\tfrac{\sqrt{3}}{\sqrt{2}}x$$
$$= x^2 - \left(\int_{-1}^{1} x^2\cdot\tfrac{1}{\sqrt{2}}\,dx\right)\tfrac{1}{\sqrt{2}} - \underbrace{\left(\int_{-1}^{1} x^2\,\tfrac{\sqrt{3}}{2}x\,dx\right)}_{\text{Zero because } x^3 \text{ is odd}}\tfrac{\sqrt{3}}{\sqrt{2}}x$$
$$= x^2 - \tfrac{1}{\sqrt{2}}\left(\tfrac{x^3}{3}\right)_{-1}^{1}\tfrac{1}{\sqrt{2}} = x^2 - \tfrac{1}{3}.$$

Now,

$$\|u_3\| = \sqrt{\int_{-1}^{1}\left(x^2-\tfrac{1}{3}\right)^2 dx} = \sqrt{2\int_{0}^{1}\left(x^2-\tfrac{1}{3}\right)^2 dx}$$
$$= \frac{\sqrt{8}}{\sqrt{45}}.$$

$$\therefore \omega_3 = \frac{x^2 - 1/3}{\frac{\sqrt{8}}{\sqrt{45}}} = \frac{\sqrt{45}}{\sqrt{8}}\left(x^2 - 1/3\right).$$

The orthonormal basis for $P_2([-1,1])$ is

$$\left\{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}x, \frac{\sqrt{45}}{\sqrt{8}}\left(x^2-1/3\right)\right\}$$ //

Orthonormal basis by Gram-schmidt process:-

$v_1 = (1,1,1,1)$; $v_2 = (1,2,0,1)$, $v_3 = (2,2,4,0)$

$u_1 = v_1$; $\omega_1 = \frac{u_1}{\|u_1\|}$

$\|u_1\| = \|(1,1,1,1)\| = \sqrt{1^2+1^2+1^2+1^2} = \sqrt{4}$.

$$\therefore \omega_1 = \left(\frac{1}{\sqrt{4}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{4}}\right).$$

$$u_2 = v_2 - \langle v_2, \omega_1\rangle\omega_1 = \underbrace{(1,2,0,1)}_{v_2} - \langle (1,2,0,1), \underbrace{\left(\tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{4}}\right)}_{\omega_1}\rangle \underbrace{\left(\tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{4}}\right)}_{\omega_1}$$

$$= (1,2,0,1) - \left[\tfrac{1}{\sqrt{4}} + \tfrac{2}{\sqrt{4}} + \tfrac{1}{\sqrt{4}}\right]\left(\tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{4}}\right)$$
$$= (1,2,0,1) - \tfrac{4}{\sqrt{4}}\left(\tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{4}}\right)$$
$$u_2 = (1,2,0,1) - (1,1,1,1) = (0,1,-1,0)$$

$$\|u_2\| = \sqrt{0^2+1^2+(-1)^2+0^2} = \sqrt{2}.$$

$$\therefore \omega_2 = \frac{u_2}{\|u_2\|} = \left(0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right).$$

Gram Schmidt Orthogonalization Process:- Let V be an inner product space with basis $\{v_1, v_2, \dots v_n\}$. Consider the following vectors

$u_1 = v_1$; $\omega_1 = \frac{u_1}{\|u_1\|}$

$u_2 = v_2 - \langle v_2, \omega_1\rangle\omega_1$; $\omega_2 = \frac{u_2}{\|u_2\|}$.

$u_3 = v_3 - \langle v_3, \omega_1\rangle\omega_1 - \langle v_3, \omega_2\rangle\omega_2$; $\omega_3 = \frac{u_3}{\|u_3\|}$.

$\vdots$

$$\boxed{u_n = v_n - \langle v_n, \omega_1\rangle\omega_1 - \langle v_n, \omega_2\rangle\omega_2 - \langle v_n, \omega_3\rangle\omega_3 - \cdots \langle v_n, \omega_{n-1}\rangle\omega_{n-1}}$$

$\omega_n = \frac{u_n}{\|u_n\|}$.

The collection $\{\omega_1, \omega_2, \omega_3, \dots \omega_n\}$ forms an orthonormal basis for V.

$$u_3 = v_3 - \langle v_3, \omega_1\rangle\omega_1 - \langle v_3, \omega_2\rangle\omega_2$$
$$= (2,2,4,0) - \langle (2,2,4,0), \left(\tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{4}}\right)\rangle\left(\tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{4}}\right)$$
$$- \langle (2,2,4,0)\left(0, \tfrac{1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}, 0\right)\rangle\left(0, \tfrac{1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}, 0\right)$$

$$= (2,2,4,0) - \frac{8}{\sqrt{4}}\left(\tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{4}}\right) - \frac{-2}{\sqrt{2}}\left(0, \tfrac{1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}, 0\right)$$

$$= (2,2,4,0) - (2,2,2,2) - (0,-1,1,0)$$

$$u_3 = (0,1,1,-2)$$

$$\|u_3\| = \sqrt{0^2+1^2+1^2+(-2)^2} = \sqrt{6}$$

$$\therefore \omega_3 = \frac{u_3}{\|u_3\|} = \left(0, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$$

Thus an orthonormal basis for column space is

$$\left\{\underbrace{\left(\tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{4}}, \tfrac{1}{\sqrt{4}}\right)}_{\omega_1}, \underbrace{\left(0, \tfrac{1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}, 0\right)}_{\omega_2}, \underbrace{\left(0, \tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, \tfrac{-2}{\sqrt{6}}\right)}_{\omega_3}\right\}$$

Pb:- Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 0 & 4 \\ 1 & 1 & 0 \end{bmatrix}$. Find an orthonormal basis for column space of A.

Soln:- Basis for Column space of A:-

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 0 & 4 \\ 1 & 1 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{matrix} R_3 \to R_3 + R_2 \\ R_4 \to \left(\frac{-1}{2}\right)R_4 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} R_4 \to R_4 - \tfrac{1}{3}R_3 \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to \tfrac{1}{3}R_3$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ leading 1s are in 1^{st}, 2^{nd} and 3^{rd} column.

Hence A basis for column space is

$$\left\{ v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}; v_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}; v_3 = \begin{pmatrix} 2 \\ 2 \\ 4 \\ 0 \end{pmatrix} \right\}$$

Orthogonal Matrix:- A Square matrix A is called an orthogonal matrix if A satisfies one of the following statements

(a) The column vectors of A are orthonormal.

(b) $A^T A = I_n$ (I_n - identity matrix of order n)

(c) $A A^T = I_n$

(d) $A^T = A^{-1}$

(e) The row vectors of A are orthonormal.

Examples:-

Consider the matrix

$$A = \begin{bmatrix} 1/\sqrt{2} & \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Here,

$$A^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \therefore A \text{ is orthogonal.}$$

Definition

Let V and W be two inner product spaces. A linear transformation $T: V \to W$ is called an isometry (or) an orthogonal transformation if it preserves the length of the vectors, that is

$\|T(x)\| = \|x\|$ for all $x \in V$.

Consider the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ for } 0 \le \theta \le \pi/2.$$

Then

$$A^T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A \text{ is orthogonal.}$$

Orthogonal transformation:-

Consider the linear transformation

$T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x,y) = \left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right)$

We calculate length of $T(x,y)$

$$\|T(x,y)\| = \sqrt{\left(\frac{x+y}{\sqrt{2}}\right)^2 + \left(\frac{x-y}{\sqrt{2}}\right)^2}$$

$$= \sqrt{\frac{x^2+2xy+y^2}{2} + \frac{x^2-2xy+y^2}{2}}$$

$$= \sqrt{\frac{2x^2+2y^2}{2}} = \sqrt{x^2+y^2}$$

$$= \|(x,y)\|$$

$\therefore$ T is orthogonal.

QR Factorization:-

Theorem:- If A is a $m\times n$ matrix of rank n then A can be factored into a product QR where Q is an $m\times n$ matrix with orthonormal column vectors and R is an invertible upper triangular matrix.

Procedure for QR factorization:-

Let $A = [v_1 \; v_2 \dots v_n]$ be a $m\times n$ matrix. Here $v_1, v_2, \dots v_n$ are columns of A. By Gram Schmidt process, we find the orthonormal basis for column space of A.

$$w_1 = \frac{v_1}{\|v_1\|}$$

$$w_2 = \frac{v_2 - \langle v_2, w_1\rangle w_1}{\|v_2 - \langle v_2, w_1\rangle w_1\|}$$

$$w_3 = \frac{v_3 - \langle v_3, w_1\rangle w_1 - \langle v_3, w_2\rangle w_2}{\|v_3 - \langle v_3, w_1\rangle w_1 - \langle v_3, w_2\rangle w_2\|}$$

$$\vdots$$

$$w_n = \frac{v_n - \langle v_n, w_1\rangle w_1 - \langle v_n, w_2\rangle w_2 - \dots - \langle v_n, w_{n-1}\rangle w_{n-1}}{\|v_n - \langle v_n, w_1\rangle w_1 - \langle v_n, w_2\rangle w_2 - \dots \langle v_n, w_{n-1}\rangle w_{n-1}\|}$$

The QR factorization of A is

$$A = [w_1 \; w_2 \dots w_n]\begin{bmatrix} \langle v_1, w_1\rangle & \langle v_2, w_1\rangle & \langle v_3, w_1\rangle \dots & \langle v_n, w_1\rangle \\ 0 & \langle v_2, w_2\rangle & \langle v_3, w_2\rangle \dots & \langle v_n, w_2\rangle \\ 0 & 0 & \langle v_3, w_3\rangle \dots & \langle v_n, w_3\rangle \\ \vdots & \vdots & \vdots & \\ 0 & 0 & 0 \dots & \langle v_n, w_n\rangle \end{bmatrix}$$

Pb:1 Find QR factorization of $A = \begin{bmatrix} 6 & 6 & 1 \\ 3 & 6 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

Soln:- Take $v_1 = (6,3,2)$; $v_2 = (6,6,1)$; $v_3 = (1,1,1)$

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{(6,3,2)}{\sqrt{6^2+3^2+2^2}} = \frac{(6,3,2)}{\sqrt{49}} = \left(\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right)$$

$$w_2 = \frac{v_2 - \langle v_2, w_1\rangle w_1}{\|v_2 - \langle v_2, w_1\rangle w_1\|} = \frac{(6,6,1) - \langle (6,6,1), (\frac{6}{7}, \frac{3}{7}, \frac{2}{7})\rangle (\frac{6}{7}, \frac{3}{7}, \frac{2}{7})}{\|(6,6,1) - \langle (6,6,1), (\frac{6}{7}, \frac{3}{7}, \frac{2}{7})\rangle (\frac{6}{7}, \frac{3}{7}, \frac{2}{7})\|}$$

$$= \frac{(6,6,1) - \left[\frac{36}{7} + \frac{18}{7} + \frac{2}{7}\right]\left(\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right)}{\|(6,6,1) - (\frac{36}{7} + \frac{18}{7} + \frac{2}{7})(\frac{6}{7}, \frac{3}{7}, \frac{2}{7})\|}$$

$$= \frac{(6,6,1) - 8\left(\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right)}{\|(6,6,1) - 8\left(\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right)\|} = \frac{\left(\frac{-6}{7}, \frac{18}{7}, \frac{-9}{7}\right)}{\|\left(\frac{-6}{7}, \frac{18}{7}, \frac{-9}{7}\right)\|}$$

$$w_2 = \frac{\left(\frac{-6}{7}, \frac{18}{7}, \frac{-9}{7}\right)}{3} = \left(\frac{-2}{7}, \frac{6}{7}, \frac{-3}{7}\right).$$

$$v_3 - \langle v_3, w_1\rangle w_1 - \langle v_3, w_2\rangle w_2$$

$$= (1,1,1) - \langle (1,1,1), (\tfrac{6}{7}, \tfrac{3}{7}, \tfrac{2}{7})\rangle (\tfrac{6}{7}, \tfrac{3}{7}, \tfrac{2}{7}) - \langle (1,1,1), (\tfrac{-2}{7}, \tfrac{6}{7}, \tfrac{-3}{7})\rangle (\tfrac{-2}{7}, \tfrac{6}{7}, \tfrac{-3}{7})$$

$$= (1,1,1) - \left[\tfrac{6}{7} + \tfrac{3}{7} + \tfrac{2}{7}\right](\tfrac{6}{7}, \tfrac{3}{7}, \tfrac{2}{7}) - \left[\tfrac{-2}{7} + \tfrac{6}{7} + \tfrac{-3}{7}\right](\tfrac{-2}{7}, \tfrac{6}{7}, \tfrac{-3}{7})$$

$$= (1,1,1) - \tfrac{11}{7}(\tfrac{6}{7}, \tfrac{3}{7}, \tfrac{2}{7}) - \tfrac{1}{7}(\tfrac{-2}{7}, \tfrac{6}{7}, \tfrac{-3}{7})$$

$$= \left(1 - \tfrac{66}{49} + \tfrac{2}{49}, \; 1 - \tfrac{33}{49} - \tfrac{6}{49}, \; 1 - \tfrac{22}{49} + \tfrac{3}{49}\right)$$

$$= \left(\tfrac{-15}{49}, \tfrac{10}{49}, \tfrac{30}{49}\right)$$

$$\text{norm} = \sqrt{\left(\tfrac{-15}{49}\right)^2 + \left(\tfrac{10}{49}\right)^2 + \left(\tfrac{30}{49}\right)^2} = \tfrac{35}{49}.$$

$$\therefore w_3 = \frac{\left(\frac{-15}{49}, \frac{10}{49}, \frac{30}{49}\right)}{\frac{35}{49}} = \left(\tfrac{-3}{7}, \tfrac{2}{7}, \tfrac{6}{7}\right)$$

$$\langle v_1, w_1\rangle = \langle (6,3,2), (\tfrac{6}{7}, \tfrac{3}{7}, \tfrac{2}{7})\rangle = \frac{36+9+4}{7} = 7.$$

$$\langle v_2, w_1\rangle = \langle (6,6,1), (\tfrac{6}{7}, \tfrac{3}{7}, \tfrac{2}{7})\rangle = \frac{36+18+2}{7} = \frac{56}{7} = 8$$

$$\langle v_3, w_1\rangle = \langle (1,1,1), (\tfrac{6}{7}, \tfrac{3}{7}, \tfrac{2}{7})\rangle = 11/7.$$

$$\langle v_2, w_2\rangle = \langle (6,6,1), (\tfrac{-2}{7}, \tfrac{6}{7}, \tfrac{-3}{7})\rangle = \frac{-12+36-3}{7} = 3$$

$$\langle v_3, w_2\rangle = \langle (1,1,1), (\tfrac{-2}{7}, \tfrac{6}{7}, \tfrac{-3}{7})\rangle = 1/7$$

$$\langle v_3, w_3\rangle = \langle (1,1,1), (\tfrac{-3}{7}, \tfrac{2}{7}, \tfrac{6}{7})\rangle = 5/7.$$

$$A = [\omega_1 \ \ \omega_2 \ \ \omega_3]\begin{bmatrix} \langle v_1,\omega_1\rangle & \langle v_2,\omega_1\rangle & \langle v_3,\omega_1\rangle \\ 0 & \langle v_2,\omega_2\rangle & \langle v_3,\omega_2\rangle \\ 0 & 0 & \langle v_3,\omega_3\rangle \end{bmatrix}$$

$$A = \begin{bmatrix} 6/7 & -2/7 & -3/7 \\ 3/7 & 6/7 & 2/7 \\ 2/7 & -3/7 & 6/7 \end{bmatrix}\begin{bmatrix} 7 & 8 & 11/7 \\ 0 & 3 & 1/7 \\ 0 & 0 & 5/7 \end{bmatrix}$$

$\langle v_1,\omega_1\rangle = \langle (1,1,1,1), (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})\rangle = 2.$

$\langle v_2,\omega_1\rangle = \langle (-1,4,4,-1), (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})\rangle = \frac{-1}{2}+2+2-\frac{1}{2} = 3$

$\langle v_3,\omega_1\rangle = \langle (4,-2,2,0), (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})\rangle = 2-1+1+0 = 2$

$\langle v_2,\omega_2\rangle = \langle (-1,4,4,-1), (\frac{-1}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{2})\rangle = +\frac{1}{2}+2+2+\frac{1}{2} = 5.$

$\langle v_3,\omega_2\rangle = \langle (4,-2,2,0), (\frac{-1}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{2})\rangle = -2-1+1 = -2$

$\langle v_3,\omega_3\rangle = \langle (4,-2,2,0), (\frac{1}{2},\frac{-1}{2},\frac{1}{2},\frac{-1}{2})\rangle = 2+1+1 = 4.$

$\therefore$ QR factorization of A is

$$A = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

<u>Q:-</u> Find QR factorization for $A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$

$\ \ \downarrow v_1 \ \ \downarrow v_2 \ \ \downarrow v_3$

<u>Soln:-</u> Take $v_1 = (1,1,1,1)$; $v_2 = (-1,4,4,-1)$ and $v_3 = (4,-2,2,0)$

$\omega_1 = \frac{v_1}{\|v_1\|} = \frac{(1,1,1,1)}{\sqrt{1^2+1^2+1^2+1^2}} = \frac{(1,1,1,1)}{\sqrt{4}} = \frac{(1,1,1,1)}{2} = (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$

$v_2 - \langle v_2,\omega_1\rangle\omega_1 = (-1,4,4,-1) - \langle (-1,4,4,-1), (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})\rangle (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$

$= (-1,4,4,-1) - [\frac{-1}{2}+2+2-\frac{1}{2}](\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$

$= (-1,4,4,-1) - 3(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$

$= (-1-\frac{3}{2}, 4-\frac{3}{2}, 4-3/2, -1-\frac{3}{2}) = (\frac{-5}{2},\frac{5}{2},5/2,-5/2)$

$\|v_2 - \langle v_2,\omega_1\rangle\omega_1\| = \sqrt{(\frac{-5}{2})^2+(\frac{5}{2})^2+(\frac{5}{2})^2+(\frac{-5}{2})^2} = \sqrt{4(\frac{5}{2})^2} = 5$

Hence $\omega_2 = \frac{v_2 - \langle v_2,\omega_1\rangle\omega_1}{\|v_2 - \langle v_2,\omega_1\rangle\omega_1\|} = \frac{(\frac{-5}{2},\frac{5}{2},\frac{5}{2},\frac{-5}{2})}{5}$

$\omega_2 = (\frac{-1}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{2}).$

<u>Pb:2</u> Find QR factorization of $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ -1 & -2 & 2 \end{bmatrix}$

<u>Soln:-</u>

Take $v_1 = (1,-1,-1)$; $v_2 = (0,2,-2)$; $v_3 = (2,0,2)$

$\omega_1 = \frac{v_1}{\|v_1\|} = \frac{(1,-1,-1)}{\sqrt{1^2+(-1)^2+(-1)^2}} = (\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}).$

$v_2 - \langle v_2,\omega_1\rangle\omega_1 = (0,2,-2) - \langle (0,2,-2)\ (\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}})\rangle(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}})$

$= (0,2,-2) - [0+(\frac{-2}{\sqrt{3}})+(\frac{2}{\sqrt{3}})](\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}})$

$= (0,2,-2) - 0\cdot(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}})$

$= (0,2,-2)$

$\|v_2 - \langle v_2,\omega_1\rangle\omega_1\| = \sqrt{0^2+2^2+(-2)^2} = 2\sqrt{2}.$

$\omega_2 = \frac{v_2 - \langle v_2,\omega_1\rangle\omega_1}{\|v_2 - \langle v_2,\omega_1\rangle\omega_1\|} = (0,\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}}).$

$v_3 - \langle v_3,\omega_1\rangle\omega_1 - \langle v_3,\omega_2\rangle\omega_2$

$= (4,-2,2,0) - \langle (4,-2,2,0), (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})\rangle(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$

$\quad - \langle (4,-2,2,0), (\frac{-1}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{2})\rangle(\frac{-1}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{2})$

$= (4,-2,2,0) - [2-1+1+0](\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}) - [-2-1+1](\frac{-1}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{2})$

$= (4,-2,2,0) - (2)(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}) - (-2)(\frac{-1}{2},\frac{1}{2},\frac{1}{2},\frac{-1}{2})$

$= (4,-2,2,0) - (1,1,1,1) - (1,-1,-1,1)$

$= (2,-2,2,-2)$

$\|v_3 - \langle v_3,\omega_1\rangle\omega_1 - \langle v_3,\omega_2\rangle\omega_2\| = \sqrt{2^2+(-2)^2+2^2+(-2)^2} = 4.$

Hence $\omega_3 = \frac{v_3 - \langle v_3,\omega_1\rangle\omega_1 - \langle v_3,\omega_2\rangle\omega_2}{\|v_3 - \langle v_3,\omega_1\rangle\omega_1 - \langle v_3,\omega_2\rangle\omega_2\|} = (\frac{1}{2},\frac{-1}{2},\frac{1}{2},\frac{-1}{2}).$

$v_3 - \langle v_3,\omega_1\rangle\omega_1 - \langle v_3,\omega_2\rangle\omega_2$

$= (2,0,2) - \langle (2,0,2), (\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}})\rangle(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}})$

$\quad - \langle (2,0,2), (0,\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}})\rangle(0,\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}})$

$= (2,0,2) - [\frac{2}{\sqrt{3}}+0-\frac{2}{\sqrt{3}}](\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}})$

$\quad - (0+0-\frac{2}{\sqrt{2}})(0,\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}})$

$= (2,0,\cancel{2}) + (0,1,-1) = (2,1,1)$

$\|v_3 - \langle v_3,\omega_1\rangle\omega_1 - \langle v_3,\omega_2\rangle\omega_2\| = \sqrt{2^2+1^2+1^2} = \sqrt{6}$

$\therefore \omega_3 = (2/\sqrt{6},\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}).$

$\langle v_1, w_1 \rangle = \langle (1,-1,-1), (\frac{1}{\sqrt3}, \frac{-1}{\sqrt3}, \frac{-1}{\sqrt3}) \rangle = \sqrt3.$

$\langle v_2, w_1 \rangle = \langle (0,2,-2), (\frac{1}{\sqrt3}, \frac{-1}{\sqrt3}, \frac{-1}{\sqrt3}) \rangle = 0$

$\langle v_3, w_1 \rangle = \langle (2,0,2), (\frac{1}{\sqrt3}, \frac{-1}{\sqrt3}, \frac{-1}{\sqrt3}) \rangle = 0$

$\langle v_2, w_2 \rangle = \langle (0,2,-2), (0, \frac{1}{\sqrt2}, \frac{-1}{\sqrt2}) \rangle = 2\sqrt2.$

$\langle v_3, w_2 \rangle = \langle (2,0,2), (0, \frac{1}{\sqrt2}, \frac{-1}{\sqrt2}) \rangle = -\sqrt2$

$\langle v_3, w_3 \rangle = \langle (2,0,2), (\frac{2}{\sqrt6}, \frac{1}{\sqrt6}, \frac{1}{\sqrt6}) \rangle = \sqrt6.$

The QR factorization

$$A = \begin{bmatrix} \frac{1}{\sqrt3} & 0 & 2/\sqrt6 \\ -1/\sqrt3 & 1/\sqrt2 & 1/\sqrt6 \\ -\frac{1}{\sqrt3} & -1/\sqrt2 & 1/\sqrt6 \end{bmatrix} \begin{bmatrix} \sqrt3 & 0 & 0 \\ 0 & 2\sqrt2 & -\sqrt2 \\ 0 & 0 & \sqrt6 \end{bmatrix} //$$

Theorem:- Let A be an $m\times n$ matrix and let $b \in \mathbb{R}^m$ be any vector. Then a vector $x_0 \in \mathbb{R}^n$ is a least square solution of $Ax=b$ if and only if x_0 is a solution of the normal equation

$$\boxed{A^TAx = A^Tb}$$

Fundamental Subspaces:- Let A be a $m\times n$ matrix. Then

$R(A)$ = Row space of A (Spanned by row vectors of A)

$C(A)$ = column space of A (Spanned by column vectors of A

$N(A)$ = Null space of A [$\{x \in \mathbb{R}^n : Ax = 0\}$.]

$N(A^T)$ = Null space of A^T

These subspaces are known as fundamental subspaces.

Theorem:- (i) $N(A) = R(A)^\perp$ hence $R(A) = N(A)^\perp$

and (ii) $N(A^T) = C(A)^\perp$ Hence $C(A) = N(A^T)^\perp$

Theorem:- For any $m\times n$ matrix A, we have

(i) $N(A) \oplus R(A) = \mathbb{R}^n$;

(ii) $N(A^T) \oplus C(A) = \mathbb{R}^m$.

Pb:1 Find all least square solutions of $AX=b$ where $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 5 & 0 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $b = \begin{bmatrix} 4 \\ -11 \\ 19 \end{bmatrix}$

Soln:- To find least square Solution, we need to Solve $A^TA\,x = A^Tb$

Here $A^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 5 \\ 1 & 2 & 0 \end{bmatrix}$; $A^TA = \begin{bmatrix} 3 & 6 & 3 \\ 6 & 30 & 0 \\ 3 & 0 & 5 \end{bmatrix}$

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 5 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -11 \\ 19 \end{bmatrix} = \begin{bmatrix} 12 \\ 114 \\ -18 \end{bmatrix}$$

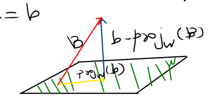
Least Square Solution:-

Consider a $m\times n$ matrix A, $m\times 1$ matrix B and the system

$$AX = B.$$

Suppose the above system is inconsistent we attempt to find a $\hat{x}$ such that $\|A\hat{x} - B\|$ is as small as possible.

The closest $\hat{x}$ is $A\hat{x} = \mathrm{proj}_W(B)$ where W is the column space of A. In this case we say $\hat{x}$ is the least square solution of $Ax = b$



The system $A^TAX = A^Tb$ is

$$\begin{bmatrix} 3 & 6 & 3 \\ 6 & 30 & 0 \\ 3 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 114 \\ -18 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x + 2y + z &= 4 \\ x + 5y + 0z &= 19 \\ 3x + 0y + 5z &= -18 \end{aligned}$$

we solve the above system by Gauss elimination method

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 1 & 5 & 0 & 19 \\ 3 & 0 & 5 & -18 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 3 & -1 & 15 \\ 0 & -6 & 2 & -30 \end{array}\right]$$

$$R_3 \to R_3 + 2R_2 \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 3 & -1 & 15 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The reduced system is $x + 2y + z = 4$
$3y - z = 15$

$\therefore$ All least square solutions are

$$\boxed{x = -6 - \frac{5t}{3};\ y = 5 + \frac{t}{3};\ z = t}$$

Theorem:- Let A be $m\times n$ matrix and let $b \in \mathbb{R}^m$ be any vector. Then a vector $x_0 \in \mathbb{R}^n$ is a least square solution of $Ax = b$ if and only if x_0 is a solution of the equation

$$\boxed{A^T A x = A^T b}$$

Find the least square solution of $Ax = b$ where

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & -1 \\ -1 & 1 & 2 \\ 3 & -5 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Soln:-

$$A^T = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & -3 & 1 & -5 \\ 1 & -1 & 2 & 0 \end{bmatrix}$$

Now, $$A^T b = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & -3 & 1 & -5 \\ 1 & -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

and $$A^T A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & -3 & 1 & -5 \\ 1 & -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & -1 \\ -1 & 1 & 2 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 15 & -24 & -3 \\ -24 & 39 & 3 \\ -3 & 3 & 6 \end{bmatrix}$$

The system $A^T A x = A^T b$ is

$$\begin{bmatrix} 15 & -24 & -3 \\ -24 & 39 & 3 \\ -3 & 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} 5x - 8y - z &= 0 \\ -8x + 13y + z &= -1/3 \\ x - y - 2z &= -1 \end{aligned}$$

To solve this system we proceed by Gauss elimination.

$$\left[\begin{array}{ccc|c} 5 & -8 & -1 & 0 \\ -8 & 13 & 1 & -1/3 \\ 1 & -1 & -2 & -1 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -1 \\ -8 & 13 & 1 & -1/3 \\ 5 & -8 & -1 & 0 \end{array}\right] \begin{array}{l} R_2 \to R_2 + 8R_1 \\ R_3 \to R_3 - 5R_1 \end{array}$$

$$\left[\begin{array}{cccc} 1 & -1 & -2 & -1 \\ 0 & 5 & -15 & \frac{-25}{3} \\ 0 & -3 & 9 & 5 \end{array}\right] \xrightarrow{R_3 \to \frac{R_3}{-3}} \left[\begin{array}{cccc} 1 & -1 & 2 & -1 \\ 0 & 5 & -15 & \frac{-25}{3} \\ 0 & 1 & -3 & -5/3 \end{array}\right] R_3 \to R_3 - \frac{R_2}{5}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 0 & 5 & -15 & -25/3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The row reduced equation is

$$x - y + 2z = -1$$
$$5y - 15z = -25/3$$

All the least square solutions are

$$\boxed{x = \frac{-8}{3} + t;\ y = \frac{-5}{3} + 3t;\ z = t} \quad t \in \mathbb{R},$$

Pb 2 Find all least square solutions of $Ax = b$ where

$$A = \begin{bmatrix} 1 & 3 & -3 \\ 2 & 4 & -2 \\ 0 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Soln:-

$$A^T = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 4 & -1 & 2 \\ -3 & -2 & 2 & -1 \end{bmatrix}$$

; Now, $$A^T b = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 4 & -1 & 2 \\ -3 & -2 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 4 & -1 & 2 \\ -3 & -2 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -3 \\ 2 & 4 & -2 \\ 0 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 13 & -8 \\ 13 & 30 & -21 \\ -8 & -21 & 18 \end{bmatrix}$$

System $A^T A x = A^T b$

$$\Rightarrow \begin{bmatrix} 6 & -13 & -8 \\ 13 & 30 & -21 \\ -8 & -21 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$$

; we proceed by Gauss elimination,

$$\left[\begin{array}{ccc|c} 6 & -13 & -8 & 2 \\ 13 & 30 & -21 & 5 \\ -8 & -21 & 18 & -4 \end{array}\right] \begin{array}{l} R_2 \to R_2 - \frac{13}{6} R_1 \\ R_3 \to R_3 + \frac{8}{6} R_1 \end{array} \left[\begin{array}{ccc|c} 6 & -13 & -8 & 2 \\ 0 & \frac{349}{6} & \frac{83}{6} & \frac{4}{6} \\ 0 & \frac{-230}{6} & \frac{44}{6} & \frac{-8}{6} \end{array}\right]$$

(Complete!)

AN INTRODUCTION TO CODING THEORY

The coding and decoding of secret messages has been important in times of warfare, of course, but it is also quite valuable in peacetime for keeping government and business secrets under tight security. Throughout history, many ingenious coding mechanisms have been proposed. One of the simplest is the substitution cipher, in which an array of symbols is used to assign each character of a given text (plaintext) to a corresponding character in coded text (ciphertext). For example, consider the cipher array in the following

R	W	E	H	S	K	T	Q	X	B	Z	F	P	U	M	C	Y	I	G	V	O	D	A	N	L	J
[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]	[12]	[13]	[14]	[15]	[16]	[17]	[18]	[19]	[20]	[21]	[22]	[23]	[24]	[25]	[26]
↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z

AN INTRODUCTION TO CODING THEORY

Using, the above cipher array the message

LINEAR ALGEBRA IS EXCITING

is encoded as

FXUSRI RFTSWIR XG SNEXVXUT.

The above encoded message can be decoded by reversing the process. In fact, we can create an "inverse" array, or decipher array, by the following way

W	J	P	V	C	L	S	D	R	Z	F	Y	O	X	U	M	H	A	E	G	N	T	B	I	Q	K
[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]	[12]	[13]	[14]	[15]	[16]	[17]	[18]	[19]	[20]	[21]	[22]	[23]	[24]	[25]	[26]
↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑	↑
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z

AN INTRODUCTION TO CODING THEORY

Hill Substitution

Consider an nonsingular $n \times n$ matrix A. (Usually A is chosen with integer entries). Split the message into blocks of n symbols each and replace each symbol with an integer value. To simplify things, we replace each letter by its position in the alphabet. The last block may have to be "padded" with random values to ensure that each block contains exactly n integers. In effect,we are creating a set of n-vectors that we can label as x_1, x_2, and so on. We then multiply the matrix A by each of these vectors in turn to produce the following new set of $n-$vectors: Ax_1, Ax_2, and so on. When these vectors are concatenated together, they form the coded message. The matrix A used in the process is often called the key matrix, or encoding matrix.

AN INTRODUCTION TO CODING THEORY

Hill Substitution

Suppose we wish to encode the message LINEAR ALGEBRA IS EXCITING using the key matrix

$$\mathbf{A} = \begin{bmatrix} -7 & 5 & 3 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \end{bmatrix}.$$

Since we are using a 3×3 matrix, we break the characters of the message into blocks of length 3 and replace each character by its position in the alphabet. This procedure gives

$$\underbrace{\begin{array}{c} \text{L} \\ \text{I} \\ \text{N} \end{array}\begin{bmatrix} 12 \\ 9 \\ 14 \end{bmatrix}}_{\mathbf{x}_1}, \quad \underbrace{\begin{array}{c} \text{E} \\ \text{A} \\ \text{R} \end{array}\begin{bmatrix} 5 \\ 1 \\ 18 \end{bmatrix}}_{\mathbf{x}_2}, \quad \underbrace{\begin{array}{c} \text{A} \\ \text{L} \\ \text{G} \end{array}\begin{bmatrix} 1 \\ 12 \\ 7 \end{bmatrix}}_{\mathbf{x}_3}, \quad \underbrace{\begin{array}{c} \text{E} \\ \text{B} \\ \text{R} \end{array}\begin{bmatrix} 5 \\ 2 \\ 18 \end{bmatrix}}_{\mathbf{x}_4},$$

$$\underbrace{\begin{array}{c} \text{A} \\ \text{I} \\ \text{S} \end{array}\begin{bmatrix} 1 \\ 9 \\ 19 \end{bmatrix}}_{\mathbf{x}_5}, \quad \underbrace{\begin{array}{c} \text{E} \\ \text{X} \\ \text{C} \end{array}\begin{bmatrix} 5 \\ 24 \\ 3 \end{bmatrix}}_{\mathbf{x}_6}, \quad \underbrace{\begin{array}{c} \text{I} \\ \text{T} \\ \text{I} \end{array}\begin{bmatrix} 9 \\ 20 \\ 9 \end{bmatrix}}_{\mathbf{x}_7}, \quad \underbrace{\begin{array}{c} \text{N} \\ \text{G} \\ \text{-} \end{array}\begin{bmatrix} 14 \\ 7 \\ 27 \end{bmatrix}}_{\mathbf{x}_8},$$

AN INTRODUCTION TO CODING THEORY

Hill Substitution

where the last entry of the last vector was chosen outside the range from 1 to 26. Now, forming the products with $\mathbf{A}$, we have

$$\mathbf{A}\mathbf{x}_1 - \begin{bmatrix} 7 & 5 & 3 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \end{bmatrix}\begin{bmatrix} 12 \\ 9 \\ 14 \end{bmatrix} - \begin{bmatrix} 3 \\ -10 \\ 4 \end{bmatrix},$$

$$\mathbf{A}\mathbf{x}_2 = \begin{bmatrix} -7 & 5 & 3 \\ 3 & 2 & 2 \\ 3 & -2 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \\ 18 \end{bmatrix} = \begin{bmatrix} 24 \\ 23 \\ -5 \end{bmatrix}, \text{ and so on.}$$

The final encoded text is

$$\begin{array}{cccccccccccc} 3 & -10 & 4 & 24 & -23 & -5 & 74 & -35 & -28 & 29 & -25 & -7 \\ 95 & -53 & -34 & 94 & -39 & -36 & 64 & -31 & -22 & 18 & -26 & 1. \end{array}$$

AN INTRODUCTION TO CODING THEORY

Hill Substitution

The code produced by a Hill substitution is much harder to break than a simple substitution cipher, since the coding of a given letter depends not only on the way the text is broken into blocks, but also on the letters adjacent to it. (Nevertheless, there are techniques to decode Hill substitutions using high-speed computers.) However, a Hill substitution is easy to decode if you know the inverse of the key matrix.

$$\mathbf{A}^{-1} = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{bmatrix}.$$

Breaking the encoded text back into 3-vectors and multiplying $\mathbf{A}^{-1}$ by each of these vectors in turn restores the original message. For example,

$$\mathbf{A}^{-1}(\mathbf{A}\mathbf{x}_1) = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -10 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 9 \\ 14 \end{bmatrix} = \mathbf{x}_1,$$

which represents the first three letters LIN.

Problem 1:

The alphabets A to Z are encoded using $A \leftrightarrow 0, B \leftrightarrow 1, \ldots Z \leftrightarrow 25$. The encrypted cipher text is the sequence of numbers 19,95, 209,10, 26, 54. The matrix used to encrypt is $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 3 & 2 \\ 6 & 7 & 4 \end{bmatrix}$. Find the original message.

Soln:- Given $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 3 & 2 \\ 6 & 7 & 4 \end{bmatrix}$

$Ax_1 = \begin{bmatrix} 19 \\ 95 \\ 209 \end{bmatrix}$; $Ax_2 = \begin{bmatrix} 10 \\ 26 \\ 54 \end{bmatrix}$

Here $A^{-1} = \begin{bmatrix} -2 & 7 & -3 \\ 0 & -2 & 1 \\ 3 & -7 & 3 \end{bmatrix}$

$A^{-1}(Ax_1) = \begin{bmatrix} -2 & 7 & -3 \\ 0 & -2 & 1 \\ 3 & -7 & 3 \end{bmatrix} \begin{bmatrix} 19 \\ 95 \\ 209 \end{bmatrix} = \begin{bmatrix} 0 \\ 19 \\ 19 \end{bmatrix}$

$A^{-1}(Ax_2) = \begin{bmatrix} -2 & 7 & -3 \\ 0 & -2 & 1 \\ 3 & -7 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 26 \\ 54 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 10 \end{bmatrix}$

Original coded message is

0	19	19	0	2	10
A	T	T	A	C	K

Problem 2:

The alphabets A to Z are encoded using $A \leftrightarrow 0, B \leftrightarrow 1, \ldots Z \leftrightarrow 25$. The encrypted cipher text is the sequence of numbers 19,95, 209,10, 26, 54. The matrix used to encrypt is $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 3 & 2 \\ 6 & 7 & 4 \end{bmatrix}$. Find the original message.

Problem 3:

Decode the message "22, 53, 51, 27, 75, 62, 9, 40, 36, 18, 55, 46, 20, 65, 72" by using the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

(A-1, B-2, ..., Z-26, Blank Space – 27).

Soln:- $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$

$Ax_1 = \begin{bmatrix} 22 \\ 53 \\ 51 \end{bmatrix}$; $Ax_2 = \begin{bmatrix} 27 \\ 75 \\ 62 \end{bmatrix}$; $Ax_3 = \begin{bmatrix} 9 \\ 40 \\ 36 \end{bmatrix}$

$Ax_4 = \begin{bmatrix} 18 \\ 55 \\ 46 \end{bmatrix}$; $Ax_5 = \begin{bmatrix} 20 \\ 65 \\ 72 \end{bmatrix}$

$A^{-1}(Ax_1) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 22 \\ 53 \\ 51 \end{bmatrix} = \begin{bmatrix} 22 \\ 9 \\ 20 \end{bmatrix}$

$A^{-1}(Ax_2) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 27 \\ 75 \\ 62 \end{bmatrix} = \begin{bmatrix} 27 \\ 21 \\ 14 \end{bmatrix}$

$A^{-1}(Ax_3) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 22 \\ 5 \end{bmatrix} = \begin{bmatrix} 9 \\ 22 \\ 5 \end{bmatrix}$

$A^{-1}(Ax_4) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 18 \\ 55 \\ 46 \end{bmatrix} = \begin{bmatrix} 18 \\ 19 \\ 9 \end{bmatrix}$

$A^{-1}(Ax_5) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ 65 \\ 72 \end{bmatrix} = \begin{bmatrix} 20 \\ 25 \\ 27 \end{bmatrix}$

$\therefore$ The original message

22	9	20	27	21	14	9	22	5	18	19	9	20	25	27
V	I	T		U	N	I	V	E	R	S	I	T	Y	

(Try!)

The alphabets A to Z are encoded using $A \leftrightarrow 0$; $B \leftrightarrow 1$, $C \leftrightarrow 2, \ldots, Z \leftrightarrow 25$. The encrypted cipher text is the sequence of numbers 19, 45, 26, 13, 36, 41. The matrix used to encrypt is $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Find the original message.

Ans:- THANKS

Application of linear algebra into Differential equations:-

Consider a differential equation $\frac{dy}{dx} + ay = 0$.

$$\frac{dy}{dx} = -ay$$

$$\Rightarrow \int \frac{dy}{y} = -\int a\,dx$$

$$\Rightarrow \log y = -ax$$

$$\boxed{y = e^{-ax}}$$

Is it the only solution to the above differential equation?

Let $k \in \mathbb{R}$, y be a solution of $\frac{dy}{dx} + ay = 0$ then

$$\frac{d(ky)}{dx} + a(ky) = k\frac{dy}{dx} + kay = k\left(\frac{dy}{dx} + ay\right)$$

$$= 0$$

$\therefore$ ky is also solution

Hence $\{y(x) \in C^\infty(\mathbb{R}) \mid \frac{dy}{dx} + ay = 0\}$ is a subspace of $C^\infty(\mathbb{R})$.

Consider a differential equation

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

Auxillary equation is $m^2 + am + b = 0$

solve this; Suppose $a^2 - 4b > 0$

then the above equation has real distinct roots, say r_1, r_2

$$y(x) = Ae^{r_1 x} + Be^{r_2 x}$$

Does y have any other solution?

Q:- Prove that $\{y(x) \in C^\infty(\mathbb{R}) \mid \frac{dy}{dx} + ay = 0\}$ is a one dimensional subspace.

Soln:- Take $y(x) = e^{-ax}$ then

$$\frac{d(e^{-ax})}{dx} + a(e^{-ax}) = -ae^{-ax} + ae^{-ax} = 0$$

Hence $y(x)$ is a solution of $\frac{dy}{dx} + ay = 0$

Let $p(x)$ be any other solution of $\frac{dy}{dx} + ay = 0$

Now, Define $z(x) = p(x)e^{ax}$

$$\frac{dz}{dx} = p(x)ae^{ax} + e^{ax}p'(x) \qquad [p'(x) + ap(x) = 0$$
$$= p(x)ae^{ax} - ap(x)e^{ax} \qquad p'(x) = -ap(x)]$$
$$= 0$$

$$\Rightarrow z(x) = c$$

$$\Rightarrow p(x)e^{ax} = c \Rightarrow p(x) = ce^{-ax}.$$

Hence the given subspace is one dimensional

Consider a vector space

$$C^\infty(\mathbb{R}) = \{f \mid f: \mathbb{R} \to \mathbb{R}, \text{ infinitely many times differentiable}\}$$

is a vector space over $\mathbb{R}$ with respect to addition and scalar multiplication

$$(f+g)(x) = f(x) + g(x)$$
$$(kf)(x) = kf(x)$$

Q:- Prove that the set

$\{y(x) \in C^\infty(\mathbb{R}) \mid \frac{dy}{dx} + ay = 0\}$ is a subspace of $C^\infty(\mathbb{R})$.

Soln:- The function $y(x) = 0$ is always a solution of $\frac{dy}{dx} + ay = 0$.

Let $y_1(x)$, $y_2(x)$ be two solutions of

$$\frac{dy}{dx} + ay = 0$$

$$\frac{d(y_1+y_2)}{dx} + a(y_1+y_2) = \frac{dy_1}{dx} + \frac{dy_2}{dx} + ay_1 + ay_2$$

$$= 0$$

thus $y_1 + y_2$ is also a solution

Q:- Consider a subset $W = \{y(x) \in C^\infty(\mathbb{R}) \mid \frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0\}$ then W is a subspace of $C^\infty(\mathbb{R})$.

Soln:- $y(x) = 0$ is always a solution of $\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

Let $y_1, y_2 \in W$ then

$$\frac{d^2(y_1+y_2)}{dx} + b\frac{d(y_1+y_2)}{dx} + c(y_1+y_2)$$
$$= \frac{d^2y_1}{dx} + \frac{d^2y_2}{dx} + b\frac{dy_1}{dx} + b\frac{dy_2}{dx} + cy_1 + cy_2$$
$$= \left(\frac{d^2y_1}{dx} + b\frac{dy_1}{dx} + cy_1\right) + \left(\frac{d^2y_2}{dx} + b\frac{dy_2}{dx} + cy_2\right)$$
$$= 0 + 0 = 0$$

Hence $y_1 + y_2 \in W$.

Let $k \in \mathbb{R}$, $y \in W$ then

$$\frac{d^2(ky)}{dx} + b\left(\frac{ky}{dx}\right) + cky$$
$$= k\frac{d^2y}{dx^2} + kb\frac{dy}{dx} + kcy$$
$$= k\underbrace{\left(\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy\right)}_{0}$$
$$= k \cdot 0 = 0$$

Hence W is a subspace.

Application of linear algebra in Differential equations:-

$C^{\infty}(\mathbb{R}) = \{f \mid f:\mathbb{R}\to\mathbb{R},\ f$ is infinitely many times differentiable$\}$

$W = \left\{ y(x) \in C^{\infty}(\mathbb{R}) \;\middle|\; \frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \right\}$

is a subspace of $C^{\infty}(\mathbb{R})$

Consider the subspace

$$W = \left\{ y(x) \in C^{\infty}(\mathbb{R}) \;\middle|\; \frac{d^2y}{dx^2} + b\frac{dy}{dx} + c\,y(x) = 0 \text{ with } b^2-4c>0 \right\}$$

T.P W is a 2-dimensional subspace.

Step 1:- consider the quadratic equation

$$m^2 + bm + c = 0 \rightarrow (1)$$

Since $b^2-4c>0$, the above equation has real distinct roots r_1, r_2. Thus, $r_1^2+br_1+c=0$ and $r_2^2+br_2+c=0$

Step 2:- We prove e^{r_1x}, e^{r_2x} is a solution of given differential equation

$$\frac{d^2}{dx^2}(e^{r_1x}) + b\frac{d}{dx}(e^{r_1x}) + c(e^{r_1x})$$

$$= r_1^2 e^{r_1x} + b r_1 e^{r_1x} + c e^{r_1x}$$

$$= \underbrace{(r_1^2 + br_1 + c)}\, e^{r_1x}$$

$$= 0 \cdot e^{r_1x} = 0$$

Hence e^{r_1x} is a solution, similarly e^{r_2x} is also a solution.

Step 3:- For the differential equation

$$\frac{dy}{dx} - r_2 y = e^{r_1x}$$

has a solution $y = \dfrac{e^{r_1x}}{r_1-r_2}$.

proof:-

$$\frac{d}{dx}\left(\frac{e^{r_1x}}{r_1-r_2}\right) - r_2\left(\frac{e^{r_1x}}{r_1-r_2}\right)$$

$$= \frac{r_1 e^{r_1x}}{r_1-r_2} - \frac{r_2 e^{r_1x}}{r_1-r_2}$$

$$= e^{r_1x} \,//.$$

Step 4:- W is a 2-dimensional subspace.

Suppose $p(x)$ be some solution of $\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ then

$$\frac{d^2p(x)}{dx^2} + b\frac{dp}{dx} + cp = 0$$

$$\Rightarrow \left(\frac{d}{dx} - r_1\right)\left[\left(\frac{d}{dx} - r_2\right)p\right] = 0$$

we have prove $\frac{dy}{dx} - r_1 y = 0$ is a one dimensional subspace.

Hence

$$\left(\frac{d}{dx} - r_2\right)p = \alpha e^{r_1x} = \alpha\left(\frac{d}{dx} - r_2\right)\left(\frac{r_1 e^{r_1x}}{r_1-r_2}\right) \quad \text{(Step 3)}$$

$$\Rightarrow \left(\frac{d}{dx} - r_2\right)\left(p - \frac{\alpha r_1 e^{r_1x}}{r_1-r_2}\right) = 0$$

$$p - \frac{\alpha r_1 e^{r_1x}}{r_1-r_2} = \beta e^{r_2x}$$

$$\Rightarrow p = \frac{\alpha r_1}{r_1-r_2} e^{r_1x} + \beta e^{r_2x}$$

$$\Rightarrow p = \alpha' e^{r_1x} + \beta e^{r_2x}.$$

$\therefore$ W is a 2-dimensional subspace.

Application of linear algebra in error correcting codes:-

Binary operation we transmit a word or number with a string which is of number either 0 (or 1).

For example:-

$7 = 1\ 1\ 1$

$5 = 1\ 0\ 1$

Addion in binary operation:-

+	0	1
0	0	1
1	1	0

multiplication is binary

×	0	1
0	0	0
1	0	1

meaning of -1 in binary operation is 1

$1 + 1 = 0$

$1 = -1$

[

Encoding:– Encoding a message is transforming a word (or) number W of length $\underline{n}$ into another word C of length $n+m$ by augmenting W with additional bits called parity check bits.

Example:– Simplest parity check code.

If the number of ones in the word is
- even: add 0 to the word
- odd: Add 1 to word.

Examples:– Using parity check code encode the following

(a) $W = (1\ 0\ 0\ 0\ 1\ 1)$

(b) $W = (1\ 1\ 1\ 0\ 0\ 1)$

Ans:– (a) encoded words $C = (1\ 0\ 0\ 0\ 1\ 1\ \boxed{1})$

(b) encoded word $C = (1\ 1\ 1\ 0\ 0\ 1\ \boxed{0})$

Pb. 2 Encode the word $W = (0\ 0\ 1\ 1)$ by using Hamming (7,4) code.

Soln:–

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore$ Encoding word $C = (1\ 0\ 0\ 0\ 0\ 1\ 1)$

$$\boxed{\text{message}} \xrightarrow{\text{encoding}} \boxed{\text{code word}} \xrightarrow{\text{transmission}} \overset{\text{Noise}}{\Big\}}\boxed{\text{received message}} \xrightarrow{\text{decoding}} \boxed{\text{decoded message (or) error indication}}$$

$W = (1\ 1\ 1\ 0\ 0\ 1\ 1)$

$= (0\ 0\ 1\ 1\ 1\ 0\ 0)$

Hamming (7,4) code:–

Let $W = (w_1, w_2, w_3\ w_4)$ is a 4-bit word using Hamming idea we encode the above word by the following way

$C = (c_1\ c_2\ w_1\ c_3\ w_2\ w_3\ w_4)$

where c_1, c_2, c_3 are parity check bits

$c_1 = w_1 + w_2 + w_4$

$c_2 = w_1 + w_3 + w_4$

$c_3 = w_2 + w_3 + w_4$.

Error detection:–

$$\left.\begin{aligned} c_3 &= w_2 + w_3 + w_4 \\ c_2 &= w_1 + w_3 + w_4 \\ c_1 &= w_1 + w_2 + w_4 \end{aligned}\right\} \Rightarrow \underbrace{\begin{aligned} c_3 - w_2 - w_3 - w_4 &= 0 \\ c_2 - w_1 - w_3 - w_4 &= 0 \\ c_1 - w_1 - w_2 - w_4 &= 0 \end{aligned}}_{\Downarrow}$$

$c_3 + w_2 + w_3 + w_4 = 0$

$c_2 + w_1 + w_3 + w_4 = 0$

$c_1 + w_1 + w_2 + w_4 = 0$

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}}_{H} \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ w_1 \\ c_3 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}}_{C^T} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{=0}$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$$

Pb: 1 Encode the word $W = (1\ 0\ 1\ 1)$ using Hamming (7,4) code.

Soln:–

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+0+0+1 \\ 1+0+1+1 \\ 0+0+1+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Encoded word is $C = (0\ 1\ 1\ 0\ 0\ 1\ 1)$

Assume the received message is has a single error. How to identify it.

we have $HC^T = 0$

Suppose the received message is R with single error. Let $E = (e_1\ e_2\ e_3\ e_4\ e_5\ e_6\ e_7)$

The entries of E are

$$e_i = \begin{cases} 1 & \text{if the noise changes } i\text{th bit} \\ 0 & \text{if noise does not change the } i\text{th bit.} \end{cases}$$

clearly

$$\begin{aligned} R &= C + E \\ R^T &= C^T + E^T \\ HR^T &= H(C^T + E^T) \\ &= \underbrace{HC^T}_{0} + HE^T \\ &= HE^T \end{aligned}$$

$$HF^T = \begin{pmatrix} e_4 + e_5 + e_6 + e_7 \\ e_2 + e_3 + e_6 + e_7 \\ e_1 + e_3 + e_5 + e_7 \end{pmatrix}$$

$$= e_1\begin{pmatrix}0\\0\\1\end{pmatrix} + e_2\begin{pmatrix}0\\1\\0\end{pmatrix} + \boxed{e_3}\begin{pmatrix}0\\1\\1\end{pmatrix} + e_4\begin{pmatrix}1\\0\\0\end{pmatrix} + e_5\begin{pmatrix}1\\0\\1\end{pmatrix} + e_6\begin{pmatrix}1\\1\\0\end{pmatrix} + e_7\begin{pmatrix}1\\1\\1\end{pmatrix}$$

Find: HR^T

Suppose you received the message

$$R = (1\ 0\ 0\ 1\ 0\ 1\ 0)$$

$$HR^T = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix}1\\0\\0\\1\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix}$$

$\therefore$ Error is on 3^{rd} bit

$\therefore$ Original message is $(1\ 0\ \boxed{1}\ 1\ 0\ 1\ 0)$