MODULE-4

(Applications of Multi Variable Calculus)

TAYLOR'S THEOREM FOR FUNCTIONS OF TWO VARIABLES

Taylor's theorem for single variable

$$f(x+h) = f(x) + h\,\frac{\partial f}{\partial x} + \frac{h^2}{\lfloor 2}\frac{\partial^2 f}{\partial x^2} + \frac{h^3}{\lfloor 3}\frac{\partial^3 f}{\partial x^3} + \ldots$$

Let $f(x, y)$ be a function of two independent variables x and y. If the function $f(x, y)$ and its partial derivatives up to nth order are continuous throughout the domain centred at a point (x, y). Then

$$f(a + h, b + k) = f(a, b) + \left[h\frac{\partial f(a,b)}{\partial x} + k\frac{\partial f(a,b)}{\partial y}\right]$$

$$+ \frac{1}{\lfloor 2}\left[h^2\frac{\partial^2 f(a,b)}{\partial x^2} + 2hk\frac{\partial^2 f(a,b)}{\partial x \partial y} + k^2\frac{\partial^2 f(a,b)}{\partial y^2}\right]$$

$$+ \frac{1}{\lfloor 3}\left[h^3\frac{\partial^3 f(a,b)}{\partial x^3} + 3h^2k\frac{\partial^3 f(a,b)}{\partial x^2 \partial y} + 3hk^2\frac{\partial^3 f(a,b)}{\partial x \partial y^2} + k^3\frac{\partial^3 f(a,b)}{\partial y^3}\right] + \ldots$$

Or

$$f(a + h, b + k) = f(a, b) + \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]f(a,b) + \frac{1}{\lfloor 2}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^2 f(a, b)$$

$$+ \frac{1}{\lfloor 3}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^3 f(a,b) + \ldots$$

Proof. Suppose $P(x, y)$ and $Q(x + h, y + k)$ be two neighbouring points. Then $f(x + h, y + k)$, the value of f at Q can be expressed in terms of f and its derivatives at P.

Here, we treat $f(x + h, y + k)$ as a function of single variable x and keeping y as a constant. Expanded as follows using Taylor's theorem for single variable.*

$$f(x+h, y+k) = f(x, y+k) + h\frac{\partial f(x,y+k)}{\partial x} + \frac{h^2}{\lfloor\underline{2}}\frac{\partial^2 f(x,y+k)}{\partial x^2} + \ldots \qquad \ldots(i)$$

Now expanding all the terms on the R.H.S. of (i) as function of y, keeping x as constant.

$$f(x+h, y+k) = \left[f(x,y) + k\frac{\partial f(x,y)}{\partial y} + \frac{k^2}{\lfloor\underline{2}}\frac{\partial^2 f(x,y)}{\partial y^2} + \ldots\right]$$

$$+ h\frac{\partial}{\partial x}\left[f(x,y) + k\frac{\partial f(x,y)}{\partial y} + \frac{k^2}{\lfloor\underline{2}}\frac{\partial^2 f(x,y)}{\partial y^2} + \ldots\right]$$

$$+ \frac{h^2}{\lfloor\underline{2}}\frac{\partial^2}{\partial x^2}\left[f(x,y) + k\frac{\partial f(x,y)}{\partial y} + \frac{k^2}{\lfloor\underline{2}}\frac{\partial^2 f(x,y)}{\partial y^2} + \ldots\right] + \ldots$$

$$f(x+h, y+k) = f(x, y) + \left[h\frac{\partial f(x,y)}{\partial x} + k\frac{\partial f(x,y)}{\partial y}\right]$$

$$+ \frac{1}{\lfloor 2}\left[h^2\frac{\partial^2 f(x,y)}{\partial x^2} + 2hk\frac{\partial^2 f(x,y)}{\partial x \partial y} + k^2\frac{\partial^2 f(x,y)}{\partial y^2}\right] + \ldots$$

For any point (a, b) putting $x = a$, $y = b$ in above equation then, we get

$$f(a+h, b+k) = f(a, b) + \left[h\frac{\partial f(a,b)}{\partial x} + k\frac{\partial f(a,b)}{\partial y}\right]$$

$$+ \frac{1}{\lfloor 2}\left[h^2\frac{\partial^2 f(a,b)}{\partial x^2} + 2hk\frac{\partial^2 f(a,b)}{\partial x \partial y} + k^2\frac{\partial^2 f(a,b)}{\partial y^2}\right] + \ldots$$

Or

$$f(a+h, b+k) = f(a, b) + \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]f(a,b) + \frac{1}{\lfloor 2}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^2 f(a, b)$$

$$+ \frac{1}{\lfloor 3}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^3 f(a,b) + \ldots$$

Hence proved.

Alternative form:

Putting $a + h = x \Rightarrow h = x - a$

$b + k = y \Rightarrow k = y - b$

then $f(x, y) = f(a, b) + \left[(x-a)\dfrac{\partial}{\partial x} + (y-b)\dfrac{\partial}{\partial y}\right] f(a, b) + \dfrac{1}{\lfloor 2}\left[(x-a)\dfrac{\partial}{\partial x} + (y-b)\dfrac{\partial}{\partial y}\right]^2 f(a,b) + \ldots \quad \ldots(ii)$

Maclaurin's Series Expansion

It is a special case of Taylor's series when the expansion is about the origin (0, 0).

So, putting $a = 0$ and $b = 0$ in equation (2), we get

$$f(x, y) = f(0, 0) + \left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right] f(0,0) + \frac{1}{\lfloor 2}\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right]^2 f(0,0) + \ldots$$

Example

Expand $e^x \cos y$ about the point $\left(1, \frac{\pi}{4}\right)$

Sol. We have $f(x, y) = e^x \cos y$ ...(*i*)

and $a = 1, b = \dfrac{\pi}{4}, f\left(1, \dfrac{\pi}{4}\right) = e \cos \dfrac{\pi}{4} = \dfrac{e}{\sqrt{2}}$

$\therefore$ From (1)

$$\frac{\partial f}{\partial x} = e^x \cos y \Rightarrow \frac{\partial f\left(1, \frac{\pi}{4}\right)}{\partial x} = e \cos \frac{\pi}{4} = \frac{e}{\sqrt{2}}$$

$$\frac{\partial f}{\partial y} = -e^x \sin y \Rightarrow \frac{\partial f\left(1, \frac{\pi}{4}\right)}{\partial y} = -e \sin \frac{\pi}{4} = -\frac{e}{\sqrt{2}}$$

$$\frac{\partial^2 f}{\partial x^2} = e^x \cos y \Rightarrow \frac{\partial^2 f\left(1, \frac{\pi}{4}\right)}{\partial x^2} = \frac{e}{\sqrt{2}}, \frac{\partial^2 f}{\partial x \partial y} = -e^x \sin y$$

$$\frac{\partial^2 f\left(1, \frac{\pi}{4}\right)}{\partial x \partial y} = -\frac{e}{\sqrt{2}}, \frac{\partial^2 f}{\partial y^2} = -e^x \cos y = -\frac{e}{\sqrt{2}}.$$

By Taylor's theorem, we have

$$f\left(1+h,\frac{\pi}{4}+k\right) = f\left(1,\frac{\pi}{4}\right) + \left(h\frac{\partial f\left(1,\frac{\pi}{4}\right)}{\partial x}+k\frac{\partial f\left(1,\frac{\pi}{4}\right)}{\partial y}\right) + \ldots \qquad \ldots(ii)$$

Let $1 + h = x \Rightarrow h = x - 1$ and $\frac{\pi}{4} + k = y \Rightarrow k = y - \frac{\pi}{4}$, equation (2) reduce in the form

$$f((x, y) = e^x \cos y = \frac{e}{\sqrt{2}} + (x-1)\cdot\frac{e}{\sqrt{2}} + \left(y-\frac{\pi}{4}\right)\left(-\frac{e}{\sqrt{2}}\right) + \frac{1}{\lfloor 2}\left[(x-1)^2\cdot\frac{e}{\sqrt{2}}\right.$$

$$\left. + 2(x-1)\left(y-\frac{\pi}{4}\right)\left(-\frac{e}{\sqrt{2}}\right)+\left(y-\frac{\pi}{4}\right)^2\left(-\frac{e}{\sqrt{2}}\right)\right] + \ldots$$

$$\Rightarrow f(x, y) = \frac{e}{\sqrt{2}}\left[1+(x-1)-\left(y-\frac{\pi}{4}\right)+\frac{(x-1)^2}{2}-(x-1)\left(y-\frac{\pi}{4}\right)-\left(y-\frac{\pi}{4}\right)^2+\ldots\right].$$

Example

Expand $f(x, y) = e^y \log(1 + x)$ in powers of x and y about $(0, 0)$

Sol. We have $f(x, y) = e^y \log(1 + x)$

Here, $a = 0$ and $b = 0$, then $f(0, 0) = e^0 \log 1 = 0$

Now,
$$\frac{\partial f}{\partial x} = \frac{e^y}{1+x} \Rightarrow \frac{\partial f}{\partial x}(0,0) = 1$$

$$\frac{\partial f}{\partial y} = e^y \log(1 + x) \Rightarrow \frac{\partial f(0,0)}{\partial x} = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{e^y}{1+x} \Rightarrow \frac{\partial^2 f(0,0)}{\partial x \partial y} = 1$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{e^y}{(1+x)^2} \Rightarrow \frac{\partial^2 f(0,0)}{\partial x^2} = -1$$

$$\frac{\partial^2 f}{\partial y^2} = e^y \log(1 + x) \Rightarrow \frac{\partial^2 f(0,0)}{\partial y^2} = 0$$

Now, applying Taylor's theorem, we get

$$f(0+h, 0+k) = f(h, k) = f(0, 0) + \left(h\frac{\partial f(0,0)}{\partial x} + k\frac{\partial f(0,0)}{\partial y}\right) + \frac{1}{\underline{|2}}\left(h^2\frac{\partial^2 f(0,0)}{\partial x^2}\right.$$

$$\left. + 2hk\,\frac{\partial^2 f(0,0)}{\partial x \partial y} + k^2\frac{\partial^2 f(0,0)}{\partial y^2}\right) + \ldots$$

Let $h = x$, $k = y$, then, we get

$$f(x, y) = e^y \log(1+x) = f(0,0) + \left(x\frac{\partial f(0,0)}{\partial x} + y\frac{\partial f(0,0)}{\partial y}\right) + \ldots$$

$$= 0 + (x \times 1 + y \times 0) + \frac{1}{\underline{|2}}\,[x^2(-1) + 2xy \times 1 + y^2 \times 0] + \ldots$$

$$\Rightarrow \qquad e^y \log(1+x) = x - \frac{x^2}{2} + xy + \ldots .$$

CHECK YOUR PROGRESS

1. Expand $f(x, y) = x^2 + xy + y^2$ in powers of $(x - 1)$ and $(y - 2)$.

   [**Ans.** $f(x, y) = 7 + 4(x - 1) + 5(y - 2) + (x - 1)^2 + (x - 1)(y - 2) + (y - 2)^2 + \ldots$]

2. Evaluate $\tan^{-1}\left(\dfrac{0.9}{1.1}\right)$. [**Ans.** 0.6904]

3. Expand $f(x, y) = \sin(xy)$ about the point $(1, \pi/2)$ up to and second degree term.

   $$\left[\textbf{Ans.}\ \ f(x,y)=1-\frac{\pi^2}{8}(x-1)^2-\frac{\pi}{2}(x-1)\left(y-\frac{\pi}{2}\right)-\frac{1}{2}\left(y-\frac{\pi}{2}\right)^2\right]+\ldots$$

4. Obtain Taylor's expansion of $x^2y + 3y - 2$ in powers of $(x - 1)$ and $(y + 2)$.

   [**Ans.** $f(x, y) = -10 - 4(x - 1) + 4(y + 2) - 2(x - 1)^2 + \ldots$]

5. Expand e^{xy} in powers of $(x - 1)$ and $(y - 1)$.

   $$\left[\textbf{Ans.}\ \ e\left\{1+(x-1)+(y-1)+\frac{(x-1)^2}{2}+(x-1)(y-1)+\frac{(y-1)^2}{2}+\ldots\right\}\right]$$

6. Expand $\cos x \cos y$ in powers of x and y.

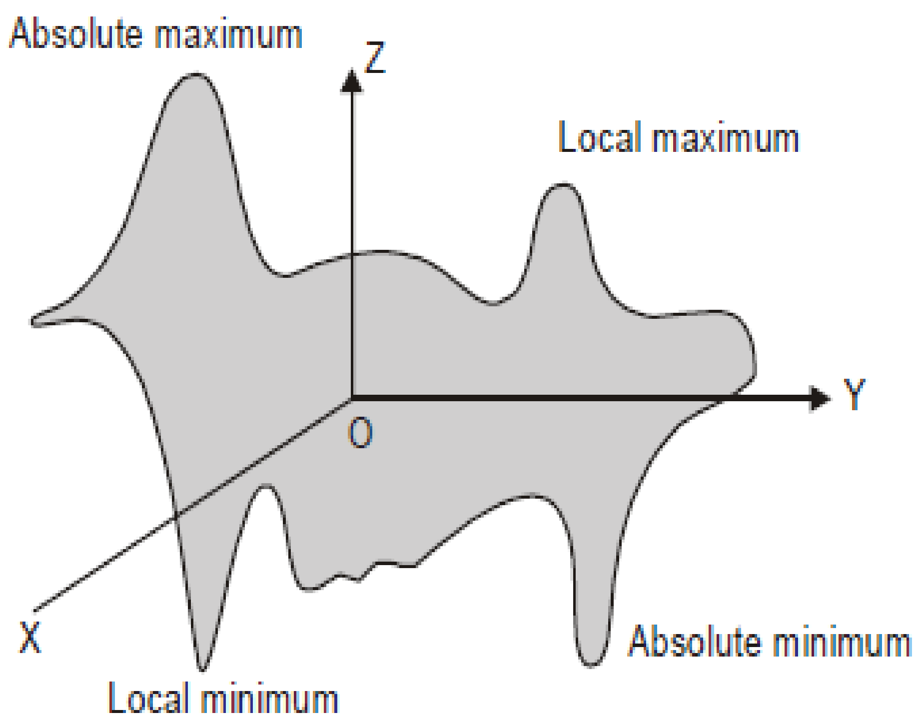
   $$\left[\textbf{Ans.}\ \ f(x,y)=1-\frac{1}{2}(x^2+y^2)+\frac{1}{24}(x^4+6x^2y^2+y^4)+\ldots\right]$$

7. Expand $f(x, y) = e^{2x}\cos 3y$ up to second degree. [**Ans.** $1 + 2x + 2x^2 - \dfrac{9}{2}y^2 + \ldots$]

EXTREMA OF FUNCTION OF SEVERAL VARIABLES

Introduction

In some practical and theoretical problems, it is required to find the largest and smallest values of a function of two variables where the variables are connected by some given relation or condition known as a constraint. For example, if we plot the function $z = f(x, y)$ to look like a mountain range, then the mountain tops or the high points are called local maxima of $f(x, y)$ and valley bottoms or the low points are called local minima of $f(x, y)$. The highest mountain and lowest valley in the entire range are said to be absolute maximum and absolute minimum. The graphical representation is as follows.



Definition

Let $f(x, y)$ be a function of two independent variables x, y such that it is continuous and finite for all values of x and y in the neighbourhood of their values a and b (say) respectively.

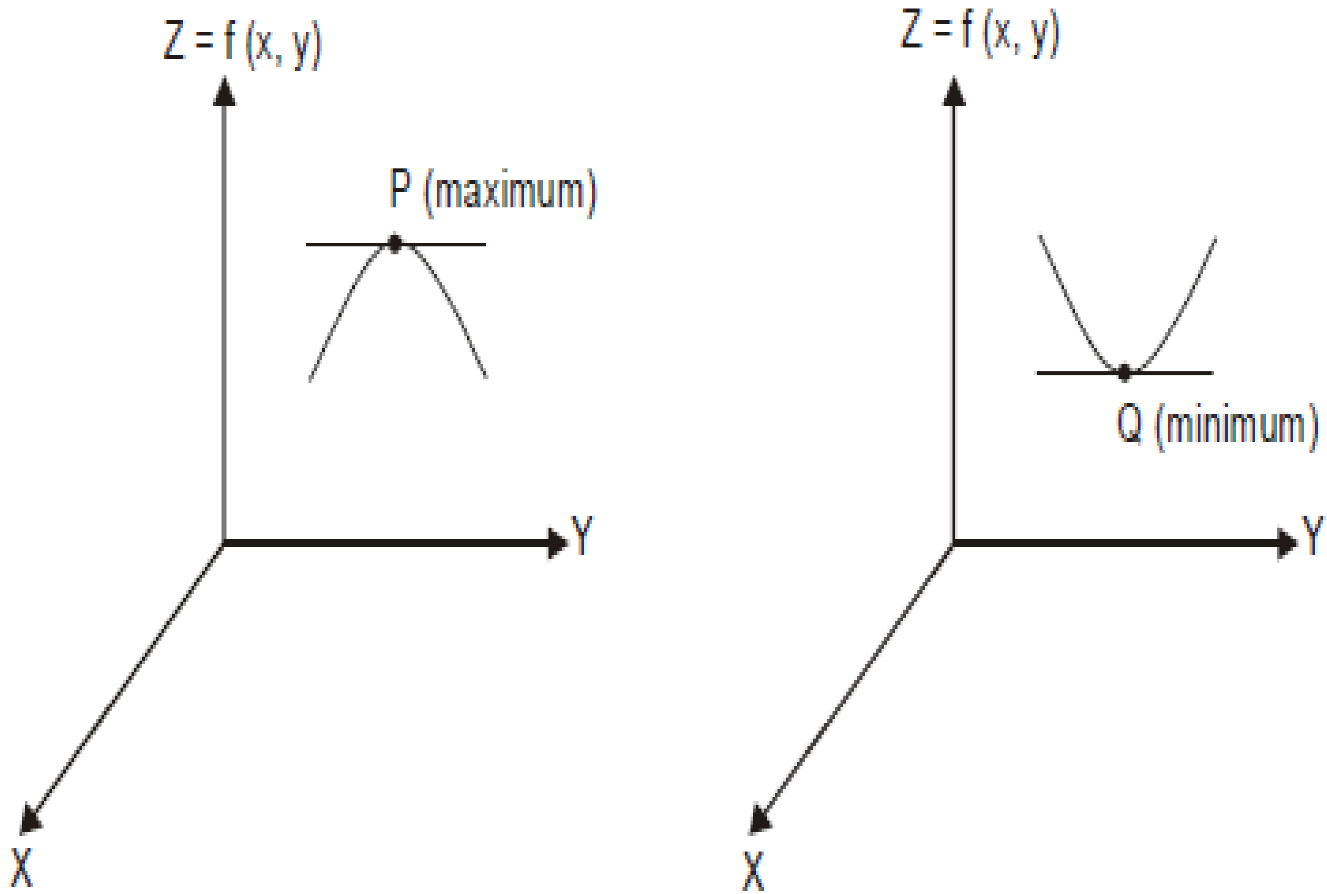
Maximum value: $f(a, b)$ is called maximum value of $f(x, y)$ if $f(a, b) > f(a + h, b + k)$. For small positive or negative values of h and k *i.e.*, $f(a, b)$ is greater than the value of function $f(x, y,)$ at all points in some small *nbd* of (a, b).

Minimum value: $f(a, b)$ is called minimum value of $f(x, y)$ if $f(a, b) < f(a + h, b + k)$.

Note: $f(a + h, b + k) - f(a, b)$ = positive, for Minimum value.
$f(a + h, b + k) - f(a, b)$ = negative, for Maximum value.

Extremum: The maximum or minimum value of the function $f(x, y)$ at any point $x = a$ and $y = b$ is called the extremum value and the point is called "extremum point".

Geometrical representation of maxima and minima: The function $f(x, y)$ represents a surface. The maximum is a point on the surface (hill top). The minimum is a point on the surface (bottom) from which the surface ascends (climbs up) in every direction.



Saddle point: It is a point where function is neither maximum nor minimum. At such point f is maximum in one direction while minimum in another direction.

Condition for the Existence of Maxima and Minima (Extrema)

By Taylor's theorem

$$f(a+h, b+k) = f(a, b) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{(a,b)} + \frac{1}{\underline{|2}}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)} + \ldots \quad \ldots(i)$$

Neglecting higher order terms of h^2, hk, k^2, etc. Since h, k are small, the above expansion reduce to

$$f(a+h, b+k) = f(a, b) + h\frac{\partial f(a,b)}{\partial x} + k\frac{\partial f(a,b)}{\partial y}$$

$$\Rightarrow f(a+h, b+k) - f(a, b) = h\frac{\partial f(a,b)}{\partial x} + k\frac{\partial f(a,b)}{\partial y} \quad \ldots(ii)$$

The necessary condition for a maximum or minimum value (L.H.S. of eqn (*ii*) negative or positive) is

$$h\,\frac{\partial f(a,b)}{\partial x} + k\,\frac{\partial f(a,b)}{\partial y} = 0$$

$$\Rightarrow \qquad \frac{\partial f(a,b)}{\partial x} = 0,\ \frac{\partial f(a,b)}{\partial y} = 0 \ \big|\, h \text{ and } k \text{ can take both } +\text{ve and } -\text{ve value} \quad \ldots(iii)$$

The conditions (*iii*) are necessary conditions for a maxmium or a minimum value of $f(x, y)$.

Note: The conditions given by (*iii*) are not sufficient for existence of a maximum or a minimum value of $f(x, y)$.

Lagrange's Conditions for Maximum or Minimum (Extrema)

Using the conditions (*iii*) in equation (*i*) neglecting the higher order term $h^3, k^3, h^2 k$ etc.

we get

$$f(a+h, b+k) - f(a, b) = \frac{1}{\lfloor 2}\left[h^2 \frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2}\right]_{(a,b)}$$

Putting $\dfrac{\partial^2 f}{\partial x^2} = r,\ \dfrac{\partial^2 f}{\partial x \partial y} = s,\ \dfrac{\partial^2 f}{\partial y^2} = t$, then

$$f(a+h, b+k) - f(a, b) = \frac{1}{2}\left[h^2 r + 2hks + k^2 t\right]$$

$$= \frac{1}{2}\left[\frac{h^2 r^2 + 2hkrs + k^2 tr}{r}\right]$$

$$f(a+h, b+k) - f(a, b) = \frac{1}{2}\left[\frac{(hr+ks)^2 + k^2\left(rt - s^2\right)}{r}\right] \quad \ldots(iv)$$

If $rt - s^2 > 0$ then the numerator in R.H.S. of (*iv*) is positive. Here sign of L.H.S. = sign of r.

Thus, if $rt - s^2 > 0$ and $r < 0$, then $f(a + h, b + k) - f(a, b) < 0$

if $rt - s^2 > 0$ and $r > 0$, then $f(a + h, b + k) - f(a, b) > 0$.

Therefore, **the Lagrange's conditions for maximum or minimum are:**

1. If $rt - s^2 > 0$ and $r < 0$, then $f(x, y)$ has maximum value at (a, b).
2. If $rt - s^2 > 0$ and $r > 0$, then $f(x, y)$ has minimum value at (a, b).
3. If $rt - s^2 < 0$, then $f(x, y)$ has neither a maximum nor minimum *i.e.*, (a, b) is saddle point.
4. If $rt - s^2 = 0$, then case fail and here again investigate more for the nature of function.

Method of Finding Maxima or Minima

1. Solve $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$, for the values of x and y. Let $x = a$, $y = b$.

   The point $P(a, b)$ is called critical or stationary point.

2. Find r, s and t at $x = a$, $y = b$.

3. Now check the following conditions:

   (*i*) If $rt - s^2 > 0$ and $r < 0$, $f(x, y)$ has maximum at $x = a$, $y = b$.

   (*ii*) If $rt - s^2 > 0$ and $r > 0$, $f(x, y)$ has minimum at $x = a$, $y = b$.

   (*iii*) If $rt - s^2 < 0$, $f(x, y)$ has neither maximum nor minimum.

   (*iv*) If $rt - s^2 = 0$, case fail.

Example

Discuss the maximum or minimum values of u when $u = x^3 + y^3 - 3axy$.

Sol. $\dfrac{\partial u}{\partial x} = 3x^2 - 3ay$; $\dfrac{\partial u}{\partial y} = 3y^2 - 3ax$; $r = \dfrac{\partial^2 u}{\partial x^2} = 6x$;

$$s = \frac{\partial^2 u}{\partial x \partial y} = -3a, \; t = \frac{\partial^2 u}{\partial y^2} = 6y.$$

Now for maximum or minimum, we must have $\dfrac{\partial u}{\partial x} = 0, \dfrac{\partial u}{\partial y} = 0$

So from $\dfrac{\partial u}{\partial x} = 0$, we get $x^2 - ay = 0$ ...(*i*)

and from $\dfrac{\partial u}{\partial y} = 0$, we get $y^2 - ax = 0$ ...(*ii*)

Solving (*i*) and (*ii*), we get $(y^2/a)^2 - ay = 0$

or $y^4 - a^3 y = 0$ or $y(y^3 - a^3) = 0$ or $y = 0, a$.

Now from (*i*), we have when $y = 0$, $x = 0$, and when $y = a$, $x = \pm a$.

But $x = -a$, $y = a$, do not satisfy (ii), here are not solutions.

Hence the solutions are $x = 0$, $y = 0$; $x = a$, $y = a$;

At $x = 0$, $y = 0$, we have $r = 0$, $s = -3a$, $t = 0$.

$\therefore$ $rt - s^2 = 0 - (-3a)^2 =$ negative and there is neither maximum nor minimum at $x = 0$, $y = 0$.

At $x = a$, $y = a$, we get $r = 6a$, $s = -3a$, $t = 6a$

$\therefore$ $rt - s^2 = (6a)(6a) - (-3a)^2 = 36a^2 - 9a^2 > 0$

Also $r = 6a > 0$ if $a > 0$ and $r < 0$ if $a < 0$.

Hence there is maximum or minimum according as $a < 0$ or $a > 0$. The maximum or minimum value of $u = -a^3$ according as $a < 0$ or $a > 0$.

Example $f(x,y) = e^{-(x^2+y^2)}$

The first and second order partial derivatives of this function are:

$$\begin{aligned}
f_x &= -2xe^{-(x^2+y^2)} \\
f_y &= -2ye^{-(x^2+y^2)} \\
f_{xx} &= -2e^{-(x^2+y^2)}(1-2x^2) \quad \text{by the product rule} \\
f_{yy} &= -2e^{-(x^2+y^2)}(1-2y^2) \\
f_{xy} &= 4xye^{-(x^2+y^2)}
\end{aligned}$$

Stationary points are when $f_x = 0$ and $f_y = 0$ and so there is only one stationary point, at $(x,y) = (0,0)$. Substituting $(x,y) = (0,0)$ into the expressions for f_{xx}, f_{yy} and f_{xy} gives

$$f_{xx} = -2, \quad f_{yy} = -2, \quad f_{xy} = 0$$

Therefore

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - 0^2 = 4 > 0$$

so that $(0,0)$ is either a min or a max. Since $f_{xx} < 0$ and $f_{yy} < 0$ it is a maximum.

Example

$$f(x, y) = 2 - x^2 - xy - y^2$$

For this function

$$f_x = -2x - y$$
$$f_y = -x - 2y$$
$$f_{xx} = -2$$
$$f_{yy} = -2$$
$$f_{xy} = -1$$

For stationary points, $-2x - y = 0$ and $-x - 2y = 0$ so again the only possibility is $(x, y) = (0, 0)$. We have

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - (-1)^2 = 3 > 0$$

so that $(0, 0)$ is either a max or a min. Since $f_{xx} < 0$ and $f_{yy} < 0$ it is a maximum.

Example

$$f(x,y) = 2x^3 + 6xy^2 - 3y^3 - 150x$$

The function in this example has four stationary points.

The first and second order partial derivatives of this function are

$$\begin{aligned}
f_x &= 6x^2 + 6y^2 - 150 \\
f_y &= 12xy - 9y^2 \\
f_{xx} &= 12x \\
f_{yy} &= 12x - 18y \\
f_{xy} &= 12y
\end{aligned}$$

For stationary points we need

$$6x^2 + 6y^2 - 150 = 0 \quad \text{and} \quad 12xy - 9y^2 = 0$$

i.e.

$$x^2 + y^2 = 25 \quad \text{and} \quad y(4x - 3y) = 0$$

The second of these equation s implies either that $y = 0$ or that $4x = 3y$ and both of these possibilities now need to be considered. If $y = 0$ then the first equation implies that $x^2 = 25$ so that $x = \pm 5$ giving $(5, 0)$ and $(-5, 0)$ as stationary points.
If $4x = 3y$ then $x = \frac{3}{4}y$ and so the first equation becomes

$$\frac{9}{16}y^2 + y^2 = 25$$

so that $y = \pm 4$. $y = 4$ gives $x = 3$ and $y = -4$ gives $x = -3$, so we have two further stationary points $(3, 4)$ and $(-3, -4)$.
Thus in total there are four stationary points $(5, 0)$, $(-5, 0)$, $(3, 4)$ and $(-3, -4)$.
Each of these must now be classified into max, min or saddle.

- Lets start with $(5,0)$. For this stationary point, $f_{xx}f_{yy} - f_{xy}^2 = 60^2 > 0$ so it is either a max or a min. But $f_{xx} = 60 > 0$ and $f_{yy} = 60 > 0$. Hence $(5,0)$ is a minimum.

- Now deal with $(-5,0)$. For this stationary point, $f_{xx}f_{yy} - f_{xy}^2 = (-60)^2 > 0$ so it is either a max or a min. But $f_{xx} = -60 < 0$ and $f_{yy} = -60 < 0$. Hence $(-5,0)$ is a maximum.

- Now deal with $(3,4)$. For this stationary point, $f_{xx}f_{yy} - f_{xy}^2 = -3600 < 0$ so $(3,4)$ is a saddle.

- Now deal with $(-3,-4)$. For this stationary point, $f_{xx}f_{yy} - f_{xy}^2 = -3600 < 0$ so $(-3,-4)$ is a saddle.

Example

A container with an open top is to have 10 m^3 capacity and be made of thin sheet metal. Calculate the dimensions of the box if it is to use the minimum possible amount of metal.

Solution: Let A be the total area of metal used to make the box, and let x and y be the length and width and z the height. Then

$$A = 2xz + 2yz + xy$$

Also

$$xyz = 10$$

because the volume is 10 m^3. This implies that $z = \frac{10}{xy}$. Putting this into the formula for A gives A as a function of x and y only:

$$\begin{aligned} A &= 2x\left(\frac{10}{xy}\right) + 2y\left(\frac{10}{xy}\right) + xy \\ &= \frac{20}{y} + \frac{20}{x} + xy \end{aligned}$$

We shall apply our techniques to this function. Now

$$\frac{\partial A}{\partial x} = -\frac{20}{x^2} + y, \quad \frac{\partial A}{\partial y} = -\frac{20}{y^2} + x$$

and for a stationary point we need $\partial A/\partial x = \partial A/\partial y = 0$. this gives

$$y = \frac{20}{x^2} \quad \text{and} \quad x = \frac{20}{y^2}.$$

Therefore

$$y = \frac{20}{(20/y^2)^2} = \frac{y^4}{20}$$

Since the zero root $y = 0$ is obviously not consistent with having a volume of 10 m^3 we reject $y = 0$ and conclude that $y^3 = 20$ so that $y = 20^{1/3} = 2.714$ metres. From $x = 20/y^2$ we conclude $x = 2.714$ metres also. To find z, use $z = \frac{10}{xy}$ so that $z = 1.357$ m.

We have to show that these values do indeed give a **minimum**. Now

$$\frac{\partial^2 A}{\partial x \partial y} = 1, \quad \frac{\partial^2 A}{\partial x^2} = \frac{40}{x^3}, \quad \frac{\partial^2 A}{\partial y^2} = \frac{40}{y^3}$$

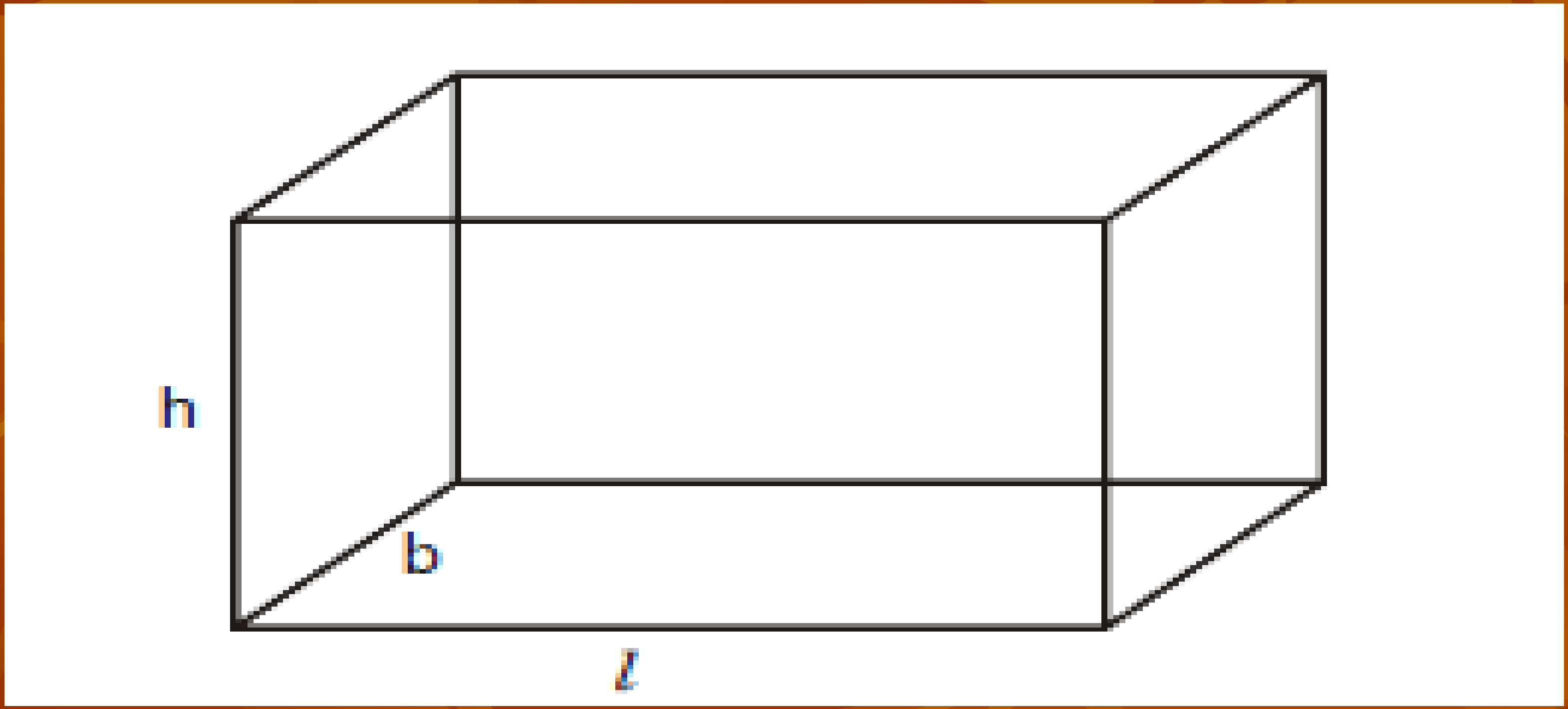
So, when $(x, y) = (2.714, 2.714)$,

$$A_{xx}A_{yy} - A_{xy}^2 = (2)(2) - 1^2 = 3 > 0$$

so it is either a max or a min. But $A_{xx} > 0$ and $A_{yy} > 0$ so it is a minimum. Our conclusion is that the box should have length 2.714 m, width 2.714 m and height 1.357 m. The actual area of metal used will then (from the formula for A) be 22.1 m^2.

Example

A rectangular box, open at the top, is to have a volume of 32 c.c. Find the dimensions of the box requiring least material for its construction.



Sol. $V = 32$ c.c.

Let length $= l$, breadth $= b$ and height $= h$

Total surface area $S = 2lh + 2bh + lb$ ...(*i*)

$$S = 2(l + b)h + lb$$

Now volume $V = lbh = 32 \Rightarrow b = \dfrac{32}{lh}$ ...(*ii*)

Putting the value of '*b*' in equation (*i*)

$$S = 2\left(l + \frac{32}{lh}\right)h + l\left(\frac{32}{lh}\right)$$

$$S = 2lh + \frac{64}{l} + \frac{32}{h}$$

$$\therefore \quad \frac{\partial S}{\partial l} = 2h - \frac{64}{l^2}, \quad \frac{\partial S}{\partial h} = 2l - \frac{32}{h^2}$$

For minimum S, we get

$$\frac{\partial S}{\partial l} = 0 \Rightarrow 2h - \frac{64}{l^2} = 0 \Rightarrow h = \frac{32}{l^2} \qquad ...(iv)$$

and
$$\frac{\partial S}{\partial h} = 0 \Rightarrow 2l - \frac{32}{h^2} = 0 \Rightarrow l = \frac{16}{h^2} \qquad ...(v)$$

From (iv) and (v), we get

$$h = \frac{32 \times h^4}{256} \Rightarrow h^3 = 8 \Rightarrow h = 2$$

Putting $h = 2$, in equation (v), we get $l = \dfrac{16}{4} = 4$

From (ii) $\qquad b = \dfrac{32}{4 \times 2} = 4$

Now, $$\frac{\partial^2 S}{\partial l^2} = \frac{128}{l^3} = \frac{128}{64} = 2 \Rightarrow r = 2 > 0$$

and $\dfrac{\partial^2 S}{\partial l \partial h} = 2 \Rightarrow s = 2$ and $\dfrac{\partial^2 S}{\partial h^2} = \dfrac{64}{h^3} = \dfrac{64}{8} = 8 \Rightarrow t = 8$

$$\therefore \quad rt - s^2 = 2 \times 8 - 4 = 12 > 0$$

$\Rightarrow \; rt - s^2 > 0$ and $r > 0$

Hence, S is minimum, for least material

$$l = 4,\ b = 4,\ h = 2.$$

Example

Find the maximum and minimum values of the function

$$z = \sin x \sin y \sin (x + y)$$

Sol. Given $z = \sin x \sin y \sin (x + y)$

$$= \frac{1}{2} [2 \sin x \sin y] \sin (x + y)$$

$$= \frac{1}{2} [\cos (x - y) - \cos (x + y)] \sin (x + y)$$

$$= \frac{1}{4} [2 \sin (x + y) \cos (x - y) - 2 \sin (x + y) \cos (x + y)]$$

or $$z = \frac{1}{4} [\sin 2x + \sin 2y - \sin(2x + 2y)]$$

$$\therefore \qquad \frac{\partial z}{\partial x} = \frac{1}{2}\,[\cos 2x - \cos(2x + 2y)]$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}\,[\cos 2y - \cos(2x + 2y)]$$

$$r = \frac{\partial^2 z}{\partial x^2} = -\sin 2x + \sin(2x + 2y) \qquad \ldots(A)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \sin(2x + 2y) \qquad \ldots(B)$$

$$t = \frac{\partial^2 z}{\partial y^2} = -\sin 2y + \sin(2x + 2y) \qquad \ldots(C)$$

For maximum or minimum, we must have $\dfrac{\partial z}{\partial x} = 0,\ \dfrac{\partial z}{\partial y} = 0$

From $\dfrac{\partial z}{\partial x} = 0$, we get $\cos 2x - \cos(2x + 2y) = 0$ ...(*i*)

From $\dfrac{\partial z}{\partial y} = 0$, we get $\cos 2y - \cos(2x + 2y) = 0$ ...(*ii*)

Solving (*i*) and (*ii*), we get $\cos 2x = \cos 2y$ which gives

$$2x = 2n\pi \pm 2y. \text{ In particular } 2x = 2y \text{ or } x = y$$

When $x = y$, from (*i*), we get $\cos 2x - \cos 4x = 0$

$$\cos 2x - (2\cos^2 2x - 1) = 0 \quad \because \cos 2\theta = 2\cos^2\theta - 1$$

$$2\cos^2 2x - \cos 2x - 1 = 0$$

or $\quad \cos 2x = \dfrac{1 \pm \sqrt{(1+8)}}{4} = \dfrac{1 \pm 3}{4} = 1, -\dfrac{1}{2}$

or $\quad 2x = 2n\pi \pm 0,\ 2m\pi \pm \dfrac{2\pi}{3}$, where m, n are zero or any integers

or $\quad x = n\pi,\ m\pi \pm \dfrac{\pi}{3}$

In particular $x = \dfrac{\pi}{3}$

When $x = \dfrac{\pi}{3}$, we have $y = x = \dfrac{\pi}{3}$

and then $r = -\sin\dfrac{2\pi}{3} + \sin\dfrac{4\pi}{3}$, from ($A$)

$$= -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}\ ;$$

$$s = \sin\frac{4\pi}{3}, \text{ from } (B)$$

or $\quad s = -\dfrac{\sqrt{3}}{2}$

and $\quad t = -\sin\dfrac{2\pi}{3} + \sin\dfrac{4\pi}{3}$, from (C)

$$= -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$$

$$\therefore \qquad rt - s^2 = \left(-\sqrt{3}\right)\left(-\sqrt{3}\right) - \left(-\frac{\sqrt{3}}{2}\right)^2 = 3 - \frac{3}{4} = \frac{9}{4}$$

$$= \text{positive.}$$

Thus at $x = \frac{\pi}{3} = y$, $rt - s^2 > 0$, $r < 0$, so there is a maximum at $x = \frac{\pi}{3} = y$.

Hence, maximum value $= \sin\frac{\pi}{3} \,.\, \sin\frac{\pi}{3} \,.\, \sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right)$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}.$$

If we take $\quad x = -\frac{\pi}{3}$, then $y = x = -\frac{\pi}{3}$

$$r = \sqrt{3}, \quad s = \frac{1}{2}\sqrt{3}, \; t = \sqrt{3}$$

$$\therefore \qquad rt - s^2 = \frac{9}{4} > 0 \,,\; r > 0$$

There is a minimum at $x = -\frac{\pi}{3} = y$.

Hence, the minimum value $= \sin\left(-\frac{\pi}{3}\right) \sin\left(-\frac{\pi}{3}\right) \sin\left\{\left(-\frac{\pi}{3}\right) + \left(-\frac{\pi}{3}\right)\right\}$

$$= -\sin\frac{\pi}{3}\cdot\sin\frac{\pi}{3}\ \sin\frac{2\pi}{3}$$

$$= -\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{8}.$$

LAGRANGE'S* METHOD OF UNDETERMINED MULTIPLIERS

Let $\phi\,(x, y, z)$ is a function of three independent variables, where x, y, z are related by a known constraint $g(x, y, z) = 0$

Thus the problem is Extrema of

$$u = f(x, y, z) \qquad \text{...}(i)$$

Subject to $\quad g\,(x, y, z) = 0 \qquad \text{...}(ii)$

For stationary point $\quad \dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = \dfrac{\partial f}{\partial z} = 0$

$$\therefore \qquad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \qquad \text{...}(iii)$$

From (ii)

$$dg = \frac{\partial g}{\partial x}\,dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}\,dz = 0 \qquad \text{...}(iv)$$

Multiplying eqn. (iv) by λ and adding to (iii), we obtain

$$\left(\frac{\partial f}{\partial x}+\lambda\frac{\partial g}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}+\lambda\frac{\partial g}{\partial y}\right)dy+\left(\frac{\partial f}{\partial z}+\lambda\frac{\partial g}{\partial z}\right)dz = 0 \qquad \text{...}(v)$$

Since x, y, z are independent variables

$$\therefore \quad \frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \qquad \text{...}(vi)$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \qquad \text{...}(vii)$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial g}{\partial z} = 0 \qquad \text{...}(viii)$$

On solving (*ii*), (*vi*), (*vii*) and (*viii*), we can find x, y, z and λ for which $f(x, y, z)$ has maximum or minimum.

Notes: 1. The Lagrange's method of undetermined multiplier's introduces an additional unknown constant λ known as Lagrange's multiplier.

2. Nature of stationary points cannot be determined by Lagranges method.

Example

Determine the maxima and minima of $x^2 + y^2 + z^2$ when $ax^2 + by^2 + cz^2 = 1$.

Sol. Let $$f(x, y, z) = x^2 + y^2 + z^2 \quad ...(i)$$

and $$g(x, y, z) \equiv ax^2 + by^2 + cz^2 - 1 = 0 \quad ...(ii)$$

From (i) $$\frac{\partial f}{\partial x} = 2x, \ \frac{\partial f}{\partial y} = 2y, \ \frac{\partial f}{\partial z} = 2z$$

From (ii) $$\frac{\partial g}{\partial x} = 2ax, \ \frac{\partial g}{\partial y} = 2by, \ \frac{\partial g}{\partial z} = 2cz.$$

Now from Lagrange's equations, we get

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 2x + \lambda . 2ax = 0 \Rightarrow 2x(1 + \lambda a) = 0 \Rightarrow x(1 + \lambda a) = 0 \quad ...(iii)$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \Rightarrow 2y + \lambda \cdot 2by = 0 \Rightarrow 2y(1 + \lambda b) = 0 \Rightarrow y(1 + \lambda b) = 0 \quad ...(iv)$$

and $$\frac{\partial f}{\partial z} + \lambda \frac{\partial g}{\partial z} = 0 \Rightarrow 2z + \lambda \cdot 2cz = 0 \Rightarrow 2z(1 + \lambda c) = 0 \Rightarrow z(1 + \lambda b) = 0 \quad ...(v)$$

Multiplying these equations by x, y, z respectively and adding, we get

$$x^2(1+\lambda a)+y^2(1+\lambda b)+z^2(1+\lambda c)=0$$

or $$(x^2+y^2+z^2)+\lambda(ax^2+by^2+cz^2)=0 \qquad ...(vi)$$

Using (i) and (ii) in above equation, we get

$$f+\lambda=0 \Rightarrow \lambda=-f$$

Putting $\lambda=-f$ in equations (iii), (iv) and (v), we get

$$x(1-fa)=0,\ y(1-fb)=0,\ z(1-fc)=0$$

$\Rightarrow$ $$1-fa=0,\ 1-fb=0,\ 1-fc=0$$

i.e., $f=\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$. These give the max. and min. values of f.

Example

Find the extreme value of $x^2 + y^2 + z^2$, given that $ax + by + cz = p$.

Sol. Let $$u = x^2 + y^2 + z^2$$

Given $$ax + by + cz = p.$$

For max. or min. from (*i*), we have

$$du = 2x\,dx + 2y\,dy + 2z\,dz = 0. \quad \ldots(iii)$$

Also from (*ii*), $$a\,dx + b\,dy + c\,dz = 0. \quad \ldots(iv)$$

Multiplying (*iv*) by λ and adding in (*iii*), we get $(x\,dx + y\,dy + z\,dz) + \lambda\,(a\,dx + b\,dy + c\,dz) = 0$.

Equating the coefficients of dx, dy and dz to zero, we get

$$x + \lambda a = 0,\ y + \lambda b = 0,\ z + \lambda c = 0 \quad \ldots(v)$$

These are Lagrange's equations.

Multiplying these by x, y, z respectively and adding, we get

$$x\,(x + \lambda a) + y\,(y + \lambda b) + z\,(z + \lambda c) = 0$$

or $$(x^2 + y^2 + z^2) + \lambda\,(ax + by + cz) = 0$$

or $$u + \lambda p = 0 \text{ or } \lambda = -\,u/p.$$

$\therefore$ From (v), we get

$$x - \left(\frac{au}{p}\right) = 0, \quad y - \left(\frac{bu}{p}\right) = 0, \quad z - \left(\frac{cu}{p}\right) = 0$$

or $$\frac{x}{a} = \frac{u}{p} = \frac{y}{b} = \frac{z}{c} \quad \text{or} \quad \frac{x}{a} = \frac{y}{b} = \frac{z}{c} \qquad \ldots(vi)$$

From (ii), we get $a^2\left(\frac{x}{a}\right) + b^2\left(\frac{y}{b}\right) + c^2\left(\frac{z}{c}\right) = p$

or $$a^2\left(\frac{x}{a}\right) + b^2\left(\frac{x}{a}\right) + c^2\left(\frac{x}{a}\right) = p, \text{ from } (vi)$$

or $$(a^2 + b^2 + c^2)\left(\frac{x}{a}\right) = p \quad \text{or} \quad x = \frac{ap}{a^2+b^2+c^2}$$

Similarly, $$y = \frac{bp}{a^2+b^2+c^2}, \quad z = \frac{cp}{a^2+b^2+c^2}$$

These give the minimum value of u.

Hence minimum value of u is

$$u = \frac{a^2p^2}{(a^2+b^2+c^2)^2} + \frac{b^2p^2}{(a^2+b^2+c^2)^2} + \frac{c^2p^2}{(a^2+b^2+c^2)^2}$$

$$= \frac{(a^2+b^2+c^2)p^2}{(a^2+b^2+c^2)^2} = \frac{p^2}{(a^2+b^2+c^2)}.$$