1 Elementary Properties

Exercise 1.1. Find the Laplace transform of each of the following functions $f(t)$ using the linearity:

(a) $\mathscr{L}\{\exp^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t\}$

(b) $3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t$

(c) $3e^{3t} + 5t^4 - 4\cos 3t + 3\sin 4t$

(d) $e^{-3t} + 5e^t + 6\sin 2t - 5\cos 2t$

(e) $7e^{2t} + 9e^{-2t} + 5\cos t + 7t^3 + 5\sin 3t + 2$

(f) $4e^{-3t} - 2\sin 5t + 3\cos 2t - 2t^3 + 3t^4$

(g) $3t^2 + 6t + 4 + (5e^{2t})^2$

(h) $(t^2+1)^2 + 3\cosh 5t - 4\sinh t$

(i) $2e^{5t} + e^{-3t} + 5e^t + 5t - 2$, $t^2 - 5t - \sin 2t + e^{3t}$

(j) $\sin^3 at$, $\cos^3 at$

(k) $t + \sin at$, $t - \cos at$, $\sin\sqrt{t}$

(l) $\sin at \sin bt$; $\sin 2t \cos 3t$, $\sin t \cos t$

(m) $\sin^2 at$, $\cos^2 at$, $(\sin at + \cos at)^2$

(n) $\cosh at - \cos at$, $\sinh at + \sin at$

(o) $\sin(at+b)$, $\cos(at+b)$

2 Multiplication of $f(t)$ by t

Exercise 2.1. Find the Laplace transform of $t\sin at$ and hence show that $\int_0^\infty e^{-bt}dt = \frac{2ab}{(a^2+b^2)^2}$, where $b > 0, a > 0$.

Exercise 2.2. Find the Laplace transform of $t\sin t$ and evaluate $\int_0^\infty te^{-2t}\sin t\,dt, \int_0^\infty te^{-3t}\sin t\,dt$

Exercise 2.3. Find $\mathscr{L}[t^2\sin 3t]$ and then evaluate $\int_0^\infty t^2e^{-t}\sin 3t\,dt$

Exercise 2.4. Find $\mathscr{L}[t^3\sin t]$ and then evaluate $\int_0^\infty t^3e^{-t}\sin t\,dt$

Exercise 2.5. Compute the Laplace transform of $t\cos at$ and then show that
$\int_0^\infty te^{-bt}\cos at\,dt = \frac{(b^2-a^2)}{(b^2+a^2)^2}, s>0$

Exercise 2.6. $\mathscr{L}[t^2\cos at] = \frac{2s(s^2-3a^2)}{(s^2+a^2)^3}, s>0$

Exercise 2.7. Evaluate the Laplace transform of $t\sinh at$, $t\cosh at$, $t(a\sin bt - b\cos bt)$

3 Laplace transform of Periodic Functions

A real valued function $f(t)$ is said to be a periodic function, if there exists a positive real number τ such that

$$f(t+\tau) = f(t) \text{ for all } t. \tag{3.1}$$

The least τ is called the period of f. Let $f(t)$ be a periodic function with period $\tau > 0$. Then its graph is repeated in regular intervals of length τ. Then the Laplace transform of f is given by

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-s\tau}}\int_0^\tau e^{-st}f(t)dt \tag{3.2}$$

Example 3.1. Find the Laplace transform of the periodic function f with period a, whose definition in one period is given by:

$$f(t) = \begin{cases} 1, & 0 < t < \frac{a}{2} \\ -1, & \frac{a}{2} < t < a. \end{cases} \tag{3.3}$$

Solution. Write $\tau = a$ in (3.2). Then

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \frac{1}{1-e^{-sa}}\int_0^a e^{-st}f(t)dt = \frac{1}{1-e^{-sa}}\left[\int_0^{a/2} e^{-st}\cdot 1dt + \int_{a/2}^a e^{-st}\cdot(-1)dt\right] \\
&= \frac{1}{1-e^{-sa}}\left[\left|-\frac{e^{-st}}{s}\right|_{t=0}^{a/2} - \left|-\frac{e^{-st}}{s}\right|_{t=a/2}^{a} dt\right] \\
&= \frac{1}{1-e^{-sa}}\left[\frac{1-e^{-sa/2}}{s} + \frac{e^{-sa}-e^{-sa/2}}{s}\right] \\
&= \frac{1}{s}\cdot\frac{1-2e^{-sa/2}+e^{-sa}}{1-e^{-sa}} \\
&= \frac{1}{s}\cdot\frac{(1-e^{-sa/2})^2}{\left(1-e^{-sa/2}\right)\left(1+e^{-sa/2}\right)} = \frac{1-e^{-sa/2}}{s\left(1+e^{-sa/2}\right)}
\end{aligned}$$

Multiplying the numerator and denominator by $e^{sa/4}$, we find that

$$\mathscr{L}\{f(t)\} = \frac{1}{s}\cdot\frac{e^{sa/4}-e^{-as/4}}{e^{sa/4}+e^{-as/4}} = \frac{1}{s}\cdot\tanh\left(\frac{sa}{4}\right)$$

Example 3.2. Find the Laplace transform of half-wave rectified sinusoidal signal f with period $2\pi/\omega$, whose definition in one period is given by:

$$f(t) = \begin{cases} \sin\omega t, & 0 < t < \frac{\pi}{\omega} \\ 0, & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega}. \end{cases} \tag{3.4}$$

Solution. Write $\tau = a$ in (3.2). Then

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \frac{1}{1-e^{-sa}}\int_0^{2\pi/\omega} e^{-st}f(t)dt = \frac{1}{1-e^{-2s\pi/\omega}}\int_0^{\pi/\omega}\sin\omega t\cdot e^{-st}dt \\
&= \frac{1}{1-e^{-2s\pi/\omega}}\cdot\left|\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t-\omega\cos\omega t)\right|_{t=0}^{\pi/\omega} \\
&= \frac{1}{1-e^{-2s\pi/\omega}}\cdot\frac{\omega(1+e^{-s\pi/\omega})}{s^2+\omega^2} \\
&= \frac{\omega}{s^2+\omega^2}\cdot\frac{1+e^{-s\pi/\omega}}{1-e^{-2s\pi/\omega}} = \frac{\omega}{s^2+\omega^2}\cdot\frac{1}{1-e^{-s\pi/\omega}}
\end{aligned}$$

Integral Formula: $\int\sin At.e^{Bt}dt = \dfrac{e^{Bt}}{A^2+B^2}(B\sin At - A\cos At)$

Example 3.3. Find the Laplace transform of full-wave rectified sinusoidal signal f with period $2\pi/\omega$, whose definition in one period is given by:

$$f(t) = \begin{cases} \sin\omega t, & 0 < t < \frac{\pi}{\omega} \\ -\sin\omega t, & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega}. \end{cases} \tag{3.5}$$

Solution. Write $\tau = a$ in (3.2). Then

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \frac{1}{1-e^{-sa}}\int_0^{2\pi/\omega} e^{-st}f(t)dt \\
&= \frac{1}{1-e^{-2s\pi/\omega}}\left\{\int_0^{\pi/\omega}\sin\omega t\cdot e^{-st}dt - \int_{\pi/\omega}^{2\pi/\omega}\sin\omega t\cdot e^{-st}dt\right\} \\
&= \frac{1}{1-e^{-2s\pi/\omega}}\left\{\left|\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t-\omega\cos\omega t)\right|_{t=0}^{\pi/\omega}\right. \\
&\qquad \left. - \left|\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t-\omega\cos\omega t)\right|_{t=\pi/\omega}^{2\pi/\omega}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{1-e^{-2s\pi/\omega}} \cdot \left\{ \frac{\omega(1+e^{-s\pi/\omega})}{s^2+\omega^2} + \frac{\omega e^{-s\pi/\omega}(1+e^{-s\pi/\omega})}{s^2+\omega^2} \right\} \\
&= \frac{\omega}{s^2+\omega^2} \cdot \frac{1+e^{-s\pi/\omega}}{1-e^{-s\pi/\omega}} = \frac{\omega}{s^2+\omega^2} \cdot \frac{e^{s\pi/2\omega}+e^{-s\pi/2\omega}}{e^{s\pi/2\omega}-e^{-s\pi/2\omega}} \\
&= \frac{\omega}{s^2+\omega^2} \cdot \coth(s\pi/2\omega)
\end{aligned}
$$

Exercise 3.1. Find the Laplace transform of half-wave the saw-tooth wave $f(t) = kt/\pi$ with period π

Exercise 3.2. Find the Laplace transform of the signal $f(t) = x(l-x)$ with period l

Exercise 3.3. Find the Laplace transform of the triangular wave the saw-tooth wave $f(t)$ with period $2a$, whose definition in one period is given by:

$$
f(t) = \begin{cases} t, & 0 < t < a \\ 2a - t, & a < t < 2a. \end{cases}
$$

4 Inverse Laplace Transform

If $\bar{F}(s) = \mathscr{L}\{f(t)\}$ is the Laplace transform of $f(t)$, $t \geq 0$, then $f(t)$ is called the *inverse* of the Laplace transform $\bar{F}(s)$ or simply the *inverse Laplace transform*, and we write

$$
f(t) = \mathscr{L}^{-1}\left\{\bar{F}(s)\right\}. \tag{4.1}
$$

Inverse Laplace Transform of Elementary Functions

(a) $\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$ for $n = 1, 2, 3, \ldots$

(b) $\mathscr{L}^{-1}\left\{\frac{1}{s^p}\right\} = \frac{t^{p-1}}{\Gamma(p)}$ for real $p > 0$

(c) $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

(d) $\mathscr{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a} \cdot \sin at$

(e) $\mathscr{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

(f) $\mathscr{L}^{-1}\left\{\frac{e^{as}}{s}\right\} = H(t-a)$, where $a \geq 0$

Linearity of the Inverse Laplace Transform

Let $f(t) = \mathscr{L}^{-1}\{\bar{F}(s)\}$, $g(t) = \mathscr{L}^{-1}\{\bar{G}(s)\}$. Then for any scalars a and b, we have

$$\mathscr{L}^{-1}\{a\bar{F}(s) + b\bar{G}(s)\} = af(t) + bg(t). \tag{4.2}$$

Hyperbolic (Sine and Cosine) as the Inverses

(a) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2 - a^2}\right\} = \dfrac{1}{a} \cdot \sinh at = \dfrac{e^{at} - e^{-at}}{2a}$

(b) $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2 - a^2}\right\} = \cosh at = \dfrac{e^{at} + e^{-at}}{2}$

5 Inverse Shifting

(a) *First Shifting*: If $\mathscr{L}^{-1}\{\bar{F}(s)\} = f(t)$, then $\mathscr{L}^{-1}\{\bar{F}(s-a)\} = e^{at} \cdot f(t)$. That is,

$$\mathscr{L}^{-1}\{\bar{F}(s-a)\} = e^{at} \cdot \mathscr{L}^{-1}\{\bar{F}(s)\} \tag{5.1}$$

Example 5.1.

$$(a)\quad \mathscr{L}^{-1}\left\{\frac{1}{(s-a)^n}\right\} = e^{at}\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\} = e^{at} \cdot \tfrac{t^{n-1}}{(n-1)!},\ n = 1,2,,\ldots$$

$$(b)\quad \mathscr{L}^{-1}\left\{\frac{1}{(s+a)^n}\right\} = e^{-at}\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\} = e^{-at} \cdot \tfrac{t^{n-1}}{(n-1)!},\ n = 1,2,,\ldots$$

$$(c)\quad \mathscr{L}^{-1}\left\{\frac{1}{(s-a)^p}\right\} = e^{at}\mathscr{L}^{-1}\left\{\frac{1}{s^p}\right\} = e^{at} \cdot \tfrac{t^{p-1}}{\Gamma(p)} \text{ for real } p > 0$$

$$(d)\quad \mathscr{L}^{-1}\left\{\frac{1}{(s+a)^p}\right\} = e^{-at}\mathscr{L}^{-1}\left\{\frac{1}{s^p}\right\} = e^{-at} \cdot \tfrac{t^{p-1}}{\Gamma(p)} \text{ for real } p > 0$$

Example 5.2.

$$(a)\quad \mathscr{L}^{-1}\left\{\frac{1}{(s-a)^2 + b^2}\right\} = e^{at}\mathscr{L}^{-1}\left\{\frac{1}{s^2 + b^2}\right\} = e^{at} \cdot \tfrac{\sin bt}{b}$$

$$(b)\quad \mathscr{L}^{-1}\left\{\frac{1}{(s+a)^2 + b^2}\right\} = e^{-at}\mathscr{L}^{-1}\left\{\frac{1}{s^2 + b^2}\right\} = e^{-at} \cdot \tfrac{\sin bt}{b}$$

$$(c)\quad \mathscr{L}^{-1}\left\{\frac{1}{(s-a)^2 - b^2}\right\} = e^{at}\mathscr{L}^{-1}\left\{\frac{1}{s^2 - b^2}\right\} = e^{at} \cdot \tfrac{\sinh bt}{b}$$

$$(d)\quad \mathscr{L}^{-1}\left\{\frac{1}{(s+a)^2 - b^2}\right\} = e^{-at}\mathscr{L}^{-1}\left\{\frac{1}{s^2 - b^2}\right\} = e^{-at} \cdot \tfrac{\sinh bt}{b}$$

Example 5.3.

$$
\begin{aligned}
(a)\quad & \mathscr{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\mathscr{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} = e^{at}\cdot\cos bt\\
(b)\quad & \mathscr{L}^{-1}\left\{\frac{s}{(s-a)^2+b^2}\right\} = \mathscr{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} + \mathscr{L}^{-1}\left\{\frac{a}{(s-a)^2+b^2}\right\}\\
& \qquad = e^{at}\left[\mathscr{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} + \mathscr{L}^{-1}\left\{\frac{a}{s^2+b^2}\right\}\right]\\
& \qquad = e^{at}\left[\cos bt + \tfrac{a}{b}\cdot\sin bt\right]\\
(c)\quad & \mathscr{L}^{-1}\left\{\frac{s-a}{(s-a)^2-b^2}\right\} = e^{at}\mathscr{L}^{-1}\left\{\frac{s}{s^2-b^2}\right\} = e^{at}\cdot\cosh bt\\
(d)\quad & \mathscr{L}^{-1}\left\{\frac{s}{(s-a)^2-b^2}\right\} = \mathscr{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} + \mathscr{L}^{-1}\left\{\frac{a}{(s-a)^2-b^2}\right\}\\
& \qquad = e^{at}\left[\mathscr{L}^{-1}\left\{\frac{s}{s^2-b^2}\right\} + \mathscr{L}^{-1}\left\{\frac{a}{s^2-b^2}\right\}\right]\\
& \qquad = e^{at}\left[\cosh bt + \tfrac{a}{b}\cdot\sinh bt\right]\\
(e)\quad & \mathscr{L}^{-1}\left\{\frac{s+a}{(s+a)^2+b^2}\right\} = e^{-at}\mathscr{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} = e^{-at}\cdot\cos bt\\
(f)\quad & \mathscr{L}^{-1}\left\{\frac{s}{(s+a)^2+b^2}\right\} = \mathscr{L}^{-1}\left\{\frac{s+a}{(s+a)^2+b^2}\right\} - \mathscr{L}^{-1}\left\{\frac{a}{(s+a)^2+b^2}\right\}\\
& \qquad = e^{-at}\left[\mathscr{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} - \mathscr{L}^{-1}\left\{\frac{a}{s^2+b^2}\right\}\right]\\
& \qquad = e^{-at}\left[\cos bt - \tfrac{a}{b}\cdot\sin bt\right]\\
(g)\quad & \mathscr{L}^{-1}\left\{\frac{s+a}{(s+a)^2-b^2}\right\} = e^{-at}\mathscr{L}^{-1}\left\{\frac{s}{s^2-b^2}\right\} = e^{-at}\cdot\cosh bt\\
(h)\quad & \mathscr{L}^{-1}\left\{\frac{s}{(s+a)^2-b^2}\right\} = \mathscr{L}^{-1}\left\{\frac{s+a}{(s+a)^2+b^2}\right\} - \mathscr{L}^{-1}\left\{\frac{a}{(s+a)^2-b^2}\right\}\\
& \qquad = e^{-at}\left[\mathscr{L}^{-1}\left\{\frac{s}{s^2-b^2}\right\} - \mathscr{L}^{-1}\left\{\frac{a}{s^2-b^2}\right\}\right]\\
& \qquad = e^{-at}\left[\cosh bt - \tfrac{a}{b}\cdot\sinh bt\right]
\end{aligned}
$$

(b) *Second Shifting*: If $\mathscr{L}^{-1}\{\bar{F}(s)\} = f(t)$, then

$$\mathscr{L}^{-1}\left\{e^{-as}\bar{F}(s)\right\} = f(t-a)\cdot H(t-a) \tag{5.2}$$

where $a \geq 0$.

Example 5.4.

$$
\begin{aligned}
(a)\quad & \mathscr{L}^{-1}\left\{\frac{e^{-as}}{s^n}\right\} = \left|\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\}\right|_{t\to(t-a)} H(t-a) = \tfrac{(t-a)^{n-1}}{(n-1)!}\cdot H(t-a)\\
(b)\quad & \mathscr{L}^{-1}\left\{\frac{e^{-as}}{s-b}\right\} = \left|\mathscr{L}^{-1}\left\{\frac{1}{s-b}\right\}\right|_{t\to(t-a)} H(t-a) = e^{b(t-a)}\cdot H(t-a)
\end{aligned}
$$

Example 5.5.

$$(a)\quad \mathscr{L}^{-1}\left\{\frac{e^{-as}}{s^2+b^2}\right\} = \left|\mathscr{L}^{-1}\left\{\frac{1}{s^2+b^2}\right\}\right|_{t\to(t-a)} H(t-a)$$
$$= \left|\frac{\sin bt}{b}\right|_{t\to(t-a)} H(t-a) = \frac{\sin b(t-a)}{b}\cdot H(t-a)$$

$$(b)\quad \mathscr{L}^{-1}\left\{\frac{e^{-as}}{s^2-b^2}\right\} = \left|\mathscr{L}^{-1}\left\{\frac{1}{s^2-b^2}\right\}\right|_{t\to(t-a)} H(t-a)$$
$$= \left|\frac{\sinh bt}{b}\right|_{t\to(t-a)} H(t-a) = \frac{\sinh b(t-a)}{b}\cdot H(t-a)$$

$$(c)\quad \mathscr{L}^{-1}\left\{\frac{se^{-as}}{s^2+b^2}\right\} = \left|\mathscr{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\}\right|_{t\to(t-a)} H(t-a)$$
$$= H(t-a)\,|\cos bt|_{t\to(t-a)} = H(t-a)\cos b(t-a)$$

$$(d)\quad \mathscr{L}^{-1}\left\{\frac{se^{-as}}{s^2-b^2}\right\} = \left|\mathscr{L}^{-1}\left\{\frac{s}{s^2-b^2}\right\}\right|_{t\to(t-a)} H(t-a)$$
$$= H(t-a)\,|\cosh bt|_{t\to(t-a)} = H(t-a)\cosh b(t-a)$$

Exercise 5.1. Find the inverse of each of the following Laplace transforms:

(a) $\frac{1}{s}+\frac{s}{s^2+4}$ (b) $\frac{s^2-1}{s^3}$ (c) $\frac{(2+s)^2}{s^5}$
(d) $\frac{s^2-3s+4}{s^3}$ (e) $\frac{s}{s^2+16}+\frac{2}{s-3}+\frac{s+1}{s^3}$ (f) $\frac{3(s^2-2)^2}{2s^5}$
(g) $\frac{3s-1}{(s+1)^4}$ (h) $\frac{1}{s+4}-\frac{6}{(s-4)^2}$ (i) $\frac{3}{s-7}+\frac{1}{s^3}$

Exercise 5.2. Find the inverse of each of the following Laplace transforms:

(a) $\frac{2s-5}{4s^2+25}+\frac{4s-18}{9-s^2}$ (b) $\frac{s}{s^2+6s+13}$ (c) $\frac{3}{s^2+6s+18}$
(d) $\frac{s}{2s^2+2s+1}$ (e) $\frac{3s+7}{s^2-2s-3}+\frac{2s-3}{s^2+4s+13}$ (f) $\frac{s+1}{s^2+4s+16}$
(g) $\frac{3s-1}{s^2-6s+2}$ (h) $\frac{s-4}{s^2-8s+10}$

6 Inverse Laplace Transform by Partial Fractions

Case (a) *Rational Fractions with Linear Factors in the Denominator*

Example 6.1.

$$\mathscr{L}^{-1}\left\{\frac{1}{(s-1)(s+2)(s-3)}\right\} = \left|\frac{1}{(s+2)(s-3)}\right|_{s=1}\cdot\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\}$$
$$+\left|\frac{1}{(s-1)(s-3)}\right|_{s=-2}\cdot\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\}$$
$$+\left|\frac{1}{(s-1)(s+2)}\right|_{s=3}\cdot\mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\}$$
$$= \frac{1}{(3)(-2)}\cdot e^{t}+\frac{1}{(-3)(-5)}\cdot e^{-2t}+\frac{1}{(2)(5)}\cdot e^{3t}$$

$$= -\frac{1}{6}\cdot e^t + \frac{1}{15}\cdot e^{-2t} + \frac{1}{10}\cdot e^{3t}$$

Example 6.2.

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{s-1}{(s-2)(s-3)(s-4)}\right\} &= \left|\frac{s-1}{(s-3)(s-4)}\right|_{s=2}\cdot\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\}\\
&\quad+\left|\frac{s-1}{(s-2)(s-4)}\right|_{s=3}\cdot\mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\}\\
&\quad+\left|\frac{s-1}{(s-2)(s-3)}\right|_{s=4}\cdot\mathscr{L}^{-1}\left\{\frac{1}{s-4}\right\}\\
&= \frac{1}{(-1)(-2)}\cdot e^t + \frac{2}{(1)(-1)}\cdot e^{-2t} + \frac{3}{(2)(1)}\cdot e^{3t}\\
&= \frac{1}{2}\cdot e^t - 2e^{-2t} + \frac{3}{2}\cdot e^{3t}
\end{aligned}$$

Example 6.3.

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{s^2+1}{s(s-1)(s+1)(s-2)}\right\} &= \left|\frac{s^2+1}{(s-1)(s+1)(s-2)}\right|_{s=0}\cdot\mathscr{L}^{-1}\left\{\frac{1}{s}\right\}\\
&\quad+\left|\frac{s^2+1}{s(s+1)(s-2)}\right|_{s=1}\cdot\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\}\\
&\quad+\left|\frac{s^2+1}{s(s-1)(s-2)}\right|_{s=-1}\cdot\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}\\
&\quad+\left|\frac{s^2+1}{s(s-1)(s+1)}\right|_{s=2}\cdot\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\}\\
&= \frac{1}{2} - e^t - \frac{1}{3}\cdot e^{-t} + \frac{5}{6}\cdot e^{2t}
\end{aligned}$$

Exercise 6.1. Find the inverse of each of the following Laplace transforms:

(a)	$\frac{3s}{(s-1)(s^2-4)}$	(b)	$\frac{1}{(s+3)(s+7)}$	(c)	$\frac{s-1}{(s+1)(s-3)}$
(d)	$\frac{s-8}{s^2-3s-4}$	(e)	$\frac{s^2-7s+5}{(s-1)(s-2)(s-3)}$	(f)	$\frac{s^2+s-2}{s(s-2)(s+3)}$
(g)	$\frac{s+10}{s^2-s-2}$	(h)	$\frac{s-4}{s^2-4}$	(i)	$\frac{1-7s}{(s-3)(s-1)(s+2)}$
(j)	$\frac{1}{s^3-s}$	(k)	$\frac{6}{(s+2)(s-4)}$	(l)	$\frac{s^2+9s-9}{s^3-9s}$
(m)	$\frac{s^2-6s+4}{s^3-3s^2+2s}$	(n)	$\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$		

Case (b) *Rational Fractions with Quadratic Factors in the Denominator*

Example 6.4. We find $f(t) = \mathscr{L}^{-1}\left\{\frac{1}{(s-1)(s-2)^2}\right\}$. Write

$$\bar{F}(s) = \frac{1}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}.$$

Then $A = \left|\frac{1}{(s-2)^2}\right|_{s=1} = 1$, $C = \left|\frac{1}{s-1}\right|_{s=2} = 1$. To find B, we clear the fractions in $\bar{F}(s)$, so that

$$A(s-2)^2 + [B(s-2)+C](s-1) = 1.$$

Comparing the coefficients of s^2 on both sides, we get $A + B = 0$ or $B = -A = -1$. Hence with $A = C = 1$ and $B = -1$, we get

$$\begin{aligned} f(t) &= \mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\} \\ &= e^t - e^{2t} + e^{2t}\cdot\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^t - e^{2t} + te^{2t} \end{aligned}$$

Exercise 6.2. Find the inverse of each of the following Laplace transforms:

(a)	$\frac{s^2+1}{s(s-1)(s+2)^2}$	(b)	$\frac{4s}{(s-1)(s+1)^2}$	(c)	$\frac{s+1}{s^2(s+1)}$
(d)	$\frac{1}{s^3(s+1)}$	(e)	$\frac{s}{(s-1)^2(s+2)^2}$	(f)	$\frac{4s+5}{(s-1)^2(s+2)}$

Case (c) *Rational Fractions with Simple Quadratic Factors in the Denominator*

Example 6.5.

$$\begin{aligned} \mathscr{L}^{-1}\left\{\frac{s^2+6}{(s^2+1)(s^2+4)}\right\} &= \left|\frac{1}{s^2+4}\right|_{s^2=-1}\cdot\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \left|\frac{1}{s^2+1}\right|_{s^2=-4}\cdot\mathscr{L}^{-1}\left\{\frac{1}{s^2+4}\right\} \\ &- \frac{1}{3}\cdot\sin t + \frac{1}{3}\cdot\frac{\sin 2t}{2} \end{aligned}$$

Example 6.6.

$$\begin{aligned} \mathscr{L}^{-1}\left\{\frac{s}{(s^2+a^2)(s^2+b^2)}\right\} &= \mathscr{L}^{-1}\left\{\frac{s[(s^2+b^2)-(s^2+a^2)]}{(b^2-a^2)(s^2+a^2)(s^2+b^2)}\right\} = \frac{1}{b^2-a^2}\cdot\mathscr{L}^{-1}\left\{\frac{s}{s^2+a^2} - \frac{s}{s^2+a^2}\right\} \\ &= \frac{1}{b^2-a^2}\left[\mathscr{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} - \mathscr{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\}\right] = \frac{1}{b^2-a^2}[\cos at - \cos bt] \end{aligned}$$

Exercise 6.3. Find the inverse of each of the following Laplace transforms:

(a)	$\frac{s^2-8}{(s^2+5)(s^2-7)}$	(b)	$\frac{s^3}{s^4-a^4}$	(c)	$\frac{s}{s^4+s^2+1}$
(d)	$\frac{1}{(s^2-a^2)^2}$	(e)	$\frac{s}{s^4+1}$	(f)	$\frac{s}{s^2(s^2+9)}$
(g)	$\frac{1}{s^4-16}$	(h)	$\frac{s}{(s^2-1)^2}$		

Case (d) *Linear and Quadratic Factors in the Denominator*

Example 6.7. We find $f(t) = \mathscr{L}^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\}$. Write

$$\bar{F}(s) = \frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}.$$

Then $A = \left|\frac{5s+3}{s^2+2s+5}\right|_{s=1} = \frac{8}{8} = 1$. To find B and C, we clear the fractions in $\bar{F}(s)$ so that

$$A(s^2+2s+5)^2 + (Bs+C)(s-1) = 1 \text{ or } 8(s^2+2s+5)^2 + (Bs+C)(s-1) = 1.$$

(a) Comparing the coefficients of s^2 on both sides, we get $1+B=0$ or $B=-1$

(b) Comparing the constant terms on both sides, we get $5-C=3$ or $C=5-2=3$

Hence

$$\begin{aligned} f(t) &= \mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathscr{L}^{-1}\left\{\frac{-s+2}{s^2+2s+5}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathscr{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} + 3\cdot\mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} \\ &= e^t - e^{2t} - e^{-t}\cdot\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{3}{2}\cdot e^{-t}\cdot\mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = e^t - e^{-t}\left[\cos 2t - \frac{3}{2}\cdot\sin 2t\right] \end{aligned}$$

Exercise 6.4. Find the inverse of each of the following Laplace transforms:

(a)	$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$	(b)	$\frac{s+1}{(s^2+1)(s^2+4)}$	(c)	$\frac{5s^2-7s+17}{(s-1)(s^2+4)}$
(d)	$\frac{s^2+2s-4}{(s^2+9)(s-5)}$	(e)	$\frac{s}{(s^2+2s+5)(s-7)}$	(f)	$\frac{5s-7}{(s+3)(s^2+2)}$
(g)	$\frac{s}{(s-1)(s^2+2s+2)}$	(h)	$\frac{36}{s(s^2+1)(s^2+9)}$	(i)	$\frac{1}{(s+1)(s+2)(s^2+2s+2)}$

Case (e) *Inverse of the Derivative of the Transform* let $\bar{F}(s) = \mathscr{L}\{f(t)\}$. Then we know that $\mathscr{L}\{tf(t)\} = -\frac{d\bar{F}}{ds}$. Hence, inverting this we get $\mathscr{L}^{-1}\left\{\frac{d\bar{F}}{ds}\right\} = -tf(t)$

Example 6.8. Let $I = \frac{s}{(s^2+a^2)^2}$. If $\bar{F}(s) = \frac{1}{s^2+a^2}$, then $f(t) = \mathscr{L}^{-1}\{\bar{F}(s)\} = \frac{1}{a}\cdot\sin at$. But

$$\frac{d\bar{F}}{ds} = \frac{d}{ds}\left(\frac{1}{s^2+a^2}\right) = -\frac{1}{(s^2+a^2)^2}\cdot(2s) \text{ or } \frac{s}{(s^2+a^2)^2} = -\frac{1}{2}\cdot\frac{d\bar{F}}{ds}$$

Then applying the inverse Laplace transform both sides, and using the above formula, we get $I = tf(t) = \frac{1}{2a}\cdot\sin at$

Example 6.9. Let $I = \frac{s^-a^2}{(s^2+a^2)^2}$. If $\bar{F}(s) = \frac{s}{s^2+a^2}$, then $f(t) = \mathscr{L}^{-1}\{\bar{F}(s)\} = \cos at$. But

$$\frac{d\bar{F}}{ds} = \frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = \frac{(s^2+a^2)(1)-s(2s)}{(s^2+a^2)^2} \text{ or } \frac{s^2-a^2}{(s^2+a^2)^2} = -\frac{d\bar{F}}{ds}$$

Then applying the inverse Laplace transform both sides, and using the above formula, we get $I = tf(t) = t\cos at$

Case (f) *Inverse by Differentiation of the Transform*

Example 6.10. Let $f(t) = \mathscr{L}^{-1}\{\bar{F}(s)\}$, where $\bar{F}(s) = \log\left(\frac{s+1}{s-1}\right)$. Multiplication by t-property yields

$$\begin{aligned} \mathscr{L}\{tf(t)\} &= -\frac{d\bar{F}}{ds} = -\frac{d}{ds}\left[\log\left(\frac{s+1}{s-1}\right)\right] = -\frac{d}{ds}\left[\log(s+1) - \log(s-1)\right] \\ &= -\left(\frac{1}{s+1} - \frac{1}{s-1}\right) = \frac{1}{s-1} - \frac{1}{s+1}. \end{aligned}$$

Then applying the inverse Laplace transform both sides, we get

$$tf(t) = \mathscr{L}^{-1}\left\{\tfrac{1}{s-1}\right\} - \mathscr{L}^{-1}\left\{\tfrac{1}{s+1}\right\} = e^t - e^{-t} = 2\sinh t \text{ or } f(t) = \tfrac{2\sinh t}{t}.$$

Example 6.11. Let $f(t) = \mathscr{L}^{-1}\left\{\bar{F}(s)\right\}$, where $\bar{F}(s) = \log\left(1 + \frac{a^2}{s^2}\right)$. Then by the multiplication by t-property,

$$\begin{aligned}\mathscr{L}\{tf(t)\} &= -\tfrac{d\bar{F}}{ds} = -\tfrac{d}{ds}\left[\log\left(1+\tfrac{a^2}{s^2}\right)\right] = -\tfrac{d}{ds}\left[\log\left(s^2+a^2\right) - 2\log(s)\right]\\ &= -\left(\tfrac{2s}{s^2+a^2} - \tfrac{2}{s}\right) = \tfrac{2}{s} - \tfrac{2s}{s^2+a^2}.\end{aligned}$$

Then applying the inverse Laplace transform both sides, we get

$$tf(t) = \mathscr{L}^{-1}\left\{\tfrac{2}{s}\right\} - \mathscr{L}^{-1}\left\{\tfrac{2s}{s^2+a^2}\right\} = 2.1 - 2\cos at = 4\sin^2(at/2) \text{ or } f(t) = \tfrac{4\sin^2(at/2)}{t}.$$

Example 6.12. Let $f(t) = \mathscr{L}^{-1}\left\{\bar{F}(s)\right\}$, where $\bar{F}(s) = \tan^{-1}\left(\frac{2}{s}\right)$. Multiplication by t-property gives

$$\mathscr{L}\{tf(t)\} = -\tfrac{d\bar{F}}{ds} = -\tfrac{d}{ds}\left[\tan^{-1}\left(\tfrac{2}{s}\right)\right] = -\tfrac{1}{(2/s)^2+1}\cdot\left\{-\tfrac{2}{s^2}\right\} = \tfrac{2}{s^2+4}.$$

Then applying the inverse Laplace transform both sides, we get

$$tf(t) = \mathscr{L}^{-1}\left\{\tfrac{2}{s^2+4}\right\} = \sin 2t \text{ or } f(t) = \tfrac{\sin 2t}{t}.$$

Exercise 6.5. Find the inverse of each of the following Laplace transforms:

(a) $\log\left(\frac{s+a}{s+b}\right)$ (b) $\log\left(1-\frac{a^2}{s^2}\right)$ (c) $\log\left(1+\frac{1}{s}\right)$

(d) $\log\left(\frac{s^2+a^2}{s^2+b^2}\right)$ (e) $\cot^{-1}\left(\frac{s}{a}\right)$

7 Inverse of the Laplace Transform by Convolution Theorem

Let f and g be piece-wise continuous on the interval $[0,\infty)$, then the special product $f * g$, defined by the integral

$$(f*g)(t) = \int_0^t f(v)g(t-v)dv \tag{7.1}$$

is called the *convolution integral* or simply *convolution* of f and g.

Theorem 7.1 (Convolution Theorem)**.** *Let f and g be piece-wise continuous on the interval $[0,\infty)$, and have exponential orders, then the Laplace transform of the convolution of f*

and g equals the product of the Laplace transforms of f and g, that is

$$\mathscr{L}(f*g)(t)=\mathscr{L}\left\{\int_0^t f(v)g(t-v)dv\right\}=\mathscr{L}\{f(t)\}\cdot\mathscr{L}\{g(t)\} \tag{7.2}$$

The inverse form *of the convolution theorem states that*

$$\mathscr{L}^{-1}\left\{\hat{F}(s)\cdot\hat{G}(s)\right\}=(f*g)(t)=\int_0^t f(v)g(t-v)dv, \tag{7.3}$$

where $f(t)=\mathscr{L}^{-1}\left\{\hat{F}(s)\right\}$ and $g(t)=\mathscr{L}^{-1}\left\{\hat{G}(s)\right\}$

Remark 7.1. Division by s Write $\hat{G}(s)=1/s$ in the Convolution theorem, we have

$$\mathscr{L}^{-1}\left\{\frac{\hat{F}(s)}{s}\right\}=\int_0^t f(v)dv, \tag{7.4}$$

where $f(t)=\mathscr{L}^{-1}\left\{\hat{F}(s)\right\}$. In other words,

$$\mathscr{L}\left\{\int_0^t f(v)dv\right\}=\frac{\hat{F}(s)}{s}, \tag{7.5}$$

Example 7.1. Use convolution theorem to evaluate $\mathscr{L}^{-1}\left\{\frac{1}{(s+a)s}\right\}$

Solution. Write $\frac{1}{(s+a)s}=\frac{1}{s+a}\cdot\frac{1}{s}$. Identify that $\mathscr{L}^{-1}\left\{\frac{1}{s+a}\right\}=e^{-at}$ and $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\}=1$. Then by convolution theorem, we have

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{s+a}\cdot\frac{1}{s}\right\}&=\mathscr{L}^{-1}\left\{\frac{1}{s+a}\right\}*\mathscr{L}^{-1}\left\{\frac{1}{s}\right\}\\&=e^{-at}*1\\&=\int_0^t e^{-av}.1\,dv=\left|-\frac{e^{-av}}{a}\right|_{v=0}^{t}=\frac{1}{a}\left(1-e^{-at}\right)\end{aligned}$$

Example 7.2. Use convolution theorem to evaluate $\mathscr{L}^{-1}\left\{\frac{1}{s^2(s+a)}\right\}$

Solution. Write $\frac{1}{s^2(s+a)}=\frac{1}{s^2}\cdot\frac{1}{s+a}$. Since $\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\}=t$ and $\mathscr{L}^{-1}\left\{\frac{1}{s+a}\right\}=e^{-at}$, convolution theorem gives

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{s+a}\right\}=t*e^{-at}=\int_0^t ve^{-a(t-v)}\,dv=e^{-at}\int_0^t ve^{av}\,dv$$

$$= e^{-at}\left|(v)\left\{\frac{e^{av}}{a}\right\} - (1)\left\{\frac{e^{av}}{a^2}\right\}\right|_{v=0}^{t}$$
$$= e^{-at}\left[\frac{te^{at}}{a} - \frac{e^{at}}{a^2} + \frac{e^0}{a^2}\right] = \frac{t}{a} + \left(\frac{e^{-at}-1}{a^2}\right)$$

Example 7.3. Use convolution theorem to evaluate $\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)s}\right\}$

Solution. Write $\frac{1}{(s^2+1)s} = \frac{1}{s^2+1}\cdot\frac{1}{s}$. Note that $\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$, $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$. Then

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\cdot\frac{1}{s}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} * \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = \sin t * 1$$
$$= \int_0^t \sin v.1\, dv = |-\cos v|_{v=0}^{t} = 1 - \cos t$$

Example 7.4. Use convolution theorem to evaluate $\mathscr{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$

Solution. We write $\frac{s}{(s^2+1)^2} = \frac{1}{s^2+1}\cdot\frac{s}{s^2+1}$. Since $\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$, $\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t$, by convolution theorem, we obtain

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\cdot\frac{s}{s^2+1}\right\} = \sin t * \cos t = \int_0^t \sin v\cos(t-v)\, dv$$
$$= \frac{1}{2}\int_0^t [\sin t + \sin(2v-t)]\,dv$$
$$= \frac{1}{2}\left[t\sin t - \left|\frac{\cos(2v-t)}{2}\right|_{v=0}^{t}\right]$$
$$= \frac{1}{2}\left[t\sin t - \frac{1}{2}(\cos t - \cos(-t))\right] = \frac{t\sin t}{2}$$

Example 7.5. Find $\mathscr{L}^{-1}\left\{\frac{1}{(s^2+4)(s-1)}\right\}$

Solution. We write $\frac{1}{(s^2+4)(s-1)} = \frac{1}{s^2+4}\cdot\frac{1}{s-1}$. Since $\mathscr{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{\sin 2t}{2}$, $\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t$, by convolution theorem,

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+4}\cdot\frac{1}{s-1}\right\} = \frac{1}{2}\sin 2t * e^t = \frac{1}{2}\int_0^t (\sin 2v)e^{t-v}\, dv = \frac{e^t}{2}\int_0^t e^{-v}\sin 2v dv$$

We know that $\int e^{Av}\sin Bv dv = \frac{e^{Av}}{A^2+B^2}(A\sin Bv - B\cos Bv)$. With $A=-1$ and $B=2$

so that $A^2 + B^2 = 5$, and we have

$$\begin{aligned}
\int_0^t e^{-v} \sin 2v dv &= \left| \frac{e^{-v}}{5}(-\sin 2v - 2\cos 2v) \right|_{v=0}^{t} \\
&= \frac{1}{5}[(\sin 0 + 2\cos 0) - e^{-t}(\sin 2t + 2\cos 2t)] \\
&= \frac{1}{5}[2 - e^{-t}(\sin 2t + 2\cos 2t)]
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{1}{s^2+4} \cdot \frac{1}{s-1}\right\} &= \frac{e^t}{2}\left[\frac{1}{5}[2 - e^{-t}(\sin 2t + 2\cos 2t)]\right] \\
&= \frac{e^t}{5} - \frac{\sin 2t + 2\cos 2t}{10}
\end{aligned}$$

Exercise 7.1 (Think About It)**.** Which method is convenient to find

(a) $\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)s^2}\right\}$?

(b) $\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$?

Also, find the inverse transform in each case.