CAT-II MODEL QUESTIONS (MAT1011)

1. Investigate the continuity of the function $f(x,y) = \begin{cases} \frac{x\,y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

2. Suppose that the temperature of the water at the point on a river where a nuclear power plant discharges its hot waste water is approximated by

$\theta(T,P) = 2T + P + TP + 0.01T^2P^2 - 40$

where T represents the temperature of the river water (in degrees Celsius) before it reaches the power plant and P is the number of megawatts (in hundreds) of electricity being produced by the plant. Find (i) $\theta_T(9,5)$ and (ii) $\theta_P(9,5)$ hence interpret approximate change in temperature resulting from a 1-unit increase in (i) temperature of the river water and (ii) production of electricity.

3. A short length of blood vessel is in the shape of a right circular cylinder. The length of the vessel is measured as 42 mm, and the radius is measured as 2.5 mm.

   (i) If the maximum error in the measurement of the length is 0.9 mm, with an error not more than 0.2 mm in the measurement of the radius, find the maximum possible error in calculating the volume of the blood vessel.

   (ii) If the errors in measuring the radius and length of the vessel are at most 1% and 3%, respectively, estimate the maximum percent error in calculating the surface area.

4. If x increases at the rate of 2 cm/sec at the instant when $x = 3$ cm. and $y = 1$ cm., at what rate must y be changing in order that the function $2xy - 3x^2y$ shall be neither increasing nor decreasing?

5. If $z = f(x, y)$ and $x = e^u cosv$, $y = e^u sinv$, prove that $x\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v} = e^{2u}\frac{\partial z}{\partial y}$

6. If $u = \frac{x^3y^3z^3}{x^3+y^3+z^3} + \log\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)$, find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

7. If $u = x\log(xy)$ where $x^3 + y^3 + 3xy = 1$, find $\frac{du}{dx}$.

8. If $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$, evaluate $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

9. If $u = xyz,\ v = x^2 + y^2 + z^2,\ w = x + y + z,$ find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

10. If $u = 3x + 2y - z,\ v = x - 2y + z,\ w = x(x + 2y - z),$ show that they are functionally related, and find the relation.

11. Find the quadratic approximation of the wind chill factor

$$W(v,T) = 91.4 - \frac{(10.45+6.68\sqrt{v}-0.447v)(457-5T)}{110}$$

where T is the temperature and v is the speed of the wind. Hence estimate W when the temperature is $40°F$ and the wind speed is 20 mph. Compare the estimated value with the exact value and state the reason of error, if any.

12. An approximate relationship between yield Y of corn and the amounts of nitrogen N and phosphorus P fertilizer used is given by

$$Y = -0.075 + 0.584N + 0.664P - 0.158N^2 - 0.18P^2 + 0.081PN$$

where Y, N, and P are in appropriate units. Find the amounts of nitrogen and phosphorus that will maximize the corn yield according to this model.

13. A company is developing a new soft drink. The cost in dollars to produce a batch of the drink is approximated by $C(S, F) = 2200 + 27S^3 + 72SF + 8F^2$ where S is the number of kilograms of sugar per batch and F is the number of grams of flavoring per batch. Find the amounts of sugar and flavoring that result in the minimum cost per batch. What is the minimum cost?

14. Suppose we have a Cobb-Douglass production function $f(x, y) = 15x^{1/3}y^{2/3}$ where x is the number of units of labor, y is the number of units of capital, and f is the number of units of a certain product that is produced. If each unit of labor costs \$200, each unit of capital costs \$100, and the total expense for both is limited to \$7,500,000, find the number of units of labor and capital needed to maximize production.

15. An architect is attempting to fit a rectangular closet of maximal volume into a chopped-off corner of a building. She determines that the available space is the constricting plane which is given by $2x + 4y + 3Z = 72$ with the dimensions given in feet. What are the dimensions of the closet of maximum volume?

16. An industrial plant is located at the precise center of a town shaped as a square with each side of length 4 miles. If the plant is placed at the point (0,0) of the *xy*-plane, then certain pollutants are dispersed is such a manner that the concentration at any point (x, y) in town is given by $C(x, y) = 1000(24 - 3x^2 - 3y^2)$ where $C(x, y)$ is the number of particles of pollutants per square mile of surface per day at a point (x, y) in town. What is the average concentration of these pollutants in this town?

17. The density of privately owned vehicles in a city is given by $P(x, y) = xy$, where x is miles in the east-west direction, y is miles in the north-south direction, and P is the number of privately owned vehicles per square mile, in thousands. The city limits are bounded by the line $x = 5 - y$ and curve $x = \sqrt{25 - y^2}$, the lines $y = 0$ and $y = 5$.

Sketch the region of integration and write an equivalent integral with the order of integration reversed. What is the total number of privately owned vehicles in the city?

18. A product design consultant for a cosmetics company has been asked to design a bottle for the company's newest perfume. The thickness of the glass is to vary so that the outside of the bottle has straight sides and the inside has curved sides, with flat ends shaped like parabolas on the 4 cm sides. Before presenting the design to management, the consultant needs to make a reasonably accurate estimate of the amount each bottle will hold. If the base of the bottle is to be 4 cm by 3 cm, and if a cross section of its interior is to be a parabola of the form $z = 4y - y^2$, what is its internal volume?

19. By changing the order of integration of $\int_0^\infty \int_0^\infty e^{-xy} \sin(px)\, dx\, dy$, show that $\int_0^\infty \frac{\sin(px)}{x}\, dx = \frac{\pi}{2}$.

20. Evaluate $\int_0^1 \int_{e^x}^{e} \frac{dx\, dy}{\log y}$ by changing the order of integration.

21. Changing the order of integration in $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\, dx$ and hence evaluate.