MODULE-1
SYSTEM OF LINEAR EQUATIONS

MAT3004

APPLIED LINEAR ALGEBRA

VIT

Topics to be discussed

System of Linear Equations:

- Gaussian elimination
- Gauss Jordan methods
- Elementary matrices
- Permutation matrix
- Inverse matrices
- System of linear equations
- LU factorizations
- LDU factorizations

List of Definition and terms

- **Linear equation (LE)**
- **System of linear equation (SLE)**
- **Matrix representation of a SLE ($\boldsymbol{AX = B}$)**
- **Homogeneous and nonhomogeneous system**
- **Solution, Trivial solution, Unique solution, Infinite solution, No solution**
- **Consistency and Inconsistency of a system**
- **Augmented matrix, variable matrix, solution matrix**
- **Pivot element or key element**
- **Rows and Columns of a matrix, Elimination and reduction of a rows and columns**
- **10. Forward and backward elimination**
- **11. Row equivalent, Row reduction of 1^{st}, 2^{nd}, 3^{rd} kinds**
- **ECHELON FORM of a matrix**
- **Triangular form of a matrix (Upper and Lower triangular form)**
- **Elementary matrix, Permutation matrix**
- **Block matrix, submatrix**
- **Inverse of a matrix, left and right inverse, invertible, non-singular, singular**

Some problem types

1. Solve the following (given) system of equations using
    - Gauss Elimination method
    - Gauss Jordan Elimination method
    - Matrix inversion method
    - LU factorization method
    - LDU factorization method
2. Find the conditions for the consistency of a (given) system $A\boldsymbol{X} = B$.
3. Find the values of a, b for which a (given) system $A\boldsymbol{X} = B$ has infinite solutions, no solution, unique solution.
4. Reduce the matrix (given) to Echelon form.
5. Find the inverse of a (given) matrix by Gauss Jordan method.
6. Find the LU and LDU decomposition of a given matrix.
7. Find the elementary matrices in I_n.
8. Write the matrix A as a product of elementary matrices.

Some Theorems

1. If a $n \times n$ square matrix A has a left inverse B and a right inverse C, then B and C are equal i.e.$B = C$. **Proof required**.
2. The product of invertible matrices is also invertible, whose inverse is the product of the individual inverses in reversed order i.e.$(AB)^{-1} = B^{-1}A^{-1}$. **Proof required**.
3. Let A be a $n \times n$ matrix. The following are equivalent :
   a) A has a left inverse;
   b) $A\boldsymbol{X} = \boldsymbol{0}$ has only the trivial solution $\boldsymbol{X} = \boldsymbol{0}$;
   c) A is row equivalent to I_n;
   d) A is a product of elementary matrices;
   e) A is invertible;
   f) A has a right inverse. **Proof required.**

4. A triangular matrix is invertible if and only if it has no zero diagonal entry.

Statement only without proof.

Linear equation:-

Consider the equation $a_1x_1 + a_2x_2 + \cdots a_nx_n = b$ — (1)

where $a_1, a_2, \ldots a_n$ and b are constants is called linear equation.

System of linear equation:- (Group of linear equations)

A general system of m linear equations with n unknowns can be written as

$$\left.\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots\cdots a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots\cdots a_{2n}x_n = b_2 \\
a_3 \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots\cdots a_{mn}x_n = b_m
\end{array}\right\} \longrightarrow (2)$$

where $x_1, x_2, \ldots x_n$ are the unknowns,

$a_{11}, a_{12}, \ldots a_{mn}$ are the coefficients of the system

and $b_1, b_2, \ldots b_m$ are the constant terms.

The system ② is said to be homogeneous if all the constant terms are zero, i.e. $b_1 = b_2 = \ldots b_m = 0$

otherwise ② is known as non-homogeneous.

Every homogeneous eq/sol has trivial solution.

⇓ all sol = 0

$3x - 5y = 0 \Rightarrow x = 0, y = 0 \Rightarrow 0 = 0$

In general, a linear system may have in any one of three possible ways:

i) The system has a single (unique) solution.

ii) The system has infinitely many solution.

iii) The system has no solution.

The system is (2) is said to be <u>consistent</u> if it has atleast one solution and <u>inconsistent</u> if it has no solution.

The system (2) can be expressed as

$$AX = B$$

where $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & & a_{mn} \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

$$[A : B] = \left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & & a_{mn} & b_m \end{array}\right]$$

is called <u>argumented</u> <u>matrix</u>.

Elementary operations :-

The following operations on a argumented matrix (system of linear equations) are called elementary operations.

i) Interchange two rows (equations)

i.e. $R_i \longleftrightarrow R_j$ ($l_i \longleftrightarrow l_j$)

ii) Multiply a non-zero constant throughout a row (an equation)

i.e. $R_i \longrightarrow aR_i$ ($l_i \rightarrow a l_i$) $a \neq 0$

iii) Add a constant multiple of an equation to another equation

i.e. $R_i \rightarrow aR_i + R_j$ (or) $R_j \rightarrow aR_j + R_i$

Row-echelon form of a matrix :- (Gauss elimination method)

Reduced row-echelon form (Gauss Jordan elimination)

A matrix is said to be in row-echelon form if it satisfies the following:

(i) The zero rows, if they exist, come last in the ordered rows.

(ii) The first non-zero entries, in the non-zero rows are 1, called leading ones.

(iii) In each column containing a leading non-zero element, the entries below that leading non-zero element are 0.

The reduced row-echelon form of an argumented matrix is of the form:

(iv) Above each leading is a column of zero's in addition to the row-echelon form

Gauss - elimination method :-

The gauss - elimination algorithm is as follows : (i) Write the argumented matrix of the system of linear equations.

(ii) Find an echelon form of the argumented matrix using elementary row operations.

(iii) Write the system of equations corresponding to the echelon form.

(iv) Use back substitution to get the solution.

Pivot element :-

The left most non-zero entries in the non-zero rows are called pivots.

Example: Solve the system of linear equations $2y+4z=2$, $x+2y+2z=3$, $3x+4y+6z=-1$ and write the pivots.

Sol:- The system of linear equations

$$0.x+2y+4z=2$$
$$x+2y+2z=3$$
$$3x+4y+6z=-1$$

Argumented matrix $[A \vdots B] = \begin{bmatrix} 0 & 2 & 4 & 2 \\ 1 & 2 & 2 & 3 \\ 3 & 4 & 6 & -1 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 2 & 4 & 2 \\ 3 & 4 & 6 & -1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & -2 & 0 & -10 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 4 & -8 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2/2 \\ R_3 \to R_3/4 \end{matrix}$$

pivots of $x=1$, $y=2$, $z=4$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

[Back substitution method]

$\Rightarrow \quad 1.x + 2y + 2z = 3$ —— ①

$\Rightarrow \quad 0x + y + 2z = 1$ —— ②

$0x + 0y + z = -2$ —— ③

From equation ③, we have $\boxed{z = -2}$

Sub $z = -2$ in equation ②, we get

$y + 2(-2) = 1$

$\boxed{y = 5}$

Substitute $y = 5$, $z = -2$ in equation ①, we get

$x + 10 - 4 = 3$

$\boxed{x = -3}$

$\therefore \quad x = -3$

$y = 5$

$z = -2$

1. Solve the following system of equations by gaussian elimination. What are the pivots

(i) $-x+y+2z=0$, $3x+4y+z=0$, $2x+5y+3z=0$

(ii) $2y-z=1$, $4x-10y+3z=5$, $3x-3y=6$

(iii) $w+x+y=3$, $-3w-17x+y+2z=1$, $4w-17x+8y-5z=1$, $-5x-2y+z=1$

(iv) $x_1+2x_2+3x_3+2x_4=-1$, $-x_1-2x_2-2x_3+x_4=2$,

$2x_1+4x_2+8x_3+12x_4=4$

(i) Sol :- The system of equations

$$-x+y+2z=0$$
$$3x+4y+z=0$$
$$2x+5y+3z=0$$

Argumented matrix $[A \vdots B] = \begin{bmatrix} -1 & 1 & 2 & 0 \\ 3 & 4 & 1 & 0 \\ 2 & 5 & 3 & 0 \end{bmatrix}$ $R_1 \to R_1 \times (-1)$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 3 & 4 & 1 & 0 \\ 2 & 5 & 3 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 7 & 7 & 0 \\ 0 & 7 & 7 & 0 \end{bmatrix} R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 7 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2 \to R_2/7$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow x - y - 2z = 0$ ——— ①

$y + z = 0$ ——— ②

Let $\boxed{z = t}$ sub in equ ②

$y + t = 0$

$\boxed{y = -t}$

Substitute $y = -t$, $z = t$ in equation ①

$x + t - 2t = 0$

$x = 2t - t$

$\boxed{x = t}$

$\therefore$ $x = t$

$y = -t$

$z = t$

2. Determine the condition on b_i so that the following system has a solution.

(i) $x+2y+6z=b_1,\ 2x-3y-2z=b_2,\ 3x-y+4z=b_3$

(ii) $x+3y-2z=b_1;\ 2x-y+3z=b_2,\ 4x+2y+z=b_3$

(i) Sol: The system of equations

$$x+2y+6z=b_1$$

$$2x-3y-2z=b_2$$

$$3x-y+4z=b_3$$

Argumented matrix $[A \vdots B] = \begin{bmatrix} 1 & 2 & 6 & b_1 \\ 2 & -3 & -2 & b_2 \\ 3 & -1 & 4 & b_3 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 6 & b_1 \\ 0 & -7 & -14 & b_2-2b_1 \\ 0 & -7 & -14 & b_3-3b_1 \end{bmatrix} R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 6 & b_1 \\ 0 & -7 & -14 & b_2-2b_1 \\ 0 & 0 & 0 & b_3-b_2-b_1 \end{bmatrix} R_2 \to R_2/-7$$

$$\sim \begin{bmatrix} 1 & 2 & 6 & b_1 \\ 0 & 1 & 1 & \frac{1}{7}(2b_1-b_2) \\ 0 & 0 & 0 & b_3-b_2-b_1 \end{bmatrix}$$

Since $\rho(A)=2$, $\rho(A \vdots B)=2$ if $b_3-b_2-b_1=0$

$\therefore$ $\rho(A)=2=\rho(A \vdots B) <$ no of variables (3)

$\therefore$ The given system has an infinite no of solution when $b_3-b_1-b_2=0$

(ii) Sol : The system of equations

$$x+3y-2z=b_1$$
$$2x-y+3z=b_2$$
$$4x+2y+z=b_3$$

Argumented matrix $[A \,\vdots\, B] = \begin{bmatrix} 1 & 3 & -2 & b_1 \\ 2 & -1 & 3 & b_2 \\ 4 & 2 & 1 & b_3 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 3 & -2 & b_1 \\ 0 & -7 & 7 & b_2-2b_1 \\ 0 & -10 & 9 & b_3-4b_1 \end{bmatrix}$$

$R_2 \to R_2/-7$

$$\sim \begin{bmatrix} 1 & 3 & -2 & b_1 \\ 0 & 1 & -1 & 2b_1-b_2 \\ 0 & -10 & 9 & b_3-4b_1 \end{bmatrix} R_3 \to R_3 + 10R_2$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & b_1 \\ 0 & 1 & -1 & 2b_1-b_2 \\ 0 & 0 & -1 & b_3-10b_2+16b_1 \end{bmatrix}$$

Since $\rho(A) = 3$, $\rho(A \vdots B) = 3$

$\therefore \rho(A) = 3 = \rho(A \vdots B) =$ no of variables(3)

$\therefore$ The given system has an unique solution

1. (ii) Sol: The system of equations

$$2y - z = 1$$
$$4x - 10y + 3z = 5$$
$$3x - 3y = 6$$

Argumented matrix $[A \vdots B] = \begin{bmatrix} 0 & 2 & -1 & 1 \\ 4 & -10 & 3 & 5 \\ 3 & -3 & 0 & 6 \end{bmatrix}$ $R_3 \to R_3/3$

$$\sim \begin{bmatrix} 0 & 2 & -1 & 1 \\ 4 & -10 & 3 & 5 \\ 1 & -1 & 0 & 2 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 4 & -10 & 3 & 5 \\ 0 & 2 & -1 & 1 \end{bmatrix} \quad R_2 \to R_2 - 4R_1.$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & -6 & 3 & -3 \\ 0 & 2 & -1 & 1 \end{bmatrix} R_2 \to R_2/3$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & -2 & 1 & -1 \\ 0 & 2 & -1 & 1 \end{bmatrix} R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2 \to R_2/-2$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad x - y = 2 \quad \text{——} \ (1)$$

$$y - \frac{1}{2}z = \frac{1}{2} \quad \text{——} \ (2)$$

Let $\boxed{z = t}$ substitute in equation (2)

$$y - \frac{1}{2}t = \frac{1}{2}$$

$$\frac{2y - t}{2} = \frac{1}{2}$$

$$2y - t = 1$$

$$2y = 1 + t$$

$$\boxed{y = \frac{1+t}{2}}$$

Put $y = \frac{1+t}{2}$, $z = t$ substitute in equation (1)

$$x - \frac{1+t}{2} = 2$$

$$x = 2 + \frac{1+t}{2}$$

$$\boxed{x = \frac{5+t}{2}}$$

$$\therefore \quad x = \frac{5+t}{2}$$

$$y = \frac{1+t}{2}$$

$$z = t$$

(iii) Sol: The system of equations

$$w + x + y = 3$$

$$-3w - 17x + y + 2z = 1$$

$$4w - 17x + 8y - 5z = 1$$

$$-5x - 2y + z = 1$$

Argumented matrix $[A ; B] = \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ -3 & -17 & 1 & 2 & 1 \\ 4 & -17 & 8 & -5 & 1 \\ 0 & -5 & -2 & 1 & 1 \end{bmatrix}$ $R_2 \to R_2 + 3R_1$

$R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ 0 & -14 & 4 & 2 & 10 \\ 0 & -21 & 4 & -5 & -11 \\ 0 & -5 & -2 & 1 & 1 \end{bmatrix} \quad R_2 \to R_2/-2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ 0 & 7 & -2 & -1 & -5 \\ 0 & -21 & 4 & -5 & -11 \\ 0 & -5 & -2 & 1 & 1 \end{bmatrix} \quad R_3 \to R_3 + 3R_2$$

$R_4 \to 7R_4 + 5R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ 0 & 7 & -2 & -1 & -5 \\ 0 & 0 & -2 & -8 & -26 \\ 0 & 0 & -24 & 2 & -18 \end{bmatrix} \begin{array}{l} R_3 \to R_3/-2 \\ R_4 \to R_4/-2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ 0 & 7 & -2 & -1 & -5 \\ 0 & 0 & +1 & 4 & 13 \\ 0 & 0 & +12 & -1 & 9 \end{bmatrix} \begin{array}{l} \cancel{R_4 \to R_4} \\ R_4 \to R_4 - 12R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ 0 & 7 & -2 & -1 & -5 \\ 0 & 0 & 1 & 4 & 13 \\ 0 & 0 & 0 & -49 & -147 \end{bmatrix}$$

-12×4

-48

$$\Rightarrow \quad w+x+y+0z=3 \quad \text{——} \ (1)$$

$$0w+7x-2y-z=-5 \quad \text{——} \ (2)$$

$$y+4z=13 \quad \text{——} \ (3)$$

$$-49z=-147 \quad \text{——} \ (4)$$

$$\boxed{z=3}$$

Put $z=3$ in equation (3)

$$y+12=13$$

$$\boxed{y=1}$$

Put $z=3, y=1$ in equation (2)

$$7x-2-3=-5$$

$$7x-5=-5$$

$$\boxed{x=0}$$

Substitute $x=0, y=1, z=3$ in equation (1)

$$w+0+1=3$$

$$\boxed{w=2}$$

$\therefore$ $w=2$

$x=0$

$y=1$

$z=3$

(iv) Sol: The system of equations

$$x_1+2x_2+3x_3+2x_4=-1$$

$$-x_1-2x_2-2x_3+x_4=2$$

$$2x_1+4x_2+8x_3+12x_4=4$$

Argumented matrix $[A \,;\, B] = \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ -1 & -2 & -2 & 1 & 2 \\ 2 & 4 & 8 & 12 & 4 \end{bmatrix} R_3 \to R_3/2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ -1 & -2 & -2 & 1 & 2 \\ 1 & 2 & 4 & 6 & 2 \end{bmatrix} \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 4 & 3 \end{bmatrix} R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\Rightarrow x_1 + 2x_2 + 3x_3 + 2x_4 = -1 \quad \text{——— (1)}$$

$$x_3 + 3x_4 = 1 \quad \text{——— (2)}$$

$$\boxed{x_4 = 2}$$

Substitute $x_4 = 2$ in equation (2)

$$x_3 + 6 = 1$$

$$\boxed{x_3 = -5}$$

Substitute $x_3 = -5$, $x_4 = 2$ in equation (1)

$$x_1 + 2x_2 - 15 + 4 = -1$$

$$x_1 + 2x_2 = +10$$

Let $\boxed{x_2 = t}$

$$x_1 + 2t = 10$$

$$\boxed{x_1 = 10 - 2t}$$

Note :-

(i) If $\rho(A) = \rho(A \vdots B)$ = number of variables, then the system has a unique solution

(ii) If $\rho(A) = \rho(A \vdots B) <$ number of variables, then the system has an infinite number of solutions

(iii) If $\rho(A) \neq \rho(A \vdots B)$, then the system has no solution

1. Determine all values of b_i that make the following system $x+y-z=b_1$, $2y+z=b_2$, $y-z=b_3$ consistent

2. Determine the condition b_i so that the following system has no solution $2x+y+7z=b_1$, $6x-2y+11z=b_2$,

$2x-y+3z=b_3$

3. Which of the following system has a non-trivial solution.

(i) $x+2y+3z=0$

$2y+2z=0$

$x+2y+3z=0$

(ii) $2x+y-z=0$

$x-2y-3z=0$

$3x+y-2z=0$

4. For which values of "a" does each of the following system have no solution, exactly one solution or infinitely many solution

(i) $x+2y-3z=4$, $3x-y+5z=2$, $4x+y+(a^2-14)z=a+2$

(ii) $x-y+z=1$, $x+3y+2z=2$, $2x+ay+3z=3$

4. (i)

Argumented matrix $[A \vdots B] = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}$$

$R_2 \to R_2 / -7$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}$$

i) If $a^2-16=0$ and $a-4=0$ then $\rho(A)=2$, $\rho(A|B)=2$

i.e $a=\pm 4$ then the given system has an infinite number of solutions $[\because \rho(A)=2=\rho(A|B)<3]$

ii) If $a^2-16\neq 0$ and $a-4\neq 0$ then $\rho(A)=3$, $\rho(A|B)=3$

i.e $a\neq\pm 4$ then the given system has an unique solution ($\because \rho(A)=3=\rho(A|B)=$ no. of unknowns)

iii) If $a=-4$, then the given system has no solution

*** Gauss - Jordan elimination :-

(i) Write the argumented matrix for the given system of linear equations.

(ii) Derive the reduced row-echelon form for the argumented matrix by using elementary row operations.

(iii) Write the system of equations corresponding to the reduced row-echelon form. This system gives the solution.

Solve the following system of linear equations by gauss jordan elimination.

$x_1 + 3x_2 - 2x_3 = 3$, $2x_1 + 6x_2 - 2x_3 + 4x_4 = 18$, $x_2 + x_3 + 3x_4 = 10$

Argumented matrix $[A \mid B] = \begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 2 & 6 & -2 & 4 & 18 \\ 0 & 1 & 1 & 3 & 10 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 0 & 0 & 2 & 4 & 12 \\ 0 & 1 & 1 & 3 & 10 \end{bmatrix} R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 0 & 1 & 1 & 3 & 10 \\ 0 & 0 & 2 & 4 & 12 \end{bmatrix} R_3 \to R_3/2$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 0 & 1 & 1 & 3 & 10 \\ 0 & 0 & 1 & 2 & 6 \end{bmatrix} R_1 \to R_1 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -5 & -9 & -27 \\ 0 & 1 & 1 & 3 & 10 \\ 0 & 0 & 1 & 2 & 6 \end{bmatrix} \begin{matrix} R_1 \to R_1 + 5R_3 \\ R_2 \to R_2 - R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 2 & 6 \end{bmatrix}$$

$\Rightarrow$ $x_1 + 0x_2 + 0x_3 + x_4 = 3$ —— ①

$0x_1 + x_2 + 0x_3 + x_4 = 4$ —— ②

$0x_1 + 0x_2 + x_3 + 2x_4 = 6$ —— ③

$x_1 + x_4 = 3$ —— ④

$x_2 + x_4 = 4$ —— ⑤

$x_3 + 2x_4 = 6$ —— ⑥

choose $x_4 = t$

From equ (4), we have $x_1 = 3 - t$

From equ (5), we have $x_2 = 4 - t$

From equ (6), we have $x_3 = 6 - 2t$

$$\therefore \quad x_1 = 2 - t$$
$$x_2 = 4 - t$$
$$x_3 = 6 - 2t$$
$$x_4 = t \qquad t \in R$$

2. Solve the following system of linear equation by Gauss-Jordan elimination.

(i) $2x - 3y = 8,\ 4x - 5y + z = 15,\ 2x + 4z = 1$

(ii) $x_1 + x_2 + x_3 - x_4 = -2,\ 2x_1 - x_2 + x_3 + x_4 = 0,$
$3x_1 + 2x_2 - x_3 - x_4 = 1,\ x_1 + x_2 + 3x_3 - 3x_4 = -8$

1. Argumented matrix $[A:B] = \begin{bmatrix} 1 & 1 & -1 & b_1 \\ 0 & 2 & 1 & b_2 \\ 0 & 1 & -1 & b_3 \end{bmatrix}$ $R_3 \Rightarrow 2R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & -1 & b_1 \\ 0 & 2 & 1 & b_2 \\ 0 & 0 & -3 & 2b_3 - b_2 \end{bmatrix} \begin{array}{l} R_2 \Rightarrow R_2/2 \\ R_3 \Rightarrow R_3/-3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & b_1 \\ 0 & 1 & \frac{1}{2} & \frac{b_2}{2} \\ 0 & 0 & 1 & \frac{b_2 - 2b_3}{3} \end{bmatrix}$$

$\rightarrow x + y - z = b_1$ —— (1)

$y + \frac{1}{2} z = \frac{b_2}{2}$ —— (2)

$$\boxed{z = \frac{b_2 - 2b_3}{3}}$$

Sub $z = \frac{b_2 - 2b_3}{3}$ in equ (2)

$$y + \frac{b_2 - 2b_3}{6} = \frac{b_2}{2}$$

$$y + b_2 2b_3 = 3b_2$$

$$y = \frac{2b_2 + 2b_3}{6}$$

$$\boxed{y = \frac{b_2+b_3}{3}}$$

Sub $y = \frac{b_2+b_3}{3}$, $x = \frac{b_2-2b_3}{3}$ in equ (1)

$$x + \frac{b_2+b_3}{3} - \frac{b_2+2b_3}{3} = b_1$$

$$x + \frac{\cancel{b_2}+b_3-\cancel{b_2}+2b_3}{3} = b_1$$

$$x + \frac{\cancel{3}b_3}{\cancel{3}} = b_1$$

$$\boxed{x = b_1 - b_3}$$

2. Argumented matrix $[A|B] = \begin{bmatrix} 2 & 1 & 7 & b_1 \\ 6 & -2 & 11 & b_2 \\ 2 & -1 & 3 & b_3 \end{bmatrix} R_1 \Rightarrow R_1/2$

$$\sim \begin{bmatrix} 1 & 1/2 & 7/2 & b_1/2 \\ 6 & -2 & 11 & b_2 \\ 2 & -1 & 3 & b_3 \end{bmatrix} \begin{matrix} R_2 \Rightarrow R_2 - 6R_1 \\ R_3 \Rightarrow R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1/2 & 7/2 & b_1/2 \\ 0 & -5 & -10 & b_2 - 3b_1 \\ 0 & -2 & -4 & b_3 - b_1 \end{bmatrix} \begin{matrix} R_2 \Rightarrow R_2/-5 \\ R_3 \Rightarrow R_3/-2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1/2 & 7/2 & b_1/2 \\ 0 & 1 & 2 & \frac{-b_2+3b_1}{5} \\ 0 & 1 & 2 & \frac{-b_3+b_1}{2} \end{bmatrix} R_3 \Rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1/2 & 7/2 & b_1/2 \\ 0 & 1 & 2 & \frac{3b_1-b_2}{5} \\ 0 & 0 & 0 & \frac{-b_1+2b_2+5b_3}{10} \end{bmatrix}$$

Since $\rho(A)=2$, $\rho(A\,|\,B)=3$

$\therefore$ $\rho(A) \neq \rho(A\,|\,B)$

$\therefore$ The given system has no solution.

3.(i) Argumented matrix $[A\,|\,B] = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix}$ $R_2 \Rightarrow R_2/2$, $R_3 \Rightarrow R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow x+2y+3z=0 \quad \text{——} (1)$$

$$y+z=0 \quad \text{——} (2)$$

sub $\boxed{z=t}$ in equ (2)

$$y+t=0$$

$$\boxed{y=-t}$$

sub $y=-t, z=t$ in equ (1)

$$x-2t+3t=0$$

$$x+t=0$$

$$\boxed{x=-t}$$

$$\therefore x=-t$$

$$y=-t$$

$$z=t$$

(ii) Argumented matrix $[A \mid B] = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & -2 & -3 & 0 \\ 3 & 1 & -2 & 0 \end{bmatrix}$ $R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & -2 & -3 & 0 \\ 2 & 1 & -1 & 0 \\ 3 & 1 & -2 & 0 \end{bmatrix} \begin{matrix} R_2 \Rightarrow R_2 - 2R_1 \\ R_3 \Rightarrow R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 7 & 7 & 0 \end{bmatrix} \begin{matrix} R_2 \Rightarrow R_2/5 \\ R_3 \Rightarrow R_3/7 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} R_3 \Rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow x - 2y - 3z = 0$ —— (1)

$y + z = 0$ —— (2)

sub $\boxed{z = t}$ in equ (2)

$\boxed{y = -t}$

sub $y = -t$, $z = t$ in equ (1)

$x + 2t - 3t = 0$

$x - t = 0$

$\boxed{x = t}$

$\therefore x = t$

$y = -t$

$z = t$

2(ii) Argumented matrix $[A|B] = \begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 2 & -1 & 1 & 1 & 0 \\ 3 & 2 & -1 & -1 & 1 \\ 1 & 1 & 3 & -3 & -8 \end{bmatrix}$

$R_2 \Rightarrow R_2 - 2R_1$
$R_3 \Rightarrow R_3 - 3R_1$
$R_4 \Rightarrow R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 0 & -3 & -1 & 3 & 4 \\ 0 & -1 & -4 & 2 & 7 \\ 0 & 0 & 2 & -2 & -6 \end{bmatrix} \quad R_2 \Leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 0 & 0 & 2 & -2 & -6 \\ 0 & -1 & -4 & 2 & 7 \\ 0 & -3 & -1 & 3 & 4 \end{bmatrix} \quad \begin{matrix} R_2 \Rightarrow R_2/2 \\ R_4 \rightarrow R_4 - 3R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & -1 & -4 & 2 & 7 \\ 0 & 0 & 11 & -3 & -17 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 0 & -1 & -4 & 2 & 7 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 11 & -3 & -17 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 \\ R_4 \to R_4 - 11R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 4 & -2 & -7 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 8 & 16 \end{bmatrix} \quad R_4 \to R_4/8$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 4 & -2 & -7 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad R_1 \Rightarrow R_1 - R_2$$

$$\cancel{1 \quad 1 \quad 1 \quad -1 \quad -2}$$

$$\sim \begin{bmatrix} 1 & 0 & -3 & 1 & 5 \\ 0 & 1 & 4 & -2 & -7 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 + 3R_3 \\ R_2 \to R_2 - 4R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2 & -4 \\ 0 & 1 & 0 & 2 & +5 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 + 2R_4 \\ R_2 \to R_2 - 2R_4 \\ R_3 \to R_3 + R_4 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$\therefore \quad x_1 = 0$

$x_2 = 1$

$x_3 = -1$

$x_4 = 2$

2 i) Augmented matrix $[A \,;\, B] = \begin{bmatrix} 2 & -3 & 0 & 8 \\ 4 & -5 & 1 & 15 \\ 2 & 0 & 4 & 1 \end{bmatrix} R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 2 & 0 & 4 & 1 \\ 4 & -5 & 1 & 15 \\ 2 & -3 & 0 & 8 \end{bmatrix} R_1 \rightarrow R_1/2$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1/2 \\ 4 & -5 & 1 & 15 \\ 2 & -3 & 0 & 8 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1/2 \\ 0 & -5 & -7 & 13 \\ 0 & -3 & -4 & 7 \end{bmatrix} \quad R_2 \to -R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1/2 \\ 0 & 5 & 7 & -13 \\ 0 & -3 & -4 & 7 \end{bmatrix} \quad R_3 \to 5R_3 + 3R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1/2 \\ 0 & 5 & 7 & -13 \\ 0 & 0 & 1 & -4 \end{bmatrix} \quad R_2 \to R_2/5$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & \frac{1}{2} \\ 0 & 1 & \frac{7}{5} & -\frac{13}{5} \\ 0 & 0 & 1 & -4 \end{bmatrix} R_1 \to R_1 - 2R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & \frac{17}{2} \\ 0 & 1 & \frac{7}{5} & -\frac{13}{5} \\ 0 & 0 & 1 & -4 \end{bmatrix} R_2 \to R_2 - \frac{7}{5}R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & \frac{17}{2} \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

$\therefore \; x = \frac{17}{2}$

$y = 3$

$z = -4$

Inverse matrix :-

An $n \times n$ square matrix A is said to be invertible or non-singular if there exists a square matrix B of the same size such that

$$AB = I_n = BA$$

Such that a matrix B is called the inverse of A, and is denoted by A^{-1}

Note:- i) A matrix 'A' is said to be singular if it is not invertible.

ii) Let A be an invertible matrix and K be any non-zero scalar, then (a) A^{-1} is invertible and $(A^{-1})^{-1} = A$

(b) the matrix KA is invertible and $(KA)^{-1} = \frac{1}{K} A^{-1}$

(c) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

(d) $AA^{-1} = A^{-1}A = I$

iii) Any matrix with a zero row or zero column cannot be invertible.

iv) The product of invertible matrices is also invertible whose inverse is the product of the individual inverses in reversed order. i.e $(AB)^{-1} = B^{-1}A^{-1}$

*** Finding the inverse of a matrix by using elementary row operations (Gauss-Jordan elimination):-

Let A be an $n \times n$ matrix, then

i) Write the matrix $[A \mid I_n]$

ii) Compute the reduced echelon form of $[A \vdots I_n]$

iii) If the reduced echelon form is of the type $[I_n \vdots B]$, then B is the inverse of A.

*** **Ex:** Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 0 & 2 \end{bmatrix}$ by using Gauss - Jordan elimination.

Consider $[A \vdots I_{3\times3}] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 5 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$

$R_2 \to R_2 - 2R_1$

$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right]$ $R_3 \to R_3 - R_1$

$R_2 \to R_2$

$$\sim\left[\begin{array}{ccc|ccc}1 & 2 & 3 & 1 & 0 & 0\\ 0 & 1 & 1 & 2 & -1 & 0\\ 0 & -2 & -1 & -1 & 0 & 1\end{array}\right] R_3 \to R_3 + 2R_2$$

$$\sim\left[\begin{array}{ccc|ccc}1 & 2 & 3 & 1 & 0 & 0\\ 0 & 1 & 1 & 2 & -1 & 0\\ 0 & 0 & 1 & 3 & -2 & 1\end{array}\right] R_1 \to R_1 - 2R_2$$

$$\sim\left[\begin{array}{ccc|ccc}1 & 0 & 1 & -3 & 2 & 0\\ 0 & 1 & 1 & 2 & -1 & 0\\ 0 & 0 & 1 & 3 & -2 & 1\end{array}\right] \begin{array}{l} R_1 \to R_1 - R_3\\ R_2 \to R_2 - R_3\end{array}$$

$$\sim\left[\begin{array}{ccc|ccc}1 & 0 & 0 & -6 & 4 & -1\\ 0 & 1 & 0 & -1 & 1 & -1\\ 0 & 0 & 1 & 3 & -2 & 1\end{array}\right]$$

$$\sim \left[I_n \,\vdots\, B \right]$$

$$\therefore \quad B = \begin{bmatrix} -6 & 4 & -1 \\ -1 & 1 & -1 \\ 3 & -2 & 1 \end{bmatrix}$$

i.e $A^{-1} = B = \begin{bmatrix} -6 & 4 & -1 \\ -1 & 1 & -1 \\ 3 & -2 & 1 \end{bmatrix}$

---

1. Find the inverse of the following matrices

i) $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 2 & -1 & 4 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$

iv) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 2 & -1 & 4 \end{bmatrix}$

Consider $[A \mid I_{3\times 3}] = \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 1 & 2 & 7 & 0 & 1 & 0 \\ 2 & -1 & 4 & 0 & 0 & 1 \end{array}\right]$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & -3 & -6 & -2 & 0 & 1 \end{array}\right] \quad R_3 \to R_3 + 3R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -5 & 3 & 1 \end{array}\right]$$

Since one row is zero so inverse of A does not exist.

i) consider $[A \mid I_{3\times3}] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -3 & -5 & 0 & 1 & 0 \\ -1 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{array}$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -9 & -2 & 1 & 0 \\ 0 & 2 & 7 & 1 & 0 & 1 \end{array}\right] R_3 \to R_3 + 2R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -9 & -2 & 1 & 0 \\ 0 & 0 & -11 & -3 & 2 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2/-1 \\ R_3 \to R_3/-11 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 9 & 2 & -1 & 0 \\ 0 & 0 & 1 & 3/11 & -2/11 & -1/11 \end{array}\right] R_1 \to R_1 + R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 11 & 3 & -1 & 0 \\ 0 & 1 & 9 & 2 & -1 & 0 \\ 0 & 0 & 1 & 3/11 & -2/11 & -1/11 \end{array}\right] \begin{array}{l} R_1 \to R_1 - 11R_3 \\ R_2 \to R_2 - 9R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -5/11 & 7/11 & 9/11 \\ 0 & 0 & 1 & 3/11 & -2/11 & -1/11 \end{array}\right]$$

$$\sim [I_n : B]$$

$$B = \begin{bmatrix} 0 & 1 & 1 \\ -5/11 & 7/11 & 9/11 \\ 3/11 & -2/11 & -1/11 \end{bmatrix}$$

i.e $A^{-1} = B = \begin{bmatrix} 0 & 1 & 1 \\ -5/11 & 7/11 & 9/11 \\ 3/11 & -2/11 & -1/11 \end{bmatrix}$

iii) Consider $[A \,\vdots\, I_{3\times 3}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right]$ $\begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{array}$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right] \quad R_3 \to R_3 + R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2/-1 \\ R_3 \to R_3/-1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - 2R_3 \\ R_2 \to R_2 - R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & -1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right]$$

$$\sim [I_n \,\vdots\, B]$$

$$B = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & -1 \\ 6 & -1 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = B = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & -1 \\ 6 & -1 & -1 \end{bmatrix}$$

iv) Consider $[A \,|\, I_{3\times3}] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right]$ $R_3 \to R_3 - 5R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2/2 \\ R_3 \to R_3/-4 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array}\right] R_1 \to R_1 - R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1/2 & 1 & -1/2 & 0 \\ 0 & 1 & 3/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array}\right] \begin{array}{l} R_1 \to R_1 + \frac{1}{2}R_3 \\ R_2 \to R_2 - \frac{3}{2}R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 13/8 & -1/2 & -1/8 \\ 0 & 1 & 0 & -15/8 & 1/2 & 3/8 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array}\right]$$

$$\sim [I_n \,|\, B]$$

$$B = \begin{bmatrix} 13/8 & -1/2 & -1/8 \\ -15/8 & 1/2 & 3/8 \\ 5/4 & 0 & -1/4 \end{bmatrix}$$

$$\therefore A^{-1} = B = \begin{bmatrix} 13/8 & -1/2 & -1/8 \\ -15/8 & 1/2 & 3/8 \\ 5/4 & 0 & -1/4 \end{bmatrix}$$

Block Matrix :-

A sub matrix "A" is a matrix obtained from A by deleting certain rows and/or columns of A.

Consider a matrix $A = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]$

divided up into four blocks (sub matrices) by the dotted lines shown.

Now, if we write $A_{11} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$, $A_{12} \begin{bmatrix} a_{14} \\ a_{24} \end{bmatrix}$

$A_{21} = \begin{bmatrix} a_{31} & a_{32} & a_{33} \end{bmatrix}$, $A_{22} = \begin{bmatrix} a_{34} \end{bmatrix}$ then

A can be written as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ called a block matrix.

Product of block matrices:-

If $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ are block matrices and the number of columns in A_{ik} is equal to the number of rows in B_{kj}, then

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

Example: Compute AB using block multiplication where

$$A = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -3 & 4 & 0 & 1 \\ \hline 0 & 0 & 2 & -1 \end{array}\right] \qquad B = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ \hline 2 & 3 & 4 \\ 3 & -2 & 1 \end{array}\right].$$

consider $A_{11} = \begin{bmatrix}1 & 2\\ -3 & 4\end{bmatrix}$ $A_{12} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ $A_{21} = \begin{bmatrix}0 & 0\end{bmatrix}$ $A_{22} = \begin{bmatrix}2 & -1\end{bmatrix}$

$B_1 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ $B_{12} = \begin{bmatrix}2\\ 3\end{bmatrix}$ $B_{21} = \begin{bmatrix}2 & 3\\ 3 & -2\end{bmatrix}$ $B_{22} = \begin{bmatrix}4\\ 1\end{bmatrix}$

$$A_{11}B_{11} = \begin{bmatrix}1 & 2\\ -3 & 4\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$A_{11}B_{11} = \begin{bmatrix}1 & 2\\ -3 & 4\end{bmatrix}$$

$$A_{12}B_{21} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}2 & 3\\ 3 & -2\end{bmatrix}$$

$$A_{12}B_{21} = \begin{bmatrix}2 & 3\\ 3 & -2\end{bmatrix}$$

$$A_{11}B_{12} = \begin{bmatrix}1 & 2\\ -3 & 4\end{bmatrix}\begin{bmatrix}2\\ 3\end{bmatrix}$$

$$A_{11}B_{12} = \begin{bmatrix}8\\ 6\end{bmatrix}$$

$$A_{12}B_{22} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}4\\ 1\end{bmatrix}$$

$$= \begin{bmatrix}4\\ 1\end{bmatrix}$$

$$A_{21}B_{11} = \begin{bmatrix}0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\end{bmatrix}$$

$$A_{22}B_{21} = \begin{bmatrix}2 & -1\end{bmatrix}\begin{bmatrix}2 & 3\\ 3 & -2\end{bmatrix}$$

$$A_{22}B_{21} = \begin{bmatrix}1 & 8\end{bmatrix}$$

$$A_{21}B_{12} = \begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \end{bmatrix}$$

$$A_{22}B_{22} = \begin{bmatrix} 2 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 & 5 & 12 \\ 0 & 2 & 7 \\ 1 & 8 & 7 \end{bmatrix}$$

<u>Elementary matrix</u> :-

An <u>elementary matrix</u> is a matrix, which is obtained from the identity matrix I_n by executing only one elementary row operation.

Example :-

$$\begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 5R_2$ $\quad$ $R_2 \leftrightarrow R_4$ $\quad$ $R_1 \to R_1 + 3R_3$

Properties :-

i) If E denotes an elementary matrix and $E'(E^{-1})$ denotes the elementary matrix corresponding to the inverse elementary row operation on E, then,

$$EE' = I$$

ii) If E multiplies a row by $c \neq 0$, then E' multiplies the same row by $\frac{1}{c}$

iii) If E interchanges two rows, then E'

interchanges them again

iv) If E adds a multiple of one row to another, then $E^{'}$ subtracts it from the same row.

v) Every elementary matrix is invertible and inverse matrix $E^{'} = E^{-1}$ is also an elementary matrix.

Example :-

1. If $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{c} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. If $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ then $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

3. If $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then $E^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Express the following matrices as a product of elementary matrices

i) $A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$

✓ ii) $A = \begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

iv) $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 8 \\ -3 & -1 & 2 \end{bmatrix}$

i) $A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$ $R_2 \to R_2 + 2R_1$

$\sim \begin{bmatrix} 1 & -3 \\ 0 & -2 \end{bmatrix}$ $R_2 \to R_2/-2$

$\sim \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$ $R_1 \to R_1 + 3R_2$

$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Since $R_2 \to R_2 + 2R_1$

$E_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

Since $R_2 \to R_2 - 2R_1$

$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

Since $R_2 \to R_2/-2$

$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$

Since $R_2 \to -2R_2$

$E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

Since $R_1 \to R_1 + 3R_2$

$$E_3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

Since $R_1 \to R_1 - 3R_2$

$$E_3^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \; A = E_1 \cdot E_2 \cdot E_3$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 2 & 11/2 \end{bmatrix}$$

$$A = E_1^{-1} \cdot E_2^{-1} \cdot E_3^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \checkmark$$

iii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ $R_1 \to R_1 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 + 5R_3 \\ R_2 \to R_2 - 4R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $R_1 \to R_1 + 2R_2$ (inverse)

$$E_1^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $R_1 \to R_1 - 5R_3$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $R_2 \to R_2 + 4R_3$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = E_1^{-1} \cdot E_2^{-1} \cdot E_3^{-1}$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

iv) $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 8 \\ -3 & -1 & 2 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$
$R_3 \to R_3 + 3R_1$

$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 2 & 8 \end{bmatrix}$ $R_3 \to R_3 - 2R_2$

$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

The matrix A cannot be expressed as the product of elementary matrices since the third row is zero

Permutations :-

A permutation matrix is a square matrix obtained from the identity matrix if permuting <changing the order> the rows

Example

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ is a permutation matrix but

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$ is not a permutation matrix.

Properties:-

i) Every permutation matrix is a elementary matrix but every elementary matrix need not be a permutation matrix.

ii) The product of any two permutation matrices is again a permutation matrix.

iii) The transpose of a permutation matrix is also a permutation matrix.

iv) Every permutation matrix P is invertible and $P^{-1} = P^{T}$

v) A permutation matrix is the product of a finite number of elementary matrices each of which corresponds to the row interchanging elementary row operation.

ii) $A = \begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix}$ $R_2 \to R_2 - 5R_1$

$\sim \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $R_2 \to R_2/2$

$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Since $R_2 \to R_2 + 5R_1$

$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

Since $R_2 \to 2R_2$

$E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$A = E_1^{-1} \cdot E_2^{-1}$

$= \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix}$

LU Factorization:-

Let A be a square matrix that can be factorized into the form $A = LU$, where L is a lower triangular matrix & U is an upper triangular matrix. This factorization is called an LU factorization or LU decomposition of A

Note: i) Every matrix has an LU factorization and when it exists, It is not unique.

ii) If the matrix A is invertible & if the permutation matrix P is fixed then the matrix PA has a unique LDU factorization.

system of linear equation

PA has a unique ...

Solving method for a given system of linear equations by LU factorization :-

Let $AX=B$ be a system of "n" linear equations in "n" unknowns then

i) Find the LU decomposition of A

ii) Solve $LY=B$ by forward substitution

iii) Solve $UX=Y$ by back substitution

Example: 1. Solve the following system of equations using LU decomposition

$$2x_1+x_2+3x_3=-1$$
$$4x_1+x_2+7x_3=5$$
$$-6x_1-2x_2-12x_3=-2$$

The given system of linear equations can be expressed as $AX = B$, where

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 7 \\ -6 & -2 & -12 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 7 \\ -6 & -2 & -12 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 3R_1 \end{matrix}$$

$$\sim \cancel{\begin{matrix} 1 & 1/2 & 3/2 \\ 4 & 1 & 7 \end{matrix}}$$

$$\sim \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 1 & -3 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} = U$$

The inverse elementary matrices that corresponds to the row operations.

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 \to R_2 + 2R_1$$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \quad R_3 \to R_3 - 3R_1$$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$L = E_1^{-1} \cdot E_2^{-1} E_3^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix}$$

$A = LU$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 7 \\ -6 & -2 & -12 \end{bmatrix}$$

Consider $LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix}$$

$\Rightarrow \boxed{y_1 = -1}$ —— (1)

$2y_1 + y_2 = 5$ —— (2)

$-3y_1 - y_2 + y_3 = -2$ —— (3)

sub $y_1 = -1$ in equ (2)

$-2 + y_2 = 5$

$\boxed{y_2 = 7}$

Sub $y_1=-1,\ y_2=7$ in equ (3)

$$3-7+y_3=-2$$

$$-4+y_3=-2$$

$$\boxed{y_3=2}$$

$y_1=-1,\ y_2=7,\ y_3=2$

Since $UX=Y$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix}$$

$$2x_1+x_2+3x_3=-1 \quad \text{--- (1)}$$

$$-x_2+x_3=7 \quad \text{--- (2)}$$

$$-2x_3=2 \quad \text{--- (3)}$$

$$(3) \Rightarrow \boxed{x_3=-1}$$

Sub $x_3=-1$ in equ (2)

$$-x_2-1=7$$

$$\boxed{x_2=-8}$$

Sub $x_2=-8,\ x_3=-1$ in equ (1)

$$2x_1-8-3=-1$$

$$2x_1-11=-1$$

$$2x_1=10$$

$$\boxed{x_1=5}$$

$x_1=5,\ x_2=-8,\ x_3=-1$

Find the LDU factorization for each of the following matrices

i) $A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

i) $A = \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} \quad R_2 \to R_2 - 4R_1$

$\sim \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = U$

Since $R_2 \to R_2 - 4R_1$

$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

$L = E^{-1} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

$A = LU$

$= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$

$A = LDU$

ii) $A = \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix}$ $R_2 \to R_2 - 8R_1$

$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = U$

Since $R_2 \to R_2 - 8R_1$

$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix}$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$R_2 \to R_2 + 8R_1$

$L = E^{-1} = \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix}$

$A = LU$

$= \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A = LDU$

$\hat{A}$ PA = (

If $A = \begin{bmatrix} \boxed{0} & 1 & 2 \\ \boxed{0} & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is any permutation matrix then express $\boxed{PA}$ as LDU factorization

$$\boxed{PA} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = U$$

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$R_1 \leftrightarrow R_3$

$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Since $R_3 \to R_3 - R_2$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$L = E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

- $PA = LU$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A) LDU

For all possible permutation matrices P, find the LDU factorization of PA for $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$ $\qquad$ $R_1 \leftrightarrow R_3$ $\qquad$ $R_2 \leftrightarrow R_3$

$$P_1 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P_1 A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 2 & 4 & 2\end{bmatrix}\begin{matrix}R_2 \Rightarrow R_2 - R_1\\ R_3 \Rightarrow R_3 - 2R_1\end{matrix}$$

$$\sim\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 2\\ 0 & 2 & 0\end{bmatrix} R_3 \Rightarrow R_3 - 2R_2$$

$$\sim\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 2\\ 0 & 0 & -4\end{bmatrix} = U$$

$R_1 \Leftrightarrow R_3$

$$E_1^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - R_1$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - 2R_1$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - 2R_2$

$$E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$L = E_1^{-1} \cdot E_2^{-1} \cdot E_3^{-1} \cdot E_4^{-1}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$P_1A = LU$

$$= \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$P_1A = LDU$

$$P_2A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{matrix} R_2 \Rightarrow R_2 - 2R_1 \\ R_3 \Rightarrow R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} R_3 \Leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix} R_3 \Rightarrow R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{bmatrix} = U$$

$R_2 \Rightarrow R_2 - 2R_1$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - R_1$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$R_3 \Leftrightarrow R_2$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - 2R_2$

$$E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$L = E_1^{-1} \cdot E_2^{-1} \cdot E_3^{-1} \cdot E_4^{-1}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$P_2 A = L U$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$P_2 A = L D U$

$$P_3A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{bmatrix} \begin{matrix} R_2 \Rightarrow R_2 - R_1 \\ R_3 \Rightarrow R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -4 \end{bmatrix} = U$$

$R_2 \Rightarrow R_2 - R_1$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - 2R_1$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$L = E_1^{-1} \cdot E_2^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$P_3A = LU$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & +1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_3A = LDU$$

Gauss elimination

4. ii) $x-y+z=1,\ x+3y+az=2,\ 2x+ay+3z=3$

$$[A\,;B]=\begin{bmatrix}1 & -1 & 1 & 1\\ 1 & 3 & a & 2\\ 2 & a & 3 & 3\end{bmatrix} R_3 \to R_3-2R_1$$

$$\sim\begin{bmatrix}1 & -1 & 1 & 1\\ 0 & 4 & a-1 & 1\\ 0 & a+2 & 1 & 1\end{bmatrix} R_3 \to -4R_3+(a+2)R_2$$

$$\sim\begin{bmatrix}1 & -1 & 1 & 1\\ 0 & 4 & a-1 & 1\\ 0 & 0 & -4+(a+2)(a-1) & a-2\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & -1 & 1 & 1\\ 0 & 4 & a-1 & 1\\ 0 & 0 & (a+3)(a-2) & a-2\end{bmatrix}$$

i) If $a\neq -3,\ a\neq 2$

$\rho(A)=\rho(A,B)=$ no of unknowns

$\therefore$ It has unique solution

ii) If $a\neq -3,\ a=2$

$\rho(A)=\rho(A,B)\neq$ no of unknowns

$\therefore$ It has infinite no of solutions

iii) If $a=-3,\ a\neq 2$

$\rho(A)\neq\rho(A,B)$

$\therefore$ It has no solution

UNIT-2 VECTOR SPACE

of

will

Ring $(R, +, \cdot)$

i) $(R, +)$ abelian group

ii) $(R, \cdot)$ semi group

iii) Distributive properties

$a(b+c) = a\cdot b + a\cdot c$ (LDL)

$(b+c)\cdot a = b\cdot a + c\cdot a$ (RDL)

Field $(F, +, \cdot)$

i) $(F, +)$ abelian group

ii) $(F, \cdot)$ semi group

iii) Distributive properties

iv) Identity axiom w.r.t multiplication

v) Inverse axiom w.r.t multiplication for non zero elements

vi) Abelian group w.r.t multiplication

Vector space :-

Let V be a non-empty set of vectors and F be a field, then $(V(F), +, \cdot)$ is said to be a vector space if it satisfies the following axioms.

i) $(V, +)$ is an abelian group

i.e. i) $\bar{a} + \bar{b} \in V \; \forall \bar{a}, b \in V$ [closure axiom]

ii) $(\bar{a}+\bar{b})+\bar{c} = \bar{a} + (\bar{b}+\bar{c}), \; \forall \bar{a}, \bar{b}, \bar{c} \in V$ [Associative axiom]

iii) $\forall \bar{a} \in V, \exists \bar{e} = \bar{0} \in V$ such that [Identity axiom]

$\bar{a} + \bar{e} = \bar{e} + \bar{a} = \bar{a}$

where $\bar{e} = \bar{0}$ is called the identity element

iv) $\forall \bar{a} \in V, \exists \bar{b} \in V$ such that [Inverse axiom]

$\bar{a} + \bar{b} = \bar{b} + \bar{a} = \bar{e}$

where $\bar{b}$ is called the inverse element of $\bar{a}$

v) $\bar{a} + \bar{b} = \bar{b} + \bar{a}, \; \forall \bar{a}, \bar{b} \in V$ [Abelian (commutative) axiom]

19

2, 19, 45

2) If $k_1, k_2 \in F$ and $\bar{a}, \bar{b} \in V$, then

$$k_1 \bar{a} \in V$$
$$k_1(\bar{a}+\bar{b}) = k_1\bar{a} + k_1\bar{b}$$
$$(k_1+k_2)\bar{a} = k_1\bar{a} + k_2\bar{a}$$
$$k_1(k_2 \cdot \bar{a}) = (k_1 k_2)\bar{a}$$
$$1 \cdot \bar{a} \in V$$

NOTE:-

If the field F is a set of real numbers, then the vector space is called real vector space and simply it is denoted by V(R).

If the field F is a set of complex numbers, then the vector space is called complex vector space and simply it is denoted by V(C).

· Examples:-

1. Let V be the set of all pairs (x, y) of real number defined $(x,y) + (x_1, y_1) = (x+x_1, y+y_1)$

$$k.(x, y) = (kx, ky)$$

Is the set V a vector space under these operations. Justify your answer

*** 2. Let V be the set of all pairs (x, y) of real numbers. Suppose that an addition and scalar multiplication of pairs are defined by

$$(x, y) + (u, v) = (x+2u, y+2v), \; k(x, y) = (kx, ky)$$

Is the set V a vector space under these operations? Justify your answer

3. Let $C(R)$ denote the set of real valued functions defined on real line R. For two functions f and g and a real number K, the sum $f+g$ and the scalar multiplication Kf of them are defined by

$$(f+g)(x) = f(x)+g(x)$$

$$(Kf)(x) = K f(x)$$

Then prove that $C(R)$ is a vector space under these operations.

2. $V = \{(x,y) : x, y \in R\}$

and $(x,y)+(u,v) = (x+2u, y+2v)$, $K(x,y) = (Kx, Ky)$

i) Closure axiom:-

Let $\bar{a} = (x,y)$, $\bar{b} = (u,v)$

then $\bar{a}+\bar{b} = (x,y)+(u+v)$

$= (x+2v, y+2v) \in V$

i.e., $\bar{a}+\bar{b} \in V$, $\forall \bar{a}, \bar{b} \in V$

$\therefore$ V satisfies closure axiom.

ii) Associative axiom:-

Let $\bar{a} = (x_1, y_1)$, $\bar{b} = (x_2, y_2)$, $\bar{c} = (x_3, y_3)$:

$x_1, x_2, x_3, y_1, y_2, y_3 \in R$

Then $(\bar{a}+\bar{b})+\bar{c} = \{(x_1, y_1)+(x_2, y_2)\}+(x_3, y_3)$

$= (x_1+2x_2, y_1+2y_2)+(x_3, y_3)$

$= ((x_1+2x_2)+2x_3, (y_1+2y_2)+2y_3)$

$= (x_1+2x_2+2x_3, y_1+2y_2+2y_3)$

$$\bar{a}+(\bar{b}+\bar{c}) = (x_1,y_1)+\{(x_2,y_2)+(x_3,y_3)\}$$

$$= (x_1,y_1)+(x_2+2x_3, y_2+2y_3)$$

$$= (x_1+2(x_2+2x_3), y_1+2(y_2+2y_3))$$

$$= (x_1+2x_2+4x_3, y_1+2y_2+4y_3)$$

$\therefore (\bar{a}+\bar{b})+\bar{c} \neq \bar{a}+(\bar{b}+\bar{c})$

$\therefore$ V is not a vector space, since it is not satisfies associative axiom

1. iii) <u>Identity axiom</u> :-

Let $\bar{a} \in V$

Then $\bar{a} = (x_1,y_1)$; $x_1,y_1 \in R$

Since $0 \in R$

$\therefore (0,0) \in V$

i.e. $\bar{e} = (0,0) \in V$

Now $\bar{a}+\bar{e} = (x_1,y_1)+(0,0)$

$= (x_1+0, y_1+0)$

$= (x_1, y_1)$

$= \bar{a}$

Similarly $\bar{e}+\bar{a} = (0,0)+(x_1,y_1)$

$= (0+x_1, 0+y_1)$

$= (x_1,y_1)$

$= \bar{a}$

i.e, $\forall \bar{a} \in V$, $\exists \bar{e} \in V$ such that

$\bar{a}+\bar{e} = \bar{e}+\bar{a} = \bar{a}$

$\therefore$ V satisfies identity axiom

iv) ... not

iv) Inverse axiom:-

Let $\bar{a} \in V$

Then $\bar{a} = (x_1, y_1)$; $x_1, y_1 \in R$

Since $x_1, y_1 \in R$

$\therefore -x_1, -y_1 \in R$

$\Rightarrow (-x_1, -y_1) \in V$

i.e., $\bar{b} = (-x_1, -y_1) \in V$

Now $\bar{a} + \bar{b} = (x_1, y_1) + (-x_1, -y_1)$

$= (x_1 - x_1, y_1 - y_1)$

$= (0, 0) = \bar{e}$

Also $\bar{b} + \bar{a} = (-x_1, -y_1) + (x_1, y_1)$

$= (-x_1 + x_1, -y_1 + y_1)$

$= (0, 0) = \bar{e}$

i.e., $\forall \bar{a} \in V, \exists \bar{b} \in V$ such that

$\bar{a} + \bar{b} = \bar{b} + \bar{a} = \bar{e}$

$\therefore$ V satisfies inverse axiom

v) Abelian (commutative) axiom:-

Let $\bar{a}, \bar{b} \in V$

Then $\bar{a} = (x_1, y_1)$, $\bar{b} = (x_2, y_2)$; $x_1, x_2, y_1, y_2 \in R$

Now $\bar{a} + \bar{b} = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2)$

Now $\bar{b} + \bar{a} = (x_2, y_2) + (x_1, y_1)$

$= (x_2 + x_1, y_2 + y_1)$

$= (x_1 + x_2, y_1 + y_2)$

i.e., $\bar{a}+\bar{b}=\bar{b}+\bar{a}$, $\forall \bar{a},\bar{b}\in V$

$\therefore$ V satisfies abelian axiom and hence $(V,+)$ is an abelian group.

Let K_1, K_2 be any two scalars and $\bar{a},\bar{b}\in V$

Then $\bar{a}=(x_1,y_1)$, $\bar{b}=(x_2,y_2)$; $x_1,x_2,y_1,y_2\in R$

vi) $K_1\bar{a}=K_1(x_1,y_1)$
$=(K_1x_1,K_1y_1)\in V$

vii) $(K_1+K_2)\bar{a}=(K_1+K_2)(x_1,y_1)$
$=((K_1+K_2)x_1,(K_1+K_2)y_1)$
$=(K_1x_1+K_2x_1,K_1y_1+K_2y_1)$
$=(K_1x_1,K_1y_1)+(K_2x_1,K_2y_1)$
$=K_1(x_1,y_1)+K_2(x_1,y_1)$
$=K_1\bar{a}+K_2\bar{a}$

viii) $K_1\cdot(K_2\bar{a})=K_1\cdot(K_2(x_1,y_1))$
$=K_1\cdot(K_2x_1,K_2y_1)$
$=(K_1(K_2x_1),K_1(K_2y_1))$
$=((K_1K_2)x_1,(K_1K_2)y_1)$
$=(K_1K_2)(x_1,y_1)$
$=(K_1K_2)\bar{a}$

ix) $K_1(\bar{a}+\bar{b})=K_1\cdot((x_1,y_1)+(x_2,y_2))$
$=K_1\cdot(x_1+x_2,y_1+y_2)$
$=(K_1(x_1+x_2),K_1(y_1+y_2))$

$= (K_1x_1 + K_1x_2, K_1y_1 + K_1y_2)$

$= (K_1x_1, K_1y_1) + (K_1x_2, K_1y_2)$

$= K_1(x_1, y_1) + K_1(x_2, y_2)$

$= K_1\bar{a} + K_1\bar{b}$

x) $1\cdot\bar{a} = 1\cdot(x_1, y_1)$

$= (1\cdot x_1, 1\cdot y_1)$

$= (x_1, y_1) \in V$

$\therefore$ V satisfies all the properties of vector space and hence it is a vector space w.r.t addition and multiplication.

i) closure axiom:-

Let $\bar{a} = (x, y)$, $\bar{b} = (x_1, y_1)$

then $\bar{a}+\bar{b} = (x, y) + (x_1, y_1)$

$= (x+x_1, y+y_1) \in V$

i.e, $\bar{a}+\bar{b} \in V$, $\forall \bar{a}, \bar{b} \in V$

$\therefore$ V satisfies closure axiom

ii) Associative axiom:-

Let $\bar{a} = (x_1, y_1)$, $\bar{b} = (x_2, y_2)$, $\bar{c} = (x_3, y_3)$: $x_1, x_2, x_3, y_1, y_2, y_3 \in R$

Then $(\bar{a}+\bar{b})+\bar{c} = \{(x_1, y_1) + (x_2, y_2)\} + (x_3, y_3)$

$= (x_1+x_2, y_1+y_2) + (x_3, y_3)$

$= ((x_1+x_2)+x_3, (y_1+y_2)+y_3)$

$= (x_1 + x_2 + x_3 , y_1 + y_2 + y_3)$

$\bar{a} + (\bar{b} + \bar{c}) = (x_1, y_1) + \{(x_2, y_2) + (x_3, y_3)\}$

$= (x_1, y_1) + (x_2 + x_3 , y_2 + y_3)$

$= (x_1 + (x_2 + x_3), y_1 + (y_2 + y_3))$

$= (x_1 + x_2 + x_3 , y_1 + y_2 + y_3)$

$\therefore (\bar{a} + \bar{b}) + \bar{c} = \bar{a} + (\bar{b} + \bar{c})$

$\therefore$ V satisfies associative axiom.

<u>Subspace</u> :-

Let $(V, +, \cdot)$ be a vector space and W be a non-empty set then W is called a subspace of V if W itself is a vector space and subset of V under the addition and the scalar multiplication defined in V.

<u>Note</u> :-

i) A vector space V itself and the zero vector $\{\bar{0}\}$ are trivially subspaces

ii) A non empty subset W of a vector space V is a subspace if and only if $\bar{x} + \bar{y}$ and $k\bar{x}$ are contained in W (or equivalently $\bar{x} + k\bar{y} \in W$) for any vectors $\bar{x}$ and $\bar{y}$ in W and any scalar $k \in R$

Which of the following sets are ... of

3-space (R^3)? Justify your answer.

i) $W = \{(x,y,z) \in R^3 : ax+by+cz=0\}$

ii) $W = \{(x,y,z) \in R^3 : xyz=0\}$

iii) $W = \{(2t, 3t, 4t) \in R^3 : t \in R\}$

iv) $W = \{(x,y,z) \in R^3 : x^2+y^2-z^2=0\}$

v) $W = \{x \in R^3 : x^T\bar{u} = \bar{0} = x^T\bar{v}\}$ where $\bar{u}$ and $\bar{v}$ are any two fixed non zero vectors in R^3

V = 3-space (R^3)

$= \{(x,y,z) : x,y,z \in R\}$

iv) clearly $W \neq \phi$ and $W \subseteq V$

Let $\alpha, \beta \in W$ and K be any scalar

Then $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2)$

where $x_1^2+y_1^2-z_1^2=0$ and $x_2^2+y_2^2-z_2^2=0$

$\alpha + K\beta = (x_1,y_1,z_1) + K(x_2,y_2,z_2)$

$= (x_1,y_1,z_1) + (Kx_2, Ky_2, Kz_2)$

$= (x_1+Kx_2, y_1+Ky_2, z_1+Kz_2) \in R^3$

now $(x_1+Kx_2)^2 + (y_1+Ky_2)^2 - (z_1+Kz_2)^2 =$

$= x_1^2 + K^2x_2^2 + 2Kx_1x_2 + y_1^2 + K^2y_2^2 + 2Ky_1y_2 - z_1^2 - K^2z_2^2 - 2Kz_1z_2$

$= (x_1^2+y_1^2-z^2) + K^2(x_2^2+y_2^2-z_2^2) + 2K(x_1x_2+y_1y_2 - z_1z_2)$

$= 0 + 2K(x_1x_2+y_1y_2-z_1z_2)$

$= 2K(x_1x_2 + y_1y_2 - z_1z_2)$

$\neq 0$

$\therefore \alpha + K\beta \notin W$ and hence W is not a subspace.

1. Let $V = C(R)$ be the vector space of all conditional function on R. which of the following sets W are subspaces? Justify your answer

i) W is the set of all differentiable functions on R

ii) W is the set of all bounded continuous functions on R

iii) W is the set of all continuous non negative-valued functions on R i.e. $f(x) \geq 0$ for any $x \in R$

iv) W is the set of all continuous odd functions on R. i.e. $f(-x) = -f(x)$ for any $x \in R$. subspace

v) W is the set of all polynomials with integer coefficients

iv) Let $V = C(R)$ be the vector space of all continuous function on R
→ continuous function

$= \{ f(x) : f(x) \text{ is continuous on } R \}$

and W be the set of all continuous and odd function on R.

i.e., $W = \{ f(x) : f(x) \text{ is continuous and } f(x) \text{ is odd function} \}$

Clearly $W \neq \phi$, ~~$0 \in$~~ $W \subseteq V$

Let $\alpha, \beta \in W$ and K be scalar

Then $\alpha = f(x)$, $f(x)$ is continuous, $f(-x) = -f(x)$

$\beta = g(x)$, $g(x)$ is continuous, $g(-x) = -g(x)$

Now $\alpha + K\beta = f(x) + Kg(x)$

Here $f(x) + Kg(x)$ is always continuous i.e, $\alpha + K\beta$ is continuous

[$\because$ Sum of two continuous functions is always continuous and f be continuous and c be any scalar, the cf is also continuous]

Now $(\alpha + K\beta)(-x) = (f + Kg)(-x)$

$= f(-x) + (Kg)(-x)$

$= -f(x) + Kg(-x)$

$= -f(x) - Kg(x)$

$= -(f(x) + Kg(x))$

$= -(f + Kg)(x)$

$= -(\alpha + K\beta)(x)$

$\therefore$ $\alpha + K\beta$ is an odd function

$\therefore$ $\alpha + K\beta \in W$ and hence W is a subspace of V

$\oplus \rightarrow$ Direct sum

Def:- Let U and W be the subspaces of a vector space V, then

i) the sum of U and W is defined by

$$U + W = \{\bar{u} + \bar{w} : \bar{u} \in U, \bar{w} \in W\}$$

ii) A vector space V is called the direct sum of two subspaces U and W, written as

$$V = U \oplus W \text{ if } V = U + W \text{ and } U \cap W = \{\bar{0}\}$$

Theorem :-

Let U and W be the two subspaces of a vector space V, then

i) $U \cap W$ is a subspace of V

ii) $U + W$ is a subspace of V

Proof :-

Let U and W are two subspaces of a vector space V

i) Then to prove that $U \cap W$ is a subspace of V.

Since U and W are subspaces of V

$\therefore U \neq \phi$, $W \neq \phi$ and $U \subseteq V$, $W \subseteq V$

i.e $\bar{0} \in V$, $\bar{0} \in W$ and $U \subseteq V$, $W \subseteq V$.

$\Rightarrow \bar{0} \in U \cap W$ and $U \cap W \subseteq V$

i.e, $U \cap W \neq \phi$ and $U \cap W \subseteq V$

Let $\alpha, \beta \in U \cap W$ and K be any scalar.

Then $\alpha, \beta \in U$ and $\alpha, \beta \in W$.

Now $\alpha, \beta \in U$, K be any scalar and U is a subspace of V.

$\Rightarrow \alpha + K\beta \in U$ — ①

Also $\alpha, \beta \in W$, K be any scalar and W is a subspace of V

$\Rightarrow \alpha + K\beta \in W$ — ②

For eqn ① & ② we get

$\alpha + K\beta \in U \cap W$

Hence $U \cap W$ is a subspace of V

ii) Then to prove that $U+W$ is a subspace of V

Since U and W are subspaces of V

Since $U \neq \phi, \bar{W} \neq \phi$ and $U \subseteq V, W \subseteq V$

$\therefore U \neq \phi, \bar{w} \neq \phi$ and $U \subseteq V, W \subseteq V$

i.e $\bar{0} \in U, \bar{0} \in W$ and $U \subseteq V, W \subseteq V$

$\Rightarrow \bar{0} \in U+W$ and $U+W \subseteq V$

i.e. $U+W \neq \phi$ and $U+W \subseteq V$

Let $\alpha, \beta \in U+W$ and K be any scalar.

Since $\alpha \in U+W = \{\bar{u}_1 + \bar{w}_1 : \bar{u}_1 \in U, \bar{w}_1 \in W\}$

$\therefore \alpha = \bar{u}_1 + \bar{w}_1 ; \bar{u}_1 \in U, \bar{w}_1 \in W$

$\beta \in U+W = \{\bar{u}_2 + \bar{w}_2 : \bar{u}_2 \in U, \bar{w}_2 \in W\}$

$\beta = \bar{u}_2 + \bar{w}_2 ; \bar{u}_2 \in U, \bar{w}_2 \in W$

Now $\alpha + K\beta = (\bar{u}_1 + \bar{w}_1) + K(\bar{u}_2 + \bar{w}_2)$

$= \bar{u}_1 + \bar{w}_1 + K\bar{u}_2 + K\bar{w}_2$

$= (\bar{u}_1 + K\bar{u}_2) + (\bar{w}_1 + K\bar{w}_2)$

$\in U+W$

$\therefore$ ~~U~~ $U+W$ is a subspace of V.

Theorem:-

Let V be a vector space and let $\bar{x}, \bar{y}$ be vectors in V. Then

i) $\bar{x} + \bar{y} = \bar{y}$ implies $\bar{x} = \bar{0}$;

ii) $0\bar{x} = \bar{0}$

iii) $K\bar{0} = \bar{0}$ for any $K \in R$

iv) $-\bar{x}$ is unique and $-\bar{x} = (-1)\bar{x}$

v) If $K\bar{x} = \bar{0}$, then $K = 0$ or $\bar{x} = \bar{0}$

Let V be a vector space and $\bar{x}, \bar{y} \in V$

i) Suppose $\bar{x}+\bar{y} = \bar{y}$

Now $\bar{x} = \bar{x}+\bar{0}$

$= \bar{x}+(\bar{y}-\bar{y})$

$= \bar{x}+(\bar{y}+(-\bar{y}))$

$= (\bar{x}+\bar{y})+(-\bar{y})$, by associative axiom

$= \bar{y}+(-\bar{y})$ $\langle \because \bar{x}+\bar{y}=\bar{y} \rangle$

$= \bar{y}-\bar{y}$

$= \bar{0}$

ii) $0\bar{x} = 0\bar{x}+\bar{0}$ $\qquad 0\bar{x} = (0+0)\bar{x}$

$= 0\bar{x}+(\bar{x}-\bar{x})$ $\qquad 0\bar{x}+0\bar{x}$

$= 0\bar{x}+(\bar{x}+(-\bar{x}))$ $\qquad 0\bar{x}+\bar{0} = 0\bar{x}+0\bar{x}$

$\qquad \bar{0} \quad 0\bar{x}$

$= (0\bar{x}+\bar{x})+(-\bar{x})$, by associative axiom

$= (0\bar{x}+1.\bar{x})+(-\bar{x})$

$= (0+1)\bar{x}+(-\bar{x})$

$= 1.\bar{x}+(\bar{x})$

$= \bar{x}-\bar{x}$

$= \bar{0}$

iii) $K\bar{0} = K(\bar{0}+\bar{0})$

$= K\bar{0}+K\bar{0}$

$K\bar{0}+\bar{0} = K\bar{0}+K\bar{0}$

$\bar{0} = K\bar{0}$

i.e., $K\bar{0} = \bar{0}$

iv)

iv) Suppose $\bar{x}_1$ is another negative of $\bar{x}$

Then $\bar{x} + \bar{x}_1 = \bar{0}$ —— ①

Now $-\bar{x} = -\bar{x} + \bar{0}$

$= -\bar{x} + (\bar{x} + \bar{x}_1)$ by equation ①

$-(-\bar{x} + \bar{x}) + \bar{x}_1$

$= \bar{0} + \bar{x}_1$

$= \bar{x}_1$

$\therefore -\bar{x}$ is unique

Now $\bar{x} + (-1)\bar{x} = 1.\bar{x} + (-1)\bar{x}$

$= (1 + (-1))\bar{x}$

$= (1-1)\bar{x}$

$= 0.\bar{x}$

$= \bar{0}$

$\Rightarrow (-1)\bar{x} = -\bar{x}$

i.e, $-\bar{x} = (-1)\bar{x}$

v) Suppose $K\bar{x} = \bar{0}$

if $K = 0$, then there is nothing to prove

Suppose $K \neq 0$

Now $\bar{x} = \frac{1}{K}(K\bar{x})$

$= \frac{1}{K}(\bar{0})$

$\bar{x} = \bar{0}$

If $K\bar{x} = \bar{0}$, then $K = 0$ & $\bar{x} = \bar{0}$

$\therefore (\alpha+\beta)p = \alpha p+\beta p$ is true.

vii) $p = a_0 + a_1 x + \dots + a_{n-1}x^{n-1} \in V$

$\alpha, \beta \in F$

$(\alpha\beta)p = \alpha\beta(a_0 + a_1 x + \dots + a_{n-1}x^{n-1})$

$= \alpha(\beta p)$

$\therefore (\alpha\beta)p = \alpha(\beta p)$ is true

ix) $p = a_0 + a_1 x + \dots + a_{n-1}x^{n-1} \in V$

$1 \in F$

$1 \cdot p = 1(a_0 + a_1 x + \dots + a_{n-1}x^{n-1})$

$1 \cdot p = p$ is true

Conclusion:-

Every polynomial is of degree $< n$ is a vector space

Row and column spaces :-

Let A be an $m \times n$ matrix with row vectors $\{\bar{r}_1, \bar{r}_2, \dots \bar{r}_m\}$ and column vectors $\{\bar{c}_1, \bar{c}_2, \dots \bar{c}_n\}$

i) The row space of A is the subspace in $\mathbb{R}^n$ spanned by the row vectors $\{\bar{r}_1, \bar{r}_2, \dots \bar{r}_m\}$, denoted by $R(A)$

ii) The column space of A is the subspace in $\mathbb{R}^m$ spanned by the column vectors $\{\bar{c}_1, \bar{c}_2, \dots \bar{c}_n\}$ denoted by $C(A)$.

iii) The solution set of the homogeneous equation $Ax=0$ is called <u>null space</u> of A, denoted by $N(A)$

***

1. Find the bases for the row, column and null space of A, where
$$A = \begin{pmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{pmatrix} \quad \begin{array}{l} R_2 \to R_2 + 2R_1 \\ 4\times5 \quad R_4 \to R_4 - 3R_1 \end{array}$$

$$\sim \begin{pmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & -3 & 3 & 4 & 1 \\ 0 & 0 & 0 & -13 & -13 \end{pmatrix} \quad \begin{array}{l} R_3 \Rightarrow R_3 - 3R_2 \\ R_4 \Rightarrow R_4/-13 \end{array}$$

$$\sim \begin{pmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad \begin{array}{l} R_2 \Rightarrow R_2/-1 \\ R_3 \Rightarrow R_3/-5 \end{array}$$

$$\sim \begin{pmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad R_1 \to R_1 - 2R_2$$

$$\sim \begin{pmatrix} 1 & 0 & 2 & 8 & 9 \\ 0 & 1 & -1 & -3 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad \begin{array}{l} R_1 \to R_1 - 8R_3 \\ R_2 \to R_2 + 3R_3 \end{array}$$

$$\sim \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad R_4 \to R_4 - R_3$$

$$\sim \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccccc} & & c_3 & & c_5 \end{array}$$

Since the non-zero row vectors of U are

$\bar{V}_1 = (1,0,2,0,1)$ $\quad \bar{V}_2 = (0,1,-1,0,1)$

$\bar{V}_3 = (0,0,0,1,1)$ are linearly independent

and they form a basis for the row space

Basis for $R(A) = \{\bar{V}_1, \bar{V}_2, \bar{V}_3\}$

<u>Basis for N(A)</u>:-

For this, we solve the homogenious

solution system $UX=0$, we have

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = 0$$

$\Rightarrow x_1 + 2x_3 + x_5 = 0$

$x_2 - x_3 + x_5 = 0$

$x_4 + x_5 = 0$

choose $x_3 = s$ and $x_5 = t$

Then $x_4 = -t$.

$x_2 = s - t$

$x_1 = -2s - t$

$x_1 = -2s - t = -2s - t$
$x_2 = s - t = s - t$
$x_3 = s = s + 0t$
$x_4 = -t = 0.s + (-1)t$
$x_5 = t = 0.s + t$

Since any solution X is a linear combination of u_s and u_t.

$\therefore$ The set $\{u_s, u_t\}$ is a basis to the null space

$\therefore$ Basis of $N(A) = \{u_s, u_t\} = \{(-2, 1, 1, 0, 0), (-1, -1, 0, -1, 1)\}$

<u>Basis for $C(A)$:-</u>

Let C_1, C_2, C_3, C_4 & C_5 denotes the column vectors of A. Since these column vectors of A can span $C(A)$, we only need to discard some of the columns that can be expressed as a linear combination of others.

$AX = 0$

$$[C_1 \; C_2 \; C_3 \; C_4 \; C_5] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$$

By taking $X = u_s = (-2, 1, 1, 0, 0)$ then

$-2C_1 + C_2 + C_3 = 0$ —— ①

Similarly consider $X = u_t = (-1, -1, 0, -1, 1)$ then

$-C_1 - C_2 - C_4 + C_5 = 0$ —— ②

From equ ①, we have $C_3 = 2C_1 - C_2$

From equ ②, we have $C_5 = C_1 + C_2 + C_4$

Hence the column vectors C_3 & C_5 corresponding to the free variable in $AX = 0$ can be expressed as

$$C_3 = 2C_1 - C_2$$

$$C_5 = C_1 + C_2 + C_4$$

i.e., the column vectors C_3, C_5 of A are linearly dependent.

Hence $\{C_1, C_2, C_4\}$ spans the column space $C(A)$

i.e., $C(A) = \{(1,-2,0,3), (2,-5,-3,6), (2,-1,4,-7)\}$

*** 8M

2. Find the basis for the row, column & null space for each of the following matrices

i) $A = \begin{pmatrix} 1 & 2 & 1 & 5 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & -1 & 1 \end{pmatrix}$

ii) $A = \begin{pmatrix} 0 & 2 & 1 & -5 \\ 1 & 1 & -2 & 2 \\ 1 & 5 & 0 & 0 \end{pmatrix}$

iii) $A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 3 & 9 & 6 \end{pmatrix}$

iv) $A = \begin{pmatrix} 0 & 1 & -1 & -2 & 1 \\ 1 & 1 & -1 & 3 & 1 \\ 2 & 1 & -1 & 8 & 3 \\ 0 & 0 & -2 & 2 & 1 \\ 3 & 5 & -5 & 5 & 10 \end{pmatrix}$

v) $A = \begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{pmatrix}$

iv) $A = \begin{pmatrix} 0 & 1 & -1 & -2 & 1 \\ 1 & 1 & -1 & 3 & 1 \\ 2 & 1 & -1 & 8 & 3 \\ 0 & 0 & -2 & 2 & 1 \\ 3 & 5 & -5 & 5 & 10 \end{pmatrix}$ $R_1 \leftrightarrow R_2$

$\sim \begin{pmatrix} 1 & 1 & -1 & 3 & 1 \\ 0 & 1 & -1 & -2 & 1 \\ 2 & 1 & -1 & 8 & 3 \\ 0 & 0 & -2 & 2 & 1 \\ 3 & 5 & -5 & 5 & 10 \end{pmatrix}$ $R_3 \to R_3 - 2R_1$, $R_5 \to R_5 - 3R_1$

$\sim \begin{pmatrix} 1 & 1 & -1 & 3 & 1 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & -1 & 1 & 2 & 1 \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 2 & -2 & -4 & 7 \end{pmatrix}$ $R_3 \to R_3 + R_2$, $R_5 \to R_5 + 2R_2$

$\sim \begin{pmatrix} 1 & 1 & -1 & 3 & 1 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 9 \end{pmatrix}$ $R_3 \leftrightarrow R_4$

$\sim \begin{pmatrix} 1 & 1 & -1 & 3 & 1 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 9 \end{pmatrix}$ $R_3 \to R_3 / -2$, $R_4 \to R_4 / 2$, $R_5 \to R_5 / 9$

$$\sim \begin{pmatrix} 1 & +1 & -1 & 3 & 1 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad R_5 \to R_5 - R_4$$

$$\sim \begin{pmatrix} 1 & +1 & -1 & 3 & 1 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad R_1 \to R_1 \mp R_2$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} \cancel{R_1} \to \cancel{R_1} + \cancel{2R_3} \\ R_2 \to R_2 + R_3 \end{matrix}$$

$1 - \frac{1}{2}$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & -3 & \frac{1}{2} \\ 0 & 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} R_2 \Rightarrow R_2 - \frac{1}{2} R_4 \\ R_3 \Rightarrow R_3 - \frac{1}{2} R_4 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since the non-zero row vectors of U are

$\bar{v}_1 = (1, 0, 0, 5, 0)$ $\bar{v}_2 = (0, 1, 0, -3, 0)$

$\bar{V}_3 = (0,0,1,-1,0)$ $\bar{V}_4 = (0,0,0,0,1)$ are linearly independent and they form a basis for row space.

$\therefore$ Basis for $R(A) = \{\bar{V}_1, \bar{V}_2, \bar{V}_3, \bar{V}_4\}$

Basis for N(A):-

$$UX = 0$$

$$\begin{pmatrix} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = 0$$

$\Rightarrow$ $x_1 + 5x_4 = 0$ —— (1)

$x_2 - 3x_4 = 0$ —— (2)

$x_3 - x_4 = 0$ —— (3)

$x_5 = 0$

$x_4 = t$ (1) $\Rightarrow$ $x_1 = -5t$

$x_2 = 3t$

$x_3 = t$

$x_4 = t$, $x_5 = 0$

$\therefore$ Solution X is n_t

$\therefore$ The set $\{n_t\}$ is a basis for null space

$\therefore$ Basis for $N(A) = \{\cancel{n_s}, n_t\} = \{-5, 3, 1, 1, 0\}$

Basis for C(A):-

For this, we consider

$$[c_1 \ c_2 \ c_3 \ c_4 \ c_5] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$$

By taking $x = u_4 = (-5, 3, 1, 1, 0)$ then

$$-5c_1 + 3c_2 + c_3 + c_4 = 0$$

$$c_4 = 5c_1 - 3c_2 - c_3$$

i.e., the column vector c_4 of A is linearly dependent.

Hence $\{c_1, c_2, c_3, c_5\}$ spans the column space $C(A)$

i.e., $C(A) = \{(0,1,2,0,3), (1,1,1,0,5), (-1,-1,-1,-2,-5), (1,1,3,1,10)\}$

3. Find a bases for the null space & column space of

$$U = \begin{bmatrix} 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis of $R(A) = \{(1,0,0,2,2), (0,1,0,-1,3), (0,0,1,4,-1)\}$

<u>Basis</u> of <u>N(A)</u> :-

For this, we consider $U\bar{x} = \bar{0}$

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_4 + 2x_5 = 0$

$x_2 - x_4 + 3x_5 = 0$

$x_3 + 4x_4 - x_5 = 0$

Consider $x_4 = s$, $x_5 = t$

$x_1 = -2s - 2t$

$x_2 = s - 3t$

$x_3 = -4s + t$, $x_4 = s$, $x_5 = t$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s-2t \\ s-3t \\ -4s+t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ -4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$= s n_s + t n_t$

where $n_s = (-2, 1, -4, 1, 0)$.

$n_t = (-2, -3, 1, 0, 1)$

$\therefore$ Basis of $N(A) = \{n_s, n_t\}$

$= \{(-2, 1, -4, 1, 0)\ (-2, 3, 1, 0, 1)\}$

$\therefore$ dim $N(A) = 2$ = nullity of A

Basis of $C(A)$:-

For this, we consider

$[c_1\ c_2\ c_3\ c_4\ c_5]\bar{x} = \bar{0}$

$$\Rightarrow [c_1\ c_2\ c_3\ c_4\ c_5]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \bar{0} \quad \text{--- (1)}$$

Suppose $\bar{x} = (x_1, x_2, x_3, x_4, x_5) = n_s$

$$\text{Then } [c_1\ c_2\ c_3\ c_4\ c_5]\begin{bmatrix} -2 \\ 1 \\ -4 \\ 1 \\ 0 \end{bmatrix} = 0$$

$\Rightarrow -2c_1 + c_2 - 4c_3 + c_4 = 0$ ——— (2)

Suppose $\bar{x} = (x_1, x_2, x_3, x_4, x_5) = n_t$

$$\text{Then } [c_1\ c_2\ c_3\ c_4\ c_5]\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 0$$

$\Rightarrow -2c_1 - 3c_2 + c_3 + c_5 = 0$ ——— (3)

From equ (2), we have

$c_4 = 2c_1 - c_2 + 4c_3$

From equ (3), we have

$$C_5 = 2C_1 + 3C_2 - C_3$$

Hence basis of $C(A) = \{C_1, C_2, C_3\}$

$\therefore \dim C(A) = 3$

<u>Note</u> :-

i) The dimension of the null space of A is called the nullity of A.

ii) ~~&~~ The row vectors of A are just the column Column vectors of its transpose A^T and the column vectors of A are the row vectors of A^T, the row (column) space of A is just the column (row) space of A^T

i.e., $R(A) = C(A^T)$ and $C(A) = R(A^T)$

<u>Example</u> :-

*** Find bases for $R(A)$ and $N(A)$ of the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

Also find a basis for $C(A)$ by finding a basis for $R(A^T)$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix} \quad \begin{array}{l} R_2 \Rightarrow R_2 - 2R_1 \\ R_3 \Rightarrow R_3/5 \\ R_4 \Rightarrow R_4 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 10 & 18 & 8 & 0 \end{bmatrix} \quad R_3 \Rightarrow R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 10 & 18 & 8 & 0 \end{bmatrix} \quad \begin{array}{l} R_2 \Rightarrow R_2/-1 \\ \cancel{R_3 \Rightarrow R_3/-2} \\ R_4 \Rightarrow R_4/2 \end{array}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 9 & 4 & 0 \end{bmatrix} \quad \begin{array}{l} \cancel{R_1 \Rightarrow R_1 + 2R_2} \\ R_3 \Leftrightarrow R_4 \end{array}$$

$$\cancel{\sim \begin{bmatrix} 1 & 0 & 6 & 4 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 5 & 9 & 4 & 0 \end{bmatrix} \quad R_1 \Rightarrow R_1 -}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 5 & 9 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \Rightarrow R_3 - 5R_2$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & -6 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \Rightarrow R_3/-6$$

$$\frac{-15}{9}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 \Rightarrow R_1 + 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 6 & 4 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 6R_3 \\ R_2 \to R_2 - 3R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2 & 3 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Basis of $R(A) = \{(1,0,0,-2,3), (0,1,0,-1,0), (0,0,1,1,0)\}$

$\dim R(A) = 3$

<u>Basis for $N(A)$</u>:-

For this, we consider $UX = \bar{0}$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 3 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - 2x_4 + 3x_5 = 0$$

$$x_2 - x_4 = 0$$

$$x_3 + x_4 = 0$$

$x_4 = t$

$x_5 = s$

$$x_3 = -t$$

$$x_2 = t$$

$$x_1 = 2t - 3s$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2t-3s \\ t \\ -t \\ t \\ s \end{bmatrix} = s\begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} +2 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= s n_s + t n_t$$

where $n_s = (-3,0,0,0,1)$

$n_t = (2,1,-1,1,0)$

$\therefore$ Basis of $N(A) = \{n_s, n_t\}$ where

$n_s = (2,1,-1,1,0)$, $n_t = (-3,0,0,0,1)$

dim $N(A) = 2 =$ nullity of A

Basis of C(A):-

For this, we consider

$[c_1 \; c_2 \; c_3 \; c_4 \; c_5]\,\bar{x} = \bar{0}$

$$\Rightarrow [c_1 \; c_2 \; c_3 \; c_4 \; c_5]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \bar{0} \quad \text{—— (1)}$$

Suppose $\bar{x} = (x_1, x_2, x_3, x_4, x_5) = n_s$

Then $$[c_1 \; c_2 \; c_3 \; c_4 \; c_5]\begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$2c_1 + c_2 \cancel{+} - c_3 + c_4 = 0 \quad \text{—— (1)}$$

Suppose $\bar{x} = (x_1, x_2, x_3, x_4, x_5) = n_t$

Then $[C_1 \; C_2 \; C_3 \; C_4 \; C_5]\begin{bmatrix}-3\\0\\0\\0\\1\end{bmatrix} = 0$

$-3C_1 + C_5 = 0$ —— (3)

$C_5 = 3C_1$

$C_4 = -2C_1 - C_2 + C_3$

Hence basis of $C(A) = \{C_1, C_2, C_3\}$

$\therefore$ dim $C(A) = 3$

Note :-

i) dim $R(A)$ = dim $C(A)$

ii) dim $N(A)$ = dim $N(U)$

= The number of free variables in $U\bar{x} = \bar{0}$

iii) dim $R(A)$ = dim $R(U)$

= The number of non-zero rows of U

= The maximal number of linearly independent row vector of A

= The number of free variables in $U\bar{x} = \bar{0}$

= The maximal number of linearly independent column vector of A

= dim $C(A)$.

Linear Transformations

Let V and W are two vector spaces, then
$T: V \to W$ is called a linear transformation if it
satisfies the following conditions

i) $T(\bar{x}+\bar{y}) = T(\bar{x})+T(\bar{y})$

ii) $T(k\bar{y}) = kT(\bar{y})$, $\forall\ \bar{x}, \bar{y} \in V$ and k be any scalar

or

$T(\bar{x}+k\bar{y}) = T(\bar{x})+kT(\bar{y})$, $\forall\ \bar{x}, \bar{y} \in V$ and k be any scalar

Verify whether the following functions are linear transformations

i) $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$

ii) $g: \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2 - x$

iii) $h: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $h(x,y) = (x-y, 2x)$

iv) $K: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $K(x,y) = (xy, x^2+1)$

v) $T(x,y) = (x^2-y^2, x^2+y^2)$

vi) $T(x,y,z) = (x+y, 0, 2x+4z)$

vii) $T(x,y) = (\sin x, y)$

viii) $T(x,y) = (x+1, 2y, x+y)$

**ix) $T(x,y,z) = (|x|, 0)$

iv) $K: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $h(x,y) = (xy, x^2+1)$

Let $\alpha, \beta \in \mathbb{R}^2$ and a be any scalar

Then $\alpha = (x_1, y_1)$, $\beta = (x_2, y_2)$

$$K(\alpha+a\beta) = K((x_1,y_1)+a(x_2,y_2))$$
$$= K((x_1,y_1)+(ax_2,ay_2))$$
$$= K(x_1+ax_2,\ y_1+ay_2)$$
$$= ((x_1+ax_2)(y_1+ay_2),\ (x_1+ax_2)^2+1)$$
$$= (x_1y_1+ax_1y_2+ax_2y_1+a^2x_2y_2,\ x_1^2+a^2x_2^2+2ax_1x_2+1) \quad \text{—— (1)}$$

Now $K(\alpha)+aK(\beta) = K(x_1,y_1)+aK(x_2,y_2)$
$$= (x_1y_1,\ x_1^2+1)+a(x_2y_2,\ x_2^2+1)$$
$$= (x_1y_1,\ \cancel{a}x_1^2+1)+(ax_2y_2,\ ax_2^2+a)$$
$$= (x_1y_1+ax_2y_2,\ x_1^2+1+ax_2^2+a)$$
$$= (x_1y_1+ax_2y_2,\ x^2+ax_2^2+a+1) \quad \text{—— (2)}$$

From equ (1) & (2) we get,

$K(\alpha+a\beta) \neq K(\alpha)+aK(\beta)$

$\therefore$ K is not a linear transformation.

ix) $T:\mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by $T(x,y,z) = (|x|, 0)$

Let $\alpha, \beta \in \mathbb{R}^3$ and K be any scalar.

Then $\alpha = (x_1,y_1,z_1)$ $\beta = (x_2,y_2,z_2)$

$$T(\alpha+K\beta) = T((x_1,y_1,z_1)+K(x_2,y_2,z_2))$$
$$= T((x_1,y_1,z_1)+(Kx_2,Ky_1,Kz_2))$$
$$= T(x_1+Kx_2,\ y_1+Ky_2,\ z_1+Kz_2)$$
$$= (|x_1+Kx_2|, 0) \quad \text{—— (1)}$$

Now $T(\alpha)+KT(\beta) = T(x_1,y_1,z_1)+KT(x_2,y_2,z_2)$

$$= ((|x_1|,0)+K(|x_2|,0))$$

$$= ((|x_1|,0)+K|x_2|,0)$$

$$= (|x_1|+K|x_2|, 0+0)$$

$$= (|x_1|+K|x_2|, 0) \quad \text{--- } ②$$

From equ ① & ② we get

$$T(\alpha+K\beta) \neq (\alpha)+TK(\beta)$$

$\therefore$ T is not a linear transformation.

vii) $T(x,y) = (\sin x, y)$

Let $\alpha, \beta \in \mathbb{R}^2$ and K be any scalar.

Then $\alpha = (x_1,y_1)$ $\beta = (x_2,y_2)$

$$T(\alpha+K\beta) = T((x_1,y_1)+K(x_2,y_2))$$

$$= T((x_1,y_1)+(Kx_2,Ky_2))$$

$$= T(x_1+Kx_2, y_1+Ky_2)$$

$$= (\sin(x_1+Kx_2), y_1+Ky_2) \quad \text{--- } ①$$

Also $T(\alpha)+KT(\beta) = T(x_1,y_1)+KT(x_2,y_2)$

$$= (\sin x_1, y_1)+K(\sin x_2, y_2)$$

$$= ((\sin x_1, y_1)+K\sin x_2, Ky_2)$$

$$= (\sin x_1+K\sin x_2, y_1+Ky_2) \quad \text{--- } ②$$

From equ ① & ② we get

$$T(\alpha+K\beta) \neq (\alpha)+TK(\beta)$$

$\therefore$ T is not a linear transformation.

Note :-

i) For an $m \times n$ matrix, the transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\bar{x}) = A\bar{x}$ is a linear transformation.

ii) Let $T : V \to W$ be a linear transformation, then a) $T(\bar{0}) = \bar{0}$

b) For any $\bar{x}_1, \bar{x}_2, \ldots \bar{x}_n \in V$ and scalars $K_1, K_2 \ldots\ldots K_n$,

$$T(K_1\bar{x}_1 + K_2\bar{x}_2 + \ldots + K_n\bar{x}_n) = K_1T(\bar{x}_1) + K_2T(\bar{x}_2) + \ldots + K_nT(\bar{x}_n)$$

iii) For a vector space V, the identity transformation $id : V \to V$ is defined by $id(\bar{x}) = \bar{x}$ for all $\bar{x} \in V$

iv) Let V and W are two vector spaces, the zero transformation $T_0 : V \to W$ is defined by $T_0(\bar{x}) = \bar{0}$ for all $\bar{x} \in V$.

Definition :-

Let V and W be two vector spaces and let $T : V \to W$ be a linear transformation from V into W,

i) $Ker(T) = \{\bar{x} \in V : T(\bar{x}) = \bar{0}\} \subseteq V$ is called the Kernel of T.

ii) $Im(T) = \{T(\bar{x}) \in W : \bar{x} \in V\} = T(V) \subseteq W$ is called the image of T.

Theorem :-

Let V and W are two vector spaces and $T: V \to W$ be a linear transformation, then

*** 3M i) $Ker(T)$ is a subspace of V

ii) $Im(T)$ is a subspace of W

Proof :-

Let V and W are two vector spaces and $T: V \to W$ is a linear transformation

i) Then to prove that $Ker(T)$ is a subspace of V

Since $T(\bar{0}) = \bar{0}$

$\therefore \bar{0} \in Ker(T)$

i.e., $Ker(T) \neq \phi$

Also $Ker(T) = \{\bar{x} \in V : T(\bar{x}) = \bar{0}\} \subseteq V$

i.e., $Ker(T) \neq \phi$ and $Ker(T) \subseteq V$

Let $\bar{x}, \bar{y} \in Ker(T)$ and K be the scalar.

Then $T(\bar{x}) = \bar{0}$, $T(\bar{y}) = \bar{0}$

Now $T(\bar{x} + K\bar{y}) = T(\bar{x}) + KT(\bar{y})$, by linear transformation

$= \bar{0} + K\bar{0}$

$= \bar{0}$

$\Rightarrow \bar{x} + K\bar{y} \in Ker(T)$

$\therefore Ker(T)$ is a subspace of V.

Proof :-

Let V and W are two vector spaces and $T: V \to W$ is a linear transformation.

ii) Then to prove that $I_m(T)$ is a subspace of W.

Since $T(\bar{0}) = \bar{0}$

$\therefore T(\bar{0}) \in I_m(T)$

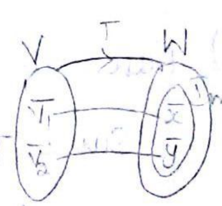
$\Rightarrow I_m(T) \neq \phi$

Also $I_m(T) = \{T(\bar{x}) \in W : \bar{x} \in V\} = T(V) \subseteq W$

$\therefore I_m(T) \neq \phi$ and $I_m(T) \subseteq W$.

Let $\bar{x}, \bar{y} \in I_m(T)$ and K be the scalar

Now T(



Then $\bar{x} = T(\bar{v}_1)$, $\bar{y} = T(\bar{v}_2)$

Now $\bar{x} + K\bar{y} = T(\bar{v}_1) + KT(\bar{v}_2)$

$\bar{x}+K\bar{y}$ is the image of

$= T(\bar{v}_1 + K\bar{v}_2) \in I_m(T)$

$\therefore I_m(T)$ is a subspace of W.

Definition:-

i) Let θ denote the angle between the x-axis and a fixed vector in $\mathbb{R}^2$. Then the matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ defines a linear transformation on $\mathbb{R}^2$ that rotates any vector in $\mathbb{R}^2$ through the angle θ about the origin. It is called a <u>rotation</u> by the angle θ.

ii) The <u>projection</u> on the x-axis is the linear transformation $P : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$P(\bar{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} \text{ for } \bar{x} = (x, y) \in \mathbb{R}^2$$

iii) The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by,

$$T(\bar{x}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} \text{ for } \bar{x} = (x, y) \in \mathbb{R}^2$$

is the <u>reflection</u> about the x-axis.

<u>Problem</u> :-

Let $w_1 = (1, 0)$, $w_2 = (2, -1)$, $w_3 = (4, 3)$ be three vectors in $\mathbb{R}^2$.

i) Let $\alpha = \{e_1, e_2, e_3\}$ be the standard basis for the 3-space $\mathbb{R}^3$ and let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by $T(e_1) = w_1$, $T(e_2) = w_2$, $T(e_3) = w_3$.

Find a formula for $T(x_1, x_2, x_3)$ and then we use it to compute $T(2, -3, 5)$.

ii) Let $\beta = \{v_1, v_2, v_3\}$ be another basis for $\mathbb{R}^3$, where $v_1 = (1, 1, 1)$, $v_2 = (1, 1, 0)$, $v_3 = (1, 0, 0)$ and let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by $T(v_1) = w_1$, $T(v_2) = w_2$, $T(v_3) = w_3$

Find a formula for $T(x_1, x_2, x_3)$ and then we use it to compute $T(2, -3, 5)$

i) Let $x \in \mathbb{R}^3$

Then $\bar{x} = (x_1, x_2, x_3)$

Now $\bar{x} = (x_1, x_2, x_3)$

$= x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$

$= x_1 e_1 + x_2 e_2 + x_3 e_3$

i.e., $(x_1, x_2, x_3) = x_1 e_1 + x_2 e_2 + x_3 e_3$

$T(x_1, x_2, x_3) = T(x_1 e_1 + x_2 e_2 + x_3 e_3)$

$= x_1 T(e_1) + x_2 T(e_2) + x_3 T(e_3)$

$= x_1 w_1 + x_2 w_2 + x_3 w_3$

$= x_1(1,0) + x_2(2,-1) + x_3(4,3)$

$= (x_1, 0) + (2x_2, -x_2) + (4x_3, 3x_3)$

$= (x_1 + 2x_2 + 4x_3, -x_2 + 3x_3)$

i.e., $T(x_1, x_2, x_3) = (x_1 + 2x_2 + 4x_3, -x_2 + 3x_3)$

$T(2,-3,5) = (2 + 2(-3) + 4(5), -(-3) + 3(5))$

$= (2 - 6 + 20, 3 + 15)$

$T(2,-3,5) = (16, 18)$

ii) Let $\bar{x} \in \mathbb{R}^3$

Then $\bar{x} = (x_1, x_2, x_3)$

Now $\bar{x} = (x_1, x_2, x_3)$

$= x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$

$= x_1 V_1 + x_2 V_2 + x_3 V_3$

i.e., $(x_1, x_2, x_3) = x_1 V_1 + x_2 V_2 + x_3 V_3$

$T(x_1, x$

ii) Let $\bar{x}$

Then $\bar{x}$

Now $\bar{x}$

$(x_1, x_2,$

$\Rightarrow K_1+$

K_1

$\Rightarrow K_2$

$\Rightarrow K_3$

K_3

$\therefore (x_1, x_2$

$\Rightarrow T(x_1, x$

$T(x_1, \cancel{x_2, x_3}) = T(x_1V_1 + x_2V_2 + x_3V_3)$

$\cancel{= x_1T(V_1) + x_2T}$

ii) Let $\bar{x} \in \mathbb{R}^3$

Then $\bar{x} = (x_1, x_2, x_3)$

Now $\bar{x} = (x_1, x_2, x_3)$

$= K_1(1,1,1) + K_2(1,1,0) + K_3(1,0,0)$

$= (K_1, K_1, K_1) + (K_2, K_2) + K_3 \ldots$

$= (K_1 + K_2 + K_3, K_1 + K_2, K_1)$

$(x_1, x_2, x_3) = (K_1 + K_2 + K_3, K_1 + K_2, K_1)$

$\Rightarrow K_1 + K_2 + K_3 = x_1$

$K_1 + K_2 = x_2$

$K_1 = x_3$

$\Rightarrow K_2 = x_2 - x_3$

$\Rightarrow K_3 = x_1 - (x_2 - x_3) - x_3$

$K_3 = x_1 - x_2$

$\therefore (x_1, x_2, x_3) = x_3(1,1,1) + (x_2 - x_3)(1,1,0) + (x_1 - x_2)(1,0,0)$

$= x_3V_1 + (x_2 - x_3)V_2 + (x_1 - x_2)V_3$

$\Rightarrow T(x_1, x_2, x_3) = T(x_3V_1 + (x_2 - x_3)V_2 + (x_1 - x_2)V_3)$

$= x_3T(V_1) + (x_2 - x_3)(V_2) + (x_1 - x_2)T(V_3)$

$= x_3\omega_1 + (x_2 - x_3)\omega_2 + (x_1 - x_2)\omega_3$

$= x_3(1,0) + (x_2 - x_3)(2,-1) + (x_1 - x_2)(4,3)$

$= (x_3, 0) + (2x_2 - 2x_3, -x_2 + x_3) + (4x_1 - 4x_2,$
$3x_1 - 3x_2)$

$= (x_3 + 2x_2 - 2x_3 + 4x_1 - 4x_2, 0 - x_2 + x_3 + 3x_1 - 3x_2)$

$T(x_1, x_2, x_3) = (4x_1 - 2x_2 - x_3,\ 3x_1 - 4x_2 + x_3)$

$T(2,-3,5) = (4(2) - 2(-3) - 5,\ 3(2) - 4(-3) + 5)$

$= (8 + 6 - 5,\ 6 + 12 + 5)$

$= (9, 23)$

$\therefore T(2,-3,5) = (9, 23)$

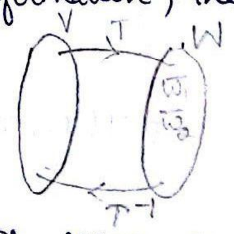
Invertible linear transformation :-

A function f from a set x to a set y is said to be invertible if there is a function g from y to x such that $g \circ f = id$ & $f \circ g = id$.

Note :- A function $f: x \to y$ is invertible if and only if it is bijective.

Theorem :-

If let V and W be two vector spaces. If $T: V \to W$ is an invertible linear transformation, then its inverse $T^{-1}: W \to V$ is also linear.



Proof :-

Let V and W be two vector spaces and $T: V \to W$ is a invertible linear transformation

Then to prove that $T^{-1}: W \to V$ is a linear transformation

For this let $\bar{w}_1, \bar{w}_2 \in W$ and K be the scalar

Since T is invertible

$\therefore$ There exists $\bar{v}_1, \bar{v}_2 \in V$ such that

$T^{-1}(\bar{w}_1) = \bar{V}_1 \;,\; T^{-1}(\bar{w}_2) = \bar{V}_2$

$\Rightarrow \bar{w}_1 = T(\bar{V}_1) \;,\; \bar{w}_2 = T(\bar{V}_2)$

Now $T^{-1}(w_1 + K\bar{w}_2) = T^{-1}\left(T(\bar{V}_1) + KT(\bar{V}_2)\right)$

$= T^{-1}\left[T(\bar{V}_1 + K\bar{V}_2)\right]$ [$\because$ T is a linear transformation]

$= (T^{-1} \circ T)(\bar{V}_1 + K\bar{V}_2)$ [$\because f(g(x)) = (f \circ g)(x)$]

$= (id)(\bar{V}_1 + K\bar{V}_2)$ [$\because f^{-1} \circ f = id$]

$= \bar{V}_1 + K\bar{V}_2$ [$\because id(\bar{x}) = \bar{x}$]

$= T^{-1}(\bar{w}_1) + KT^{-1}(\bar{w}_2)$

$\Rightarrow T^{-1}$ is a linear transformation

*** Show that each of the following linear transformations T on $\mathbb{R}^3$ is invertible, and find a formula for T^{-1}!

i) $T(x, y, z) = (3x, x-y, 2x+y+z)$

ii) $T(x, y, z) = (2x, 4x-y, 2x+3y-z)$

ii) <u>T is one-one:-</u> (distinct elements have distinct images)

For this let $T(\alpha_1) = T(\alpha_2)$ where $\alpha_1 = (x_1, y_1, z_1)$
$\alpha_2 = (x_2, y_2, z_2)$

Then $T(x_1, y_1, z_1) = T(x_2, y_2, z_2)$

$\Rightarrow (2x_1, 4x_1 - y_1, 2x_1 + 3y_1 - z_1) = (2x_2, 4x_2 - y_2, 2x_2 + 3y_2 - z_2)$

$2x_1 = 2x_2$	$4x_1 - y_1 = 4x_2 - y_2$	$2x_1 + 3y_1 - z_1 = 2x_2 + 3y_2 - z_2$
$x_1 = x_2$	$4x_2 - y_1 = 4x_2 - y_2$	$2x_2 + 3y_2 - z_1 = 2x_2 + 3y_2 - z_2$
	$y_1 = y_2$	$z_1 = z_2$

$\Rightarrow (x_1, y_1, z_1) = (x_2, y_2, z_2)$

$\alpha_1 = \alpha_2$

i.e., $T(\alpha_1) = T(\alpha_2) \Rightarrow \alpha_1 = \alpha_2$

$\therefore$ T is one-one

<u>T is onto</u> :- (domain & codomain are same)

For this let $\alpha \in \mathbb{R}^3$ <co-domain>

Then $\alpha = (2x, 4x-y, 2x+3y-z) \in \mathbb{R}^3$

$\Rightarrow x, y, z \in \mathbb{R}$

$\Rightarrow (x, y, z) \in \mathbb{R}^3$ <domain>

Also $T(x,y,z) = (2x, 4x-y, 2x+3y-z)$

i.e., $\forall \alpha \in \mathbb{R}^3$ (codomain), $\exists (x,y,z) \in \mathbb{R}^3$ (domain) such that

$T(x,y,z) = (2x, 4x-y, 2x+3y-z)$

$\therefore$ T is onto and hence T is invertible.

<u>$T^{-1}(x,y,z)$</u> :-

For this let $T(x,y,z) = (p, q, r)$

Then $(2x, 4x-y, 2x+3y-z) = (p, q, r)$

$\Rightarrow 2x = p, \quad 4x - y = q, \quad 2x + 3y - z = r$

$x = \frac{p}{2}, \quad \overset{2}{\cancel{4}}\left(\frac{p}{\cancel{2}}\right) - y = q, \quad \cancel{2}\left(\frac{p}{\cancel{2}}\right) + 3(2p-q) - z = r$

$+y = 2p - q \qquad z = p + 6p - 3q - r$

$z = 7p - 3q - r$

Since $T(x,y,z) = (P,q,r)$ and T is invertible,

$$T^{-1}(P,q,r) = (x,y,z)$$

$$\Rightarrow T^{-1}(P,q,r) = \left(\frac{P}{2}, 2P-q, 7P-3q-r\right)$$

i.e., $T^{-1}(x,y,z) = \left(\frac{x}{2}, 2x-y, 7x-3y-z\right)$.

i) one-one & onto

$\underline{T^{-1}(x,y,z)}$:-

For this let $T(x,y,z) = (P,q,r)$

Then $(3x, x-y, 2x+y+z) = (P,q,r)$

$$\Rightarrow 3x = P, \quad x-y = q, \quad 2x+y+z = r$$

$$x = \frac{P}{3} \qquad \frac{P}{3} - y = q \qquad 2\left(\frac{P}{3}\right) + \frac{1}{3}(P-3q) + z = r$$

$$P - 3y = 3q \qquad \frac{2P}{3} + \frac{P-3q}{3} + z = r$$

$$-3y = 3q - P \qquad 2P + P - 3q + 3z = 3r$$

$$y = \frac{1}{3}(P-3q) \qquad 3z = -3P + 3q + 3r$$

$$z = -P + q + r$$

Since $T(x,y,z) = (P,q,r)$ and T is invertible

$$T^{-1}(P,q,r) = (x,y,z)$$

$$T^{-1}(P,q,r) = \left(\frac{P}{3}, \frac{P-3q}{3}, -P+q+r\right)$$

i.e., $T^{-1}(x,y,z) = \left(\frac{x}{3}, \frac{x-3y}{3}, -x+y+z\right)$.

Definition :-

Let V and W are two vector spaces and $T: V \to W$ be a linear transformation, then T is called an <u>isomorphism</u> if ~~is~~ T is invertible

In this case, we say that V and W are isomorphic to each other and is represented by $V \cong W$

***
__Theorem :-__

Let $T: V \to W$ be an isomorphism, and let $\{\bar{v}_1, \bar{v}_2, \dots \bar{v}_n\}$ be a basis for V, then

Two vector spaces V and W are isomorphic if and only if $\dim V = \dim W$.

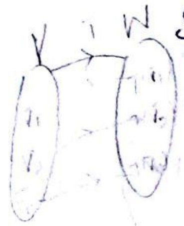
__Proof :-__ Let $T: V \to W$ be an isomorphism.

Let $\{\bar{v}_1, \bar{v}_2, \dots v_n\}$ be a basis for V.

Then to prove that V and W are isomorphic $\Leftrightarrow$ $\dim V = \dim W$.

__First suppose__ V and W are isomorphic.

__Then to prove that__ $\dim V = \dim W$.



Since $\{\bar{v}_1, \bar{v}_2, \dots \bar{v}_n\}$ be a basis for V.

Then to prove $\{T(\bar{v}_1), T(\bar{v}_2), \dots T(\bar{v}_n)\}$ is a basis for W.

So that $\dim V = \dim W$.

i) __Linearly independent__ :-

Since T is invertible.

$\therefore\ c_1 T(\bar{v}_1) + c_2 T(\bar{v}_2) + \dots + c_n T(\bar{v}_n) = \bar{0}$

$\Rightarrow T[c_1 \bar{v}_1 + c_2 \bar{v}_2 + \dots + c_n \bar{v}_n] = \bar{0}$

$\Rightarrow T[c_1 \bar{v}_1 + c_2 \bar{v}_2 + \dots + c_n \bar{v}_n] = T(\bar{0})$

$\Rightarrow c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_n\bar{v}_n = \bar{0}$

Since $\{\bar{v}_1, \bar{v}_2, \ldots \bar{v}_n\}$ is a basis of V

$\therefore c_1 = c_2 = \ldots\ldots = c_n = 0$

$\therefore \{T(\bar{v}_1), T(\bar{v}_2), \ldots T(\bar{v}_n)\}$ is linearly independent

ii) <u>Span</u> :-

Let $\bar{y} \in W$ be any element

Since T is onto, then $\exists$ an $\bar{x} \in V$ such that

$T(\bar{x}) = \bar{y}$; where $\bar{x} = \sum_{i=1}^{n} a_i \bar{v}_i$, for some scalars $a_i, 1 \leq i \leq n$.

$$\Rightarrow T(\bar{x}) = \sum_{i=1}^{n} a_i T(\bar{v}_i)$$

$$\Rightarrow \bar{y} = a_1 T(\bar{v}_1) + a_2 T(\bar{v}_2) + \ldots + a_n T(\bar{v}_n)$$

i.e., $\bar{y}$ is a linear combination of $T(\bar{v}_1), T(\bar{v}_2), \ldots\ldots, T(\bar{v}_n)$

$\therefore \{T(\bar{v}_1), T(\bar{v}_2), \ldots T(\bar{v}_n)\}$ is a basis of W

$\therefore \dim W = n = \dim V$

Conversely,

if $\dim V = \dim W = n$

Then we choose $\{\bar{v}_1, \bar{v}_2, \ldots \bar{v}_n\}$ and $\{\bar{w}_1, \bar{w}_2, \ldots \bar{w}_n\}$ are basis for V & W respectively.

Then there exist a linear transformation $T: V \to W$ and

$S: W \to V$ such that

$T(v_i) = \overline{w_i}$ and $S(\overline{w_i}) = \overline{v_i}$ for $1 \le i \le n$.

$\Rightarrow S(T(\overline{v_i})) = S(\overline{w_i})$

Clearly, we have $(S \circ T)(\overline{v_i}) = \overline{v_i}$

$\Rightarrow T(S(\overline{w_i})) = T(\overline{v_i})$

and $(T \circ S)(\overline{w_i}) = \overline{w_i}$, for $1 \le i \le n$

which implies that $S \circ T$, $T \circ S$ are the identity transformations on V and W respectively and they are unique.

Hence T and S are invertible

$\therefore$ They are isomorphisms so that V and W are isomorphic to each other.

<u>Note</u> :-

The matrix of reflection $= R_\theta \circ T \circ R_{-\theta}$ where θ is the reflection/rotation about the x-axis and T is the reflection matrix about the x-axis

<u>Example</u>:-

Find the matrix of reflection about the line (i) $y = x$ in $\mathbb{R}^2$ (ii) $y = \sqrt{3}x$ in $\mathbb{R}^2$

i) The given line is $y = x$

slope $m = 1$

$\Rightarrow m = \tan\theta = 1$

$\Rightarrow \theta = \frac{\pi}{4}$

Since $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$R_{\frac{\pi}{4}} = \begin{bmatrix} \cos\frac{\pi}{4} \\ \sin\frac{\pi}{4} \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$

$R_{-\theta} = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$

$R_{-\frac{\pi}{4}} = \begin{bmatrix} \cos\frac{\pi}{4} \\ -\sin\frac{\pi}{4} \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$

The matrix

is $R_{\frac{\pi}{4}}$

P. B. Anki Reddy

Unit - 5

Inner Product Space

The dot product (Euclidean inner Product) of two vectors $\bar{x} = (x_1, x_2, x_3)$ $\bar{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$ is a number defined by the formula

$$\bar{x}\,\bar{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= \bar{x}^T \bar{y}$$

The length (or magnitude or norm) of a vector $\bar{X} = (x_1, x_2 x_3)$ is defined by

$$\|\bar{x}\| = (\bar{X}\cdot\bar{X})^{1/2}$$

$$= \sqrt{x_1^2 + x_2^2 + x_3^2}$$ and the Euclidean distance between two vectors $\bar{X}$ & $\bar{Y}$ in $\mathbb{R}^3$ is defined by $d(\bar{x}, \bar{y}) = \|\bar{x} - \bar{Y}\|$

$$= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

The angle "θ" between two vectors $\bar{X}$ and $\bar{Y}$ in $\mathbb{R}^3$ is measured by

$$\bar{X}\cdot\bar{y} = \|\bar{X}\|\|\bar{Y}\| \cos\theta \quad , \quad 0 \le \theta \le \pi$$

· Inner Product :-

~~$\bar{X}, \bar{Y}, \bar{Z}$~~ An inner product on a real vector space V is a function that associates a real number $\langle \bar{X}, \bar{Y} \rangle$ to each pair of vectors $\bar{X}$ & $\bar{Y}$

in V, in such a way that the following rules are satisfied

i) $\langle \bar{x}, \bar{y} \rangle = \langle \bar{y}, \bar{x} \rangle$ (Symmetric)

ii) $\langle \bar{x}+\bar{y}, \bar{z} \rangle = \langle \bar{x}, \bar{z} \rangle + \langle \bar{y}, \bar{z} \rangle$ (Additivity)

iii) $\langle k\bar{x}, \bar{y} \rangle = k\langle \bar{x}, \bar{y} \rangle$ (homogeneity)

iv) $\langle \bar{x}, \bar{x} \rangle \geq 0$ and $\langle \bar{x}, \bar{x} \rangle = 0$

iff $\bar{x} = \bar{0}$ (positive definiteness)

A pair $(V, \langle , \rangle)$ of a vector space V and inner product $\langle , \rangle$ is called a <u>inner product space</u>.

1. *** Let $V = C[0,1]$ be the vector space of real valued continuous functions on $[0,1]$. For any two functions f and g in V define $\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx$. Then prove that $(V, \langle , \rangle)$ is an inner product space.

i) $\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx$

$= \int_0^1 g(x)f(x)\,dx$

$= \langle g, f \rangle$

ii) $\langle f+g, h \rangle = \int_0^1 (f+g)(x) \cdot h(x)\,dx$

$$= \int_0^1 (f(x)+g(x)) \cdot h(x)\,dx$$

$$= \int_0^1 [f(x)\,h(x)+g(x)\,h(x)]\,dx$$

$$= \int_0^1 f(x)\,h(x)\,dx + \int_0^1 g(x)\,h(x)\,dx$$

$$= \langle f,h\rangle + \langle g,h\rangle$$

iii) $\langle kf, g\rangle = \int_0^1 kf(x) \cdot g(x)\,dx$

$$= \int_0^1 kf(x)\,g(x)\,dx$$

$$= k\int_0^1 f(x) \cdot g(x)\,dx$$

$$= k\langle f,g\rangle$$

iv) $\langle f, f\rangle = \int_0^1 f(x) \cdot f(x)\,dx$

$$= \int_0^1 [f(x)]^2\,dx$$

$$\geq 0$$

Also $\langle f,f\rangle = 0 \Leftrightarrow \int_0^1 f(x)\,f(x)\,dx = 0$

$$\Leftrightarrow \int_0^1 [f(x)]^2\,dx = 0$$

$$\Leftrightarrow f(x) = 0$$

$$\Leftrightarrow f = 0$$

$\therefore \langle\, ,\rangle$ is an inner product in V and hence $(V, \langle\, ,\rangle)$ is an inner product space.

***

Decide which of the following functions on $\mathbb{R}^2$ are inner products and which are not.

For $\bar{x} = (x_1, x_2), \bar{y} = (y_1, y_2)$ in $\mathbb{R}^2$

i) $\langle \bar{x}, \bar{y} \rangle = x_1 y_1 x_2 y_2$

ii) $\langle \bar{x}, \bar{y} \rangle = 4x_1 y_1 + 4x_2 y_2 - x_1 y_2 - x_2 y_1$

iii) $\langle \bar{x}, \bar{y} \rangle = x_1 y_1 - x_2 y_1$

iv) $\langle \bar{x}, \bar{y} \rangle = x_1 y_1 + 3x_2 y_2$

v) $\langle \bar{x}, \bar{y} \rangle = x_1 y_1 - x_1 y_2 - x_2 y_1 + 3x_2 y_2$

i) $\langle \bar{x}, \bar{y} \rangle = x_1 y_1 x_2 y_2$

$\bar{x} = (x_1, x_2) \quad \bar{y} = (y_1, y_2) \quad \bar{z} = (z_1, z_2)$

i) $\langle \bar{x}, \bar{y} \rangle = \langle (x_1, x_2), (y_1, y_2) \rangle$

$= x_1 y_1 x_2 y_2$

$= y_1 x_1 y_2 x_2$

$= \langle (y_1, y_2), (x_1, x_2) \rangle$

$= \langle \bar{y}, \bar{x} \rangle$

ii) $\langle \bar{x}+\bar{y}, \bar{z} \rangle = \langle (x_1, x_2) + (y_1, y_2), (z_1, z_2) \rangle$

$= \langle (x_1 + y_1, x_2 + y_2), (z_1, z_2) \rangle$

$= (x_1 + y_1) z_1 \cdot (x_2 + y_2) z_2$

$= (x_1 z_1 + y_1 z_1)(x_2 z_2 + y_2 z_2)$

$= x_1 z_1 x_2 z_2 + x_1 z_1 y_2 z_2 + y_1 z_1 x_2 z_2 + y_1 z_1 y_2 z_2 \quad — (1)$

New $\langle \bar{x}, \bar{z} \rangle + \langle \bar{y}, \bar{z} \rangle = \langle (x_1, x_2), (z_1, z_2) \rangle + \langle (y_1, y_2), (z_1, z_2) \rangle$

$= x_1 z_1 x_2 z_2 + y_1 z_1 y_2 z_2$ —— ②

From equ ① & ②

$\langle \bar{x}+\bar{y}, \bar{z}\rangle \neq \langle \bar{x}, \bar{z}\rangle + \langle \bar{y}, \bar{z}\rangle$

$\therefore \langle \bar{x}, \bar{y}\rangle = x_1 y_1 x_2 y_2$ is not an inner product.

ii) $\langle \bar{x}, \bar{y}\rangle = 4x_1y_1 + 4x_2y_2 - x_1y_2 - x_2y_1$

$\bar{x} = (x_1, x_2)$ $\bar{y} = (y_1, y_2)$ $\bar{z} = (z_1, z_2)$

i) $\langle \bar{x}, \bar{y}\rangle = \langle (x_1, x_2), (y_1, y_2)\rangle$

$= 4x_1y_1 + 4x_2y_2 - x_1y_2 - x_2y_1$

$= 4y_1x_1 + 4y_2x_2 - y_2x_1 - y_1x_2$

$= \langle (y_1, y_2), (x_1, x_2)\rangle$

$= \langle \bar{y}, \bar{x}\rangle$

ii) $\langle \bar{x}+\bar{y}, \bar{z}\rangle = \langle (x_1, x_2) + (y_1, y_2), (z_1, z_2)\rangle$

$= \langle (x_1+y_1, x_2+y_2), (z_1, z_2)\rangle$

$= 4(x_1+y_1)z_1 + 4(x_2+y_2)z_2 - (x_1+y_1)z_2 - (x_2+y_2)z_1$

$= 4x_1z_1 + 4y_1z_1 + 4x_2z_2 + 4y_2z_2 - x_1z_2 - y_1z_2 - x_2z_1 - y_2z_1$ —— ①

$\langle \bar{x}, \bar{z}\rangle + \langle \bar{y}, \bar{z}\rangle = \langle (x_1, x_2), (z_1, z_2)\rangle + \langle (y_1, y_2), (z_1, z_2)\rangle$

$= 4x_1z_1 + 4x_2z_2 - x_1z_2 - x_2z_1 + 4y_1z_1 + 4y_2z_2 - y_1z_2 - y_2z_1$ ←— ②

$\langle \bar{x}+\bar{y}, \bar{z}\rangle = \langle \bar{x}, \bar{z}\rangle + \langle \bar{y}, \bar{z}\rangle$

①

iii) $\langle K\bar{x},\bar{y}\rangle = K\langle \bar{x},\bar{y}\rangle$

$$\langle K\bar{x},\bar{y}\rangle = \langle K(x_1,x_2),(y_1,y_2)\rangle$$
$$= \langle (Kx_1,Kx_2),(y_1,y_2)\rangle$$
$$= 4Kx_1y_1+4Kx_2y_2-Kx_1y_2-Kx_2y_1$$
$$= K(4x_1y_1+4x_2y_2-x_1y_2-x_2y_1)$$
$$= K\langle (x_1,x_2),(y_1,y_2)\rangle$$
$$= K\langle \bar{x},\bar{y}\rangle$$

iv) $\langle \bar{x},\bar{x}\rangle = \langle (x_1,x_2),(x_1,x_2)\rangle$

$$= 4x_1x_1+4x_2x_2-x_1x_2-x_2x_1$$
$$= 4x_1^2+4x_2^2-2x_1x_2 \geq 0$$

Also $\langle \bar{x},\bar{x}\rangle = 0 \Leftrightarrow \langle (x_1,x_2),(x_1,x_2)\rangle = 0$

$$\Leftrightarrow 4x_1^2+4x_2^2-2x_1x_2=0$$
$$\Leftrightarrow x_1=0 \ \& \ x_2=0$$
$$\Leftrightarrow (x_1,x_2)=(0,0)$$
$$\Leftrightarrow \bar{x}=\bar{0}$$

$\therefore \langle \bar{x},\bar{y}\rangle = 4x_1y_1+4x_2y_2-x_1y_2-x_2y_1$ is an inner product.

iii) $\langle \bar{x},\bar{y}\rangle = x_1y_1-x_2y_1$

$\bar{x}=(x_1,x_2)$ $\quad \bar{y}=(y_1,y_2)$ $\quad \bar{z}=(z_1,z_2)$

i) $\langle \bar{x},\bar{y}\rangle = \langle (x_1,x_2),(y_1,y_2)\rangle$

$$= x_1y_1-x_2y_1$$
$$= y_1x_1-y_1x_2$$
$$= \langle (y_1,y_2),(x_1,x_2)\rangle$$

$$\langle \bar{x}, \bar{y} \rangle \neq \langle \bar{y}, \bar{x} \rangle$$

Hence $\langle \bar{x}, \bar{y} \rangle = x_1y_1 - x_2y_1$ is not an inner product

iv) $\langle \bar{x}, \bar{y} \rangle = x_1y_1 + 3x_2y_2$

i) $\langle \bar{x}, \bar{y} \rangle = \langle (x_1, x_2), (y_1, y_2) \rangle$

$= x_1y_1 + 3x_2y_2$

$= y_1x_1 + 3y_2x_2$

$= \langle (y_1, y_2), (x_1, x_2) \rangle$

$= \langle \bar{y}, \bar{x} \rangle$

ii) $\langle \bar{x}+\bar{y}, \bar{z} \rangle = \langle (x_1, x_2) + (y_1, y_2), (z_1, z_2) \rangle$

$= \langle (x_1+y_1, x_2+y_2), (z_1, z_2) \rangle$

$= (x_1+y_1)z_1 + 3(x_2+y_2)z_2$

$= x_1z_1 + y_1z_1 + 3x_2z_2 + 3y_2z_2$ —— ①

$\langle \bar{x}, \bar{z} \rangle + \langle \bar{y}, \bar{z} \rangle = \langle (x_1, x_2), (z_1, z_2) \rangle +$

$\langle (y_1, y_2), (z_1, z_2) \rangle$

$= x_1z_1 + 3x_2z_2 + y_1z_1 + 3y_2z_2$

iii) $\langle k\bar{x}, \bar{y} \rangle = \langle k(x_1, x_2), (y_1, y_2) \rangle$

$= \langle (kx_1, kx_2), (y_1, y_2) \rangle$

$= kx_1y_1 + 3kx_2y_2$

$= k(x_1y_1 + 3x_2y_2)$

$= k\langle (x_1, x_2), (y_1, y_2) \rangle$

$= k\langle \bar{x}, \bar{y} \rangle$

iv) $\langle \bar{x}, \bar{x} \rangle = \langle (x_1, x_2), (x_1, x_2) \rangle$

$= x_1x_1 + 3x_2x_2$

$= x_1^2 + 3x_2^2 \geq 0$

Also $\langle \bar{x}, \bar{x} \rangle = 0 \Leftrightarrow \langle (x_1, x_2), (x_1, x_2) \rangle = 0$

$$\Leftrightarrow x_1^2 + 3x_2^2 = 0$$

$$\Leftrightarrow x_1 = 0 \text{ \& } x_2 = 0$$

$$\Leftrightarrow (x_1, x_2) = (0, 0)$$

$$\Leftrightarrow \bar{x} = \bar{0}$$

$\therefore \langle \bar{x}, \bar{y} \rangle = x_1y_1 + 3x_2y_2$ is an inner product

v) $\langle \bar{x}, \bar{y} \rangle = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$

i) $\langle \bar{x}, \bar{y} \rangle = \langle (x_1, x_2), (y_1, y_2) \rangle$

$$= x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$$

$$= y_1x_1 - y_2x_1 - y_1x_2 + 3y_2x_2$$

$$= \langle (y_1, y_2), (x_1, x_2) \rangle$$

$$= \langle \bar{y}, \bar{x} \rangle$$

ii) $\langle \bar{x} + \bar{y}, \bar{z} \rangle = \langle (x_1, x_2) + (y_1, y_2), (z_1, z_2) \rangle$

$$= \langle (x_1 + y_1, x_2 + y_2), (z_1, z_2) \rangle$$

$$= (x_1 + y_1)z_1 - (x_1 + y_1)z_2 - (x_2 + y_2)z_1 + 3(x_2 + y_2)z_2$$

$$= x_1z_1 + y_1z_1 - x_1z_2 - y_1z_2 - x_2z_1 - y_2z_1 + 3x_2z_2 + 3y_2z_2 \quad \text{--- (1)}$$

$\langle \bar{x}, \bar{z} \rangle + \langle \bar{y}, \bar{z} \rangle = \langle (x_1, x_2), (z_1, z_2) \rangle + \langle (y_1, y_2), (z_1, z_2) \rangle$

$$= x_1z_1 - x_1z_2 - x_2z_1 + 3x_2z_2 + y_1z_1 - y_1z_2 - y_2z_1 + 3y_2z_2 \quad \text{--- (2)}$$

$$\langle \bar{x} + \bar{y}, \bar{z} \rangle = \langle \bar{x}, \bar{z} \rangle + \langle \bar{y}, \bar{z} \rangle$$

iii) $\langle k\bar{x}, \bar{y} \rangle =$

$\langle k\bar{x}, \bar{y} \rangle =$

iv) $\langle \bar{x}, \bar{x} \rangle =$

Also $\langle \bar{x}, \bar{x}$

$\therefore \langle \bar{x}, \bar{y}$

iii) $\langle K\bar{x}, \bar{y} \rangle = K\langle \bar{x}, \bar{y} \rangle$

$$\langle K\bar{x}, \bar{y} \rangle = \langle K(x_1, x_2), (y_1, y_2) \rangle$$
$$= \langle (Kx_1, Kx_2), (y_1, y_2) \rangle$$
$$= Kx_1y_1 - Kx_1y_2 - Kx_2y_1 + 3Kx_2y_2$$
$$= K(x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2)$$
$$= K\langle (x_1, x_2), (y_1, y_2) \rangle$$
$$= K\langle \bar{x}, \bar{y} \rangle$$

iv) $\langle \bar{x}, \bar{x} \rangle = \langle (x_1, x_2), (x_1, x_2) \rangle$

$$= x_1x_1 - x_1x_2 - x_2x_1 + 3x_2x_2$$
$$= x_1^2 - 2x_1x_2 + 3x_2^2 \geq 0$$

Also $\langle \bar{x}, \bar{x} \rangle = 0 \Leftrightarrow \langle (x_1, x_2), (x_1, x_2) \rangle = 0$

$$\Leftrightarrow x_1^2 - 2x_1x_2 + 3x_2^2 = 0$$
$$\Leftrightarrow x_1 = 0 \ \& \ x_2 = 0$$
$$\Leftrightarrow (x_1, x_2) = (0, 0)$$
$$\Leftrightarrow \bar{x} = \bar{0}$$

$\therefore \langle \bar{x}, \bar{y} \rangle = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$ is an inner product.

Cauchy-Schwarz inequality :-

Statement :-

If $\bar{x}, \bar{y}$ are vectors in an inner product space V, then $\langle \bar{x}, \bar{y} \rangle^2 \leq \langle \bar{x}, \bar{x} \rangle \langle \bar{y}, \bar{y} \rangle$ (or)

$$\langle \bar{x}, \bar{y} \rangle \leq \|\bar{x}\| \, \|\bar{y}\|$$

Proof :-

Let $\bar{x}, \bar{y}$ are two vectors in V.

Then to prove that

$$\langle \bar{x}, \bar{y} \rangle^2 \leq \langle \bar{x}, \bar{x} \rangle \langle \bar{y}, \bar{y} \rangle$$

Suppose $\bar{x} = \bar{0}$, then the result is trivial.

Assume $\bar{x} \neq \bar{0}$ for any scalar t, we have

$$\langle t\bar{x} + \bar{y}, t\bar{x} + \bar{y} \rangle \geq 0$$

$$\Rightarrow \langle \bar{x}, \bar{x} \rangle t^2 + 2t \langle \bar{x}, \bar{y} \rangle + \langle \bar{y}, \bar{y} \rangle \geq 0,$$

This inequality implies that the polynomial $\langle \bar{x}, \bar{x} \rangle t^2 + 2\langle \bar{x}, \bar{y} \rangle t + \langle \bar{y}, \bar{y} \rangle$ in t, has either no real roots or a repeated real root.

Therefore, its discriminant ($b^2 - 4ac$) must be non-positive.

$$4\langle \bar{x}, \bar{y} \rangle^2 - 4\langle \bar{x}, \bar{x} \rangle \langle \bar{y}, \bar{y} \rangle \leq 0.$$

÷ by 4

$$\Rightarrow \langle \bar{x}, \bar{y} \rangle^2 \leq \langle \bar{x}, \bar{x} \rangle \langle \bar{y}, \bar{y} \rangle \text{ (or)}$$

$$\langle \bar{x}, \bar{y} \rangle^2 \leq \|\bar{x}\|^2 \|\bar{y}\|^2$$

$$\langle \bar{x}, \bar{y} \rangle \leq \|\bar{x}\| \|\bar{y}\|$$

1. In $\mathbb{R}^2$, equipped with an inner product $\langle \bar{x}, \bar{y} \rangle$ $\langle \bar{x}, \bar{y} \rangle = 2x_1y_1 + 3x_2y_2$ find the angle between $\bar{x} = (1,1)$ $\bar{y} = (1,0)$.

The given inner product is

$$\langle \bar{x}, \bar{y} \rangle = 2x_1y_1 + 3x_2y_2$$

i.e., $\langle (x_1, x_2), (y_1, y_2) \rangle = 2x_1y_1 + 3x_2y_2$

We know that

$$\cos\theta = \frac{\langle \bar{x}, \bar{y} \rangle}{\|\bar{x}\| \|\bar{y}\|}$$

$$\text{Now } \langle \bar{x}, \bar{y} \rangle = \langle (1,1), (1,0) \rangle$$

$$= 2(1)(1) + 3(1)(0) = 2$$

$$\langle \bar{x}, \bar{x} \rangle = \langle (1,1), (1,1) \rangle$$

$$= 2(1)(1) + 3(1)(1)$$

$$= 2 + 3 = 5$$

$$\|\bar{x}\|^2 = 5$$

$$\|\bar{x}\| = \sqrt{5}$$

$$\langle \bar{y}, \bar{y} \rangle = \langle (1,0), (1,0) \rangle$$

$$= 2(1)(1) + 3(0)(0) = 2$$

$$\|\bar{y}\|^2 = 2$$

$$\|\bar{y}\| = \sqrt{2}$$

$$\text{Hence } \cos\theta = \frac{2}{\sqrt{5}\sqrt{2}} = \frac{2}{\sqrt{10}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{10}}\right)$$

2. In $\mathbb{R}^2$, equipped with an inner product $\langle \bar{x}, \bar{y} \rangle = x_1y_1 + x_2y_2$ find the angle between $\bar{x} = (1,1)$ & $\bar{y} = (1,0)$

The given inner product is

$$\langle \bar{x}, \bar{y} \rangle = x_1y_1 + x_2y_2$$

i.e., $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1y_1 + x_2y_2$

We know that $\cos\theta = \dfrac{\langle \bar{x}, \bar{y} \rangle}{\|\bar{x}\| \|\bar{y}\|}$

Now $\langle \bar{x}, \bar{y} \rangle = \langle (1,1), (1,0) \rangle$

$$= (1)(1) + (1)(0)$$
$$= 1$$

$$\langle \bar{x}, \bar{x} \rangle = \langle (1,1), (1,1) \rangle$$
$$= (1)(1) + (1)(1)$$
$$= 2$$

$\|\bar{x}\|^2 = 2$

$\|\bar{x}\| = \sqrt{2}$

$$\langle \bar{y}, \bar{y} \rangle = \langle (1,0), (1,0) \rangle$$
$$= (1)(1) + (0)(0)$$
$$= 1$$

$\|\bar{y}\|^2 = 1$

$\|\bar{y}\| = 1$

Hence $\cos\theta = \dfrac{1}{\sqrt{2} \cdot 1} = \dfrac{1}{\sqrt{2}}$

$\theta = \cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$ $\quad \theta = \dfrac{\pi}{4} = 45^\circ$

3. Let V= C

real valued

with the

<f

4. For the

compute

numbers

(i) f

(ii)

(iii)

(i) <f

<f,

Theorem:- A vector space V is the direct sum of subspaces U and W, i.e., $V = U \oplus W$, if and only if for any $\bar{v} \in V$ there exist unique $\bar{u} \in U$ and $\bar{w} \in W$ such that $\bar{v} = \bar{u} + \bar{w}$

Proof:-

Let V be a vector space and U, W are subspaces of V.

Then to prove that $V = U \oplus W$

$V = U \oplus W \Leftrightarrow$ for any $\bar{v} \in V$ there exist unique $\bar{u} \in U$ and $\bar{w} \in W$ such that $\bar{v} = \bar{u} + \bar{w}$

First Suppose, $V = U \oplus W$

By the definition of direct sum, we have

$$V = U + W \text{ and } U \cap W = \{\bar{0}\}$$

$$U + W = \{\bar{u} + \bar{w} : \bar{u} \in U \text{ and } \bar{w} \in W\}$$

Since $V = U + W$

$\therefore$ For any $\bar{v} \in V$ such that

~~$\bar{v} \in V$ such that~~

$\bar{v} = \bar{u} + \bar{w}$ where $\bar{u} \in U, \bar{w} \in W$

To prove $\bar{v} = \bar{u} + \bar{w}$ is unique

For this assume $\bar{v} = \bar{u}_1 + \bar{w}_1$

Then $\bar{u} + \bar{w} = \bar{u}_1 + \bar{w}_1$

$\Rightarrow \bar{u} - \bar{u}_1 = \bar{w}_1 - \bar{w} \in U \cap W$

Since $U \cap W = \{\bar{0}\}$

$\therefore\ \bar{u}-\bar{u}_1=\bar{0}$ and $\bar{w}_1-\bar{w}=\bar{0}$

$\Rightarrow \bar{u}=\bar{u}_1$ and $\bar{w}=\bar{w}_1$

$\therefore\ \bar{v}=\bar{u}+\bar{w}$ is unique

Conversely, suppose for any $\bar{v}\in V$ there exists unique $\bar{u}\in U$ and $\bar{w}\in W$ such that $\bar{v}=\bar{u}+\bar{w}$

Then to prove that, $V=U\oplus W$

Since for any $\bar{v}\in V$, there exists $\bar{u}\in U$ and $\bar{w}\in W$ such that $\bar{v}=\bar{u}+\bar{w}$

$$V=U+W$$

Next to prove, $U\cap W=\{\bar{0}\}$

For this let $U\cap W$ consisting a non zero vector say $\bar{v}_1$

Then $\bar{v}_1=\bar{0}+\bar{v}_1=\bar{v}_1+\bar{0}=\frac{1}{2}\bar{v}_1+\frac{1}{2}\bar{v}_1=\frac{1}{3}v_1+\frac{2}{3}v_1=\dots$

i.e., $\bar{v}_1$ can be expressed in different ways, which is a contradiction to for any $\bar{v}\in V$, $\exists$ unique $\bar{u}\in U$ and $\bar{w}\in W$ such that

$$\bar{v}=\bar{u}+\bar{w}$$

so, our assumption $U\cap W$ consisting a non-zero vector is wrong.

$$\therefore\ U\cap W=\{\bar{0}\}$$

Hence $V=U\oplus W$

$\therefore$ The condition is sufficient

** Example:-

3) Give any example for sum but not direct sum.

Let $U = \{a\bar{i} + c\bar{k} : a, c \in R\}$ and

$W = \{b\bar{j} + c\bar{k} : b, c \in R\}$

Then U and W are subspaces of R^3 and a vector in $U+W$ is of the form

$$(a\bar{i} + c_1\bar{k}) + (b\bar{j} + c_2\bar{k}) = a\bar{i} + b\bar{j} + c_1\bar{k} + c_2\bar{k}$$

$$= a\bar{i} + b\bar{j} + (c_1 + c_2)\bar{k}$$

$$= a\bar{i} + b\bar{j} + c\bar{k} \text{ where } c = c_1 + c_2$$

i.e., $U + W = R^3$

Since for $\bar{k} \in U \cap W$ is not zero vector, because

$$\bar{k} = \frac{1}{2}\bar{k} + \frac{1}{2}\bar{k} = \frac{1}{3}\bar{k} + \frac{2}{3}\bar{k}$$

$\therefore U \oplus W \neq \mathbb{R}^3$

Bases :-

Definition:-

Let V be a vector space and let $\{\bar{x}_1, \bar{x}_2, \ldots \bar{x}_m\}$ be a set of vectors in V, then a vector $\bar{y}$ in V of the form

$$\bar{y} = a_1\bar{x}_1 + a_2\bar{x}_2 + \ldots a_m\bar{x}_m$$ is called the

linear combination of the vectors $\bar{x}_1, \bar{x}_2, \ldots \bar{x}_m$, where $a_1, a_2, \ldots a_m$ are scalars.

Definition:-

A set of vectors $\{\bar{x}_1, \bar{x}_2, \ldots \bar{x}_m\}$ in a vector space V is said to be linearly independent if the vector equation, called the linear dependence of $\bar{x}_i$'s,

$$c_1\bar{x}_1 + c_2\bar{x}_2 + \ldots c_m\bar{x}_m = \bar{0}$$

has only the trivial solution $c_1=0, c_2=0, \ldots c_m=0$. Otherwise it is said to be linearly dependent

Example :-

1. $(x_1, x_2, x_3) = x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$

2. $(1,2,3) = 1(1,0,0) + 2(0,1,0) + (-3)(0,0,1)$

3. Express the given functions as a linear combination of functions in the given set Q

i) $P(x) = -1-3x+3x^2$ & $Q(x) = \{P_1(x), P_2(x), P_3(x)\}$ where $P_1(x) = 1+2x+x^2$, $P_2(x) = 2+5x$, $P_3(x) = 3+8x-2x^2$

ii) $P(x) = -2-4x+x^2$ & $Q = \{P_1(x), P_2(x), P_3(x), P_4(x)\}$ where $P_1(x) = 1+2x^2+x^3$, $P_2(x) = 1+x+2x^3$, $P_3(x) = -1-3x-4x^3$, $P_4(x) = 1+2x-x^2+x^3$

Consider $P(x) = a_1P_1(x) + a_2P_2(x) + a_3P_3(x)$

$$\Rightarrow -1-3x+3x^2 = a_1\{1+2x+x^2\} + a_2\{2+5x\} + a_3\{3+8x-2x^2\}$$

$$-1-3x+3x^2 = a_1 + a_1 2x + a_1x^2 + 2a_2 + 5a_2x + 3a_3 + 8a_3x - 2a_3x^2$$

$$-1-3x+3x^2 = (a_1+2a_2+3a_3) + (2a_1+5a_2+8a_3)x + (a_1-2a_3)x^2$$

Compare like like terms on both sides, we get

$$a_1 + 2a_2 + 3a_3 = -1 \quad \text{—— (1)}$$

$$2a_1 + 5a_2 + 8a_3 = -3 \quad \text{—— (2)}$$

$$a_1 - 2a_3 = 3 \quad \text{—— (3)}$$

$$(1) \times 5 \Rightarrow 5a_1 + \cancel{10a_2} + 15a_3 = -5$$

$$(2) \times 2 \Rightarrow 4a_1 + 10a_2 + 16a_3 = -6$$

$$a_1 \quad - a_3 = 1 \quad \text{—— (4)}$$

$$(3) \Rightarrow \cancel{a_1} - 2a_3 = 3$$

$$(4) \Rightarrow a_1 - a_3 = 1$$

$$-a_3 = 2$$

$$\boxed{a_3 = -2} \text{ in equ (3)}$$

$$a_1 + 4 = 3$$

$$\boxed{a_1 = -1}$$

$$a_1 = -1, a_3 = -2 \text{ in equ (1)}$$

$$-1 + 2a_2 - 6 = -1$$

$$2a_2 - 7 = -1$$

$$2a_2 = 6$$

$$\boxed{a_2 = 3}$$

$$a_1 = -1, \; a_2 = 3, \; a_3 = -2$$

$$-1 - 3x + 3x^2 = -1\{1 + 2x + x^2\} + 3\{2 + 5x\} - 2\{3 + 8x - 2x^2\}$$

1. Is $\{\cos^2 x, \sin^2 x, 1, e^x\}$ linearly independent in the vector space $C(\mathbb{R})$?

$$\begin{vmatrix} \cos^2 x & \sin^2 x & 1 & e^x \\ -\sin 2x & \sin 2x & 0 & e^x \\ -2\cos 2x & 2\cos 2x & 0 & e^x \\ 4\sin 2x & -4\sin 2x & 0 & e^x \end{vmatrix} \quad C_1 \to C_1 + C_2$$

$$= \begin{vmatrix} 1 & \sin^2 x & 1 & e^x \\ 0 & \sin 2x & 0 & e^x \\ 0 & 2\cos 2x & 0 & e^x \\ 0 & -4\sin 2x & 0 & e^x \end{vmatrix}$$

$= 0$

$\therefore \{\cos^2 x, \sin^2 x, 1, e^x\}$ is linearly dependent.

2. Verify the given sets of functions are linearly independent in the vector space $C[-\pi, \pi]$

i) $\{1, x, x^2, x^3, x^4\}$

ii) $\{1, e^x, e^{2x}, e^{3x}\}$

iii) $\{\bar{V}_1, \bar{V}_2, \bar{V}_3, \bar{V}_4\}$ where $\bar{V}_1 = (1,1,2,4)$, $\bar{V}_2 = (2,-1,-5,2)$, $\bar{V}_3 = (1,-1,-4,0)$, $\bar{V}_4 = (2,1,1,6)$

$+8^2x - 2x^2\}$

iv) $\{x, \cos x, \sin x\}$ L.I

v) $\{x, e^x, e^{-x}\}$ L.I

vi) $\{x|x|, x^2\}$ L.D

vii) $\{(1,2,3), (3,2,1)\}$

i). $a_1 \cdot 1 + a_2 \cdot x + a_3 \cdot x^2 + a_4 \cdot x^3 + a_5 x^4 = \bar{0}$

$a_1 \cdot 1 + a_2 \cdot x + a_3 \cdot x^2 + a_4 \cdot x^3 + a_5 x^4 = 0.1 + 0.x + 0.x^2 + 0.x^3 + 0.x^4$

$a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0$. $\therefore$ It is linearly independent

iii) Let $a_1\bar{V}_1 + a_2\bar{V}_2 + a_3\bar{V}_3 + a_4\bar{V}_4 = \bar{0}$

Then $a_1(1,1,2,4) + a_2(2,-1,-5,2) + a_3(1,-1,-4,0) + a_4(2,1,1,6) = (0,0,0,0)$

$\Rightarrow (a_1, a_1, 2a_1, 4a_1) + (2a_2, -a_2, -5a_2, 2a_2) + (a_3, -a_3, -4a_3, 0) + (2a_4, a_4, a_4, 6a_4) = (0,0,0,0)$

$\Rightarrow (a_1 + 2a_2 + a_3 + 2a_4,\ a_1 - a_2 - a_3 + a_4,\ 2a_1 - 5a_2 - 4a_3 + a_4,\ 4a_1 + 2a_2 + 0 + 6a_4) = (0,0,0,0)$

$\Rightarrow a_1 + 2a_2 + a_3 + 2a_4 = 0$ —— (1)

$a_1 - a_2 - a_3 + a_4 = 0$ —— (2)

$2a_1 - 5a_2 - 4a_3 + a_4 = 0$ —— (3)

$4a_1 + 2a_2 + 6a_4 = 0$ —— (4)

(1) $\Rightarrow a_1 + 2a_2 + \cancel{a_3} + 2a_4 = 0$

(2) $\Rightarrow a_1 - a_2 - \cancel{a_3} + a_4 = 0$

$2a_1 + a_2 + 3a_4 = 0$ —— (5)

(4) $\Rightarrow 4a_1 + \cancel{2a_2} + 6a_4 = 0$

(5)$\times 2$ $\Rightarrow 4a_1 + \cancel{2a_2} + 6a_4 = 0$

$8a_1 + 12a_4 = 0$ —— (6)

(1) $\times 2 \Rightarrow 2a_1 + 2a_2 + 2a_3 + 4a_4 = 0$

(3) $\Rightarrow 2a_1 - 5a_2 - 4a_3 + a_4 = 0$

$\quad - \quad + \quad + \quad -$

$7a_2 + 6a_3 + 3a_4 = 0$ —— (6)

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 4R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & -9 & -6 & -3 \\ 0 & -6 & -4 & -2 \end{bmatrix} \quad R_3 \to R_3 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -6 & -4 & -2 \end{bmatrix} \quad R_4 \to R_4 - 2R_2 \quad \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ It has one zero row it is linearly dependent.

$$① \Rightarrow a_1 + 2a_2 + a_3 + 2a_4 = 0$$

$$② \Rightarrow a_1 - a_2 - a_3 + a_4 = 0$$

$$2a_1 + a_2 + 3a_4 = 0 \quad \text{——} ⑤$$

$$④ \Rightarrow 4a_1 + 2a_2 + 6a_4 = 0$$

$$⑤ \times 2 \Rightarrow 4a_1 + 2a_2 + 6a_4 = 0$$

$\cancel{8a_1}$

Since the two equations give answer zero it is linearly dependent

(4) $\Rightarrow$ $-a_2+3a_3=3$

$a_2+2a_3=7$

$5a_3=10$

$\boxed{a_3=2}$

(4) $\Rightarrow$ $-a_2+3(2)=3$

$-a_2+6=3$

$-a_2=3-6$

$\boxed{a_2=3}$

(1) $\Rightarrow$ $a_1+3+2(2)=1$

$a_1+7=1$

$\boxed{a_1=-6}$

$\alpha=(-6)e_1+(3)e_2+(2)e_3$

3. Express $\alpha = (1,-2,5)$ as a linear combination of the vectors $e_1 = (1,1,1)$, $e_2 = (1,2,3)$, $e_3 = (2,-1,1)$

Let $\alpha = a_1e_1 + a_2e_2 + a_3e_3$

Then $(1,-2,5) = a_1(1,1,1) + a_2(1,2,3) + a_3(2,-1,1)$

$= (a_1, a_1, a_1) + (a_2, 2a_2, 3a_2) + (2a_3, -a_3, a_3)$

$(1,-2,5) = (a_1+a_2+2a_3), (a_1+2a_2-a_3), (a_1+3a_2+a_3)$

$\Rightarrow a_1+a_2+2a_3 = 1$ —— ①

$a_1+2a_2-a_3 = -2$ —— ②

$a_1+3a_2+a_3 = 5$ —— ③

① $\Rightarrow a_1+a_2+2a_3 = 1$

② $\Rightarrow a_1+2a_2-a_3 = -2$

$\quad - \quad - \quad + \quad (+)$

$-a_2+3a_3 = 3$ —— ④

① $\Rightarrow a_1+a_2+2a_3 = 1$

③ $\Rightarrow a_1+3a_2+a_3 = 5$

$\quad - \quad - \quad - \quad -$

$-2a_2+a_3 = -4$ —— ⑤

④ $\Rightarrow -a_2+a_3 = 3$

⑤ $\Rightarrow -2a_2+a_3 = -4$

$\quad (+) \quad (-) \quad (+)$

$a_2 = 7$

② $\Rightarrow a_1+2a_2-a_3 = -2$

③ $\Rightarrow a_1+3a_2+a_3 = 5$

$-a_2-2a_3 = -7$ —— ⑤

4. Is the vector $\alpha = (2,-5,3)$ can be expressed as a linear combination of the vectors $e_1 = (1,-3,2)$, $e_2 = (2,-4,-1)$, $e_3 = (1,-5,-7)$

Let $\alpha = a_1e_1 + a_2e_2 + a_3e_3$

Then $(2,-5,3) = a_1(1,-3,2) + a_2(2,-4,-1) + a_3(1,-5,-7)$

$= (a_1,-3a_1,2a_1) + (2a_2,-4a_2,-a_2) + (a_3,-5a_3,-7a_3)$

$(2,-5,3) = (a_1+2a_2+a_3), (-3a_1-4a_2-5a_3), (2a_1-a_2-7a_3)$

$a_1 + 2a_2 + a_3 = 2$

$-3a_1 - 4a_2 - 5a_3 = -5$

$2a_1 - a_2 - 7a_3 = 3$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ -3 & -4 & -5 & -5 \\ 2 & -1 & -7 & 3 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & -5 & -9 & -1 \end{bmatrix} \quad R_3 \to 2R_3 + 5R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & -28 & 3 \end{bmatrix}$$

$\Rightarrow a_1 + 2a_2 + a_3 = 2$ —— (1)

$2a_2 - 2a_3 = 1$ —— (2)

$-28a_3 = 3$ —— (3)

(3) $\Rightarrow$ $\boxed{a_3 = \frac{-3}{28}}$

(2) $\Rightarrow$ $2a_2 - \cancel{2}\left(\frac{-3}{\cancel{28}_{14}}\right) = 1$

linear

-1)

-5,-7)

$-a_3, -7a_3)$

$a_3 - 7a_3)$

$$2a_2 + \frac{3}{14} = 1$$

$1 - \frac{3}{14}$

$$2a_2 = \frac{11}{14}$$

$$\boxed{a_2 = \frac{11}{28}}$$

$$① \Rightarrow \quad a_1 + 2\left(\frac{11}{28}\right) - \frac{3}{28} = 2$$

$$a_1 + \frac{22}{28} - \frac{3}{28} = 2$$

$2 - \frac{19}{28}$

$$a_1 + \frac{19}{28} = 2.$$

$56 - 19$

37

$$\boxed{a_1 = \frac{37}{28}}$$

$$\alpha = \left(\frac{37}{28}\right)e_1 + \left(\frac{11}{28}\right)e_2 + \left(\frac{-3}{28}\right)e_3$$

NOTE :-

i) A set of vectors is linearly dependent if and only if atleast one of the vectors in the set can be written as a linear combination of the others.

ii) The non zero rows of a matrix in row-echelon form are linearly independent, and so are the columns that containing leading 1's.

iii) If $n > m$, any set of n vectors in m-space $\mathbb{R}^m$ is linearly dependent.

iv) If U is the reduced row-echelon form of A, then the columns of A are linearly independent if and only if the columns of A are linearly independent

Span :- Let $\bar{x}_1, \bar{x}_2, \dots \bar{x}_m$ be vectors in a vector space V. Then the set $W = \{a_1\bar{x}_1 + a_2\bar{x}_2 + \dots + a_m\bar{x}_m\}$ of all linear combinations of $\bar{x}_1, \bar{x}_2, \dots \bar{x}_m$ is a subspace of V. It is called the subspace of V spanned by $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m$ or $\bar{x}_1, \bar{x}_2, \dots \bar{x}_m$ <u>span</u> the subspace W.

<u>Basis</u> :-

Let V be a vector space. A <u>basis</u> for V is a set of linearly independent vectors that spans V.

For example,

the set $\{\bar{e}_1, \bar{e}_2, \dots \bar{e}_n\}$ is a standard basis for the n-space $\mathbb{R}^n$. $\underline{L(S) = V}$

3M- It will be dependent

<u>Example</u> :-

8M 1. Verify the set of vectors (1,1,0) (0,-1,1) and (1,0,1) is a basis or not for $\mathbb{R}^3$.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$= 1(-1) - 1(-1)$

$= -1 + 1$

$= 0$

$\therefore$ The given vectors are linearly dependent & hence it does not form a basis for $\mathbb{R}^3$.

2. Verify the set of vectors $\{(1,1,1), (0,1,1), (0,0,1)\}$ are form a basis or not for $\mathbb{R}^3$

Let $S = \{(1,1,1), (0,1,1), (0,0,1)\}$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$\{(1,1,0), (0,-1,1), (1,0,1)\}$

$= 1(1-0)$

$= 1 \neq 0$

$\therefore$ S is linearly independent

Let $(x,y,z) \in \mathbb{R}^3$

Then $(x,y,z) = a(1,1,1) + b(0,1,1) + c(0,0,1)$

$= (a,a,a) + (0,b,b) + (0,0,c)$

$= (a, a+b, a+b+c)$

$\Rightarrow a = x, \ a+b = y, \ a+b+c = z$

$b = y - a \qquad c = z - a - b$

$b = y - x \qquad = z - x - (y-x)$

$= z - x - y + x$

$c = z - y$

$\therefore (x,y,z) = x(1,1,1) + (y-x)(0,1,1) + (z-y)(0,0,1)$

$L(S) = V$

Hence S is a basis for $\mathbb{R}^3$

***3. Show that the vectors $\bar{V}_1 = (1,2,1)$, $\bar{V}_2 = (2,9,0)$ & $\bar{V}_3 = (3,3,4)$ in the 3-space $\mathbb{R}^3$ form a basis

Let $S = \{(1,2,1), (2,9,0), (3,3,4)\}$

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 9 & 0 \\ 3 & 3 & 4 \end{vmatrix} = 1(36-0) - 2(8-0) + 1(6-27)$$

$$= 36 - 16 - 21$$

$$= -1 \neq 0$$

$\therefore S$ is linearly independent.

Let $(x, y, z) \in \mathbb{R}^3$

Then $(x,y,z) = V_1(1,2,1) + V_2(2,9,0) + V_3(3,3,4)$

$$= (V_1, 2V_1, V_1) + (2V_2, 9V_2, 0) + (3V_3, 3V_3, 4V_3)$$

$$= (V_1 + 2V_2 + 3V_3,\ 2V_1 + 9V_2 + 3V_3,\ V_1 + 4V_3)$$

$V_1 + 2V_2 + 3V_3 = x$ — ① $\quad 2V_1 + 9V_2 + 3V_3 = y$ — ② $\quad V_1 + 4V_3 = z$ — ③

$① \times 9 \Rightarrow 9V_1 + \cancel{18V_2} + 27V_3 = 9x$

$② \times 2 \Rightarrow 4V_1 + \cancel{18V_2} + 6V_3 = 2y$

$\quad (-) \quad (-) \quad (-) \quad (-)$

$5V_1 \quad + 21V_3 = 9x - 2y$ — ④

$③ \times 5 \Rightarrow \cancel{5V_1} + 20V_3 = 5z$

$④ \Rightarrow \cancel{5V_1} + 21V_3 = 9x - 2y$

$\quad (-) \quad (-) \quad (-)$

$-V_3 = 5z - 9x + 2y$

$$\boxed{V_3 = 9x - 2y - 5z}$$

$$③ \Rightarrow \quad V_1 + 36x - 8y - 20z = z$$

$$\boxed{V_1 = -36x + 8y + 21z}$$

$$① \Rightarrow \quad -36x + 8y + 21z + 2V_2 + 27x - 6y - 15z = x$$

$$-9x + 2y + 6z + 2V_2 = x$$

$$2V_2 = 10x - 2y - 6z$$

$$\boxed{V_2 = 5x - y - 3z}$$

$$(x, y, z) = (-36x + 8y + 21z)(1,2,1) + (5x - y - 3z)(2,1,0)$$
$$+ (9x - 2y - 5z)(3,3,4)$$

$L(S) = V$

$\therefore$ S is a basis for $\mathbb{R}^3$

4. Let w_1 and w_2 be two subspace for $\mathbb{R}^4$ given by $w_1 = \{(a,b,c,d) : b - 2c + d = 0\}$,
$w_2 = \{(a,b,c,d) : a = d, b = 2c\}$ then find the basis for w_1, w_2 & $w_1 \cap w_2$

$$w_1 = \{(a,b,c,d) : b - 2c + d = 0\}$$
$$= \{(a,b,c,d) : b = 2c - d\}$$
$$= \{(a, 2c - d, c, d)\}$$
$$= \{a(1,0,0,0) + c(0,2,1,0) + d(0,-1,0,1)\}$$

Hence $\{(1,0,0,0), (0,2,1,0), (0,-1,0,1)\}$ is a basis for w_1.

Definition :-

For an $m \times n$ matrix A, the rank of A is defined to be the dimension of the row space (or column space) and is denoted by rank A.

Note :-

i) If A is an $m \times n$ matrix, then ~~rank (A) ≤~~

$\text{rank}(A) \le \min\{m, n\}$

ii) For any $m \times n$ matrix A,

$$\dim R(A) + \dim N(A) = \text{rank}(A) + \text{nullity}(A) = n$$ no of column / variables

$$\dim C(A) + \dim N(A^T) = \text{rank}(A) + \text{nullity}(A^T) = m$$

< Rank - nullity theorem >

** Find the nullity and the rank of each of the following matrices

i) $A = \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ -1 & -2 & 1 & 1 & 0 \\ 1 & 2 & -3 & -7 & 2 \\ 1 & 2 & -2 & -4 & 3 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & 3 & 1 & 7 \\ 2 & 3 & -1 & 9 \\ -1 & -2 & 0 & -5 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 5 & -2 \end{bmatrix}$

i) $A = \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ -1 & -2 & 1 & 1 & 0 \\ 1 & 2 & -3 & -7 & 2 \\ 1 & 2 & -2 & -4 & 3 \end{bmatrix}$ $\begin{matrix} R_2 \Rightarrow R_2 + R_1 \\ R_3 \Rightarrow R_3 - R_1 \\ R_4 \Rightarrow R_4 - R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & -3 & -9 & 1 \\ 0 & 0 & -2 & -6 & 2 \end{bmatrix}$ $\begin{matrix} R_3 \Rightarrow R_3 + 3R_2 \\ R_4 \Rightarrow R_4 + 2R_2 \end{matrix}$

$\sim \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$ $\begin{matrix} R_3 \Rightarrow R_3/4 \\ R_4 \Rightarrow R_4/4 \end{matrix}$

$\sim \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ $R_4 \Rightarrow R_4 - R_3$

$\sim \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{matrix} R_1 \Rightarrow R_1 - R_3 \\ R_2 \Rightarrow R_2 - R_3 \end{matrix}$

$\sim \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

ii)

$\therefore$ Basis of $R(A) = \{(1,2,0,2,0),(0,0,1,3,0),$
$(0,0,0,0,1)\}$

$\therefore$ dim $R(A) = \text{rank}(A) = 3$

Since $\text{rank}(A) + \text{nullity}(A) = 5$

$\therefore$ $\text{nullity}(A) = 5 - 3 = 2$

ii) $A = \begin{bmatrix} 1 & 3 & 1 & 7 \\ 2 & 3 & -1 & 9 \\ -1 & -2 & 0 & -5 \end{bmatrix}$ $R_2 \Rightarrow R_2 - 2R_1$, $R_3 \Rightarrow R_3 + R_1$

$-\frac{14}{9}$

$\sim \begin{bmatrix} 1 & 3 & 1 & 7 \\ 0 & -3 & -3 & -5 \\ 0 & 1 & 1 & 2 \end{bmatrix}$ $R_3 \Rightarrow 3R_3 + R_2$

$\sim \begin{bmatrix} 1 & 3 & 1 & 7 \\ 0 & -3 & -3 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $R_2 \Rightarrow R_2/-3$

$\sim \begin{bmatrix} 1 & 3 & 1 & 7 \\ 0 & 1 & 1 & 5/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $R_1 \Rightarrow R_1 - 3R_2$

$\sim \begin{bmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & 1 & 5/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $R_1 \Rightarrow R_1 - 2R_2$, $R_2 \Rightarrow R_2 - 5/3 R_3$

$\sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\therefore$ Basis of $R(A) = \{(1,0,-2,0), (0,1,1,0), (0,0,0,1)\}$

$\therefore$ $\dim R(A) = \text{rank}(A) = 3$

$\therefore$ $\text{rank}(A) + \text{nullity of } (A) = 4$

$\text{nullity}(A) = 4 - 3 = 1$

iii)

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 5 & -2 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -1 & 1 & -2 \\ 0 & -3 & 3 & -6 \end{bmatrix} R_3 \to R_3 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2 \to R_2/-1$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_1 \to R_1 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Basis of $R(A) = \{(1,0,3,-2)(0,1,-1,2)\}$

$\dim R(A) = \text{rank}(A) = 2$

$\therefore$ rank (A) + nullity $(A) = 4$

$\therefore$ nullity $(A) = 4 - 2$

$= 2$

** 3M **Theorem** :-

Let A be an $n \times n$ square matrix, then A is invertible if and only if rank$(A) = n$

Proof :-

Let A be an $n \times n$ square matrix

Then to prove that A is invertible $\Leftrightarrow$ rank $(A) = n$

A is invertible $\Leftrightarrow$ A is non singular

$\Leftrightarrow |A| \neq 0$

$\Leftrightarrow$ All these row vectors (n) are linearly independent

$\Leftrightarrow$ set of all these vectors (n) forms a basis for row space of A

$\Leftrightarrow$ number of elements in the basis = dim $R(A)$

$\Leftrightarrow$ rank$(A) = n$.

Note :

i) rank $(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$

ii) Let A be an invertible square matrix, then for any matrix B, rank $(AB) = $ rank $(B) = $ rank(BA)

Bases for subspaces :-

Let $\alpha = \{\bar{v}_1, \bar{v}_2 \ldots \bar{v}_k\}$ & $\beta = \{\bar{w}_1, \bar{w}_2 \ldots \bar{w}_l\}$ be bases for V and W respectively.

Let Q be the $n \times (k+l)$ matrix whose columns are these basis vectors

i.e, $Q = \{\bar{v}_1\ \bar{v}_2 \ldots \bar{v}_k\ \bar{w}_1\ \bar{w}_2 \ldots \bar{w}_l\}$

****
Theorem :-

Let V and W be two subspaces of $\mathbb{R}^n$, and Q be the matrix whose columns are these basis vectors of V and W.

Then i) $C(Q) = V+W$, so that a basis for the column space $C(Q)$ is a basis for $V+W$

ii) $N(Q)$ can be identified with $V \cap W$ so that

$\dim(V \cap W) = \dim N(Q)$

Proof :-

i) It is clear that $C(Q) = V+W$

ii) Let $x = (a_1, \ldots, a_k ; b_1, \ldots, b_l) \in N(Q) \subseteq \mathbb{R}^{k+l}$ then

$$Q(x) = a_1 v_1 + \ldots + a_k v_k + b_1 w_1 + \ldots + b_l w_l = 0$$

from which we get

$$a_1 v_1 + \ldots + a_k v_k = -(b_1 w_1 + \ldots + b_l w_l)$$

if we set

$$y = a_1 v_1 + \ldots + a_k v_k$$
$$= -(b_1 w_1 + \ldots + b_l w_l)$$

then $y \in V \cap W$ since the first right hand side $a_1v_1 + \dots + a_kv_k$ is in V as a linear combination of the basis vectors in α and the second right hand side $-(b_1w_1 + \dots b_lw_l)$ is in W as a linear combination of the basis vectors in β that is to each $x \in N(Q)$, there corresponds a vector y in $V \cap W$.

On the other hand, if $y \in V \cap W$, then y can be written in two linear combinations by the bases for V and W separately as

$$y = a_1v_1 + \dots + a_kv_k \in V$$

$$y = b_1w_1 + \dots + b_lw_l \in W$$

for some $a_1, \dots, a_k$ and $b_1, \dots, b_l$. Let $x = (a_1, \dots, a_k, -b_1, \dots, -b_l) \in \mathbb{R}^{k+l}$ then if it is quite clear that $Qx = 0$ i.e, $x \in N(Q)$ therefore, the correspondence of x in $N(Q) \subseteq \mathbb{R}^{k+l}$ to a vector y in $V \cap W \subseteq \mathbb{R}^n$ gives us a one to one correspondence between the sets $N(Q)$ and $V \cap W$

Moreover, if $x_i, i = 1, 2$ correspond to y_i, then one can easily check that $x_1 + x_2$ correspond to $y_1 + y_2$ and kx_1 corresponds to ky_1 this means that the two vector spaces $N(Q)$ and $V \cap W$ can be identified as vector spaces

In particular, for a basis for $N(Q)$, the corresponding set in $V \cap W$ is a basis for $V \cap W$ that is, if the set of vectors

$$\begin{cases} x_1 = (a_{11}, \dots, a_{1K}, b_{11}, \dots, b_{1\ell}) \\ \vdots \\ x_s = (a_{s1}, \dots, a_{sK}, b_{s1}, \dots, b_{s\ell}) \end{cases}$$

is a basis for $N(Q)$, then the set of vectors

$$\begin{cases} y_1 = a_{11}V_1 + \dots + a_{1K}V_K \\ \vdots \\ y_s = a_{s1}V_1 + \dots + a_{sK}V_K \end{cases} \quad \text{or} \quad \begin{cases} y_1 = -(b_{11}W_1 + \dots + b_{1\ell}W_\ell) \\ \vdots \\ y_s = -(b_{s1}W_1 + \dots + b_{s\ell}W_\ell) \end{cases}$$

is a basis for $V \cap W$, and vice versa. this implies that $\dim N(Q) = \dim(V \cap W)$

(i) Find an interpolating polynomial for $(0,3)$, $(1,0)$, $(-1,2)$, $(3,6)$

(ii) Find a polynomial $p(x) = a + bx + cx^2 + dx^3$ that satisfies $p(0) = 1$, $p'(0) = 2$, $p(1) = 4$, $p'(1) = 4$

(iii) Let $f(x) = \sin x$ that at $x = 0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{3\pi}{4}, \pi$, the values of f are $y = 0, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, 0$. Find the polynomial $p(x)$ of degree ≤ 4 that passes through these five points

Let $y = f(x) = a + bx + cx^2 + dx^3 + ex^4$

Since $x = 0$, $y = 0$

$\therefore 0 = a$ —— ①

$x = \frac{\pi}{4}, \; y = \frac{1}{\sqrt{2}}$

$$\frac{1}{2} = a + \frac{b\pi}{4} + c\frac{\pi^2}{16} + d\frac{\pi^3}{64} + \frac{e\pi^4}{256} \quad \text{——— (2)}$$

$x = \frac{\pi}{3}, \; y = \frac{\sqrt{3}}{2}$

$$\frac{\sqrt{3}}{2} = a + \frac{b\pi}{3} + \frac{c\pi^2}{9} + \frac{d\pi^3}{27} + \frac{e\pi^4}{81} \quad \text{——— (3)}$$

$x = \frac{3\pi}{4}, \; y = \frac{1}{\sqrt{2}}$

$$\frac{1}{\sqrt{2}} = a + \frac{3\pi b}{4} + \frac{9\pi^2 c}{16} + \frac{27 d\pi^3}{64} + \frac{81 e\pi^4}{256} \quad \text{——— (4)}$$

$x = \pi, \; y = 0$

$$\therefore \; 0 = a + b\pi + c\pi^2 + d\pi^3 + e\pi^4 \quad \text{——— (5)}$$

$$(2) \Rightarrow a + \frac{b\pi}{4} + \frac{c\pi^2}{16} + \frac{d\pi^3}{64} + \frac{e\pi^4}{256} = \frac{1}{2}$$

$$(3) \Rightarrow a + \frac{b\pi}{3} + \frac{c\pi^2}{9} + \frac{d\pi^3}{27} + \frac{e\pi^4}{81} = \frac{\sqrt{3}}{2}$$

$$- \quad - \quad - \quad - \quad - \quad -$$

$$-\frac{\pi}{12}b - \frac{7\pi^2}{144}c - \frac{37\pi^3}{1728}d -$$

Basis (1)

$\therefore \; 1(-1) - 1(-1)$

$= -1 + 1$

$= 0$

(2)

$1(1) - 1(0 \cdot 0) + 1(0)$

$= 1 \neq 0$

$$③ \Rightarrow \quad V_1 + 36x - 8y - 20z = z$$

$$\boxed{V_1 = -36x + 8y + 21z}$$

$$① \Rightarrow \quad -36x + 8y + 21z + 2V_2 + 27x - 6y - 15z = x$$

$$-9x + 2y + 6z + 2V_2 = x$$

$$2V_2 = 10x - 2y - 6z$$

$$\boxed{V_2 = 5x - y - 3z}$$

$$(x, y, z) = (-36x + 8y + 21z)(1,2,1) + (5x - y - 3z)(2,1,0)$$
$$+ (9x - 2y - 5z)(3,3,4)$$

$L(S) = V$

$\therefore$ S is a basis for $\mathbb{R}^3$

4. Let W_1 and W_2 be two subspace for $\mathbb{R}^4$ given by $W_1 = \{(a,b,c,d) : b - 2c + d = 0\}$,
$W_2 = \{(a,b,c,d) : a = d,\ b = 2c\}$ then find the basis for W_1, W_2 & $W_1 \cap W_2$

$$W_1 = \{(a,b,c,d) : b - 2c + d = 0\}$$
$$= \{(a,b,c,d) : b = 2c - d\}$$
$$= \{(a, 2c-d, c, d)\}$$
$$= \{a(1,0,0,0) + c(0,2,1,0) + d(0,-1,0,1)\}$$

Hence $\{(1,0,0,0), (0,2,1,0), (0,-1,0,1)\}$ is a basis for W_1.

$W_2 = \{ (a,b,c,d) : a=d,\ b=2c \}$

$= \{ (d, 2c, c, d) \}$

$= \{ c(0,2,1,0) + d(1,0,0,1) \}$

Hence $\{ (1,0,0,1)\ (0,2,1,0) \}$ is a basis of W_2.

$W_1 \cap W_2 \quad (0,2,1,0)$

5. Show that the set $\{1, x, x^2, \ldots x^n\}$ is a basis for $P_n(\mathbb{R})$, the vector space of all polynomials of degree $\leq n$ with real coefficients.

$a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 + \ldots + a_n x^n = \bar{0}$

$\Rightarrow a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 + \ldots + a_n x^n = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + \ldots + 0 \cdot x^n$

$\Rightarrow a_0 = 0,\ a_1 = 0,\ a_2 = 0, \ldots a_n = 0$

$\{1, x, x^2, \ldots x^n\}$ is linearly independent

Then $(x, y, z) \in$

Consider $1 \cdot x^2 - 2 \cdot x^{10} \in P_n(\mathbb{R})$

Then $1 \cdot x^2 - 2 \cdot x^{10} = 0 \cdot 1 + 0 \cdot x + 1 \cdot x^2 + 0 \cdot x^3 + \ldots +$
$(-2)\, x^{10} + 0 \cdot x^{11} + \ldots + 0 \cdot x^n$

i.e. Every element of $P_n(\mathbb{R})$ can be expressed as a linear combination of $\{1, x, x^2, \ldots x^n\}$

Hence $\{1, x, x^2, \ldots x^n\}$ is a basis for $P_n(\mathbb{R})$.

Example: In the 3-space $\mathbb{R}^3$, let W be the set of all vectors (x_1, x_2, x_3) that satisfies the equation $x_1 - x_2 - x_3 = 0$. Prove that W is a subspace of $\mathbb{R}^3$. Find a basis for the subspace W

$$(x_1, x_2, x_3) = x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$$

It is standard element

$\therefore (x_1, x_2, x_3)$ is linearly independent.

Basis :-

$$W = \{(x_1, x_2, x_3) : x_1 - x_2 - x_3 = 0\}$$

$$= \{(x_1, x_2, x_3) : x_1 = x_2 + x_3\}$$

$$= \{(x_2 + x_3, x_2, x_3)\}$$

$$W = \{x_2(1,1,0) + x_3(1,0,1)\}$$

Hence $\{(1,1,0)\ (1,0,1)\}$ is a basis of W

Subspace :-

$$W = \{(x_1, x_2, x_3) : x_1 - x_2 - x_3 = 0\}$$

let $\alpha, \beta \in W$

$\alpha = (x_1, x_2, x_3) \quad \beta = (y_1, y_2, y_3)$

where $x_1 - x_2 - x_3 = 0$

$y_1 - y_2 - y_3 = 0$

$$\alpha + K\beta = (x_1 + Ky_1, x_2 + Ky_2, x_3 + Ky_3)$$

Now $(x_1 + Ky_1) - (x_2 + Ky_2) - (x_3 + Ky_3) = (x_1 - x_2 - x_3) + K(y_1 - y_2 - y_3)$

$= 0 + K(0)$

$= 0$

$\therefore \alpha + K\beta \in W$

W is a subspace

Dimension :

Definition :

The dimension of vector space V is the number, say n, of vectors in a basis for V, denoted by $\dim V = n$. When V has a basis of a finite no. of vectors, V is said to be finite dimensional.

Note :-

i) If V has only the zero vector i.e., $V = \{\bar{0}\}$ then $\dim V = 0$

ii) If $V = \mathbb{R}^n$, then the standard basis $\{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n\}$ for V implies $\dim \mathbb{R}^n = n$.

iii) If $V = P_n(\mathbb{R})$ of all polynomials of degree $\le n$, then $\dim P_n(\mathbb{R}) = n+1$, since $\{1, x, x^2, \dots, x^n\}$ is a basis for V.

iv) If $V = M_{m \times n}(\mathbb{R})$ of all $m \times n$ matrices, then $\dim(M_{m \times n}(\mathbb{R})) = m.n$

v) If $V = C(\mathbb{R})$ of all real-valued continuous functions defined on the real line, the V is that finite dimensional i.e., $\dim(C(\mathbb{R})) = \infty$.

vi) Let $V = M_{n \times n}(\mathbb{R})$ of all $n \times n$ matrices, then

a) the dimension of the subspace of all $n \times n$ diagonal matrices whose traces are zero is $n-1$.

b) the dimension of the subspace of all $n \times n$ symmetric matrices $= \frac{n(n+1)}{2}$

c) the dimension of the subspace of all $n \times n$ skew-symmetric matrix $= \frac{n(n-1)}{2}$

Note:- v(i) Let V be a finite dimensional vector space,

i) any linearly independent set in V can be extended to a basis by adding more vectors if necessary

ii) Any set of vectors that spans V can be reduced to a basis by discarding vectors if necessary.

Note:- Let V be a vector space of dimension n, then

i) any set of n vectors that spans V is a basis

ii) any set of n linearly independent vectors is a basis for V.

Example:-

1. Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors $\bar{x}_1 = (1,-2,5,-3)$, $\bar{x}_2 = (0,1,1,4)$, $\bar{x}_3 = (1,0,1,0)$. Find a basis for W and extend it to a basis of $\mathbb{R}^4$.

Let $A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ $R_3 \Rightarrow R_3 - R_1$

$\sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & +2 & -4 & 3 \end{bmatrix}$ $R_3 \Rightarrow R_3 - 2R_2$

$\sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & -6 & -5 \end{bmatrix}$ $R_3 \Rightarrow -R_3$

$\sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 6 & 5 \end{bmatrix}$

The above matrix have three non-zero rows vectors are clearly linearly independent

$\therefore \{(1,-2,5,-3),(0,1,1,4),(1,0,1,0)\}$ is a basis for W

To extend it to a basis of $\mathbb{R}^4$, just add any non-zero vector of the form $\bar{x}_4 = (0,0,0,t)$

2 Let V_1 & V_2 be the subspaces of $\mathbb{R}^4$ generated by $\{(1,1,0,-1),(1,2,3,0),(2,3,3,-1)\}$ and $\{(1,2,2,-2),(2,3,2,-3),(1,3,4,-3)\}$ respectively. Find the dimension of i) V_1 ii) V_2 iii) V_1+V_2 iv) $V_1 \cap V_2$

$V_1 = \{(1,1,0,-1),(1,2,3,0),(2,3,3,-1)\}$

$V_2 = \{(1,2,2,-2),(2,3,2,-3),(1,3,4,-3)\}$

$\underline{V_1}$:-

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \end{bmatrix} \begin{array}{l} R_2 \Rightarrow R_2 - R_1 \\ R_3 \Rightarrow R_3 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} R_3 \Rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Dimension of $V_1 = 2$

V_2 :-

$$A = \begin{bmatrix} 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix} \quad \begin{matrix} R_2 \Rightarrow R_2 - 2R_1 \\ R_3 \Rightarrow R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & +1 \\ 0 & 1 & 2 & -1 \end{bmatrix} \quad R_3 \Rightarrow R_2 + R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \Rightarrow -R_3 \\ R_3 \Rightarrow -R_3/2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Dimension of $V_2 = 2$

$V_1 + V_2$:-

$$V_1 + V_2 = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \\ 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \\ R_5 \to R_5 - 2R_1 \\ R_6 \to R_6 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - R_2 \\ R_5 \to R_5 - R_4 \\ R_6 \to R_6 / 2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 \end{bmatrix} \quad R_6 \to R_6 - R_4$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

~~dim(V+W~~

$\dim(V+W) = 3$

iv) $\dim(V+W) + \dim(V \cap W) = \dim(V) + \dim(W)$

$3 + \dim(V \cap W) = 2 + 2$

$\dim(V \cap W) = 4 - 3$

$\dim(V \cap W) = 1$

Every polynomial is of degree $< n$ is a vector space over R.

Note: $(R,+)$ is a vector space

Proof:- Let $P(x)$ or $P = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1}$ be the polynomial of degree $< n$

where $a_0, a_1, a_2, \dots a_{n-1} \in R$

i) closure axiom:-

Let $P = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1}$

$q = b_0 + b_1x + b_2x^2 + \dots + b_{n-1}x^{n-1} \in V$

$p+q = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \dots + (a_{n-1}+b_{n-1})x^{n-1} \in V$

closure axiom is true.

ii) Associative axiom:-

Associative axiom is always true over R.

iii) Identity axiom:-

$P = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} \in V$

$\exists e = 0 + 0x + 0x^2 + \dots + 0x^{n-1} \in V$

$\therefore P + e = P$

$\therefore$ Identity axiom is true

iv) Inverse axiom:-

Let $P = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} \in V$

$\exists P^{-1} = a_0^{-1} + a_1^{-1}x + \dots + a_{n-1}^{-1}x^{n-1}$

Such that $P + P^{-1} = e \in V$

$\therefore$ Inverse axiom is true

v) Commutative axiom :-

let $p = a_0 + a_1x + \dots + a_{n-1}x^{n-1} \in V$

$q = b_0 + b_1x + \dots + b_{n-1}x^{n-1} \in V$

$\therefore p+q = q+p$ is true.

Commutative axiom is true.

$\therefore (V, +)$ is an abelian group.

vi) $p = a_0 + a_1x + \dots + a_{n-1}x^{n-1} \in V$

$q = b_0 + b_1x + \dots + b_{n-1}x^{n-1} \in V$

$\alpha \in F$ (Field of scalar)

$\alpha(p+q) = \alpha((a_0+b_0) + (a_1+b_1)x + \dots + (a_{n-1}+b_{n-1})x^{n-1})$

$= \alpha(a_0+b_0) + \alpha(a_1+b_1)x + \dots + \alpha(a_{n-1}+b_{n-1})x^{n-1}$

$\therefore \alpha(p+q) = \alpha p + \alpha q$ is true

vii) $(\alpha+\beta)p$

$p = a_0 + a_1x + \dots a_{n-1}x^{n-1} \in V$

$\alpha, \beta \in F$

$(\alpha+\beta)p = (\alpha+\beta)(a_0 + a_1x + \dots + a_{n-1}x^{n-1})$

$= a_0(\alpha+\beta) + a_1(\alpha+\beta)x + \dots + a_{n-1}(\alpha+\beta)x^{n-1}$

$= \cancel{a_0\alpha + a_0\beta + a_1}$

$= a_0\alpha + a_1\alpha x + \dots a_{n-1}\alpha x^{n-1}$

$+ \beta a_0 + \beta a_1 x + \dots + \beta a_{n-1}x^{n-1} \in V$

$\therefore (\alpha+\beta)p = \alpha p+\beta p$ is true.

vii) $p = a_0 + a_1 x + \dots + a_{n-1} x^{n-1} \in V$

$\alpha, \beta \in F$

$(\alpha\beta)p = \alpha\beta(a_0 + a_1 x + \dots + a_{n-1}x^{n-1})$

$= \alpha(\beta p)$

$\therefore (\alpha\beta)p = \alpha(\beta p)$ is true

ix) $p = a_0 + a_1 x + \dots + a_{n-1} x^{n-1} \in V$

$1 \in F$

$1 \cdot p = 1(a_0 + a_1 x + \dots + a_{n-1} x^{n-1})$

$1 \cdot p = p$ is true

<u>Conclusion</u>:-

Every polynomial is of degree $< n$ is a vector space

**Row and column spaces :-

Let A be an $m \times n$ matrix with row vectors $\{\bar{r}_1, \bar{r}_2, \dots \bar{r}_m\}$ and column vectors $\{\bar{c}_1, \bar{c}_2, \dots \bar{c}_n\}$

i) The <u>row space</u> of A is the subspace in $\mathbb{R}^n$ spanned by the row vectors $\{\bar{r}_1, \bar{r}_2, \dots \bar{r}_m\}$, denoted by $R(A)$.

ii) The <u>column space</u> of A is the subspace in $\mathbb{R}^m$ spanned by the column vectors $\{\bar{c}_1, \bar{c}_2, \dots \bar{c}_n\}$ denoted by $C(A)$.

Unit IV

Vector spaces of linear transformations

Let V and W be two vector spaces of dimensions n and m. Let $L(V;W)$ denote set of all linear transformations from V to W, i.e.,

$L(V;W)$ $\{T : T$ is a linear transformation from V to W$\}$

For any two linear transformation S and T in $L(V,W)$ and $\lambda \in \mathbb{R}$,

We define the sum $S+T$ and the scalar multiplication λS by $(S+T)(v) = S(\bar{v}) + T(\bar{v})$ and

$(\lambda S)(\bar{v}) = \lambda(S(\bar{v}))$

for any $\bar{v} \in V$. clearly, the sum $S+T$ and the scalar multiplication λS are also linear and satisfy the operational rules of a vector space, so that $L(V;W)$ becomes a vector space.

Theorem:

Let V and W be vector spaces with ordered bases α and β respectively, and let $S, T : V \to W$ be linear, then we have

$$[S+T]_\alpha^\beta = [S]_\alpha^\beta + [T]_\alpha^\beta$$

and $[KS]_\alpha^\beta = K[S]_\alpha^\beta$

Proof :-

Let $\alpha = \{\bar{v}_1, \dots, \bar{v}_n\}$ and $\beta = \{\bar{w}_1, \dots, \bar{w}_m\}$

$S(\bar{v}_1) = \bar{w}_1 \qquad S(\bar{v}_j) = S(a_1\bar{v}_1 + a_2\bar{v}_2 + \dots + a_n\bar{v}_n) = a_1 S(\bar{v}_1) + a_2 S(\bar{v}_2) + \dots a_n S(\bar{v}_n) = a_1(\bar{w}_1) + \dots a_n(\bar{w}_n)$

Then $S(\bar{v}_j) = \sum_{i=1}^{m} a_{ij}\bar{w}_i$ and $T(\bar{v}_j) = \sum_{i=1}^{m} b_{ij}\bar{w}_i$

for each $1 \le j \le n$, so that $[S]_\alpha^\beta = [a_{ij}]$ and

$[T]_\alpha^\beta = [b_{ij}]$.

$S(\bar{v}_j) + T(\bar{v}_j)$

Hence $(S+T)(\bar{v}_j) = \sum_{i=1}^{m} a_{ij}\bar{w}_i + \sum_{i=1}^{m} b_{ij}\bar{w}_i$

$$= \sum_{i=1}^{m}(a_{ij} + b_{ij})\bar{w}_i$$

Thus $[S+T]_\alpha^\beta = [S]_\alpha^\beta + [T]_\alpha^\beta$

|||ly $[KS]_\alpha = K[S]_\alpha$ $\quad K\,S(\bar{v}_j) = K\sum_{i=1}^{m} a_{ij}\bar{w}_i$

$K.S(\bar{v}_j) = \sum_{i=1}^{m}(K a_{ij})\bar{w}_i$

*** 8M

1\. Let α be the standard bases for $\mathbb{R}^3$ and let $S, T : \mathbb{R}^3 \to \mathbb{R}^3$ be two linear transformation given by $S(\bar{e}_1) = (2,2,1)$, $S(\bar{e}_2) = (0,1,2)$, $S(\bar{e}_3) = (-1,2,1)$. $T(\bar{e}_1) = (1,0,1)$, $T(\bar{e}_2) = (0,1,1)$, $T(\bar{e}_3) = (1,1,2)$. Compute $[S+T]_\alpha$, $[2T-S]_\alpha$ and $[T \circ S]_\alpha$

$$\alpha = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\} = \{(1,0,0),(0,1,0),(0,0,1)\}$$

$$S(\bar{e}_1) = (2,2,1) = 2(1,0,0) + 2(0,1,0) + 1(0,0,1)$$

$$S(\bar{e}_2) = (0,1,2) = 0(1,0,0) + 1(0,1,0) + 2(0,0,1)$$

$$S(\bar{e}_3) = (-1,2,1) = (-1)(1,0,0) + 2(0,1,0) + 1(0,0,1)$$

$$[S]_\alpha = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$T(\bar{e}_1) = (1,0,1) = 1(1,0,0) + 0(0,1,0) + 1(0,0,1)$$

$$T(\bar{e}_2) = (0,1,1) = 0(1,0,0) + 1(0,1,0) + 1(0,0,1)$$

$$T(\bar{e}_3) = (1,1,2) = 1(1,0,0) + 1(0,1,0) + 2(0,0,1)$$

$$[T]_\alpha = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$[S+T]_\alpha = [S]_\alpha + [T]_\alpha$$

$$= \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$[S+T]_\alpha = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix}$$

$$[2T-S]_\alpha = 2[T]_\alpha - [S]_\alpha$$

$$= 2\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 2 & 4 \end{bmatrix} + \begin{bmatrix} -2 & 0 & +1 \\ -2 & -1 & -2 \\ -1 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

$$[T \circ S]_\alpha = [T]_\alpha \cdot [S]_\alpha$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (2+0+1) & 2 & -1+1 \\ 2+1 & 1+2 & 2+1 \\ 2+2+2 & 1+4 & -1+2+2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 0 \\ 3 & 3 & 3 \\ 6 & 5 & 3 \end{bmatrix}$$

2. Let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation defined by $T(f) = (3+x)f' + 2f$, and $S: P_2(\mathbb{R}) \to \mathbb{R}^3$ be the one defined by $S(a+bx+cx^2) = (a-b, a+b, c)$. For a basis $\alpha = \{1, x, x^2\}$ for $P_2(\mathbb{R})$ and the standard basis $\beta = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ for $\mathbb{R}^3$, compute $[S]_\alpha^\beta$, $[T]_\alpha$ and $[S \circ T]_\alpha^\beta$

$S(a+bx+cx^2) = (a-b, a+b, c)$

$S(1) = S(1+0.x+0.x^2) = (1,1,0)$

$= 1(1,0,0) + 1(0,1,0) + 0(0,0,1)$

$S(x) = S(0+1.x+0.x^2) = (-1,1,0)$

$= -1(1,0,0) + 1(0,1,0) + 0(0,0,1)$

$S(x^2) = S(0+0.x+1.x^2) = (0,0,1)$

$= 0(1,0,0) + 0(0,1,0) + 1(0,0,1)$

$$[S]^{\beta}_{\alpha} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T(f) = (3+x)f' + 2f$$

$$T(1) = (3+x)(0) + 2(1)$$

$$T(1) = 2 \equiv 2\cdot 1 + 0\cdot x + 0\cdot x^2$$

$$T(x) = (3+x)(1) + 2x$$
$$= 3+x+2x = 3+3x = 3\cdot 1 + 3\cdot x + 0\cdot x^2$$

$$T(x^2) = (3+x)(2x) + 2(x^2) = 6x + 4x^2$$
$$= 0\cdot 1 + 6\cdot x + 4\cdot x^2$$

$$[T]_{\alpha} = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix}$$

$$[S\circ T]^{\beta}_{\alpha} = [S]^{\beta}_{\alpha} \cdot [T]^{\beta}_{\alpha}$$
$$= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -6 \\ 2 & 6 & 6 \\ 0 & 0 & 4 \end{bmatrix}$$

$$T(1) = 2 = 2(1,0,0) + 0(0,1,0) + 0(0,0,1)$$

$$T(x) = 3+3x = 3(1,0,0) + 3(0,1,0) + 0(0,0,1)$$

$$T(x^2) = 6x+4x^2 = 0(1,0,0) + 6(0,1,0) + 4(0,0,1)$$

$$[T]^{\beta}_{\alpha} = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix}$$

*** 3M

Theorem:-

Let V and W be vector spaces with ordered bases α & β respectively, and let $T: V \to W$ be a isomorphism, then $[T^{-1}]_{\beta}^{\alpha} = \left([T]_{\alpha}^{\beta}\right)^{-1}$

Proof :-

Let $T: V \to W$ be an isomorphism.

Then T is invertible, dim V = dim W.

Also the matrices $[T]_{\alpha}^{\beta}$ and $[T^{-1}]_{\beta}^{\alpha}$ are square and of the same size.

Thus, $[T]_{\alpha}^{\beta} [T^{-1}]_{\beta}^{\alpha} = [T \circ T^{-1}]_{\beta} = [id]_{\beta}$

$f \cdot g = I$
$g \cdot f = I$
$f^{-1} = g / g^{-1} = f$

Hence, $[T^{-1}]_{\beta}^{\alpha} = \left([T]_{\alpha}^{\beta}\right)^{-1}$.

1. For the vector spaces $P_1(\mathbb{R})$ and $\mathbb{R}^2$, choose the bases $\alpha = \{1, x\}$ for $P_1(\mathbb{R})$ and $\beta = \{\bar{e}_1, \bar{e}_2\}$ for $\mathbb{R}^2$ respectively. Let $T: P_1(\mathbb{R}) \to \mathbb{R}^2$ be the linear transformation defined by $T(a+bx) = (a, a+b)$
   i) Show that T is invertible
   ii) Find $[T]_{\alpha}^{\beta}$ and $[T^{-1}]_{\beta}^{\alpha}$

T is one-one :-

For this let $T(\alpha_1) = T(\alpha_2)$ where

$$T(a_1 + b_1 x) = T(a_2 + b_2 x)$$

$$(a_1, a_1 + b_1) = (a_2, a_2 + b_2)$$

$$\boxed{a_1 = a_2}, \quad a_1 + b_1 = a_2 + b_2$$

$$a_2 + b_1 = a_2 + b_2 \Rightarrow \boxed{b_1 = b_2}$$

Theore

T is onto

Let

The

Al

i

$T(a+b$

$Ker(T) = \{$

$=$

$=$

$\therefore$ T is

Therefore $(a_1, a_1+b_1) = (a_2, a_2+b_2)$

$\Rightarrow \alpha_1 = \alpha_2$ i.e, $T(\alpha_1) = T(\alpha_2) \Rightarrow \alpha_1 = \alpha_2$

$\Rightarrow$ $\therefore$ T is one-one

<u>T is onto</u> :-

Let $(a, a+b) \in \mathbb{R}^2$ (co-domain)
$a, b \in \mathbb{R}$

Then $(a+bx) \in P_1(\mathbb{R})$ (domain)

Also $T(a+bx) = (a, a+b)$

i.e, For every $(a, a+b) \in \mathbb{R}^2$

There exists $(a+bx) \in P_1(\mathbb{R})$

Such that $T(a+bx) = (a, a+b)$

Therefore T is ~~on-to~~ onto and hence

T is invertible.

$\left\{ \begin{array}{l} T: V \to W \\ Ker(T) = \{\bar{x} : T(\bar{x}) = \bar{0}\} \end{array} \right\}$

(or)

$T(a+bx) = (a, a+b)$

$Ker(T) = \{\alpha \in P_1(\mathbb{R}) : T(\alpha) = \bar{0}\}$

$= \{(a+bx) \in P_1(\mathbb{R}) : T(a+bx) = \bar{0}\}$

$= \{(a+bx) \in P_1(\mathbb{R}) : (a, a+b) = (0,0)\}$

$= \{(a+bx) \in P_1(\mathbb{R}) : a=0, a+b=0\}$

$= \{(a+bx) \in P_1(\mathbb{R}) : a=0, b=0\}$

$= \{(0+0x) \in P_1(\mathbb{R})\}$

$= \{0\}$

$\left\langle \because Ker(T) = \{0\} \Leftrightarrow \text{T is one-one \& T is invertible} \right\rangle$

$\therefore$ T is invertible

$$[T]_\alpha^\beta$$

$$T(a+bx) = (a, a+b)$$

$$\alpha = \{1, x\}$$

$$T(1) = T(\overset{a+bx}{1+0\cdot x}) = (1,1) = 1(1,0) + 1(0,1)$$

$$T(x) = T(0+1\cdot x) = (0,1) = 0(1,0) + 1(0,1)$$

$$[T]_\alpha^\beta = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$[T^{-1}]_\beta^\alpha = \left[[T]_\alpha^\beta\right]^{-1} \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\qquad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$= \left[\begin{array}{cc|cc} 1 & \cancel{0} & 1 & \cancel{0} \\ \cancel{0} & 1 & \cancel{1} & 1 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] R_2 \Rightarrow R_2 - R_1$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$$[T^{-1}]_\beta^\alpha = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Change of bases :-

The matrix representation $[id]_\beta^\alpha$ of the identity transformation $id : V \longrightarrow V$ with respect to any two ordered bases α and β is called the basis - change matrix or the coordinate change matrix from β to α.

1. Find the basis
when (i) $\alpha = \{(2,$
(ii) $\alpha = \{(5,1)$
(iii) $\alpha = \{(1,1$
(iv) $\alpha = \{t,$

(iii) $\beta = \{(2,0,$

$id\,(2,0,$

$(2,0,$

$id\,(2,0,$

$id\,(-1,$

$(-1,$

1. Find the basis change matrix $[id]_\alpha^\beta$ from α to β,

when (i) $\alpha = \{(2,3)\ (0,1)\}$, $\beta = \{(6,4), (4,8)\}$

(ii) $\alpha = \{(5,1), (1,2)\}$, $\beta = \{(1,0), (0,1)\}$

(iii) $\alpha = \{(1,1,1), (1,1,0), (1,0,0)\}$, $\beta = \{(2,0,3), (-1,4,1), (3,2,5)\}$

(iv) $\alpha = \{t, 1, t^2\}$, $\beta = \{3+2t+t^2, t^2-4, 2+t\}$

(iii) $\beta = \{(2,0,3), (-1,4,1), (3,2,5)\}$

$$id(2,0,3) = (2,0,3)$$

$$= a_1(1,1,1) + b_1(1,1,0) + c_1(1,0,0)$$

$$(2,0,3) = (a_1, a_1, a_1) + (b_1, b_1) + (c_1)$$

$$a_1 + b_1 + c_1 = 2$$

$$a_1 + b_1 = 0$$

$$a_1 + \cancel{b_1} = 3$$

$$\boxed{a_1 = 3}$$

$$\boxed{b_1 = -3}$$

$$\boxed{c_1 = 2}$$

$$id(2,0,3) = 3(1,1,1) + (-3)(1,1,0) + 2(1,0,0)$$

$$id(-1,4,1) = (-1,4,1)$$

$$= a_2(1,1,1) + b_2(1,1,0) + c_2(1,0,0)$$

$$(-1,4,1) = (a_2+b_2+c_2), (a_2+b_2), (a_2)$$

$$\boxed{a_2 = 1}$$

$$a_2 + b_2 = 4$$

$$\boxed{b_3 = 3}$$

$$a_2 + b_2 + c_2 = -1$$

$$\boxed{c_2 = -5}$$

$$= 1(1,1,1) + 3(1,1,0) + (-5)(1,0,0)$$

$$id\ (3,2,5) = (3,2,5)$$

$$= a_3(1,1,1) + b_3(1,1,0) + c_3(1,0,0)$$

$$(3,2,5) = (a_3, a_3, a_3) + (b_3, b_3) + c_3$$

$$\cancel{a_3}\ \cancel{\boxed{c_3 = 5}}$$

$$\boxed{a_3 = 5}$$

$$a_3 + b_3 = 2$$

$$\boxed{b_3 = -3}$$

$$a_3 + b_3 + c_3 = 3$$

$$\boxed{c_3 = 1}$$

$$= 5(1,1,1) + (-3)(1,1,0) + 1(1,0,0)$$

$$[id]_\beta^\alpha = \begin{bmatrix} 3 & 1 & 5 \\ -3 & 3 & -3 \\ 2 & -5 & 1 \end{bmatrix}$$

(i) $\alpha = \{(2,3),(0,1)\}$ $\beta = \{(6,4),(4,8)\}$

$$id\,(6,4) = (6,4)$$

$$= a_1(2,3) + b_1(0,1)$$

$$= (2a_1), (3a_1 + b_1)$$

$$2a_1 = 6$$

$$\boxed{a_1 = 3}$$

$$3a_1 + b_1 = 4$$

$$\boxed{b_1 = -5}$$

$$id(6,4) = 3(2,3) + (-5)(0,1)$$

$$\text{id}(4,8) = (4,8)$$
$$= a_2(2,3) + b_2(0,1)$$
$$= (2a_2), (3a_2 + b_2)$$

$$2a_2 = 4$$
$$\boxed{a_2 = 2}$$

$$3a_2 + b_2 = 8$$
$$\boxed{b_2 = 2}$$

$$\text{id}(4,8) = 2(2,3) + 2(0,1)$$

$$[\text{id}]_\beta^\alpha = \begin{bmatrix} 3 & 2 \\ -5 & 2 \end{bmatrix}$$

(iv) $\alpha = \{t, 1, t^2\}$ $\quad \beta = \{3+2t+t^2,\ t^2-4,\ 2+t\}$

$$\text{id}(3+2t+t^2) = 3+2t+t^2$$
$$= 2.t + 3.1 + 1.t^2$$

$$\text{id}(t^2-4) = t^2 - 4$$
$$= 0.t + (-4).1 + 1.t^2$$

$$\text{id}(2+t) = 2+t$$
$$= 1.t + 2.1 + 0.t^2$$

$$[\text{id}]_\beta^\alpha = \begin{bmatrix} 2 & 0 & \cancel{2}1 \\ 3 & -4 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

2. Find the basis-change matrix from a basis α to another basis β for the 3-space $\mathbb{R}^3$, where

$\alpha = \{(1,0,1), (1,1,0), (0,1,1)\}$

$\beta = \{(2,3,1), (1,2,0), (2,0,3)\}$

$id(2,3,1) = (2,3,1)$

$$= a_1(1,0,1) + b_1(1,1,0) + c_1(0,1,1)$$

$$(2,3,1) = (a_1+b_1;\ b_1+c_1,\ a_1+c_1)$$

$$a_1+b_1 = 2 \quad \text{—} ①$$
$$b_1+c_1 = 3 \quad \text{—} ②$$
$$a_1+c_1 = 1 \quad \text{—} ③$$

$$① \Rightarrow \cancel{a_1}+b_1 = 2$$
$$③ \Rightarrow \underline{\cancel{a_1}+\underset{-}{c_1} = \underset{-}{1}}$$
$$b_1 - c_1 = 1$$
$$② \Rightarrow \underline{b_1+c_1 = 3}$$
$$2b_1 = 4$$
$$\boxed{b_1 = 2}$$

$$① \Rightarrow \boxed{a_1 = 0}$$

$$② \Rightarrow \boxed{c_1 = 1}$$

$$= 0(1,0,1) + 2(1,1,0) + 1(0,1,1)$$

$id(1,2,0) = (1,2,0)$

$$(1,2,0) = (a_2+b_2,\ b_2+c_2 + a_2+c_2)$$

$$a_2+b_2 = 1 \quad \text{—} ①$$
$$b_2+c_2 = 2 \quad \text{—} ②$$
$$a_2+c_2 = 0 \quad \text{—} ③$$

$$① \Rightarrow a_2+b_2 = 1$$
$$③ \Rightarrow \underline{\underset{-}{a_2}+\underset{-}{c_2} = \underset{-}{0}}$$
$$b_2 - c_2 = 1$$

$$b_2 - c_2 = 1$$
$$b_2 + c_2 = 2$$
$$2b_2 = 3$$
$$\boxed{b_2 = \frac{3}{2}}$$

$$a_2 + b_2 = 1$$
$$a_2 + \frac{3}{2} = 1$$
$$2a_2 + 3 = 2$$
$$2a_2 = -1$$
$$\boxed{a_2 = \frac{-1}{2}}$$

$$b_2 + c_2 = 2$$
$$\frac{3}{2} + c_2 = 2 \qquad 2 - \frac{3}{2}$$
$$\boxed{c_2 = \frac{1}{2}}$$

$$= -\frac{1}{2}(1,0,1) + \frac{3}{2}(1,1,0) + \left(\frac{1}{2}\right)(0,1,1)$$

$$id(2,0,3) = (2,0,3)$$
$$(2,0,3) = (a_3 + b_3,\ b_3 + c_3,\ a_3 + c_3)$$
$$a_3 + b_3 = 2$$
$$b_3 + c_3 = 0$$
$$a_3 + c_3 = 3$$

$$b_3 - c_3 = -1$$
$$b_3 + c_3 = 0$$
$$2b_3 = -1$$
$$\boxed{b_3 = \frac{-1}{2}}$$

$$a_3 + \frac{-1}{2} = 2$$
$$\boxed{a_3 = \frac{5}{2}}$$

$$b_3 + c_3 = 0$$
$$\boxed{c_3 = \frac{1}{2}}$$

$$= \frac{5}{2}(1,0,1) + \left(-\frac{1}{2}\right)(1,1,0) + \frac{1}{2}(0,1,1)$$

$$[id]_{\beta}^{\alpha} = \begin{bmatrix} 0 & -1/2 & 5/2 \\ 2 & 3/2 & -1/2 \\ 1 & 1/2 & 1/2 \end{bmatrix}$$

*** 8M

Consider a curve $xy=1$ on the plane $\mathbb{R}^2$. Find the quadratic equation of the curve which is obtained from the curve $xy=1$ by rotating around the origin clockwise through an angle $\frac{\pi}{4}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$x = x'\cos\theta - y'\sin\theta$$

$$y = x'\sin\theta + y'\cos\theta$$

$$= \begin{bmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x'/\sqrt{2} - y'/\sqrt{2} \\ x'/\sqrt{2} + y'/\sqrt{2} \end{bmatrix}$$

$$x = x'/\sqrt{2} - y'/\sqrt{2}$$

$$y = x'/\sqrt{2} + y'/\sqrt{2}$$

$$xy = 1$$

$$\left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right) = 1$$

$$\frac{x'^2}{2} - \frac{y'^2}{2} = 1$$

$$x'^2 - y'^2 = 2$$

Similarity:-

For

to be simi

matrix Q

Note:-(i) If

to

(ii) Le

a vector

vector spac

[T]

where

change

(iii)

vector

for V.

from

Similarity :

For any square matrices A and B, A is said to be <u>similar to B</u> if there exists a non singular matrix Q such that $\boxed{B = Q^{-1} A Q}$

<u>Note</u> :- (i) If A is similar to B, then B is also similar to A.

(ii) Let $T : V \to W$ be a linear transformation from a vector space V with bases α and α' to another vector space W with bases β and β', then

$$[T]_{\alpha'}^{\beta'} = P^{-1} [T]_{\alpha}^{\beta} Q,$$

where $Q = [id]_{\alpha'}^{\alpha}$ and $P = [id_W]_{\beta'}^{\beta}$ are the basis change matrices.

(iii) Let $T : V \to V$ be a linear transformation on a vector space V and let α and β be ordered bases for V let $Q = [id]_{\beta}^{\alpha}$ be the basis change matrix from β to α, then

(i) Q is invertible, and $Q^{-1} = [id]_{\alpha}^{\beta}$

(ii) for any $\bar{x} \in V$, $[\bar{x}]_{\alpha} = Q[\bar{x}]_{\beta}$

(iii) $[T]_{\beta} = Q^{-1} [T]_{\alpha} Q$

1. Let $\beta = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ be a basis for the 3-space $\mathbb{R}^3$ consisting of $\bar{v}_1 = (1,1,0)$, $\bar{v}_2 = (1,0,1)$ and $\bar{v}_3 = (0,1,1)$. Let T be the linear transformation on $\mathbb{R}^3$ given by the matrix $[T]_\beta = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \\ -1 & 1 & 1 \end{bmatrix}$

Let $\alpha = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ be the standard basis. Find the basis change matrix $[id]_\alpha^\beta$ and $[T]_\alpha$

$[id]_\alpha^\beta$:-

$\alpha = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$

$id(1,0,0) = (1,0,0)$

$(1,0,0) = a(1,1,0) + b(1,0,1) + c(0,1,1)$

$(1,0,0) = (a+b, a+c, b+c)$

$a+b = 1$

$a+c = 0$

$b+c = 0$

$(-)$

$a-b = 0$

$a+b = 1$

$2a = 1$

$\boxed{a = \frac{1}{2}}$

$a+b = 1$ $\quad 1-\frac{1}{2}$

$\boxed{b = \frac{1}{2}}$

$a+c = 0$

$\boxed{c = -\frac{1}{2}}$

$$\mathrm{id}(1,0,0) = \frac{1}{2}(1,1,0) + \frac{1}{2}(1,0,1) - \frac{1}{2}(0,1,1)$$

$$\mathrm{id}(0,1,0) = (0,1,0)$$

$$(0,1,0) = a(1,1,0) + b(1,0,1) + c(0,1,1)$$

$$(0,1,0) = (a+b,\ a+c,\ b+c)$$

$$a+b=0$$
$$a+c=1$$
$$b+c=0$$

$$\begin{array}{l} a+b=0 \\ a+c=1 \\ \hline b-c=1 \\ b+c=0 \\ \hline 2a=.1 \end{array}$$

$$\boxed{a=\frac{1}{2}}$$

$$a+c=1$$

$$\boxed{c=\frac{1}{2}}$$

$$b+c=0$$

$$\boxed{b=\frac{-1}{2}}$$

$$\mathrm{id}(0,1,0) = \frac{1}{2}(1,1,0) - \frac{1}{2}(1,0,1) + \frac{1}{2}(0,1,1)$$

$$\mathrm{id}(0,a,1) = (0,0,1)$$

$$(0,0,1) = a(1,1,0) + b(1,0,1) + c(0,1,1)$$

$$(0,0,1) = (a+b,\ a+c,\ b+c)$$

$$a+b=0$$
$$a+c=0$$
$$b+c=1$$

$$a+b=0$$
$$a+c=0$$
$$\overline{\quad b-c=0 \quad}$$
$$b+c=1$$
$$\overline{\quad 2b=1 \quad}$$
$$\boxed{b=\frac{1}{2}}$$

$$a+b=0$$
$$\boxed{a=-\frac{1}{2}}$$

$$b+c=1$$
$$\boxed{c=\frac{1}{2}}$$

$$id(0,0,1)=-\frac{1}{2}(1,1,0)+\frac{1}{2}(1,0,1)+\frac{1}{2}(0,1,1)$$

$$[id]^{\beta}_{\alpha}=\begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix}$$

$$[id]^{\alpha}_{\beta}=\left([id]^{\beta}_{\alpha}\right)^{-1}$$

$$=\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$[T]_\alpha = [id$

$[T]_\alpha =$

2. Let T
defined b
Let $\alpha = \{e$
and let
Consisting
Find the
show th
Vectors i
of $[T]_\beta$

$[T]_\alpha$

$$[T]_\alpha = [id]_\beta^\alpha \, [T]_\beta \, [id]_\alpha^\beta$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix}$$

$$[T]_\alpha = \begin{bmatrix} 2 & 1 & 1 \\ 3/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 7/2 \end{bmatrix}$$

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(x_1, x_2, x_3) = (2x_1 + x_2,\ x_1 + x_2 + 3x_3,\ -x_2)$. Let $\alpha = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ be the standard ordered basis for $\mathbb{R}^3$, and let $\beta = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ be another ordered basis consisting of $\bar{v}_1 = (-1,0,0)$, $\bar{v}_2 = (2,1,0)$ and $\bar{v}_3 = (1,1,1)$. Find the associated matrices $[T]_\alpha$ and $[T]_\beta$ for T. Also, show that $T(\bar{v}_j)$ is the linear combination of basis vectors in β with the entries of the j-th column of $[T]_\beta$ as its coefficients for $j = 1, 2, 3$.

$[T]_\alpha$:-

$$\alpha = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$$

$$\alpha = \{(1,0,0), (0,1,0), (0,0,1)\}$$

$$T(1,0,0) = (2, 1, 0)$$

$$= 2(1,0,0) + 1(0,1,0) + 0(0,0,1)$$

$T(0,1,0) = (1,1,-1)$

$= 1(\cancel{0}1,0,0) + 1(0,1,0) + (-1)(0,0,1)$

$T(0,0,1) - (0,3,0)$

$= 0(1,0,0) + 3(0,1,0) + 0(0,0,1)$

$$[T]_\alpha = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & -1 & 0 \end{bmatrix}$$

$\underline{[T]_\beta :-}$

$\beta = \{\bar{V}_1, \bar{V}_2, \bar{V}_3\}$

$T(x_1, x_2, x_3) = (2x_1 + x_2,\ x_1 + x_2 + 3x_3,\ -x_2)$

$T(-1,0,0) = (-2,-1,0)$

$(-2,-1,0) = a(-1,0,0) + b(2,1,0) + c(1,1,1)$

$(-2,-1,0) = (-a+2b+c,\ b+c,\ c)$

$-a+2b+c = -2$

$b+c = -1$

$\boxed{c=0}$

$\boxed{b=-1}$

$-a + \cancel{(2)}+0 = -2$

$-a = -0$

$\boxed{a=0}$

$T(-1,0,0) = 0(-1,0,0) \cancel{+1}(2,1,0) + 0(1,1,1)$

$T(2,1,0)$

$(5,+3,-1)$

$(5,13,-1)$

$-$

$T(2,1,0$

$T(1,1,$

$(3,5,-$

$(3,5,$

$T(2,1,0) = (4+1,\ 2+1,\ -1)$

$= (5,\ 3,-1)$

$(5,+3,-1) = a(-1,0,0)+b(2,1,0)+c(1,1,1)$

$(5,+3,-1) = (-a+2b+c\ ,\ b+c,\ c)$

$-a+2b+c = 5$

$b+c = 3$

$\boxed{c=-1}$

$\boxed{b=4}$

$-a+2b+c=5$

$-a+8-1=5$

$-a+7=5$

$-a=-2$

$\boxed{a=2}$

$T(2,1,0) = 2(-1,0,0)+4(2,1,0)+(-1)(1,1,1)$

$T(1,1,1) = (3,\ 5,-1)$

$(3,5,-1) = a(-1,0,0)+b(2,1,0)+c(1,1,1)$

$(3,5,-1) = (-a+2b+c,\ b+c,\ c)$

$-a+2b+c=3$

$b+c=5$

$\boxed{c=-1}$

$b+c=5$

$\boxed{b=6}$

$-a+12-1=3$

$-a=-8$

$\boxed{a=8}$

$T(1,1,0) = 8(-1,0,0)+6(2,1,0)+(-1)(1,1,1)$

$$[T]_\beta = \begin{bmatrix} 0 & 2 & 8 \\ -1 & 4 & 6 \\ 0 & -1 & -1 \end{bmatrix}$$

Consider $j=2$, then

$$\begin{aligned} T(\bar{V}_2) &= T(2,1,0) \\ &= (5,3,-1) \\ &= 2\bar{V}_1 + 4\bar{V}_2 - \bar{V}_3 \end{aligned}$$

$$\begin{aligned} T(\bar{V}_1) &= T(-1,0,0) \\ &= (-2,-1,0) \\ &= 0\bar{V}_1 + (-1)\bar{V}_2 + 0\bar{V}_3 \end{aligned}$$

$$\begin{aligned} T(\bar{V}_3) &= T(1,1,1) \\ &= (3,5,-1) \\ &= 8\bar{V}_1 + 6\bar{V}_2 + (-1)\bar{V}_3 \end{aligned}$$

3. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(x_1,x_2,x_3) = (x_1+2x_2+x_3,\ -x_2,\ x_1+4x_3)$. Let α be the standard basis and let $\beta = \{\bar{V}_1, \bar{V}_2, \bar{V}_3\}$ be another ordered basis consisting of $\bar{V}_1 = (1,0,0)$, $\bar{V}_2 = (1,1,0)$ and $\bar{V}_3 = (1,1,1)$ for $\mathbb{R}^3$. Find the associated matrix of T with respect to α and the associated matrix of T with respect to β. Are they similar?

$[T]_\alpha$:-

$\alpha = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$

$T(1,0,0) = (1,0,1)$

$= 1(1,0,0) + 0(0,1,0) + 1(0,0,1)$

$T(0,1,0) = (2,-1,0)$

$= 2(1,0,0) - 1(0,1,0) + 0(0,0,1)$

$T(0,0,1) = (1,0,4)$

$= 1(1,0,0) + 0(0,1,0) + 4(0,0,1)$

$$[T]_\alpha = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

$[T]_\beta$:- $\beta = \{\bar{V}_1, \bar{V}_2, \bar{V}_3\}$

$T(1,0,0) = (1,0,1)$

$(1,0,1) = a(1,0,0) + b(1,1,0) + c(1,1,1)$

$(1,0,1) = (a+b+c, b+c, c)$

$a+b+c = 1$

$b+c = 0$

$\boxed{c = 1}$

$\boxed{b = -1}$

$a+b+c = 1$

$a - 1 + 1 = 1$

$\boxed{a = 1}$

$(1,0,1) = 1(1,0,0) - 1(1,1,0) + 1(1,1,1)$

$T(1,1,0) = (3,-1,1)$

$(3,-1,1) = a(1,0,0) + b(1,1,0) + c(1,1,1)$

$(3,-1,1) = (a+b+c,\ b+c,\ c)$

$a+b+c = 3$

$b+c = -1$

$\boxed{c=1}$

$\boxed{b=-2}$

$a+b+c = 3$

$a-2+1 = 3$

$\boxed{a=4}$

$= 4(1,0,0) - 2(1,1,0) + 1(1,1,1)$

$T(1,1,1) = (4,-1,5)$

$(4,-1,5) = a(1,0,0) + b(1,1,0) + c(1,1,1)$

$(4,-1,5) = (a+b+c,\ b+c,\ c)$

$a+b+c = 4$

$b+c = -1$

$\boxed{c=5}$

$\boxed{b=-6}$

$a+b+c = 4$ $\quad = 5(1,0,0) - 6(1,1,0)$

$a-6+5 = 4$ $\quad + 5(1,1,1)$

$\boxed{a=5}$

$$[T]_\beta = \begin{bmatrix} 1 & 4 & 5 \\ -1 & -2 & -6 \\ 1 & 1 & 5 \end{bmatrix}$$

$[id]_{\alpha}^{\beta}$:-

$id(1,0,0) = (1,0,0)$

$(1,0,0) = a(1,0,0) + b(1,1,0) + c(1,1,1)$

$(1,0,0) = (a+b+c,\ b+c,\ c)$

$a+b+c = 1$

$b+c = 0$

$\boxed{c=0}$

$\boxed{b=0}$

$\boxed{a=1}$

$= 1(1,0,0) + 0(1,1,0) + 0(1,1,1)$

$id(0,1,0) = (0,1,0)$

$(0,1,0) = a(1,0,0) + b(1,1,0) + c(1,1,1)$

$(0,1,0) = (a+b+c,\ b+c,\ c)$

$a+b+c = 0$

$b+c = 1$

$\boxed{c=0}$

$\boxed{b=1}$

$\boxed{a=-1}$

$= -1(1,0,0) + 1(1,1,0) + 0(1,1,1)$

$id(0,0,1) = (0,0,1)$

$(0,0,1) = a(1,0,0) + b(1,1,0) + c(1,1,1)$

$(0,0,1) = (a+b+c,\ b+c,\ c)$

$a+b+c=0$

$b+c=0$

$\boxed{c=1}$

$\boxed{b=-1}$

$a+b+c=0$

$a-1+1=0$

$\boxed{a=0}$

$= 0(1,0,0)-1(1,1,0)+1(1,1,1)$

$$[id]_{\alpha}^{\beta} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$[id]_{\beta}^{\alpha}$:-

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_1 \to R_1+R_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1+R_3 \\ R_2 \to R_2+R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad \cancel{R_2 \to R_2+R_3}$$

$$[id]_{\beta}^{\alpha} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$[T]_{\gamma} = [id]_{\beta}^{\alpha}$ …

$= \begin{bmatrix} 1 & 1 & \dots \\ 0 & 1 & \dots \\ 0 & 0 & \dots \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & \dots \\ 0 & \dots \\ 0 & \dots \end{bmatrix}$

$[T]_{\alpha} = \begin{bmatrix} 1 & \dots \\ 0 & \dots \\ 1 & \dots \end{bmatrix}$

Hence …

Note:-

1) Let A …

then …

(i) det(…

(ii) tr(…

(iii) ran …

2) Let A …

to B then, …

$$[T]_\alpha = [id]_\beta^\alpha [T]_\beta [id]_\alpha^\beta$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \\ -1 & -2 & -6 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$-4+5$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ -1 & -1 & -4 \\ 1 & 0 & 4 \end{bmatrix}$$

$1-2$

$2-6$

$-1+5$

$1-1+1$

$3-1$

$$[T]_\alpha = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

Hence $[T]_\alpha$ and $[T]_\beta$ are similar.

Note:-

1) Let A and B are similar $n \times n$ matrices

then

(i) $\det(A) = \det(B)$

(ii) $tr(A) = tr(B)$

(iii) $rank(A) = rank(B)$

2) Let A & B be $n \times n$ matrices. If A is similar to B then, A^2 is similar to B^2

A^3 is similar to B^3

$\vdots$

A^n is similar to B^n, $n \geq 2$.

4. Let $\alpha = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ be the standard basis of $\mathbb{R}^3$ and $\beta = \{\bar{v}_1 = (1,1,1), \bar{v}_2 = (1,1,0), \bar{v}_3 = (1,0,0)\}$.

i) Find the basis-change matrix P from α to β

ii) Find the basis-change matrix Q from β to α

iii) Verify $Q = P^{-1}$

i) $P = [id]_\beta^\alpha$ $\quad \beta = \{(1,1,1), (1,1,0), (1,0,0)\}$

$\alpha = \{(1,0,0), (0,1,0), (0,0,1)\}$

$id(1,1,1) = (1,1,1)$

$= 1(1,0,0) + 1(0,1,0) + 1(0,0,1)$

$id(1,1,0) = (1,1,0)$

$= 1(1,0,0) + 1(0,1,0) + 0(0,0,1)$

$id(1,0,0) = (1,0,0)$

$= 1(1,0,0) + 0(0,1,0) + 0(0,0,1)$

$$[id]_\beta^\alpha = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$Q = [id]_\alpha^\beta$

$id(1,0,0) = (1,0,0)$

$= a_1(1,1,1) + a_2(1,1,0) + a_3(1,0,0)$

$(1,0,0) = (a_1 + a_2 + a_3, a_1 + a_2, a_1)$

$a_1 + a_2 + a_3 = 1$

$a_1 + a_2 = 0$

$\boxed{a_1 = 0}$

$\boxed{a_2 = 0}$

$\boxed{a_3 = 1}$

R^3

$id(1,0,0) = 0(1,1,1) + 0(1,1,0) + 1(1,0,0)$

$id(0,1,0) = (0,1,0)$

$= a_1(1,1,1) + a_2(1,1,0) + a_3(1,0,0)$

$(0,1,0) = (a_1+a_2+a_3, a_1+a_2, a_1)$

$a_1+a_2+a_3 = 0$

$a_1+a_2 = 1$.

$\boxed{a_1 = 0}$

$\boxed{a_2 = 1}$

$\boxed{a_3 = -1}$

$id(0,1,0) = 0(1,1,1) + 1(1,1,0) + (-1)(1,0,0)$

$id(0,0,1) = (0,0,1)$

$(0,0,1) = (a_1+a_2+a_3, a_1+a_2, a_1)$

$a_1+a_2+a_3 = 0$

$a_1+a_2 = 0$

$\boxed{a_1 = 1}$

$\boxed{a_2 = -1}$

$\boxed{a_3 = 0}$

$id(0,0,1) = 1(1,1,1) - 1(1,1,0) + 0(1,0,0)$

$$[id]_\alpha^\beta = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$P^{-1} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

151-152 Definitions

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right] R_2 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right] \begin{array}{l} R_2 \to -R_2 \\ R_3 \to -R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right] R_1 \to R_1 - R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & +1 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right] R_2 \to R_2 - R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]$$

$$P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$P^{-1} = Q$$