Change of order of integration in Double integral.

Problems (12)

1. Change the order of integration and evaluate

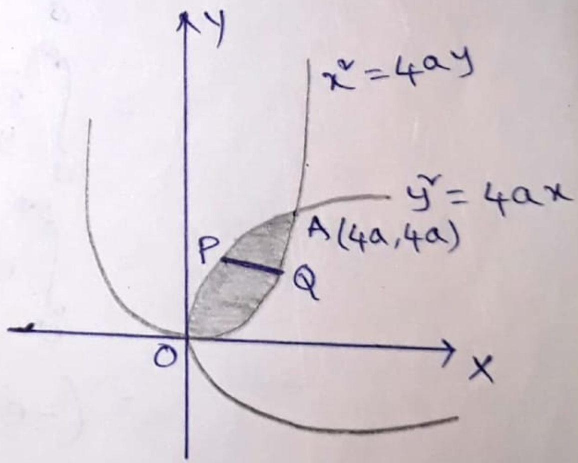
$$\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}}$$

Sol. In this integral for a fixed x, y varies from $\frac{x^2}{4a}$ to $2\sqrt{ax}$ and then x varies from 0 to $4a$.

Let us draw the curves $y = \frac{x^2}{4a}$ i.e $x^2 = 4ay$ and $y = 2\sqrt{ax}$ i.e $y^2 = 4ax$.

These two parabolas intersect at $(0,0)$ and $(4a,4a)$

In changing the order of integration, for a fixed y, x varies from $y^2/4a$ to $\sqrt{4ay}$ and then y varies from 0 to $4a$.



$$\int_{y=0}^{4a}\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\,dy = \int_{y=0}^{4a}\left[\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\right]dy$$

$$= \int_0^{4a}\left[x\right]_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$= \int_0^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right)dy = \left(2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\right)_0^{4a}$$

$$= 2\sqrt{a}\,\frac{4a\sqrt{4a}}{3/2} - \frac{64a^3}{12a} = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2$$

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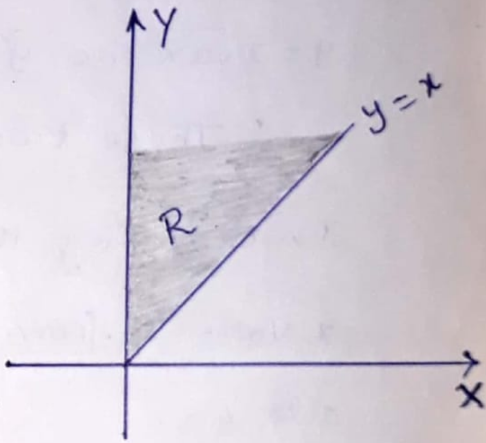
2. Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$, by changing the order of integration. (13)

Sol. In the given integral x increases from 0 to ∞ and for each x, y increases from x to ∞. Thus, the lower value of y lies on the line $y = x$.

Therefore, the region of integration is the region in the first quadrant that lies above the line $y = x$.

In changing the order of integration, for a fixed y, x varies from 0 to y and then y varies from 0 to ∞



$$\therefore \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx = \int_{y=0}^\infty \left[\int_{x=0}^{y} \frac{e^{-y}}{y}\, dx \right] dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} \left[\int_0^y dx \right] dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} \cdot y\, dy$$

$$= \int_0^\infty e^{-y}\, dy$$

$$= \left(-e^{-y}\right)_0^\infty$$

$$= -(0-1)$$

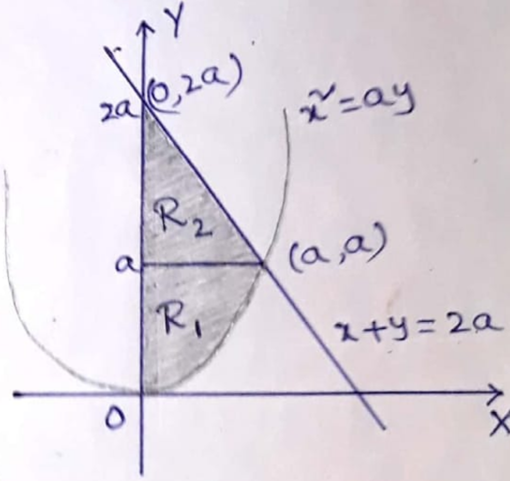
$$= 1$$

• 3. Evaluate the following integral by changing the order of integration: $\int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx.$ (14)

Sol. In the given integral, for a fixed x, y varies from x^2/a to $2a-x$ and then x varies from 0 to a.

Let us draw the curves $y = \frac{x^2}{a}$ i.e $x^2 = ay$ and $y = 2a - x$ i.e the line $x+y=2a$.

The parabola $x^2 = ay$ and the line $x+y=2a$ intersect at (a,a).



The shaded region R is the region of integration. Observe that R is made up of two parts R_1 and R_2.

In R_1, for a fixed y, x varies from 0 to $\sqrt{ay}$ and then y varies from 0 to a.

In R_2, for a fixed y, x varies from 0 to $2a-y$ and then y varies from a to $2a$.

$$\therefore \int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx = \int_{y=0}^{a}\left[\int_{x=0}^{\sqrt{ay}} xy\,dx\right]dy + \int_{y=a}^{2a}\left[\int_{x=0}^{2a-y} xy\,dx\right]dy$$

$$= \int_0^a y\left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} dy + \int_a^{2a} y\left[\frac{x^2}{2}\right]_0^{2a-y} dy$$

$$= \int_0^a y\left(\frac{ay}{2}\right)dy + \int_a^{2a} y\,\frac{(2a-y)^2}{2}\,dy$$ (15)

$$= \frac{a}{2}\int_0^a y^2\,dy + \int_a^{2a} \frac{y}{2}\left(4a^2-4ay+y^2\right)dy$$

$$= \frac{a}{2}\left[\frac{y^3}{3}\right]_0^a + \frac{1}{2}\int_a^{2a}\left(4a^2y-4ay^2+y^3\right)dy$$

$$= \frac{a}{2}\left(\frac{a^3}{3}\right) + \frac{1}{2}\left[2a^2y^2-4a\,\frac{y^3}{3}+\frac{y^4}{4}\right]_a^{2a}$$

$$= \frac{a^4}{6} + \frac{1}{2}\left[2a^2(4a^2-a^2) - \frac{4a}{3}(8a^3-a^3) + \frac{1}{4}(16a^4-a^4)\right]$$

$$= \frac{a^4}{6} + \frac{1}{2}\left[6a^4 - \frac{28}{3}a^4 + \frac{15}{4}a^4\right]$$

$$= \frac{a^4}{6} + \frac{1}{2}\left(\frac{5a^4}{12}\right)$$

$$= \frac{9a^4}{24}$$

$$= \frac{3}{8}a^4$$

4. Change the order of integration in the integral (16)

$\int_0^1 \int_{\sqrt{y}}^{2-y} xy\,dx\,dy$ and hence evaluate it.

Sol. In the given integral, for a fixed y,

x varies from $\sqrt{y}$ to $2-y$ and then y varies

from 0 to 1.

Let us draw the curves $x=\sqrt{y}$ (i.e the parabola $x^2=y$)

and $x=2-y$ (i.e the line $x+y=2$)

The parabola $x^2=y$ and the line $x+y=2$ intersect

at (1, 1).

The shaded region R is the region of integration.

observe that R is made up of two parts R_1 and R_2.

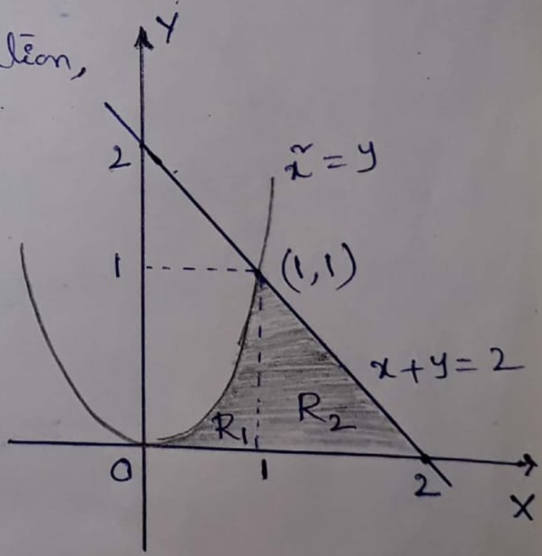
To change the order of integration,

in R_1, for a fixed x,

y varies from 0 to x^2 and

then x varies from

0 to 1.



In R_2, for a fixed x,

y varies from 0 to $2-x$

and then x varies from

1 to 2.

$$\therefore \int_0^1 \int_{\sqrt{y}}^{2-y} xy\,dx\,dy = \int_{x=0}^{1}\left[\int_{y=0}^{x^2} xy\,dy\right]dx + \int_{x=1}^{2}\left[\int_{y=0}^{2-x} xy\,dy\right]dx$$

$$= \int_0^1 x\left[\frac{y^2}{2}\right]_0^{x^2} dx + \int_1^2 x\left[\frac{y^2}{2}\right]_0^{2-x} dx$$

$$= \int_0^1 \frac{x^5}{2}\,dx + \int_1^2 \frac{x}{2}\left[(2-x)^2\right]dx$$

$$= \int_0^1 \frac{x^5}{2}\,dx + \frac{1}{2}\int_1^2 x\left(4-4x+x^2\right)dx$$

$$= \left(\frac{x^6}{12}\right)_0^1 + \frac{1}{2}\int_1^2 \left(4x-4x^2+x^3\right)dx$$

$$= \frac{1}{12} + \frac{1}{2}\left[8-\frac{32}{3}+4-\left(2-\frac{4}{3}+\frac{1}{4}\right)\right]$$

$$= \frac{1}{12} + \frac{1}{2}\left[\frac{4}{3}-\frac{11}{12}\right]$$

$$= \frac{1}{12} + \frac{1}{2}\left(\frac{5}{12}\right)$$

$$= \frac{1}{12} + \frac{5}{24}$$

$$= \frac{7}{24}$$

Problems

① Evaluate the following

(i) $$\int_1^4 \int_0^{\sqrt{4-x}} xy \, dy \, dx$$

(ii) $$\int_0^1 \int_0^{x^2} e^{y/x} \, dy \, dx$$

(iii) $$\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y \, dx \, dy$$

(iv) $$\int_0^1 \int_0^1 \frac{1}{\sqrt{1-x^2}\sqrt{1-y^2}} \, dx \, dy$$

(v) If A is the area of the region bounded by the lines $x=0$, $x=1$, $y=0$, $y=2$, then evaluate

$$\iint_A (x^2+y^2) \, dx \, dy$$

(vi) If R is the rectangular region with vertices $(0,0), (2,0), (2,3)$, evaluate

$$\iint_R x^2 y^2 \, dx \, dy.$$

(2) Evaluate $\iint_R (x+y)^2 \, dx\, dy$, where R is the region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

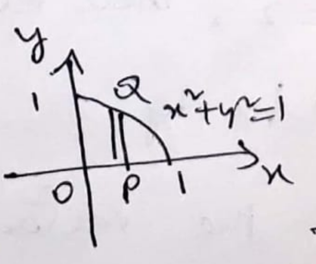
Hint: $$\iint_R (x+y)^2 \, dx\, dy = \int_{x=-a}^{a} \left(\int_{y=-b\sqrt{1-\left(\frac{x^2}{a^2}\right)}}^{b\sqrt{1-\left(\frac{x^2}{a^2}\right)}} (x+y)^2 \, dy \right) dx$$

$$= \frac{\pi}{4} ab \left(a^2+b^2\right)$$

(3) If R is the region bounded by the ~~curve~~ circle $x^2+y^2=1$ in the first quadrant, evaluate

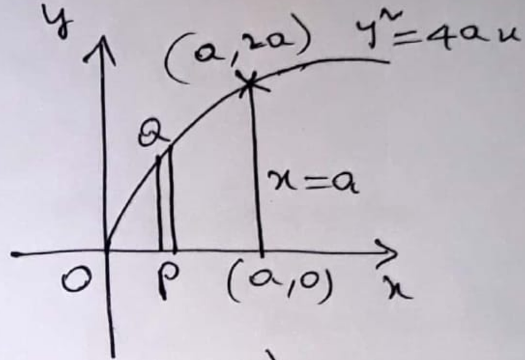
$$\iint_R \frac{xy}{\sqrt{1-y^2}} \, dx\, dy$$

Hint: $$\iint_R \frac{xy}{\sqrt{1-y^2}} \, dx\, dy = \int_{x=0} \left(\int_{y=0}^{\sqrt{1-x^2}} \frac{xy}{\sqrt{1-y^2}} \, dy \right) dx$$



(4) Evaluate $\int_0^a \int_0^{2\sqrt{xa}} x^2\, dy\, dx \ (a>0)$, by changing the order of integration.

Hint: Given



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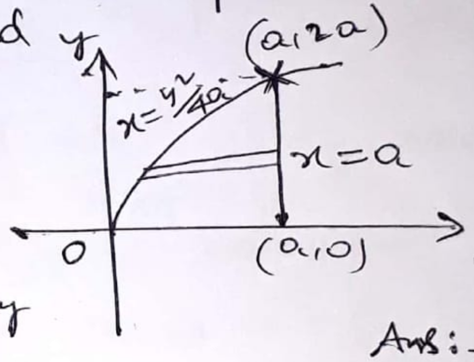


$$\therefore \int_{x=0}^{a}\int_{y=0}^{2\sqrt{ax}} x^2\, dy\, dx$$

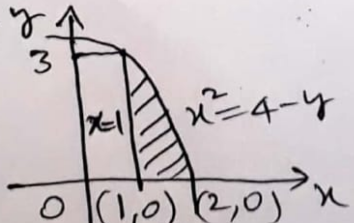
$$= \int_{y=0}^{2a}\left(\int_{x=\frac{y^2}{4a}}^{a} x^2\, dx\right) dy$$

Ans: $\frac{4}{7}a^4$.

(5) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\, dy\, dx$ by changing the order of integration. $\boxed{\text{Ans: } \frac{\pi}{16}}$

(6) Evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\, dx\, dy$, by changing the order of integration.

$\boxed{\text{Ans: } \frac{241}{60}}$



(7) Evaluate $\int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx$, by changing the order of the integration.

(8) Evaluate $\int_0^\infty \int_0^x x\,e^{-\frac{x^2}{2}}\,dy\,dx$

(9) Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2 \sqrt{x^2+y^2}\,dy\,dx$ by transforming to polar co-ordinates.

(10) Evaluate $\iint_R \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ over the annular region R between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ with $b>a$, by transforming to polar co-ordinates.

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