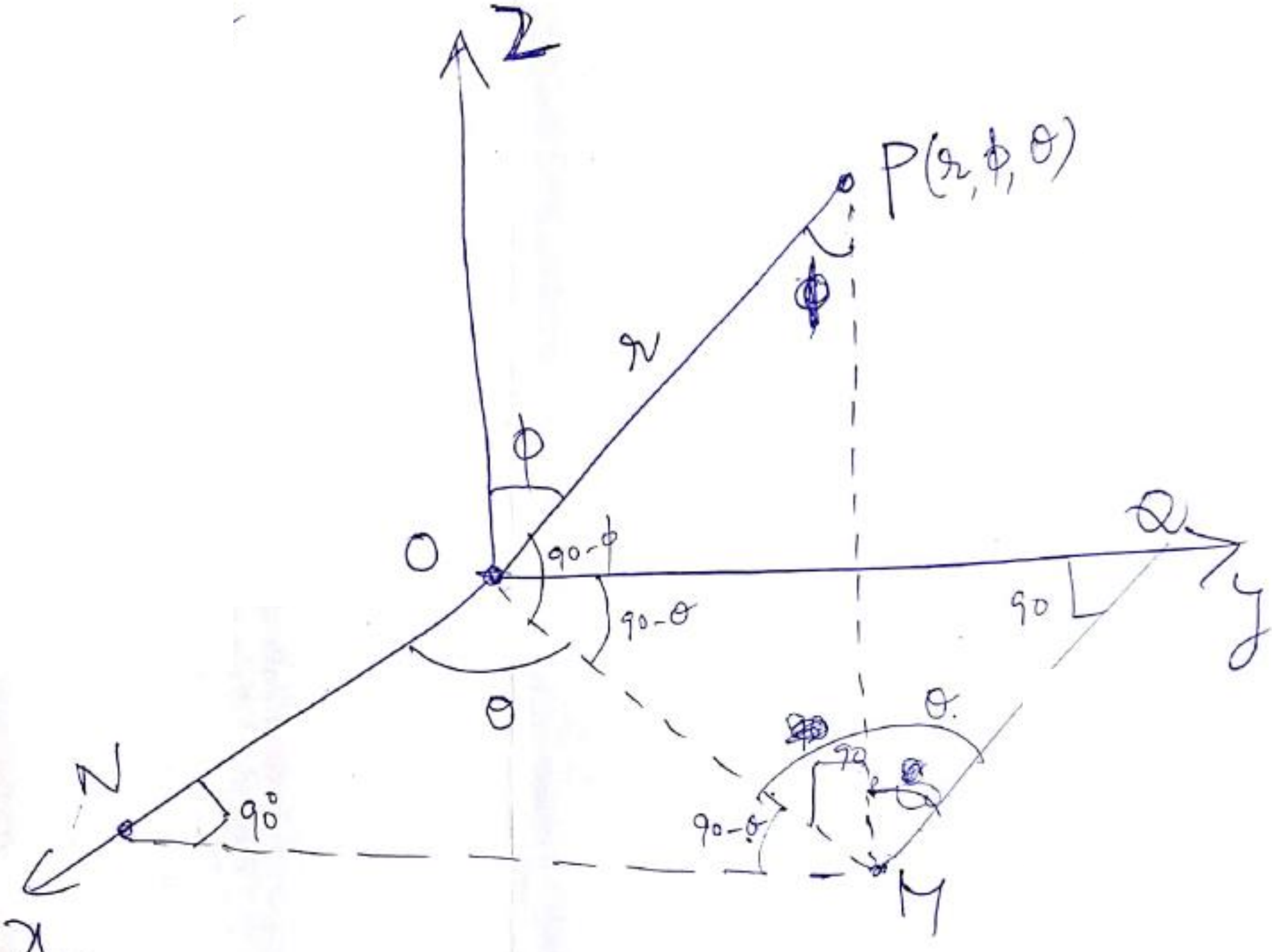
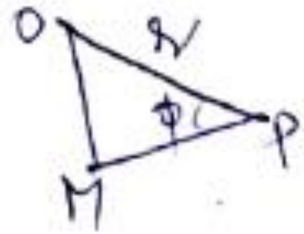
Conversion of Cartesian in to spherical polar coordinates


Z
P(r, φ, θ)
r
φ
O
Q
y
90 − φ
90 − θ
90
θ
N
90°
90 − θ
M
x

$$\boxed{\begin{array}{c} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi \end{array}}$$

From Δ^{le} OPM



$$\sin\phi = \frac{OM}{OP}$$

$$\Rightarrow \boxed{OM = r\sin\phi}$$

$$\cos\phi = \frac{MP}{OP}$$

$$\Rightarrow MP = r\cos\phi$$

$$\boxed{z = r\cos\phi}$$

From $\triangle$le MNO :

$$\sin\theta = \frac{MN}{MO}$$

$$MN = MO\sin\theta$$

$$MN = r\sin\phi\sin\theta$$

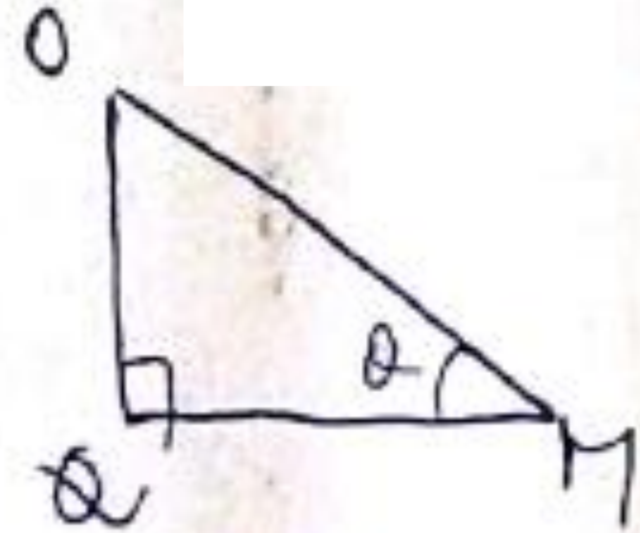
$$\boxed{y = r\sin\phi\sin\theta}$$

From $\triangle^{le}$ OMQ

$$\cos\theta = \frac{QM}{OM}$$

$$QM = OM\cos\theta$$

$$\boxed{x = h\sin\phi\cos\theta}$$



$$\boxed{\begin{aligned} x &= r\sin\phi\cos\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\phi \end{aligned}}$$

$$dx\,dy\,dz = \underline{|J|}\,.\, dr\,d\phi\,d\theta$$

$$\begin{vmatrix} x_r & x_\phi & x_\theta \\ y_r & y_\phi & y_\theta \\ z_r & z_\phi & z_\theta \end{vmatrix}$$

$$\boxed{dx\,dy\,dz = r^2 \sin\phi \; dr\,d\phi\,d\theta}$$

Using spherical polar coordinates

find the volume of the sphere $x^2+y^2+z^2=a^2$

Sol: put $x = r\sin\phi\cos\theta$

$y = r\sin\phi\sin\theta$

$z = r\cos\phi$

$dx\,dy\,dz = r^2\sin\phi\,dr\,d\phi\,d\theta$

$r \to 0$ to a

$\phi \to 0$ to π

$\theta \to 0$ to 2π

$$\text{Vol} = \iiint dx\,dy\,dz = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{r=0}^{a} r^2 \sin\phi \, dr\, d\phi\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \left(\frac{r^3}{3}\right)_0^a \sin\phi \, d\phi\, d\theta$$

$$= \frac{a^3}{3}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \sin\phi\, d\phi\, d\theta$$

$$= \frac{a^3}{3}\int_{\theta=0}^{2\pi} \left(-\cos\phi\right)_0^{\pi} d\theta = -\frac{a^3}{3}\int_0^{2\pi}\left[-1-1\right] d\theta$$

$$= \frac{2a^3}{3}\left[\theta\right]_0^{2\pi} = \boxed{\frac{4\pi a^3}{3}}$$

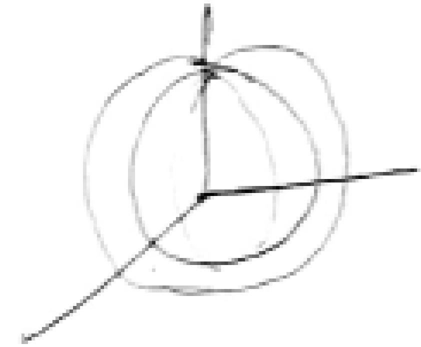
Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dx\, dy\, dz$

by changing into spherical polar coordinates

Sol. put $x = r\sin\phi\cos\theta$

$y = r\sin\phi\sin\theta$

$z = r\cos\phi$



$$dx\,dy\,dz = r^2\sin\phi\,dr\,d\phi\,d\theta$$

The region of integration is bounded by

$z = 0$; $z = \sqrt{1-x^2-y^2}$

$y = 0$; $y = \sqrt{1-x^2}$

$x = 0$; $x = 1$

$r \longrightarrow 0$ to 1

$\phi \longrightarrow 0$ to $\pi/2$

$\theta \longrightarrow 0$ to $\pi/2$

$$I = \int_{r=0}^{1} \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \frac{r^2 \sin\phi \, dr \, d\phi \, d\theta}{\sqrt{1-r^2}}$$

$$\int_{r=0}^{1} \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \frac{r^2+1-1}{\sqrt{1-r^2}} \sin\phi \, dr \, d\phi \, d\theta$$

$$\int_{r=0}^{1} \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \frac{-(1-r^2)+1}{\sqrt{1-r^2}} \, dr \, d\phi \, d\theta$$

$$\int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \left(-\frac{\pi}{4} + \frac{\pi}{2}\right) \sin\phi \, d\phi \, d\theta$$

$$\Rightarrow \frac{\pi}{4}\int_{\theta=0}^{\pi/2}\left(-\cos\phi\right)_0^{\pi/2} d\theta$$

$$= \frac{\pi}{4}\int_{\theta=0}^{\pi/2} -(0-1)\, d\theta$$

$$= \frac{\pi}{4}\left[\theta\right]_0^{\pi/2}$$

$$= \boxed{\frac{\pi^2}{8}}$$

Evaluate $\iiint (x^2+y^2+z^2)\,dx\,dy\,dz$ taken over the volume enclosed by the sphere $x^2+y^2+z^2=1$ by transforming into spherical polar coordinates

Sol: Let $x = r\sin\phi\cos\theta$

$y = r\sin\phi\sin\theta$

$z = r\cos\phi$

$dx\,dy\,dz = r^2\sin\phi\,dr\,d\phi\,d\theta$

$r \to 0$ to 1

$\phi \to 0$ to π

$\theta \to 0$ to 2π

$x^2+y^2+z^2 = r^2\sin^2\phi\cos^2\theta + r^2\sin^2\phi\sin^2\theta + r^2\cos^2\phi$

$= r^2\sin^2\phi + r^2\cos^2\phi$

$= r^2$

$$\iiint (x^2+y^2+z^2)\, dx\,dy\,dz = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{r=0}^{1} r^2\; r^2\sin\phi\; dr\,d\phi\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{r=0}^{1} r^4 \sin\phi\; dr\,d\phi\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \left(\frac{r^5}{5}\right)_0^1 \sin\phi\; d\phi\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{1}{5}\left(-\cos\phi\right)_0^{\pi} d\theta$$

$$= \frac{1}{5}\int_{\theta=0}^{2\pi} -\left[-1-1\right] d\theta = \frac{2}{5}\left(\theta\right)_0^{2\pi} = \boxed{\frac{4\pi}{5}}$$

Using spherical polar coordinates, evaluate $\iiint \frac{1}{x^2+y^2+z^2}\,dx\,dy\,dz$

taken over the volume

bounded by the sphere $x^2+y^2+z^2=a^2$

Sol

Let $x = r\sin\phi\cos\theta$

$y = r\sin\phi\sin\theta$

$z = r\cos\phi$

$$dx\,dy\,dz = r^2\sin\phi\,dr\,d\phi\,d\theta$$

$$x^2+y^2+z^2 = r^2\sin^2\phi\cos^2\theta + r^2\sin^2\phi\sin^2\theta + r^2\cos^2\phi$$

$$= r^2\sin^2\phi + r^2\cos^2\phi = r^2$$

$$G.I = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{r=0}^{a} \frac{1}{x^2+y^2+z^2}\,dx\,dy\,dz$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{r=0}^{a} \frac{1}{\cancel{r^2}}\cdot\cancel{r^2}\sin\phi\,dr\,d\phi\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} (r)_0^a \sin\phi\,d\phi\,d\theta$$

$$= a\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \sin\phi\,d\phi\,d\theta$$

$$= a\int_{\theta=0}^{2\pi} (-\cos\phi)_0^{\pi}\,d\theta$$

$$= a\left[-\int_0^{2\pi}(-1-1)\,d\theta\right.$$

$$= 2a[\theta]_0^{2\pi} = \boxed{4\pi a}$$

using spherical polar coordinates,

evaluate $\iiint xyz \, dx \, dy \, dz$

taken over the volume

bounded by the sphere $x^2+y^2+z^2=a^2$

in the first octant

Sol :

put $x = r \sin\phi \cos\theta$

$y = r \sin\phi \sin\theta$

$z = r \cos\phi$

$dx\,dy\,dz = r^2 \sin\phi \, dr\, d\phi\, d\theta$

$r \rightarrow 0$ to a

$\phi \rightarrow 0$ to $\pi/2$

$\theta \rightarrow 0$ to $\pi/2$

$$\iiint xyz \, dx\, dy\, dz =$$

$$\iiint r^3 \sin^2\phi \cos\theta \sin\theta \cos\phi \cdot r^2 \sin\phi \, dr\, d\phi\, d\theta$$

$$\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2}\int_{r=0}^{a} r^5 \sin^3\phi \cos\theta \sin\theta \cos\phi \, dr\, d\phi\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \left(\frac{r^6}{6}\right)_0^a \sin^3\phi \cos\theta \sin\theta \cos\phi \, d\phi \, d\theta$$

$$= \frac{a^6}{6}\left[\int_{\theta=0}^{\pi/2} \left(\frac{\sin^4\theta}{4}\right)_0^{\pi/2} \cos\theta \sin\theta \, d\theta\right]$$

$$= \frac{a^6}{6} \times \frac{1}{4} \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta$$

$$= \frac{a^6}{24} \times \frac{1}{2} \int_0^{\pi/2} 2\sin\theta \cos\theta \, d\theta$$

$$= \frac{a^6}{48} \int_0^{\pi/2} \sin 2\theta \, d\theta$$

$$= \frac{a^6}{48}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{a^6}{48} \times \frac{1}{2}\left[-(-1-1)\right]$$

$$= \boxed{\frac{a^6}{48}}$$