MAT1011

CALCULUS FOR ENGINEERS

Module . 1

0. Differentiation.
1. Extrema on an Interval, Rolle's & Lagrange's Mean Value theorem
2. First and Second derivative tests
3. Maxima, Minima and Concavity
4. Areas and Volumes
5. Beta, Gamma functions.

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Differentiation

Definition: Let y be a continuous function of x, then an increase in the value of x will produce an increase (or) decrease in the value of y. These increments are generally denoted by the symbols Δx, Δy respectively; Δy is positive (or) negative according as y increases or decreases and similarly for Δx. If Δx, the increment in x indefinitely small, Δy the corresponding increment in y, will also be indefinitely small since y is a continuous function of x.

Usually, the average rate of change of 'y' w.r.to 'x' i.e. the ratio $\frac{\Delta y}{\Delta x}$ tends to a definite limit as Δx tends to zero. This limit when it exists, is called the differential coefficient of y w.r.to x and is denoted by the symbol $\frac{dy}{dx}$

So, if $y = f(x)$, then $y + \Delta y = f(x + \Delta x)$

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$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Note ① The differential coefficient is also called the derivative and is sometimes denoted by the symbol $D_x y$, where $D_x \equiv \frac{d}{dx}$.

② In general, the differential coefficient of $f(x)$ is generally written as $\frac{d}{dx}\left[f(x)\right]$ (or) $f'(x)$.

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Ex: A Construction worker accidently drops a hammer from a height of 90 m while working on the roof of a new apartment building. The height of the hammer s in meters after t seconds is modelled by the function

$s(t) = 90 - 4.9t^2$; $t \geq 0$. Then

(i) Determine the average velocity of the hammer between 1 sec and 4 sec

(ii) Explain the significance of the sign of your result in part (i)

(iii) Determine the velocity of the hammer at 1 sec. and 4 sec.

(iv) When will the hammer hit the ground?

(v) Determine the impact velocity of hammer

(vi) Determine the acceleration function. What do you notice? Interpret the meaning for this situation

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Solution :

(i) Average velocity $= \frac{\Delta s}{\Delta t}$

$$= \frac{s(4) - s(1)}{4-1}$$

$$= \frac{[90-(4.9)4^2] - [90-(4.9)1^2]}{4-1}$$

$$= \frac{73.5}{3} = -24.5$$

The average velocity of the hammer between 1s and 4s is -24.5 m/sec.

(ii) In this type of problem, the movement in the upward direction is commonly assigned positive values. Therefore, the negative answer indicates that the motion of the hammer is downward. Note that the speed of the hammer is 24.5 m/sec.

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(iii) $v(t) = s'(t)$

$$= \frac{d}{dt}\left[90 - 4.9t^2\right]$$

$$= -9.8t$$

Substitute $t=1$ and $t=4$

$v(1) = -9.8(1) = -9.8$

$v(4) = -9.8(4) = -39.2$

The velocity of the hammer at 1 s is -9.8 m/sec and at 4 s is -39.2 m/sec

The negative answers indicate downward movement.

(iv) The hammer hits the ground when the displacement is zero. i.e. $S(t)=0$

$\Rightarrow 90 - 4.9t^2 = 0$

$\Rightarrow t = \pm 4.29$

Therefore, the hammer takes approximately 4.3 sec. to hit the ground.

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(V) The impact velocity of the hammer is the velocity of the hammer when it hits the ground.

$$V(4.3) = (-9.8)(4.3) = -42.14$$

The impact velocity of the hammer is about 42 m/s.

(Vi) The acceleration function is the derivative of velocity function.

$$a(t) = V'(t)$$

$$= \frac{d}{dt}\left[-9.8t\right] = -9.8.$$

The hammer falls at a constant acceleration of $-9.8 \ m/sec^2$.

This value is the acceleration due to gravity for any falling object on earth (when air resistance is ignored).

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Exercise :

① Determine the velocity and acceleration at $t=2$ for each position function $S(t)$ where S is in meters and t is in seconds

(i) $S(t) = t^3 - 3t^2 + t - 1$

(ii) $S(t) = -4.9t^2 + 15t + 1$

(iii) $S(t) = t(3t+5)(1-2t)$

(iv) $S(t) = (t^2-2)(t^2+2)$

② Find the rate of change of area of a circle per second with respect to it's radius $r = 5$ cm

③ Find the rate of change of the area of the circle per second when it's radius of 6 cm is expanding at rate of 1 cm per second.

④ Suppose that $y = x^2 - 3$

(i) Find the average rate of change of y w.r.to x over the interval $[0,2]$

(ii) Find the instantaneous rate of change of y w.r.to x at the point $x = -1$

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Sol. (i) Given that $f(x) = x^2 - 3$

also, $x_0 = 0$; $x_1 = 2$

Average rate of change with $f(x)$ is

$$m = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(2) - f(0)}{2 - 0} = 2$$

i.e. The average rate of change over the interval [0, 2] is 2 units of increase in y for each unit of increase in 'x'.

(ii) Given $f(x) = x^2 - 3$

So, $f'(x) = 2x$

Therefore the instantanous rate of change of y with respect to x at $x = -1$ is

$$m = f'(-1) = -2.$$

This means that the instantanous rate of change is negative. i.e, y is decreasing at $x = -2$.

It is decreasing at a rate of 2 units in y for each unit change in 'x'.

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