Module-5:

Multiple Integrals

1. Evaluation of Double Integrals

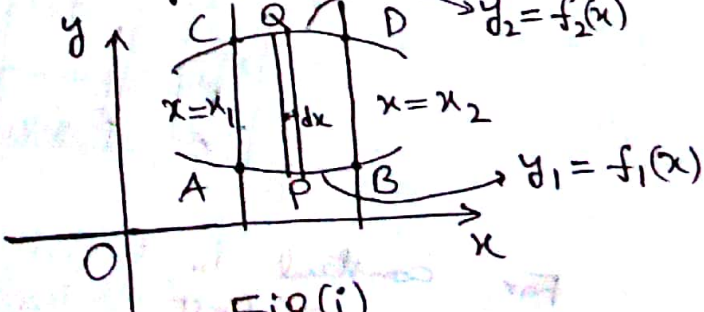
The double integral of $f(x,y)$ over the region A of the xy-plane is written as $\iint_A f(x,y)\,dA$ and expressed as

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dx\,dy.$$

(i) When y_1, y_2 are functions of x and x_1, x_2 are constants; $f(x,y)$ is first integrated w.r.t. y keeping x fixed between limits y_1, y_2 and then the resulting expression is integrated w.r.t. x within the limits x_1, x_2

$$\text{i.e.,}\quad I_1 = \int_{x_1}^{x_2}\left(\int_{y_1}^{y_2} f(x,y)\,dy\right)dx$$

Fig (i) illustrates this process (geometrically illustrated)

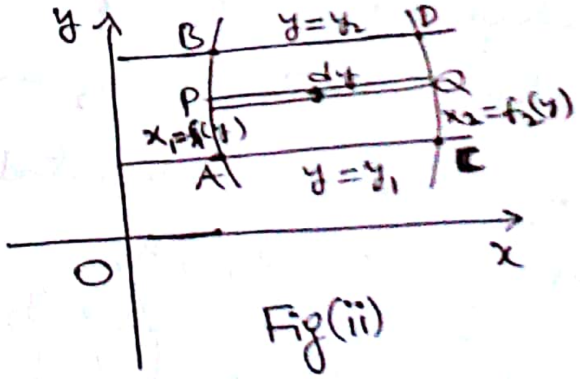


Fig(i)

(ii) When x_1, x_2 are functions of y and y_1, y_2 are constants, $f(x,y)$ is first integrated w.r.t. x keeping y fixed, within the limits x_1, x_2 and the resulting expression is integrated w.r.t. y between the limits y_1, y_2.

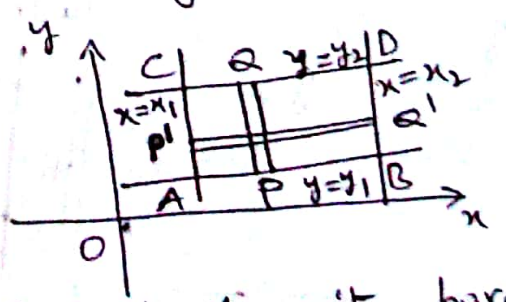
$$\text{i.e., } I_2 = \int_{y_1}^{y_2} \left(\int_{x_1}^{x_2} f(x,y)\,dx \right) dy$$

Fig (ii) illustrates this process geometrically.



Fig(ii)

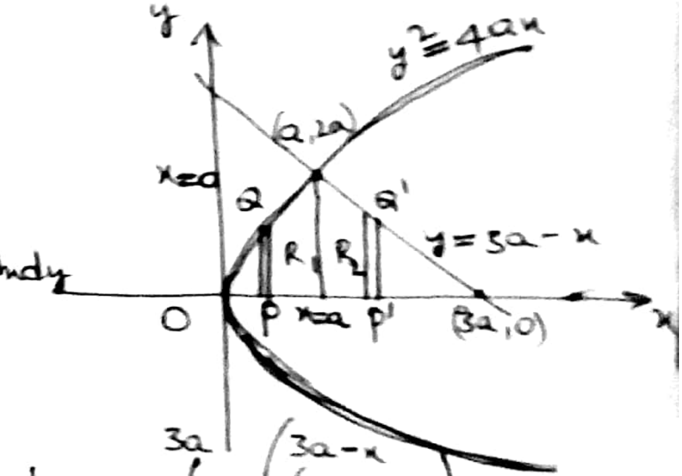
(iii) When both pairs of limits are constants, the region of integration is the rectangle ABCD



For constant limits, it hardly matters whether we first integrate w.r.t. x and then w.r.t. y or vice versa.

1. Using double integral Evaluate the area of the region bounded by the curves $y^2 = 4ax$, $x+y=3a$ and $y=0$

Sol: $\text{Area} = \iint_R dx\,dy$



$$= \iint_{R_1} dx\,dy + \iint_{R_2} dx\,dy$$

$$= \int_{x=0}^{a} \left(\int_{y=0}^{2\sqrt{ax}} dy \right) dx + \int_{x=a}^{3a} \left(\int_{y=0}^{3a-x} dy \right) dx$$

$$= \frac{4a^2}{3} + 2a^2 = \frac{10a^2}{3}$$

2. Evaluate $\iint_R xy\,dx\,dy$, where R is the domain bounded by x-axis, $x=2a$ and the curve $x^2=4ay$

3. Evaluate $\iint_R (4xy-y^2)\,dx\,dy$ where **R** is the region (rectangle) bounded by $x=1$, $x=2$, $y=0$ and $y=3$.

Problems

1. Evaluate the following integrals

i) $\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx$ ii) $\int_0^5 \int_0^{y^2} x(x^2+y^2)\, dx\, dy$

Sol. i) $\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx = \int_{x=0}^1 \left[\int_{y=0}^{x^2} e^{y/x}\, dy \right] dx$

$$= \int_0^1 \left[x e^{y/x} \right]_0^{x^2} dx$$

$$= \int_0^1 x(e^x - 1)\, dx$$

$$= \int_0^1 (x e^x - x)\, dx$$

$$= \left[e^x(x-1) - \frac{x^2}{2} \right]_0^1$$

$$= -\frac{1}{2} - (-1) = \frac{1}{2}$$

ii) $\int_0^5 \int_0^{y^2} x(x^2+y^2)\, dx\, dy = \int_{y=0}^5 \left[\int_{x=0}^{y^2} (x^3 + xy^2)\, dx \right] dy$

$$= \int_0^5 \left[\frac{x^4}{4} + \frac{x^2 y^2}{2} \right]_0^{y^2} dy$$

$$= \int_0^5 \left[\frac{(y^2)^4}{4} + \frac{(y^2)^2 y^2}{2} \right] dx$$

$$= \int_0^5 \left(\frac{y^8}{4} + \frac{y^6}{2} \right) dy$$

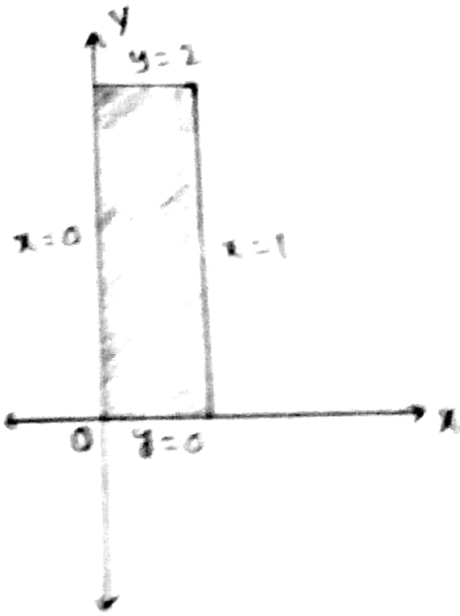
$$= \left(\frac{y^9}{36} + \frac{y^7}{14} \right)_0^5$$

$$= \frac{5^9}{36} + \frac{5^7}{14}$$

$$= \frac{5^7}{4} \left(\frac{25}{9} + \frac{2}{7} \right)$$

2. If A is the area of the rectangular region bounded by the lines $x=0$, $x=1$, $y=0$, $y=2$, evaluate $\int_A (x^2+y^2)\,dA$

Sol. Here, x varies from 0 to 1 and for each x, y varies from 0 to 2.



$$\therefore \int_A (x^2+y^2)\,dA = \int_{x=0}^{1} \left[\int_{y=0}^{2} (x^2+y^2)\,dy \right] dx$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^2 dx$$

$$= \int_0^1 \left(2x^2 + \frac{8}{3} \right) dx$$

$$= \left(2\frac{x^3}{3} + \frac{8}{3}x \right)_0^1$$

$$= \frac{2}{3} + \frac{8}{3}$$

$$= \frac{10}{3}.$$

3. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$ ③

Sol. Given Integral $= \int_{x=0}^{1}\left[\int_{y=0}^{\sqrt{1+x^2}} \frac{dy}{(1+x^2)+y^2}\right]dx$

$= \int_0^1\left[\int_0^p \frac{dy}{p^2+y^2}\right]dx$ where $p=\sqrt{1+x^2}$

$= \int_0^1 \left[\frac{1}{p}\tan^{-1}\left(\frac{y}{p}\right)\right]_0^p dx$

$= \int_0^1 \frac{1}{p}\left[\tan^{-1}(1)-\tan^{-1}(0)\right]dx$

$= \int_0^1 \frac{1}{p}\left[\frac{\pi}{4}-0\right]dx$

$= \frac{\pi}{4}\int_0^1 \frac{dx}{\sqrt{1+x^2}}$ $\quad (\because p=\sqrt{1+x^2})$

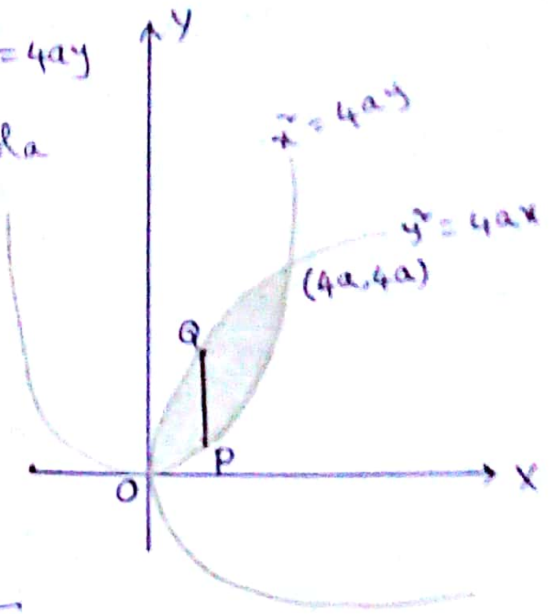
$= \frac{\pi}{4}\left[\log\left(x+\sqrt{x^2+1}\right)\right]_0^1$

$= \frac{\pi}{4}\log(1+\sqrt{2})$

4. Evaluate $\iint_R y\,dx\,dy$, where R is the region bounded by $y^2=4ax$ and $x^2=4ay$, $a>0$.

sol. Given parabolas $y^2=4ax$ and $x^2=4ay$ intersect at $(0,0)$ and $(4a,4a)$. The region bounded by these parabolas is shown in the figure.

In this region, y increases from a point P on the parabola $x^2=4ay$ to a point Q on the parabola $y^2=4ax$.



At P, $y=\frac{x^2}{4a}$ and at Q $y=\sqrt{4ax}$.

x increases from 0 to $4a$.

$$\therefore \iint_R y\,dx\,dy = \int_{x=0}^{4a}\left[\int_{y=\frac{x^2}{4a}}^{\sqrt{4ax}} y\,dy\right]dx$$

$$= \int_0^{4a}\left[\frac{y^2}{2}\right]_{\frac{x^2}{4a}}^{\sqrt{4ax}} dx$$

$$= \int_0^{4a}\frac{1}{2}\left[4ax-\frac{x^4}{16a^2}\right]dx$$

$$= \frac{1}{2}\left[2ax^2-\frac{1}{16a^2}\left(\frac{x^5}{5}\right)\right]_0^{4a}$$

$$= \frac{1}{2}\left[32a^3-\frac{1}{16a^2}\frac{(4a)^5}{5}\right]$$

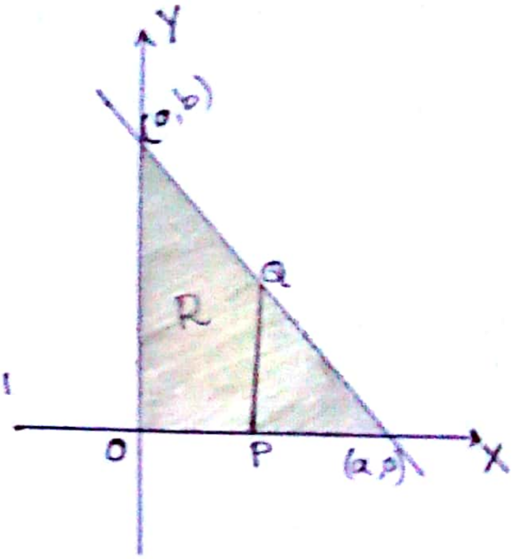
$$= \frac{1}{2}\left[32a^3-\frac{64a^3}{5}\right]$$

$$= \frac{1}{2}\left[\frac{96a^3}{5}\right] = \frac{48}{5}a^3$$

5. Evaluate $\iint xy\,dx\,dy$ over the region in the positive quadrant for which $\frac{x}{a} + \frac{y}{b} \le 1$ (5)

Sol. The shaded region is the region of integration.

Inthis region, for a fixed x, y varies from 0 to $b(1 - \frac{x}{a})$ and then x varies from 0 to a. $(\because \frac{x}{a} + \frac{y}{b} = 1$



$$\iint_R xy\,dx\,dy = \int_{x=0}^{a}\left[\int_{y=0}^{b(1-\frac{x}{a})} y\,dy\right] x\,dx$$

$$= \int_0^a \left[\frac{y^2}{2}\right]_0^{b(1-\frac{x}{a})} x\,dx$$

$$= \frac{1}{2}\int_0^a b^2\left(1 - \frac{x}{a}\right)^2 x\,dx$$

$$= \frac{b^2}{2}\int_0^a \left(1 - \frac{2x}{a} + \frac{x^2}{a^2}\right) x\,dx$$

$$= \frac{b^2}{2}\int_0^a \left(x - \frac{2x^2}{a} + \frac{x^3}{a^2}\right) dx$$

$$= \frac{b^2}{2}\left[\frac{x^2}{2} - \frac{2x^3}{3a} + \frac{x^4}{4a^2}\right]_0^a$$

$$= \frac{b^2}{2}\left[\frac{a^2}{2} - \frac{2a^3}{3a} + \frac{a^4}{4a^2}\right] = \frac{a^2b^2}{24}$$