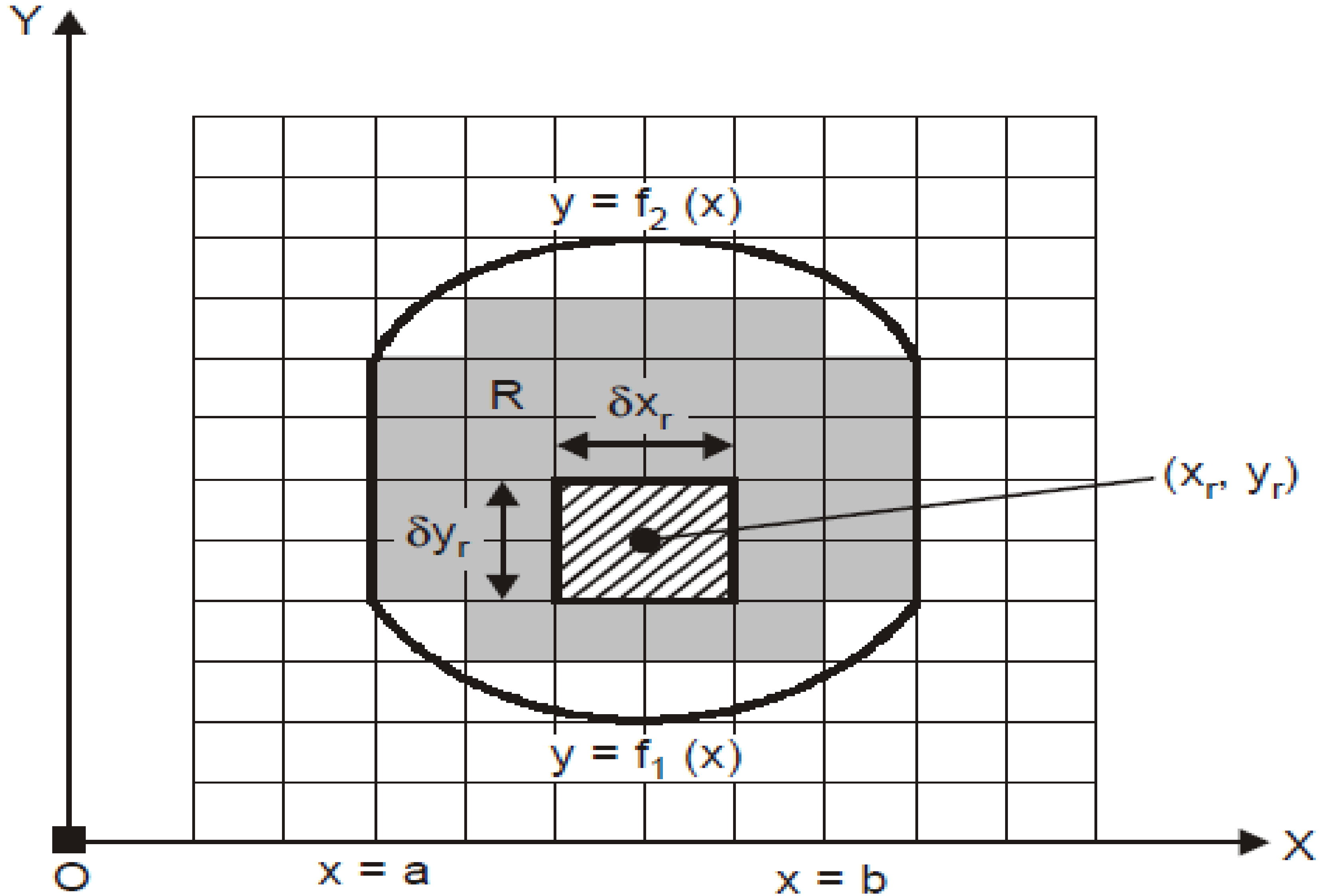
# Module-5

## (Multiple Integrals)

# MULTIPLE INTEGRALS

The process of integration for one variable can be extended to the functions of more than one variable. The generalization of definite integrals is known as **"multiple integral"**.

# DOUBLE INTEGRALS



Consider the region $R$ in the $x, y$ plane we assume that $R$ is a closed*, bounded** region in the $x$ , $y$ plane, by the curve $y = f_1(x)$, $y = f_2(x)$ and the lines $x = a$, $x = b$.

Let us lay down a rectangular

grid on $R$ consisting of a finite number of lines

parallel to the coordinate axes. The $N$ rectangles

lying entirely within $R$

Let $(x_r, y_r)$ be an arbitrarily selected point in the $r$th partition rectangle for each $r = 1, 2, ..., N$. Then denoting the area $\delta x_r \cdot \delta y_r = \delta S_r$

Thus, the total sum of areas

$$S_N = \sum_{r=1}^{N} f(x_r, y_r)\ \delta S_r$$

Let the maximum linear dimensions of each portion of areas approach zero, and $n$ increases indefinitely then the sum $S_N$ will approach a limit, “namely the double integral $\iint\limits_R f(x, y)\, dS$ and the value of this limit is given by

$$\iint\limits_R f(x,y)\, dS = \int_a^b \left[ \int_{f_1(x)}^{f_2(x)} f(x,y)\, dy \right] dx$$

Similarly if the region is bounded by $y = c$, $y = d$ and by the curves $x = f_1(y)$, $x = f_2(y)$, then

$$\iint\limits_R f(x,y)\, dS = \int_c^d \left[ \int_{f_1(y)}^{f_2(y)} f(x,y)\, dx \right] dy$$

**Example 1.** Evaluate $\int_1^{\log 8} \int_0^{\log y} e^{x+y}\, dx\, dy.$

**Sol.** We have

$$
\begin{aligned}
I &= \int_1^{\log 8}\left[\int_0^{\log y}\left(e^x dx\right)\right] e^y\, dy \\
&= \int_1^{\log 8}\left[e^x\right]_0^{\log y} e^y\, dy \\
&= \int_1^{\log 8} e^y\left[e^{\log y} - e^0\right] dy \\
&= \left[e^y(y-1)\right]_1^{\log 8} - \left[e^y\right]_1^{\log 8} \\
&= e^{\log 8}\cdot(\log 8 - 1) - 0 - (e^{\log 8} - e) \\
&= 8\log 8 - 8 - 8 + e \\
&= 8\log 8 - 16 + e.
\end{aligned}
$$

**Example 2.** Evaluate $\int_0^{\pi} \int_0^{x} \sin y \, dy \, dx$.

**Sol.** We have

$$I = \int_0^{\pi}\left[\int_0^{x} \sin y \, dy\right] dx$$

$$= \int_0^{\pi}[-\cos y]_0^{x} \, dx, \text{ treating } x \text{ as constant}$$

$$= -\int_0^{\pi} (\cos x - \cos 0) \, dx$$

$$= -\int_0^{\pi} \cos x \, dx + \int_0^{\pi} dx$$

$$= -[\sin x]_0^{\pi} + [x]_0^{\pi} = \pi.$$

**Example 3.** Evaluate $\int_0^{\sqrt{2}} \int_{-\sqrt{(4-2y^2)}}^{\sqrt{(4-2y^2)}} y \; dx \; dy$.

**Sol.** We have

$$I = \int_0^{\sqrt{2}} \left[ \int_{-\sqrt{(4-2y^2)}}^{\sqrt{(4-2y^2)}} dx \right] y \; dy$$

$$= \int_0^{\sqrt{2}} [x]_{-\sqrt{(4-2y^2)}}^{\sqrt{(4-2y^2)}} y \; dy$$

$$= \int_0^{\sqrt{2}} 2\sqrt{\left(4-2y^2\right)} \; y \; dy$$

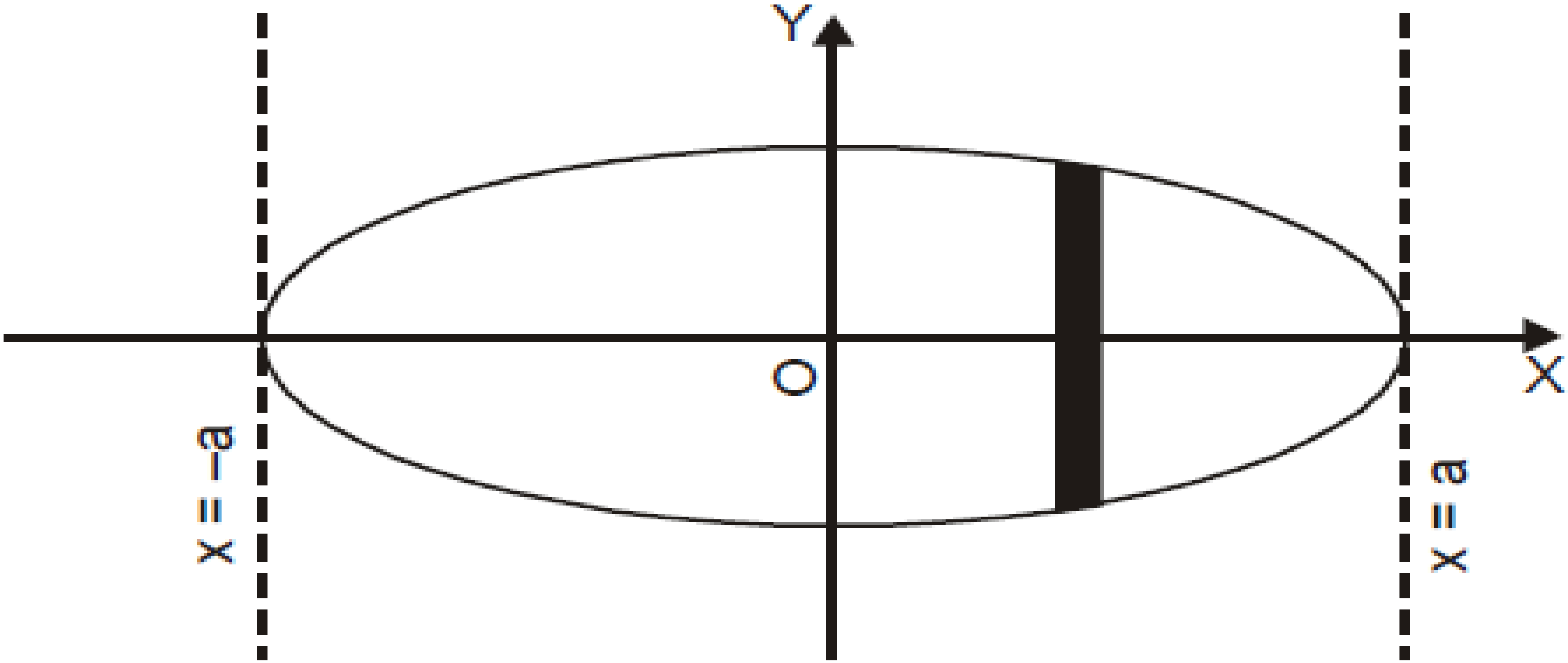
$$= -\frac{1}{2} \int_4^0 \sqrt{t} \; dt \qquad \Big| \text{ by putting } 4 - 2y^2 = t, \; -4y \; dy = dt$$

$$= -\frac{1}{2} \cdot \frac{2}{3} \left[ t^{\frac{3}{2}} \right]_4^0$$

$$= \frac{1}{3} \cdot 8 = \frac{8}{3}.$$

**Example 4.** Evaluate $\iint (x+y)^2 \ dx\,dy$ over the area bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



**Sol.** We have $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\Rightarrow \quad \frac{y}{b} = \pm \sqrt{1-\frac{x^2}{a^2}}$$

$$\Rightarrow \quad y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$$

$$\therefore \quad \iint (x+y)^2 \, dx \, dy = \iint \left(x^2 + y^2 + 2xy\right) dx \, dy$$

$$= \int_{-a}^{a} \int_{\left(\frac{-b}{a}\right)\sqrt{a^2-x^2}}^{\left(\frac{b}{a}\right)\sqrt{a^2-x^2}} \left(x^2 + y^2 + 2xy\right) dx \, dy$$

$$= \int_{-a}^{a} \int_{\left(\frac{-b}{a}\right)\sqrt{a^2-x^2}}^{\left(\frac{b}{a}\right)\sqrt{a^2-x^2}} \left(x^2+y^2\right) \; dx \; dy \; + \; 2\int_{-a}^{a} \int_{\left(-\frac{b}{a}\right)\sqrt{a^2-x^2}}^{\left(\frac{b}{a}\right)\sqrt{a^2-x^2}} xy \; dy \; dx$$

$$= \int_{-a}^{a} 2\int_{0}^{\left(\frac{b}{a}\right)\sqrt{a^2-x^2}} \left(x^2+y^2\right) dy \; dx + 0$$

$$\left| \text{As } \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \left[\text{when } f(x) \text{ even}\right] = 0 \left[\text{when } f(x) \text{ odd}\right]\right.$$

$$= \int_{-a}^{a} \left[2\left(x^2 y + \frac{y^3}{3}\right)\right]_{0}^{\left(\frac{b}{a}\right)\sqrt{a^2-x^2}} dx$$

$$= 2\int_{-a}^{a} \left[x^2 \times \frac{b}{a}\sqrt{a^2-x^2} + \frac{1}{3}\frac{b^3}{a^3}\left(a^2-x^2\right)^{\frac{3}{2}}\right] dx$$

$$= 4\int_0^a \left[\frac{b}{a}x^2\sqrt{a^2-x^2}+\frac{b^3}{3a^3}\left(a^2-x^2\right)^{\frac{3}{2}}\right] dx \text{ (Again by definite integral)}$$

[On putting $x = a \sin \theta$ and $dx = a \cos \theta \, d\theta$]

$$= 4\int_0^{\frac{\pi}{2}}\left(\frac{b}{a}\cdot a^2 \sin^2\theta . a\cos\theta+\frac{b^3}{3a^3}a^3\cos^3\theta\right) \times a \cos \theta \, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}}\left(a^3 b\sin^2\theta\cos^2\theta \, d\theta+\frac{ab^3}{3}\cos^4\theta\right) d\theta$$

$$= 4a^3b \int_0^{\frac{\pi}{2}}\sin^2\theta\cos^2\theta \, d\theta+\frac{4ab^3}{3}\int_0^{\frac{\pi}{2}}\sin^0\theta\cos^4\theta \, d\theta$$

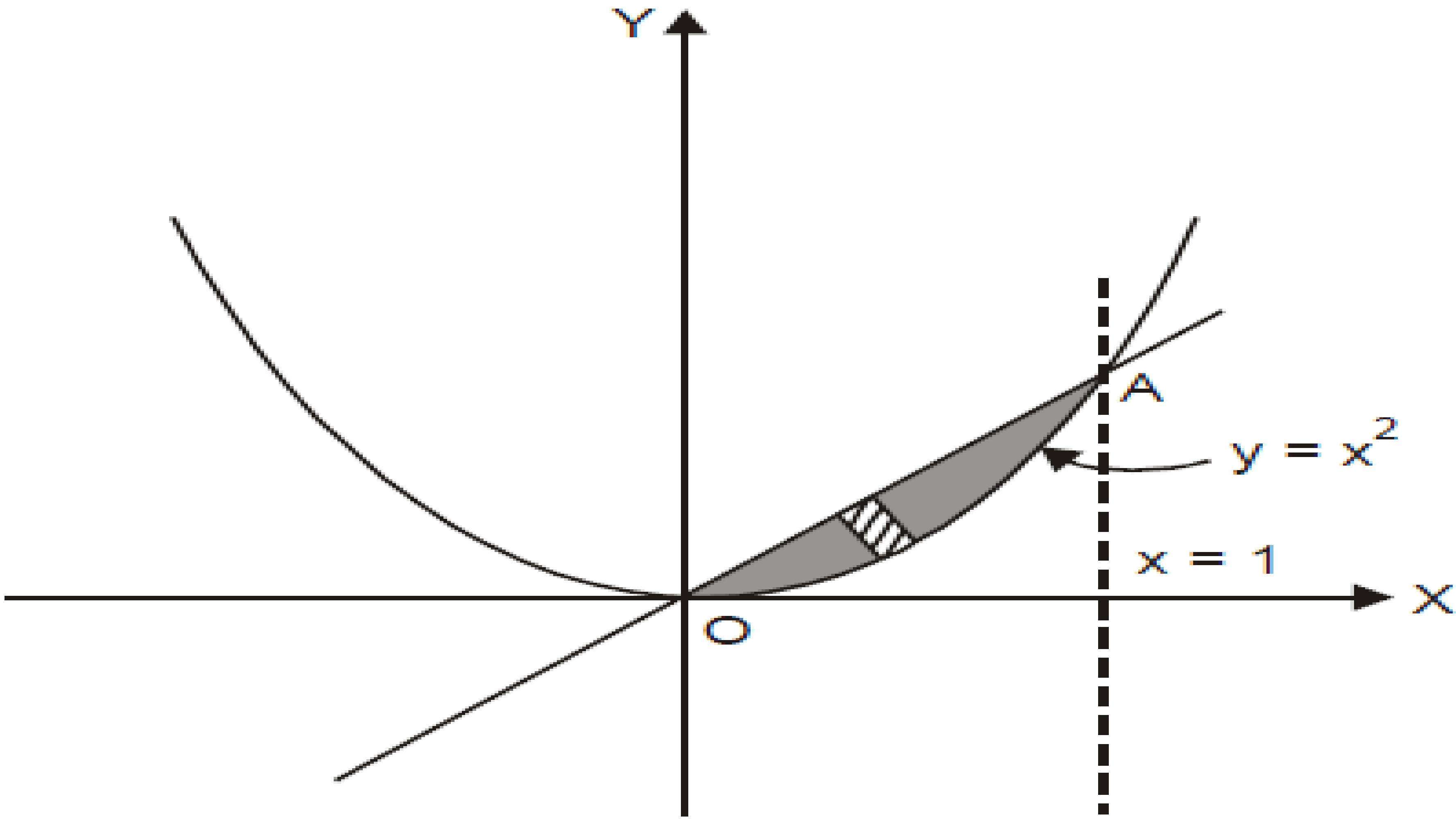
$$= 4a^3b\cdot\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\,\Gamma\left(\frac{2+2+2}{2}\right)}+\frac{4ab^3}{3}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{2\,\Gamma\left(\frac{0+4+2}{2}\right)} \quad \left| \text{As} \int_0^{\frac{\pi}{2}}\sin^m\theta\ \cos^n\theta\ d\theta=\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)}\right.$$

$$= 4a^3b.\ \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}.\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(3)} + \frac{4ab^3}{3}.\ \frac{\Gamma\left(\frac{1}{2}.\frac{3}{2}.\frac{1}{2}.\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(3)}$$

$$= 4a^3b\cdot\frac{\pi}{16} + \frac{4ab^3}{3}\cdot\frac{3\pi}{16} \qquad \left|\text{As } \Gamma(n) = (n-1)\Gamma(n-1) \text{ and } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right.$$

$$= \frac{\pi a^3 b}{4} + \frac{\pi ab^3}{4} = \frac{\pi}{4}\cdot ab\left(a^2+b^2\right).$$

**Example 5.** Evaluate $\iint xy\ (x+y)\,dx\,dy$ over the area between $y = x^2$ and $y = x$.



**Sol.** The area is bounded by the curves $y = f_1(x) = x^2$, $y = f_2(x) = x$.

When $f_1(x) = f_2(x)$,

$$x^2 = x, \text{ i.e., } x(x-1) = 0$$

or $x = 0, x = 1$

*i.e.*, the area of integration is bounded by

$$y = x^2, y = x, x = 0, x = 1$$

$$\therefore \quad = \iint_A xy(x+y)\,dx\,dy$$

$$= \int_{x=0}^{1}\left[\int_{y=x^2}^{y=x} xy(x+y)\,dy\right] dx$$

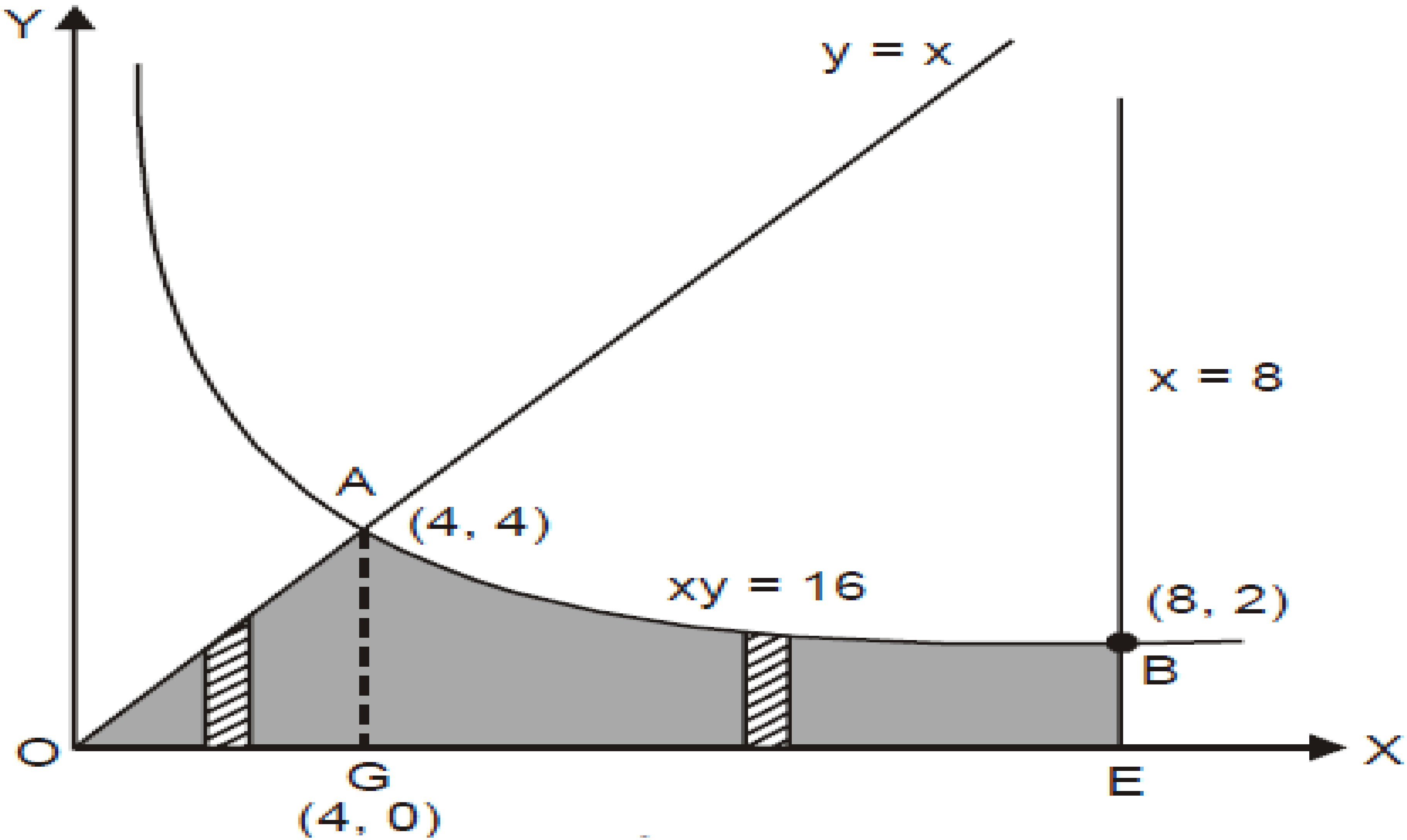
$$= \int_0^1\left[\int_{x^2}^{x}(x^2y + xy^2)\,dy\right] dx$$

$$= \int_0^1\left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{x^2}^{x} dx$$

$$= \int_0^1\left[\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3}\right] dx$$

$$= \left[\frac{x^5}{6} - \frac{x^7}{14} - \frac{x^8}{24}\right]_0^1 = \left[\frac{1}{6} - \frac{1}{14} - \frac{1}{24}\right] = \frac{3}{56}.$$

**Example 6.** Find $\iint_D x^2 dx\, dy$ where $D$ is the region in the first quadrant bounded by the hyperbola $xy = 16$ and the lines $y = x$, $y = 0$ and $x = 8$.



**Sol.** We have

$$xy = 16 \quad \text{...}(i)$$

$$y = x \quad \text{...}(ii)$$

$$y = 0 \quad \text{...}(iii)$$

$$x = 8 \quad \text{...}(iv)$$

From eqns. (*i*) and (*ii*), we get $x = 4$, $y = 4$

*i.e.*, intersection point of curve and the line

$$y = x = (4, 4).$$

Similarly intersection point of (*i*) and (*iv*) = (8, 2)

To evaluate the given integral, we divide the area *OABEO* into two parts by *AG* as shown in the Figure

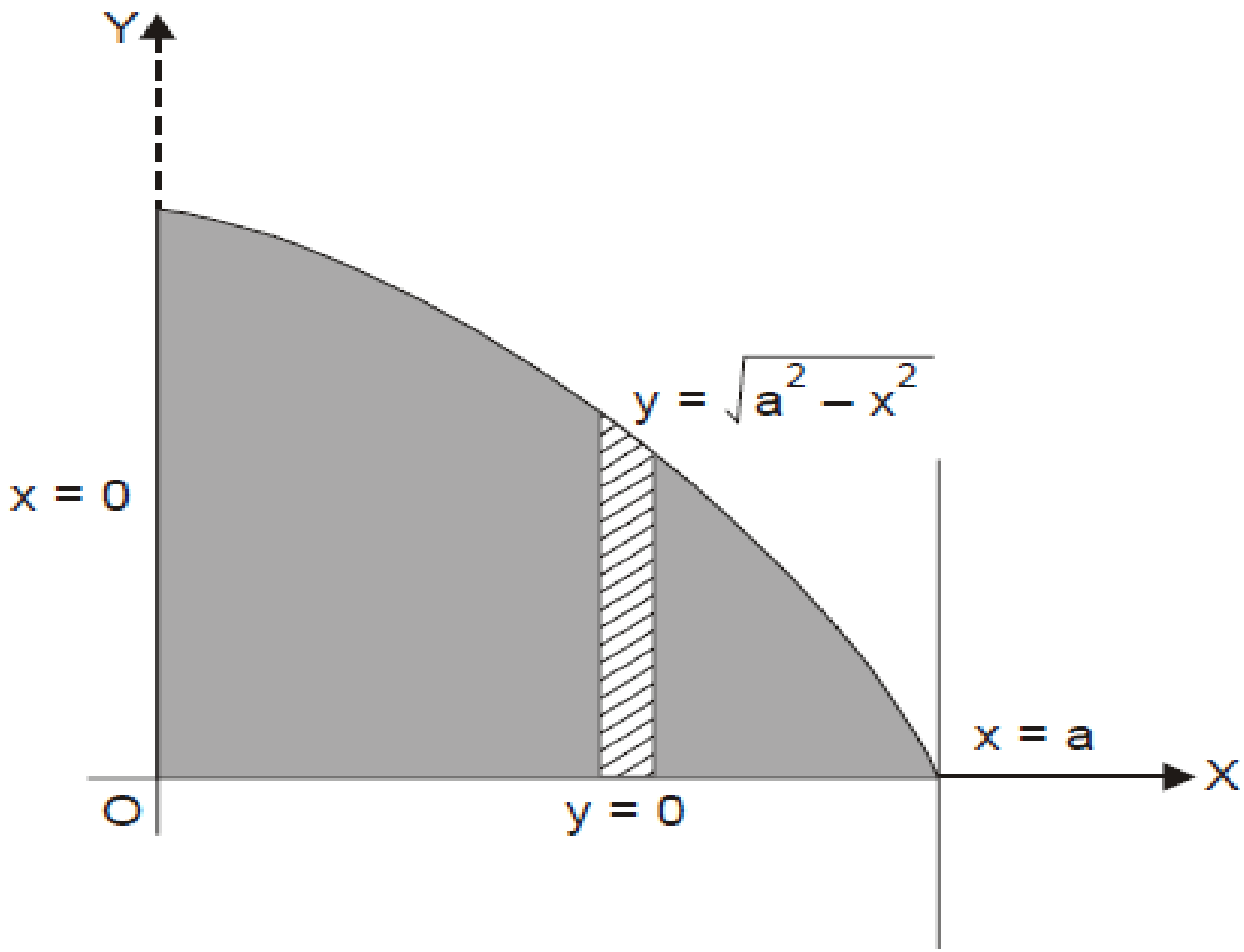
Then, $\iint_D x^2 dx\, dy = \int_{x=0}^{x=4}\int_{y=0}^{y=x} x^2 dx\, dy + \int_{x=4}^{x=8}\int_{y=0}^{y=\frac{16}{x}} x^2 dx\, dy$

$= \int_0^4 x^2 dx \int_0^x dy + \int_4^8 x^2 dx \int_0^{\frac{16}{x}} dy = \int_0^4 x^2 (y)_0^x dx + \int_4^8 x^2 (y)_0^{\frac{16}{x}} dx$

$= \int_0^4 x^3 dx + \int_4^8 16x\, dx = \left[\frac{x^4}{4}\right]_0^4 + \left[8x^2\right]_4^8$

$= 64 + 8(64 - 16) = 64 + 384 = 448.$

**Example 7.** Evaluate $\iint_A xy\,dx\,dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$.



**Sol.** Here the region of integration is positive quadrant of circle $x^2 + y^2 = a^2$, where $x$ varies from 0 to $a$ and $y$ varies from 0 to $\sqrt{(a^2 - x^2)}$.

Here, $$\iint_A xy\,dx\,dy = \int_0^a \int_0^{\sqrt{(a^2-x^2)}} xy\,dx\,dy$$

$$= \int_0^a \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} x\,dx \Bigg]\,dx$$

$$= \int_0^a \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} x\,dx$$

$$= \frac{1}{2}\int_0^a x\left(a^2 - x^2\right)\,dx$$

$$= \frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{1}{8}a^4.$$

**Example 8.** Find $\iint_D \left(x^2+y^2\right) dx\, dy$ where $D$ is bounded by $y = x$ and $y^2 = 4x$.

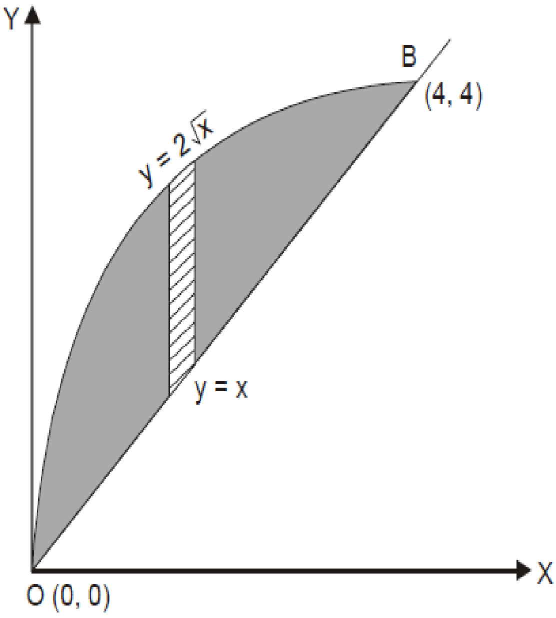
**Sol.**

$$\iint_D \left(x^2+y^2\right) dx\, dy = \int_0^4 \int_x^{2\sqrt{x}} \left(x^2+y^2\right)\, dy\, dx$$

$$= \int_0^4 \left[x^2 y + \frac{y^3}{3}\right]_x^{2\sqrt{x}} dx$$

$$= \int_0^4 \left(2x^{5/2} + \frac{8}{3}x^{3/2} - \frac{4}{3}x^3\right) dx$$

$$= \frac{768}{35}.$$



**Example 9.** Find $\iint_D x^3 y\, dx\, dy$ where $D$ is the region enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ in the first quadrant.

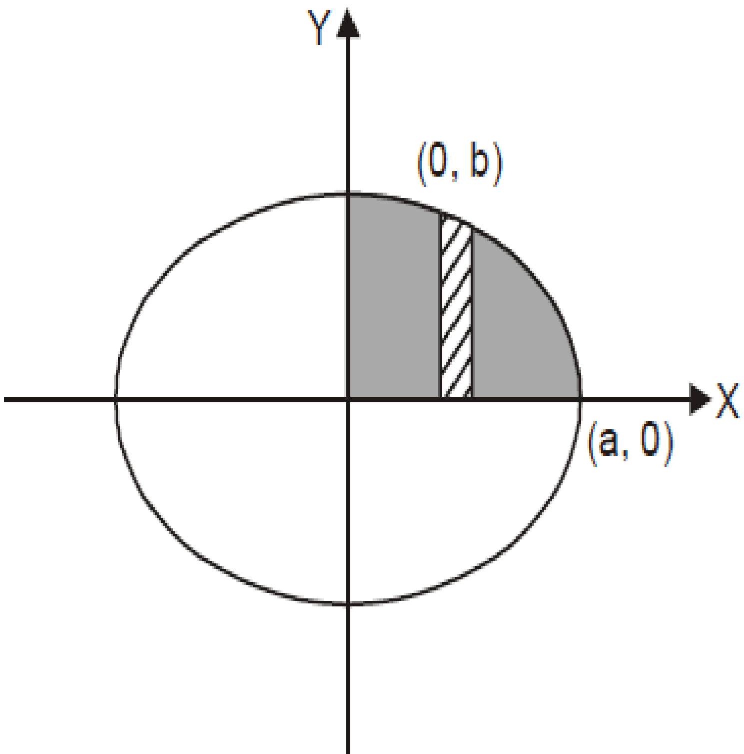
**Sol.**

$$\iint_D x^3 y\, dx\, dy = \int_0^a \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} x^3 y\, dy\, dx$$

$$= \int_0^a \left[\frac{x^3 y^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}}$$

$$= \frac{b^2}{2a^2} \int_0^a \left(a^2 x^3 - x^5\right) dx$$

$$= \frac{b^2}{2a^2}\left[\frac{a^2 x^4}{4} - \frac{x^6}{6}\right]_0^a = \frac{b^2 a^4}{24}.$$



**Example 10.** Evaluate $I = \int_0^{2\pi} \int_{a\sin\theta}^{a} r dr d\theta$.

**Sol.**

$$I = \int_0^{2\pi} \int_{r=a\sin\theta}^{a} r dr d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2}\right]_{a\sin\theta}^{a} d\theta = \frac{1}{2}\int_0^{2\pi} \left(a^2 - a^2 \sin^2\theta\right) d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi} \cos^2\theta \, d\theta = \frac{a^2}{2}\int_0^{2\pi} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$I = \frac{a^2}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{\pi a^2}{2}.$$

**Example 11.** Evaluate $\iint_A r^2 \sin\theta \, d\theta \, dr$ over the area of cordioid $r = a(1 + \cos\theta)$ above the initial line.

**Sol.** The region of integration $A$ can be covered by radial strips whose ends are at $r = 0$, $r = a(1 + \cos\theta)$.

The strips lie between $\theta = 0$ and $\theta = \pi$

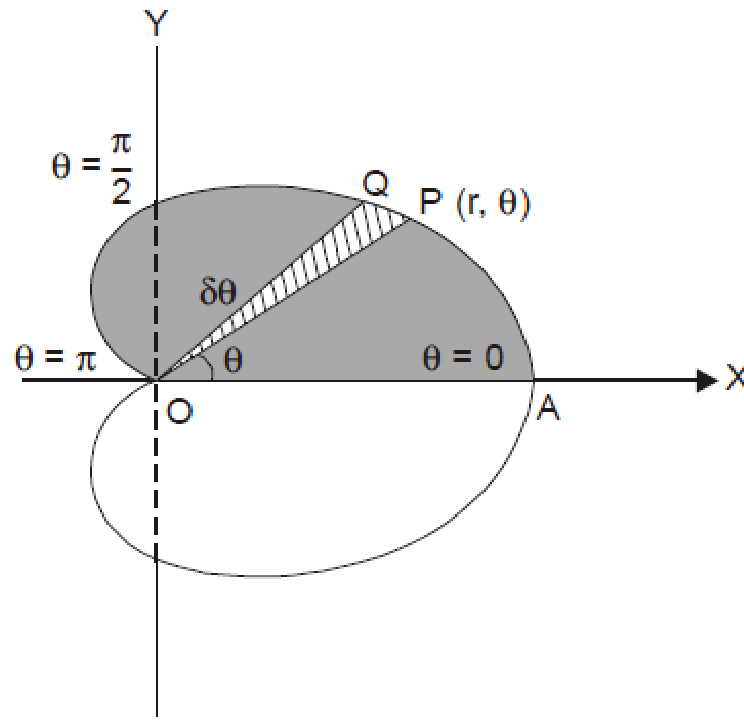



**Fig. 4.9**

Thus $\iint_A r^2 \sin\theta \, d\theta \, dr = \int_0^{\pi} \int_0^{a(1+\cos\theta)} r^2 \sin\theta \, d\theta \, dr$

$$= \int_0^{\pi} \sin\theta \left[ \int_0^{a(1+\cos\theta)} r^2 dr \right] d\theta$$

$$= \int_0^{\pi} \sin\theta \left[ \frac{r^3}{3} \right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi} (1+\cos\theta)^3 \sin\theta \, d\theta$$

$$= \frac{16a^3}{3} \int_0^{\pi} \cos^7 \frac{\theta}{2} \sin\frac{\theta}{2} d\theta$$

$$= \frac{16}{3} a^3 \int_0^{\frac{\pi}{2}} \sin\phi \cos^7 \phi \cdot 2d\phi \qquad \left| \text{Put } \frac{\theta}{2} = \phi \right.$$

$$= 2 \times \frac{16}{3} a^3 \left[ \frac{-\cos^8 \phi}{8} \right]_0^{\frac{\pi}{2}} = \frac{4}{3} a^3.$$

**Example 12.** Evaluate $\int_0^{\frac{\pi}{2}} \int_{a(1-\cos\theta)}^{a} r^2 dr d\theta$.

**Sol.** $\int_0^{\frac{\pi}{2}} d\theta \int_{a(1-\cos\theta)}^{a} r^2 dr = \int_0^{\frac{\pi}{2}} d\theta \left[\frac{r^3}{3}\right]_{a(1-\cos\theta)}^{a}$

$$= \int_0^{\frac{\pi}{2}} d\theta \left(\frac{a^3}{3} - \frac{a^3(1-\cos\theta)^3}{3}\right)$$

$$= \frac{a^3}{3} \int_0^{\frac{\pi}{2}} \left[1-(1-\cos\theta)^3\right] d\theta$$

$$= \frac{a^3}{3} \int_0^{\frac{\pi}{2}} \left[1-\left(1-3\cos\theta+3\cos^2\theta-\cos^3\theta\right)\right] d\theta$$

$$= \frac{a^3}{3} \int_0^{\frac{\pi}{2}} \left(3\cos\theta-3\cos^2\theta+\cos^3\theta\right) d\theta$$

$$= \frac{a^3}{3} \left[\left[3\sin\theta\right]_0^{\frac{\pi}{2}} - 3\frac{1}{2}\frac{\pi}{2} + \frac{2}{3\cdot 1}\right]$$

$$= \frac{a^3}{3} \left[3 - \frac{3\pi}{4} + \frac{2}{3}\right]$$

$$= \frac{a^3}{36} [44 - 9\pi].$$

**Example 13.** Evaluate $\int_0^{\pi} \int_0^{a\theta} r^3 d\theta\, dr$

**Sol.** We have

$$I = \int_0^{\pi} \int_0^{a\theta} r^3 d\theta\, dr$$

$$= \int_0^{\pi} \left[ \int_0^{a\theta} r^3 dr \right] d\theta$$

$$= \int_0^{\pi} \left[ \frac{r^4}{4} \right]_0^{a\theta} d\theta$$

$$= \frac{1}{4} \int_0^{\pi} a^4 \theta^4 d\theta$$

$$= \frac{a^4}{4} \left[ \frac{\theta^5}{5} \right]_0^{\pi} = \frac{a^4 \pi^5}{20}.$$

**Example 14.** Evaluate $\int_0^{\pi} \int_0^{a(1+\cos\theta)} r^3 \sin\theta \cos\theta \, d\theta \, dr$.

**Sol.** We have

$$I = \int_0^{\pi} \sin\theta\cos\theta \left[\int_0^{a(1+\cos\theta)} r^3 dr\right] d\theta$$

$$\int_0^{\pi} \sin\theta\cos\theta \left[\frac{r^4}{4}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{a^4}{4} \int_0^{\pi} (1+\cos\theta)^4 \sin\theta \cos\theta \, d\theta$$

Put $1 + \cos\theta = t$ and $-\sin\theta \, d\theta = dt$

$$= \frac{a^4}{4} \int_2^{0} t^4 (t-1)(-dt)$$

$$= \frac{a^4}{4} \int_0^{2} (t^5 - t^4) \, dt = \frac{16}{15} a^4.$$

**Double Integrals**

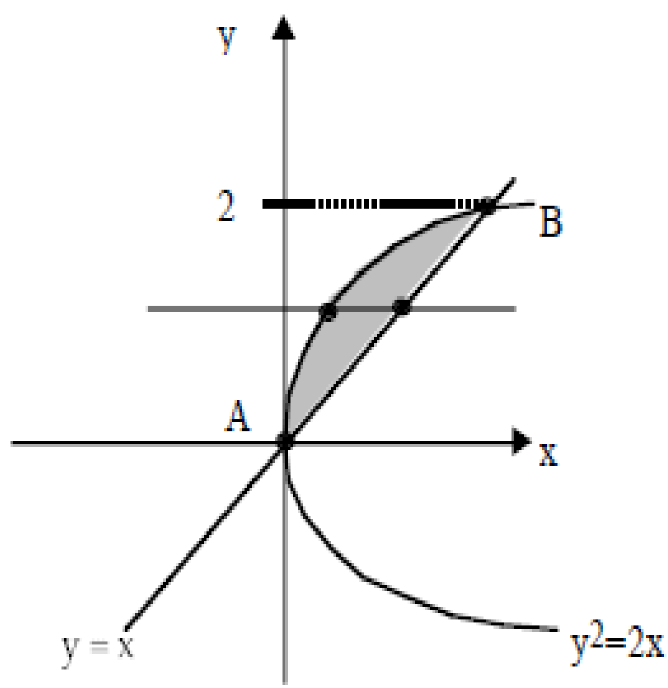
**1.** Sketch the region $R$ in the $xy$-plane bounded by the curves $y^2 = 2x$ and $y = x$, and find its area.

**Solution**

The region $R$ is bounded by the parabola $x = \dfrac{1}{2}y^2$ and the straight line $y = x$. The points of intersection of the two curves are given by

$$y = \frac{y^2}{2} \iff y^2 - 2y = 0 \iff y(y-2) = 0 \iff y = 0, 2.$$

This gives the two points $A = (0,0)$ and $B = (2,2)$.



The region $R$ is a Type II region, and can be described by

$$R: \quad 0 \le y \le 2$$
$$\frac{y^2}{2} \le x \le y.$$

Then,

$$\begin{aligned}
\text{area}(R) &= \int\int_R 1\, dA = \int_0^2 \int_{y^2/2}^{y} 1\, dx\, dy \\
&= \int_0^2 y - \frac{y^2}{2}\, dy = \left(\frac{y^2}{2} - \frac{y^3}{6}\right)\Big|_0^2 \\
&= 2 - \frac{8}{6} = \frac{2}{3}.
\end{aligned}$$

**2.** Evaluate the integral

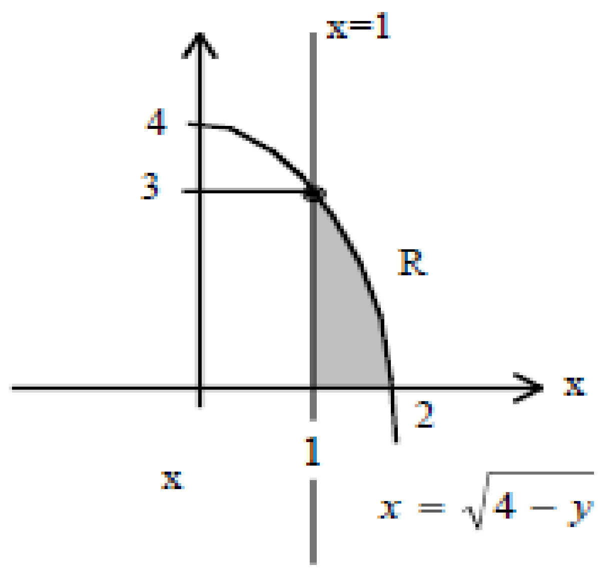
$$\int_{y=0}^{3}\int_{x=1}^{\sqrt{4-y}}(x+y)\,dx\,dy$$

by interchanging the order of integration.

## Solution

The region of integration is the Type II region $R$

$$R: \qquad \begin{aligned} &0 \le y \le 3 \\ &1 \le x \le \sqrt{4-y}. \end{aligned}$$



We have

$$x = \sqrt{4-y} \quad \Rightarrow \quad x^2 = 4 - y \quad \Leftrightarrow \quad y = 4 - x^2.$$

Then, from the drawing above, we can rewrite the region $R$ as the Type I region

$$R: \qquad \begin{aligned} &1 \le x \le 2 \\ &0 \le y \le 4 - x^2. \end{aligned}$$

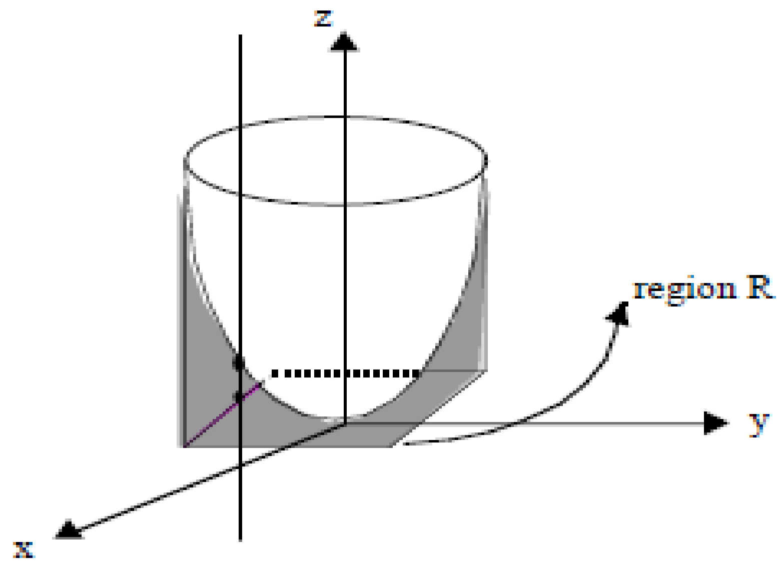
Then,

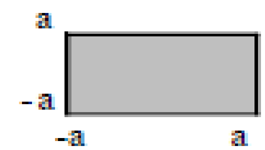
$$\begin{aligned}
\int_{y=0}^{3} \int_{x=1}^{\sqrt{4-y}} (x+y)\, dx\, dy &= \int_{x=1}^{2} \int_{y=0}^{4-x^2} (x+y)\, dy\, dx \\
&= \int_1^2 \left( xy + \frac{y^2}{2} \right) \Big|_0^{4-x^2} \\
&= \int_1^2 4x - x^3 + \frac{16 - 8x^2 + x^4}{2}\, dx \\
&= \left( 2x^2 - \frac{x^4}{4} + 8x - \frac{4x^3}{3} + \frac{x^5}{10} \right) \Big|_1^2 = \frac{241}{60}.
\end{aligned}$$

3. Find the volume of the region of $\mathbb{R}^3$ bounded by the paraboloid $z = x^2 + y^2$ and by the planes $z = 0$, $x = -a$, $x = a$, $y = -a$ and $y = a$.

**Solution**

Let $S$ be the 3D region bounded by the paraboloid and the planes.




Then,

$$\text{volume}(S) = \int\int_R (x^2 + y^2 - 0) \; dA,$$

where $R$ is the projection of $S$ in the $xy$-plane, i.e.

$$R: \qquad \begin{array}{l} -a \le x \le a \\ -a \le y \le a. \end{array}$$

Then,

$$\begin{aligned}
\text{volume}(S) &= \int\int_R x^2 + y^2 \, dA = \int_{-a}^{a}\int_{-a}^{a} x^2 + y^2 \, dy \, dx \\
&= 4\int_0^a \int_0^a x^2 + y^2 \, dy \, dx = 4\int_0^a \left(x^2 y + \frac{y^3}{3}\right)\Big|_0^a \, dx \\
&= 4\int_0^a x^2 a + \frac{a^3}{3} \, dx = 4\left(\frac{x^3 a}{3} + \frac{a^3 x}{3}\right)\Big|_0^a \\
&= 4\left(\frac{2a^4}{3}\right) = \frac{8a^4}{3}.
\end{aligned}$$

**4.** Evaluate $\displaystyle\iint\limits_R \sqrt{x^2+y^2}\,dA$, where $R$ is the region of the plane given by $x^2+y^2 \le a^2$.

**Solution**

The region $R$ and the integrand $\sqrt{x^2+y^2}$ are best described with polar coordinates $(r, \theta)$. In those coordinates, the region $R$, which is the region inside the circle $x^2+y^2=a^2$, becomes

$$\begin{aligned} R: \quad & 0 \le r \le a \\ & 0 \le \theta \le 2\pi. \end{aligned}$$

Then,

$$\begin{aligned} \int\int_R \sqrt{x^2+y^2}\,dA &= \int_0^{2\pi}\int_0^a r\,r\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^a r^2\,dr\,d\theta \\ &= \int_0^{2\pi} \frac{a^3}{3}\,d\theta = \frac{2\pi a^3}{3}. \end{aligned}$$

**5.** Evaluate $\displaystyle\iint\limits_R e^{-(x^2+y^2)}\,dA$, where $R$ is the region of **4.** above.

**Solution**

Using polar coordinates again, we write

$$\begin{aligned}
\iint_R e^{-(x^2+y^2)}\,dA &= \int_0^{2\pi}\int_0^a e^{-r^2}\,r\,dr\,d\theta \\
&= \int_0^{2\pi} \frac{e^{-r^2}}{-2}\Big|_0^a\,d\theta \\
&= \int_0^{2\pi} \frac{1}{2}(1-e^{-a^2})\,d\theta = \pi(1-e^{-a^2}).
\end{aligned}$$

**6.** Evaluate the integral $\int_0^1 \int_{y^2}^1 y e^{x^2}\, dx\, dy$.    **Hint:** First reverse the order of integration.

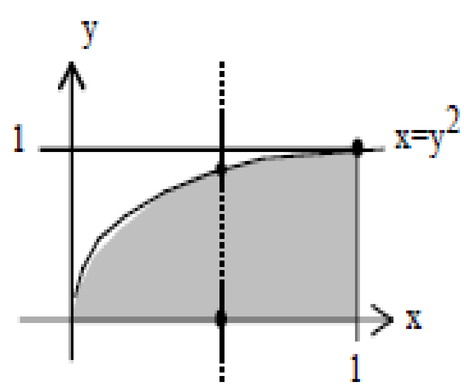
**Solution**

If we try to evaluate the integral as written above, then the first step is to compute the indefinite integral

$$\int e^{x^2}\, dx.$$

But $e^{x^2}$ does not have an indefinite integral that can be written in terms of elementary functions. Then, we will fist reverse the order of integration. The region of integration is the Type II region

$$R: \quad 0 \le y \le 1$$
$$\quad\quad y^2 \le x \le 1.$$



Then, $R$ can also be described as the Type I region

$$R: \qquad \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le \sqrt{x}. \end{array}$$

This gives

$$\begin{aligned} \int_0^1 \int_{y^2}^1 y e^{x^2}\, dx\, dy &= \int_0^1 \int_0^{\sqrt{x}} y e^{x^2}\, dy\, dx \\ &= \int_0^1 \left( \frac{y^2}{2} e^{x^2} \right) \Big|_0^{\sqrt{x}} dx \\ &= \frac{1}{2} \int_0^1 x e^{x^2}\, dx \\ &= \frac{1}{2} \left( \frac{e^{x^2}}{2} \right) \Big|_0^1 = \frac{1}{4}(e-1). \end{aligned}$$

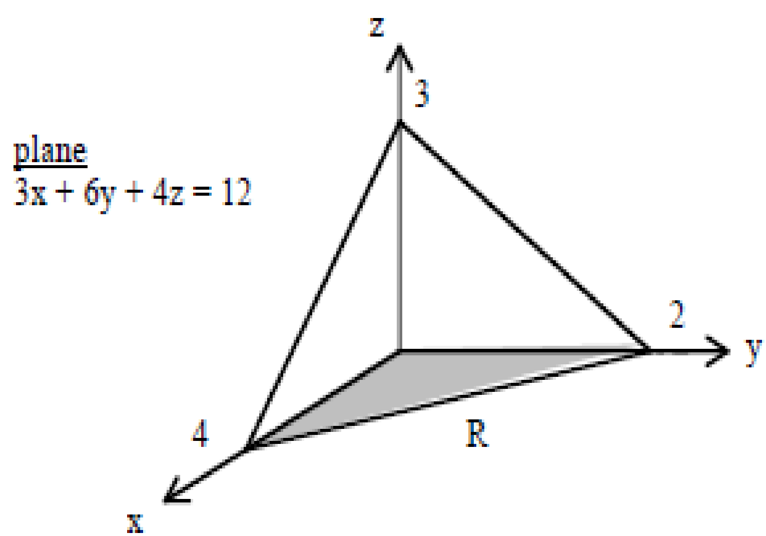
7. Find the volume of the tetrahedron bounded by the coordinate axes and the plane $3x + 6y + 4z = 12$.

**Solution**

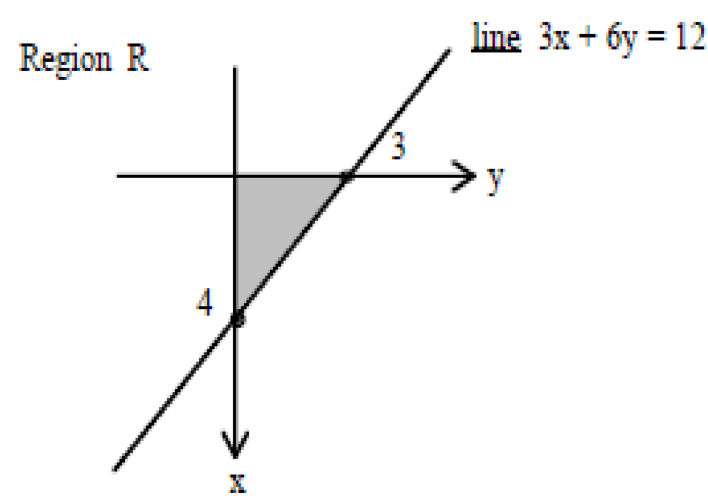
We have to find the volume of the tetrahedron $S$ bounded by the plane

$$3x + 6y + 4z = 12 \iff z = \frac{12 - 3x - 6y}{4}$$

and the coordinate axes. This is the portion of the plane in the first octant, as one can see from Picture (1).




(1)




(2)

Then, we have

$$\text{volume}(S) = \int\int_R \frac{12 - 3x - 6y}{4}\, dA,$$

where $R$ is the projection of the tetrahedron in the $xy$-plane. Then, $R$ is the Type I region (see Picture (2))

$$R: \qquad \begin{array}{l} 0 \le x \le 4 \\ 0 \le y \le \dfrac{12 - 3x}{6}. \end{array}$$

Finally, this gives

$$\begin{aligned}
\text{volume}(S) &= \frac{1}{4}\int_0^4 \int_0^{12-3x/6} (12 - 3x - 6y)\, dy\, dx \\
&= \frac{1}{4}\int_0^4 \left((12 - 3x)y - 3y^2\right)\Big|_0^{2-\frac{x}{2}}\, dx \\
&= \frac{1}{4}\int_0^4 (12 - 3x)\left(2 - \frac{x}{2}\right) - 3\left(2 - \frac{x}{2}\right)^2\, dx \\
&= \frac{1}{4}\int_0^4 \frac{3}{4}x^2 - 6x + 12\, dx \\
&= \frac{1}{4}\left(\frac{x^3}{4} - 3x^2 + 12x\right)\Big|_0^4 = 4.
\end{aligned}$$

**8.** Evaluate the integral

$$\int_0^1 \int_{\sqrt{3}\,y}^{\sqrt{4-y^2}} \sqrt{x^2+y^2}\,dx\,dy.$$

**Hint:** Use polar coordinates.

**Solution**

The region $R$ of integration is the Type II region

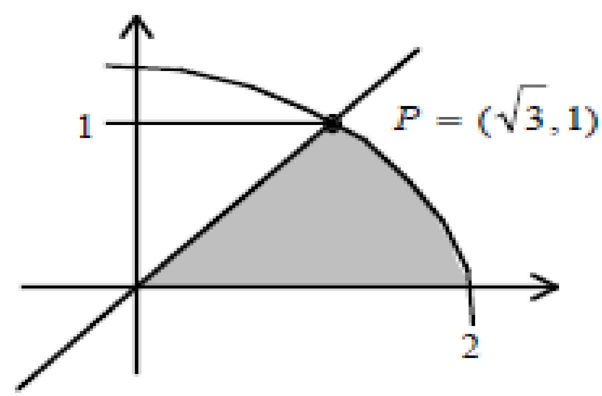
$$R: \quad \begin{array}{l} 0 \le y \le 1 \\ \sqrt{3}\,y \le x \le \sqrt{4-y^2} \end{array}$$

We have

$$x = \sqrt{4-y^2} \quad \Rightarrow \quad x^2 = 4 - y^2 \quad \Leftrightarrow \quad x^2 + y^2 = 4.$$

Then, $x$ varies between the straight line $x = \sqrt{3}\,y$ and the circle $x^2 + y^2 = 4$.

The region R is



In polar coordinates, the region $R$ is

$$R: \qquad \begin{aligned} &0 \le r \le 2 \\ &0 \le \theta \le \alpha, \end{aligned}$$

where $\alpha$ is the angle made by the straight line $x = \sqrt{3}\, y$. The straight line and the cicle meet at the points

$$\left(\sqrt{3}y\right)^2 + y^2 = 4 \iff 4y^2 = 4 \iff y^2 = 1 \iff y = \pm 1.$$

The intersection point in the first quadrant is then $P = (\sqrt{3}, 1) = (2\cos\alpha, 2\sin\alpha)$. Then,

$$\alpha = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}.$$

Finally, the integrand $\sqrt{x^2+y^2}$ is $r$ in polar coordinates. This gives

$$\begin{aligned} \int_0^1 \int_{\sqrt{3}y}^{\sqrt{4-y^2}} \sqrt{x^2+y^2}\, dx\, dy &= \int_0^{\pi/6} \int_0^2 r\, r\, dr\, d\theta \\ &= \int_0^{\pi/6} \frac{r^3}{3}\Big|_0^2\, d\theta \\ &= \frac{\pi}{6}\,\frac{8}{3} = \frac{4\pi}{9}. \end{aligned}$$

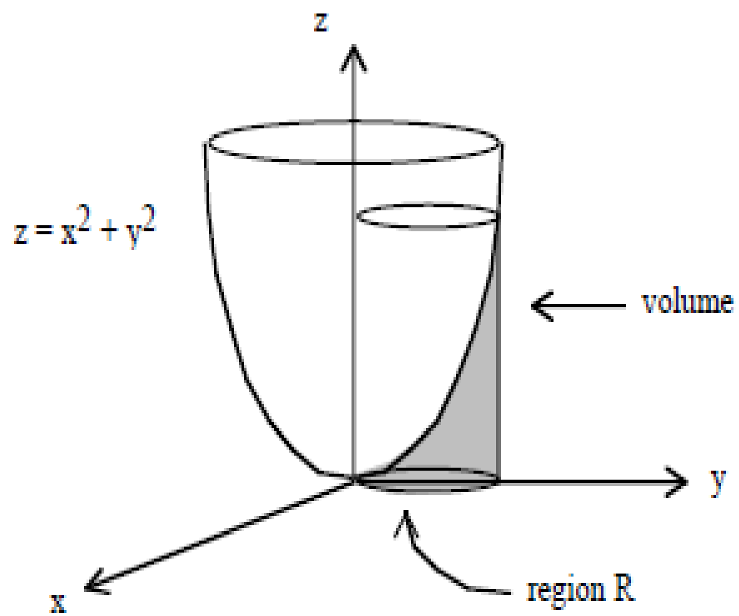
9. Find the volume below the surface $z = x^2 + y^2$, above the plane $z = 0$, and inside the cylinder $x^2 + y^2 = 2y$.

**Solution**

Completing the squares, we rewrite the equation of the cylinder as

$$x^2 + y^2 = 2y \iff x^2 + (y^2 - 2y) = 0 \iff x^2 + (y-1)^2 = 1.$$

The base of the cylinder is then the circle of radius $1$ centered at $(0, 1)$. Then, we have to find the volume of the 3D region:



From the picture above, we write

$$V = \int\int_R x^2 + y^2 \, dA,$$

where $R$ is the projection of the 3D region in the plane, i.e. the circle $x^2 + y^2 = 2y$. Using polar coordinates, this gives

$$V = \int\int_R r^2 r \, dr \, d\theta.$$

In polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, the circle writes as

$$x^2 + y^2 = 2y \iff r^2 = 2r\sin\theta \iff r = 2\sin\theta,$$

and $R$ is the region

$$\begin{aligned} R: \qquad & 0 \le \theta \le \pi \\ & 0 \le r \le 2\sin\theta. \end{aligned}$$

Then,

$$\begin{aligned} V &= \int_0^\pi \int_0^{2\sin\theta} r^3 \, dr \, d\theta = \int_0^\pi \frac{r^4}{4}\Big|_0^{2\sin\theta} d\theta \\ &= 4\int_0^\pi \sin^4\theta \, d\theta = 4\left(\frac{3\theta}{8} - \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta \Big|_0^\pi\right) \\ &= 4\left(\frac{3\pi}{8}\right) = \frac{3\pi}{2}. \end{aligned}$$

# 1 Double Integral

The double integral of $f(x, y)$ over a region $\mathscr{D}$ in the $xy$-plane is denoted by

$$\iint_{\mathscr{D}} f(x, y)\, dA, \tag{1.1}$$

where $dA$ is called an **area element** of $\mathscr{D}$. The double integral (1.1) gives the **volume** of the region bounded by the region $\mathscr{D}$ under the surface $z = f(x, y)$ .

## 2 Double Integral over a Rectangle

Let $f(x,y)$ be continuous on the rectangular region $\mathscr{R} = \{a \le x \le b, c \le y \le d\}$ in the $xy$-plane. Then the double integral of $f(x,y)$ over $\mathscr{R}$ is determined through iterated integrals:

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left\{ \int_{y=c}^{d} f(x,y)dy \right\} dx, \tag{2.1}$$

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left\{ \int_{x=a}^{b} f(x,y)dx \right\} dy. \tag{2.2}$$

The *limits of integration are all constant* in (2.1) and (2.2).

The iterated integral (2.1) or (2.2) gives the **volume** of the region, under the surface $z = f(x,y)$, bounded below by the rectangle $\mathscr{R}$.

If $f(x,y) = g(x)h(y)$, then the iterated integral (2.1) can be written as

$$I = \left\{ \int_a^b g(x)\,dx \right\} \left\{ \int_c^d h(y)\,dy \right\} \tag{2.3}$$

**Example 2.1.** Compute $I = \int_0^2 \int_{-1}^1 x^2 y^2 \, dx dy$

*Solution.* The double integral is written as the product of two individual integrals, as

$$I = \left\{ \int_{-1}^{1} x^2 \, dx \right\} \left\{ \int_{0}^{2} y^2 \, dy \right\} = 2 \left\{ \int_{0}^{1} x^2 \, dx \right\} \left| \frac{y^3}{3} \right|_0^2 = 2 \cdot \frac{1}{3} \cdot \frac{8}{3} = \frac{16}{9}$$

**Example 2.2.** Find $I = \int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$

*Solution.* We write the double integral as the product of two individual integrals:

$$I = \left\{ \int_0^1 \frac{dx}{\sqrt{1-x^2}} \right\} \left\{ \int_0^1 \frac{dy}{\sqrt{1-y^2}} \right\} = \left|\sin^{-1} x\right|_{x=0}^{1} \left|\sin^{-1} y\right|_{y=0}^{1}$$

$$= \left(\sin^{-1} 1\right)^2 = \left(\tfrac{\pi}{2}\right)^2 = \frac{\pi^2}{4}$$

**Example 2.3.** Evaluate $I = \displaystyle\int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1}\, dy\, dx$

*Solution.* We write

$$
\begin{aligned}
I &= \left\{ \int_0^1 \frac{x}{x^2+1}\, dx \right\} \left\{ \int_{-3}^3 y^2\, dy \right\} \\
&= \left| \frac{1}{2} \log(x^2+1) \right|_0^1 \times \left| \frac{2y^3}{3} \right|_0^3 \\
&= \frac{1}{2}(\log 2 - \log 1) \times 2\left(\frac{27}{3}\right) = 9 \log 2
\end{aligned}
$$

**Exercise 2.1.** Evaluate the following double integrals:

(a) $\int_0^{\log 4}\int_0^{\log 3} e^{x+y}\,dx\,dy$

(b) $\int_{-1}^{2}\int_0^{\pi/2} y\sin x\,dx\,dy$

(c) $\int\limits_0^3\int\limits_{-2}^0 (x^2y-2xy)\,dy\,dx$

**Ans.** (a) 6 (b) 3/2 (c) 0

**Exercise 2.2.** Evaluate the double integral of the following functions over the given rectangles:

(a) $f(x,y)=\frac{\sqrt{x}}{y^2}$, $\mathscr{R}: 0\le x\le 4,\ 1\le y\le 2$

(b) $f(x,y)=xye^{x^2+y^2}$, $\mathscr{R}: 0\le x\le 1,\ 0\le y\le 1$

**Ans.** (a) 8/3 (b) $(e-1)^2$

**Example 2.4.** Evaluate $I = \int\limits_0^{\pi/2}\int\limits_0^{\pi/2} \sin(x+2y)\,dx\,dy$

*Solution.* We use the partial integration:

$$I = \int\limits_0^{\pi/2}\left\{\int\limits_0^{\pi/2} \sin(x+2y)\,dx\right\}dy = \int\limits_0^{\pi/2} \Big|-\cos(x+2y)\Big|_{x=0}^{\pi/2} dy$$

$$= \int\limits_0^{\pi/2} (\sin 2y - \cos 2y)\,dy = \tfrac{1}{2}\Big|-\cos 2y - \sin 2y\Big|_{y=0}^{\pi/2}$$

$$= -\tfrac{1}{2}\big[(\cos\pi + \sin\pi) - (\cos 0 + \sin 0)\big] = 1$$

**Exercise 2.3.** Evaluate the double integral of the following functions over the given rectangles:

(a) $f(x,y) = xy\cos(x^2y)$, $\mathscr{R}: 0 \le x \le 2,\ 0 \le y \le 1$

(b) $f(x,y) = 1 - \dfrac{x^2+y^2}{2}$, $\mathscr{R}: 0 \le x \le 1,\ 0 \le y \le 1$

(c) $f(x,y) = y^2\cos x + y$, $\mathscr{R}: 0 \le x \le \pi/2,\ -3 \le y \le 3$

(d) $f(x,y) = \dfrac{x}{y} + \dfrac{y}{x}$, $\mathscr{R}: 1 \le x \le 4,\ 1 \le y \le 2$

(e) $f(x,y) = ye^{-xy}$, $\mathscr{R}: 0 \le x \le 2,\ 0 \le y \le 3$

**Ans.** (a) $1/2$ (b) $2/3$ (c) $8$ (d) $\frac{15}{16} + 15\log 2$ (e) $\frac{1}{2e^6} + \frac{5}{2}$

## 3 Finding Volumes

**Example 3.1.** Find the volume of the region beneath the plane $z = x + y + 1$, which is bounded below by the rectangle $\mathscr{R} = \{-1 \le x \le 1, -1 \le y \le 0\}$.

*Solution.* Using (2.2), the required volume is

$$V = \int_{-1}^{0}\int_{-1}^{1} (x+y+1)\,dx\,dy = \int_{-1}^{0}\left\{\int_{-1}^{1} x\,dx\right\}dy + \int_{-1}^{0}\left\{\int_{-1}^{1} (y+1)\,dx\right\}dy$$

$$= \int_{-1}^{0}\left\{\int_{x=-1}^{1} \overbrace{x}^{\text{odd}}\,dx\right\}dy + \int_{-1}^{0}(y+1)\left\{\int_{x=-1}^{1} \underbrace{(1)}_{\text{even}}\,dx\right\}dy$$

$$= 0 + \int_{-1}^{0}(y+1)\left\{2\int_{0}^{1} dx\right\}dy = \int_{-1}^{0} 2(y+1)\,dy = \left|(y+1)^2\right|_{y=-1}^{0}$$

$$= (0+1)^2 - (-1+1)^2 = 1.$$

**Example 3.2.** Find the volume of the region beneath the plane $z = x + 2y$, which is bounded below by the rectangle $\mathscr{R} = \{1 \le x \le 5, 2 \le y \le 3\}$.

*Solution.* Using (2.2), the required volume is

$$V = \int_2^3 \int_1^5 (x + 2y)\,dx\,dy = \int_2^3 \left| \frac{(x+2y)^2}{2} \right|_{x=1}^5 dy$$

$$= \frac{1}{2} \int_2^3 \{(5+2y)^2 - (1+2y)^2\}\,dy$$

$$= \frac{1}{2} \int_2^3 (16y + 24)\,dy = \int_2^3 (8y + 12)\,dy$$

$$= \left|4y^2 + 12y\right|_{y=2}^3 = (36 + 36) - (16 + 24) = 32.$$

**Example 3.3.** Find the volume of the region under the surface $z = 6x^2y - 2x$, which is bounded below by the rectangle $\mathscr{R} = \{1 \le x \le 4, 0 \le y \le 2\}$.

*Solution.* Using (2.1), the required volume is

$$V = \int_1^4 \int_0^2 \left(6x^2y - 2x\right) dy\, dx = \int_1^4 \left|3x^2y^2 - 2xy\right|_{y=0}^{2} dx = \int_1^4 \left(12x^2 - 4x\right)\, dx$$

$$= \left|4x^3 - 2x^2y\right|_{y=1}^{4} = (256 - 32) - (4 - 2) = 222.$$

**Example 3.4.** Find the volume of the region bounded above by the surface $z = x - y^2$, which is bounded below by the rectangle $\mathscr{R} = \{2 \le x \le 3, 1 \le y \le 2\}$.

*Solution.* Using (2.1), the required volume is

$$V = \int_2^3 \int_1^2 (x - y^2)\, dy dx = \int_2^3 \left| xy - \frac{y^3}{3} \right|_{y=1}^{2} dx = \int_2^3 \left[ \left(2x - \frac{8}{3}\right) - \left(x - \frac{1}{3}\right) \right] dx$$

$$= \int_2^3 \left(x - \frac{7}{3}\right) dx = \left| \frac{x^2}{2} - \frac{7x}{3} \right|_{x=2}^{3} = \left(\frac{9}{2} - \frac{21}{3}\right) - \left(\frac{4}{2} - \frac{14}{3}\right) = \frac{1}{6}$$

**Example 3.5.** Find the volume enclosed by the paraboloid $z = x^2 + y^2$, over the rectangle $\mathscr{R} = \{-1 \le x \le 1, -1 \le y \le 1\}$.

*Solution.* Using (2.2), the required volume is

$$V = \int_{-1}^{1} \left\{ \int_{-1}^{1} \overbrace{(x^2 + y^2)}^{\text{even}} \, dx \right\} dy = 2 \int_{-1}^{1} \left\{ \int_{0}^{1} (x^2 + y^2) \, dx \right\} dy$$

$$= 2 \int_{-1}^{1} \left| \tfrac{x^3}{3} + x y^2 \right|_{x=0}^{1} dx = 2 \int_{-1}^{1} \underbrace{\left(\tfrac{1}{3} + y^2\right)}_{\text{even}} dy$$

$$= 4 \int_{0}^{1} \left(\tfrac{1}{3} + y^2\right) \, dy = 4 \left| \frac{y}{3} + \frac{y^3}{3} \right|_{y=0}^{1} = \frac{8}{3}$$

**Exercise 3.1.** Find the volume bounded above by the surface $z = 2\sin x \cos y$, above the rectangle $\mathscr{R} = \{0 \le x \le \pi/2, 0 \le y \le \pi/4\}$.

**Ans.** $\sqrt{2}$

**Exercise 3.2.** Find the volume of the region bounded above by the surface $z = \frac{1}{xy}$, above the square $\mathscr{S} = \{1 \le x \le 2, 1 \le y \le 2\}$.

**Ans.** $\sqrt{2}$

**Exercise 3.3.** Find the volume under the plane $z = 3x + 4y$, over the rectangle $\mathscr{R} = \{1 \le x \le 2, 0 \le y \le 3\}$.

**Ans.** $63/2$

**Exercise 3.4.** Find the volume under the surface $z = x\sin(x + y)$, over the rectangle $\mathscr{R} = \{0 \le x \le \pi/6, 0 \le y \le \pi/3\}$.

**Ans.** $\pi/12$

**Exercise 3.5.** Find the volume under the elliptical paraboloid $z = 16 - x^2 - y^2$, above the square $\mathscr{S} = \{0 \le x \le 2, 0 \le y \le 2\}$.

**Ans.** $160/3$

## 4 Double Integral Over General Regions – Limits Specified

**Example 4.1.** Sketch the region $\mathscr{D}$ of integration through the following inequalities:

(a) $0 \leq x \leq 3, 0 \leq y \leq 2x$

**Ans.** $\mathscr{D}$ is enclosed by the triangle whose sides are the lines $y = 0$, $x = 3$ and $y = 2x$

(b) $-1 \leq x \leq 2, x - 1 \leq y \leq x^2$

**Ans.** $\mathscr{D}$ is enclosed by the parabola $y = x^2$, the line $y = x - 1$ and the ordinates $x = -1, x = 2$

(c) $y^2 \leq x \leq 4, -2 \leq y \leq 2$

**Ans.** $\mathscr{D}$ is enclosed by the parabola $x = y^2$, the lines $y = \pm 2$ and the ordinates $x = 4$

(d) $y \leq x \leq 2y, 0 \leq y \leq 1$

**Ans.** $\mathscr{D}$ is the triangular region bounded by the lines $x = y$, $x = 2y$ and $y = 1$

(e) $\frac{y}{4} \leq x \leq y^{1/3}, 0 \leq y \leq 8$

**Ans.** $\mathscr{D}$ is enclosed by the straight line $y = 4x$ and the cubical parabola $y = x^3$

**Example 4.2.** Sketch the region $\mathscr{D}$ of integration and compute the following double integrals:

(a) $I = \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} x^3 y \, dy dx$

*Solution.* The region $\mathscr{D}$, defined by the inequalities $0 \le x \le a$ and $0 \le y \le b\sqrt{1-x^2/a^2}$ is enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the first quadrant. Now,

$$I = \int_{x=0}^a \left\{ \int_{y=0}^{b\sqrt{1-x^2/a^2}} x^3 y \, dy \right\} dx = \int_{x=0}^a x^3 \left\{ \left| y^2 \right|_{y=0}^{b\sqrt{1-x^2/a^2}} \right\} dx$$

$$= b^2 \int_0^a x^3 \left( 1 - \frac{x^2}{a^2} \right) dx = b^2 \int_0^a \left( x^3 - \frac{x^5}{a^2} \right) dx$$

$$= b^2 \left| \frac{x^4}{4} - \frac{x^6}{6a^2} \right|_0^a = b^2 \left( \frac{a^4}{4} - \frac{a^6}{6a^2} \right) = \frac{a^4 b^2}{12}.$$

The double integral gives the volume under the surface $z = x^3 y$, bounded below by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the first quadrant.

(b) $I = \int_0^1 \int_0^{x^2} x\left(x^2 + y^2\right)\, dy dx$

*Solution.* The region $\mathscr{D}$, defined by $0 \le x \le 1$ and $0 \le y \le x^2$ is enclosed by the parabola $y = x^2$, the $x$-axis and the line $x = 1$. Then

$$I = \int_{x=0}^{1} \left\{ \int_{y=0}^{x^2} \left(x^3 + xy^2\right)\, dy \right\} dx = \int_{x=0}^{1} \left| x^3 y + \tfrac{xy^3}{3} \right|_{y=0}^{x^2} dx$$

$$= \int_0^1 \left(x^5 - \frac{x^7}{3}\right) dx = \left| \frac{x^6}{6} - \frac{x^8}{24} \right|_0^1 = \frac{1}{6} - \frac{1}{24} = \frac{1}{8}.$$

The double integral gives the volume under the surface $z = x^3 + xy^2$, bounded below the plane region $\mathscr{D}$.

(c) $I = \int_0^{\pi} \int_0^{x} x \sin y \, dy dx$

*Solution.* The region $\mathscr{D}$, described by the inequalities $0 \le x \le \pi$, $0 \le y \le x$ is bounded by the triangle with sides $y = x$, $y = 0$ and $x = \pi$, and

$$I = \int_{x=0}^{\pi} x \left\{ \int_{y=0}^{x} \sin y \, dy \right\} dx = \int_{x=0}^{\pi} x \left| -\cos y \right|_{y=0}^{x} dx$$

$$= \int_0^{\pi} x(1 - \cos x) dx = \left| (x)(x - \sin x) - (1)\left(\frac{x^2}{2} + \cos x\right) \right|_{x=0}^{\pi}$$

$$= \left\{ (\pi)(\pi - \sin \pi) - \frac{\pi^2}{2} - \cos \pi \right\} - \{0 - (0 + \cos 0)\} = 2 + \frac{\pi^2}{2}.$$

The double integral gives the volume under the surface $z = x \sin y$, bounded below the plane region $\mathscr{D}$.

(d) $I = \int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}}\, dy dx$

*Solution.* The region $\mathscr{D}$ is defined by $1 \le x \le 4$, $0 \le y \le \sqrt{x}$ is enclosed by the parabola $y^2 = x$, the $x$-axis and the ordinates $x = 1$ and $x = 4$. And

$$I = \frac{3}{2}\int_{x=1}^{4}\left\{\int_{y=0}^{\sqrt{x}} e^{y/\sqrt{x}}\, dy\right\} dx = \frac{3}{2}\int_{x=0}^{4}\left|\sqrt{x}e^{y/\sqrt{x}}\right|_{y=0}^{\sqrt{x}} dx$$

$$= \frac{3}{2}\int_0^4 \sqrt{x}(e-1)\, dx = \frac{3(e-1)}{2}\left|\frac{2}{3}x^{3/2}\right|_{x=0}^{4} = 8(e-1)$$

(e) $I = \int_1^2 \int_x^{2x} \frac{x}{y} dy dx$

*Solution.* The region $\mathscr{D}$, defined by $1 \leq x \leq 2$, $x \leq y \leq 2x$, is bounded by the quadrilateral whose sides are $x = 1$, $x = 2$, $y = x$, $y = 2x$, and

$$I = \int_{x=1}^{2} \left\{ \int_{y=x}^{2x} \left( \frac{x}{y} \right) dy \right\} dx = \int_{x=1}^{2} x \Big| \log y \Big|_{y=x}^{2x} dx$$

$$= \int_1^2 x(\log 2) dx = (\log 2) \left| \frac{x^2}{2} \right|_{x=1}^{2} = \frac{3 \log 2}{2}$$

**Exercise 4.1.** Sketch the region $\mathscr{D}$ of integration and compute the following double integrals:

(a) $I = \int_0^{\pi} \int_0^{\sin x} y \, dy dx$

**Ans.** The region $\mathscr{D}$ is described by $0 \le x \le \pi$, $0 \le y \le \sin x$, which is enclosed by the sine curve $y = \sin x$ between $x = 0$ and $x = \pi$; $I = \pi/4$

(b) $I = \int_1^{2} \int_y^{y^2} dx dy$

**Ans.** The region $\mathscr{D}$, described by $1 \le y \le 2$, $y \le x \le y^2$, is enclosed by the lines $y = 2$ and $y = x$ and the parabola $y^2 = x$; $I = 5/6$

(c) $I = \int_0^{1} \int_0^{y^2} 3y^3 e^{xy} dx dy$

**Ans.** The region $\mathscr{D}$, described by $0 \le y \le 1$, $0 \le x \le y^2$, is enclosed by the $y$-axis, the line $y = 1$ and the parabola $y^2 = x$; $I = e - 2$

(d) $I = \int_0^4 \int_0^{\sqrt{4-x^2}} xy \, dy dx$

**Ans.** The region $\mathscr{D}$ is described by $0 \le x \le 4$, $0 \le y \le \sqrt{4-x^2}$, which is enclosed by the circle $x^2 + y^2 = 4$ and the coordinate axes in the first quadrant; $I = -16$

(e) $\int_0^1 \int_0^{\sqrt{1-x^2}} 8y \, dy dx$

**Ans.** The region $\mathscr{D}$, defined by $0 \le x \le 1$, $0 \le y \le \sqrt{1-x^2}$, is enclosed by the unit circle $x^2 + y^2 = 1$ and the coordinate axes in the first quadrant; $I = 8/3$

**Exercise 4.2.** Sketch the region $\mathscr{D}$ of integration and compute the following double integrals:

(a) $\displaystyle\int_0^{\pi/2}\int_0^{\cos y} e^x \sin y\, dx\, dy$
Ans. $e-2$

(b) $\displaystyle\int_1^2\int_0^y dx\, dy$
Ans. $5/4$

(c) $\displaystyle I=\int_0^1\int_{x^2}^{x^3}\left(x^2+y^2\right) dy\, dx$
Ans. $-\frac{1}{21}$

# 6 Finding Volumes

**Exercise 6.1.** Compute the volume of the solid

($a$) bounded above by the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by the lines $y = x$, $x = 0$ and $x + y = 2$ in the $xy$-plane

**Ans.** $V = \int\limits_0^1 \int\limits_x^{2-x} (x^2 + y^2)\ dydx = \frac{4}{3}$

($b$) in the first octant bounded by the coordinate planes, the plane $x = 3$, and the parabolic cylinder $z = 4 - y^2$.

**Ans.** $V = \int\limits_0^2 \int\limits_0^3 (4 - y^2)\ dxdy = 16$

($c$) in the first octant bounded by the planes $x = 2$ and $y = 4$ and the cylinder $z = y^2$.

**Ans.** $V = \int\limits_0^2 \int\limits_0^4 y^2\ dydx = 128/3$

($d$) enclosed by the planes: $y = 0, z = 0$, $y = 3$, $z = x$ and $x + z = 4$

**Ans.** Given $x$ and $y$, we observe that $z$ varies fron $x$ to $z = 4 - x$. Therefore $V = \int\limits_0^3 \int\limits_0^2 (4 - x - x)\ dxdy = \int\limits_0^3 \int\limits_0^2 (4 - 2x)\ dxdy = 12$

($e$) enclosed by the tetrahedron with faces: $x = 0, y = 0, z = 0$ and the plane $x + y + z = 1$

**Ans.** $V = \int\limits_0^1 \int\limits_0^{1-x} (1 - x - y)\ dydx = 1/6$

# 8 Double Integral in Polar Form

**Limits Specified:** $I = \int_{\theta=\alpha}^{\beta} \int_{r=r_1}^{r_2} f(r,\theta)\, dr\, d\theta$ or $\int_{\theta=\alpha}^{\beta} \int_{r=f_1(\theta)}^{f_2(\theta)} f(r,\theta)\, dr\, d\theta$

**Exercise 8.1.** Evaluate the following integrals:

(a) $\int\limits_{0}^{\frac{\pi}{4}} \int\limits_{0}^{a\sin\theta} \frac{r}{\sqrt{a^2-r^2}}\, dr\, d\theta$

(b) $\int\limits_{0}^{\frac{\pi}{2}} \int\limits_{a\cos\theta}^{a} r^4 dr\, d\theta$

(c) $\int\limits_{0}^{\pi} \int\limits_{0}^{1} r\sin\theta\, dr\, d\theta$

(d) $\int\limits_{0}^{\pi} \int\limits_{0}^{a} r^3 \sin\theta \cos\theta\, dr\, d\theta$

**Limits Not Specified:** To evaluate $I = \iint\limits_R f(r,\theta)\, dr\, d\theta$:

$(a)$ Sketch the region $R$ and label the bounding curves

$(b)$ Imagine a radius vector $L$ from the pole in the direction of increasing $r$, passing through $R$

$(c)$ Locate the points $P$ and $Q$, where $L$ enters and leaves the region $R$ respectively. Express $r$ in terms of $\theta$ at these points, say $r = \phi_1(\theta)$ at $P$ and $r = \phi_2(\theta)$ at $Q$

$(d)$ Choose the (constant) $\theta$-limits that include all such radii, say $\theta = \alpha$ to $\theta = \beta$

$(e)$ Thus the limits of integration are $r = \phi_1(\theta)$ to $\phi_2(\theta)$, and $\theta = \alpha$ to $\beta$

$$I = \int\limits_{\theta=\alpha}^{\beta} \left\{ \int\limits_{y=\phi_1(\theta)}^{\phi_2(\theta)} f(r,\theta)\, dr \right\} d\theta$$

**Exercise 8.2.** Evaluate the double integrals of following functions $f(r,\theta)$ over the given region $\mathscr{R}$:

(a) $r^2$ over the region $R$ enclosed by the circles $r = a\sin\theta$ and $r = b\sin\theta$ where $a < b$

(b) $r^2\sin\theta$ bounded by the semicircle $r = 2a\cos\theta$ above the initial line

(c) $r\sin\theta$ bounded by the cardioid $r = a(1-\sin\theta)$ above the initial line

(d) $\frac{r}{\sqrt{r^2+a^2}}$ over the region $R$ enclosed by one loop of the lemniscate $r^2 = a^2\cos 2\theta$

## 9 Change of Variable - Cartesian into Polar:

The conversion of Cartesian double integral to polar form is governed by the following equation:

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta,$$

where $G$ is the Cartesian region $R$, expressed in terms of polar coordinates.

**Example 9.1.** Evaluate the following double integrals through the polar transformation: $x = r\cos\theta, y = r\sin\theta$:

$$(a)\ I = \int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-x^2-y^2}\, dydx$$

*Solution.* The region $R$ is described by the inequalities $0 \le x \le a$, $0 \le y \le \sqrt{a^2 - x^2}$, which is enclosed by the circle $x^2 + y^2 = a^2$ in the first quadrant. The polar limits are $r = 0$ to $a$; $\theta = 0$ to $\frac{\pi}{2}$.

Therefore, $I = \int_0^{\pi/2} \int_0^a e^{-r^2}\, r dr d\theta = \frac{\pi\left(1-e^{-a^2}\right)}{4}$

(b) $\displaystyle\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy$

*Solution.* The region $R$ is described by the inequalities $0 \le y \le a$, $0 \le x \le \sqrt{a^2-y^2}$, which is enclosed by the circle $x^2+y^2=a^2$ in the first quadrant. The polar limits are $r=0$ to $a$; $\theta=0$ to $\frac{\pi}{2}$.

Therefore, $I = \int_0^{\pi/2}\int_0^a r^2\, r\,dr\,d\theta = \frac{\pi a^4}{8}$

(c) $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} y\sqrt{x^2+y^2}\,dy\,dx$

*Solution.* The region $R$ is described by the inequalities $0 \le x \le 1$, $0 \le y \le \sqrt{1-x^2}$, which is enclosed by the unit circle $x^2+y^2=1$ in the first quadrant. The polar limits are $r=0$ to $1$; $\theta=0$ to $\frac{\pi}{2}$.

Therefore, $I = \int_0^{\pi/2}\int_0^1 (r\sin\theta)\cdot r\,r\,dr\,d\theta = \left\{\int_0^{\pi/2}\sin\theta\,d\theta\right\}\left\{\int_0^1 r^3 dr\right\} = \frac{1}{4}$

(d) $$\int_0^1 \int_x^{\sqrt{1-x^2}} \frac{x}{x^2+y^2}\, dy dx$$

*Solution.* The region $R$ is described by the inequalities $0 \le x \le 1$, $x \le y \le \sqrt{1-x^2}$, which is the sector enclosed by arc of the unit circle $x^2+y^2=1$, the line $y=x$ and the $y$-axis in the first quadrant. The polar limits are $r=0$ to $1$; $\theta = \pi/4$ to $\frac{\pi}{2}$.

Therefore, $I = \int_{\pi/4}^{\pi/2} \int_0^1 \frac{r^2 \cos^2\theta}{r^2} \cdot r dr d\theta = \left\{ \int_{\pi/4}^{\pi/2} \cos^2\theta \, d\theta \right\} \left\{ \int_0^1 r dr \right\} = \frac{1}{4}$

(e) $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}} dx dy$

*Solution.* The region $R$ is described by the inequalities $0 \le y \le a$, $y \le x \le a$, which is enclosed by the triangle with edges $y = x$, $x = a$ and $y = 0$. The polar limits are $r = 0$ to $a\sec\theta$; $\theta = 0$ to $\pi/4$. Therefore,

$$I = \int_0^{\pi/4} \int_0^{a\sec\theta} r^2 \cos^2\theta \, dr d\theta = \int_0^{\pi/4} \left| \frac{r^3}{3} \right|_{r=0}^{a\sec\theta} \cos^2\theta \, d\theta = \frac{a^3}{3} \log(1+\sqrt{2})$$

(f) $\int_{-1}^{1} \int_0^{\sqrt{1-x^2}} dy dx$

*Solution.* The region $R$ is described by the inequalities $-1 \le x \le 1$, $0 \le y \le \sqrt{1-x^2}$, which is enclosed by the upper semicircular arc $x^2 + y^2 = 1$ and the $x$-axis. The polar limits are $r = 0$ to $1$; $\theta = 0$ to $\pi$; $I = \int_0^{\pi} \int_0^{1} r dr d\theta = \frac{\pi}{2}$

(g) $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{1-x^2}} dy\,dx$

*Solution.* The region $R$ is described by the inequalities $-1 \le x \le 1$, $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$, which is enclosed by the circle $x^2+y^2=a^2$. The polar limits are $r=0$ to $a$; $\theta=0$ to $2\pi$; $I=\int_0^{2\pi}\int_0^{a} r\,dr\,d\theta=\pi a^2$

(h) $\displaystyle\int_{\sqrt{2}}^{2}\int_{\sqrt{4-y^2}}^{y} dx\,dy$

*Solution.* The region $R$ is described by the inequalities $\sqrt{2} \le y \le 2$, $\sqrt{4-y^2} \le x \le y$, which is enclosed by arc of the circle $x^2+y^2=4$ and the lines $y=x$, $y=2$. The polar limits are $r=2$ to $2\csc\theta$; $\theta=\pi/4$ to $\pi/2$

Therefore $I=\int_{\pi/4}^{\pi/2}\int_{2}^{2\csc\theta} r\,dr\,d\theta = 2\int_{\pi/4}^{\pi/2}(\csc^2\theta-1)\,d\theta=2$

(i) $$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{(1+x^2+y^2)^2}\,dydx$$

*Solution.* The region $R$ is described by the inequalities $-1 \le x \le 1,\ -\sqrt{1-x^2} \le y \le 0$, which is enclosed by the lower semicircular arc $x^2+y^2=1$ and the $x$-axis. The polar limits are $r=0$ to $1$; $\theta=\pi$ to $2\pi$

Therefore $I = \int_{\pi}^{2\pi}\int_{0}^{1} \frac{2r}{1+r^2}\,drd\theta = \pi\log 2$

(j) $$\int_{1}^{2}\int_{0}^{\sqrt{2x-x^2}} dydx$$

*Solution.* The region $R$ is described by the inequalities $1 \le x \le 2,\ 0 \le y \le \sqrt{1-x^2}$, which is enclosed by the arc of the circle $(x-1)^2+y^2=1$, the $x$-axis and the line $x=1$. The polar limits are $r=\sec\theta$ to $2\cos\theta$; $\theta=0$ to $\pi/4$

Therefore $I = \int_{0}^{\pi/4}\int_{\sec\theta}^{2\cos\theta} r\,drd\theta = \pi/4$

## 11 Applications of Double Integral

### Areas of Plane Regions

$(a)$ Cartesian Form: $A = \iint\limits_R dxdy = \iint\limits_R dydx$

$(b)$ Polar Form: $A = \iint\limits_R rdrd\theta$

**Example 11.1.** Evaluate the area of the region

($a$) enclosed between the coordinate axes and the line $x + y = 2$

**Ans.** $\int_0^2 \int_0^{2-x} dy dx = 2$

($b$) enclosed by the parabola $y^2 = x$ and the straight line $y = x$

**Ans.** $A = \int_0^1 \int_{y^2}^{y} dx dy = 1/6$

($c$) bounded by the the straight lines $y = x$, $3y = x$ and $y = 2$

**Ans.** $A = \int_0^2 \int_y^{3y} dx dy = 4$

($d$) enclosed by the circle $x^2 + y^2 = a^2$ in the first quadrant using the polar coordinates

**Ans.** $A = \int_0^{\pi/2} \int_0^a r dr d\theta = \pi a^2/4$

(*e*) enclosed by the arc of the unit circle $x^2+y^2=1$, the $y$ axis and the line $y=x$ in the first quadrant using the polar coordinates

**Ans.** $A=\int_{\pi/4}^{\pi/2}\int_{0}^{1} r\,dr\,d\theta=\pi/8$

**Average value** of $f(x,y)$ over a plane region $R$ is given by

$$f_{\text{ave}}=\frac{1}{\text{area of } R}\iint_R f(x,y)\,dA.$$

## Mass, Moments, Centre of Mass, Moments of Inertia, Centroid

Let $f(x,y)$ be the linear density of a thin metal sheet bounding a region $R$. Then

Mass of the plate: $M = \iint\limits_R f(x,y)\,dA$

First Moments about the coordinate axes:

$$M_x = \iint\limits_R y f(x,y)\,dA,\; M_y = \iint\limits_R x f(x,y)\,dA$$

Centre of Mass $(x_c, y_c)$, where $x_c = \dfrac{M_y}{M},\; y_c = \dfrac{M_x}{M}$

Moments of Inertia

(a) about the $x$-axis: $I_x = \iint\limits_R y^2 f(x,y)\,dA$

(b) about the $y$-axis is $I_y = \iint\limits_R x^2 f(x,y)\,dA$

(c) about a line $L$ is $I_L = \iint\limits_R r^2(x,y) f(x,y)\,dA$, where $r(x,y)$ is the distance from $(x,y)$ to $L$

(d) about the origin (polar moment) is $I_0 = I_x + I_y$

**Centrod:** When the density of the metal plate is constant, $f(x,y)$ cancels out of the numerator and denominator of the formulas of $x_c$ and $y_c$. As such, the centre of mass becomes a feature of the object's shape and not of the material of which it is made. In this case, the centre of mass is referred to as the **centroid** of the shape $(x_C, y_C)$, where

$$x_C = \frac{M_y}{M} = \frac{\iint\limits_R x\,dA}{\iint\limits_R dA} \quad \text{and} \quad y_C = \frac{M_x}{M} = \frac{\iint\limits_R y\,dA}{\iint\limits_R dA}$$

.

## Triple Integrals

**10. Evaluate** $$\int_{x=0}^{1}\int_{y=0}^{1}\int_{z=\sqrt{x^2+y^2}}^{2} xyz\;dz\;dy\;dx.$$

**Solution**

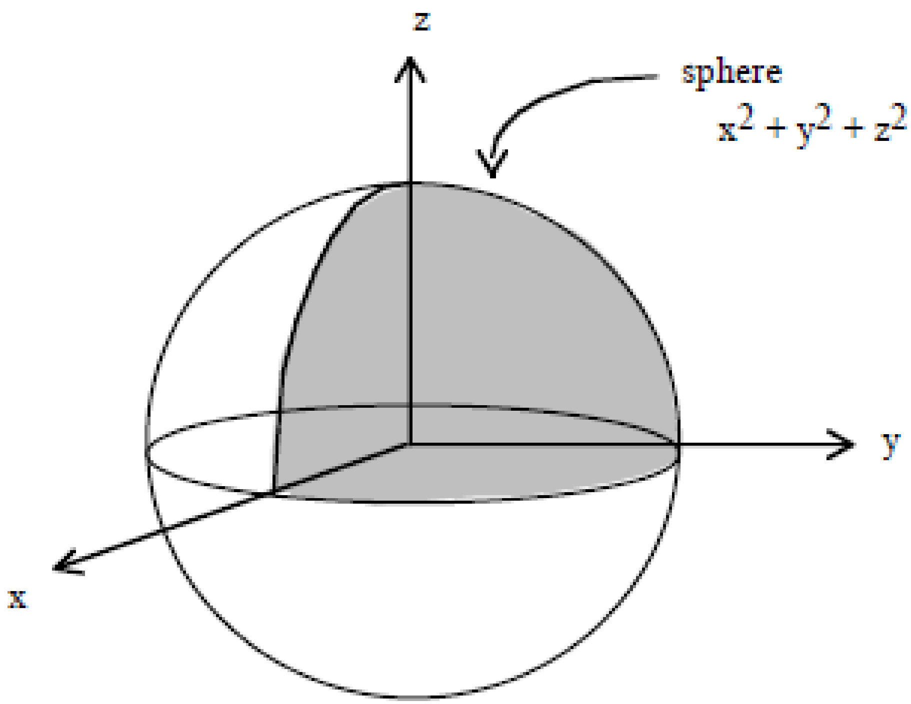
$$\begin{aligned}
\int_0^1\int_0^1\int_{\sqrt{x^2+y^2}}^{2} xyz\;dz\;dy\;dx &= \int_0^1\int_0^1 \frac{xyz^2}{2}\Big|_{z=\sqrt{x^2+y^2}}^{2} dy\;dx \\
&= \int_0^1\int_0^1 2xy - \frac{xy(x^2+y^2)}{2}\;dy\;dx \\
&= \int_0^1\int_0^1 2xy - \frac{x^3y}{2} - \frac{y^3x}{2}\;dy\;dx \\
&= \int_0^1 \left(xy^2 - \frac{x^3y^2}{4} - \frac{y^4x}{8}\right)\Big|_0^1 dx \\
&= \int_0^1 x - \frac{x^3}{4} - \frac{x}{8}\;dx \\
&= \int_0^1 \frac{7x}{8} - \frac{x^3}{4}\;dx \\
&= \left(\frac{7x^2}{16} - \frac{x^4}{16}\right)\Big|_0^1 \\
&= \frac{7}{16} - \frac{1}{16} = \frac{3}{8}.
\end{aligned}$$

**11.** Find the mass of the 3D region $B$ given by $x^2 + y^2 + z^2 \leq 4,\ x \geq 0,\ y \geq 0,\ z \geq 0,$

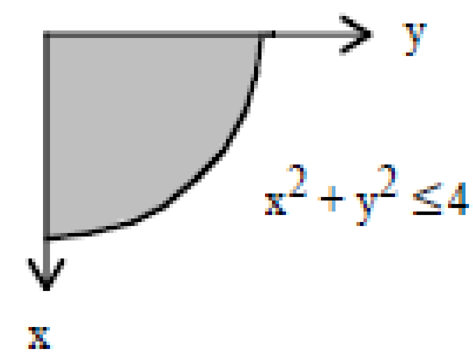
if the density is equal to $xyz$.

**Solution** We have $\text{mass}(B) = \int\int\int_B xyz\, dV,$

and the region $B$ is the portion of the sphere of radius 2 in the first octant.




3D region $B$




2D region $R$

Then, $B$ can be described as $0 \leq z \leq \sqrt{4 - x^2 - y^2}$, for all $(x, y) \in R$,

where $R$ is the projection of $B$ in the $xy$-plane. Describing $R$ as a Type I region, this gives

$$B: \quad \begin{array}{l} 0 \leq x \leq 2 \\ 0 \leq y \leq \sqrt{4 - x^2} \\ 0 \leq z \leq \sqrt{4 - x^2 - y^2}. \end{array}$$

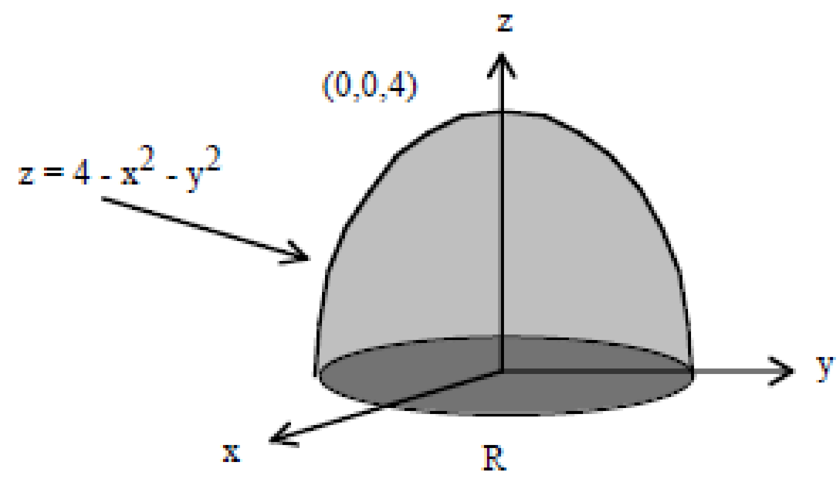
$$\begin{aligned} \text{Then, } \operatorname{mass}(B) &= \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} xyz \, dz \, dy \, dx \\ &= \int_0^2 \int_0^{\sqrt{4-x^2}} \left( xy\frac{z^2}{2} \right) \Big|_0^{\sqrt{4-x^2-y^2}} \, dy \, dx \\ &= \frac{1}{2} \int_0^2 \int_0^{\sqrt{4-x^2}} xy(4 - x^2 - y^2) \, dy \, dx \\ &= \frac{1}{2} \int_0^2 \int_0^{\sqrt{4-x^2}} 4xy - x^3y - y^3x \, dy \, dx \\ &= \frac{1}{2} \int_0^2 \left( 2xy^2 - \frac{x^3y^2}{2} - \frac{y^4x}{4} \right) \Big|_0^{\sqrt{4-x^2}} \, dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^2 2x(4-x^2) - \frac{x^3(4-x^2)}{2} - \frac{(4-x^2)^2x}{4}\,dx \\
&= \frac{1}{2}\int_0^2 \frac{x^5}{4} - 2x^3 + 4x\,dx \\
&= \frac{1}{2}\left(\frac{x^6}{24} - \frac{x^4}{2} + \frac{4x^2}{2}\right)\Big|_0^2 \\
&= \frac{1}{2}\left(\frac{64}{24} - 8 + 8\right) = \frac{4}{3}.
\end{aligned}
$$

**12.** Find the volume of the region $B$ bounded by the paraboloid $z = 4 - x^2 - y^2$ and the $xy$-plane.

**Solution**

We have $\text{volume}(B) = \int\int\int_B 1\ dV.$




Then, $B$ can be described as

$$0 \le z \le 4 - x^2 - y^2, \quad \text{for all } (x, y) \in R,$$

where $R$ is the projection of $B$ in the $xy$-plane. Then, $R$ is the interior of the circle $x^2 + y^2 = 4$. In polar coordinates, the region $R$ is $\quad R: \ 0 \le \theta \le 2\pi \quad 0 \le r \le 2,$

and in cylindrical coordinates, the region $B$ is

$$B: \begin{array}{l} 0 \le z \le (4 - r^2) \\ 0 \le \theta \le 2\pi \\ 0 \le r \le 2. \end{array}$$

$$\begin{aligned} \text{Then,} \quad \text{volume}(B) &= \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} r\ dz\ dr\ d\theta \\ &= \int_0^{2\pi} \int_0^2 (4 - r^2) r\ dr\ d\theta \\ &= 2\pi \int_0^2 4r - r^3\ dr \ = \ 2\pi \left( 2r^2 - \frac{r^4}{4} \Big|_0^2 \right) = 2\pi(4) = 8\pi \end{aligned}$$

**13.** Find the center of gravity of the region in **12.**, assuming constant density $\sigma$.

**Solution**

By symmetry, $\overline{x} = \overline{y} = 0$. Also, as the density is $d(x, y, z) = \sigma$,

$$\begin{aligned}\overline{z} &= \frac{\int\int\int_B d(x,y,z)z\, dV}{\int\int\int_B d(x,y,z)\, dV} = \frac{\sigma\int\int\int_B z\, dV}{\sigma\int\int\int_B 1\, dV} \\ &= \frac{\int\int\int_B z\, dV}{\text{volume}(B)} = \frac{1}{8\pi}\int\int\int_B z\, dV.\end{aligned}$$

Using the description of the region $B$ in cylindrical coordinates of **12.**, we get

$$\begin{aligned}
\int\int\int_B z\, dV &= \int_0^{2\pi}\int_0^2\int_0^{4-r^2} z\ r\ dz\ dr\ d\theta \\
&= \int_0^{2\pi}\int_0^2 \frac{(4-r^2)^2}{2} r\ dr\ d\theta \\
&= \frac{2\pi}{2}\int_0^2 16r - 8r^3 + r^5\ dr \\
&= \pi\left(8r^2 - 2r^4 + \frac{r^6}{6}\Big|_0^2\right) = \frac{32}{3}\pi
\end{aligned}$$

Then,

$$\bar{z} = \frac{1}{8\pi}\left(\frac{32}{3}\pi\right) = \frac{4}{3}.$$

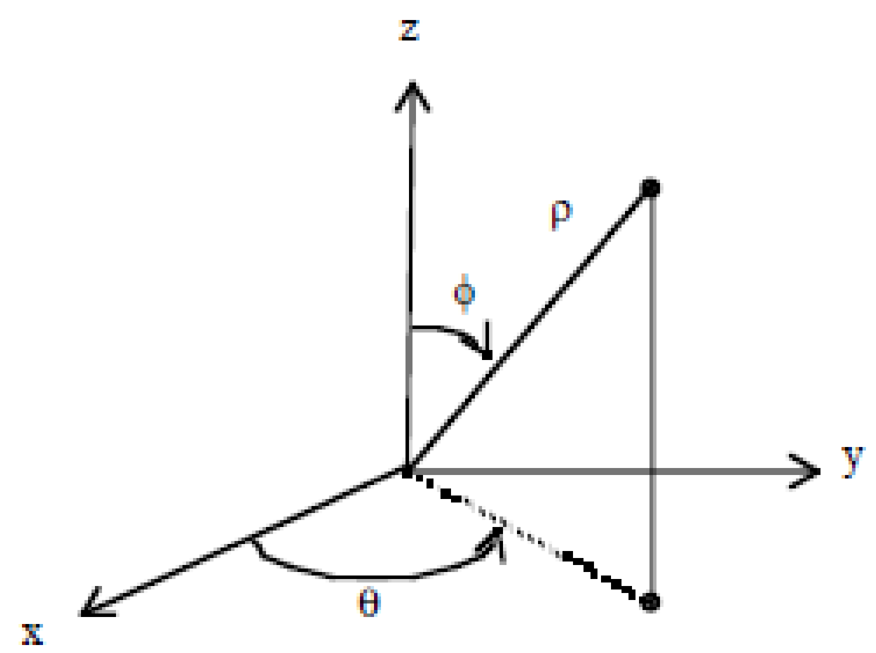
**14.** Evaluate

$$\int\int\int_B \sqrt{x^2+y^2+z^2}\, dV,$$

where $B$ is the region bounded by the plane $z = 3$ and the cone $z = \sqrt{x^2+y^2}$.

**Solution**

We will use the spherical coordinates



to describe the region $B$. In those coordinates,

$$\begin{aligned} x &= \rho\,\sin\phi\,\cos\theta \\ y &= \rho\,\sin\phi\,\sin\theta \\ z &= \rho\,\cos\phi. \end{aligned}$$

Then, the cone $z = \sqrt{x^2 + y^2}$ writes as

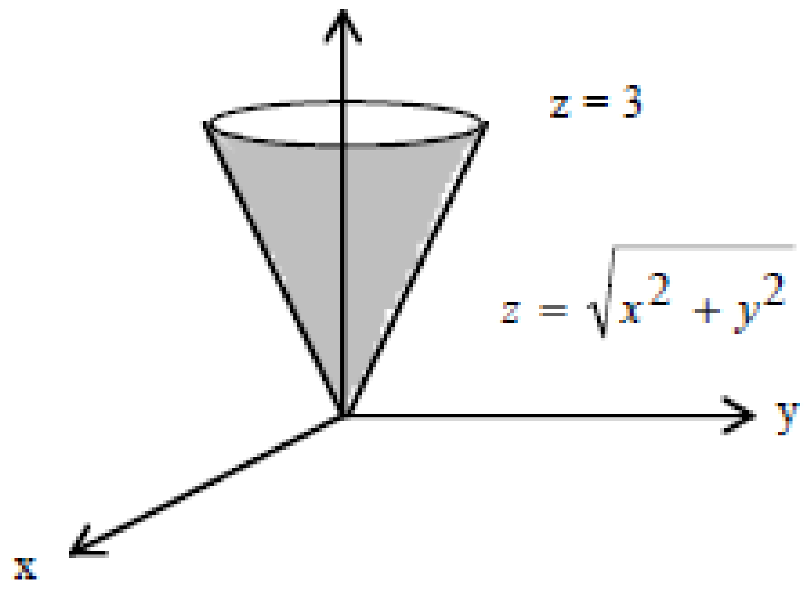
$$\rho\cos\phi = \rho\sin\phi \iff \frac{\sin\phi}{\cos\phi} = \tan\phi = 1 \iff \phi = \frac{\pi}{4},$$

the plane $z = 3$ as

$$3 = \rho\cos\phi \iff \rho = \frac{3}{\cos\phi},$$

and the region $B$ can be described as

$$B: \qquad \begin{aligned} &0 \le \phi \le \frac{\pi}{4} \\ &0 \le \phi \le 2\pi \\ &0 \le \rho \le \frac{3}{\cos\phi}. \end{aligned}$$



Finally, in spherical coordinates, $\sqrt{x^2+y^2+z^2}=\rho$.
Then,

$$\begin{aligned}
\int\int\int_B \sqrt{x^2+y^2+z^2}\,dV &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{3/\cos\phi} \rho\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= 2\pi\int_0^{\pi/4}\sin\phi\int_0^{3/\cos\phi}\rho^3\,d\rho\,d\phi \\
&= 2\pi\int_0^{\pi/4}\sin\phi\left(\left.\frac{\rho^4}{4}\right|_0^{3/\cos\phi}\right)\,d\phi \\
&= 2\pi\frac{3^4}{4}\int_0^{\pi/4}\frac{\sin\phi}{\cos^4\phi}\,d\phi \\
&= \frac{81\pi}{2}\left(\left.\frac{(\cos\phi)^{-3}}{3}\right|_0^{\pi/4}\right) \\
&= \frac{81\pi}{6}\left(\left(\frac{\sqrt{2}}{2}\right)^{-3}-1\right) = \frac{27\pi}{2}\left(2\sqrt{2}-1\right)
\end{aligned}$$

**15.** Evaluate

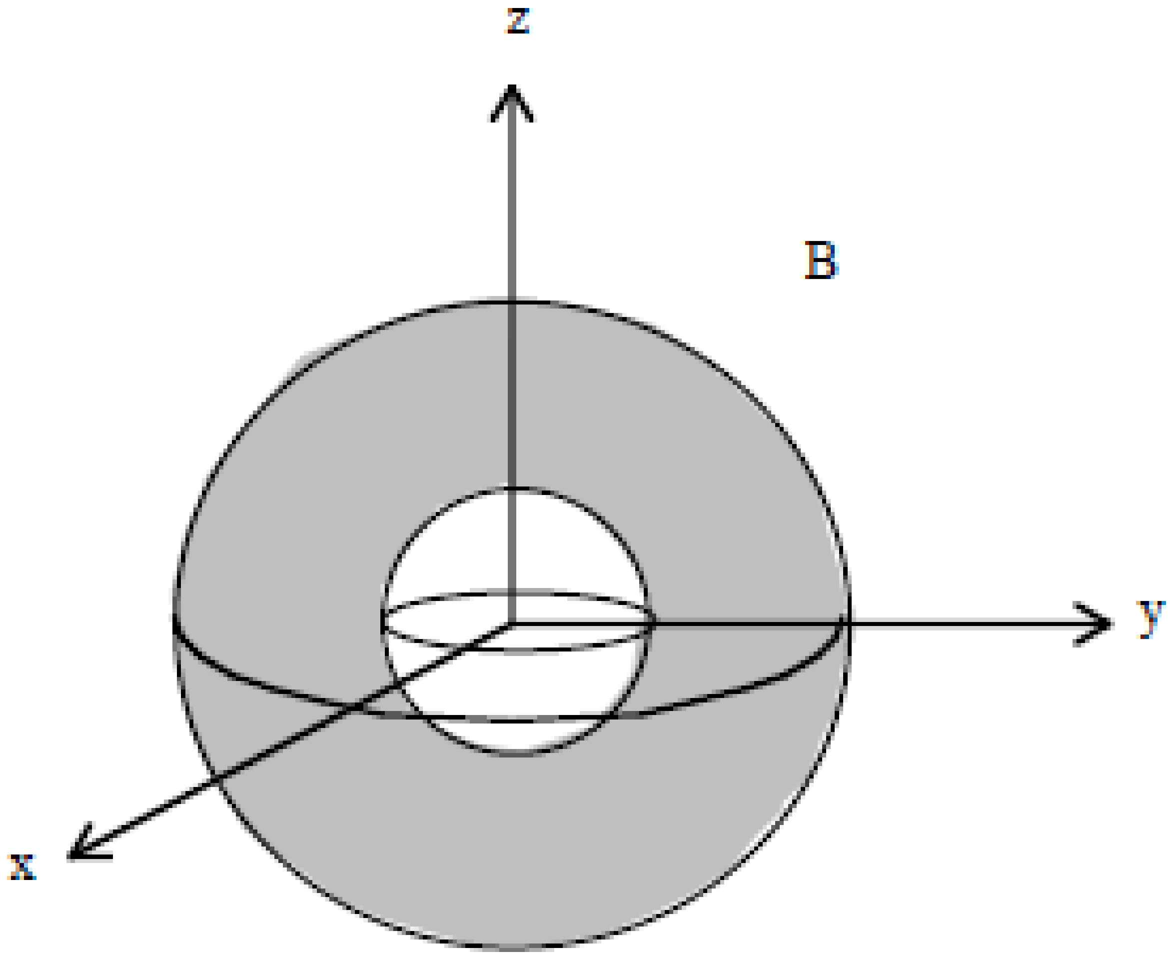
$$\int\int\int_B \left(x^2+y^2+z^2\right)^{-3/2}\, dV,$$

where $B$ is the region bounded by the spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$, where $a>b>0$.

**Solution**

Using spherical coordinates, the region $B$ between the 2 spheres can be described as

$$\begin{aligned} B: \qquad & 0 \le \theta \le 2\pi \\ & 0 \le \phi \le \pi \\ & a \le \rho \le b. \end{aligned}$$


z
B
y
x

$$
\begin{aligned}
\int\int\int_B (x^2+y^2+z^2)^{-3/2}\,dV &= \int_0^{2\pi}\int_0^{\pi}\int_a^b \rho^{-3}\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= 2\pi\int_0^{\pi}\int_a^b \frac{1}{\rho}\sin\phi\,d\rho\,d\phi \\
&= 2\pi\int_0^{\pi}\left(\ln\rho\Big|_a^b\right)\sin\phi\,d\phi \\
&= 2\pi\ln\left(\frac{b}{a}\right)\int_0^{\pi}\sin\phi\,d\phi \\
&= 2\pi\ln\left(\frac{b}{a}\right)\left(-\cos\phi\Big|_0^{\pi}\right) = 4\pi\ln\left(\frac{b}{a}\right)
\end{aligned}
$$

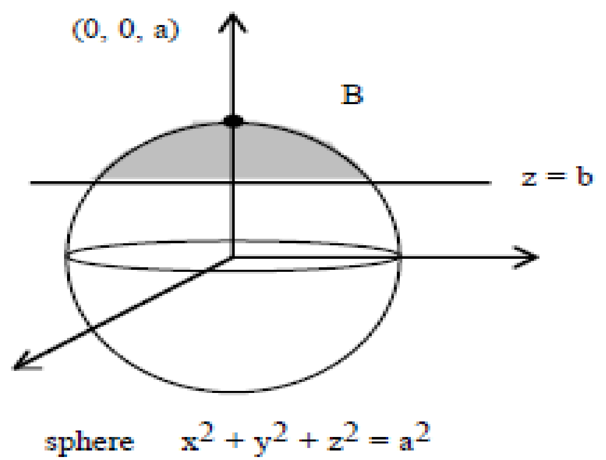
16. Find the volume of the region $B$ bounded above by the sphere $x^2+y^2+z^2=a^2$ and below by the plane $z=b$, where $a>b>0$.

**Solution**

The region $B$ can be described as the set of $(x,y,z)\in\mathbb{R}^3$ such that

$$b\le z\le\sqrt{a^2-x^2-y^2}$$

for all $(x,y)$ in the plane region bounded by the circle $x^2+y^2+b^2=a^2 \iff x^2+y^2=a^2-b^2$.




The region $B$ is best described in cylindrical coordinates, and this gives

$$B:\quad \begin{aligned} &b\le z\le\sqrt{a^2-r^2}\\ &0\le r\le\sqrt{a^2-b^2}\\ &0\le\theta\le 2\pi. \end{aligned}$$

$$
\begin{aligned}
\text{volume}(B) &= \int\int\int_B 1\, dV \\
&= \int_0^{2\pi}\int_0^{\sqrt{a^2-b^2}}\int_b^{\sqrt{a^2-r^2}} r\, dz\, dr\, d\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{a^2-b^2}} (\sqrt{a^2-r^2}-b) r\, dr\, d\theta \\
&= 2\pi \int_0^{\sqrt{a^2-b^2}} \sqrt{a^2-r^2}\, r - br\, dr \\
&= 2\pi \left( \frac{(a^2-r^2)^{3/2}}{(3/2)(-2)} - b\frac{r^2}{2} \Big|_0^{\sqrt{a^2-b^2}} \right) \\
&= 2\pi \left( -\frac{1}{3}\left((b^2)^{3/2}-(a^2)^{3/2}\right) - \frac{b}{2}(a^2-b^2) \right) \\
&= 2\pi \left( \frac{a^3-b^3}{3} - \frac{a^2b}{2} + \frac{b^3}{2} \right) = 2\pi \left( \frac{a^3}{3} - \frac{a^2b}{2} + \frac{b^3}{6} \right)
\end{aligned}
$$

**17.** Sketch the region $B$ whose volume is given by the triple integral

$$\int_0^4 \int_0^{(4-x)/2} \int_0^{(12-3x-6y)/4} 1 \, dz \, dy \, dx.$$

Rewrite the triple integral using the order of integration $dV = dy\,dx\,dz$.

**Solution**

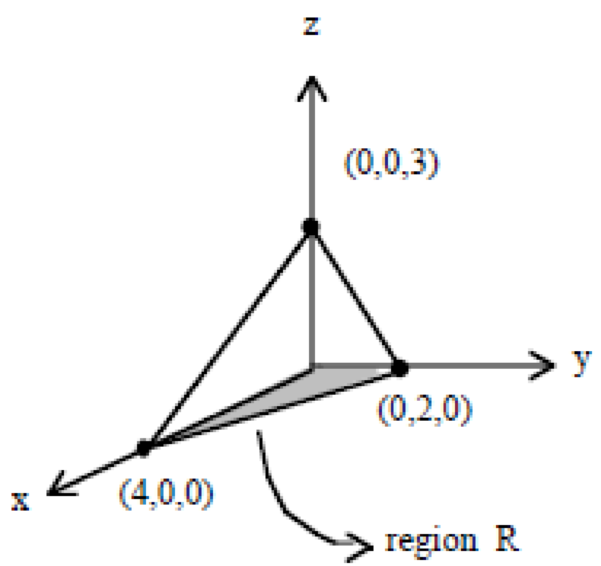
From the triple integral, the region $B$ is described by

$$0 \le z \le \frac{12 - 3x - 6y}{4},$$

i.e. $z$ varies between the planes $z = 0$ and $z = \dfrac{12 - 3x - 6y}{4} \iff 3x + 6y + 4z = 12$. Furthermore, $(x, y)$ are in the region $R$ described by

$$R: \quad \begin{aligned} &0 \le y \le \frac{4-x}{2} \\ &0 \le x \le 4, \end{aligned}$$

which is the projection of the plane $3x + 6y + 4z = 12$ in the $xy$-plane.



We now use the ordre of integration $dV = dy\,dx\,dz$. The region $B$ be can be described as

$$0 \leq y \leq \frac{12 - 3x - 4x}{6}$$

for all $(x, z)$ in the region $R$ which is the projection of $B$ in the $xz$-plane. Then, $R$ can be described as

$$R: \qquad 0 \leq z \leq 3$$
$$0 \leq x \leq \frac{(12 - 4z)}{3}.$$

This gives

$$\int_0^4 \int_0^{(4-x)/2} \int_0^{(12-3x-6y)/4} 1\,dz\,dy\,dx = \int_0^3 \int_0^{(12-4z)/3} \int_0^{(12-3x-4z)/6} 1\,dy\,dx\,dz.$$

**18.** Evaluate

$$\int\int\int_B \sqrt{x^2+y^2}\,dV,$$

where $B$ is the region lying above the $xy$-plane, and below cone $z = 4 - \sqrt{x^2+y^2}$.

**Solution**

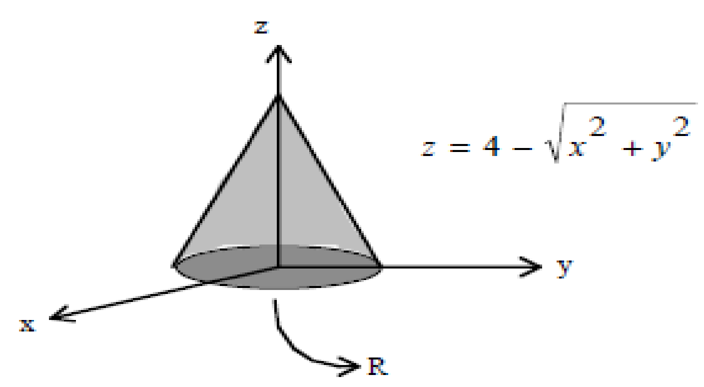
The region $B$ be can be described as

$$0 \leq z \leq 4 - \sqrt{x^2 - y^2}, \quad \text{for all } (x, y) \in R,$$

where $R$ is the projection of $B$ in the $xy$-plane. Then, $R$ is the region inside the curve

$$0 = 4 - \sqrt{x^2+y^2} \iff x^2 + y^2 = 16,$$

which is a circle of radius $4$.



We then use cylindrical coordinates to describe the region $B$. This gives

$$B: \quad \begin{aligned} &0 \le z \le 4-r \\ &0 \le r \le 4 \\ &0 \le \theta \le 2\pi. \end{aligned}$$

Then,

$$\begin{aligned} \int\int\int_B \sqrt{x^2+y^2}\, dV &= \int_0^{2\pi}\int_0^4\int_0^{4-r} r\, r\, dz\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^4 r^2(4-r)\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^4 4r^2 - r^3\, dr \\ &= \int_0^{2\pi} \frac{4r^3}{3} - \frac{r^4}{4}\Big|_0^4\, d\theta \\ &= \frac{64}{3}\int_0^{2\pi} d\theta = \frac{64}{3}(2\pi). \end{aligned}$$

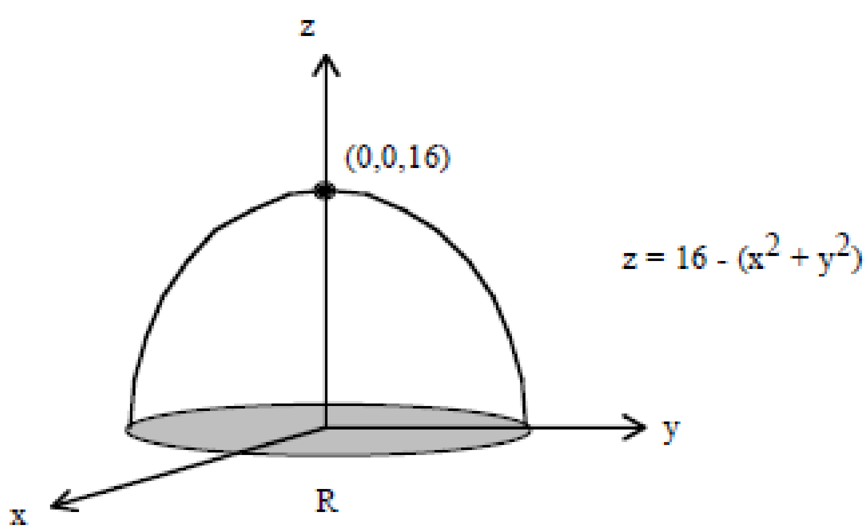
19. Evaluate the integral

$$\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{(16-x^2-y^2)} \sqrt{x^2+y^2}\, dz\, dy\, dx.$$

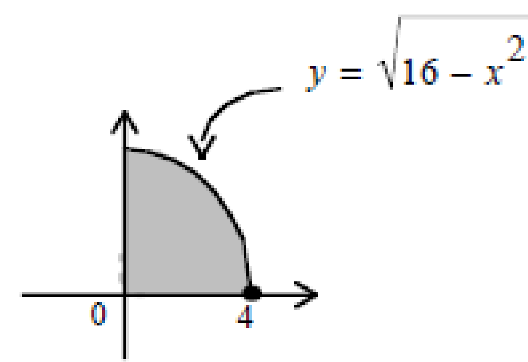
**Hint:** First convert to cylindrical coordinates.

**Solution**

The region $B$ described by the integral is the region given by $0 \le z \le (16 - x^2 - y^2)$, i.e. bounded below by the plane $z = 0$ and above by the paraboloid $z = 16 - x^2 - y^2$, for all $(x, y) \in R$. For the region $R$, we have $0 \le y \le \sqrt{16 - x^2}$, i.e. $y$ varies between the straight line $y = 0$ and the top part of the circle $x^2 + y^2 = 16$. Similarly, $x$ varies between the straight lines $x = 0$ and $x = 4$. Then, $R$ is the portion of the circle $x^2 + y^2 = 16$ in the first quadrant.




3D region $B$




2D region $R$

The region $B$ is best described in cylindrical coordinates as

$$B: \qquad \begin{aligned} &0 \le z \le 16 - r^2 \\ &0 \le \theta \le \pi/2 \\ &0 \le r \le 4. \end{aligned}$$

Then,

$$\begin{aligned}
\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{16-x^2-y^2} \sqrt{x^2+y^2}\, dz\, dy\, dx &= \int_0^{\pi/2} \int_0^4 \int_0^{16-r^2} r\, r\, dz\, dr\, d\theta \\
&= \frac{\pi}{2} \int_0^4 r^2(16-r^2)\, dr \\
&= \frac{\pi}{2} \int_0^4 16r^2 - r^4\, dr \\
&= \frac{\pi}{2} \left( \frac{16r^3}{3} - \frac{r^5}{5} \Big|_0^4 \right) \\
&= \frac{\pi}{2} \left( \frac{2048}{15} \right) = \frac{1024\pi}{15}.
\end{aligned}$$

**20.** Evaluate

$$\int\int\int_B \sqrt{x^2+y^2+z^2}\, dV,$$

where $B$ is the region above the $xy$-plane bounded by the cone $z^2 = 3(x^2+y^2)$ and by the sphere $x^2+y^2+z^2=1$.

**Solution**

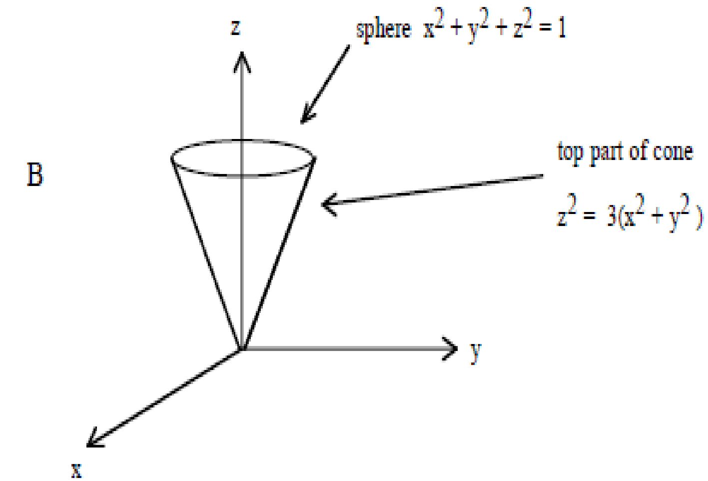
In spherical coordinates $(\rho, \theta, \phi)$, the sphere is

$$x^2+y^2+z^2=1 \iff \rho^2=1 \iff \rho=1,$$

and the cone is

$$z^2=3(x^2+y^2) \iff \rho^2\cos^2\phi = 3\rho^2\sin^2\phi \iff \tan^2\phi = \frac{1}{3} \iff \tan\phi = \pm\frac{1}{\sqrt{3}} \iff \phi = \frac{\pi}{6} \text{ or } \frac{4\pi}{6}.$$

The part of the cone above the $xy$-plane corresponds to $\phi = \frac{4\pi}{6}$.



Then, in spherical coordinates, the region $B$ is

$$B: \qquad \begin{aligned} &0 \le \rho \le 1 \\ &0 \le \phi \le \frac{\pi}{6} \\ &0 \le \theta \le 2\pi. \end{aligned}$$

Then,

$$\begin{aligned}
\int\int\int \sqrt{x^2+y^2+z^2}\, dV &= \int_0^{2\pi}\int_0^{\pi/6}\int_0^1 \rho\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/6} \sin\phi \left( \frac{\rho^4}{4}\bigg|_0^1 \right) d\phi\, d\theta \\
&= \frac{1}{4}\int_0^{2\pi}\int_0^{\pi/6} \sin\phi\, d\phi\, d\theta \\
&= \frac{1}{4}\int_0^{2\pi} \left( -\cos\phi\bigg|_0^{\pi/6} \right) d\theta \\
&= \frac{1}{4}\int_0^{2\pi} 1 - \frac{\sqrt{3}}{2}\, d\theta \\
&= \left( 1 - \frac{\sqrt{3}}{2} \right) \frac{\pi}{2}.
\end{aligned}$$

## Limits of Integration Specified - Limits all constants

The general forms are

$$(A) \quad \int_e^f \int_c^d \int_a^b f(x,y,z)\, dx dy dz,$$

$$(B) \quad \int_a^b \int_c^d \int_e^f f(x,y,z)\, dz dy dx,$$

$$(C) \quad \int_a^b \int_e^f \int_c^d f(x,y,z)\, dx dz dy$$

and so on.

The limits will define a rectangular box $R = \{a \le x \le b, c \le y \le d\}$.

Evaluate the following triple integrals:

(a) $$\int_0^1 \int_0^1 \int_0^1 e^{x+y+z}\, dx dy dz$$

(b) $$\int_0^1 \int_0^2 \int_0^2 x^2 yz\, dz dy dx$$

(c) $$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \cos(x+y+z)\, dx dyz$$

## Limits of Integration Specified - Noncontant Inner Limits

Evaluate the following:

(a) $$\int_0^1 \int_0^{1-z} \int_0^{1-y-z} xyz \, dx\,dy\,dz$$

(b) $$\int_0^1 \int_{y^2}^1 \int_1^{1-x} x \, dz\,dx\,dy$$

(c) $$\int_{-1}^1 \int_0^{z} \int_{x-z}^{x+z} (x+y+z) \, dx\,dy\,dz$$

(d) $$\int_0^1 \int_0^{1-x} \int_0^{x+y} e^z \, dz\,dy\,dx$$

(e) $$\int_0^{\log 2} \int_0^{x} \int_0^{x+y} e^{x+y+z} \, dz\,dy\,dx$$

## Limits of Integration Not Given

Evaluate the triple integral of each of the following functions $f(x,y,z)$ over the given volume $V$:

(a) $xy+yz+zx$; $V$ is enclosed by the rectangular box with edges $x=0$, $x=1$; $y=0$, $y=1$; $z=0$, $z=1$

(b) $xyz$; $V$ is enclosed by the coordinate planes and the plane $x+y+z=1$

(c) $1$; $V$ is enclosed by the planes $x=0, y=0, z=0, \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

## Triple Integral using Spherical Polar Coordinates

Evaluate the following triple integrals through the spherical polar coordinate transformation $x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta, z = \rho \cos\phi$:

1. $$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 \frac{1}{\sqrt{x^2+y^2+z^2}}\, dz\, dy\, dx$$
   The volume is enclosed by the right circular cone $z = \sqrt{a^2 - y^2}$ with height $z = 1$ in the first octant

2. $$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\, dz\, dy\, dx$$
   The volume is enclosed by the positive octant of the unit sphere

3. The triple integral of $f(x,y,z) = x+y+z$ over the volume enclosed by the sphere $x^2 + y^2 + z^2 = a^2$

4. The triple integral of $f(x,y,z) = x^2 + y^2 + z^2$ over the volume enclosed by the upper hemisphere $x^2 + y^2 + z^2 = a^2, z \geq 0$

## Triple Integral using Cylindrical Polar Coordinates

Using the cylindrical polar coordinates $x = r\cos\theta, y = r\sin\theta, z = z$, evaluate:

1. $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{0}^{x^2+y^2} z^2\,dz\,dy\,dx$

2. $\iiint (x+y+z)\,dV$ over the volume bounded by the coordinate planes, tha plane $z = h$ and the cylinder $x^2 + y^2 = 1$

3. $\iiint z(x^2+y^2)dV$ over the volume bounded by the planes $z = 2$, $z = 3$, and the cylinder $x^2 + y^2 = 1$

4. $\iiint (x^2+y^2+z^2)dV$ over the volume bounded by the planes $z = 0$, $z = 1$, and the cylinder $x^2 + y^2 = 1$

## Applications of Triple Integral

### Volumes of Solids

- The volume of a closed, bounded region $R$ in space is $V = \iiint\limits_R dV$
  - Cartesian: $V = \iint\limits_R dx\,dy\,dz$
  - Spherical polar: $V = \iint\limits_R \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$
  - Cylindrical polar: $V = \iint\limits_R r \, dr\, d\theta\, dz$
  - General: $V = \iint\limits_R J\left(\frac{x,y,z}{u,v,w}\right) du\,dv\,dw$

Compute the volume of

1. the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = a$
2. the sphere $x^2 + y^2 + z^2 = a^2$ using the spherical polar coordinates
3. the cylinder $x^2 + y^2 = a^2$ bounded by the planes $z = 0$ and $z = h$ using the cylindrical polar coordinates
4. the portion of the sphere $x^2 + y^2 + z^2 = 1$ lying inside the cylinder $x^2 + y^2 = y$ using the cylindrical polar coordinates
5. the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
   [Hint: Use $x = a\rho\sin\phi\cos\theta, y = b\rho\sin\phi\sin\theta, z = c\rho\cos\phi$]

## Mass, Moments, Centre of Mass, Moments of Inertia, Centroid

Let $f(x,y,z)$ be the density a solid enclosed by a region $R$. Then

- Mass of the plate is $M = \iiint\limits_R f(x,y,z)\,dV$
- First Moments about the coordinate planes are

$$M_{yz} = \iiint\limits_R xf(x,y,z)\,dV,$$
$$M_{zx} = \iiint\limits_R yf(x,y,z)\,dV,$$
$$M_{xy} = \iiint\limits_R zf(x,y,z)\,dV$$

- Centre of Mass is $(\overline{x},\overline{y},\overline{z})$, where $\overline{x} = \frac{M_{yz}}{M}$, $\overline{y} = \frac{M_{zx}}{M}$, $\overline{z} = \frac{M_{xy}}{M}$
- Moments of Inertia (Second Moments)
  - about the $x$-axis is $I_x = \iiint\limits_R (y^2+z^2)f(x,y,z)\,dV$

- about the $y$-axis is $I_y = \iiint\limits_R (z^2 + x^2) f(x,y,z)\, dV$
- about the $z$-axis is $I_y = \iiint\limits_R (x^2 + y^2) f(x,y,z)\, dV$
- about a line $L$ is $I_L = \iiint\limits_R r^2(x,y,z) f(x,y)\, dV$, where $r(x,y,z)$ is the distance from $(x,y,z)$ to $L$

• Centrod of the shape $(x_C, y_C, z_C)$, where

$$x_C = \frac{M_{yz}}{M} = \frac{\iiint\limits_R x\, dV}{\iiint\limits_R dV}, \quad y_C = \frac{M_{zx}}{M} = \frac{\iiint\limits_R y\, dA}{\iiint\limits_R dV}, z_C = \frac{M_{xy}}{M} = \frac{\iiint\limits_R z\, dA}{\iiint\limits_R dV}$$

## CHECK YOUR PROGRESS

1. Evaluate $\int_1^{\log 8}\int_0^{\log y} e^{x+y}\, dx\, dy.$ **[Ans.** $8 \log 8 - 16 + e$**]**

2. Evaluate $\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dy\, dx}{1+x^2+y^2}.$ $\left[\textbf{Ans. } \frac{\pi}{4}\log\left(\sqrt{2}+1\right)\right]$

3. $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos(x+y)\, dy\, dx.$ **[Ans.** $-2$**]**

4. $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy.$ $\left[\textbf{Ans. } \frac{\pi a^3}{6}\right]$

5. $\int_0^1 \int_0^{x^2} e^{\frac{x}{y}}\, dx\, dy.$ $\left[\textbf{Ans. } \frac{1}{2}\right]$

6. Evaluate $\iint_S \sqrt{xy-y^2}\, dy\, dx$ where $S$ is the triangle with vertices (0, 0), (10, 1) and (1, 1).

**[Ans.** 6**]**

7. Evaluate $\iint_S \left(x^2+y^2\right) dx\, dy$, where $S$ is the area enclosed by the curves, $y = 4x$, $x + y = 3$, $y = 0$ and $y = 2$. $\left[\textbf{Ans. } \frac{463}{48}\right]$

8. Evaluate $\iint x^2y^2\ dx\ dy$ over the region $x^2 + y^2 \le 1$. $\left[\textbf{Ans. } \frac{\pi}{24}\right]$

9. Evaluate $\iint (x^2 + y^2)\ dx\ dy$ over the region bounded by $x = 0,\ y = 0,\ x + y = 1$.

$\left[\textbf{Ans. } \frac{1}{6}\right]$

10. Evaluate $\iint_A \frac{xy}{\sqrt{1-y^2}}\ dx\ dy$, where the region of integration is the positive quadrant of the circle $x^2 + y^2 = 1$. $\left[\textbf{Ans. } \frac{1}{6}\right]$

11. Evaluate $\iint xy\ dx\ dy$ over the region in the positive quadrant for which $x + y \le 1$.

$\left[\textbf{Ans. } \frac{1}{24}\right]$

12. Evaluate $\int_{-1/2}^{1} \int_{-x}^{1+x} (x^2 + y)\ dy\ dx$. $\left[\textbf{Ans. } \frac{63}{32}\right]$

13. $\iint_D (4xy - y^2)\ dx\ dy$, where $D$ is the rectangle bounded by $x = 1,\ x = 2,\ y = 0,\ y = 3$.

**[Ans.** 18**]**

14. $\iint\limits_A (1+x+y)\ dx\ dy$, $A$ is the region bounded by the lines $y = -x$, $x = \sqrt{y}$, $y = 2$, $y = 0$.

$\left[\textbf{Hint:} \text{ Limits } x: -y \text{ to } \sqrt{y};\ y: 0 \text{ to } 2\right]$. $\left[\textbf{Ans. } \frac{44}{15}\sqrt{2}+\frac{13}{3}\right]$

15. $\int_0^1 \int_0^{\sqrt{1+x^2}} \left(1+x^2+y^2\right)^{-1}\ dx\ dy.$ $\left[\textbf{Ans. } \frac{\pi}{4}\log\left(1+\sqrt{2}\right)\right]$

16. Evaluate $\iint_A y\, dx\, dy$, where $A$ is bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$. $\left[\textbf{Ans. } \frac{48}{5}\right]$

17. Show that $\int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^2} dy\right] dx \neq \int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^2} dx\right] dy.$

18. Evaluate $\int_0^1 \int_{y^2}^{y} \left(1+xy^2\right)\ dx\ dy.$ $\left[\textbf{Ans. } \frac{41}{240}\right]$

19. $\iint\limits_A \left(x^2+y^2\right)\ dx\ dy$, where $A$ is bounded by $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$, where $b > a$.

$\left[\textbf{Ans. } \frac{\pi}{2}\left(b^4-a^4\right)\right]$

20. $\int_3^4 \int_1^2 \frac{dy\, dx}{(x+y)^2}.$ $\left[\textbf{Ans. } \log\left(\frac{25}{24}\right)\right]$

**Using polar coordinates evaluate the following double integral.**

**21.** $\int_0^{\pi} \int_{2\sin\theta}^{4\sin\theta} r^3 \, dr \, d\theta.$ $\left[\textbf{Ans. } \dfrac{45\pi}{2}\right]$

**22.** $\int_0^{\pi} \int_0^{\cos\theta} \rho \sin\theta \, d\rho \, d\theta.$ $\left[\textbf{Ans. } \dfrac{1}{3}\right]$

**23.** $\int_0^{\pi} \int_0^{a} r^3 \sin\theta \cos\theta \, dr \, d\theta.$ $[\textbf{Ans. } 0]$

**24.** $\iint r^3 \, dr \, d\theta$, over the area bounded between the circles $r = 2 \cos \theta$ and $r = 4 \cos \theta$.

$\left[\textbf{Ans. } \dfrac{45}{2}\pi\right]$

**25.** Show that $\iint_R r^2 \sin\theta \, dr \, d\theta = \dfrac{2a^3}{3}$, where $r$ is the region bounded by the semicircle $r = 2a \cos\theta$.

**26.** $\int_0^{\frac{\pi}{2}} \int_0^{a} r^n \sin^n\theta \cos\theta \, d\theta \, dr$, for $n + 1 > 0$. $\left[\textbf{Ans. } \dfrac{a^{n+1}}{(n+1)^2}\right]$

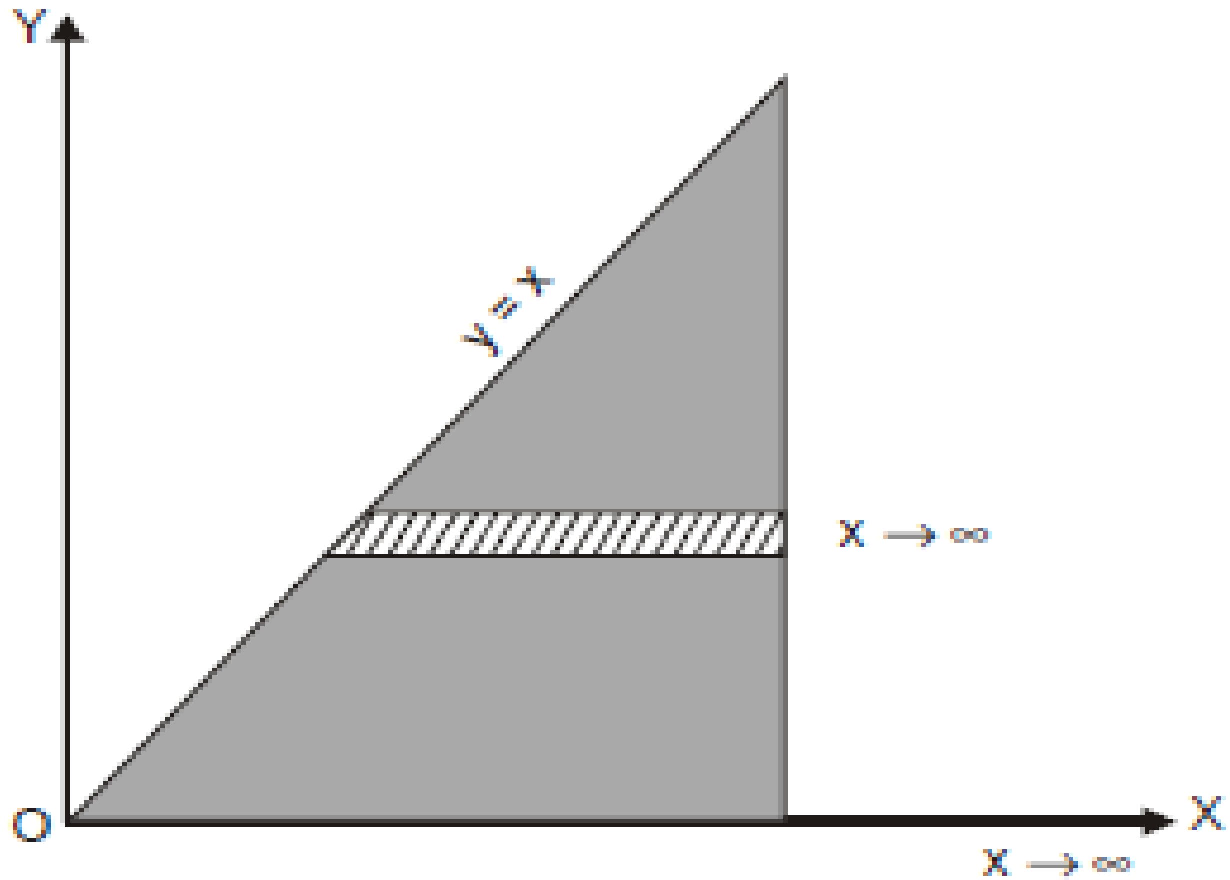
# CHANGE OF THE ORDER OF INTEGRATION

If the limits of $x$ and $y$ are constant then $\iint F(x,y)\ dx\ dy$ can be integrated in either order, but if the limits of $y$ are functions of $x$, then the new limits of $x$ in functions of $y$ are to be determined. The best method is by geometrical consideration.

Thus in several problems, the evaluation of double integral becomes easier with the change of order of integration, which of course, changes the limits of integration also.

**Example**

Evaluate the integral $\int_0^{\infty}\int_0^{x} x \exp\left(-\frac{x^2}{y}\right) dx\, dy$ by changing the order of integration.



Here $\quad y = 0$ and $y = x$

$\qquad x = 0$ and $x = \infty$

Here $x$ start from $x = y$ and goes to $x \to \infty$ and $y$ varies from $y = 0$ to $y \to \infty$

$$\therefore \int_0^\infty \int_0^x x \exp\left(-\frac{x^2}{y}\right) dx\, dy$$

$$= \int_{y=0}^{\infty} \int_{x=y}^{\infty} x e^{-\frac{x^2}{y}} \cdot dy\, dx$$

$$= \int_0^\infty \int_y^\infty -\frac{y}{2}\left(-\frac{2x}{y} e^{-\frac{x^2}{y}}\right) dy\, dx$$

$$= \int_0^\infty \left[-\frac{y}{2}e^{-\frac{x^2}{y}}\right]_y^\infty dy$$

Put $e^{-\frac{x^2}{y}} = t \Rightarrow -\frac{2x}{y}e^{-\frac{x^2}{y}}dx = dt$

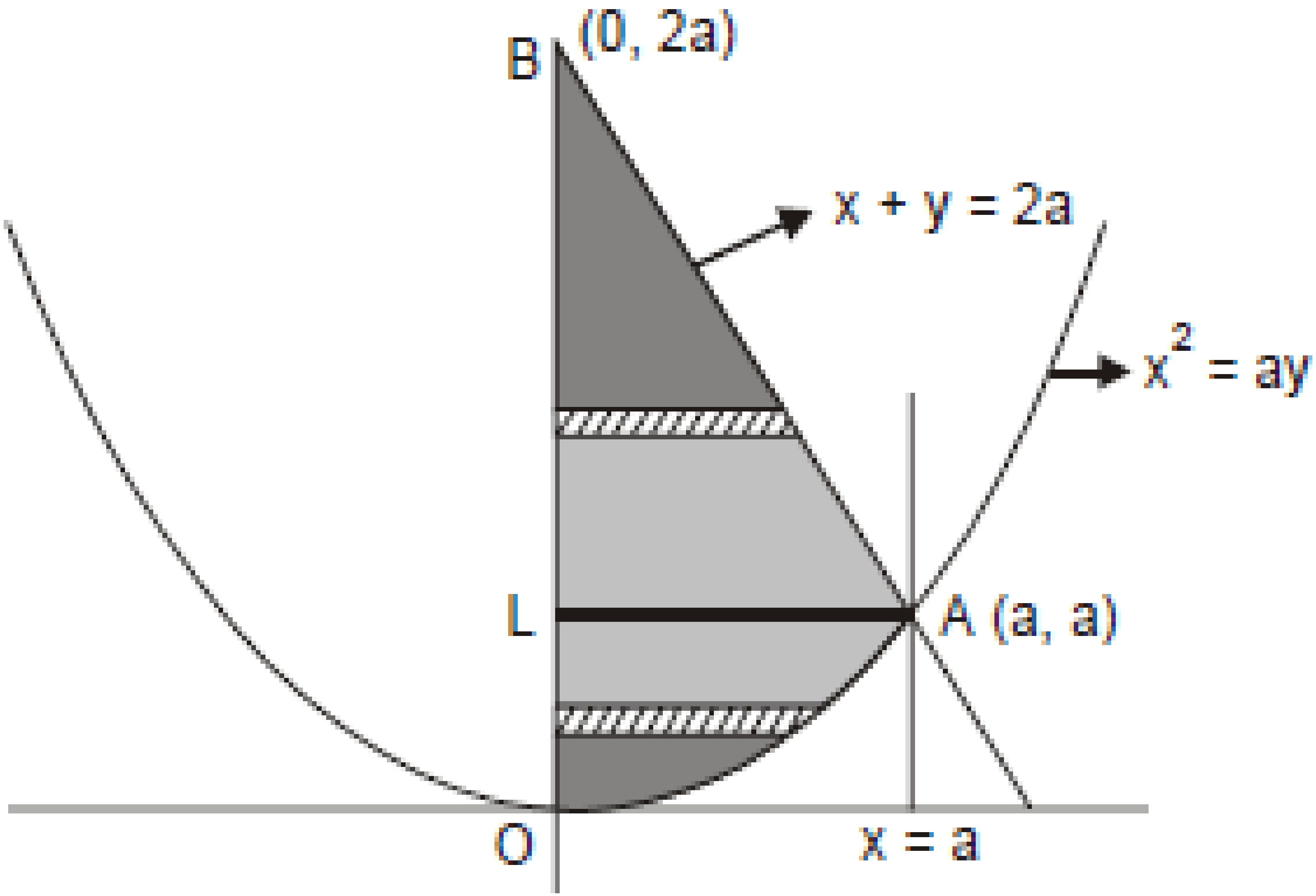
$$= \int_0^\infty \left[0+\frac{y}{2}e^{-\frac{y^2}{y}}\right] dy = \int_0^\infty \frac{y}{2}e^{-y}dy$$

$$= \left[\frac{y}{2}\left(-e^{-y}\right)-\frac{1}{2}\left(e^{-y}\right)\right]_0^\infty$$

(Integration by parts)

$$= (0-0) + \left(0+\frac{1}{2}\right) = \frac{1}{2}.$$

**Example** Change the order of integration in $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} f(x,y)\ dx\ dy.$



**Sol.** Here the limits are

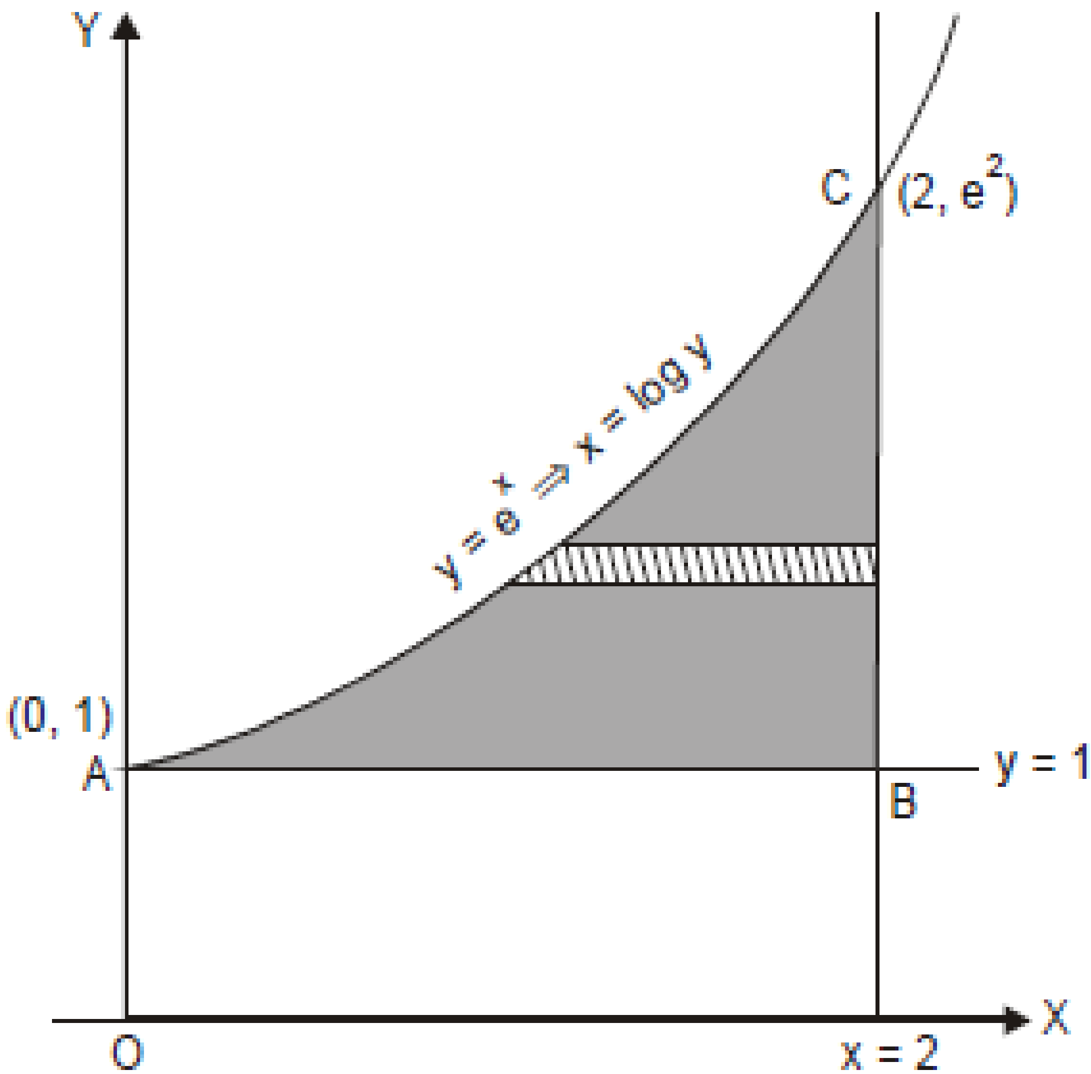
$$\begin{aligned} x^2 &= ay \text{ and } y = 2a - x \\ x^2 &= ay \text{ and } x + y = 2a \\ x &= 0 \text{ and } x = a \end{aligned}$$

Now $\displaystyle\int_0^a \int_{\frac{x^2}{a}}^{2a-x} f(x, y)\ dx\, dy$

$$= \int_0^a \int_0^{\sqrt{ay}} f(x, y)\ dy\, dx$$

$$+ \int_a^{2a} \int_0^{2a-y} f(x, y)\ dy\, dx\,.$$

**Example** Evaluate $\int_0^2 \int_1^{e^x} dx\, dy$ changing the order of integration.



**Sol.** The given limits are $x = 0$, $x = 2$, $y = 1$ and $y = e^x$.

Here $\int_0^2 \int_1^{e^x} dx\,dy = \int_1^{e^2} \int_{x=\log y}^{2} dy\,dx$ | As $x$ varies from $x = \log y$ to $x = 2$

$$= \int_1^{e^2} (2 - \log y)\ dy$$

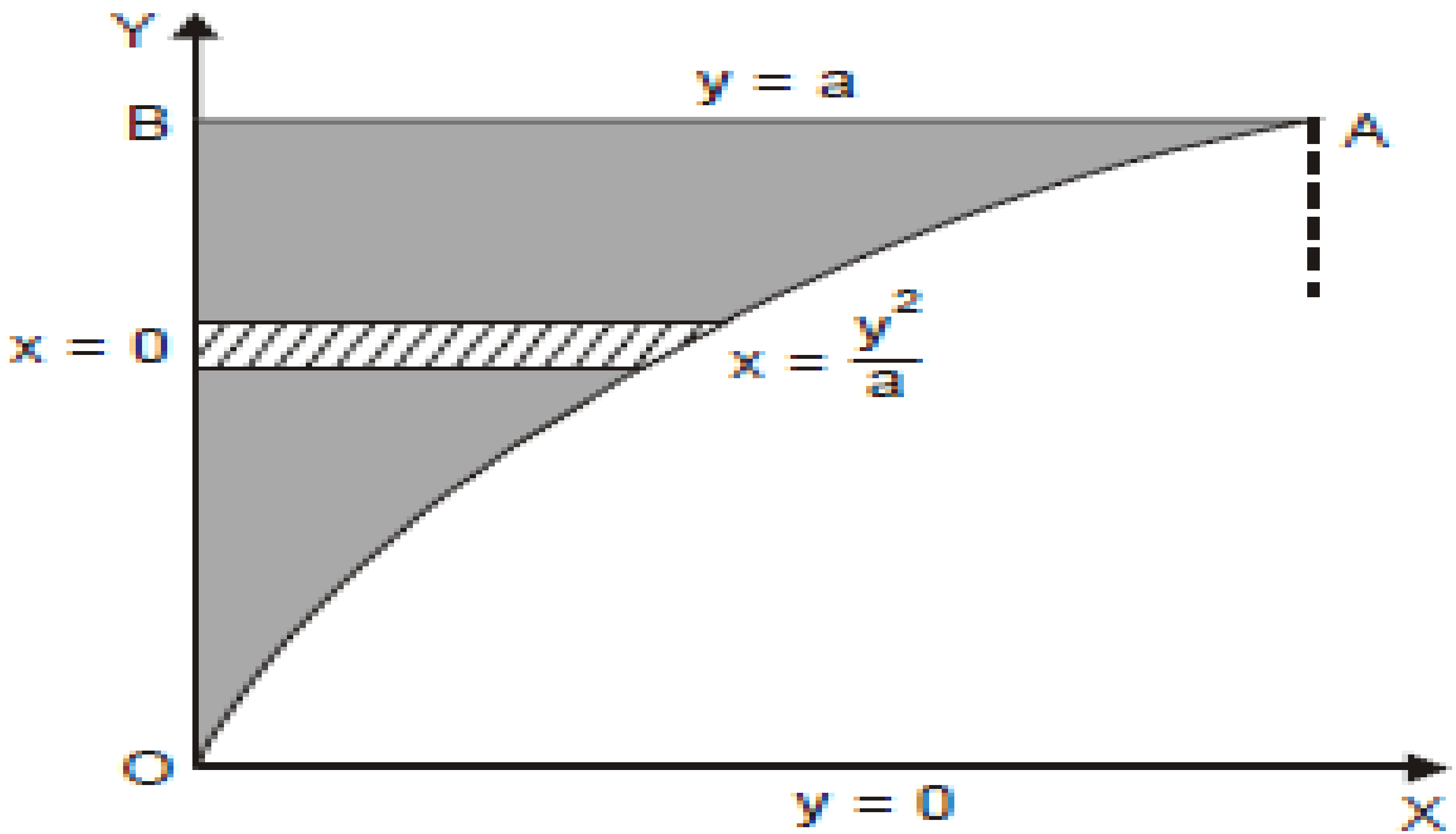
$$= (2y - y\log y + y)_1^{e^2} = (3y - y\log y)_1^{e^2}$$

$$= (3e^2 - 2e^2) - 3$$

$$= e^2 - 3.$$

**Example** Change the order of integration and hence evaluate

$$\int_0^a \int_{\sqrt{ax}}^a \frac{y^2\,dx\,dy}{\sqrt{y^4 - a^2x^2}}.$$



**Sol.** The limits are $y^2 = ax$, $y = a$ and $x = 0$, $x = a$, $y = 0$ to $y = a$ and $x$ varies from $x = 0$ to $x = \dfrac{y^2}{a}$.

By changing the order of integration. Hence, the given integral,

$$\int_{x=0}^{a}\int_{y=\sqrt{ax}}^{a}\frac{y^2\,dx\,dy}{\sqrt{y^4-a^2x^2}} = \int_{y=0}^{a}\int_{x=0}^{y^2/a}\frac{y^2\,dy\,dx}{\sqrt{y^4-a^2x^2}}$$

$$= \frac{1}{a}\int_0^a\int_0^{\frac{y^2}{a}}\frac{y^2\,dy\,dx}{\sqrt{\left(\frac{y^2}{a}\right)^2-x^2}}$$

$$= \frac{1}{a}\int_0^a y^2\left[\sin^{-1}\left(\frac{ax}{y^2}\right)\right]_0^{\frac{y^2}{a}}dy$$

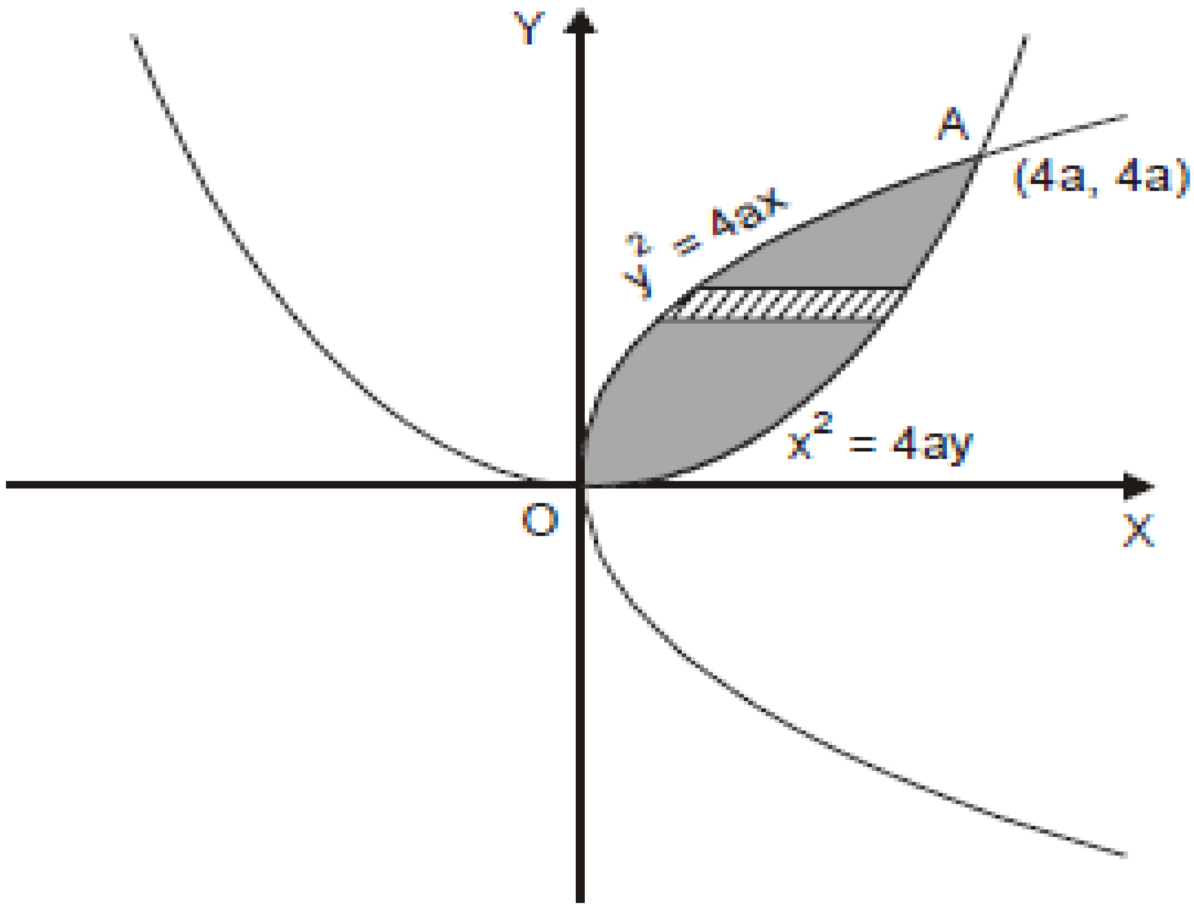
$$= \frac{1}{a}\int_0^a y^2\ [\sin^{-1}(1) - \sin^{-1}(0)]\ dy$$

$$= \frac{\pi}{2a}\int_0^a y^2 dy = \frac{\pi}{2a}\left(\frac{y^3}{3}\right)_0^a$$

$$= \frac{\pi}{6a}(a^3) = \frac{\pi a^2}{6}.$$

**Example** Change the order of integration in the following integral and evaluate:

$$\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx.$$



**Sol.** The limits are $y = \frac{x^2}{4a}$ and $y = 2\sqrt{ax}$ and $x = 0$, $x = 4a$.

From the parabola $y^2 = 4ax$ *i.e.*, $x = \frac{y^2}{4a}$ and from parabola $x^2 = 4ay$ *i.e.*, $x = 2\sqrt{ay}$ *i.e.*, $x$ varies from $\frac{y^2}{4a}$ to $2\sqrt{ay}$.

$$\therefore \qquad \int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx = \int_0^{4a}\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy\,dx$$

$$= \int_0^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right) dy$$

$$= \left[2\sqrt{a}.\frac{y^{\frac{3}{2}}}{\frac{3}{2}} - \frac{y^3}{12a}\right]_0^{4a}$$

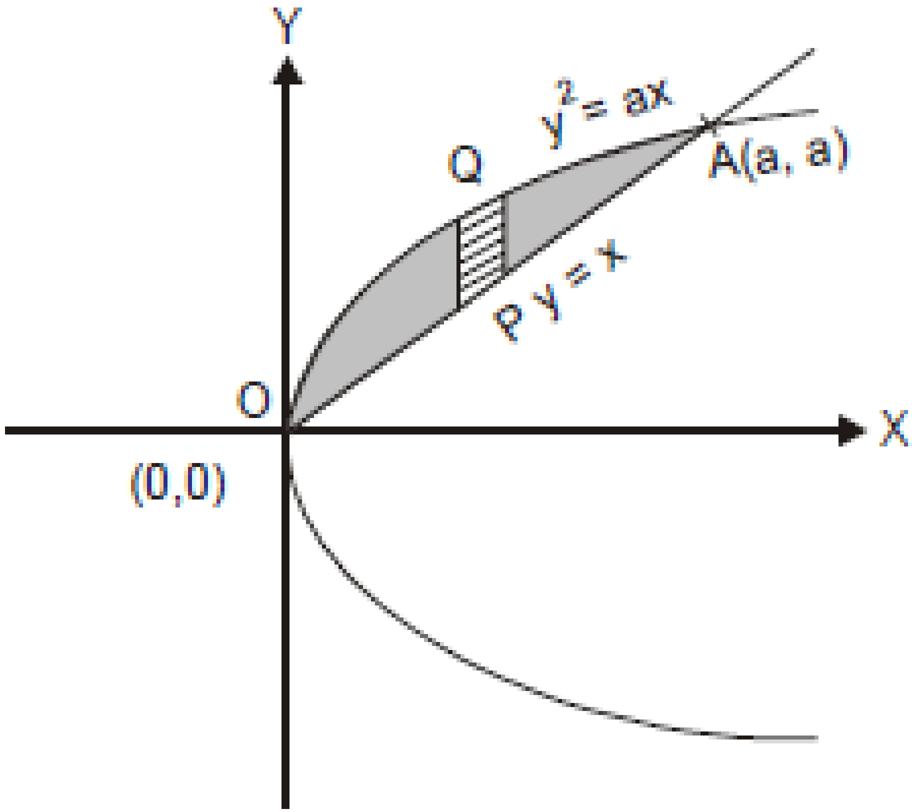
$$= \frac{4}{3}\sqrt{a}\;.\;(4a)^{\frac{3}{2}} - \frac{64a^3}{12a}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{16a^2}{3}.$$

## Example

Change the order of integration evaluate $\int_0^a \int_{y^2/a}^{y} \frac{y}{(a-x)\sqrt{ax-y^2}}\, dx\, dy$



**Sol.** Here the limits are $x = \dfrac{y^2}{a}$ *i.e.*, $y^2 = ax$, $x = y$ and $y = 0$, $y = a$ intersection point of parabola and line $y = x$ is $A(a, a)$.

Here $x$ varies from 0 to $a$ and $y$ varies from $x$ to $\sqrt{ax}$

$$\therefore \int_0^a \int_{y^2/a}^{y} \frac{y}{(a-x)\sqrt{ax-y^2}}\, dx\, dy$$

$$= \int_0^a \int_x^{\sqrt{ax}} \frac{y}{(a-x)\sqrt{ax-y^2}}\, dy\, dx$$

$$= \int_0^a \frac{1}{(a-x)}\left[-\left(ax-y^2\right)^{1/2}\right]_x^{\sqrt{ax}} dx$$

$$\begin{aligned} ax - y^2 &= t \\ -2y\,dy &= dt \\ \int \frac{y}{\sqrt{ax-y^2}}\, dy &= -\left(ax-y^2\right)^{1/2} \end{aligned}$$

$$= \int_0^a \frac{1}{a-x}\left[0+\left(ax-x^2\right)^{1/2}\right]dx$$

$$= \int_0^a \frac{\sqrt{ax-x^2}}{(a-x)}dx = \int_0^a \frac{\sqrt{x}\,\sqrt{a-x}}{(a-x)}dx = \int_0^a \frac{\sqrt{x}}{\sqrt{a-x}}dx$$

Let $x = a\sin^2\theta$ *i.e.,* $dx = 2a\sin\theta\cos\theta\, d\theta$

$$= \int_0^{\pi/2} \frac{\sqrt{a}.\sin\theta}{\sqrt{a}.\cos\theta}\cdot 2a\sin\theta\cos\theta\, d\theta$$

$$= 2a\int_0^{\pi/2}\sin^2\theta d\theta = 2a\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right) = a\int_0^{\pi/2}(1-\cos 2\theta)\,d\theta$$

$$= a\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{\pi/2} = a\left[\frac{\pi}{2}-0\right] = \frac{a\pi}{2}.$$

**Example**

By changing the order of integration of $\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx \, dy$ show that $\int_0^\infty \frac{\sin px}{x} dx = \frac{\pi}{2}$.

**Sol.** We have

$$\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx \, dy = \int_0^\infty \sin px \left\{ \int_0^\infty e^{-xy} dy \right\} dx \qquad \Big| \text{As the limits are constants.}$$

$$= \int_0^\infty \sin px \left[ \frac{e^{-xy}}{-x} \right]_0^\infty dx$$

$$= \int_0^\infty \frac{\sin px}{x} \, dx \qquad \ldots(i)$$

Again $\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx \, dy = \int_0^\infty \left[ \int_0^\infty e^{-xy} \sin px \, dx \right] dy$

$$= \int_0^\infty \left[ \frac{-e^{-xy}}{p^2 + y^2} (p \cos px + y \sin px) \right]_0^\infty dy$$

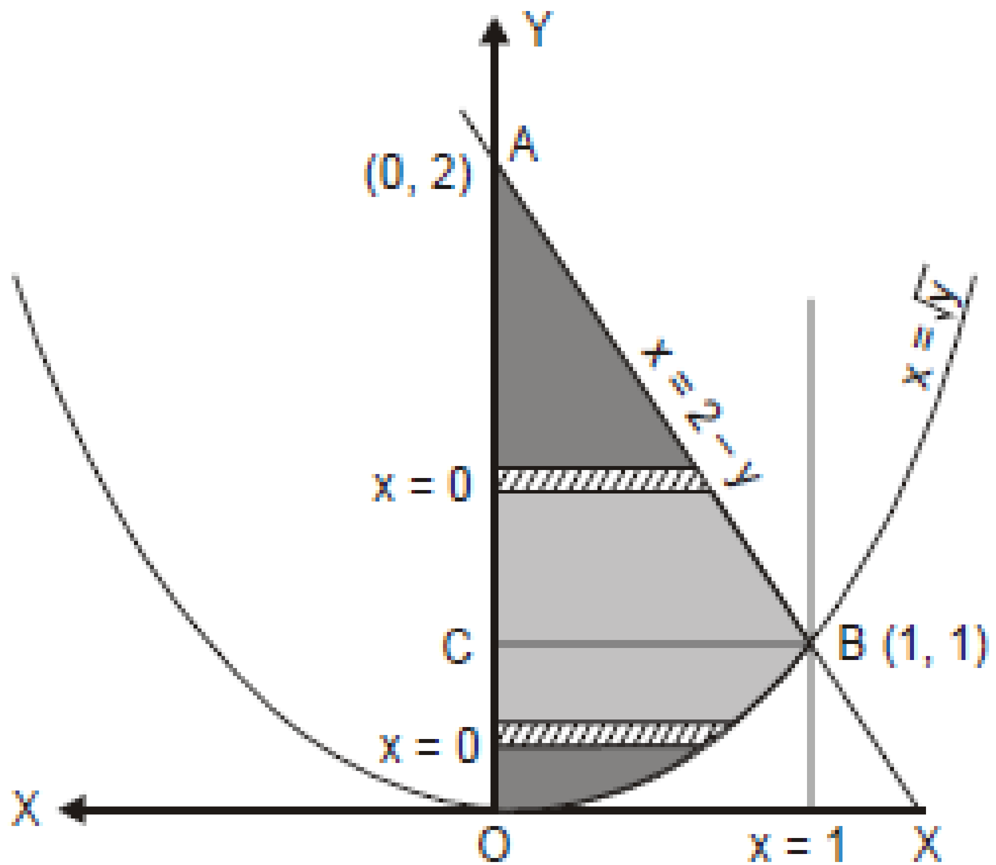
$$= \int_0^\infty \frac{p}{p^2 + y^2} \, dy = \left[ \tan^{-1} \left( \frac{y}{p} \right) \right]_0^\infty = \frac{\pi}{2} \quad ...(ii)$$

Hence from (*i*) and (*ii*)

$$\int_0^\infty \frac{\sin px}{x} \, dx = \frac{\pi}{2}.$$ **Hence proved.**

**Example**

Change the order of integration in $I = \int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy$ and hence evaluate the same.



**Sol.** The limits are $y = x^2$ and $y = 2 - x$

$x = 0$ and $x = 1$

The point of intersection of the parabola $y = x^2$ and the line $y = 2 - x$ is $B$ (1, 1).

We have taken a strip parallel to $x$ axis in the area $OBC$ and second strip in the area $ABC$.

The limits of $x$ in the area $OBC$ are 0 and $\sqrt{y}$ and the limits of $x$ in the area $ABC$ are 0 and $2 - y$.

$$\therefore \int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy = \int_0^1 y\,dy \int_0^{\sqrt{y}} x\,dx + \int_1^2 y\,dy \int_0^{2-y} x\,dx$$

$$= \int_0^1 y\,dy \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} + \int_1^2 y\,dy \left[\frac{x^2}{2}\right]_0^{2-y}$$

$$= \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y(2-y)^2\ dy = \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\int_1^2 \left(4y - 4y^2 + y^3\right)\ dy$$

$$= \frac{1}{6} + \frac{1}{2}\left[2y^2 - \frac{4}{3}y^3 + \frac{y^4}{4}\right]_1^2 = \frac{1}{6} + \frac{1}{2}\left[8 - \frac{32}{3} + 4 - 2 + \frac{4}{3} - \frac{1}{4}\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{96-128+48-24+16-3}{12}\right]$$

$$= \frac{1}{6} + \frac{5}{24}$$

$$= \frac{9}{24} = \frac{3}{8}.$$

# Change of variables between Cartesian and polar co-ordinates

- Some times it becomes easier to evaluate definite integrals by changing one system of variables to another system of variables, such as Cartesian coordinate system to polar coordinate system

**Example**

$\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\, dx\, dy$ by changing to polar coordinates hence, show that

$\int_0^{\infty} e^{-x^2}\, dx = \dfrac{\sqrt{\pi}}{2}$.

**Sol.** Let $x = r\cos\theta,\ y = r\sin\theta$

$\Rightarrow \quad x^2 + y^2 = r^2$

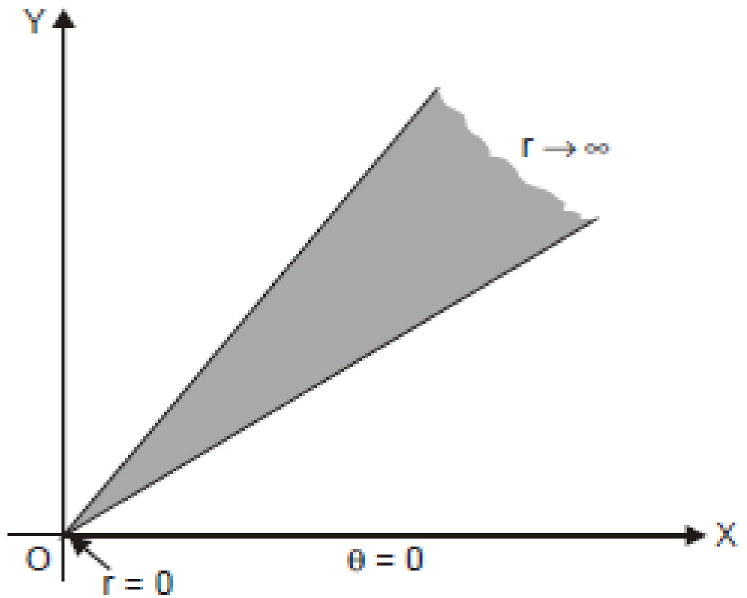
Here $r$ varies from 0 to $\infty$

and $\theta = \tan^{-1}\dfrac{y}{x}$

Now, $\theta = \tan^{-1} 0 = 0,\ \theta = \tan^{-1}\infty = \dfrac{\pi}{2}$

$\Rightarrow \quad \theta$ varies from 0 to $\dfrac{\pi}{2}$

Hence, $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\, dx\, dy = \int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2} r\, dr\, d\theta$



$$= -\frac{1}{2} \int_0^{\frac{\pi}{2}} \int_0^{\infty} (-2r) e^{-r^2} \, dr \, d\theta = -\frac{1}{2} \int_0^{\frac{\pi}{2}} \left[ e^{-r^2} \right]_0^{\infty} d\theta$$

$$= -\frac{1}{2} \int_0^{\frac{\pi}{2}} (0-1) \, d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} 1 \, d\theta = \frac{\pi}{4}.$$

Let $$I = \int_0^{\infty} e^{-x^2} \, dx \qquad \text{...}(i)$$

$$I = \int_0^{\infty} e^{-y^2} \, dy \qquad \text{...}(ii)$$

[Property of definite integrals]

Multiplying (*i*) and (*ii*), we get

$$I^2 = \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} \, dx \, dy = \frac{\pi}{4}.$$

[As obtained above]

$$I = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}.$$ **Proved.**

**Example** Evaluate the following by changing into polar coordinates

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} y^2 \sqrt{x^2+y^2}\, dy\, dx.$$

**Sol.** Here $y = 0$, $y = a$ and $x = 0$, $x = \sqrt{a^2 - y^2}$

Let $x = r\cos\theta$, $y = r\sin\theta$ $\therefore$ $x^2 + y^2 = r^2$ or $r = a$

Also, when $x = 0$, $r\cos\theta = 0 \Rightarrow r = 0$, or $\theta = \dfrac{\pi}{2}$

when $y = 0$, $r\sin\theta = 0 \Rightarrow r = 0$ or $\theta = 0$

when $y = a$, $r\sin\theta = a \Rightarrow a\sin\theta = a \Rightarrow \theta = \dfrac{\pi}{2}$

Hence, we have

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} y^2 . \sqrt{x^2+y^2}\, dy\, dx = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^2 \sin^2\theta\, r . r\, dr\, d\theta = \int_0^{\pi/2} \sin^2\theta\, d\theta . \int_0^a r^4 . dr$$

$$= \frac{a^5}{5} \int_0^{\pi/2} \left(\frac{1-\cos 2\theta}{2}\right) d\theta = \frac{a^5}{10}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{\pi a^5}{20}.$$

# TRIPLE INTEGRALS

Triple integral is a generalization of a double integral. Let a function $f(x, y, z)$ defined at every point of a three dimensional region $V$; Divide the region $V$ into $n$ elementary volumes $\delta V_1$, $\delta V_2$, ..., $\delta V_n$ and let $(x_r, y_r, z_r)$ be any point inside the $r$th sub-division $\delta V_r$.

Find the sum $\sum_{r=1}^{n} f(x_r, y_r, z_r)\delta V_r$.

Then $$\iiint_V f(x,y,z)\,dV = \lim_{\substack{n\to\infty \\ \delta V_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r, z_r)\,\delta V_r .$$

To extend definition of repeated integrals for triple integrals, consider a function $F(x, y, z)$ and keep $x$ and $y$ constant and integrate with respect to $z$ between limits in general depending upon $x$ and $y$. This would reduce $F(x, y, z)$ to a function of $x$ and $y$ only. Thus let

$$\phi(x, y) = \int_{z_1(x,y)}^{z_2(x,y)} F(x,y,z)\,dz$$

Then in $\phi\,(x, y)$ we can keep $x$ constant and integrate with respect to $y$ between limits in general depending upon $x$ this leads to a function of $x$ alone say

$$\psi\,(x) \;=\; \int_{y_1(x)}^{y_2(x)} \phi(x, y)\, dy$$

Finally $\psi(x)$ is integrated with respect to $x$ assuming that the limits for $x$ are from $a$ to $b$. Thus

$$\iiint_V F(x, y, z)\, dV \;=\; \int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} F(x, y, z)\, dx\, dy\, dz$$

$$\boxed{\iiint_V F(x, y, z)\, dV \;=\; \int_a^b \left[ \int_{y_1(x)}^{y_2(x)} \left\{ \int_{z_1(x,y)}^{z_2(x,y)} F(x, y, z) dz \right\} dy \right] dx} \;.$$

# Examples

1. Integrate the function $f(x,y,z) = xy$ over the volume enclose by the planes $z = x + y$ and $z = 0$, and between the surfaces $y = x^2$ and $x = y^2$.

SOLUTION:

Since z is expressed as a function of *(x,y)*, we should integrate in the z direction first. After this we consider the xy-plane. The two curves meet at (0,0) and (1,1). We can integrate in x or y first. If we choose y, we can see that the region begins at $y = x^2$ and ends at $y = \sqrt{x}$, and so x is between 0 and 1. Thus the integral is:

$$\int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} f(x,y,z)\, dz\, dy\, dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} xy \, dz \, dy \, dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} xy\left( {}_{z=0}^{z=x+y}[z] \right) dy \, dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} xy(x+y) dy \, dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 y + y^2 x) dy \, dx$$

$$= \int_0^1 {}_{y=x^2}^{y=\sqrt{x}}\left( \frac{x^2 y^2}{2} + \frac{y^3 x}{3} \right) dx$$

$$= \frac{1}{2} \int_0^1 \left( x^2 \sqrt{x}^2 - x^2 (x^2)^2 \right) dx + \frac{1}{3} \int_0^1 \left( \sqrt{x}^3 x - (x^2)^3 x \right) dx$$

$$= \frac{1}{2} \int_0^1 (x^3 - x^6) dx + \frac{1}{3} \int_0^1 \left( x^{\frac{5}{2}} - x^7 \right) dx$$

$$= \frac{1}{2}\left( \frac{1}{4} - \frac{1}{7} \right) + \frac{1}{3}\left( \frac{2}{7} - \frac{1}{8} \right) = \boxed{\frac{3}{28}}$$

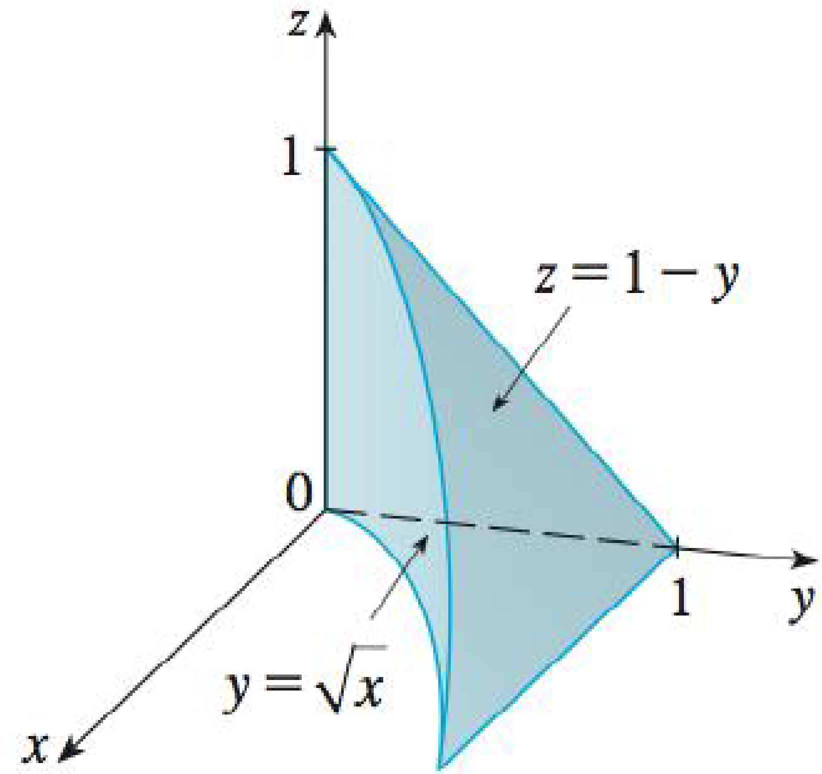
2. In the following integral, exchange the order of integration of y and z:

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, dz\, dy\, dx$$

SOLUTION:

With a problem like this, it helps to draw the figure enclose by the surfaces. From the integrals, it can be seen that *z* enters the volume at $z = 0$ and leaves through the plane $z = 1 - y$. In the $xy$ plane, $y$ enters through $y = \sqrt{x}$ and leaves through $y = 1$. These 2 conditions alone are enough to draw the figure and see that $0 \le x \le 1$.

The shape is like a tetrahedron with the vertices A(0,0,0), B(0,0,1), C(1,1,0), and D(0,1,0), but with a curved surface between vertices A, B, C.




Rewriting the equation of the plane as $y = 1 - z$, and looking at this figure, clearly, $\sqrt{x} \leq y \leq 1 - z$. Consider the projection of this surface in the $zx$ plane. It is a 3-sided figure bounded by $x = 0, z = 0,$ and $z = 1 - \sqrt{x}$. Again, it can be seen that these conditions are sufficient to obtain the limits of x as $0 \leq x \leq 1$. Thus, the integral is:

$$\boxed{\int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x, y, z)\, dy\, dz\, dx}$$