Math 113 Abstract Algebra
Fall 2022 Ryan Cottone

HW 4

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Question 1

Prove that the group Aut($\mathbb{Z}/n\mathbb{Z}$) of automormphisms of the additive group Z/nZ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{\times}$.

Solution: We begin by considering an aribtrary automormphism φ such that $\varphi(1) = a$. For some arbitrary $x \in \mathbb{Z}/n\mathbb{Z}$, $\varphi(x) = \varphi(1) \cdot x = ax$, meaning $\varphi(a)$ generates the cyclic subgroup of a in $(\mathbb{Z}/n\mathbb{Z})^{+}$. Since φ is an automormphism, this cyclic subgroup must generate the entire group, meaning a is also a member of $(\mathbb{Z}/n\mathbb{Z})^{x}$. The inverse is defined in the reverse way with the same method, so φ forms an bijection. It is trivial to see this is also a homomorphism: $\varphi_a \circ \varphi_b(x) = (ab)x = \varphi_{ab}(x)$. We conclude that these two groups are isomorphic.

Question 2

Let S_3 act on the set Ω of ordered pairs $\{(i,j)|1 \leq i,j \leq 3\}$ by $w \cdot (i,j) = (w(i), w(j))$.

1. Find the orbits of this action. For every orbit pick one element inside it and describe its stablizer.

   Solution: First consider the equal-value pairs: (1, 1), (2, 2) and (3, 3). These are in their own orbit of the action (1 2 3).

   The stablizer of (1, 1) is $\{I, (2,3), (2,4), (3,4)\}$.

   Next, consider (1, 2). Then,

   $$\begin{aligned} 1 \cdot (1,2) &= (1,2) \\ (1,2) \cdot (1,2) &= (2,1) \\ (1,3) \cdot (1,2) &= (3,2) \\ (2,3) \cdot (1,2) &= (1,3) \\ (1,3,2) \cdot (1,2) &= (3,1) \\ (1,2,3) \cdot (1,2) &= (2,3) \end{aligned}$$

   which is the rest of the elements in this set. Therefore these are the only two orbits.

   The stablizer of (1, 2) is $\{I\}$. This is evident since the order of the orbit is 8, equal to the order of S_3, which means the stabilizer must have order 1.

2. Consider the bijection $\Omega \to \{1,2,\ldots,9\}$ given by $(i,j) \to 3(i-1)+j$.Compute the cycle decomposition of $S_3 \to S_9$ (the permutation representation composed with the above isomorphism).

   Solution: We can represent S_3 as a cycle decomposition from its orbits. The new decomposition is therefore (1, 5, 9) from the (1, 1), (2, 2), (3, 3) and finally (2, 3, 4, 6, 7, 8) from the second orbit. The full decomp is thus:

   (1 5 9)(2 3 4 6 7 8)

Question 3

List all conjugacy classes in D_{10}.

Solution: There are 4 conjugacy classes in D_{10}:

$$\begin{aligned}
1,\\
r, r^{-1},\\
r^2, r^{-2},\\
s, sr, sr^2, sr^3, sr^4
\end{aligned}$$

Question 4

Let $G = S_{12}$ and $x =$(1 9)(2 5 3 4 11)(7 8 12) $\in G$. Use the orbit-stablizer theorem to compute the number of elements in $C_G(x)$.

Solution: We use the fact that

$$|C_{S_{12}}(x)| = \frac{|S_{12}|}{\texttt{orb}_{S_{12}}(x)}$$

which is derived by the orbit-stablizer theorem.

$|S_{12}| = 12!$, and $\texttt{orb}_{S_{12}}(x) = \frac{n!}{m(n-m)!}$ for an m length cycle. Since we have three cycles of length 2, 5, and 3, we have $\texttt{orb}_{S_{12}}(x) = \frac{12!}{3(n-3)!} + \frac{12!}{5(12-5)!} + \frac{12!}{2(12-2)!}$. This eventually reduces to

$$C_{S_{12}}(x) = 3(12-3)! + 5(12-5)! + 2(12-2)!$$

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Question 5

Cancelled per instructor.

Question 6

Let R be the set $\mathbb{Z}[x_1,x_2,x_3,x_4]$ of polynomials in variables x_1 through x_4 with integer coefficients. Let S_4 act on R by permuting the indices of the four variables, i.e.

$$w \cdot p(x_1,x_2,x_3,x_4) = p(x_{w(1)},x_{w(2)},x_{w(3)},x_{x(4)})$$

for $w \in S_4$, $p(x_1,x_2,x_3,x_4) \in R$.

1. List all elements in the orbit $S_4 \cdot (x_1+x_2)$ and in the stablizer of (x_1+x_2).

   Solution:
   Orbit: $\{x_1+x_2, x_1+x_3, x_1+x_4, x_2+x_3, x_2+x_4, x_3+x_4\}$
   Stabilizer: identity, (3 4), (1 2), (1 2)(3 4)

   This is because $(x_1+x_2) = (x_2+x_1)$.

2. List all elements in the orbit $S_4 \cdot (x_1x_2+x_3x_4)$. How many elements are in its stablizer?

   Solution:
   Orbit: $\{(x_1x_2+x_3x_4),(x_1x_3+x_2x_4),(x_4x_1+x_3x_2)\}$

   Since $|\texttt{Ord}| = 3$ and $|S_4| = 24$, $|\texttt{Stab}| = 8$ for this element.

3. How many elements are in the stablizer of $(x_1+x_2)(x_3+x_4)$? **Solution:** We begin by calculating the orbit of this element: $\{(x_1+x_2)(x_3+x_4),(x_1+x_3)(x_2+x_4),(x_1+x_4)(x_3+x_2)\}$
   Since the orbit has order 3, $|\texttt{Stab}| = \frac{|S_4|}{3} = 8$.

Question 7

Let $G = \mathbb{Z}/2\mathbb{Z}$. Consider the bijection $G \to \{1,2,3,4\}$ with

$$\begin{aligned}
([0],[0]) &\to 1, \\
([0],[1]) &\to 2, \\
([1],[0]) &\to 3, \\
([1],[1]) &\to 4,
\end{aligned}$$

which induces the isomorphism $S_G \approx S_4$. Under this identification, compute the images of all elements of G in S_4 for the left regular action of G on itself.

Solution: Per the identification, we can find all images of elements from S_4 on itself.

$$\begin{aligned}
(01)(10) &= I \\
(01)(1) &= (01) \\
(10)(0) &= (10)
\end{aligned}$$

Question 8

Let $A \subset G$ and $g \in G$. Prove that

$$gN_G(A)g^{-1} = N_G(gAg^{-1})$$

and

$$gC_g(A)g^{-1} = C_G(gAg^{-1})$$

Solution: For every element x in $gN_G(A)g^{-1}$:

$$\begin{gathered} x \in gN_G(A)g^{-1} \\ g^{-1}xg = N_G(A) \end{gathered}$$

which means that $(g^{-1}xg)A(g^{-1}x^{-1}g) = A$, and that $(g^{-1}x)(gAg^{-1})(x^{-1}g)$.

$$\begin{gathered} (g^{-1}xg)A(g^{-1}x^{-1}g) = A \\ x(gAg^{-1})x^{-1} = gAg^{-1} \end{gathered}$$

implying $x \in N_G(gAg^{-1})$.

Similarily,

For every element x in $gC_G(A)g^{-1}$:

$$\begin{gathered} x \in gC_G(A)g^{-1} \\ g^{-1}xg = C_G(A) \end{gathered}$$

which means that $(g^{-1}xg)a(g^{-1}x^{-1}g) = a$, and that $(g^{-1}x)(gag^{-1})(x^{-1}g)$ for all a.

$$\begin{gathered} (g^{-1}xg)a(g^{-1}x^{-1}g) = a \\ x(gag^{-1})x^{-1} = gag^{-1} \end{gathered}$$

implying $x \in C_G(gAg^{-1})$. This really follows directly since $C_G \subset N_G$.

Question 9

Suppose G acts on the left on the set X via $\rho : G \times X \to X$. Let $\tilde{p} : G \to S_X$ be the corresponding permutation representation. Define the set of fixed points X^G of G in X as

$$X^G = \{x \in X | g \cdot x = x \forall g \in G\} \subset X$$

Prove that for any $\sigma \in N_{S_X}(\tilde{\rho}(G))$ and any $x \in X^G$ we have $\sigma(x) \in X^G$.

Solution: If $\sigma \in N_{S_X}(\tilde{\rho}(G))$, then $\sigma\tilde{\rho}(G)\sigma^{-1} = \tilde{\rho}(G)$. The fixed-point definition of X^G implies that, for all $\sigma \in S_X$, $\sigma(x) = x$ for $x \in X^G$. Therefore, $\tilde{\rho}(G)(x) = x$, and therefore $\sigma(x) \in X^G$.

Question 10

Find all finite groups (up to isomorphism) which have exactly two conjugacy classes.

Solution: We posit that all finite groups with exactly two conjugacy classes are isomorphic to $(Z/2Z)^+$. To see this, consider an arbitrary finite group G with 2 conjugacy classes. One must be the identity, so the rest of the elements are in the other single conjugacy class. We see that this conjugacy clas C must have exactly $|G|-1$ elements, since we exclude only the identity.

From here, we use the orbit-stablizer theorem to see that $\frac{|G|}{\texttt{stab(C)}} = \texttt{orb(C)}$. Since $\texttt{stab(C)} = |G|-1$, we have that $|G|$ is divisble by $|G|-1$, which can only be the case for $|G|=2$. Since $|G|=2$, we can find a trivial isomorphism where the identity maps to 0 and the other elements map to 1.