Assignment1(stat)

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August 2023

Problem 1

The dataset x is produced by the following code:

```
x <- c(-20, -10, 3, 4, 5, 6, 7, 20, 39)
```

Problem 2

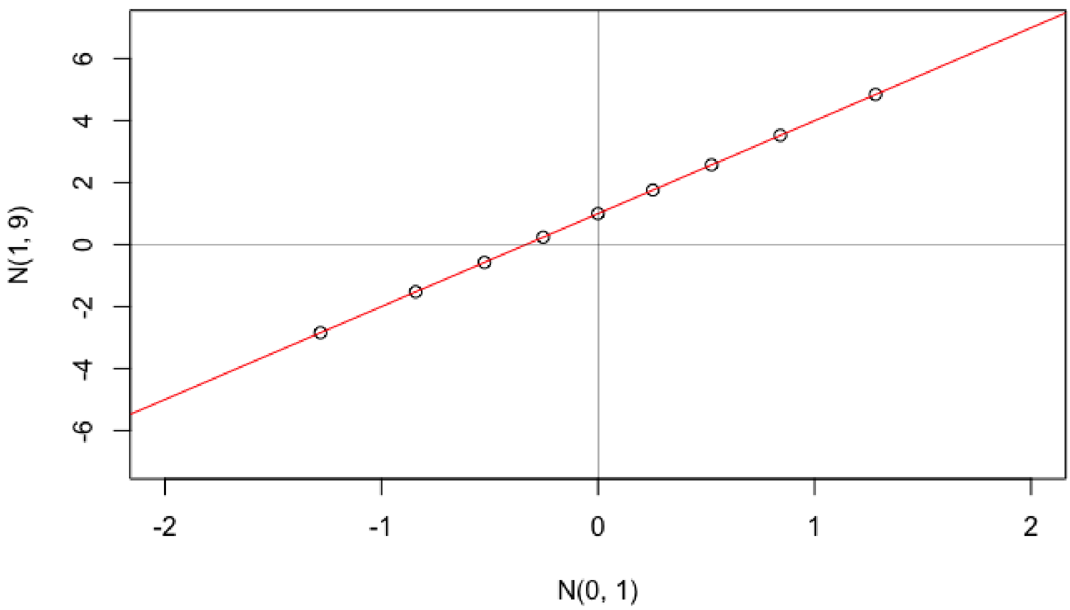
$$X \overset{d}{=} \mu + \sigma * Z$$
$$P(X \leq x) = P(\mu + \sigma * Z \leq x)$$
$$P(X \leq x) = P(Z \leq (x - \mu)/\sigma)$$

So for $x = c_{\frac{k}{n+1}}$ $(1 \leq k \leq n)$,

$$P(Z \leq (c_{\frac{k}{n+1}} - \mu)/\sigma) = P(X \leq c_{\frac{k}{n+1}})$$
$$P(Z \leq (c_{\frac{k}{n+1}} - \mu)/\sigma) = \frac{k}{n+1}$$
$$P(Z \leq (c_{\frac{k}{n+1}} - \mu)/\sigma) = P(Z \leq \Phi^{-1}(\frac{k}{n+1}))$$

So for $k = 1, ..., n$

$$(c_{\frac{k}{n+1}} - \mu)/\sigma = \Phi^{-1}(\frac{k}{n+1})$$



Problem 3

a.

$$\sum_{i=1}^{n}(X_i - \overline{X})^2 = \sum_{i=1}^{n} X_i^2 - n\overline{X}^2$$

$$E(\hat{\theta}) = aE(\sum_{i=1}^{n} X_i^2 - n\overline{X}^2) = a[nE(X^2) - nE(\overline{X}^2)]$$
$$= a[n(\mu^2 + \sigma^2) - n(\sigma^2/n + \mu^2)] = a(n-1)\sigma^2$$

$$bias(\hat{\theta}) = E(\hat{\theta}) - \sigma^2 = [a(n-1) - 1]\sigma^2$$
$$bias(\hat{\theta})^2 = E(\hat{\theta}) - \sigma^2 = [a(n-1) - 1]^2\sigma^4$$

$$var(\hat{\theta}) = a^2(n-1)^2 var(\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2)$$
$$= a^2(n-1)^2 \frac{2\sigma^4}{n-1}$$

$$= 2a^2(n-1)\sigma^4$$

$$MSE(\hat{\theta}) = 2a^2(n-1)\sigma^4 + [a(n-1)-1]^2\sigma^4$$

b.

Differentiate:

$$d/da(MSE(\hat{\theta})) = 4a(n-1)\sigma^4 + 2[a(n-1)-1](n-1)\sigma^4$$

Solve $d/da(MSE(\hat{\theta})) = 0$:

$$4a(n-1)\sigma^4 + 2[a(n-1)-1](n-1)\sigma^4 = 0$$

If $\sigma \neq 0$:

$$\Rightarrow 4a(n-1) + 2[a(n-1)-1](n-1) = 0$$
$$\Rightarrow 4a(n-1) + 2a(n-1)^2 - 2(n-1) = 0$$
$$\Rightarrow a = \frac{2(n-1)}{4(n-1) + 2(n-1)^2}$$
$$\Rightarrow a = \frac{1}{2+n-1} = \frac{1}{n+1}$$

c.

The solution from b. does not coincide with the MLE. However, since it is not asymptotically unbiased, it does not refute the optimality property of the MLE.

Problem 4

Let

$$s = \sum_{i=1}^{n} X_i$$

The likelihood function is:

$$L(p) = p^s(1-p)^{n-s}$$

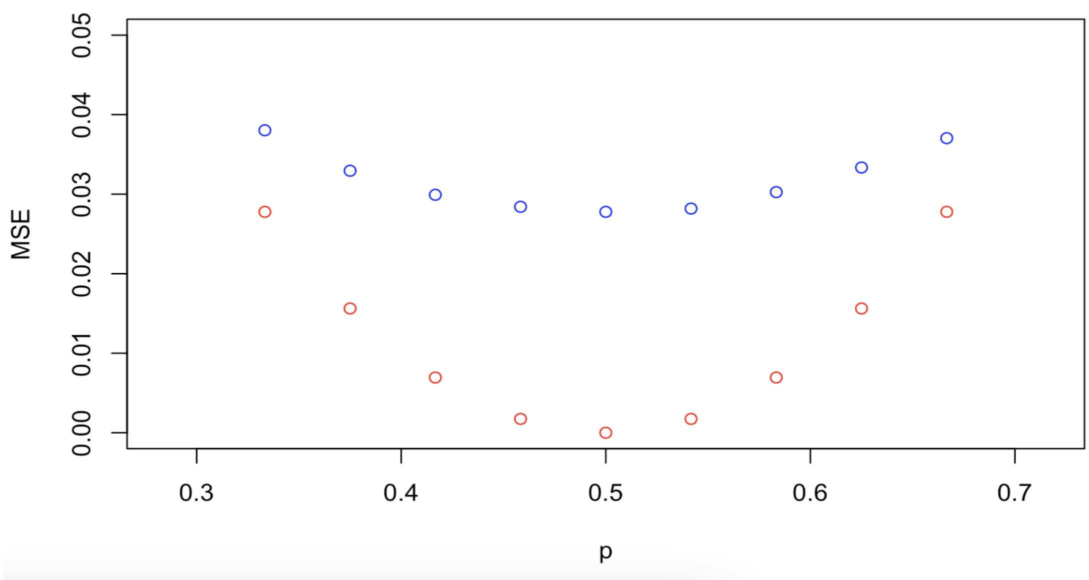
$$g(p) = s \times ln(p) + (n-s) \times ln(1-p)$$
$$s(p) = \frac{s}{p} - \frac{n-s}{1-p}$$
$$s(p) = \frac{s - sp - np + sp}{p(1-p)}$$
$$s(p) = \frac{s-np}{p(1-p)}$$

Now,

$$\begin{cases} s(p) < 0 & \forall p < \frac{s}{n} \\ s(p) = 0 & \text{if } p = \frac{s}{n} \\ s(p) > 0 & \forall p = \frac{s}{n} \end{cases}$$

So the MLE is

$$\hat{\theta}_{ML} = \begin{cases} \frac{s}{n} & \text{if } 1/3 \leq \frac{s}{n} \leq 2/3 \\ 1/3 & \text{if } \frac{s}{n} < 1/3 \\ 2/3 & \text{if } \frac{s}{n} > 2/3 \end{cases}$$



Problem 5

Finding MLE

$$P(X_i = 1) = pr + (1-p)(1-r) = 2pr - p - r + 1$$

Let

$$s = \sum_{i=1}^{n} X_i$$

So

$$L(p) = [2pr - p - r + 1]^s [p + r - 2pr]^{n-s}$$

$$g(p) = s \times ln(2pr - p - r + 1) + (n - s) \times ln(p + r - 2pr)$$
$$s(p) = \frac{s(2r-1)}{2pr - p - r + 1} + \frac{(n-s)(1-2r)}{p + r - 2pr}$$
If $r = 1/2$, $s(p) = 0 \forall p \in [0, 1]$, so all possible p values have the same likelihood.
If $r \neq 1/2$, we solve $s(p) = 0$ to find the MLE:

$$\frac{s(2r-1)}{2pr - p - r + 1} + \frac{(n-s)(1-2r)}{p + r - 2pr} = 0$$

$$\Leftrightarrow (2r-1)[\frac{s}{2pr - p - r + 1} + \frac{(s-n)}{p + r - 2pr}] = 0$$

$$\Leftrightarrow \frac{s}{2pr - p - r + 1} + \frac{(s-n)}{p + r - 2pr} = 0 \text{ (since } 2r - 1 \neq 0)$$

$$\Leftrightarrow sp + sr - 2spr + 2spr - sp - sr + s - 2npr + np + nr - n = 0$$

$$\Leftrightarrow s - 2npr + np + nr - n = 0$$

$$p(2nr - n) = nr + s - n$$

$$p = \frac{nr + s - n}{2nr - n}$$

So the MLE is
$$\hat{\Theta}_{ML} = \frac{nr + s - n}{2nr - n}$$
where
$$s = \sum_{i=1}^{n} X_i$$

Finding MME

$$M_1 = \frac{s}{n}$$

Solve $E(X) = M_1$:
$$pr + (1-p)(1-r) = \frac{s}{n}$$

$$2pr - p - r + 1 = \frac{s}{n}$$

$$p(2r - 1) = \frac{s}{n} + r - 1$$

$$p = \frac{s + rn - n}{n(2r-1)} \text{ (for } r \neq 1/2)$$

So
$$\hat{\Theta}_{MM} = \frac{s + rn - n}{n(2r-1)} \text{ (for } r \neq 1/2)$$