

## MAST20004 Assignment 4, S1 2023. Due 4pm, Friday May 19

### Question 1

A box contains an unknown number of white and black balls. We wish to estimate the proportion  $p$  of white balls in the box. To do so, we draw  $n$  successive balls with replacement. Let  $Z_n$  be the proportion of white balls obtained after  $n$  drawings.

- (a) Use Chebyshev's Inequality to show that, for all  $\varepsilon > 0$ ,

$$\mathbb{P}(|Z_n - p| \geq \varepsilon) \leq \frac{1}{4n\varepsilon^2}.$$

- (b) Using the result in part (a), find the smallest value of  $n$  such that, with probability greater than 0.95, the proportion  $Z_n$  in the sample will estimate  $p$  to within an accuracy of 0.1.

- (c) Answer the same question as in part (b) using the central limit theorem.

## Question 2

Let  $N$  be a geometric random variable (taking values in  $1, 2, \dots$ ) with parameter  $p$  and let  $(X_i)_{i \in \mathbb{N}}$  be independent exponential random variables with parameter  $\lambda$  that are independent of  $N$ . Let  $T = \sum_{i=1}^N X_i$ , and let  $M_\alpha = \inf\{i : X_i > \alpha\}$  (where  $\alpha > 0$  is a constant).

- (a) Find  $\mathbb{E}[e^{\theta T} | N = n]$ , indicating for which values of  $\theta$  your answer is valid.

- (b) Give a simple expression for  $\mathbb{E}[e^{\theta T} | N]$ .

- (c) Find  $\mathbb{E}[e^{\theta T}]$ , specifying the values of  $\theta$  for which your answer is valid.

- (d) Based on your answer above, identify the distribution of  $T$ .

- (e) Find the distribution of  $M_\alpha$ , and hence show that for some  $\alpha > 0$ ,  $M_\alpha$  and  $N$  have the same distribution.

- (f) Prove that (for each  $\alpha$ )  $\sum_{i=1}^{M_\alpha} X_i$  does not have the same distribution as  $T$ .

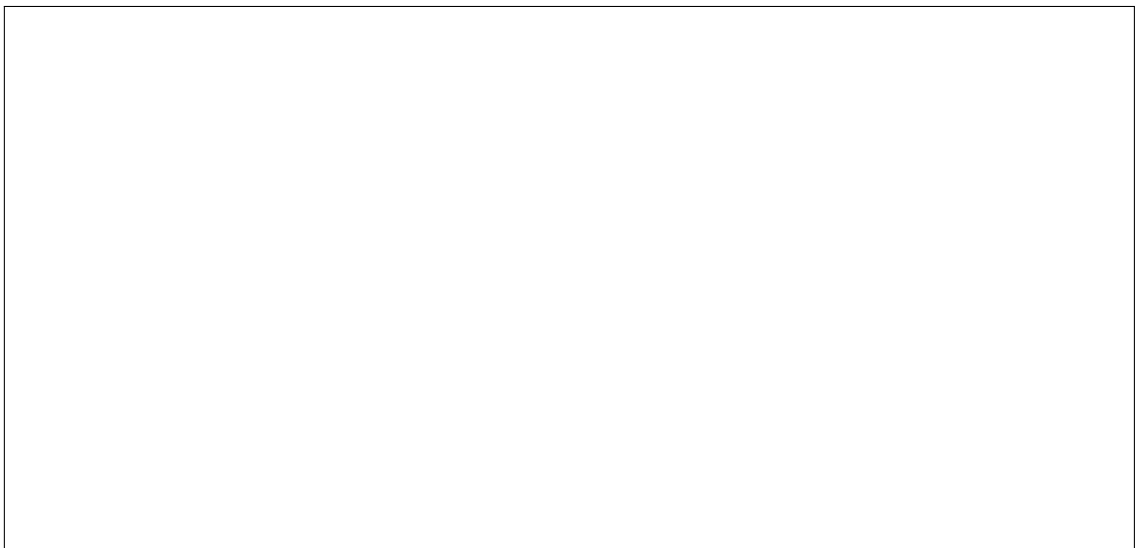
### Question 3

Let  $\beta > 1$ , and suppose that  $X$  is a random variable taking the value  $\beta^k$  with probability  $c2^{-|k|}$  for  $k \in \mathbb{Z}$  (where  $c$  is a constant). The demi-god Maui has two boats. He puts \$  $X$  in one boat and \$  $\beta X$  in the other. Which boat receives which amount is decided by a fair coin toss that is independent of  $X$ . Moana gets to look at the amount of money  $Y_1$  in boat 1. She then has to decide to take the money in boat 1 or boat 2 (without seeing how much money,  $Y_2$ , is in boat 2).

- (a) Find the value of  $c$ .



- (b) Explain intuitively why we should believe that  $\mathbb{E}[Y_1] = \mathbb{E}[Y_2]$ .



- (c) Evaluate  $\mathbb{E}[Y_1|X = \beta^k]$  and  $\mathbb{E}[Y_2|X = \beta^k]$ .

- (d) Give a simple expression for  $\mathbb{E}[Y_1|X]$  and  $\mathbb{E}[Y_2|X]$ .

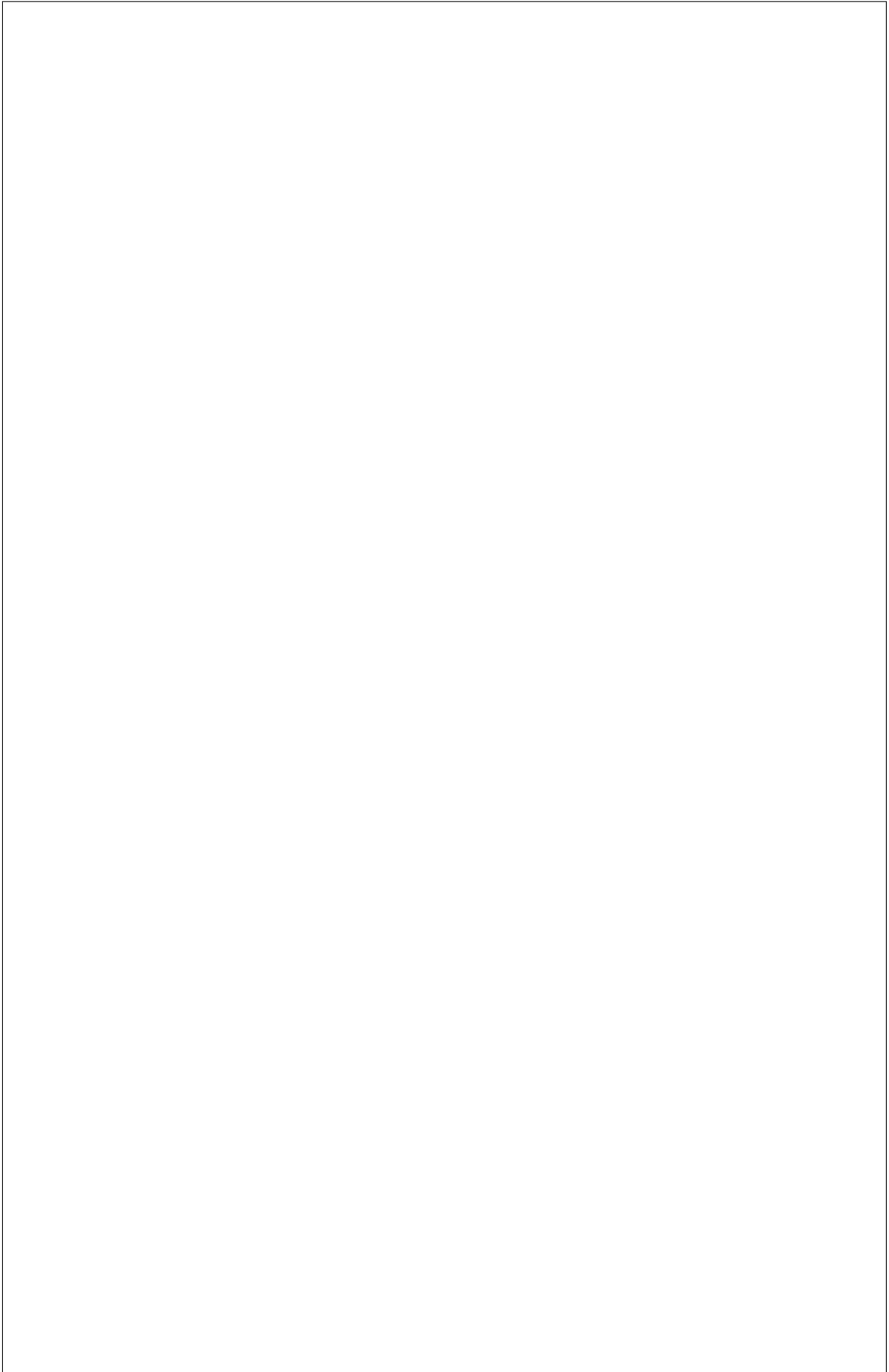
- (e) Evaluate  $\mathbb{P}(Y_1 = \beta^k)$ .

- (f) Find the conditional distribution of  $Y_2$  given that  $Y_1 = \beta^k$  (i.e. determine the possible values of  $Y_2$  and their probabilities, given that the event  $Y_1 = \beta^k$  occurs)



(g) Evaluate  $\mathbb{E}[Y_2|Y_1 = \beta^k]$  (write your answer as  $\beta^k$  times something).

(h) Show that when  $\beta > 2$ ,  $\mathbb{E}[Y_2|Y_1 = \beta^k] > \beta^k$ .



- (i) From the previous part, no matter what Moana sees in boat 1, the expected value of what is in boat 2 is larger. Moana concludes that (when  $\beta > 2$ )

(a)  $\mathbb{E}[Y_2|Y_1] > Y_1$ , and

(b) it is advantageous to choose boat 2 without even looking in boat 1, and

(c)  $\mathbb{E}[Y_2] > \mathbb{E}[Y_1]$ .

Which of the above are right, and which are wrong?

**Question 4**

Let  $X_1, \dots, X_d$  be  $d$  independent standard normal random variables. The goal of this problem is to study the sum of the squares of these random variables.

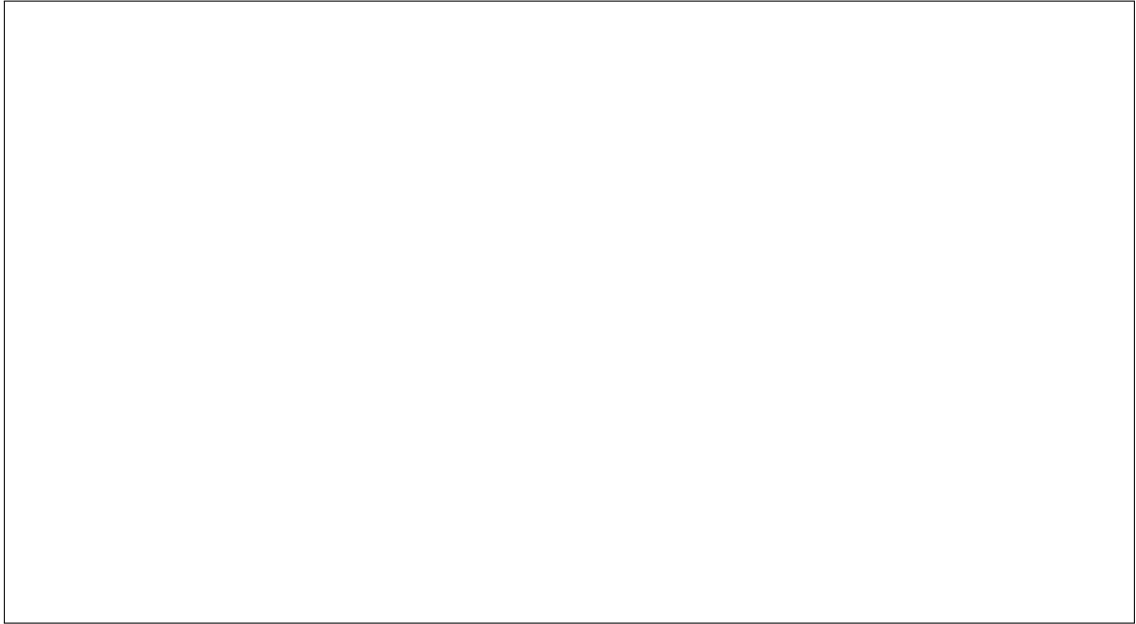
- (a) For a standard normal random variable  $X$ , determine the moment generating function  $M_{X^2}(t)$  of the random variable  $X^2$ . All working must be given.

- (b) Calculate the moment generating function  $M_{S_d}(t)$  of the sum  $S_d = \sum_{j=1}^d X_j^2$ .

- (c) The Gamma distribution  $\gamma(r, \alpha)$  with parameters  $r > 0$  and  $\alpha > 0$  has probability density function

$$f(x) = \frac{\alpha^r x^{r-1} \exp(-\alpha x)}{\Gamma(r)}$$

if  $x \geq 0$  and 0 elsewhere (where  $\Gamma(r)$  is the gamma function). If  $d$  is a positive integer,  $r = d/2$  and  $\alpha = 1/2$ , the Gamma distribution is called the *Chi-squared ( $\chi^2$ ) distribution with  $d$  degrees of freedom*. Using any relevant results from lectures, write down the moment generating function of a random variable  $X_d$  with this distribution.



- (d) Deduce that the sum  $S_d = \sum_{j=1}^d X_j^2$  of  $d$  mutually-independent standard normal random variables has a Chi-squared distribution with  $d$  degrees of freedom.

