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MAST20004 Assignment 4, S1 2023. Due 4pm, Friday May 19

Question 1

A box contains an unknown number of white and black balls. We wish to estimate the proportion p of white balls in the box. To do so, we draw n successive balls with replacement. Let Z_n be the proportion of white balls obtained after n drawings.

(a) Use Chebyshev's Inequality to show that, for all $\varepsilon > 0$,

$$\mathbb{P}(|Z_n - p| \geq \varepsilon) \leq \frac{1}{4n\varepsilon^2}.$$

(b) Using the result in part (a), find the smallest value of n such that, with probability greater than 0.95, the proportion Z_n in the sample will estimate p to within an accuracy of 0.1.

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(c) Answer the same question as in part (b) using the central limit theorem.

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Question 2

Let N be a geometric random variable (taking values in $1, 2 \dots$) with parameter p and let $(X_i)_{i \in \mathbb{N}}$ be independent exponential random variables with parameter λ that are independent of N. Let $T = \sum_{i=1}^{N} X_i$, and let $M_\alpha = \inf\{i : X_i > \alpha\}$ (where $\alpha > 0$ is a constant).

(a) Find $\mathbb{E}[e^{\theta T} | N = n]$, indicating for which values of θ your answer is valid.

(b) Give a simple expression for $\mathbb{E}[e^{\theta T} | N]$.

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(c) Find $\mathbb{E}[e^{\theta T}]$, specifying the values of θ for which your answer is valid.

(d) Based on your answer above, identify the distribution of T.

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(e) Find the distribution of M_α, and hence show that for some $\alpha > 0$, M_α and N have the same distribution.

(f) Prove that (for each α) $\sum_{i=1}^{M_\alpha} X_i$ does not have the same distribution as T.

Question 3

Let $\beta > 1$, and suppose that X is a random variable taking the value β^k with probability $c2^{-|k|}$ for $k \in \mathbb{Z}$ (where c is a constant). The demi-god Maui has two boats. He puts \$ X in one boat and \$ βX in the other. Which boat receives which amount is decided by a fair coin toss that is independent of X. Moana gets to look at the amount of money Y_1 in boat 1. She then has to decide to take the money in boat 1 or boat 2 (without seeing how much money, Y_2, is in boat 2).

(a) Find the value of c.

(b) Explain intuitively why we should believe that $\mathbb{E}[Y_1] = \mathbb{E}[Y_2]$.

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(c) Evaluate $\mathbb{E}[Y_1|X = \beta^k]$ and $\mathbb{E}[Y_2|X = \beta^k]$.

(d) Give a simple expression for $\mathbb{E}[Y_1|X]$ and $\mathbb{E}[Y_2|X]$.

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(e) Evaluate $\mathbb{P}(Y_1 = \beta^k)$.

(f) Find the conditional distribution of Y_2 given that $Y_1 = \beta^k$ (i.e. determine the possible values of Y_2 and their probabilities, given that the event $Y_1 = \beta^k$ occurs)

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(g) Evaluate $\mathbb{E}[Y_2|Y_1 = \beta^k]$ (write your answer as β^k times something).

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(h) Show that when $\beta > 2$, $\mathbb{E}[Y_2|Y_1 = \beta^k] > \beta^k$.

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(i) From the previous part, no matter what Moana sees in boat 1, the expected value of what is in boat 2 is larger. Moana concludes that (when $\beta > 2$)

(a) $\mathbb{E}[Y_2|Y_1] > Y_1$, and

(b) it is advantageous to choose boat 2 without even looking in boat 1, and

(c) $\mathbb{E}[Y_2] > \mathbb{E}[Y_1]$.

Which of the above are right, and which are wrong?

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Question 4

Let $X_1, \ldots, X_d$ be d independent standard normal random variables. The goal of this problem is to study the sum of the squares of these random variables.

(a) For a standard normal random variable X, determine the moment generating function $M_{X^2}(t)$ of the random variable X^2. All working must be given.

(b) Calculate the moment generating function $M_{S_d}(t)$ of the sum $S_d = \sum_{j=1}^{d} X_j^2$.

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(c) The Gamma distribution $\gamma(r, \alpha)$ with parameters $r > 0$ and $\alpha > 0$ has probability density function

$$f(x) = \frac{\alpha^r x^{r-1} \exp(-\alpha x)}{\Gamma(r)}$$

if $x \geq 0$ and 0 elsewhere (where $\Gamma(r)$ is the gamma function). If d is a positive integer, $r = d/2$ and $\alpha = 1/2$, the Gamma distribution is called the *Chi-squared (χ^2) distribution with d degrees of freedom.* Using any relevant results from lectures, write down the moment generating function of a random variable X_d with this distribution.

(d) Deduce that the sum $S_d = \sum_{j=1}^{d} X_j^2$ of d mutually-independent standard normal random variables has a Chi-squared distribution with d degrees of freedom.