ODD and EVEN FUNCTIONS

Defn: odd function

A function $f(x)$ is odd if $f(-x) = -f(x)$

Defn: even function

A function $f(x)$ is even if $f(-x) = f(x)$.

1. Expand $f(x) = x^2$ as a Fourier series in the interval $(-\pi < x < \pi)$ and hence deduce that

(i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \pi^2/12$

(ii) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \pi^2/6$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \pi^2/8$

Solution:

Given $f(x) = x^2$ in $(-\pi < x < \pi)$

(* While doing problem for odd/even Fourier series check the interval from (-ve to +ve), so that one can write $\int_{-a}^{a} = 2\int_0^a$ (even) but $\int_0^{2a} \neq 2\int_0^a$)

Here $f(x) = x^2$; $(-\pi < x < \pi)$ is even function.

Then, the Fourier series,

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin nx} \rightarrow (1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi}$$

$$a_0 = \frac{2}{3\pi}\pi^3 = \boxed{\frac{2\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

($x^2 \cos nx$ is even function)

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \, dx$$

$$= \frac{2}{\pi}\left\{ (x^2)\left(\cancel{\frac{\sin nx}{n}}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(\cancel{-\frac{\sin nx}{n^3}}\right)\right\}_0^{\pi}$$

$$= \frac{2}{\cancel{\pi}}\left\{ 2\cancel{\pi} \frac{\cos n\pi}{n^2}\right\}$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

[$x^2 \sin nx$ is odd function, So, $\int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0$.]

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx \, dx$$

$$= 0.$$

$\therefore$ The Fourier series is

[since $b_n = 0$]

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{2\pi^2/3}{2} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$$

$$x^2 = \pi^2/3 + 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \cdots\right]$$

(i) $\underline{x=0}$, $0 = \pi^2/3 + 4\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right]$

$$-\pi^2/3 = -4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right]$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \boxed{\frac{\pi^2}{12}}$$

(ii) $\underline{x = \pi}$ (end point)

$$\frac{(-\pi)^2 + \pi^2}{2} = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{\pi^2 - \pi^2/3}{4} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \Rightarrow \boxed{\frac{\pi^2}{6}}$$

(iii) We know that

$$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots=\frac{\pi^2}{12} \longrightarrow (A)$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6} \longrightarrow (B)$$

$$(A)+(B) \Rightarrow 2\left(\frac{1}{1^2}+\frac{1}{3^2}+\cdots\right)=\frac{\pi^2}{12}+\frac{\pi^2}{6}$$

$$\boxed{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}}$$

Formula:

* If $f(x)$ is even, then in $(-L<x<L)$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where $a_0=\frac{2}{L}\int_0^L f(x)\,dx$

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

* If $f(x)$ is odd, then in $(-L<x<L)$

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where $b_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

Practice problems:

1. Expand $f(x)=x$ in $(-\pi<x<\pi)$

Sol: $f(x)=\sum_{n=1}^{\infty} b_n \sin(nx)$, $\boxed{a_0=0,\ a_n=0}$ since $f(x)$ is odd.

$$b_n=\frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx=\frac{-2(-1)^n}{n}$$