Faculty of Engineering and Technology

Department of Mathematics

Lecture Notes

Subject Code/Title: 18MAB203T-Probability and Stochastic Processes

Course offered to: ECE IV Semester/II Year

Contents

Cross Correlation Function

Definition: If the processes $\{X(t)\}$ and $\{Y(t)\}$ are jointly wide-sense stationary,then $E\left[X(t)Y(t-\tau)\right]$ is a function of τ , denoted by $R_{xy}(\tau)$. This function $R_{xy}(\tau)$ is called the cross – correlation function of the processes $\{X(t)\}$ and $\{Y(t)\}$

Note: $R_{xy}(\tau)$ can also be defined as $E\left[X(t)Y(t+\tau)\right]$

Properties:

1. $R_{xy}(-\tau)=R_{yx}(\tau)$

Proof: $R_{xy}(\tau)=E\left[X(t)Y(t+\tau)\right]$

$$R_{xy}(-\tau)=E\left[X(t)Y(t-\tau)\right]$$

Substitute $t_1=t-\tau$

$$=E\left[Y(t_1)X(t_1+\tau)\right]$$

$$=R_{yx}(\tau)$$

2. $\left|R_{xy}(\tau)\right|\le\sqrt{R_{xx}(0)R_{yy}(0)}$

Proof: For any real number α , we know that

$$E\left[\alpha X(t)+Y(t+\tau)\right]^2\ge 0$$
$$E\left[\alpha^2X^2(t)+Y^2(t+\tau)+2\alpha X(t)Y(t+\tau)\right]\ge 0$$
$$E\left[\alpha^2X^2(t)\right]+E\left[Y^2(t+\tau)\right]+E\left[2\alpha X(t)Y(t+\tau)\right]\ge 0$$
$$\alpha^2E\left[X^2(t)\right]+E\left[Y^2(t+\tau)\right]+2\alpha E\left[X(t)Y(t+\tau)\right]\ge 0$$

Since $\{X(t)\}$ and $\{Y(t)\}$ are jointly WSS, each is a WSS process.

Hence the second order moments are constants. But $E(X^2(t))=R_{xx}(0)$ by the property of auto correlation function and $E(Y^2(t+\tau))=R_{yy}(0)$

$$\alpha^2R_{xx}(0)+R_{yy}(0)+2\alpha R_{xy}(\tau)\ge 0 \quad \forall\alpha$$

Since $R_{xx}(0)>0$ and α is any real number, the discriminant is ≤ 0.

$$4(R_{xy}(\tau))^2 - 4R_{xx}(0)R_{yy}(0) \le 0$$

$$(R_{xy}(\tau))^2 - R_{xx}(0)R_{yy}(0) \le 0$$

$$\left|R_{xy}(\tau)\right| \le \sqrt{R_{xx}(0)R_{yy}(0)}$$

3. $\left|R_{xy}(\tau)\right| \le \dfrac{R_{xx}(0)+R_{yy}(0)}{2}$

Proof: We know that $R_{xx}(0)$ and $R_{yy}(0)$ are positive numbers so their A.M ≥G.M

$$\frac{R_{xx}(0)+R_{yy}(0)}{2} \ge \sqrt{R_{xx}(0)R_{yy}(0)}$$

By property 2, $\left|R_{xy}(\tau)\right| \le \sqrt{R_{xx}(0)R_{yy}(0)}$

$$\left|R_{xy}(\tau)\right| \le \sqrt{R_{xx}(0)R_{yy}(0)} \le \frac{R_{xx}(0)+R_{yy}(0)}{2}$$

$$\left|R_{xy}(\tau)\right| \le \frac{R_{xx}(0)+R_{yy}(0)}{2}$$

4. If the processes $\{X(t)\}$ and $\{Y(t)\}$ are orthogonal, then $R_{xy}(\tau)=0$

5. If the processes $\{X(t)\}$ and $\{Y(t)\}$ are independent, then $R_{xy}(\tau)=\mu_x \times \mu_y$

Example 1:

Consider 2 random processes $X(t)=3\cos(\omega t+\theta)$ and $Y(t)=2\cos(\omega t+\theta-\pi/2)$ where θ is a random variable uniformly distributed in $(0,2\pi)$. Prove that $\left|R_{xy}(\tau)\right| \le \sqrt{R_{xx}(0)R_{yy}(0)}$

Solution:

$$R_{xx}(t,t+\tau) = E\left[X(t)X(t+\tau)\right]$$

$$= E\left[3\cos(\omega t+\theta).3\cos(\omega t+\omega\tau+\theta)\right]$$

$$= \frac{9}{2}E\left[\cos\left(2\omega t+2\theta+\omega\tau\right)+\cos(-\omega\tau)\right]$$

$$= \frac{9}{2}\int_0^{2\pi}\cos\left(2\omega t+2\theta+\omega\tau\right)\frac{1}{2\pi}d\theta + \frac{9}{2}E(\cos\omega\tau)$$

$$= \frac{9}{4\pi}\left[\frac{\sin\left(2\omega t+2\theta+\omega\tau\right)}{2}\right]_0^{2\pi} + \frac{9}{2}\cos\omega\tau$$

$$= \frac{9}{4\pi}\left[\frac{\sin(2\omega t+\omega\tau)-\sin(2\omega t+\omega\tau)}{2}\right]+\frac{9}{2}\cos\omega\tau$$

$$= \frac{9}{2}\cos\omega\tau$$

$$R_{xx}(\tau)=\frac{9}{2}\cos\omega\tau \Rightarrow R_{xx}(0)=\frac{9}{2}$$

In a similar manner **prove** $R_{yy}(\tau)=2\cos\omega\tau \Rightarrow R_{yy}(0)=2$.

$$R_{xy}(t,t+\tau)=E\left[X(t)Y(t+\tau)\right]$$

$$= E\left[3\cos(\omega t+\theta).2\cos(\omega t+\omega\tau+\theta-\pi/2)\right]$$

$$= 3E\left[\sin(2\omega t+2\theta+\omega\tau)+\sin\omega\tau\right]$$

$$= 3\int_0^{2\pi}\sin(2\omega t+2\theta+\omega\tau)\frac{1}{2\pi}d\theta+3E(\sin\omega\tau)$$

$$= \frac{3}{2\pi}\left[\frac{\cos(2\omega t+2\theta+\omega\tau)}{2}\right]_0^{2\pi}+3\sin\omega\tau$$

$$= \frac{-3}{2\pi}\left[\frac{\cos(2\omega t+\omega\tau+4\pi)-\cos(2\omega t+\omega\tau)}{2}\right]+3\sin\omega\tau$$

$$= 3\sin\omega\tau = R_{xy}(\tau)$$

Hence, $\{X(t)\}$ and $\{Y(t)\}$ are jointly WSS.

Now $R_{xx}(0)R_{yy}(0)=9 \Rightarrow \sqrt{R_{xx}(0)R_{yy}(0)}=3$

$$R_{xy}(\tau)=3\sin\omega\tau \Rightarrow \left|R_{xy}(\tau)\right|=\left|3\sin\omega\tau\right|\le 3$$

$$\therefore \left|R_{xy}(\tau)\right|\le\sqrt{R_{xx}(0)R_{yy}(0)}$$

Example2: Two random processes $\{X(t)\}$ and $\{Y(t)\}$ are defined by $X(t)=A\cos\omega t+B\sin\omega t$ & $Y(t)=B\cos\omega t-A\sin\omega t$. Show that $\{X(t)\}$ and $\{Y(t)\}$ are jointly WSS if A& B are uncorrelated random variables with zero means and the same variances and ω is a constant.

Solution:

Given $X(t)=A\cos\omega t+B\sin\omega t$ and $Y(t)=B\cos\omega t-A\sin\omega t$ where A & B are uncorrelated random variables with zero means.

So $E(A)=0, E(B)=0$ and $E(AB)=E(A)E(B)=0$

Given $Var(A) = Var(B) = \sigma^2$ (say)

Then $E(A^2) = E(B^2) = \sigma^2$

Given $X(t) = A\cos\omega t + B\sin\omega t$

$$E(X(t)) = E\left(A\cos\omega t + B\sin\omega t\right)$$

Then $E(X(t)) = E(A)\cos\omega t + E(B)\sin\omega t$

$= 0$ as $E(A) = E(B) = 0$

$$\begin{aligned}
&R_{xx}\left(t, t+\tau\right) = E\left[X(t)X(t+\tau)\right] \\
&= E\left[\left(A\cos\omega t + B\sin\omega t\right)\left(A\cos\omega\left(t+\tau\right) + B\sin\omega\left(t+\tau\right)\right)\right] \\
&= E\left[A^2\cos\omega t\cos\omega\left(t+\tau\right)\right] + E\left[AB\cos\omega t\sin\omega\left(t+\tau\right)\right] \\
&\quad + E\left[B^2\sin\omega t\sin\omega\left(t+\tau\right)\right] + E\left[AB\sin\omega t\cos\omega\left(t+\tau\right)\right] \\
&= E\left(A^2\right)\cos\omega t\cos\omega\left(t+\tau\right) + E\left(AB\right)\cos\omega t\sin\omega\left(t+\tau\right) \\
&\quad + E\left(B^2\right)\sin\omega t\sin\omega\left(t+\tau\right) + E\left(AB\right)\sin\omega t\cos\omega\left(t+\tau\right)
\end{aligned}$$

But $E(AB) = 0$ and $E(A^2) = E(B^2) = \sigma^2$

$$\begin{aligned}
R_{xx}\left(t, t+\tau\right) &= E\left[A^2\right]\cos\omega t\cos\omega\left(t+\tau\right) + E\left[B^2\right]\sin\omega t\sin\omega\left(t+\tau\right) \\
&= \sigma^2\cos(\omega t + \omega\tau - \omega t) \\
&= \sigma^2\cos\left(\omega\tau\right)
\end{aligned}$$

Hence $\left\{X(t)\right\}$ is a WSS.

In a similar manner **prove** $\left\{Y(t)\right\}$ is also a WSS.

Now to show their cross correlation is a function of τ

$$\begin{aligned}
&R_{xy}\left(t, t+\tau\right) = E\left[X(t)Y(t+\tau)\right] \\
&= E\left[\left(A\cos\omega t + B\sin\omega t\right)\left(B\cos\omega\left(t+\tau\right) - A\sin\omega\left(t+\tau\right)\right)\right] \\
&= E\left[AB\cos\omega t\cos\omega\left(t+\tau\right)\right] - E\left[A^2\cos\omega t\sin\omega\left(t+\tau\right)\right] \\
&\quad + E\left[B^2\sin\omega t\cos\omega\left(t+\tau\right)\right] - E\left[AB\sin\omega t\sin\omega\left(t+\tau\right)\right] \\
&= E\left[AB\right]\cos\omega t\cos\omega\left(t+\tau\right) - E\left[A^2\right]\cos\omega t\sin\omega\left(t+\tau\right) \\
&\quad + E\left[B^2\right]\sin\omega t\cos\omega\left(t+\tau\right) - E\left[AB\right]\sin\omega t\sin\omega\left(t+\tau\right)
\end{aligned}$$

But $E(AB) = 0$ and $E(A^2) = E(B^2) = \sigma^2$

$$R_{xy}(t,t+\tau)=\sigma^2\left(\sin\omega t\cos\omega(t+\tau)-\cos\omega t\sin\omega(t+\tau)\right)$$
$$=\sigma^2\sin(\omega t-\omega t-\omega\tau)=\sigma^2\sin(-\omega\tau)$$
$$=-\sigma^2\sin(\omega\tau)$$

This is a function of τ only.

Therefore $\{X(t)\}$ and $\{Y(t)\}$ are jointly WSS.

<u>Ergodic Process</u>

<u>Ergodicity</u>

By considering a random process as a collection of random variables indexed in time, we defined the statistical averages (ensemble averages) such as mean and auto correlation of the random process. To estimate ensemble averages, we have to compute a weighted average of all sample functions of the random process. However, if the process has access to only single sample function, we use its time average to estimate the ensemble average of the process.

In general, ensemble averages and time averages are not equal except for a very special class of random process called ergodic process. Ergodicity deals with the equality of time averages and ensemble averages. Hence ergodicity is a stronger condition than stationarity and hence all random processes that are stationary are not ergodic.

<u>Time average</u> If {X(t)} is a random process, then $\frac{1}{2T}\int_{-T}^{T}X(t)dt$ is called the time average of {X(t)} over (-T, T) and denoted by $\overline{X}_T$

<u>Ergodic Process</u> A random process {X(t)} is said to be ergodic, if its ensemble averages are equal to appropriate time averages.

<u>Mean Ergodic Process</u> If the random process {X(t)} has a constant mean E(X(t)) = μ and if

$\overline{X_T}=\frac{1}{2T}\int_{-T}^{T}X(t)dt \to \mu$ as $T\to\infty$, then {X (t)} is said to be mean – ergodic.

<u>Mean Ergodic Theorem</u> If {X (t)} is a random process with constant mean μ and if $\overline{X_T}=\frac{1}{2T}\int_{-T}^{T}X(t)dt$, then {X(t)} is mean-ergodic (or ergodic in the mean), provided $\lim_{T\to\infty}\mathrm{Var}\left(\overline{X}_T\right)=0$

Example

Prove that the random process {X(t)} with constant mean is mean ergodic if $\lim_{T\to\infty}\left[\frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}C(t_1,t_2)dt_1dt_2\right]=0$

Proof

By mean-ergodic theorem, the condition for the mean-ergodicity of the process {X(t)} is $\lim_{T\to\infty}\mathrm{Var}\left(\overline{X}_T\right)=0$.

$$\overline{X}_T = \frac{1}{2T}\int_{-T}^{T}X(t)dt \text{ and } E(\overline{X_T})=E(X(t))$$

$$\overline{X}_T{}^2 = \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}X(t_1)X(t_2)dt_1dt_2$$

$$E\left(\overline{X}_T{}^2\right)=\frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}R(t_1,t_2))dt_1dt_2$$

$$Var\left(\overline{X}_T\right) = E\left(\overline{X}_T{}^2\right)-\{E\left(\overline{X}_T\right)\}^2$$

$$= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}(R(t_1,t_2)-E(X(t_1)E(X(t_2))dt_1dt_2$$

$$= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}C(t_1,t_2)dt_1dt_2$$

Now $\lim_{T\to\infty}\mathrm{Var}\left(\overline{X}_T\right)=0 \Rightarrow \lim_{T\to\infty}\frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}C(t_1,t_2)dt_1dt_2 = 0$

Hence proved.