18MAB201T-

Transforms & Boundary value problems.

* Partial differential Equations
* Fourier series
* Applications of Partial Differential Equations
* Fourier Transform
* Z- transform.

UNIT-I - PDE

Differential Equations have wide applications in various engineering and Science disciplines.

ODE: The differential equation involving only one independent variable is called Ordinary Differential Equation

PDE: The differential equation involving more than one independent is called Partial Differential Equation:

Order: The order of the PDE is the order of the highest order derivative occurring in it.

Degree: The degree of the PDE is the power of its highest derivative occurring in it.

* Formation of Partial Differential Equations

* By eliminating arbitrary constants
* By eliminating arbitrary functions

Notations:

If $z = f(x, y)$, then

$$p = \frac{\partial z}{\partial x} \qquad q = \frac{\partial z}{\partial y}$$

$$r = \frac{\partial^2 z}{\partial x^2} \qquad t = \frac{\partial^2 z}{\partial y^2}$$

$$s = \frac{\partial^2 z}{\partial x \partial y}$$

Note:

* If the number of arbitrary constants $\leq$ number of independent ~~values~~ variables then we get first order PDE (ie, use. p, q only)

* If the number of arbitrary constants $>$ number of independent variables then we get second or more than second order PDE (we use, p, q, r, s, t)

Example:

1. Form the PDE from $z = (x^2+a)(y^2+b)$.

Hints: here 'a' & 'b' are arbitrary constants, x and y are independent variables.
z - dependent variable (depends on x, y)

our aim is to eliminate a & b

$$z = (x^2+a)(y^2+b) \quad \text{---} \quad (1)$$

eqn (1) Partially diff w.r.to 'x' we get

$$\frac{\partial z}{\partial x} = p = 2x(y^2+b) \quad \text{---} \quad (2)$$

eqn (1) partially diff w.r.to 'y' we get

$$\frac{\partial z}{\partial y} = q = (x^2+a)(2y) \quad \text{---} \quad (3)$$

from eqn (2) & (3)

$$\frac{p}{2x} = y^2+b \quad \& \quad \frac{q}{2y} = x^2+a$$

put in eqn (1) we get

$$z = \frac{p}{2x}\,\frac{q}{2y}$$

$$\boxed{4xyz = pq}$$

[Here the number of independent variable equal to number of arbitrary constants so we get first order PDE]

(2) Obtain the PDE of all spheres with centres lies on $z=0$ and whose radius is constant and equal to r.

$\therefore$ The equation of the sphere with $z=0$ is

$$(x-a)^2 + (y-b)^2 + z^2 = r^2 \quad \text{--- (1)}$$

eqn (1) P.D w.r.to 'x'

$$\cancel{2}(x-a) + \cancel{2}z\frac{\partial z}{\partial x} = 0 \qquad \left(\frac{\partial z}{\partial x} = p\right)$$

$$zp = -(x-a) \quad \text{--- (2)}$$

eqn (1) again P,D w.r.to 'y'

$$\cancel{2}(y-b) + \cancel{2}z\frac{\partial z}{\partial y} = 0 \qquad \left(\frac{\partial z}{\partial y} = q\right)$$

$$zq = -(y-b) \quad \text{--- (3)}$$

from eqn (2) & (3) in (1)

$$(-zp)^2 + (-zq)^2 + z^2 = r^2$$

$$\boxed{z^2(p^2+q^2+1) = r^2}$$

(3) Form the PDE from $\log(az-1) = x+ay+b$ (5)

$$\log(az-1) = x+ay+b \quad \text{--- (1)}$$

eqn (1) P.D.w.r.to 'x'

$$\frac{1}{az-1} ap = 1 \quad \text{--- (2)}$$

eqn (1) P.D.w.r.to 'y'

$$\frac{1}{az-1} \cancel{a} q = \cancel{a}$$

$$\frac{q}{az-1} = 1 \quad \text{--- (3)}$$

from (2) & (3) we get (ie) (2)/(3)

$$\frac{ap}{q} = 1 \Rightarrow \boxed{a = \frac{q}{p}}$$

substitute a value in eqn (3)

$$q = \cancel{z}\frac{q}{p} - 1$$

$$pq = qz - p$$

$$\boxed{p(q+1) = qz}$$ Ans//

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(4) Form a PDE by eliminating a, b and c from

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Soln:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{—— (1)}$$

eq (1) P.D w r to 'x'

$$\frac{\cancel{2}x}{a^2} + \frac{\cancel{2}z}{c^2} p = 0 \quad \text{—— (2)}$$

eqn (1) P.D w. r to 'y'

$$\frac{\cancel{2}y}{b^2} + \frac{\cancel{2}z}{c^2} q = 0 \quad \text{—— (3)}$$

eqn (2) again P.D w.r to 'y'

$$\frac{1}{c^2}\left[pq + zs\right] = 0$$

$$\therefore \boxed{pq + zs = 0} \text{ // Ans:}$$

(Here a, b & c are 3 arbitrary constants, and x & y are two independent variable)

$\therefore$ [So we get second order PDE]

Problems for practice:

(1) Form the partial differential equation by eliminating the arbitrary constants a and b from $z = a^2x + ay^2 + b$. [Ans. $4y^2p = q^2$]

(2) Find the PDE of all plane having equal intercept on the x and y axis.
$\left(\frac{x}{a} + \frac{y}{a} + \frac{z}{c} = 1\right)$ [Ans: $p = q$]

(3) Form the PDE by eliminating the arbitrary constants a & b from $(x-a)^2 + (y-b)^2 = z^2\cot^2\alpha$
[Ans: $p^2 + q^2 = \tan^2\alpha$]

(4) Find the PDE from $z = ax^n + by^n$
[Ans. $nz = px + qy$]

Elimination of arbitrary functions

Type 1: $z = f(x, y)$ where x & y are independent variables.

Type 2: $\phi(u, v) = 0$ where $u = u(x, y, z)$
$v = v(x, y, z)$ and
x, y and z are independent variables.

Type - I

1. Form the PDE by eliminating the arbitrary function from $z = f(x^2+y^2)$ — ①

eqn ① PD w.r to 'x'

$$p = f'(x^2+y^2) \cdot 2x$$

$$\frac{p}{2x} = f'(x^2+y^2) \quad \text{— ②}$$

eqn ① PD w.r to 'y'

$$q = f'(x^2+y^2) \cdot 2y$$

$$\frac{q}{2y} = f'(x^2+y^2) \quad \text{— ③}$$

from eqn ② and ③ we have

$$\frac{p}{\cancel{2}x} = \frac{q}{\cancel{2}y} \Rightarrow \boxed{py = qx}$$

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2. Form the PDE by eliminating the arbitrary function from $\cancel{z} = f\left(\frac{y}{x}\right)$

soln

$$\cancel{z} = f\left(\frac{y}{x}\right) \quad \text{--- (1)}$$

eqn (1) P.D w.r.to 'x'

$$p = f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

$$-\frac{p}{\left(\frac{y}{x^2}\right)} = f'(y/x) \quad \text{--- (2)}$$

eqn (1) P.D. w.r to 'y'

$$q = f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$\frac{q}{(1/x)} = f'(y/x) \quad \text{--- (3)}$$

from (2) & (3)

$$\frac{-p}{(y/x^2)} = \frac{q}{(1/x)}$$

$$\boxed{xp + yq = 0}$$

3) Form the PDE by eliminating the arbitrary function from $z = f(x-at) + f(x+at)$.

$$z = f(x-at) + f(x+at) \quad \text{--- } (1)$$

eqn (1) P D w.r to 'x'

$$p = f'(x-at) + f'(x+at) \quad \text{--- } (2) \qquad \left(p = \frac{\partial z}{\partial x}\right)$$

eqn (1) P D w.r to 't' $\qquad \left(q = \frac{\partial z}{\partial t}\right)$

$$q = f'(x-at)(-a) + af'(x+at) \quad \text{--- } (3)$$

again eqn (2) P D w.r to 'x' $\qquad \left(r = \frac{\partial^2 z}{\partial x^2}\right)$

$$r = f''(x-at) + f''(x+at) \quad \text{--- } (4)$$

again eqn (3) P D w.r to 't' $\qquad \left(t = \frac{\partial^2 z}{\partial t^2}\right)$

$$t = a^2 f''(x-at) + a^2 f''(x+at) \quad \text{--- } (5)$$

$$t = a^2\left[f''(x-at) + f''(x+at)\right]$$

$$\boxed{t = a^2 r} \quad \text{(or)} \quad \boxed{\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}}$$

4. Form the PDE by eliminating the arbitrary function from $z = xy + 2f(x^2+y^2+z^2)$.

Soln

$$z = xy + 2f(x^2+y^2+z^2) \quad \text{--- } (1)$$

eqn (1) P D w. r to 'x'

$$p = y + 2f'(x^2+y^2+z^2)(2x+2zp)$$

eqn (1) P D w. r to 'y'

$$q = x + 2f'(x^2+y^2+z^2)(2y+2zq)$$

$$p - y = f'(x^2+y^2+z^2)(x+zp) \quad \text{--- } (2)$$

$$q - x = f'(x^2+y^2+z^2)(y+zq) \quad \text{--- } (3)$$

from (2) & (3)

$$\boxed{\frac{p-y}{q-x} = \frac{x+zp}{y+zq}}$$

on simplification we get,

$$(y+xz)p - (x+yz)q = y^2 - x^2$$

5 Eliminate the arbitrary function from
$z = x^2 f(y) + y^2 g(x)$

Soln

$$z = x^2 f(y) + y^2 g(x) \quad \text{--- (1)}$$

eqn (1) P.D. w.r to 'x'

$$p = 2x f(y) + y^2 g'(x) \quad \text{--- (2)}$$

eqn (1) P.D w.r to 'y'

$$q = x^2 f'(y) + 2y\, g(x) \quad \text{--- (3)}$$

again eqn (2) P.D. w.r to 'x'

$$r = 2 f(y) + y^2 g''(x) \quad \text{--- (4)}$$

again eqn (3) P.D w.r to 'y'

$$t = x^2 f''(y) + 2 g(x) \quad \text{--- (5)}$$

again eq (1) P.D w.r to 'y' (x)

$$s = 2x f'(y) + 2y\, g'(x) \quad \text{--- (6)}$$

(2) $\times x \Rightarrow \quad px = 2x^2 f(y) + xy^2 g'(x)$

(3) $\times y \Rightarrow \quad qy = x^2 y f'(y) + 2y^2 g(x)$

$$px + qy = 2\left[x^2 f(y) + y^2 g(x)\right] + xy\left[xg'(x) + yf'(y)\right]$$

$$px + qy = 2z + xy\left[\frac{s}{2}\right]$$

$$\boxed{2(px+qy) = 4z + xys}$$

Problems for practice:

(1) Form the PDE by eliminating the arbitrary function from $z = g(y+x) + x f(y+x)$

Ans: $\boxed{r + t = 2s}$

(2) Eliminate the arbitrary function f and obtain the PDE from $z = e^y f(x+y)$

Ans: $\boxed{q + z + p}$

Type - II $[\phi(u,v)=0]$

Let $\phi(u,v)=0$ be given function.
Then we can construct the PDE as follows
⇒ Differentiate u and v w.r.t x, y and z.
⇒ Find
$$P=\frac{\partial(u,v)}{\partial(y,z)}=\begin{vmatrix}\frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\end{vmatrix}$$

$$Q=\frac{\partial(u,v)}{\partial(z,x)}=\begin{vmatrix}\frac{\partial u}{\partial z} & \frac{\partial u}{\partial x}\\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x}\end{vmatrix}$$

$$R=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$$

⇒ Write the PDE in the form
$$Pp+Qq=R.$$

Aliter

If $\phi(u,v)=0$ then we can write $u=f(v)$ or $v=g(u)$ then using type-I we get the required "PDE"

1. Form the PDE by eliminating the arbitrary function from ϕ from the relation
$\phi(x^2+y^2+z^2, lx+my+nz) = 0$

Soln. (method - I)

Here $u = x^2+y^2+z^2$ $\quad v = lx+my+nz$

$$P = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} 2y & 2z \\ m & n \end{vmatrix} = 2yn - 2mz$$
$$= 2(yn - mz)$$

$$Q = \frac{\partial(u,v)}{\partial(z,x)} = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} 2z & 2x \\ n & l \end{vmatrix} = 2lz - 2nx$$
$$= 2(lz - nx)$$

$$R = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ l & m \end{vmatrix} = 2(xm - ly)$$

$\therefore$ The solution (or) The Required PDE

$$Pp + Qq = R$$

$$\boxed{(yn - mz)p + (lz - nx)q = xm - ly}$$

<u>method - 2</u>

$$lx + my + nz = f(x^2+y^2+z^2) \quad \text{— (1)}$$

eq (1) P D w.r to 'x'

$$l + np = f'(x^2+y^2+z^2)(2x + 2zp)$$

$$\frac{l+np}{2(x+zp)} = f'(x^2+y^2+z^2) \quad \text{— (2)}$$

eqn (1) P D w.r to 'y'

$$m + nq = f'(x^2+y^2+z^2)(2y + 2zq)$$

$$\frac{m+nq}{2(y+zq)} = f'(x^2+y^2+z^2) \quad \text{— (3)}$$

from (2) & (3)

$$\frac{l+np}{\cancel{2}(x+zp)} = \frac{m+nq}{\cancel{2}(y+zq)}$$

$$(l+np)(y+zq) = (m+nq)(x+zp)$$

$$ly + lzq + nyp + \cancel{nzpq} = mx + mzp + nxq + \cancel{nzpq}$$

$$\boxed{(ny - mz)p + (lz - nx)q = mx - ly} \quad // \text{ Ans.}$$

[Even Some one take

$$x^2+y^2+z^2 = f(lx+my+nz) \text{ and try,}$$

surely we get same answer]

2) Form the PDE by eliminating the arbitrary function ϕ from $\phi(x^2+y^2+z^2,\ z^2-2xy)=0$

soln

$$u = x^2+y^2+z^2 \quad \text{and} \quad v = z^2 - 2xy$$

$$P = \frac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} 2y & 2z \\ -2x & 2z \end{vmatrix} = 4(yz+xz)$$

$$= 4z(x+y) \quad \text{--- (1)}$$

$$Q = \frac{\partial(u,v)}{\partial(z,x)} = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} 2z & 2x \\ 2z & -2y \end{vmatrix} = -4yz - 4xz$$

$$= -4z[x+y] \quad \text{--- (2)}$$

$$R = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ -2y & -2x \end{vmatrix} = -4x^2+4y^2$$

$$= 4[y^2-x^2] \quad \text{--- (3)}$$

$\therefore$ The Required solution. $Pp + Qq = R.$

$$\boxed{z(x+y)p - z(x+y)q = y^2-x^2} \text{ // Ans.}$$

Simplification

$$\cancel{(x+y)}[zp - zq] = \cancel{(y+x)}(y-x)$$

$$\boxed{p - q = \frac{y-x}{z}} \text{ not required.}$$

Problems for practice:

① Form the PDE by eliminating the arbitrary function ϕ from $\phi(x+y+z,\ x^2+y^2-z^2)=0$

Ans: $\boxed{(y+z)p-(x+z)q=x-y}$

② Form the PDE by eliminating the arbitrary function f from

$$f\left(z^2-xy,\ \frac{x}{z}\right)=0$$

Ans: $\boxed{x^2p+(2z^2-xy)q=xz}$

③ Form the PDE by eliminating the arbitrary function f from $f\left(\frac{x-a}{z-c},\ \frac{y-b}{z-c}\right)=0$

$\boxed{\text{Ans: } (x-a)p+(y-b)q=(z-c)}$