Lagrange's linear equation

A linear partial differential equation of the first order known as Lagrange's linear eqn. is of the form $Pp + Qq = R$.

where P, Q, R are functions of x, y, z.

$\therefore$ (i) Auxiliary eqn: $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$.

(ii) Solve these auxiliary simultaneous equations giving two independent solutions. $u = a$ & $v = b$.

(iii) write $\phi(u,v) = 0$, (or) $u = f(v)$ or $v = g(u)$.

Solution: (i) method of multipliers

(ii) method of grouping.

① Find the general solution of $px + qy = z$.

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}.$$

from first two ratio

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\log x = \log y + \log c$$

$$\boxed{\frac{x}{y} = c_1}$$

from ② & ③ ratios

$$\frac{dy}{y} = \frac{dz}{z}.$$

$$\boxed{\frac{y}{z} = c_2}$$

$$\boxed{\phi\left(x/y,\ y/z\right) = 0}$$

2 solve $(mz - ny)p + (nx - lz)q = ly - mx$.

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}.$$

Choose the multipliers l, m, n.

$$\text{each ration} = \frac{l\,dx + m\,dy + n\,dz}{\cancel{lmz} - \cancel{lny} + \cancel{mnx} - \cancel{lmz} + \cancel{lny} - \cancel{mnx}}$$

Integration $l\,dx + m\,dy + n\,dz = 0$

$$\boxed{lx + my + nz = a}$$

Choose the multipliers x, y, z.

$$\text{each ration} = \frac{x\,dx + m\,dy + n\,dz}{\cancel{xmz} - \cancel{xny} + \cancel{nyx} - \cancel{lyz} + \cancel{lyz} - \cancel{mxz}}$$

$0 = x\,dx + m\,dy + n\,dz$

Integrali

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c$$

$$\boxed{x^2 + y^2 + z^2 = b}$$

$$\phi\,(x^2 + y^2 + z^2,\ lx + my + nz) = 0.$$

(3) Find the general solution of.

$$x(z^2-y^2)p + y(x^2-z^2)q = z(y^2-x^2)$$

$$\frac{dx}{x(z^2-y^2)} = \frac{dy}{y(x^2-z^2)} = \frac{dz}{z(y^2-x^2)}$$

Choose the multipliers x, y, z

each ratio equal to $\dfrac{x\,dx + y\,dy + z\,dz}{x^2z^2 - x^2y^2 + y^2x^2 - y^2z^2 + z^2y^2 - z^2x^2}$

$$x\,dx + y\,dy + z\,dz = 0.$$

Integrating $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = a$

$$\boxed{x^2 + y^2 + z^2 = c}$$

Chose the multiplicrs $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$.

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{z^2 - y^2 + x^2 - z^2 + y^2 - x^2}$$

$$\log x + \log y + \log z = \log b$$

$$\boxed{xyz = c_2}$$

$$\phi(xyz,\ x^2+y^2+z^2) = 0.$$

<u>Practice Problems</u>: Find the general solution of.

① $p\tan x + q\tan y = \tan z$.		Ans. $\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$
② $(y-z)p + (z-x)q = x-y$	$(1,1,1)$ (x,y,z)	Ans: $\phi(x+y+z,\ x^2+y^2+z^2) = 0$
③ $x(y-z)p + y(z-x)q = z(x-y)$	$(1,1,1)$ $(\frac{1}{x}, \frac{1}{y}, \frac{1}{z})$	$\phi(xyz,\ x+y+z) = 0$.
④ $(3z-4y)p + (4x-2z)q = 2y-3x$	$(2,3,4)$ (x,y,z)	$\phi(x^2+y^2+z^2,\ 2x+3y+4z) = 0$
⑤ $(2z-y)p + (x+z)q + 2x+y = 0$	$(-1,2,1)$ (x,y,z)	$\phi(x^2+y^2+z^2,\ x+2y-x) = 0$.

(4) Solve. $\frac{y^2 z}{x} p + xzq = y^2$

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{x\cancel{z}} = \frac{dz}{y^2}$$

Comparing (1) & (3)

$$\frac{dx}{\frac{\cancel{y^2} z}{x}} = \frac{dz}{\cancel{y^2}}$$

$$x\,dx = z\,dz.$$

Integrating $\boxed{\frac{x^2}{2} - \frac{z^2}{2} = a}$

Comparing (1) & (2)

$$\frac{x\,dx}{y^2 \cancel{z}} = \frac{dy}{x\cancel{z}}$$

$$x^2 dx = y^2 dy \qquad \frac{x^3}{3} - \frac{y^3}{3} = b$$

$$\phi\left(\frac{x^2}{2} - \frac{z^2}{2}, \frac{x^3}{3} - \frac{y^3}{3}\right) = 0 \quad (\text{OR}) \quad \phi\left(x^2 - z^2, x^3 - y^3\right) = 0.$$

(5) Find the general solution of

$$(y+z)p + (z+x)q = x+y$$

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$$

each ration equal to $\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{y-x} = \frac{dy-dz}{z-y}$

Taking first two ratios

$$\frac{d(x+y+z)}{2(x+y+z)} = \frac{-d(x-y)}{x-y}$$

Integrating

$$\frac{1}{2}.\log(x+y+z) = -\log(x-y) + \log c$$

$$\log(x+y+z) = \log(x-y)^{-2} + \log k.$$

$$x+y+z = k(x-y)^{-2}$$

$$(x+y+z)(x-y)^2 = k.$$

Taking the last two ratios

$$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}.$$

$$\log(x-y) = \log(y-z) + \log b$$

$$\frac{x-y}{y-z} = b.$$

$$\therefore \quad \phi\left[(x+y+z)(x-y)^2, \frac{x-y}{y-z}\right] = 0.$$

(6) $z(x-y) = px^2 - qy^2$

$$\frac{dx}{x^2} = \frac{dy}{-y^2} = \frac{dz}{z(x-y)}$$

each ratio equal to

$$\frac{dx+dy}{(x+y)\cancel{(x-y)}} = \frac{dz}{z\cancel{(x-y)}}$$

$$\frac{d(x+y)}{x+y} = \frac{dz}{z}$$

$\log(x+y) = \log z + \log b$

$$\boxed{\frac{x+y}{z} = c}$$

Comparing $\frac{dx}{x^2} = \frac{-dy}{y^2}$

Integration

$$-\frac{1}{x} = \frac{1}{y} + k.$$

$$\frac{1}{x} + \frac{1}{y} = a$$

$$\phi\left(\frac{1}{x} + \frac{1}{y},\ \frac{x+y}{z}\right) = 0.$$

Practice problems:

① Solve $pz + qy = x$ Ans: $\phi\left(x^2 - z^2,\ \frac{y}{x+y+z}\right) = 0$

② $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$, $\phi\left(xy + yz + zx,\ \frac{x-y}{y-z}\right) = 0.$

③ $(y^2 + z^2)p - xyq + xz$ $\phi\left(\frac{y}{z},\ x^2 + y^2 + z^2\right) = 0$