18MAB203T-Probability and Stochastic Processes

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Definition

Random Variables

A Random Variable is a rule that assigns a real number to every outcome of the random experiment.

Random Process

A random process is a collection of random variables $\{X(s,t)\}$ that are functions of a real variable, namely time 't' where $s \in S$(Sample space) and $t \in T$(Parameter set or index set).

Classification of Random Process

- **Discrete Random Sequence**
  If both S and T are discrete then the random process is called Discrete Random sequence.
  Eg: No of books in Library at opening time.
- **Continuous Random Sequence**
  If S is continuous and T is discrete, then the random process is called Continuous Random sequence.
  Eg: Quantity of petrol in the petrol bank at opening time.
- **Discrete Random Process**
  If S is discrete and T is continuous then the random process is called Discrete Random Process.
  Eg: No of phone calls recieving in (0,t).
- **Continuous Random Process**
  If both S and T are continuous then the random process is called Continuous Random Process.
  Eg: Stirring sugar in coffee.

Definition

Strict Sense Stationary Process

A random process is called a Stationary Process or Strictly Stationary Process or Strict Sense Stationary Process if all its finite dimensional distributions are invariant under transition of time parameter.

Example

Example1

If the random process X(t) takes the value -1 with probability $\frac{1}{3}$ and takes the value 1 with probability $\frac{2}{3}$, Find whether X(t) is a stationaery process or not.

Solution.

Given

X(t)=n	-1	1
p_n	$\frac{1}{3}$	$\frac{2}{3}$

To prove X(t) is a SSS process

(i,e.)

1. E[X(t)] = constant
2. Var[X(t)]=constant

Example

1.

$$E[X(t)] = \sum_{n=-1}^{1} np_n$$

$$= (-1)(\frac{1}{3}) + (1)(\frac{2}{3}) = \frac{-1}{3}) + \frac{2}{3} = \frac{1}{3} = constant$$

2.

$$E[X^2(t)] = \sum_{n=-1}^{1} n^2 p_n$$

$$= (-1)^2(\frac{1}{3}) + (1)^2\frac{2}{3}) = \frac{1}{3}) + \frac{2}{3} = 1$$

$$Var[X(t)] = E[X^2(t)] - [E[X(t)]]^2$$

$$= 1 - \left(\frac{1}{3}\right)^2 = 1 - \frac{1}{9} = \frac{8}{9} = constant$$

Hence, X(t) is a SSS process.

Example

Example2

Show that, If the process $X(t) = acos\omega t_{+} bsin\omega t$ is SSS, where a and b are independent random variables, then they are normal.

Solution.

Given $X(t) = acos\omega t_{+} bsin\omega t$

$$E[a] = E[b] = 0 \quad and \tag{1}$$

$$E[ab] = E[a]E[b] \quad and \tag{2}$$

$$E[a^2] = E[b^2] = \sigma^2 \tag{3}$$

1.

$$\begin{aligned} E[X(t)] &= E[acos\omega t + bsin\omega t] \\ &= E[a]cos\omega t + E[b]sin\omega t \\ &= (0)cos\omega t + (0)sin\omega t = 0 \\ &= constant \end{aligned}$$

2.

$$\begin{aligned} E[X^2(t)] &= E[(acos\omega t + bsin\omega t)^2] \\ &= E[a^2cos^2\omega t + b^2sin^2\omega t + 2abcos\omega tsin\omega t] \\ &= E[a^2]cos^2\omega t + E[b^2]sin^2\omega t + 2E[ab]cos\omega tsin\omega t \\ &= \sigma^2cos^2\omega t + \sigma^2sin^2\omega t + 0 \quad by\ (2)\ and\ (3) \\ &= \sigma^2[cos^2\omega t + sin^2\omega t = \sigma^2(1) = \sigma^2 \\ Var[X(t)] &= E[X^2(t)] - [E[X(t)]]^2 \\ &= \sigma^2 - 0 = \sigma^2 = constant \end{aligned}$$

Hence, X(t) is a SSS process.

Definition

Wide Sense Stationary Process

A random process is called a wide sense stationary process or weekly stationary process or covariance stationary process if it satisfies the conditions

1. E[X(t)] $=$ constant.
2. $R(t_1, t_2) = E[X(t_1)X(t_2)] = R(t_1 - t_2)$

Example

Example1

The process {X(t)}, whose probability distribution under certain conditions is given by

$$P\{X(t) = n\} = \frac{(at)^{n-1}}{(1+at)^{n+1}}, n = 1, 2, 3, \ldots$$
$$= \frac{at}{1+at}, n = 0$$

Show that it is not stationary.

The probability distribution of {X(t)} is

X(t)=n:	0	1	2	3	...
P_n :	$\frac{at}{1+at}$	$\frac{1}{(1+at)^2}$	$\frac{at}{(1+at)^3}$	$\frac{(at)^2}{(1+at)^4}$	...

Example

$$E\{X(t)\} = \sum_{n=0}^{\infty} np_n$$

$$= \frac{1}{(1+at)^2} + \frac{2at}{(1+at)^3} + \frac{3(at)^2}{(1+at)^4} + \dots$$

$$= \frac{1}{(1+at)^2}\{1 + 2\alpha + 3\alpha^2 + \dots\}, where \alpha = \frac{at}{1+at}$$

$$= \frac{1}{(1+at)^2}(1-\alpha^{-2}) = \frac{1}{(1+at)^2}(1+at)^2 = 1$$

$$E\{X^2(t)\} = \sum_{n=0}^{\infty} n^2 p_n = \sum_{n=1}^{\infty} n^2 \frac{(at)^{n-1}}{(1+at)^{n+1}}$$

$$= \frac{1}{(1+at)^2}\left[\sum_{n=1}^{\infty} n(n+1)\left(\frac{at}{1+at}\right)^{n-1} - \sum_{n=1}^{\infty} n\left(\frac{at}{1+at}\right)^{n-1}\right]$$

Example

$$= \frac{1}{(1+at)^2}\left[\frac{2}{\left(1-\frac{at}{1+at}\right)^3}-\frac{1}{\left(1-\frac{at}{1+at}\right)^2}\right]$$

$$= 1 + 2at$$

$$Var\{X(t)\} = 2at$$

If {X(t)} is a stationary process, E{X(t)} and Var{X(t)} are constants. Since Var{X(t)} is a function of t, the given process is not stationary.

Example

Example2

Show that the random process $X(t) = Acos(\omega_0 t + \theta)$ is wide sense stationary, if a and ω_0 are constants and θ is a uniformly distributed RV in $(0, 2\pi)$

Since θ is uniformly distributed RV in $(0, 2\pi)$

$$
\begin{aligned}
f_0(\theta) &= \frac{1}{2\pi}, 0 \leq \theta \leq 2\pi \\
E\{X(t)\} &= E\{Acos(\omega_0 t + \theta)\} \\
&= A\int_0^{2\pi} \frac{1}{2\pi} cos(\omega_0 t + \theta) d\theta \\
&= \frac{A}{2\pi}\{sin(2\pi + \omega_0 t) - sin\omega_0 t\} \\
&= 0 = a \;\; constant
\end{aligned}
$$

Example

$$
\begin{aligned}
E\{X(t_1)\}\{X(t_2)\} &= E\{A^2 cos(\omega_0 t_1 + \theta) \times cos(\omega_0 t_2 + \theta)\} \\
&= \frac{A^2}{2} E\{cos[(t_1 + t_2)\omega_0 + 2\theta] \\
&\quad + cos(t_1 - t_2)\omega_0\} \\
&= \frac{A^2}{2} \int_0^{2\pi} \frac{1}{2\pi} \{cos[(t_1 + t_2)\omega_0 + 2\theta] \\
&\quad + cos(t_1 - t_2)\omega_0\} d\theta \\
&= \frac{A^2}{2} cos(t_1 - t_2)\omega_0 \\
R(t_1, t_2) &= afunctionof(t_1 - t_2)
\end{aligned}
$$

Therefore, {X(t)} is a WSS process.

Example

Example3

Given a RV Y with characteristic function

$$\begin{aligned} \phi(\omega) &= E\{e^{i\omega Y}\} \\ &= E\{cos\omega Y + isin\omega Y\} \end{aligned}$$

and a random process defined by $X(t) = cos(\lambda t + Y)$, show that {X(t)} is stationary in the wide sense

$$\begin{aligned} If \ \ \phi(1) &= \phi(2) = 0 \\ E\{X(t)\} &= E\{cos(\lambda t + Y)\} \\ &= cos\lambda t \times E(cosY) - sin\lambda t \times cosE(sinY) \end{aligned} \tag{1}$$

Example

Given $\phi(1) = 0$

$$
\begin{aligned}
(i,e.) \quad E\{cosY + isinY\} &= 0 && (2)\\
\therefore \quad E(cosY) = 0 &= E(sinY) \\
Using\ (2)\ in\ (1), we\ get\ \ E\{X(t)\} &= 0 && (3)
\end{aligned}
$$

$$
\begin{aligned}
E\{X(t_1) \times X(t_2)\} &= E\{cos(\lambda t_1 + Y) \times cos(\lambda t_2 + Y)\} \\
&= cos\lambda t_1 cos\lambda t_2 E(cos^2 Y) + sin\lambda t_1 sin\lambda t_2 E(sin^2 Y) \\
&\quad - sin\lambda(t_1 + t_2)E(sinYcosY) \\
&= cos\lambda t_1 cos\lambda t_2 E\left(\frac{1}{2} + \frac{1}{2}cos2Y\right) && (4)\\
&\quad + sin\lambda t_1 sin\lambda t_2 E\left(\frac{1}{2} + \frac{1}{2}cos2Y\right) \\
&\quad - \frac{1}{2}sin\lambda(t_1 + t_2)E(sin2Y)
\end{aligned}
$$

Example

Given $\phi(2) = 0$

$$\begin{aligned}
(i,e.) \quad E\{cos2Y + isin2Y\} &= 0 && (5)\\
\therefore \quad E(cos2Y) &= 0 = E(sin2Y)\\
Using\ (5)\ in\ (4),\ \ we\ get\\
R(t_1, t_2) &= E\{X(t_1) \times X(t_2)\}\\
&= \frac{1}{2}\{cos\lambda t_1 cos\lambda t_2 + sin\lambda t_1 sin\lambda t_2\}\\
&= \frac{1}{2}cos\lambda(t_1 - t_2) && (6)
\end{aligned}$$

From (3) and (6) it follows that {X(t)} is a WSS process.

Example

Example4

Show that the process $X(t) = Acos\lambda t + Bsin\lambda t$ (where A and B are RV) is wide sense stationary, if

1. E(A)=E(B)=0
2. $E(A^2) = E(B^2)$
3. E(AB)=0

$$E\{X(t)\} = cos\lambda t \times E(A) + sin\lambda t \times E(B) \tag{1}$$

If {X(t)} is to be a WSS process. E{X(t)} must be a constant (i.e, independent of t).
In 1 if E(A) and E(B) are any constants other than zero, E{X(t)} will be a function of t.

$$\therefore \quad E(A) = E(B) = 0$$

Example

$$
\begin{aligned}
R(t_1, t_2) &= E\{X(t_1) \times X(t_2)\} \\
&= E\{(Acos\lambda t_1 + Bsin\lambda t_1)(Acos\lambda t_2 + Bsin\lambda t_2)\} \\
&= E(A_2)cos\lambda t_1 cos\lambda t_2 + E(B_2)sin\lambda t_1 sin\lambda t_2 \\
&\quad + E(AB)sin\lambda(t_1 + t_2)
\end{aligned} \tag{2}
$$

If {X(t)} is to be a WS proceess, $R(t_1, t_2)$ must be a function of $(t_1 - t_2)$.
$\therefore$ In 2, E(AB)=0 and $E(A^2) = E(B^2) = k$
Then $R(t_1, t_2) = kcos\lambda(t_1 - t_2)$

Example

Example5

If $X(t) = Ycos\omega t + Zsin\omega t$, where Y and Z are two independent normal RV with $E(Y) = E(Z) = 0, E(Y^2) = E(Z^2) = \sigma^2$ and ω is a constant, prove that {X(t)} is a SSS process of order 2.

Since {X(t)} is a linear combination of Y and Z, that are independent, {X(t)} follows a normal distribution with

$$
\begin{aligned}
E\{X(t)\} &= cos\omega t E(Y) + sin\omega t E(Z) = 0 \\
\text{and} \quad Var\{X(t)\} &= cos^2\omega t E(Y^2) + sin^2\omega t E(Z^2) \\
&= \sigma^2
\end{aligned}
$$

Since $\{X(t_1)\}$ and $\{X(t_2)\}$ are each $N(0, \sigma), X(t_1)$ and $X(t_2)$ are jointly normal with the joint pdf given by

Example

$$f(x_1,x_2,t_1,t_2)=\frac{1}{2\pi\sigma^2\sqrt{1-r^2}}\exp\left\{\frac{-(x_1^2-2rx_1x_2+x_2^2)}{2(1-r^2)\sigma^2}\right\}; -\infty < x_1, x_2 < \tag{1}$$

In (1)

$$\begin{aligned} r &= correlation\ coefficient\ between; \{X(t_1)\}\ and\ \{X(t_2)\} \\ &= \frac{C(t_1,t_2)}{\sqrt{Var\{X(t_1)\}\times Var\{X(t_2)\}}} \\ &= \frac{1}{\sigma^2}E\{X(t_1)\}\times\{X(t_2)\} \\ &= \frac{1}{\sigma^2}E[(Ycos\omega t_1+Zsin\omega t_1)(Ycos\omega t_2+Zsin\omega t_2)] \\ &= \frac{1}{\sigma^2}[E(Y^2)cos\omega t_1 cos\omega t_2+E(Z^2)sin\omega t_1 sin\omega t_2] \end{aligned}$$

[since E(YZ)=0 as Y and Z are independent]

Example

Now, the joint pdf of $X(t_1 + h)$ and $X(t_2 + h)$ is given by a similar expressions as in 1, where

$$\begin{aligned} r &= cos\omega\{(t_1 + h) - (t_1 + h)\} \\ &= cos\omega(t_1 - t_2) \end{aligned}$$

Thus, the joint pdf of $\{X(t_1), X(t_2)\}$ and $\{X(t_1 + h), X(t_2 + h)\}$ are the same.
Therefore,$\{X(t_1)\}$ is a SSS process of order 2.

Example

Example6

Two random process {X(t)} and {Y(t)} are defined by $Acos(\omega_0 t + Bsin\omega_0 t$ and $Bcos(\omega_0 t - Asin\omega_0 t$. Show that {X(t)} and {Y(t)} are jointly wide-sense stationary, if A and B are uncorrelated RVs with zero means and the same variances and ω_0 is a costant.

$$E(A) = E(B) = 0; Var(A) = Var(B)$$
$$E(A^2) = E(B^2)$$

Since A and B are uncorrelated, E(AB)=0.
Therefore, by Example4, {X(t)} and {Y(t)} are individually WSS process.
Now,

$$\begin{aligned} R(t_1, t_2) &= E\{X(t_1) \times X(t_2)\} \\ &= E\{(Acos\omega_0 t_1 + Bsin\omega_0 t_1)(Bcos\omega_0 t_2 - Asin\omega_0 t_2)\} \end{aligned}$$

Example

$$= E(B^2)sin\omega_0 t_1 cos\omega_0 t_2 - E(A^2)cos\omega_0 t_1 sin\omega_0 t_2$$
$$= \sigma^2 sin\omega_0(t_1, t_2) \quad [assuming E(A^2) = E(B^2) = \sigma^2]$$
$$= a\ function\ of\ (t_1 - t_2)$$

Definition

Autocorrelation Function

If the process {X(t)} is stationary either in the strict sense or in the wide sense, E{X(t)} X(t-τ) is a function of τ denoted by $R_{xx}(\tau)$ or R(τ) or $R_x(\tau)$. This function$R(\tau)$ is called Autocorrelation function of the process {X(t)}.

Properties

1. **R(t) is an even function of τ.**
   Proof.

$$\begin{aligned} R(\tau) &= EX(t) \times X(t-\tau) \\ R(-\tau) &= EX(t) \times X(t-\tau) \\ &= EX(t+\tau) \times X(t) \\ &= R(\tau) \end{aligned}$$

2. **R(τ) is maximum at τ=0 (i,e) $|R(\tau)| \leq R(0)$.**
   Proof.
   The Cauchy-Schwarz inequlity is

$$E(XY)^2 \leq E(X)^2 \times E(Y)^2$$

Put X=X(t) and Y=X(t-τ)
Then

Properties

$$[E\{X(t) \times X(t-\tau)\}]^2 \leq E\{X^2(t)X^2(t-\tau)\}$$
$$\{R(\tau)\}^2 \leq [E\{X^2(t)\}]^2$$

[Since E{X(t)} and {X(t)} are constant for a stationary process]

$$(i, e.) \quad R(\tau)^2 \leq R(0)^2$$

Taking squaree root on both sides

$$|R(\tau)| \leq R(0)$$

[Since R(0)=E$X^2(t)$ is positive]

3. **If the autocorrelation function $R(t)$ of a real stationary process $\{X(t)\}$ is continuous at $\tau = 0$, it is continuous at every other point.**

Proof.

Consider

$$[E\{X(t) - X(t-\tau)\}]^2 = E\{X^2(t)\} + E\{X^2(t-\tau)\} - 2E\{X(t) \times X(t-\tau)$$
$$= R(0) + R(0) - 2R(\tau)$$
$$= 2[R(0) - R(\tau)] \tag{7}$$

Since $R(\tau)$ is continuous at $\tau = 0$, $\lim_{\tau \to 0} R(\tau) = R(0)$

(i,e.) $$\lim_{\tau \to 0} \{R.S.of 7\} = 0$$

$\therefore$ $$\lim_{\tau \to 0} \{L.S.of 7\} = 0$$

$$\lim_{\tau \to 0} \{X(t-\tau)\} = X(t) \tag{8}$$

(i,e.) $X(t)$ is continuous for all t

Consider $R(\tau + h) - R(\tau)$

$$= E[\{X(t) \times X\{t - (\tau + h)]\} - E\{X(t) \times X(t-\tau)]$$

$$= E[X(t)\{X(t-\tau-h)-X(t-\tau)] \tag{9}$$

Now, $\lim_{h\to 0}[X\{(t-\tau)-h\}-X(t-\tau)] = 0$, by 8

$\therefore \qquad \lim_{h\to 0}\{R.S.of 9\} = 0$

$\therefore \qquad \lim_{h\to 0}\{L.S.of 9\} = 0$

$(i,e.) \qquad \lim_{h\to 0}\{R(\tau+h)\} = R(\tau)$

(i,e.) $R(\tau)$ is continuous for all τ

4. **If R(τ) is the autocorrelation function of a stationary process X(t) with no periodic component, then $\lim_{\tau\to\infty} \mathbf{R}(\tau) = \mu_{\mathbf{x}}^{\mathbf{2}}$,provided the limit exists.**

   Proof.

$$R(\tau) = E\{X(t)\times X(t-\tau)\}$$

Whenτ is large, $X(t)$ and $X(t-\tau)$ are two sample functions of the process $\{X(t)\}$ observed at a very long interval of time.

Properties

Therefore, $X(t)$ and $X(t-\tau)$ tend to become independent [$X(t)$ and $X(t-\tau)$ may be dependent, when $X(t)$ contains a periodic component, which is not true].

$$
\begin{aligned}
\therefore \quad \lim_{\tau \to \infty} R(\tau) &= E\{X(t) \times X(t-\tau)\} \\
&= \mu_x^2 \quad [Since E\{X(t)\} \text{ is a constant}] \\
i,e. \qquad \mu_x &= \sqrt{\lim_{\tau \to \infty} R(\tau)}
\end{aligned}
$$

Example1

Check whether the following functions are valid autocorrelation functions

1. $R_{xx}(\tau) = \dfrac{25\tau^2}{4 + 5\tau^2}$
2. $R_{xx}(\tau) = \tau^3 + \tau^2$
3. $R_{xx}(\tau) = cos(\tau) + \dfrac{|\tau|}{T}$

1.

$$R_{xx}(\tau) = \frac{25\tau^2}{4 + 5\tau^2}$$

$$R_{xx}(-\tau) = \frac{25 - \tau^2}{4 + 5 - \tau^2} = \frac{25\tau^2}{4 + 5\tau^2}$$

$$R_{xx}(\tau) = R_{xx}(-\tau)$$

$\therefore$ $R_{xx}(\tau)$ is a autocorrelation function.

Example

2.

$$R_{xx}(\tau) = \tau^3 + \tau^2$$
$$R_{xx}(-\tau) = -\tau^3 + -\tau^2 = -\tau^3 + \tau^2$$
$$R_{xx}(\tau) \neq R_{xx}(-\tau)$$

$\therefore$ $R_{xx}(\tau)$ is not a autocorrelation function.

3.

$$R_{xx}(\tau) = cos(\tau) + \frac{|\tau|}{T}$$
$$R_{xx}(-\tau) = cos(-\tau) + \frac{|-\tau|}{T}$$
$$= cos(\tau) + \frac{|\tau|}{T}$$
$$R_{xx}(\tau) = R_{xx}(-\tau)$$

$\therefore$ $R_{xx}(\tau)$ is a autocorrelation function.

Definition

Cross-Correlation Function

If the processes $\{X(t)\}$ and $\{Y(t)\}$ are jointly wide-sense stationary, then $E\{X(t) \times X(t-\tau)\}$ is a function of τ, denoted by $R_{xy}(\tau)$. This function is $R_{xy}(\tau)$ is called the cross-correlation function of the processes $\{X(t)\}$ and $\{Y(t)\}$.

Properties

1. $R_{yx}(\tau) = R_{xy}(-\tau)$

   Proof.

$$
\begin{aligned}
R_{xy}(\tau) &= E[X(t)Y(t+\tau)] \\
R_{xy}(-\tau) &= E[X(t)Y(t-\tau)] \\
substitute t_1 &= t-\tau \\
&= E[Y(t_1)X(t_1+\tau)] \\
&= R_{yx}(\tau)
\end{aligned}
$$

2. $|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0) \times R_{yy}(0)}$

   Proof. For any real number α, we know that

   $E[\alpha X(t) + Y(t+\tau)]^2 \geq 0$

   $E[\alpha^2 X^2(t) + Y^2(t+\tau) + 2\alpha X(t)Y(t+\tau)] \geq 0$

   $E[\alpha^2 X^2(t)] + E[Y^2(t+\tau)] + E[2\alpha X(t)Y(t+\tau)] \geq 0$

   $\alpha^2 E[X^2(t)] + E[Y^2(t+\tau)] + 2\alpha E[X(t)Y(t+\tau)] \geq 0$

Properties

Since $\{X(t)\}$ and $\{Y(t)\}$ are jointly WSS, each is a WSS process Hence the second order moments are constants. But $E(X^2(t)) = R_{xx}(0)$ by the proprty of autocorrelation function and $E(Y^2(t+\tau)) = R_{yy}(0)$.
$\alpha^2 R_{xx}(0) + R_{yy}(0) + 2\alpha R_{xy}(\tau) \geq 0 \quad \forall \alpha$
Since $R_{xx}(0) > 0$ and α is any real number, the discriminant is ≤ 0.

$$4(R_{xy}(\tau))^2 - 4R_{xx}(0)R_{yy}(0) \leq 0$$
$$(R_{xy}(\tau))^2 - R_{xx}(0)R_{yy}(0) \leq 0$$
$$|R_{xy}(\tau) \leq \sqrt{R_{xx}(0)R_{yy}(0)}$$

3. $|R_{xy}(\tau)| \leq \frac{1}{2}\{R_{xx}(0) \times R_{yy}(0)\}$

   Proof.

   We know that $R_{xx}(0)$ and $R_{yy}(0)$ are positive numbers so their $A.M \geq G.M$
   $$\frac{R_{xx}(0) + R_{yy}(0)}{2} \geq \sqrt{R_{xx}(0) + R_{yy}(0)}$$

By Property 2, $|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0) + R_{yy}(0)}$

$$|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0) + R_{yy}(0)} \leq \frac{R_{xx}(0) + R_{yy}(0)}{2}$$

$$|R_{xy}(\tau)| \leq \frac{R_{xx}(0) + R_{yy}(0)}{2}$$

4. If the process {X(t)} and {X(t)} are orthogonal, then $R_{xy}(\tau) = 0$
5. If the process {X(t)} and {X(t)} are independent, then $R_{xy}(\tau) = \mu_x \times \mu_y$

Example

Example1

Consider 2 random processes $X(t) = 3cos(\omega t + \theta)$ and $Y(t) = 2cos(\omega t + \theta - \frac{\pi}{2})$ where θ is a random variable uniformly distributed in $(0, 2\pi)$ Prove that $|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0) + R_{yy}(0)}$

Solution.

$$R_{xx}(t, t+\tau) = E[X(t)X(t+\tau)]$$

$$= E[3cos(\omega t + \theta).3cos(\omega t + \theta)]$$

$$= \frac{9}{2}E\left[cos(2\omega t + 2\theta + \omega\tau) + cos(-\omega\tau)\right]$$

$$= \frac{9}{2}\int_0^{2\pi} E\left[cos(2\omega t + 2\theta + \omega\tau)\frac{1}{2\pi}d\theta + \frac{9}{2}E(cos\omega\tau)\right]$$

Example

$$
\begin{aligned}
&= \frac{9}{4\pi}\left[\frac{sin(2\omega t + 2\theta + \omega\tau)}{2}\right]_0^{2\pi} + \frac{9}{2}cos\omega\tau \\
&= \frac{9}{4\pi}\left[\frac{sin(2\omega t + \omega\tau) - sin(2\omega t + \omega\tau)}{2}\right] + \frac{9}{2}cos(\omega\tau) \\
&= \frac{9}{2}cos(\omega\tau) \\
R_{xx}(\tau) &= \frac{9}{2}cos\omega\tau \\
R_{xx}(0) &= \frac{9}{2}
\end{aligned}
$$

In a similar manner prove $R_{yy}(\tau) = 2cos\omega\tau \implies R_{yy}(0) = 2$

$$R_{xy}(t, t+\tau) = E[X(t)Y(t+\tau)]$$

$$= E[3cos(\omega t + \theta).2cos(\omega t + \omega\tau + \theta - \frac{\pi}{2})]$$

Example

$$= 3E[sin(2\omega t + 2\theta + \omega\tau) + sin(\omega\tau)]$$

$$= 3\int_0^{2\pi} sin(2\omega t + 2\theta + \omega\tau)\frac{1}{2\pi}d\theta + 3E(sin\omega\tau)$$

$$= \frac{3}{2\pi}\left[\frac{cos(2\omega t + 2\theta + \omega\tau)}{2}\right]_0^{2\pi} + 3sin\omega\tau$$

$$= \frac{-3}{2\pi}\left[\frac{sin(2\omega t + \omega\tau + 4\pi) - sin(2\omega t + \omega\tau)}{2}\right] + 3sin(\omega\tau)$$

$$R_{xy}(\tau) = 3sin\omega\tau$$

Hence $\{X(t)\}$ and $\{X(t)\}$ are jointly WSS.

Now $R_{xx}(0)R_{yy}(0) = 9 \implies \sqrt{R_{xx}(0)R_{yy}(0)} = 3$

$R_{xy}(\tau) = 3sin\omega\tau \implies |R_{xy}(\tau)| = |3sin\omega\tau| \leq 3$

$|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0)R_{yy}(0)}$

Example

Example2

Two random processes {X(t)} and {X(t)} are defined by $X(t) = Acos\omega t + Bsin\omega t$ and $Y(t) = Acos\omega t - Bsin\omega t$ Show that {X(t)} and {X(t)} are jointly WSS if $A\&B$ are uncorrelated random variables with zero means and the same variances and ω is a constant.

Proof.

Given $X(t) = Acos\omega t + Bsin\omega t$ and $Y(t) = Acos\omega t - Bsin\omega t$ where $A\&B$ are uncorrelated random variables with zero means.

So $E(A) = 0, E(B) = 0$ and $E(AB) = E(A)E(B) = 0$

Given $Var(A) = Var(B) = \sigma^2$

Then $E(A^2) = E(B^2) = \sigma^2$

Given $X(t) = Acos\omega t + Bsin\omega t$

$$E(X(t)) = E(Acos\omega t + Bsin\omega t)$$

Then $E(X(t)) = E(A)cos\omega t + E(B)sin\omega t$

$$= 0 \text{ as } E(A) = E(B) = 0$$

Example

$$R_{xx}(t, t+\tau) = E[X(t)X(t+\tau)]$$

$$\begin{aligned}
&= E\left[(Acos\omega t + Bsin\omega t) + (Acos\omega(t+\tau) + Bsin\omega(t+\tau)\right] \\
&= E\left[A^2 cos\omega t cos\omega(t+\tau)\right] + E\left[ABcos\omega t sin\omega(t+\tau)\right] \\
&\quad + E\left[B^2 sin\omega t sin\omega(t+\tau)\right] + E\left[ABsin\omega t cos\omega(t+\tau)\right] \\
&= E(A^2) cos\omega t cos\omega(t+\tau) + E(AB) cos\omega t sin\omega(t+\tau) \\
&\quad + E(B^2) sin\omega t sin\omega(t+\tau) + E(AB) sin\omega t cos\omega(t+\tau)
\end{aligned}$$

But $E(AB) = 0$ and $E(A^2) = E(B^2) = \sigma^2$

$$\begin{aligned}
R_{xx}(t, t+\tau) &= E(A^2) cos\omega t cos\omega(t+\tau) + E(B^2) sin\omega t sin\omega(t+\tau) \\
&= \sigma^2 cos(\omega t + \omega\tau - \omega t) \\
&= \sigma^2 cos(\omega\tau)
\end{aligned}$$

Example

Hence {X(t)} is a WSS.
In a similar manner prove {X(t)}is also a WSS. Now to show their cross correlation is a function of τ.

$$R_{xy}(t, t+\tau) = E[X(t)Y(t+\tau)]$$

$$\begin{aligned}
&= E\left[(Acos\omega t + Bsin\omega t) + (Bcos\omega(t+\tau) - Asin\omega(t+\tau)\right] \\
&= E\left[ABcos\omega tcos\omega(t+\tau)\right] - E\left[A^2cos\omega tsin\omega(t+\tau)\right] \\
&\quad + E\left[B^2sin\omega tcos\omega(t+\tau)\right] - E\left[ABsin\omega tsin\omega(t+\tau)\right] \\
&= E(AB)cos\omega tcos\omega(t+\tau) - E(A^2)cos\omega tsin\omega(t+\tau) \\
&\quad + E(B^2)sin\omega tcos\omega(t+\tau) - E(AB)sin\omega tsin\omega(t+\tau)
\end{aligned}$$

But $E(AB) = 0$ and $E(A^2) = E(B^2) = \sigma^2$

Example

$$
\begin{aligned}
R_{xy}(t, t+\tau) &= \sigma^2 \big(sin\omega t cos\omega(t+\tau) - cos\omega t sin\omega(t+\tau) \big) \\
&= \sigma^2 sin(\omega t + \omega\tau - \omega t) \\
&= \sigma^2 sin(-\omega\tau) \\
&= -\sigma^2 sin(\omega\tau)
\end{aligned}
$$

This is a function τ of only.
Therefore {X(t)} and {X(t)} are jointly WSS.

Definition

Time average

If $\{X(t)\}$ is a random process, then $\dfrac{1}{2T}\displaystyle\int_{-T}^{T} X(t)dt$ is called the time average of $\{X(t)\}$ over (-T,T) and denoted by $\overline{X}_T$.

Ergodic Process

A random process $\{X(t)\}$ is said to be ergodic, if its ensembles qverages are equal to appropriate time averages.

Mean-Ergodic Process

If the random process $\{X(t)\}$ has a constant mean $\{X(t)\} = \mu$ and if $\overline{X}_T = \dfrac{1}{2T}\displaystyle\int_{-T}^{T} X(t)dt \to \mu,\ asT \to \infty$, then $\{X(t)\}$ is said to be mean-ergodic.

Theorem

Mean-Ergodic Theorem

If $\{X(t)\}$ is a random process with constant mean μ and if $\overline{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t)dt$, then $\{X(t)\}$ is mean-ergodic, provided $\lim_{T\to\infty}\{Var\overline{X}_T\} = 0$

Proof:

$$\begin{aligned} \overline{X}_T &= \frac{1}{2T}\int_{-T}^{T} X(t)dt \\ E\overline{X}_T &= \frac{1}{2T}\int_{-T}^{T} E\{X(t)\}dt \\ &= \mu \end{aligned} \tag{10}$$

By Tchebycheff's inequality,

$$P\{|\overline{X}_T - E\overline{X}_T| \le \epsilon\} \ge 1 - \frac{Var(\overline{X}_T)}{\epsilon^2} \tag{11}$$

Theorem

Taking Limits as $T \to \infty$ and using 10 we get

$$p\{|\lim_{T\to\infty}(\overline{X}_T) - \mu| \leq \epsilon\} \geq 1 - \frac{\lim_{T\to\infty} Var(\overline{X}_T)}{\epsilon^2}$$

$\therefore$ when $\lim\limits_{T\to\infty} Var(\overline{X}_T) = 0$, 11 becomes

$$p\{|\lim_{T\to\infty}(\overline{X}_T) - \mu| \leq \epsilon\} \geq 1$$

(i,e.) $\lim\limits_{T\to\infty}(\overline{X}_T) = E\{X(t)\}$ with probability 1.

Example

Example1

Prove that the random process $\{X(t)\}$ with constant mean is mean ergodic if $\lim_{T\to\infty}[\frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C(t_1,t_2)dt_1dt_2] = 0$

Proof.

By mean-ergodic theorem, the condition for the mean-ergodicity of the process $\{X(t)\}$ is
$\lim_{T\to\infty} Var(\overline{X}_T) = 0$

$$\overline{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t) dt and E\overline{X}_T = E(X_T)$$

$$\overline{X}_T^2 = \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} X(t_1)X(t_2)dt_1dt_2$$

Example

$$E\{\overline{X}_T^2\} = \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} R(t_1, t_2)dt_1 dt_2$$

$$Var(\overline{X}_T) = E\{\overline{X}_T^2\} - E^2(\overline{X}_T^2)$$

$$= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}[R(t_1, t_2) - E\{X(t_1)E\{X(t_1)]dt_1 dt_2$$

$$= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C(t_1, t_2)dt_1 dt_2$$

Therefore, the condition $\lim_{T\to\infty} Var(\overline{X}_T) = 0$ is equivalent to the condition

$$\lim_{T\to\infty}[\frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C(t_1, t_2)dt_1 dt_2] = 0$$

Hence the result.