Transverse Vibrations of a stretched elastic string

Consider small transverse vibrations of an elastic string of length l, which is stretched and then fixed at its two ends.

Now we will study the transverse vibration of the string when no external forces act on it.

Take an end of the string as the origin and the string in the equilibrium position as the x-axis and the line through the origin and perpendicular to the x-axis as the y-axis.

We make the following assumptions:

1) The motion takes place entirely in one plane. This plane is chosen as the xy plane.
2) In this plane, each particle of the string moves in a direction perpendicular to the equilibrium position of the string.
3) The tension T caused by stretching the string before fixing it at the end points is constant at all times at all points of the deflected string.
4) The tension T is very large compared with the weight of the string and hence the gravitational force may be neglected.

5) The effect of friction is negligible.

6) The string is perfectly flexible. It can transmit only tension but not bending or shearing forces.

7) The slope of the deflection curve is small at all points and at all times.

By using second law of Newton, we can derive one dimensional wave equation:

ie, $$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

where $a^2 = \frac{\text{Tension}}{\text{mass}} = \frac{T}{m}$ (positive).

<u>Note</u>:- The displacement $y(x,t)$ is given by the equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}.$$

Solution of the wave equation:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \longrightarrow (1)$$

(By the method of seperation of variables)

Let $y = X(x) \cdot T(t)$ be a solution of equation (1), where $X(x)$ is a function of x only and $T(t)$ is a function of t only.

$$\frac{\partial y}{\partial x} = X'T \qquad \frac{\partial y}{\partial t} = XT'$$

$$\frac{\partial^2 y}{\partial x^2} = X''T \qquad \frac{\partial^2 y}{\partial t^2} = XT'' \quad \text{where}$$

$$X'' = \frac{d^2X}{dx^2} \; \& \; T'' = \frac{d^2T}{dt^2}.$$

$$XT'' = a^2 X''T$$

Hence (1) becomes,

$$\text{ie,} \quad \frac{X''}{X} = \frac{T''}{a^2T} \longrightarrow (2)$$

The L.H.S of (2) is a function of x only, whereas the R.H.S is a function of t only. But x and t are independent variables. Hence (2) is true only if each is equal to a constant.

$$\therefore \frac{X''}{X} = \frac{T''}{a^2T} = k \text{ (say) where } k \text{ is any constant.}$$

Hence $\quad X'' - kX = 0 \;\&\; T'' - a^2kT = 0.$

Solutions of these equations depend upon the nature of the value of k.

Case (i). Let $k = \lambda^2$, a positive value.

Now, the equation ③ are ① $X'' - \lambda^2 X = 0$ & $T'' - a^2\lambda^2 T = 0$.

$$X'' - \lambda^2 X = 0$$

$$\frac{d^2X}{dx^2} - \lambda^2 X = 0,$$

$$\left(\frac{d^2}{dx^2} - \lambda^2\right) X = 0. \qquad D = d/dx$$

$$(D^2 - \lambda^2) X = 0$$

The auxiliary eqn is

$$m^2 - \lambda^2 = 0$$

$$m^2 = \lambda^2$$

$$m = \pm \lambda.$$

$$T'' - a^2\lambda^2 T = 0.$$

$$\frac{d^2T}{dt^2} - a^2\lambda^2 T = 0.$$

$$\left(\frac{d^2}{dt^2} - a^2\lambda^2\right) T = 0 \qquad D = d/dt$$

$$D^2 - a^2\lambda^2 = 0.$$

The auxiliary eqn is

$$m^2 - a^2\lambda^2 = 0$$

$$m^2 = a^2\lambda^2$$

$$m = \pm \lambda a$$

$$\therefore T = C_1 e^{\lambda a t} + D_1 e^{-\lambda a t}$$

$$\therefore X = A_1 e^{\lambda x} + B_1 e^{-\lambda x}.$$

$$\therefore y = (A_1 e^{\lambda x} + B_1 e^{-\lambda x})(C_1 e^{\lambda a t} + D_1 e^{-\lambda a t})$$

Case (ii) Let $k = -\lambda^2$, a negative value.

Then the equation ③ are $X'' + \lambda^2 X = 0$ & $T'' + a^2\lambda^2 T = 0$.

$$X'' + \lambda^2 X = 0$$

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0.$$

$$(D^2 + \lambda^2) X = 0$$

The aux. eqn is

$$m^2 + \lambda^2 = 0$$

$$m^2 = -\lambda^2$$

$$m = \pm i\lambda$$

$$\therefore X = A_2 \cos \lambda x + B_2 \sin \lambda x$$

$$T'' + a^2\lambda^2 T = 0.$$

$$\frac{d^2T}{dt^2} + a^2\lambda^2 T = 0.$$

$$(D^2 + \lambda^2 a^2) T = 0$$

The aux. eqn is

$$m^2 + \lambda^2 a^2 = 0$$

$$m^2 = -\lambda^2 a^2$$

$$m = \pm i\lambda a$$

$$T = C_2 \cos \lambda a t + D_2 \sin \lambda a t$$

$$\therefore y = (A_2 \cos \lambda x + B_2 \sin \lambda x)(C_2 \cos \lambda a t + D_2 \sin \lambda a t)$$

Case (iii) Let $k=0$.

Now the equations (3) are $X''=0$ & $T''=0$.

$X''=0$

(ie) $\frac{d^2X}{dx^2}=0$

integrating twice w r to x,

$\frac{dX}{dx} = A_3$

$X = A_3x + B_3$

$T''=0$

ie, $\frac{d^2T}{dt^2}=0$

int. w.r to t, (twice)

$\frac{dT}{dt} = C_3$

$T = C_3t + D_3$.

$\therefore \ y = (A_3x+B_3)(C_3t+D_3)$

The various possible solutions of the wave equation are

$y(x,t) = (A_1e^{\lambda x} + B_1e^{-\lambda x})(C_1e^{\lambda at} + D_1e^{-\lambda at})$

$y(x,t) = (A_2\cos\lambda x + B_2\sin\lambda x)(C_2\cos\lambda at + D_2\sin\lambda at)$

$y(x,t) = (A_3x + B_3)(C_3t + D_3)$

Note :- We can choose the correct solution as follows:

Out of the above three types of solutions we have to choose the correct one which is consistent with the physical nature of the problem. Since we are dealing with problem on vibrations, y must be a periodic function of x and t.

Therefore, we choose the solution which contains the trigonometric terms since sine and cosine functions are periodic in nature. Hence the correct solution is

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)\ (C\cos\lambda at + D\sin\lambda at).$$