DEPARTMENT OF PHYSICS AND NANOTECHNOLOGY
SRM INSTITUTE OF SCIENCE AND TECHNOLOGY

18PYB101J-Electromagnetic Theory, Quantum Mechanics, Waves and Optics

Module I Lecture-9

Maxwell Equations

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Some Important Vector Results and Theorems

In electromagnetism, we shall use the following vector results:

1. $\nabla \times (\nabla \times E) = \nabla(\nabla \bullet E) - \nabla^2 E$

   curl curl $\boldsymbol{E}$ = grad div $\boldsymbol{E}$ $-\nabla^2 E$

2. div grad $\boldsymbol{S} = \nabla^2 S$

   $\nabla \bullet (\nabla \vec{S}) = \nabla^2 S$

3. div ($\boldsymbol{SV}$) = $\boldsymbol{S}$ div $\boldsymbol{V} + \boldsymbol{V} \bullet$ grad $\boldsymbol{S}$

   $\nabla \bullet (\vec{S}\vec{V}) = \vec{S}(\nabla \bullet \vec{V}) + \vec{V} \bullet (\nabla \vec{S})$

4. curl grad $\phi = 0$

   $\nabla \times (\nabla \phi) = 0$

5. *Gauss Divergence Theorem*

It relates the volume integral of the divergence of a vector $\boldsymbol{V}$ to the surface integral of the vector itself. According to this theorem, if a closed S bounds a volume τ, then

$$\int_\tau (\text{div } \boldsymbol{V})\, d\tau = \oint_s \boldsymbol{V} \bullet \boldsymbol{ds} \quad \text{(or)} \quad \int_\tau (\nabla \bullet \boldsymbol{V}) d\tau = \int_S \boldsymbol{V} \bullet \boldsymbol{ds}$$

6. *Stoke's Theorem*

It relates the surface integral of the curl of a vector to the line integral of the vector itself. According to this theorem, if a closed path C bounds a surface S,

$$\int_s (\text{curl } \boldsymbol{V}) \bullet \boldsymbol{ds} = \oint_C \boldsymbol{V} \bullet \boldsymbol{dl}$$

$$\int_s (\nabla \times \boldsymbol{V}) \bullet \boldsymbol{ds} = \oint_C \boldsymbol{V} \bullet \boldsymbol{dl}$$

Maxwell Equations

Maxwell’s equations combine the fundamental laws of electricity and magnetism and are of profound importance in the analysis of most electromagnetic wave problems. The behaviour of electromagnetic fields is studied with the help of a set of equations given by Maxwell and hence called Maxwell’s equations. These equations are the mathematical abstractions of certain experimentally observed facts and find their application to all sorts of problem in electromagnetism. Maxwell’s equations are derived from Ampere’s law, Faraday’s law and Gauss law. They are listed in the Table

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Table: Maxwell's Equations

Maxwell's Law	Differential form	Integral form
First law: (Based on Gauss law of electrostatics)	$\nabla.\vec{D}=\rho$	$\oint_s \vec{D}.\vec{ds}=\int_v \rho\, dv$
Second Law: (Based on Gauss law of magnetostatics)	$\nabla.\vec{B}=0$	$\oint_s \vec{B}.\vec{ds}=0$
Third law: (Based on the Faradays' law of electromagnetism)	$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$	$\oint \vec{E}.\,\vec{dl}=-\oint_s \frac{\partial\vec{B}}{\partial t}.\vec{ds}$
Fourth Law: (Based on the Amperes circuital law or Biot – Savart law)	$\nabla\times\vec{H}=\sigma\vec{E}+\frac{\partial\vec{D}}{\partial t}$	$\int_l \vec{H}.\vec{dl}=\oint_s(\sigma\vec{E}+\frac{\partial\vec{D}}{\partial t}).\vec{ds}$

where $\vec{D}$ = electric displacement vector (C m^{-2})

ρ = volume charge density (C m^{-3})

$\vec{B}$ = magnetic induction (Wb m^{-2})

$\vec{E}$ = electric field intensity (V m^{-1})

$\vec{H}$ = magnetic field intensity (A m^{-1})

Maxwell's equations: Derivation

Maxwell's First Law

Suppose the charge is distributed over a volume V. Let ρ be the volume density of the charge, then the charge q is given by,

$$q = \int_v \rho dv$$

The integral form of Gauss law is,

$$\varphi = \oint_s \vec{E} \bullet \vec{ds} = \frac{1}{\varepsilon_0} \int_v \rho dv \qquad (1)$$

According to Gauss divergence theorem,

$$\oint_s \vec{E} \bullet \vec{ds} = \int_v (\nabla \bullet \vec{E})\, dv \qquad (2)$$

From equations (1) and (2),

$$\int_v (\nabla \bullet \vec{E})\, dv = \frac{1}{\varepsilon_0} \int_v \rho\, dv \qquad (3)$$

Since, this is true for any volume V, integral must be equal.

$$\therefore \nabla \bullet \vec{E} = \frac{\rho}{\varepsilon_0} \qquad (4)$$

$$\text{div } \vec{E} = \frac{\rho}{\varepsilon_0} \qquad (5)$$

But electric displacement vector, $\vec{D} = \varepsilon_0 \vec{E}$ (6)

$$(5) \times \varepsilon_0 \Rightarrow$$

$$\varepsilon_0 \text{ div } \vec{E} = \frac{\rho}{\varepsilon_0} \times \varepsilon_0$$

$$\text{(or)} \qquad \text{div } (\varepsilon_0 \vec{E}) = \rho$$

$$\text{(or)} \qquad \text{div } (\vec{D}) = \rho$$

$$(\nabla \bullet \vec{D}) = \rho \qquad (7)$$

This is the differential form of Maxwell's I law.

From (1), $\oint_s \varepsilon_0 \vec{E} \bullet \vec{ds} = \int_v \rho dv$

$$\oint_s \vec{D} \bullet \vec{ds} = \int_v \rho \text{dv} \qquad (8)$$

This is the integral form of Maxwell's I law.

Maxwell's Second Law

From Biot - Savart law of electromagnetism, the magnetic induction at any point due to a current element,

$$dB = \frac{\mu_o}{4\pi} \cdot \frac{idl\sin\theta}{r^2}$$

In vector notation,

$$\vec{dB} = \frac{\mu_0}{4\pi r^3}(i\vec{dl} \times \vec{r}) = \frac{\mu_0}{4\pi r^2}(i\vec{dl} \times \hat{r})$$

Therefore, the total induction $\vec{B} = \frac{\mu_0 i}{4\pi}\int(\frac{1}{r^2}.\vec{dl} \times \hat{r})$

This is Biot - Savart law.

If we replace the current i by the current density J the current per unit area, $J = \frac{i}{A}$ then,

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{1}{r^2}.(\vec{J}\times\hat{r}).dv \qquad [\, i = J\,.\,A \text{ and } I\,.\,dl = J(A\,.\,dl) = J\,.\,dv]$$

Taking divergence on both sides,

$$\nabla \bullet \vec{B} = \frac{\mu_0}{4\pi}\int_v \nabla \bullet (\frac{1}{r^2}.\vec{J}\times\hat{r})dv$$

If the current density is assumed to be constant, then $\nabla \bullet \vec{J} = 0$

$$\therefore \nabla \bullet \vec{B} = 0$$

This is the differential form of Maxwell's' second equation.

Experiments to – date have shown that magnetic monopoles do not exist. Hence, the number of magnetic lines of force entering any arbitrary closed surface is exactly the same leaving it. Therefore the flux of magnetic induction **B** across a closed surface is zero.

By Gauss divergence theorem,

$$\int_v (\nabla \bullet \vec{B})\, dv = \oint_s B.ds = 0$$

This is the integral form of Maxwell's' second law.

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Maxwell's Third Law

By Faradays' law of electromagnetic induction,

$$e = -\frac{d\phi}{dt}$$

Now, let us consider work done on a charge, moving it through a distance dl.

$$W = \int \vec{E}.\vec{dl} \text{ which is a line integral}$$

If the work is done along a closed path, emf $= \oint \vec{E}.\vec{dl}$

The magnetic flux linked with closed area S due to the induction B = $\phi = \oint_s \vec{B}.\vec{ds}$

$$\therefore \text{emf} = e = -\frac{d\phi}{dt} = -\frac{d}{dt}\left[\oint_s \vec{B}.\vec{ds}\right] = -\oint_s \frac{d\vec{B}}{dt} \bullet \vec{ds}$$

Hence, $\oint \vec{E}.\vec{dl} = -\oint_s \frac{d\vec{B}}{dt}.\vec{ds}$

This is the Maxwell's third equation in integral form.

Using Stokes' theorem, the line integral of a vector function along a closed path $\oint \vec{E}.\vec{dl}$ can be converted to the surface integral of the normal component, the vector $\nabla \times \vec{E}$ of the enclosed surface.

(i.e) $\oint \vec{E}.\vec{dl} = \int_s (\nabla \times \vec{E}).\vec{ds}$

$$\therefore \int_s (\nabla \times \vec{E}).\vec{ds} = -\int_s \frac{d\vec{B}}{dt}.\vec{ds}$$

Hence, $(\nabla \times \vec{E}) = -\frac{\partial B}{\partial t}$

This is the Maxwell's' third equation in differential form.

Maxwell's Fourth Law

By Amperes' circuital law,

$$\int_{\ell} \vec{B}.\vec{dl} = \mu_0 i$$

But, $\mu_0 = \dfrac{B}{H}$ (or) $B = \mu_0 H$

Therefore, $\int_{i} \vec{H}.\vec{dl} = i$

But $i = \int_{s} \vec{J}.\vec{ds}$

Hence, $\int_{\ell} \vec{H}.\vec{dl} = \int_{s} \vec{J}.\vec{ds}$

But $\vec{J} = \sigma\vec{E} + \dfrac{\partial\vec{D}}{\partial t}$

$$\therefore \int_{\ell} \vec{H}.\vec{dl} = \int_{s} \sigma\vec{E}.\vec{ds} + \int_{s} \frac{\partial\vec{D}}{\partial t}.\vec{ds}$$

This is Maxwell's fourth equation in integral form.

Using Stokes theorem,

$$\int_{\ell} \vec{H}.\vec{dl} = \int_{s} (\nabla \times \vec{H}).\vec{ds}$$

Hence,

$$\int_s (\nabla \times \vec{H}).\vec{ds} = \int_s \sigma \vec{E}.\vec{ds} + \int_s \frac{\partial \vec{D}}{\partial t}.\vec{ds}$$

$$\int_s (\nabla \times \vec{H}).\vec{ds} = \left(\sigma \vec{E} + \frac{\partial \vec{D}}{\partial t} \right).\vec{ds}$$

(or) $\nabla \times \vec{H} = \text{curl}\, \vec{H} = \sigma \vec{E} + \frac{\partial \vec{D}}{\partial t}$

This is Maxwell's fourth equation in integral form.