UNIT 5: POWER SPECTRAL DENSITY (PSD)

DEFINITION.

WSS (or) SSS

Let $\{x(t)\}$ be a stationary process with autocorrelation $R_{xx}(\tau)$. The fourier transform of $F.T[R_{xx}(\tau)]$ is called power spectral density or power density spectrum.

$$F.T[R_{xx}(\tau)] = S_{xx}(\omega)$$

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-i\omega\tau} \, d\tau \quad \text{— (1)}$$ Wiener-Khinitchine relation.

If $S_{xx}(\omega)$ is given then:

$$S_{xx}(\omega) = F^{-1}[R_{xx}(\tau)]$$

$$R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) e^{i\omega\tau} \, d\omega \quad \text{— (2)}$$

$R_{xx}(\tau)$ & $S_{xx}(\omega)$ form a fourier transform pair.

NOTE:

$S_{xx}(\omega)$ gives the distribution of energy or power of the random process as a function of frequency. Hence it gets the name power spectral density

CROSS POWER SPECTAL DENSITY.

$\{x(t)\}$ $\{y(t)\}$ - Jointly Stationary process than

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-i\omega\tau} \, d\tau.$$

$$R_{xy}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(\omega) e^{i\omega\tau} \, d\omega.$$

$$S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau) e^{i\omega\tau} \cdot d\tau.$$ $R_{XY}(\tau) = R_{YX}(-\tau)$

$$R_{YX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YX}(\omega) \cdot e^{i\omega\tau} \cdot d\omega.$$

Acf (auto

PROPERTY 1.

1. $S_{XX}(0) = \int_{-\infty}^{\infty} R_{XX}(-\tau) \cdot d\tau$. (total area under

Represents the value of the Spectral density at zero freq which is equal to total area under ACF the graph of auto correlation function.

$\rightarrow E[X^2(t)] \rightarrow$ mean square

2. $R_{XX}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) \cdot d\omega.$

The mean square value of WSS is equal to the total area under the graph of spectral density function.

3. For a real Random Process $S_{XX}(\omega)$ is an even function of ω.

$$S_{XX}(-\omega) = S_{XX}(\omega).$$

PROOF:

By definition

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} \cdot d\tau.$$

$$S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{i\omega\tau} \cdot d\tau.$$

$\tau = -u.$ $\quad d\tau = -du.$ $\quad \tau \rightarrow -\infty$ to ∞

$u \rightarrow \infty$ to $-\infty$.

Second order mean is always positive.

$$= \int_{\infty}^{\infty} R_{xx}(-u)\, e^{i\omega(-u)} \cdot (-du)$$

$$= \int_{-\infty}^{\infty} R_{xx}(-u)\, e^{-i\omega u} \cdot du \, . \quad \int_a^b -dx = \int_b^a dx$$

Auto correlation fn is an even fn :

$$= \int_{-\infty}^{\infty} R_{xx}(u)\, e^{-i\omega u} \cdot du \, .$$

Spectral density is even fn of ω.

$S_{xx}(-\omega) = S_{xx}(\omega)$

even $\to$ even
odd $\to$ odd
$R_{xy} \to S_{xy}$

4. $S_{xx}(\omega) \geq 0$ for all ω.

Spectral density always positive

at $\tau = 0$ $\quad R_{xx}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\,.d\omega \quad . \; E[x^2(t)] \geq 0.$

$R_{xx}(0) = E[x^2(t)] \geq 0$. So, $S_{xx}(\omega) \geq 0$.

CROSS CORRELATION PROPERTIES.

1: $S_{xy}(-\omega) = S_{yx}(\omega)$.

$$S_{xy}(-\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{+i\omega(-\tau)} \cdot d\tau$$

$\tau = -u$. $\quad \tau \to -\infty$ to ∞

$d\tau = -du$. $\quad u \to \infty$ to $-\infty$

$$= \int_{+\infty}^{-\infty} R_{xy}(-u)\, e^{-i\omega(u)} (-du)$$

$$= -\int_{\infty}^{-\infty} R_{xy}(-u)\, e^{-i\omega u} \cdot du$$

$R_{xy}(-u) = R_{xy}(u)$

$$= \int_{-\infty}^{\infty} R_{xy}(u) \cdot e^{-i\omega u} \cdot du \, .$$

Unit vectors

$$= \int_{-\infty}^{\infty} R_{yx}(u) e^{-i\omega u} \cdot du$$

$$S_{yx}(-\omega) = S_{yx}(\omega)$$

2. $Re[S_{xy}(\omega)]$ $Re[S_{yx}(\omega)]$ are even function of ω.

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-i\omega\tau} \, d\tau.$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau) [\cos\omega\tau - i\sin\omega\tau] \cdot d\tau.$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau) \cos\omega\tau \cdot d\tau - i \int_{-\infty}^{\infty} R_{xy}(\tau) \sin\omega\tau \cdot d\tau.$$

Let $\int_{-\infty}^{\infty} R_{xy}(\tau) \cdot \cos\omega\tau \cdot d\tau = f(\omega)$

$$f(-\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) \cos(-\omega)\tau \cdot d\tau.$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau) \cos(\omega\tau) \, d\tau$$

$$f(-\omega) = f(\omega)$$

3. $Im[S_{xy}(\omega)]$ & $Im[S_{yx}(\omega)]$ odd function of ω.

$$Im[S_{xy}(\omega)] = -\int_{-\infty}^{\infty} R_{xy}(\tau) \cdot \sin\omega\tau \cdot d\tau = f(\omega).$$

$$Im[S_{xy}(-\omega)] = -\int_{-\infty}^{\infty} R_{xy}(\tau) \cdot \sin(-\omega)\tau \cdot d\tau.$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau) \sin\omega\tau \cdot d\tau$$

4. $X(t)$ & $Y(t)$ are orthogonal

$R_{xy}(\tau)=0 \qquad S_{xy}(\omega)=0.$

5. $X(t)$ & $Y(t)$ uncorrelated then.

$S_{xy}(\omega) = E[X(t)]\, E[Y(t)].$

$\delta(\omega)$ = Direct delta fn.

1. The power spectral density of a RP $\{X(t)\}$ is given by $S_{xx} = \begin{cases} \pi & |\omega|<1 \\ 0 & \text{else} \end{cases}$ find ACF.

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega).\, e^{i\omega\tau}\, d\omega.$$

$$= \frac{1}{2\pi}\int_{-1}^{1} \pi . e^{i\omega\tau}.\, d\omega.$$

$$= \frac{\pi}{2\pi}\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-1}^{1}$$

$$\frac{e^{i\theta}+e^{-i\theta}}{2} = \cos\theta$$

$$\frac{e^{i\theta}-e^{-i\tau}}{2i} = \sin\theta$$

$$= \frac{1}{2}\left[\frac{e^{i\tau}}{i\tau} - \left(\frac{e^{-i\tau}}{i\tau}\right)\right]$$

$$= \frac{1}{2}\, \frac{\cancel{e^{i\tau}}}{i\tau}\left[e^{i\tau} - e^{-i\tau}\right].$$

$$= \frac{1}{2i\tau}\left[e^{i\tau} - e^{-i\tau}\right]. = \frac{\sin\tau}{\tau}.$$

$e^{\infty} = \infty \qquad e^{0} = 1$

2. The ACF of a Random telegraph Signal process is $R(\tau) = a^2 e^{-2\lambda|\tau|}$ Determine PSD of the Signal

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) \cdot e^{-i\omega\tau} \cdot d\tau.$$

$$= \int_{-\infty}^{\infty} a^2 e^{-2\lambda|\tau|} \cdot e^{-i\omega\tau} \cdot d\tau.$$

$$= a^2 \left[\int_{-\infty}^{0} e^{-2\lambda(-\tau)} \cdot e^{-i\omega\tau} d\tau + \int_{0}^{\infty} e^{-2\lambda(\tau)} \cdot e^{-i\omega\tau} d\tau \right]$$

$$= a^2 \left[\int_{-\infty}^{0} e^{2\lambda\tau - i\omega\tau} \cdot d\tau + \int_{0}^{\infty} e^{-\tau(2\lambda + i\omega)} \cdot d\tau \right]$$

$$= a^2 \left[\int_{-\infty}^{\infty} e^{(2\lambda - i\omega)\tau} \cdot d\tau + \int_{0}^{\infty} e^{-\tau(2\lambda + i\omega)} \cdot d\tau \right]$$

$$= a^2 \left[\left[\frac{e^{(2\lambda - i\omega)\tau}}{2\lambda - i\omega} \right]_{-\infty}^{0} + \left[-\frac{e^{-\tau(2\lambda + i\omega)}}{2\lambda + i\omega} \right]_{0}^{\infty} \right]$$

$$= a^2 \left[\left[\frac{e^{0}}{2\lambda - i\omega} - \frac{e^{-\infty}}{2\lambda - i\omega} \right] + \left[-e^{-\infty} + \frac{e^{0}}{2\lambda + i\omega} \right] \right]$$

$$= \cancel{2} a^2 \left[\frac{1}{2\lambda - i\omega} + \frac{1}{2\lambda + i\omega} \right]$$

$$= a^2 \left[\frac{2\lambda + i\omega + 2\lambda - i\omega}{(2\lambda^2 - (i\omega)^2)} \right] = a^2 \frac{4\lambda}{4\lambda^2 + \omega^2}$$

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$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x).dx$ even×even = even, odd × odd = even

$\int_{-a}^{a} f(x).dx = 0 \rightarrow$ odd, odd×even = odd

3 Find PSD of the RP whose $R_{xx}(\tau) = \begin{cases} 1-|\tau| & |\tau|\le 1 \\ 0 & \text{else.} \end{cases}$

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau).e^{-i\omega\tau}.d\tau.$$

$-1 \le \tau \le 1$

$$= \int_{-1}^{1} 1-|\tau|.e^{-i\omega\tau}.d\tau.$$

$$= \int_{-1}^{1} (1-|\tau|)(\cos\omega\tau - i\sin\omega\tau).d\tau.$$

even, even, even, odd, even, odd

$$= \int_{-1}^{1}(1-|\tau|)\cos\omega\tau.d\tau - i\int_{-1}^{1}(1-|\tau|)\sin\omega\tau.d\tau.$$

$1-|\tau|$

$$= 2\int_{0}^{1}(1-(\tau))\cos\omega\tau.d\tau.$$

$$= 2\left[(1-\tau)\frac{\sin\omega\tau}{\omega} - (-1)\left(-\frac{\cos\omega\tau}{\omega^2}\right)\right]_0^1$$

$$= 2\left[-\frac{\cos\omega}{\omega^2} + \frac{1}{\omega^2}\right]$$

$1-\cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$

$$= \frac{2(1-\cos\omega)}{\omega^2}$$

$$= \frac{4\sin^2(\omega/2)}{\omega^2} = \frac{4}{\omega^2}\sin^2\left(\frac{\omega}{2}\right).$$

4. $S_{xx}(\omega) = \begin{cases} \frac{b}{a}(a-|\omega|) & |\omega|\le a \\ 0 & |\omega|>a \end{cases}$ find $R_{xx}(\tau)$.

Ans: $\frac{2b}{a\pi\tau^2}\sin^2\left(\frac{a\tau}{2}\right)$

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R(\tau).e^{-i\omega\tau}.d\tau.$$

$-a \le \omega \le a$

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega).e^{i\omega\tau}.d\omega.$$

$$= \frac{1}{2\pi}\int_{-a}^{a}\frac{b}{a}(a-|\omega|).e^{i\omega\tau}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-a}^{a}\frac{b}{a}(a-|\omega|)(\cos\omega\tau - i\sin\omega\tau).d\omega$$

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$\frac{b}{a}\omega - \frac{b}{a}\omega$

$$= \frac{1}{2\pi}\left[\int_{-a}^{a} \frac{b}{a}(a-|\omega|)\cos\omega\tau . d\omega - \int_{-a}^{a} \frac{b}{a}(a-|\omega|) i\sin\omega\tau \, d\omega\right.$$

$$= \frac{2}{2\pi}\int_{0}^{a} \frac{b}{a}(a-\omega)\cos\omega\tau . d\omega$$

$u = \frac{b}{a}(a-\omega)$ $\quad V = \cos\omega\tau \, d\omega$

$u' = -\frac{b}{a}$ $\quad V_1 = \frac{\sin\omega\tau}{\tau}$

$u'' = 0$ $\quad V_2 = \frac{-\cos\omega\tau}{\tau^2}$

$$= \frac{1}{\pi}\left[\frac{b}{a}(a-\omega)\frac{\sin\omega\tau}{\tau} - \left(-\frac{b}{a}\right) \cdot \frac{-\cos\omega\tau}{\tau^2}\right]_0^a$$

$$= \frac{1}{\pi}\left[-\frac{b}{a}\,\frac{\cos a\tau}{\tau^2} + \frac{b}{a\tau^2}\right]$$

$$= \frac{b}{\pi a\tau^2}\left[-\cos a\tau + 1\right]$$

$$= \frac{b}{\pi a\tau^2}\left[1 - \cos a\tau\right]$$

$1 - \cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$

$$= \frac{b}{\pi a\tau^2}\left[2\sin^2\left(\frac{a\tau}{2}\right)\right]$$

$$= \frac{2b}{\pi a\tau^2}\sin^2\left(\frac{a\tau}{2}\right)$$

5. The PSD of a zero mean WSS process $\{x(t)\}$ is $S_{xx}(\omega)=\begin{cases}1 & |\omega|\leq\omega_0\\ 0 & \text{elsewhere}\end{cases}$ find $R_{xx}(\tau)$. Show that $x(t)$ & $x\left(t+\frac{\pi}{\omega_0}\right)$ are uncorrelated

$$R_{xx}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{xx}(\omega)e^{i\omega\tau}\,d\omega$$

$$=\frac{1}{2\pi}\int_{-\omega_0}^{\omega_0}e^{i\omega\tau}\,d\omega$$

$$=\frac{1}{2\pi}\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-\omega_0}^{\omega_0}$$

$$=\frac{1}{2\pi}\left[\frac{e^{i\omega_0\tau}}{i\tau}-\frac{e^{-i\omega_0\tau}}{i\tau}\right]$$

$$=\frac{1}{2\pi i\tau}\left[e^{i\omega_0\tau}-e^{-i\omega_0\tau}\right]$$

$$=\frac{\sin\omega_0\tau}{\cancel{2}\pi\tau}=\frac{\sin\omega_0\tau}{\pi\tau}.$$

Un Correlated,

$\mathrm{Cov}(x,y)=0$.

$E[xy]-E[x]E[y]=0$

$E[xy]=0$

$E\left[x(t)\cdot x\left(t+\frac{\pi}{\omega_0}\right)\right]=0$

$R_{xx}\left(\frac{\pi}{\omega_0}\right)=0.$

$$R_{xx}(\tau)=\frac{\sin\omega_0\tau}{\pi\tau}.$$

$$R_{xx}\left(\frac{\pi}{\omega_0}\right)=\frac{\sin\left(\omega_0\times\frac{\pi}{\omega_0}\right)}{\pi\tau}$$

$$=\frac{\sin\pi}{\pi\left(\frac{\pi}{\omega_0}\right)}$$

$$=0.$$

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$\int_{-a}^{a} f(x).dx = 2\int_{0}^{a} f(x).dx$.

Formulas in Inverse FT.

1. $F^{-1}\left[\frac{1}{(\alpha+i\omega)^2}\right] = u(\tau).\tau e^{-\alpha\tau}$.

$u(\tau)$ is unit step fn.

2. $F^{-1}\left[\frac{1}{(\alpha-i\omega)^2}\right] = u(\tau).\tau e^{\alpha\tau}$.

3. $F^{-1}\left[\frac{2\alpha}{\alpha^2+\omega^2}\right] = e^{-\alpha|\tau|}$.

DEFINITION.

Average Power of the Process.

P_{xx} is defined as $P_{xx} = R_{xx}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{\cancel{xx}}(\omega).d\omega$

$P_{xx} \rightarrow E[x^2(t)] \rightarrow$ mean Square.

1. Find the Avg Power of the process

$S_{xx}(\omega) = \frac{4}{4+\omega^2}$

Avg Power $= P_{xx} = R_{xx}(0)$

$R_{xx}(\tau) = F^{-1}\left[\frac{4}{4+\omega^2}\right]$

$= F^{-1}\left[\frac{2\times2}{2^2+\omega^2}\right]$

$= e^{-2|\tau|}$.

$P_{xx} = R_{xx}(0) = e^{0} = 1$.

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2. $S_{xx}(\omega) = \frac{1}{9+\omega^2}$

$$P_{xx} = R_{xx}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{9+\omega^2}\, d\omega$$

$$= \frac{2}{2\pi}\int_{0}^{\infty} \frac{1}{9+\omega^2}\, d\omega$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{1}{\pi}\left[\frac{1}{3}\tan^{-1}\left[\frac{x}{3}\right]\right]_0^{\infty}$$

$$= \frac{1}{\pi}\times\frac{1}{3}\left[\tan^{-1}[\infty] - \tan^{-1}[0]\right]$$

$$= 1/3\pi\,[\pi/2]$$

$$= 1/6$$

3. Find the mean Square Value of process whose PSD is $S_{xx}(\omega) = \frac{\omega^2+9}{\omega^4+5\omega^2+4}$.

$\omega^2 = u$.

$$S_{xx}(\omega) = \frac{u+9}{u^2+5u+4} = \frac{u+9}{(u+4)(u+1)}$$

$4 < \frac{9}{5}$

$$\frac{u+9}{u^2+5u+4} = \frac{A}{(u+4)} + \frac{B}{(u+1)}$$

$$u+9 = A(u+1) + B(u+4)$$

$u = -1$

$$-1+9 = A(-1+1) + B(-1+4)$$

$$8 = B(3)$$

$$B = 8/3$$

$u = -4$.

$$-4+9 = A(-4+1) + B(-4+4)$$

$$5 = A(-3)$$

$$A = -5/3$$

$$= \frac{-5/3}{u+4} + \frac{8/3}{u+1}.$$

$$S_{xx}(\omega) = -\frac{5}{3}\left[\frac{1}{\omega^2+4}\right] + \frac{8}{3}\left[\frac{1}{\omega^2+1}\right].$$

$$MSV = E[x^2(t)] = R_{xx}(0)$$

$$R_{xx}(\tau) = F^{-1}\left[S_{xx}(\omega)\right]$$

$$= F^{-1}\left[-\frac{5}{3}\left(\frac{1}{\omega^2+4}\right) + \frac{8}{3}\left[\frac{1}{\omega^2+1}\right]\right].$$

$$= F^{-1}\left[-\frac{5}{3}\times\frac{4}{4}\left[\frac{1}{\omega^2+4}\right] + \frac{8}{3}\times\frac{2}{2}\left[\frac{1}{\omega^2+1}\right]\right]$$

$$= -\frac{5}{12}e^{-2|\tau|} + \frac{4}{3}e^{-1|\tau|}.$$

$$R_{xx}(0) = -\frac{5}{12} + \frac{4}{3}$$

$$= 11/12.$$

4. $S_{xx}(\omega) = \dfrac{\omega^2-17}{(\omega^2+49)(\omega^2+16)}$ find P_{xx}.

The cross power spectral density of

$$S_{xy}(\omega)=\begin{cases} a+ib\omega & |\omega|<1 \\ 0 & \text{elsewhere} \end{cases}$$ find cross correlation func.

$$R_{xy}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)\, e^{i\omega\tau}\, d\omega.$$

$$=\frac{1}{2\pi}\int_{-1}^{1}(a+ib\omega)\, e^{i\omega\tau}\, d\omega.$$

$$=\frac{1}{2\pi}\int_{-1}^{1}(a+ib\omega)(\cos\omega\tau+i\sin\omega\tau)\, d\omega.$$

$$=\frac{1}{2\pi}\int_{-1}^{1} a\cos\omega\tau+ai\sin\omega\tau+ib\omega\cos\omega\tau+(-1)b\omega\sin\omega\tau\, d\omega$$

$$=\frac{2}{2\pi}\left[\int_{0}^{1} a\cos\omega\tau\, d\omega-\int_{0}^{1} b\omega\sin\omega\tau\, d\omega\right].$$

$$=\frac{1}{\pi}\left[\frac{a\sin\omega\tau}{\tau}\right]_0^1-\cancel{\frac{1}{\pi}}\, b\left[\frac{\omega(-\cos\omega\tau)}{\tau}-\frac{1(-\sin\omega\tau)}{\tau^2}\right]_0^1$$

$$=\frac{a}{\pi}\left[\frac{\sin\tau}{\tau}\right]-\frac{b}{\pi}\left[\frac{-\cos\tau}{\tau}+\frac{\sin\tau}{\tau^2}-0\right].$$

$$=\frac{a}{\pi}\frac{\sin\tau}{\tau}+\frac{b}{\pi}\frac{\cos\tau}{\tau}-\frac{b}{\pi}\frac{\sin\tau}{\tau^2}.$$

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LINEAR SYSTEM WITH RANDOM INPUT.

The output $Y(t)$ can be expressed as the function of input $x(t)$.

$$Y(t) = f[x(t)].$$

Linear $f[a_1 x_1(t) + a_2 x_2(t)] = a_1 f[x_1(t)] + a_2 f[x_2(t)]$.

Time invariant

$$Y(t+\tau) = f[x(t+\tau)] \qquad \tau \in (-\infty, \infty)$$

Casual

If the value of output at time $t = t_1$ depends only on the past values of the input

Memoryless System.

The output $Y(t)$ at $t = t_1$ depends only on $x(t_1)$ and not on any other values.

Representation of the System in the form of Convolution

$$Y(t) = h(t) * x(t)$$

→ Power transfer function. → I/p function

O/p function

$$= \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du.$$

$h(t) \rightarrow$ unit impulse response (System weighing

NOTE :·

1. for a casual system we always have unit impulse response $h(t) = 0$ when $t < 0$.

The output function becomes

$$Y(t) = \int_{0}^{\infty} h(u)\, x(t-u)\, du.$$

2. $H(\omega) = F.T[h(t)]$ System transfer function (or) power transfer function.

Relationship that connects input & output power func

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega).$$

1. A linear time invariant system $h(t) = e^{-\beta t} u(t)$ find the power spectral density of output $Y(t)$ corresponding to the input $x(t)$

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega).$$

$$H(\omega) = F.T(h(t)).$$

$u(t) = 1 \quad t \geq 0$

$\quad\quad 0 \quad t < 0.$

$$= \int_{-\infty}^{\infty} e^{-\beta t} u(t) e^{-i\omega t} dt$$

$h(t) = e^{-\beta t} \quad t \geq 0$

$\quad\quad = 0 \quad t < 0.$

$$= \int_0^{\infty} e^{-\beta t} . e^{-i\omega t} . dt.$$

$$= \int_0^{\infty} e^{-t(\beta + i\omega)} . dt.$$

$\alpha = a + ib = \sqrt{a^2 + b^2}$

$$= \left[\frac{e^{-t(\beta+i\omega)}}{\beta + i\omega} \right]_0^{\infty}.$$

$$H(\omega) = \frac{1}{\beta + i\omega}.$$

$$|H(\omega)| = \frac{1}{\sqrt{\beta^2 + \omega^2}}$$

$$|H(\omega)|^2 = \frac{1}{\beta^2 + \omega^2}.$$

$$S_{yy}(\omega) = \frac{1}{\beta^2 + \omega^2} S_{xx}(\omega).$$

2. $h(t) = 2e^{-7t}$ $t \geq 0$ find $S_{yy}(\omega)$ to be the i/p to a linear system the autocorrelation is $R_{xx}(\tau) = e^{-4|\tau|}$.

$h(t) = 2e^{-7t}$

$R_{xx}(\tau) = e^{-4|\tau|}$.

$F.T[h(t)] = H(\omega)$

$F.T[R_{xx}(\tau)] = S_{xx}(\omega)$.

$$H(\omega) = \int_{-\infty}^{\infty} h(t) \cdot e^{-i\omega t} \cdot dt$$

$$= \int_{-\infty}^{\infty} 2e^{-7t} \cdot e^{-i\omega t} \cdot dt$$

$$= 2\int_{-\infty}^{\infty} e^{-(7+i\omega)t} \cdot dt = 2\left[\frac{e^{-t(7+i\omega)}}{-(7+i\omega)}\right]_{-\infty}^{\infty}$$

$$= 0 - \frac{1}{-(7+i\omega)}$$

$$= \frac{2}{7+i\omega}$$

$$|H(\omega)|^2 = \frac{2}{\sqrt{7^2+\omega^2}} \Rightarrow |H(\omega)|^2 = \frac{4}{49+\omega^2}.$$

$S_{xx}(\omega) = F.T[R_{xx}(\tau)]$.

$$= \int_{-\infty}^{\infty} R_{xx}(\tau) \cdot e^{-i\omega\tau} \cdot d\tau = \int_{-\infty}^{\infty} e^{-4|\tau|} \cdot e^{-i\omega\tau} \cdot d\tau.$$

$$= \int_{-\infty}^{0} e^{-(-\tau)4} e^{-i\omega\tau} \cdot d\tau + \int_{0}^{\infty} e^{-4(\tau)} \cdot e^{-i\omega\tau} \cdot d\tau.$$

$$= \int_{-\infty}^{0} e^{4\tau - i\omega\tau} + \int_{0}^{\infty} e^{-4\tau - i\omega\tau} \cdot d\tau.$$

$$= \left[\frac{e^{\tau(4-i\omega)}}{(4-i\omega)}\right]_{-\infty}^{0} + \left[\frac{e^{-\tau(4+i\omega)}}{-(4+i\omega)}\right]_{0}^{\infty}$$

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$$S_{YY}(\omega) = |H(\omega)|^2 \cdot S_{XX}(\omega).$$

1. ~~$h(t) = e^{-2t}$, $t \geq 0$.~~

$R_{XX}(\tau) = e^{-|\tau|}$

$$= \frac{1}{4-i\omega} - 0 + 0 + \frac{1}{4+i\omega}$$

$$= \frac{1}{4-i\omega} + \frac{1}{4+i\omega}.$$

$$= \frac{1}{4-i\omega} \times \frac{4+i\omega}{4+i\omega} + \frac{1}{4+i\omega} \times \frac{4-i\omega}{4-i\omega}.$$

$$= \frac{4+i\omega}{16-(i\omega)^2} + \frac{4-i\omega}{16-(i\omega)^2}$$

$$= \frac{4+i\omega}{16+\omega^2} + \frac{4-i\omega}{16+\omega^2}$$

$$= \frac{8}{16+\omega^2}.$$

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

$$= \frac{4}{49+\omega^2} \cdot \frac{8}{16+\omega^2}$$

$$= \frac{32}{(49+\omega^2)(16+\omega^2)}$$

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