

Unit - III. Applications of Partial Differential Equations.

Classification of Partial differential equations of the second

Order:

The most general linear partial differential equation of second order can be written as

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = 0$$

$$\text{i.e., } Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0 \quad \text{--- } \textcircled{1}$$

where A, B, C, D, E, F are in general functions of x & y .

The equation $\textcircled{1}$ of second order (linear) is said to

(i) elliptic if $B^2 - 4AC < 0$

(ii) hyperbolic if $B^2 - 4AC > 0$

(iii) parabolic if $B^2 - 4AC = 0$.

Examples:

$\textcircled{1}$ Elliptic Type: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Laplace equation in two dimension)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \quad (\text{Poisson's equation})$$

$\textcircled{2}$ Parabolic type: $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ (one-dimensional heat equation)

$\textcircled{3}$ Hyperbolic type: $\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ (one-dimensional wave equation).

Problems:- Classify the following equations:

① $xx_{xx} + yy_{yy} = 0$

$A=x \quad B=0 \quad C=1$

$B^2 - 4AC = 0 - 4(x)(1)$
 $= -4x$

The equation is elliptic if $x > 0$

The equation is hyperbolic if $x < 0$

The equation is parabolic if $x = 0$.

Note:- The same differential equation may be elliptic in one region, parabolic in another and hyperbolic in some other region.

② $\frac{\partial u}{\partial x^2} + 2 \frac{\partial u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$

$A=1, \quad B=2, \quad C=1$

$B^2 - 4AC = 4 - 4(1)(1) = 0$

Hence, the equation is parabolic at all points.

③ $x f_{xx} + y f_{yy} = 0, \quad x > 0, \quad y > 0$

$A=x \quad B=0 \quad C=y$

$B^2 - 4AC = 0 - 4xy$

\therefore It is elliptic for all $x > 0, y > 0$.

$$(4) \quad x^2 f_{xx} + (1-y^2) f_{yy} = 0$$

Here, $A = x^2$, $B = 0$, $C = 1-y^2$

$$B^2 - 4AC = 0 - 4(x^2)(1-y^2)$$

$$= -4x^2(1-y^2)$$

$$= -4x^2(y^2-1)$$

for all x , except $x=0$ (x^2 is positive)

If $-1 < y < 1$, y^2-1 is negative.

For $-\infty < x < \infty$ ($x \neq 0$), $-1 < y < 1$, the equation is elliptic.

For $-\infty < x < \infty$, $x \neq 0$, $y < -1$ or $y > 1$, the equation is hyperbolic.

For $x=0$ for all y or for all x , $y = \pm 1$, the equation is parabolic.

$$(5) \quad u_{xx} + 4u_{xy} + (x^2 + 4y^2) u_{yy} = \sin(x+y)$$

here, $A = 1$; $B = 4$; $C = x^2 + 4y^2$

$$B^2 - 4AC = 16 - 4(1)(x^2 + 4y^2)$$

$$= 4(4 - (x^2 + 4y^2))$$

$$= 4(4 - x^2 - 4y^2)$$

The equation is elliptic if $4 - x^2 - 4y^2 < 0$

$$-x^2 - 4y^2 < -4$$

$$x^2 + 4y^2 > 4$$

$$\frac{x^2}{4} + \frac{y^2}{1} > 1$$

\therefore It is elliptic in the region outside the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$.

The equation is hyperbolic if $4 - x^2 - 4y^2 > 0$

$$-x^2 - 4y^2 > -4$$

$$x^2 + 4y^2 < 4$$

$$\frac{x^2}{4} + \frac{y^2}{1} < 1$$

\therefore It is hyperbolic in the region inside the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$

The equation is parabolic if $4 - x^2 - 4y^2 = 0$

$$-x^2 - 4y^2 = -4$$

$$x^2 + 4y^2 = 4$$

$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

It is parabolic on the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$

⑥ The Laplace equation $\frac{\partial u}{\partial x^2} + \frac{\partial u}{\partial y^2} = 0$

here, $A=1$, $B=0$, $C=1$

$$\therefore B^2 - 4AC = 0 - 4 < 0$$

Hence the equation is elliptic

The poisson's equation $\frac{\partial u}{\partial x^2} + \frac{\partial u}{\partial y^2} = f(x,y)$

here also, $B^2 - 4AC = -4 < 0$

Hence the equation is elliptic

one dimensional heat equation $\frac{d^2 \partial u}{\partial x^2} = \frac{\partial u}{\partial t}$

here, $A=d^2$, $B=0$, $C=0$

$$\therefore B^2 - 4AC = 0 - 4(d^2)(0) = 0$$

Hence the equation is parabolic

one dimensional wave equation

$$\frac{d^2 \partial u}{\partial x^2} = \frac{\partial u}{\partial t^2}$$

here, $A=d^2$, $B=0$, $C=-1$

$$B^2 - 4AC = 0 - 4(d^2)(-1) = 4d^2 > 0$$

\therefore The equation is hyperbolic

Exercise problems:-

① $(x+1)u_{xx} - 2(x+2)u_{xy} + (x+3)u_{yy} = 0$

② $\frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial u}{\partial x \partial y} + 4 \frac{\partial^2 u}{\partial y^2} - 12 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + 7u = x^2 + y^2$

③ $(1+x^2)f_{xx} + (5+2x^2)f_{xy} + (4+x^2)f_{yy} = 2 \sin(x+y)$

④ $(1-x^2)f_{xx} - 2xyf_{xy} + (1-y^2)f_{yy} = 0$

⑤ prove $f_{xx} + 2f_{xy} + 4f_{yy} = 0$ is elliptic

⑥ prove $f_{xx} - 2f_{xy} + f_{yy} = 0$ and $u_{xx} = u_t$ are parabolic

The following assumptions:

1) The motion takes place entirely in one plane. This plane

is chosen as the xy plane.

2) In this plane, each particle of the string moves in a direction perpendicular to the equilibrium position of the string.

3) The tension T is constant throughout the string before being set at the end points and is constant at all times at all points of the deflected string.

4) The tension T is very large compared with the weight of the string and hence the gravitational force may be neglected.