DEPARTMENT OF PHYSICS AND NANOTECHNOLOGY
SRM INSTITUTE OF SCIENCE AND TECHNOLOGY

18PYB101J-Electromagnetic Theory, Quantum Mechanics, Waves and Optics

Module 3- Lecture-15

Hydrogen atom problem – radial, angular equation derivation and Hydrogen atom problem – solutions to radial and angular equations

SRM Institute of Science and Technology

The Schrödinger Equation to the Hydrogen Atom

- The approximation of the potential energy of the electron-proton system is electrostatic:

$$V(r) = -\frac{e^2}{4\pi\varepsilon_0 r}$$

- Rewrite the three-dimensional time-independent Schrödinger Equation.

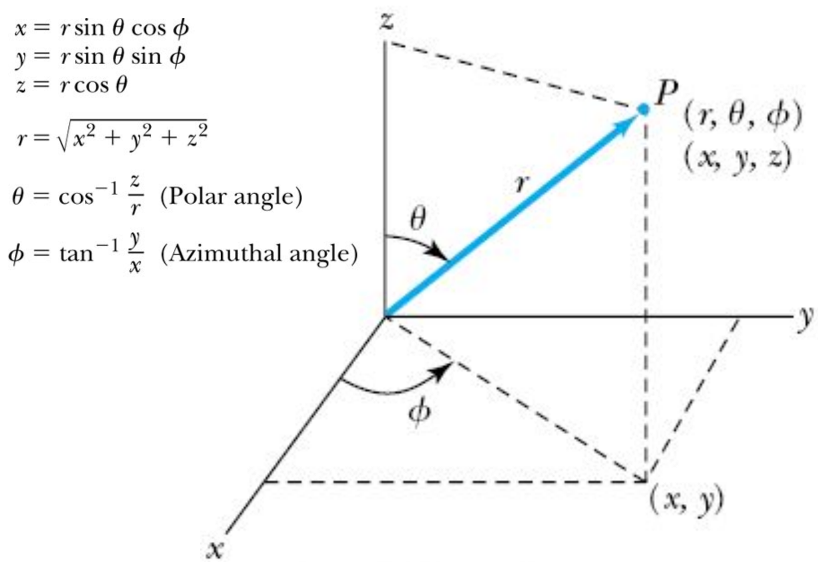
$$-\frac{\hbar^2}{2m}\frac{1}{\psi(x,y,z)}\left[\frac{\partial^2\psi(x,y,z)}{\partial x^2}+\frac{\partial^2\psi(x,y,z)}{\partial y^2}+\frac{\partial^2\psi(x,y,z)}{\partial z^2}\right] = E - V(r)$$

For Hydrogen-like atoms (He^+ or Li^{++})

- Replace e^2 with Ze^2 (Z is the atomic number).
- Use appropriate reduced mass μ.

Application of the Schrödinger Equation

- The potential (central force) *V*(*r*) depends on the distance *r* between the proton and electron.



Transform to spherical polar coordinates because of the radial symmetry.

Insert the Coulomb potential into the transformed Schrödinger equation.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{2\mu}{\hbar^2}(E - V)\psi = 0$$

Application of the Schrödinger Equation

- The wave function ψ is a function of r, θ, ϕ

→ Equation is separable.

→ Solution may be a product of three functions.

→ $$\psi(r,\theta,\phi) = R(r)f(\theta)g(\phi)$$ Equation 7.3

- We can separate Equation 7.3 into three separate differential equations, each depending on one coordinate: r, θ, or ϕ.

7.2: Solution of the Schrödinger Equation for Hydrogen

- Substitute Eq (7.4) into Eq (7.3) and separate the resulting equation into three equations: $R(r)$, $f(\theta)$, and $g(\phi)$.

Separation of Variables

- The derivatives from Eq (7.4)

$$\frac{\partial \psi}{\partial r} = fg\frac{\partial R}{\partial r} \qquad \frac{\partial \psi}{\partial \theta} = Rg\frac{\partial f}{\partial \theta} \qquad \frac{\partial^2 \psi}{\partial \phi^2} = Rf\frac{\partial^2 g}{\partial \phi^2}$$

- Substitute them into Eq (7.3)

$$\frac{fg}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{Rg}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{Rf}{r^2\sin^2\theta}\frac{\partial^2 g}{\partial \phi^2} + \frac{2\mu}{\hbar^2}(E-V)Rfg = 0$$

- Multiply both sides of Eq (7.6) by $r^2 \sin^2 \theta / Rfg$

$$-\frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2\mu}{\hbar^2}r^2\sin^2\theta(E-V) - \frac{\sin\theta}{f}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) = \frac{1}{g}\frac{\partial^2 g}{\partial \phi^2}$$

Solution of the Schrödinger Equation

- Only r and θ appear on the left side and only ϕ appears on the right side of Eq (7.7)
- The left side of the equation cannot change as ϕ changes.
- The right side cannot change with either r or θ.

- Each side needs to be equal to a constant for the equation to be true.

  Set the constant $-m_\ell^2$ equal to the right side of Eq (7.7)

$$\frac{d^2 g}{d\phi^2} = -m_\ell{}^2 g$$ -------- **azimuthal equation**

- It is convenient to choose a solution to be $e^{im_\ell\phi}$.

Solution of the Schrödinger Equation

- $e^{im_\ell \phi}$ sfies Eq (7.8) for any value of m_ℓ.
- The solution be single valued in order to have a valid solution for any ϕ, which is $g(\phi) = g(\phi + 2\pi)$
  $g(\phi = 0) = g(\phi = 2\pi) \longrightarrow e^0 = e^{2\pi i m_\ell}$
- m_ℓ to be zero or an integer (positive or negative) for this to be true.
- If Eq (7.8) were positive, the solution would not be realized.
- Set the left side of Eq (7.7) equal to $-m_\ell^2$ and rearrange it.

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{2\mu r^2}{\hbar^2}(E - V) = \frac{m_\ell{}^2}{\sin^2\theta} - \frac{1}{f\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)$$

- Everything depends on r on the left side and θ on the right side of the equation.

Solution of the Schrödinger Equation

- Set each side of Eq (7.9) equal to constant $\ell(\ell + 1)$.

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{2\mu}{\hbar^2}\left[E-V-\frac{\hbar^2}{2\mu}\frac{\ell(\ell+1)}{r^2}\right]R=0$$ **----Radial equation**

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{df}{d\theta}\right)+\left[\ell(\ell+1)-\frac{{m_\ell}^2}{\sin^2\theta}\right]f=0$$ **----Angular equation**

- Schrödinger equation has been separated into three ordinary second-order differential equations [Eq (7.8), (7.10), and (7.11)], each containing only one variable.

Solution of the Radial Equation

- The radial equation is called the **associated Laguerre equation** and the *solutions R* that satisfy the appropriate boundary conditions are called *associated Laguerre functions*.

- Assume the ground state has $\ell = 0$ and this requires $m_\ell = 0$.

  Eq (7.10) becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{2\mu}{\hbar^2}(E-V)R=0$$

- The derivative of $r^2\frac{dR}{dr}$ yields two terms.

  Write those terms and insert Eq (7.1)

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\frac{2\mu}{\hbar^2}\left(E+\frac{e^2}{4\pi\varepsilon_0 r}\right)R=0$$

Solution of the Radial Equation

- Try a solution $R = Ae^{-r/a_0}$

  A is a normalized constant.

  a_0 is a constant with the dimension of length.

  Take derivatives of *R* and insert them into Eq (7.13).

$$\left(\frac{1}{a_0^{\ 2}} + \frac{2\mu}{\hbar^2}E\right) + \left(\frac{2\mu e^2}{4\pi\varepsilon_0\hbar^2} - \frac{2}{a_0}\right)\frac{1}{r} = 0$$

- To satisfy Eq (7.14) for any *r* is for each of the two expressions in parentheses to be zero.

  Set the second parentheses equal to zero and solve for a_0.

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{\mu e^2}$$

  Set the first parentheses equal to zero and solve for *E*.

$$E = -\frac{\hbar^2}{2\mu a_0^{\ 2}} = -E_0$$

  Both equal to the Bohr result.

Quantum Numbers

- The appropriate boundary conditions to Eq (7.10) and (7.11) leads to the following restrictions on the quantum numbers ℓ and m_ℓ:
  - $\ell = 0, 1, 2, 3, \ldots$
  - $m_\ell = -\ell, -\ell + 1, \ldots, -2, -1, 0, 1, 2, . \ell . , \ell - 1, \ell$
  - $|m_\ell| \le \ell$ and $\ell < 0$.

- The predicted energy level is

$$E_n = -\frac{E_0}{n^2}$$

Hydrogen Atom Radial Wave Functions

- First few radial wave functions $R_{n\ell}$

Table 7.1 Hydrogen Atom Radial Wave Functions

n	ℓ	$R_{n\ell}(r)$
1	0	$\frac{2}{(a_0)^{3/2}} e^{-r/a_0}$
2	0	$\left(2 - \frac{r}{a_0}\right) \frac{e^{-r/2a_0}}{(2a_0)^{3/2}}$
2	1	$\frac{r}{a_0} \frac{e^{-r/2a_0}}{\sqrt{3}(2a_0)^{3/2}}$
3	0	$\frac{1}{(a_0)^{3/2}} \frac{2}{81\sqrt{3}} \left(27 - 18\frac{r}{a_0} + 2\frac{r^2}{a_0^2}\right) e^{-r/3a_0}$
3	1	$\frac{1}{(a_0)^{3/2}} \frac{4}{81\sqrt{6}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0}$
3	2	$\frac{1}{(a_0)^{3/2}} \frac{4}{81\sqrt{30}} \frac{r^2}{a_0^2} e^{-r/3a_0}$

- Subscripts on R specify the values of n and ℓ.

Solution of the Angular and Azimuthal Equations

- The solutions for Eq (7.8) are $e^{im_\ell \phi}$ or $e^{-im_\ell \phi}$
- Solutions to the angular and azimuthal equations are linked because both have m_ℓ.
- Group these solutions together into functions.

$$Y(\theta,\phi) = f(\theta)g(\phi)$$ ---- **spherical harmonics**

Normalized Spherical Harmonics

Table 7.2 Normalized Spherical Harmonics $Y(\theta, \phi)$

ℓ	m_ℓ	$Y_{\ell m_\ell}$
0	0	$\frac{1}{2\sqrt{\pi}}$
1	0	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$
1	± 1	$\mp\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\, e^{\pm i\phi}$
2	0	$\frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta - 1)$
2	± 1	$\mp\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta\, e^{\pm i\phi}$
2	± 2	$\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{\pm 2i\phi}$
3	0	$\frac{1}{4}\sqrt{\frac{7}{\pi}}(5\cos^3\theta - 3\cos\theta)$
3	± 1	$\mp\frac{1}{8}\sqrt{\frac{21}{\pi}}\sin\theta(5\cos^2\theta - 1)\, e^{\pm i\phi}$
3	± 2	$\frac{1}{4}\sqrt{\frac{105}{2\pi}}\sin^2\theta\cos\theta\, e^{\pm 2i\phi}$
3	± 3	$\mp\frac{1}{8}\sqrt{\frac{35}{\pi}}\sin^3\theta\, e^{\pm 3i\phi}$

Solution of the Angular and Azimuthal Equations

- The radial wave function *R* and the spherical harmonics *Y* determine the probability density for the various quantum states. The total wave function $\psi(r,\theta,\phi)$ depends on *n*, ℓ, and m_ℓ. The wave function becomes

$$\psi_{n\ell m_\ell}(r,\theta,\phi) = R_{n\ell}(r) Y_{\ell m_\ell}(\theta,\phi)$$

7.3: Quantum Numbers

The three quantum numbers:

- n Principal quantum number
- ℓ Orbital angular momentum quantum number
- m_ℓ Magnetic quantum number

The boundary conditions:

- $n = 1, 2, 3, 4, \ldots$ Integer
- $\ell = 0, 1, 2, 3, \ldots, n - 1$ Integer
- $m_\ell = -\ell, -\ell + 1, \ldots, 0, 1, \ldots, \ell - 1, \ell$ Integer

The restrictions for quantum numbers:

- $n > 0$
- $\ell < n$
- $|m_\ell| \le \ell$

Principal Quantum Number *n*

- It results from the solution of $R(r)$ in Eq (7.4) because $R(r)$ includes the potential energy $V(r)$.

  The result for this quantized energy is

$$E_n = \frac{-\mu}{2}\left(\frac{e^2}{4\pi\varepsilon_0 \hbar}\right)^2 \frac{1}{n^2} = -\frac{E_0}{n^2}$$

- The negative means the energy *E* indicates that the electron and proton are bound together.

Orbital Angular Momentum Quantum Number ℓ

- It is associated with the $R(r)$ and $f(\theta)$ parts of the wave function.

- Classically, the orbital angular momentum $\vec{L} = \vec{r} \times \vec{p}$ ı $L = mv_{\text{orbital}}r$.

- ℓ is related to L by

$$L = \sqrt{\ell(\ell+1)}\hbar$$

- In an $\ell = 0$ state,

$$L = \sqrt{0(1)}\hbar = 0$$

It disagrees with Bohr's semiclassical "planetary" model of electrons orbiting a nucleus $L = n\hbar$.

Orbital Angular Momentum Quantum Number ℓ

- A certain energy level is **degenerate** with respect to ℓ when the energy is independent of ℓ.

- Use letter names for the various ℓ values.
  - ℓ = 0 1 2 3 4 5 . . .
  - Letter = *s* *p* *d* *f* *g* *h* . . .

ℓ =	0	1	2	3	4	5	. . .
Letter =	*s*	*p*	*d*	*f*	*g*	*h*	. . .

- Atomic states are referred to by their *n* and ℓ.
- A state with $n = 2$ and $\ell = 1$ is called a 2*p* state.
- The boundary conditions require $n > \ell$.

Magnetic Quantum Number m_ℓ

- The angle ϕ is a measure of the rotation about the z axis.
- The solution for $g(\phi)$ specifies that m_ℓ is an integer and related to the z component of L.

$$L_z = m_\ell \hbar$$

- The relationship of L, L_z, ℓ, and m_ℓ for $\ell = 2$.
- $L = \sqrt{\ell(\ell+1)}\hbar = \sqrt{6}\hbar$ because L_z is quantized.
- Only certain orientations of $\vec{L}$ are possible and this is called **space quantization**.