One dimensional heat flow :-

Consider the flow of heat and the accompanying variation of temperature with position and with time in conducting solids.

The following empirical laws are taken as the basis of investigation.

1. Heat flows from a higher to lower temperature.
2. The amount of heat required to produce a given temperature change in a body is proportional to the mass of the body and to the temperature change. This constant of proportionality is known as the specific heat (c) of the conducting material.
3. The rate at which heat flows through an area is proportional to the area and to the temperature gradient normal to the area. This constant of proportionality is known as the thermal conductivity (k) of the normal.

The one dimensional heat flow equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

where α^2 stands for $\frac{k}{\rho c}$.

k – thermal Conductivity

c – Specific heat

ρ – density

$\frac{k}{\rho c}$ is called the diffusivity ($cm^2/sec.$)

Solution of the heat equation by the method of Seperation of variables.

The one-dimensional heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow (1)$$

Assume a solution of the form $u(x,t) = X(x) \cdot T(t)$ $\longrightarrow (2)$

where X is a function of x alone and T is a function of t alone.

differentiating (2) partially w.r to 't', we get

$$\frac{\partial u}{\partial t} = XT' \longrightarrow (3)$$

differentiating (2) partially w.r to 'x' twice, we get

$$\frac{\partial^2 u}{\partial x^2} = X''T \longrightarrow (4)$$

Substituting (3) & (4) in eqn (1), we get

$$XT' = \alpha^2 X''T$$

Seperating the variables, we get

$$\frac{T'}{\alpha^2 T} = \frac{X''}{X} = k \text{ (say)}$$

ie, $\frac{T'}{\alpha^2 T} = k$ and $\frac{X''}{X} = k$

ie, $T' - k\alpha^2 T = 0 \longrightarrow (6)$ and $X'' - kX = 0 \longrightarrow (7)$

The equations (6) & (7) are ordinary differential equations the solution of which depend on the value of k.

<u>Case (i)</u> : Let $k = \lambda^2$, a positive number.

∴ The differential equations (6) & (7) become

$X'' - \lambda^2 X = 0$ and $T' - \lambda^2\alpha^2 T = 0$.

ie, $\frac{d^2X}{dx^2} - \lambda^2 X = 0$.

$(D^2 - \lambda^2) X = 0$ where $D = \frac{d}{dx}$

The auxiliary equation is $m^2 - \lambda^2 = 0$

$m^2 = \lambda^2$

$m = \pm\lambda$.

$X = A_1 e^{\lambda x} + B_1 e^{-\lambda x}$

ie, $\frac{dT}{dt} - \lambda^2\alpha^2 T = 0$.

$\frac{dT}{dt} = \lambda^2\alpha^2 T$

$\frac{dT}{T} = \lambda^2\alpha^2 dt$

ie, integrating on b.s, we get

$\log T = \lambda^2\alpha^2 t + \log c_1$

Taking exponential on b's, we get

$T = c_1 e^{\lambda^2\alpha^2 t}$

Substituting the values of X and T in eqn (2), we get

$$u(x,t) = \left(A_1 e^{\lambda x} + B_1 e^{-\lambda x}\right)\left(c_1 e^{\lambda^2\alpha^2 t}\right)$$

<u>Case (ii)</u> : Let $k = -\lambda^2$, a negative number.

∴ The differential equations (6) & (7) becomes

$X'' + \lambda^2 X = 0$ and

ie, $\dfrac{d^2X}{dx^2} + \lambda^2 X = 0$

$\left(\dfrac{d^2}{dx^2} + \lambda^2\right) X = 0.$

$(D^2 + \lambda^2) X = 0$

The auxiliary eqns is

$m^2 + \lambda^2 = 0$

$m^2 = -\lambda^2$

$m = \pm i\lambda$

$\therefore X = A_2 \cos\lambda x + B_2 \sin\lambda x.$

$T' + \lambda^2 \alpha^2 T = 0.$

$\dfrac{dT}{dt} + \lambda^2\alpha^2 T = 0.$

$\dfrac{dT}{dt} = -\lambda^2\alpha^2 T$

$\dfrac{dT}{T} = -\lambda^2\alpha^2\, dt$

Int. on b.s, we get

$\log T = -\lambda^2\alpha^2 t + \log c_2$

Taking exponential on b.s we get

$T = c_2 e^{-\lambda^2\alpha^2 t}$

$\therefore u(x,t) = (A_2 \cos\lambda x + B_2 \sin\lambda x)\, c_2 e^{-\lambda^2\alpha^2 t}.$

Case (iii)

Let $k = 0.$

Then equations (6) & (7) becomes,

$X'' = 0$

$\dfrac{d^2X}{dx^2} = 0$

integrating w.r to 'x' twice, we get

$X = A_3 x + B_3$

$T' = 0.$

$\dfrac{dT}{dt} = 0.$

integrating w.r to 't', we get

$T = c_3.$

$\therefore u(x,t) = (A_3 x + B_3)\, c_3.$

Hence the possible solutions of ① are

$$u(x,t) = (A_1 e^{\lambda x} + B_1 e^{-\lambda x}) \cdot c_1 e^{\alpha^2 \lambda^2 t}$$

$$u(x,t) = (A_2 \cos\lambda x + B_2 \sin\lambda x)\, c_2 e^{-\alpha^2 \lambda^2 t}$$

$$u(x,t) = (A_3 x + B_3)\, C_3 .$$

<u>Note</u>:-

① Out of these three solutions we have to choose the correct solution which satisfies the physical nature of the problem.

here, we are dealing with the problem on heat conduction. According to the law of Thermodynamics, when time 't' increases the temperature $u(x,t)$ will not increase.

Now, Consider the solution

$$\underline{u(x,t) = (A_1 e^{\lambda x} + B_1 e^{-\lambda x})\, c_1 e^{\alpha^2 \lambda^2 t}}$$

here, if t increases then $u(x,t)$ is also increases.

ie, if we allow $t \to \infty$ then $u(x,t) \to \infty$

This is in contradiction with the law of Thermodynamics. Hence this solution is not suitable for our problems on heat conduction.

② At steady state conditions only we can use the solution (ie, when the temperature no longer varies with time)

$$\underline{u(x,t) = (A_3 x + B_3)\, C_3 .}$$

③ Therefore, the correct solution which is suitable for our problems on one dimensional heat flow is

$$\underline{u(x,t) = (A_2 \cos\lambda x + B_2 \sin\lambda x)\, c_2 e^{-\alpha^2 \lambda^2 t} .}$$