UNIT-II - FOURIER SERIES

* Fourier series is used to express any periodic function into simple periodic functions like sine and cosine terms.

Defn: PERIODIC FUNCTIONS

A function $f(x)$ is said to be periodic with period T, then $\boxed{f(x+T) = f(x)}$ for every x.

(Eg): (i) $\sin\theta$ and $\cos\theta$ are periodic functions with period 2π. If $f(\theta) = \sin\theta$

i.e., $f(\theta+2\pi) = \sin(\theta+2\pi) = \sin\theta = f(\theta)$

(ii) If $f(\theta) = \tan\theta$, then

$$f(\theta+\pi) = \tan(\theta+\pi) = \tan\theta = f(\theta)$$

so, $\tan\theta$ is periodic with period π.

Fourier series:

A function $f(x)$ can be expressed in the ($c < x < c+2L$) (period $= 2L$) form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{L}\int_c^{c+2L} f(x)\,dx \\ a_n &= \frac{1}{L}\int_c^{c+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ b_n &= \frac{1}{L}\int_c^{c+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{aligned}\right\} \quad (*)$$

Note: $(c < x < c+2L) =$ c ——— c+2L (2L period)

(i) If $c = 0$, $(0 < x < 2L)$

(ii) If $c = -\pi$, $(-\pi < x < \pi)$

(iii) If $c = 0$, $L = \pi$, $(0 < x < 2\pi)$

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(i) If $c=0$, $(0<x<2L)$, Period $=2L$
i.e., $2T=2L$
$\boxed{T=L}$

Always calculate Half period is what?

Then
$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

The coeffs: a_0, a_n, b_n all are as like (*).

(ii) If $c=-\pi$, $(-\pi<x<\pi)$, period $=2\pi$, $2L=2\pi$
$\boxed{L=\pi}$

then
$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\cancel{\pi}x}{\cancel{\pi}}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\cancel{\pi}x}{\cancel{\pi}}\right)$$
$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(nx)+\sum_{n=1}^{\infty}b_n\sin(nx)$$

where
$$\left.\begin{aligned}a_0&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx\\ a_n&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx\\ b_n&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx\end{aligned}\right\}\quad (A)$$

(iii) If $c=0$, $L=\pi$, $(0<x<2\pi)$, period $=2\pi$
$\boxed{L=\pi}$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(nx)+\sum_{n=1}^{\infty}b_n\sin(nx)$$

when the Coeffs are
$$\left.\begin{aligned}a_0&=\frac{1}{\pi}\int_{0}^{2\pi}f(x)\,dx\\ a_n&=\frac{1}{\pi}\int_{0}^{2\pi}f(x)\cos nx\,dx\\ b_n&=\frac{1}{\pi}\int_{0}^{2\pi}f(x)\sin nx\,dx\end{aligned}\right\}\quad (B)$$

DID THE FOURIER SERIES CONVERGES TO $f(x)$?

Ans: yes, but the function $f(x)$ should satisfy the Dirichlet's conditions.

Dirichlet's conditions:

If a function $f(x)$ is defined in $c < x < c+2\pi$ then $f(x)$ can be expanded in the form of

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx),$$

provided the following conditions are satisfied

(i) $f(x)$ is single valued and finite in $(c, c+2\pi)$

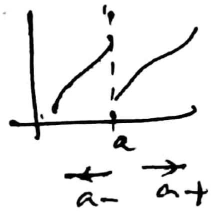
(ii) $f(x)$ is continuous (or) piecewise continuous with finite number of discontinuities in $(c, c+2\pi)$

(iii) $f(x)$ has finite number of max/min in $(c, c+2\pi)$

If these conditions are satisfied, the Fourier series will converge to $f(x)$ in $(c, c+2\pi)$

Function evaluation at the discontinuous points

(i) If $x=a$ is discontinuous point in $(c, c+2\pi)$ then $f(a) = \dfrac{f(a^-)+f(a^+)}{2}$



Function evaluation at the end points of $(c, c+2\pi)$

(i) In $(-\pi, \pi) \Rightarrow f(-\pi) = \dfrac{f(-\pi)+f(\pi)}{2}$

(ii) In $(0, 2L) \Rightarrow f(2L) = \dfrac{f(0)+f(2L)}{2}$

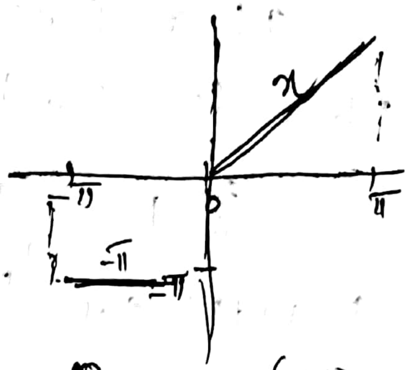
PROBLEMS:

1. Obtain the Fourier series of the periodic function

$$f(x) = \begin{cases} -\pi ; & -\pi < x < 0 \\ x ; & 0 < x < \pi \end{cases}$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Solution:



Total length $= (-\pi, \pi)$

$\therefore$ period $= 2\pi$

$2L = 2\pi$

$\boxed{L = \pi}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)} \rightarrow (1)$$

To find: a_0

$$a_0 = \frac{1}{L}\int_c^{c+2L} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} (-\pi)\,dx + \int_0^{\pi} x\,dx\right\}$$

$$= \frac{1}{\pi}\left\{(-\pi)(x)_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{(-\pi)(0-(-\pi)) + \left(\frac{\pi^2}{2} - 0\right)\right\}$$

$$= \frac{1}{\pi}\left\{-\pi^2 + \frac{\pi^2}{2}\right\} = \frac{1}{\pi}\left(\frac{-\pi^2}{2}\right) = \frac{-\pi}{2}$$

To find: a_n

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left\{ \int_{-\pi}^{0} (-\pi)\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx \right\}$$

$$= \frac{1}{\pi}\left\{ (-\pi)\cancel{\left(\frac{\sin nx}{n}\right)}_{-\pi}^{0} + \left[(x)\cancel{\left(\frac{\sin nx}{n}\right)} - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi} \right\}$$

$$= \frac{1}{\pi}\left\{ \frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2} \right\}$$

$$= \frac{1}{\pi}\left\{ \frac{(-1)^n - 1}{n^2} \right\} = \frac{1}{\pi n^2}\left\{ (-1)^n - 1 \right\}$$

To find: b_n

$$b_n = \frac{1}{L}\int_{c}^{c+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{ \int_{-\pi}^{0} (-\pi)\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx \right\}$$

$$= \frac{1}{\pi}\left\{ (-\pi)\left(-\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left[(x)\left(\frac{-\cos nx}{n}\right) - (1)\cancel{\left(\frac{\sin nx}{n^2}\right)} \right]_0^{\pi} \right\}$$

$$= \frac{1}{\pi}\left\{ \left(\frac{\pi}{n} - \frac{\pi\cos n\pi}{n}\right) + \left[\left(-\pi\frac{\cos n\pi}{n} - 0\right)\right] \right\}$$

$$= \frac{1}{\pi}\left\{ \frac{\pi}{n} - \frac{2\pi\cos n\pi}{n} \right\}$$

$$= \frac{1}{\pi n}\left(\pi - 2\pi(-1)^n\right) = \frac{1}{n}\left(1 - 2(-1)^n\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$= \frac{-\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{\pi n^2}\left((-1)^n - 1\right)\cos nx + \sum_{n=1}^{\infty} \frac{1}{n}\left(1 - 2(-1)^n\right)\sin nx.$$

$$f(x) = -\frac{\pi}{4} + \frac{1}{\pi}\left[-\frac{2}{1^2}\cos x - \frac{2}{3^2}\cos 3x - \cdots\right] +$$
$$\left[3\sin x - \frac{\sin 2x}{2} + 3\cdot\frac{\sin 3x}{3} - \cdots\right]$$

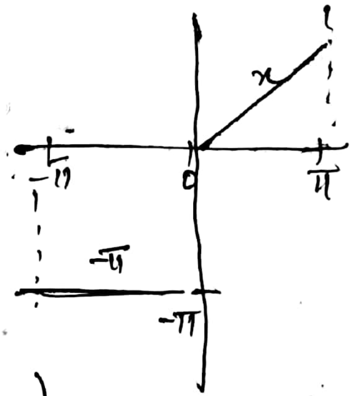
$$\boxed{f(x) = -\frac{\pi}{4} + \left(-\frac{2}{\pi}\right)\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \cdots\right] + \left[3\sin x - \frac{\sin 2x}{2} + \cdots\right]} \longrightarrow (2)$$

Deduction:

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

[It is a trial to check, at what value of x will we get the required deduction?]

For this problem, At $x=0$, it is possible to get the required deduction.



$x=0$ is discontinuous point

$$f(0) = -\frac{\pi}{4} + \left(\frac{-2}{\pi}\right)\left[\frac{1}{1^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{f(0^-) + f(0^+)}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \cdots\right)$$

$$\left(\frac{-\pi + 0}{2}\right) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \cdots\right)$$

$$-\frac{\pi}{2} + \frac{\pi}{4} = -\frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \cdots\right)$$

Simplify $\quad -\frac{\pi}{4} \times \frac{\pi}{-2} = \left(\frac{1}{1^2} + \frac{1}{3^2} + \cdots\right)$

$$\Rightarrow \boxed{\frac{\pi^2}{8}}$$

2. Find the Fourier series of $f(x)=(x+x^2)$ in $(-\pi,\pi)$ of periodicity 2π and hence deduce that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

Sol:

period $= 2L = 2\pi \Rightarrow \boxed{L=\pi}$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \longrightarrow (1)$$

where $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$$

To find: a_0

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx=\frac{1}{\pi}\left\{\frac{x^2}{2}+\frac{x^3}{3}\right\}_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left\{\left(\cancel{\frac{\pi^2}{2}}+\frac{\pi^3}{3}\right)-\left(\cancel{\frac{\pi^2}{2}}-\frac{\pi^3}{3}\right)\right\}$$

$$=\frac{1}{\pi}\left\{\frac{2\pi^3}{3}\right\}=\frac{2\pi^2}{3}$$

To find: a_n

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx$$

$$=\frac{1}{\pi}\left\{(x+x^2)\cancel{\left(\frac{\sin nx}{n}\right)}^{0}-(1+2x)\left(-\frac{\cos nx}{n^2}\right)+(2)\cancel{\left(-\frac{\sin nx}{n^3}\right)}^{0}\right\}_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left\{(1+2x)\left(\frac{\cos nx}{n^2}\right)\right\}_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left\{(1+2\pi)\frac{\cos n\pi}{n^2}-(1-2\pi)\left(\frac{\cos(-n\pi)}{n^2}\right)\right\}$$

$$=\frac{1}{\pi}\left\{\cancel{\frac{\cos n\pi}{n^2}}+\frac{2\pi\cos n\pi}{n^2}-\cancel{\frac{\cos n\pi}{n^2}}+\frac{2\pi\cos n\pi}{n^2}\right\}$$

$$= \frac{1}{\cancel{\pi}}\left\{\frac{4\cancel{\pi}(-1)^n}{n^2}\right\}$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

To find: b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{(x+x^2)\left(\frac{-\cos nx}{n}\right) - (1+2x)\left(\cancel{\frac{-\sin nx}{n^2}}\right) + (2)\left(+\frac{\cos nx}{n^3}\right)\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[(\pi+\pi^2)\left(-\frac{\cos n\pi}{n}\right) - (-\pi+\pi^2)\left(-\frac{\cos n\pi}{n}\right)\right] + 2\left(\cancel{\frac{\cos n\pi}{n^3}} - \cancel{\frac{\cos n\pi}{n^3}}\right)\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{\cos n\pi}{n}\left(\pi+\cancel{\pi^2}+\pi-\cancel{\pi^2}\right)\right\}$$

$$= \frac{1}{\cancel{\pi}}\left\{-\frac{2\cancel{\pi}\cos n\pi}{n}\right\}$$

$$\boxed{b_n = \frac{-2(-1)^n}{n}}$$

From ①

$$f(x) = \frac{2\pi^2/3}{2} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty}\frac{-2(-1)^n}{n}\sin nx$$

$$= \frac{\pi^2}{3} + 4\left[\frac{(-1)}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{(-1)}{3^2}\cos 3x + \cdots\right] + -2\left[\frac{(-1)}{1}\sin x + \frac{1}{2}\sin 2x - \cdots\right]$$

Deduction: $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$

At $x=0$, we can't get all terms are positive

At $x=\pi$, we will get the required deduction

At $x=\pi$

$$f(\pi) = \frac{\pi^2}{3} + 4\left[\frac{(-1)}{1^2}\cos\pi + \frac{1}{2^2}\cos 2\pi + \frac{(-1)}{3^2}\cos 3\pi + \cdots\right] + 0$$

$$f(\pi) = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$x=\pi$ is the end point.

$f(x) = x + x^2$

$f(-\pi) = -\pi + \pi^2$

$f(\pi) = \pi + \pi^2$

$$\frac{f(-\pi)+f(\pi)}{2} = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \cdots\right]$$

$$\frac{(-\cancel{\pi}+\pi^2)+(\cancel{\pi}+\pi^2)}{2} = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \cdots\right]$$

$$\frac{\cancel{2}\pi^2}{\cancel{2}} = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \cdots\right]$$

$$\frac{1}{4}\left(\pi^2 - \pi^2/3\right) = \frac{1}{1^2} + \frac{1}{2^2} + \cdots$$

$$\Rightarrow \boxed{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}}$$