18MAB201T
Transforms and Boundary Value Problems
Unit-3

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1. What is a Partial Differential Equation?

You've probably all seen an ordinary differential equation (ODE); for example the pendulum equation,

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0, \qquad (1.1)$$

describes the angle, θ, a pendulum makes with the vertical as a function of time, t. Here g and L are constants (the acceleration due to gravity and length of the pendulum respectively), t is the **independent variable** and θ is the **dependent variable**. This is an ODE because there is only one independent variable, here t which represents time.
A partial differential equation (PDE) relates the partial derivatives of a function of two or more independent variables together. For example, Laplace's equation for $\phi(x, y)$,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad (1.2)$$

arises in many places in mathematics and physics. For simplicity, we will use subscript notation for partial derivatives, so this equation can also be written $\phi_{xx} + \phi_{yy} = 0$.
We say a function is a **solution** to a PDE if it satisfy the equation and any side conditions given. Mathematicians are often interested in if a solution **exists** and when it is **unique**.

2. Classifying PDE's: Hyperbolic, Parabolic, Elliptic

Classify the following PDE's

1. $3u_{xx} - 4u_{xy} - 3u_y = 0$.
   Ans.: $B^2 - 4AC = 16 > 0$. $\therefore$ The PDE is hyperbolic $\forall\, x, y$.

2. $y^2u_{xx} - 2xyu_{xy} + x^2u_{yy} - 3u_x = 0$.
   Ans.: $B^2 - 4AC = 0 \;\forall\, x$ & y. $\therefore$ Parabolic in this case.

3. $x^2u_{xx} + 2xyu_{xy} + (1 + y^2)u_{yy} - 2u_x = 0$.
   Ans.: $B^2 - 4AC = -4x^2 < 0$ for $x < 0$ or $x > 0$. $\therefore$ Elliptic in this case.
   If $x = 0$, then $B^2 - 4AC = 0$. $\therefore$ Parabolic in this case.

4. $x^2u_{xx} + (1 - y^2)u_{yy} = 0$.
   Ans.: $B^2 - 4AC = 4x^2(1 - y^2) > 0 \;\; \forall x, x \neq 0$ and $y > 1, y < -1$. $\therefore$
   The PDE is hyperbolic in this case.
   $B^2 - 4AC = 4x^2(1 - y^2) < 0 \;\; \forall x, x \neq 0$ and $-1 < y < 1$. $\therefore$ The PDE is elliptic.
   If $x = 0 \;\forall y$ or $\forall x, y = \pm 1,$ then $B^2 - 4AC = 0$. $\therefore$ The PDE is parabolic in this case.

Example : 1

Classify the follow differential equations as PDE's hyperbolic, parabolic, elliptic.

1. The **diffusion equation** for $u(x,t)$:

$$u_t = c^2 u_{xx}$$

2. The **wave equation** for $w(x,t)$:

$$w_{tt} = c^2 w_{xx}$$

3. The **thin film equation** for $u(x,t)$:

$$u_t = -(u u_{xxx})_x$$

4. The **forced harmonic oscillator** for $y(t)$:

$$y_{tt} + \omega^2 y = F \cos(\Omega t)$$

5. The **Poisson Equation** for the electric potential $\phi(x,y,z)$:

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 4\pi\rho(x,y,z)$$

where $\rho(x,y,z)$ is a known charge density.

6. **Burger's equation** for $u(x,t)$:

$$u_t + u u_x = \nu u_{xx}$$

3. Examples of Wave Equations in Different Situations

As we have seen before the "classical" one-dimensional wave equation has the form:

$$u_{tt} = c^2 u_{xx}, \tag{3.1}$$

where $u = u(x,t)$ can be thought of as the vertical displacement of the vibration of a string.

The string can be fixed at both ends, or just at one end, or we can think of an "infinite" string, that is not bound at any end. Each will yield different boundary conditions for the well-posed wave equation. We can also consider the case where the string is "pushed" with an external force $h(x,t)$, or where we take under consideration the friction coefficient from the air that the string displaces. These two equations will be called "forced" and "damped" respectively. In the "forced" case, the wave equation is:

$$u_{tt} = c^2 u_{xx} + h(x,t),$$

where an example of the acting force is the gravitational force. In the "damped" case the equation will look like:

$$u_{tt} + k u_t = c^2 u_{xx},$$

where k can be the friction coefficient.

If we have more than one spatial dimension (a membrane for example), the wave equation will look a bit different. In the case of the vibrating membrane we have two spatial variables and the wave equation will look like:

$$u_{tt} = c^2 (u_{xx} + u_{yy}).$$

For $n-$dimensions (whatever THAT means...) the n-wave equation will be:

$$u_{tt} = c^2 (u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n}).$$

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The initial conditions for the one-dimensional wave equation will be:

(i) $u(x, 0) = f(x)$

(ii) $u_t(x, 0) = g(x)$.

For the finite string the boundary conditions will be:

(i) $u(0, t) = A(t)$

(ii) $u(L, t) = B(t)$.

So, altogether we have 4 conditions which are repeated below together.

(i) $u(x, 0) = f(x)$

(ii) $u_t(x, 0) = g(x)$

(iii) $u(0, t) = A(t)$

(iv) $u(L, t) = B(t)$.

4. Solution of 1-D Wave Equation (Dirichlet Problem)

If we tie the string at both ends we can have the following boundary conditions:

$$u(0,t) = A(t), u(L,t) = B(t),$$

where A, B are $\mathcal{C}^1$ piecewise functions. For example, we can have a sinusoidal function at one end and a Heaviside function at the other.
When the boundary values A and B are 0 we obtain the **Dirichlet Problem** for the wave equation:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, & 0 < x < L, t > 0 & \quad \text{DE} \\ u(0,t) = 0, u(L,t) &= 0, & t > 0 & \quad \text{BC} \\ u(x,0) = f(x), u_t(x,0) &= g(x) & 0 < x < L & \quad \text{IC.} \end{aligned}$$

As you have seen in Lecture 5 for the diffusion equation, the method of separating the variables is a very convenient way to obtain solutions for PDEs. In the case of the Dirichlet Problem we will quickly review the method.

Theorem 4.1. *A solution of the problem:*

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, & 0 < x < L, t > 0 & \quad \text{DE} \\ u(0,t) = 0, u(L,t) &= 0, & t > 0 & \quad \text{BC} \\ u(x,0) = f(x), u_t(x,0) &= g(x) & 0 < x < L & \quad \text{IC.} \end{aligned}$$

is given by:

$$u(x,t) = \sum_{n=1}^{N} \left[\frac{L}{n\pi c} \bar{A}_n \sin(\frac{n\pi ct}{L}) + B_n \cos(\frac{n\pi ct}{L}) \right] \sin(\frac{n\pi x}{L}),$$

where: $f(x) = \sum_{n=1}^{N} B_n \sin(\frac{n\pi x}{L})$, *and* $g(x) = \sum_{n=1}^{N} \bar{A}_n \sin(\frac{n\pi x}{L})$. *The coefficients* $\bar{A}_n$ *and* B_n *are given by:*

$$B_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L}), \bar{A}_n = \frac{2}{L}\int_0^L g(x)\sin(\frac{n\pi x}{L}).$$

Proof. We will use the method of separation of variables, namely think of the solution $u(x,t)$ as a product of a function that depends only on the variable t and of a function that depends only on the variable x.

Let

$$u(x,t) = X(x)T(t)$$

and substitute in the equation

$$u_{tt} = c^2 u_{xx},$$

we obtain:

$$X(x)\ddot{T}(t) = c^2\ddot{X}(x)T(t),$$

or

$$\frac{\ddot{T}(t)}{c^2T(t)} = \frac{\ddot{X}(x)}{X(x)},$$

thus the equality is one of functions of different variables, so both quotients have to be constant.

Say

$$\frac{\ddot{T}(t)}{c^2T(t)} = \frac{\ddot{X}(x)}{X(x)} = k = \pm p^2,$$

then we can solve each ordinary differential equation separately. We have the following three cases: $k = -p^2$, $k = p^2$, and $k = p = 0$.

Case 1: When the constant is $-p^2$, then the solutions for

$$\frac{\ddot{X}(x)}{X(x)} = -p^2,$$

are:

$$X(x) = c_1 \sin(px) + c_2 \cos(px),$$

and the solutions for

$$\frac{\ddot{T}(t)}{c^2 T(t)} = -p^2,$$

are:

$$T(t) = d_1 \sin(pct) + d_2 \cos(pct).$$

Then

$$u(x,t) = [d_1 \sin(pct) + d_2 \cos(pct)][c_1 \sin(px) + c_2 \cos(px)].$$

Case 2: When the constant is p^2, then the solutions for

$$\frac{\ddot{X}(x)}{X(x)} = p^2,$$

are:

$$X(x) = c_1 e^{px} + c_2 e^{-px},$$

and the solutions for

$$\frac{\ddot{T}(t)}{c^2 T(t)} = p^2,$$

are $\pm cp$, are

$$T(t) = d_1 e^{pct} + d_2 e^{-pct}.$$

Then

$$u(x,t) = (d_1 e^{pct} + d_2 e^{-pct})(c_1 e^{px} + c_2 e^{-px}).$$

Case 3: When the constant is 0, then the equations become

$$\ddot{X}(x) = \ddot{T}(t) = 0,$$

and $X(x) = c_1 x + c_2$, and $T(t) = d_1 t + d_2$. Then

$$u(x,t) = (d_1 t + d_2)(c_1 x + c_2).$$

To confirm which is most suitable solution, we may need the help of the actual nature of the solutions. To check that we may proceed with boundary conditions. Let's take a look at the boundary conditions: $u(0,t) = 0, u(L,t) = 0.$
The only solution for $u(x,t)$ that can satisfy them is

$$u(x,t) = (d_1 \sin(pct) + d_2 \cos(pct))(c_1 \sin(px) + c_2 \cos(px)),$$

and the boundary conditions translate into:

$$\begin{aligned} (d_1 \sin(pct) + d_2 \cos(pct))(c_1 \sin(0) + c_2 \cos(0)) &= 0 \\ (d_1 \sin(pct) + d_2 \cos(pct))(c_1 \sin(pL) + c_2 \cos(pL)) &= 0, \quad \forall t > 0, \end{aligned}$$

namely:

$$\begin{aligned} c_2 &= 0 \\ c_1 \sin(pL) &= 0. \end{aligned}$$

From the last condition we obtain $p = \dfrac{\pi n}{L}$, and

$$u(x,t) = \sum_{n=1}^{\infty} \left[d_{1n} \sin(\frac{\pi n}{L} ct) + d_{2n} \cos(\frac{\pi n}{L} ct) \right] c_n \sin(\frac{\pi n}{L} x).$$

Now, the conditions left to check are the initial conditions:

$$\begin{aligned} u(x,0) &= f(x) = \sum_{n=1}^{N} B_n \sin(\frac{n\pi x}{L}), \\ u_t(x,0) &= g(x) = \sum_{n=1}^{N} \bar{A}_n \sin(\frac{n\pi x}{L}). \end{aligned}$$

Then $u(x,t) = \displaystyle\sum_{n=1}^{N} \left[\frac{L}{n\pi c} \bar{A}_n \sin(\frac{n\pi ct}{L}) + B_n \cos(\frac{n\pi ct}{L}) \right] \sin(\frac{n\pi x}{L})$,

$\square$

Remark 4.0.1. *In the more general case for the Dirichlet Problem, when initial conditions (IC) change to more general homogeneous conditions:* $k_1 u(x,0) + k_2 u_x(x,0) = 0$, *we can solve the problem in the same manner, using separation of variables.*

Exercise: 4.1. *Check that* **Case 1** *is the only one that verifies the boundary conditions in the proof above.*

Exercise: 4.2. *Check that the solution found above verifies the initial conditions.*

Exercise: 4.3. *A string of length* π *is held fixed at both endpoints. Its initial position is* $f(x) = \sin(x)$ *and its initial velocity is* $g(x) = \cos x$. *Assuming that* $c = 1$, *find the position of the string* $u(x,t)$ *for every* $x \in [0,\pi]$ *and for every* $t > 0$. *Find an approximate value for* $u(x,t)$ *by adding several terms of the series. Animate the approximation and draw a* $3D$ *plot.*

Exercise: 4.4. *Solve the following problem for the string equation:*

$$\text{PDE} \qquad u_{tt} = u_{xx},$$

$$\text{BC} \qquad u(0,t) = 0, \quad u_x(\pi,t) = 0 \quad \text{for every } t > 0;$$

$$\text{IC} \qquad u(x,0) = \sin(x), \quad u_t(x,0) = 0, \quad \text{for every } x \in [0,\pi].$$

Notice the change in the boundary conditions. This will lead to different eigenvalues and eigenfunctions.

Use Maple to animate the solution you found, to draw a $3D$ *plot, and to check that the solution satisfies the conditions of the problem.*

5. The Plucked String as an Example for 1-D Wave Equation

The **plucked string** refers to the initial condition for the Dirichlet problem , where $f(x)$ looks like a "plucked string", namely:

$$\begin{array}{rcl} u_{tt} = c^2 u_{xx}, & 0 < x < L, t > 0 & \text{DE} \\ u(0,t) = 0, u(L,t) = 0, & t > 0 & \text{BC} \\ u(x,0) = f(x), u_t(x,0) = 0 & 0 < x < L & \text{IC}, \end{array}$$

where

$$f(x) = \begin{cases} \dfrac{u_0}{x_0} x, & 0 \le x \le x_0 \\ u_0 \dfrac{x - L}{x_0 - L}, & x_0 \le x \le L \end{cases}$$

We have that:

$$\dot{f}(x) = \begin{cases} \dfrac{u_0}{x_0}, & 0 \le x \le x_0 \\ \dfrac{u_0}{x_0 - L}, & x_0 \le x \le L, \end{cases}$$

and $f(0) = f(L) = 0$, thus:

$$\begin{aligned} B_n &= \frac{2}{L}\left(\frac{L}{n\pi}\right)\int_0^{x_0}\frac{u_0}{x_0}\cos\left(\frac{n\pi x}{L}\right)dx + \frac{2}{L}\left(\frac{L}{n\pi}\right)\int_{x_0}^{L}\frac{u_0}{x_0 - L}\cos\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2}{L}\left(\frac{L}{n\pi}\right)^2\left(\frac{u_0}{x_0}\sin\left(\frac{n\pi x_0}{L}\right) - \frac{u_0}{x_0 - L}\sin\left(\frac{n\pi x_0}{L}\right)\right) \\ &= \frac{2L^2 u_0}{\pi^2 x_0 (L - x_0)}\frac{1}{n^2}\sin\left(\frac{n\pi x_0}{L}\right). \end{aligned}$$

Now we can write the formal solution to the plucked string equation:

$$u(x,t) = \frac{2L^2 u_0}{\pi^2 x_0 (L - x_0)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi x_0}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right).$$

5.1. Musical Instruments (Vibrating Strings)

Many instruments produce sound by making strings vibrate; such are the harp, the piano, the harpsichord, the guitar, the violin, and others. Strings are kept fixed at the endpoints, but they way the instruments are played create different initial conditions. In instruments like the guitar, the string is plucked; this produces an initial perturbation with no initial velocity. In the piano, on the other hand, the string is hit, which creates an initial velocity but no initial perturbation from the initial position.
The oscillations of the string are described by

$$u(x,t) = \sum_{n=1}^{\infty}(A_n \cos\omega_n t + B_n \sin\omega_n t)\sin p_n t,$$

where

$$p_n = n\frac{\pi}{L} \qquad \text{and} \qquad \omega_n = cp_n = cn\frac{\pi}{L}.$$

The sound we hear is thus a combination of the main harmonic sounds (eigenfunctions)

$$u_n(x,t) = (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin p_n t.$$

The contribution of each particular harmonic is measured by its *energy*, which turns out to be equal to:

$$E_n = \frac{\omega_n^2 M}{4}(A_n^2 + B_n^2),$$

where $M = DL$ is the total mass of the string (recall that D was the density).
For the plucked string (Section 7.5), the energy is given by:

$$E_n = \frac{M u_0^2 L^2 c^2}{n^2 \pi^2 x_0^2 (L - x_0)^2} \sin^2 \frac{\pi n x_0}{L}.$$

The energy decreases as n^{-2}, so only the main tone u_1 and a few other harmonics are audible.
On the other hand, if we hit the string with a flat hammer of length 2δ with center at x_0 and producing an initial velocity v_0, the energy of the nth harmonic is

$$E_n = \frac{4MV_0^2}{n^2}\pi^2 \sin^2 \frac{\pi n x_0}{L} \sin^2 \frac{\pi n \delta}{L},$$

and the energy again decreases as n^{-2}. However, if the hammer is sufficiently narrow, letting δ tend to zero (the blade of a knife), we get the model of a string getting an impulse concentrated at a point x_0. The corresponding energy is:

$$E_n = \frac{v_0^2}{L} \sin^2 \frac{\pi n x_0}{L}.$$

Thus, for a very narrow hammer the energies of all harmonics are of the same order and the generated sound will be saturated with harmonics. This can be checked experimentally, by hitting a string with the blade of a knife. The sound will have a metallic quality.

Not all harmonics are desirable. The first ones, u_2 up to u_6, sound well together with the main harmonic u_1. However, the 7th and the first harmonics sounding together produce a sense of dissonance.

There are several ways to try to "kill" those harmonics by percussion (as in the piano).

a). The position of the hammer. The presence of the factor $\sin \dfrac{\pi n x_0}{L}$ shows that by choosing the center x_0 of the hammer at the node of the undesired harmonic we may make it disappear (make the corresponding A_n and B_n be equal to zero). In modern pianos the position of the hammer is chosen near the nodes of the 7th and the 8th harmonics, to "kill" them.

b). The shape of the hammer. In modern pianos the hammers are not flat, but rather round. One can model this situation by choosing the initial velocity to be, say, a parabola on the interval $[x_0 - \delta, x_0 + \delta]$, instead of a horizontal line. Older pianos, which had flatter and narrower hammers, produced a more piercing, shrilled sound.

c). The rigidity of the hammer. If instead of being rigid the hammer is softer. In this case the motion is not described by its initial position and velocity, but rather by a short-time acting force, which varies in time.

Exercise: 5.1. *Find the solution $u(x,t)$ of the string equation if $l = \pi$, $c = 1$, when both endpoints are fixed, the initial velocity is zero, and $u(x,t)$ is the following piecewise linear function:*

6. Uniqueness of the Solution to the 1-D Wave Equation

Theorem 6.1. *Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of the problem:*

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, & 0<x<L, t>0 && \text{DE} \\ u(0,t) = A(t), u(L,t) &= B(t), & t>0 && \text{BC} \\ u(x,0) = f(x), u_t(x,0) &= g(x) & 0<x<L. && \text{IC}, \end{aligned}$$

where A, B, f, g are $\mathcal{C}^1$ piecewise continuous. Then $u_1(x,t) = u_2(x,t)$ for all points in the domain.

Proof. Let $v(x,t) = u_1(x,t) - u_2(x,t)$, then v satisfies the wave equation with initial conditions: $u(x,0) = u_t(x,0) = 0$, and boundary conditions $u(0,t) = u(L,t) = 0$. Our goal is to prove that $v(x,t) = 0 \quad \forall x, t$.
In order to accomplish this, define:

$$H(t) = \int_0^L \left[c^2 v_x(x,t)^2 + v_t(x,t)^2\right] dx.$$

We will prove that $H(t) = 0$ first. Differentiating with respect to t we obtain:

$$\begin{aligned} \dot{H}(t) &= \int_0^L \left[c^2 2 v_x v_{xt} + 2 v_t v_{tt}\right] dx \\ &= 2c^2 \int_0^L \left[v_x v_{xt} + v_t v_{xx}\right] dx \\ &= 2c^2 \int_0^L \frac{\delta}{\delta x}(v_x v_t) dx = 2c^2 \left[v_x(x,t) v_t(x,t)\right]_0^L \\ &= 2c^2 (v_x(L,t) v_t(L,t) - v_x(0,t) v_t(0,t)) = 0. \end{aligned}$$

Since $\dot{H}(t) = 0$, $H(t)$ is constant, and as $H(0) = 0$, we conclude that $H(t) = 0$.
Then $v_t(x,t) = 0$, and $v(x,t) = v(x,t) - v(x,0) = \int_0^L v_t(x,t)dt = 0$. □

Remark 6.0.1. *The energy integral of the string at time t is:*

$$E(t) = \int_0^L \left[T_0 u_x(x,t)^2 + D u_t(x,t)^2\right] dx,$$

where D is the mass per unit length and T_0 is the constant tension when the string is straight. We can see that the energy is proportional to H if we construct H using u instead of v. So the uniqueness proof comes from the conservation of energy for the unforced string.

Example : 2

A damped string of length 1 has equation

$$u_{tt} = c^2 u_{xx} - \gamma u_t,$$

where γ is a small damping coefficient. Find the solution $u(x,t)$ assuming that both endpoints are fixed, the initial condition is $x(1-x)$ and the initial velocity is zero.
To see the nature of the by visualization, we can plot and animate the solution for the case when $c = \dfrac{1}{4}$ and $\gamma = \dfrac{1}{5}$. But that is the scope our course. So whoever interested can do it later with the help of teacher.

Example : 3

Find the solution $u(x, y, t)$ of a square membrane with side 1 fixed on the boundary, if the initial position is $u(x, y, 0) = (x - x^2)(y - y^2)$ and the initial velocity is zero.
Same way as said in the previous problem, we can animate several eigenfunctions $u_{n,m}(x, y, t)$, say, $u_{1,1}$, $u_{1,2}$, $u_{3,5}$, assuming that $c = 1$.

Example : 4

Solve the string equation $u_{tt} = c^2 u_{xx}$ for $L = 1$, with the boundary conditions $u(0, t) = 0$ and $u(1, t) = 1$, with zero initial velocity, assuming that the initial position is

(a) $u(x, 0) = \sin x$ (easier),

(b) $u(x, 0) = x^2$ (harder).

Hint: You cannot use the superposition principle, since the boundary condition at $x = 1$ is not homogeneous. Try a change of coordinates first, $v(x, t) = u(x, t) + h(x)$, where $h(x)$ is a suitable (easy) function that would guarantee that v also satisfies the string equation, now with homogeneous boundary conditions.

Example : 5

Solve the string equation $u_{tt} = c^2 u_{xx}$ for $L = 1$, with the boundary conditions $u(0,t) = 0$ and $u_x(1,t) + u(1,t) = 0$. This corresponds to the case when the left end is fixed and the right end is attached to an elastic hinge. The initial conditions are $u(x,0) = x - \frac{2}{3}x^2$ and $u_t(x,0) = x$.

Note. This exercise is hard! The eigenvalues p_n will be solutions of a transcendental equation.

We will see Heat Equation now. After that we will have few problems worked out here. Then you can go through the books and solve related problems and discuss your doubts with the teacher.

7. Heat Flow (Heat Equation or Diffusion Equation)

A classical example of the application of ordinary differential equations is Newton's Law of Cooling which, basically, answers the question *"How does a cup of coffee cool?"* Newton hypothesized that the rate at which the temperature, $U(t)$, changes is proportional to the difference with the ambient temperature, which we call $\widehat{U}$,

$$\frac{dU}{dt} = -\kappa(U - \widehat{U}). \tag{7.1}$$

Here κ is a positive rate constant (with units of inverse time) that measures how fast heat is lost from the coffee cup to the ambient environment. If we specify the initial temperature,

$$U(0) = U_0, \tag{7.2}$$

we can solve for the evolution of the temperature,

$$U(t) = \widehat{U} + (U_0 - \widehat{U})e^{-\kappa t}. \tag{7.3}$$

If we graph the temperature as a function of time, we see that it decays exponentially to the ambient temperature, $\widehat{U}$, at a rate governed by κ.
When we derived Newton's Law of cooling we made several assumptions – most importantly that the temperature in the coffee cup did not vary with location. If we account for the variation of temperature with location, we can derive a PDE called the **heat equation** or, more generally, the **diffusion equation**. If the temperature, $U(x,t)$ is a function of a single spatial variable, x, we will show that it satisfies the diffusion equation,

$$U_t = c^2 U_{xx}, \tag{7.4}$$

where c is a constant known as the thermal diffusivity. In higher dimensions, the equation can be written

$$U_t = c^2 \nabla^2 U, \tag{7.5}$$

where ∇^2 is the **Laplacian**.

8. Solution to the Heat Equation (Diffusion Equation)

We can summarize the last section by restating a well-posed problem for the diffusion equation on the interval $0 < x < L$ with Dirichlet boundary conditions,

The Dirichlet Problem for the Diffusion Equation
(Non-homogeneous Boundary Conditions)

$$\begin{array}{rlcc} & U_t = c^2 U_{xx} & 0 < x < L, t > 0 & \text{DE} \\ U(0,t) = U_0 & U(L,t) = U_1 & t > 0 & \text{BC} \\ & U(x,0) = f(x) & 0 < x < L. & \text{IC} \end{array}$$

Solving the general problem will have to wait, but we can find some specific solutions to the problem using the ideas of **Separation of Variables**. For the moment, we will restrict ourselves to homogeneous boundary conditions,

The Dirichlet Problem for the Diffusion Equation
(Homogeneous Boundary Conditions)

$$\begin{array}{rlcc} & U_t = c^2 U_{xx} & 0 < x < L, t > 0 & \text{DE} \\ U(0,t) = 0 & U(L,t) = 0 & t > 0 & \text{BC} \\ & U(x,0) = f(x) & 0 < x < L, & \text{IC} \end{array}$$

If you want, you can skip the derivation for the moment and jump ahead to Exercise 1, if you don't mind the solution appearing *deus ex machina* (a fancy term for "out of thin air").

8.1. A Solution to the Homogeneous Dirichlet Problem

Let us look for solutions to the homogeneous Dirichlet problem of the form

$$U(x,t) = X(x)T(t) \tag{8.1}$$

we find from the differential equation (DE) that

$$XT_t = c^2 X_{xx} T \tag{8.2}$$

and dividing by XT we find

$$\frac{T_t}{c^2 T} = \frac{X_{xx}}{X} = k = \pm p^2, \tag{8.3}$$

where p is to be determined.
We may write the above equation as

$$\frac{\dot{T}(t)}{c^2 T(t)} = \frac{\ddot{X}(x)}{X(x)} = k = \pm p^2,$$

Now, we have to solve each ordinary differential equation separately. We have the following three cases: $k = -p^2$, $k = p^2$, and $k = p = 0$.
Case 1: When the constant is $-p^2$, then the solutions for

$$\frac{\ddot{X}(x)}{X(x)} = -p^2,$$

are:

$$X(x) = c_1 \sin(px) + c_2 \cos(px),$$

and the solutions for

$$\frac{\dot{T}(t)}{c^2T(t)} = -p^2,$$

are:

$$T(t) = d_1 e^{-p^2c^2t}.$$

Then

$$u(x,t) = [d_1 e^{-p^2c^2t}][c_1 \sin(px) + c_2 \cos(px)].$$

Case 2: When the constant is p^2, then the solutions for

$$\frac{\ddot{X}(x)}{X(x)} = p^2,$$

are:

$$X(x) = c_1 e^{px} + c_2 e^{-px},$$

and the solutions for

$$\frac{\dot{T}(t)}{c^2T(t)} = p^2,$$

are $\pm cp$, are

$$T(t) = d_1 e^{p^2c^2t}.$$

Then

$$u(x,t) = d_1 e^{p^2c^2t}(c_1 e^{px} + c_2 e^{-px}).$$

Case 3: When the constant is 0, then the equations become

$$\ddot{X}(x) = \dot{T}(t) = 0,$$

and $X(x) = c_1 x + c_2$, and $T(t) = d_1$. Then

$$u(x,t) = d_1(c_1 x + c_2).$$

To confirm which is most suitable solution, we may need the help of the actual nature of the solutions. To check that we may proceed with boundary conditions. Let's take a look at the boundary conditions: $u(0,t) = 0, u(L,t) = 0$.
The only solution for $u(x,t)$ that can satisfy them is

$$u(x,t) = d_1 e^{-p^2 c^2 t}(c_1 \sin(px) + c_2 \cos(px)).$$

Further, we can apply the boundary conditions we will proceed to get the solution.

9. Solved Problems/Examples

Example : 6

Write one dimensional wave equation and its possible solutions.

Hints/Solution:

One dimensional wave equation is given by $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ (or) $u_{tt} = c^2 u_{xx}$

Its possible solutions are given by

$(i)\ u(x,t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{pct} + c_4 e^{-pct})$,

$(ii)\ u(x,t) = (c_5 \cos px + c_6 \sin px)(c_7 \cos pct + c_8 \sin pct)$ and

$(iii)\ u(x,t) = (c_9 x + c_{10})(c_{11} t + c_{12})$.

Example : 7

Write one dimensional heat equation and its possible solutions.

Hints/Solution:

One dimensional heat equation is given by $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ (or) $u_t = c^2 u_{xx}$

Its possible solutions are given by

$(i)\ u(x,t) = (c_1 e^{px} + c_2 e^{-px}) e^{c^2 p^2 t}$,

$(ii)\ u(x,t) = (c_3 cospx + c_4 \sin px) e^{-c^2 p^2 t}$ and

$(iii)\ u(x,t) = (c_5 x + c_6)$.

Example : 8

In one dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ (or) $u_{tt} = c^2 u_{xx}$ what does c^2 stands for?.

Hints/Solution:

In one dimensional wave equation the value of c^2 is stands for $c^2 = \frac{T}{m}$, where T is the tension and m is mass per unit length.

Example : 9

In one dimensional heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ (or) $u_t = c^2 u_{xx}$ what does c^2 stands for?.

Hints/Solution:

In one dimensional heat equation the value of c^2 is stands for $c^2 = \frac{k}{\rho h}$, where k is thermal conductivity, ρ is the density of the material and h is the specific heat of the material.

Example : 10

The ends of a rod of length 20 cm are maintained at the temperature $10^\circ C$ and $20^\circ C$ respectively until steady state prevails. Determine the steady state temperature of the rod.

Hints/Solution:

1-D heat equation is $u_t = c^2 u_{xx}$.
In steady state, $u_{xx} = 0 \implies u(x) = ax + b$.
Applying given conditions, we get

$$b = u(0) = T_0 \text{ and } a = \frac{T_l - T_0}{l} = \frac{u(l) - u(0)}{l}$$

$$\implies u(x) = \frac{T_l - T_0}{l} x + T_0 - - - (1)$$

Here, $T_0 = 10^\circ C$, $T_l = 20^\circ C$ and $l = 20$.
$\therefore$ (1) becomes $u(x) = \frac{1}{2} x + 10$.

Example : 11

When the ends of a rod of length 30 cm are maintained at the temperature $20^\circ C$ and $80^\circ C$ respectively until steady state prevails. Determine the steady state temperature of the rod.

Hints/Solution:

1-D heat equation is $u_t = c^2 u_{xx}$.
In steady state, $u_{xx} = 0 \implies u(x) = ax + b$.
Applying given conditions, we get

$$b = u(0) = T_0 \text{ and } a = \frac{T_l - T_0}{l} = \frac{u(l) - u(0)}{l}$$

$$\implies u(x) = \frac{T_l - T_0}{l} x + T_0 - - - (1)$$

Here, $T_0 = 20^\circ C,\ T_l = 80^\circ C$ and $l = 30$.
$\therefore$ (1) becomes $u(x) = 2x + 20$.

Example : 12

A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $3x(l - x)$, find the displacement.

Hints/Solution:

One dimensional wave equation is given by $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ (or) $u_{tt} = c^2 u_{xx} - - -$ (1)
The boundary conditions becomes,
$(i)\ u(0, t) = 0,\ \ t \geq 0,$
$(ii)\ u(l, t) = 0\ \ t \geq 0,$
$(iii)\ u(x, 0) = 0,\ \ 0 \leq x \leq l$

(iv) $\dfrac{\partial u}{\partial t}(x,0) = 3x(l-x), \;\; 0 \le x \le l$

The suitable solution is given by
$u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pct + c_4 \sin pct) - - - (2)$.
Applying boundary conditions (BC's) (i) and (ii) in (2), we get

$$u(x,t) = \left(c_2 \sin \frac{n\pi x}{l}\right)\left[c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l}\right] - - - (3).$$

Applying boundary conditions (BC) (iii) in (3), we get

$$u(x,t) = \left(c_3 \sin \frac{n\pi x}{l}\right)\left[c_4 \sin \frac{n\pi ct}{l}\right] = c_n \sin \frac{n\pi x}{l} \sin \frac{n\pi ct}{l} - - - (4).$$

Since c_n is an arbitrary constant, and n is any integer and since (1) is homogeneous and linear, the most general solution of (1) becomes,

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} \sin \frac{n\pi ct}{l} - - - (5).$$

Using BC (iv), we have

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{l} \sin \frac{n\pi x}{l} = 3x(l-x), \;\; 0 \le x \le l$$

Expanding $3x(l-x)$ as Fourier series in $(0,l)$, we have

$$3x(l-x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \text{ then}$$

$$b_n = c_n \frac{n\pi c}{l} = \frac{2}{l}\int_0^l 3x(l-x)\, \sin \frac{n\pi x}{l}\, dx = \frac{12l^2}{n^3\pi^3}[1-(-1)^n] = \begin{cases} 0 & \text{if } n = \text{even} \\ \dfrac{24l^2}{n^3\pi^3} & \text{if } n = \text{odd} \end{cases}$$

$$\implies \quad c_n = \begin{cases} 0 & \text{if } n = \text{even} \\ \dfrac{24l^3}{cn^4\pi^4} & \text{if } n = \text{odd} \end{cases}$$

$$u(x,t) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{24l^3}{cn^4\pi^4} \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l}$$

$$i.e.\ u(x,t) = \sum_{r=1}^{\infty} \frac{24l^3}{c(2r-1)^4\pi^4} \sin\frac{(2r-1)\pi x}{l} \sin\frac{(2r-1)\pi ct}{l}$$

Example : 13

A rod of length l through which heat flows is insulated at its sides. The ends are kept at zero temperature. If the initial temperature at the interior points of the bar is given by $k\sin^3\left(\dfrac{\pi x}{l}\right)$, find the temperature distribution in the bar after time t.

Hints/Solution:

One dimensional heat equation is given by $\dfrac{\partial u}{\partial t} = c^2\dfrac{\partial^2 u}{\partial x^2}$ (or) $u_t = c^2 u_{xx} - - - (1)$

The boundary conditions becomes,

$(i)\ u(0,t) = 0,\ \ t \geq 0,$

$(ii)\ u(l,t) = 0\ \ t \geq 0,$

$(iii)\ u(x,0) = k\sin^3\left(\dfrac{\pi x}{l}\right),\ \ 0 \leq x \leq l$

The suitable solution is given by

$$u(x,t) = (c_1 \cos px + c_2 \sin px) e^{-c^2 p^2 t} - - - (2).$$

Applying boundary conditions (BC's) (i) and (ii) in (2), we get

$$u(x,t) = \left(c_2 \sin \frac{n\pi x}{l} \right) \left[e^{-c^2 \frac{n^2\pi^2}{l^2} t} \right] = c_n \sin \frac{n\pi x}{l} \left[e^{-c^2 \frac{n^2\pi^2}{l^2} t} \right] - - - (3).$$

Since c_n is an arbitrary constant, and n is any integer and since (1) is homogeneous and linear, the most general solution of (1) becomes,

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} \left[e^{-c^2 \frac{n^2\pi^2}{l^2} t} \right] - - - (4).$$

Using BC (iii), we have

$$k \sin^3 \left(\frac{\pi x}{l} \right) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \text{ then}$$

$$\frac{k}{4} \left[\sin \left(\frac{\pi x}{l} \right) - \sin \left(\frac{3\pi x}{l} \right) \right] = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l},$$

Equating like co-efficients and solving for b_n we get

$$u(x,t) = \frac{3k}{4} \sin \frac{\pi x}{l} e^{-c^2 \frac{\pi^2}{l^2} t} - \frac{k}{4} \sin \frac{3\pi x}{l} e^{-c^2 \frac{3^2\pi^2}{l^2} t}$$

Example : 14

The ends A & B of a rod of length 'ℓ' units long have their temperature kept at $0°C$ & $120°C$ until steady state condition prevails. The temperature of the end B is suddenly reduced to $0°C$ and that of A is maintained to $0°C$. Find the temperature distribution in the rod after time t.

Hints/Solution:

One dimensional heat equation is given by $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ (or) $u_t = c^2 u_{xx} - - - (1)$

In steady state, $u_{xx} = 0 \implies u(x) = ax + b$. Applying given conditions, we get $b = u(0) = T_0$ and $a = \dfrac{T_l - T_0}{l} = \dfrac{u(l) - u(0)}{l} \implies u(x) = \dfrac{T_l - T_0}{l} x + T_0 - - - (1)$

Here, $T_0 = 0°C,\ T_l = 120°C$ and $length = l$. $\therefore$ (1) becomes $u(x) = \dfrac{120}{l} x$.

The boundary conditions becomes,

$(i)\ u(0,t) = 0,\ \ t \geq 0,$

$(ii)\ u(l,t) = 0\ \ t \geq 0,$

$(iii)\ u(x,0) = \left(\dfrac{120x}{l}\right),\ \ 0 \leq x \leq l$

The suitable solution is given by

$u(x,t) = (c_1 \cos px + c_2 \sin px) e^{-c^2 p^2 t} - - - (2).$

Applying boundary conditions (BC's) (i) and (ii) in (2), we get

$$u(x,t) = \left(c_2 \sin \frac{n\pi x}{l}\right) \left[e^{-c^2 \frac{n^2 \pi^2}{l^2} t}\right] = c_n \sin \frac{n\pi x}{l} \left[e^{-c^2 \frac{n^2\pi^2}{l^2} t}\right] - - - (3).$$

Using BC (iii) and expanding $\dfrac{120x}{l}$ as Fourier series in $(0, l)$, we have

Ans.: $u(x,t) = \sum_{n=1}^{\infty} \frac{240}{n\pi}(-1)^{n+1} \sin \frac{n\pi x}{\ell} e^{-\frac{c^2n^2\pi^2 t}{\ell^2}}$
