Mathematical Induction

Let $S(n)$ denote a mathematical statement that involves one or more occurrences of the variable n, which represents a positive integer,

a) If $S(1)$ is true

b) If, whenever $S(k)$ is true for some particular, but arbitrarily chosen $k \in Z^+$, $S(k+1)$ is also true, then $S(n)$ is true for all $n \in Z^+$.

1. Prove by mathematical induction, that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}, \quad n \geq 1.$$

Let $S(n) = 1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$, $n \geq 1$.

Initial Step or Basis step

$$S(1) = 1 = \frac{1(1+1)}{2}$$

$$1 = \frac{1 \times 2}{2}$$

$$1 = 1.$$

$\therefore$ $S(1)$ is true.

Let us assume that $S(k)$ is true.

i.e $S(k) = 1 + 2 + \dots + k = \frac{k(k+1)}{2}$

To prove that : $S(k+1)$ is true.

$1 + 2 + 3 + \dots + k + k+1$

$= \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$.

$\Rightarrow S(k+1)$ is valid.

Hence $S(n)$ is true for all $n \geq 1$.

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2. Prove by mathematical induction that

$1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{1}{3} n(2n-1)(2n+1)$.

Let $S(n) = 1^2 + 3^2 + \dots + (2n-1)^2 = \frac{1}{3} n(2n-1)(2n+1)$.

when $n=1$.

$S(1): 1^2 = \frac{1}{\cancel{3}} \times 1 \times 1 \times \cancel{3}$

$1 = 1$.

$\Rightarrow S(1)$ is true.

Let us assume $S(k)$ is true.

$1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 = \frac{1}{3} k(2k-1)(2k+1)$.

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To prove that : $S(k+1)$ is true.

$1^2+3^2+5^2+ \cdots + (2k-1)^2 + (2k+1)^2$

$= \frac{1}{3}k(2k-1)(2k+1) + (2k+1)^2$

$= \frac{1}{3}(2k+1)\left[k(2k-1) + 3(2k+1)\right]$

$= \frac{1}{3}(2k+1)\left[2k^2 - k + 6k + 3\right]$

$= \frac{1}{3}(2k+1)(2k^2+5k+3)$

$= \frac{1}{3}(2k+1)(2k+3)(k+1)$

$= \frac{1}{3}(k+1)(2k+1)(2k+3)$

$\Rightarrow S(k+1)$ is valid.

Hence $S(n)$ is true for all $n \in Z^+$.

3. Use mathematical induction to show that

$n! \geq 2^{n-1}$ for $n = 1, 2, 3, \ldots$

Let $S(n): n! \geq 2^{n-1}$.

$\therefore S(1): 1! \geq 2^{1-1}$

$\Rightarrow 1 \geq 2^0$

$\Rightarrow 1 \geq 1$

$\Rightarrow S(1)$ is true.

Let us assume $S(k)$ is true.

i.e $k! \geq 2^{k-1}$ — ①

To prove that : $S(k+1)$ is true.

Now $(k+1)! = (k+1)\,k!$

$\geq (k+1)\,2^{k-1}$ [by ①]

$\geq 2 \cdot 2^{k-1}$ since $k+1 \geq 2$

$= 2^k$.

$k=1$ $2 \geq 2$

$k=2$ $3 \geq 2$

$\vdots$

$\Rightarrow S(k+1)$ is also true.

Hence $S(n)$ is true for $n = 1, 2, 3, \ldots$

4. Use mathematical induction to prove that $n^3 + 2n$ is divisible by 3, for $n \geq 1$.

Let $S(n) : (n^3 + 2n)$ is divisible by 3.

$\therefore S(1) : 1^3 + 2.1 = 1 + 2 = 3$ is divisible by 3, which is true.

Let us assume $S(k)$ is true.

i.e $k^3 + 2k$ is divisible by 3 — ①

To prove that : $S(k+1)$ is true.

$$(k+1)^3 + 2(k+1)$$

$$= k^3 + 3k^2 + 3k + 1 + 2k + 2$$

$$= k^3 + 3k^2 + 3k + 2k + 3$$

$$= k^3 + 2k + 3k^2 + 3k + 3$$

$$= \underbrace{k^3 + 2k}_{\text{divisible by 3 from (1)}} + \underbrace{3(k^2 + k + 1)}_{\text{This is also divisible by 3}}$$

$\Rightarrow (k+1)^3 + 2(k+1)$ is divisible by 3.

$\Rightarrow S(k+1)$ is true.

Hence $S(n)$ is true for $n \geq 1$.