Module 1 (18MAB203T)

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RANDOM VARIABLE

Random Variable in One Dimensional

A **random variable** (R.V.) is a real valued function defined over the sample space of an experiment.

A **random variable** is a function $X(s) = x$, which assigns a real number (x) to every element (s) of the sample space (S) corresponding to random experiment (E), *i.e.*

$$\begin{aligned} X &: S \to R. \\ X(s) &= x \in R. \end{aligned}$$

Example

Example

Consider an experiment of tossing 2 coins simultaneously with a random variable X denoting the number of heads. Then we define the following:

Let 2 coins be tossed simultaneously. Then
E : An experiment of tossing 2 unbiased coins.
S : Outcomes of the experiment E.

i.e. Sample space $= \{HH, HT, TH, TT\}$.

X : a random variable which denotes the number of heads.

$$
\begin{aligned}
X\,(HH) &= 2 = x_1 (\text{say}) \\
X\,(HT) &= 1 = x_2 \\
X\,(TH) &= 1 = x_2 \\
X\,(TT) &= 0 = x_3
\end{aligned}
$$

Random variable X takes set values $\{0,\ 1,\ 2\}$, which is range space of X denoted as R_X.

Note: Here X is a discrete random variable.

Types of Random Variable

Types of random variables

There are two types of random variables according to their range space. They are

1. **Discrete random variable** (D.R.V.)
2. **Continuous random variable** (C.R.V.)

Discrete Random Variable

If X is a random variable (R.V.) which can take a finite number or countably infinite number of values, X is called a **Discrete Random Variable**. When the R.V. is discrete, the possible values of X may be assumed as $x_1,\ x_2,\ ...,x_n,....$ In the finite case, the list of values terminates and in the countably infinite case, the list goes upto infinity.

Continuous Random Variable

If X is an R.V. which can take all values (*i.e.* infinite number of values) in an interval, then X is called **Continuous Random Variable**.

Discrete Random Variable

Example

1. The number of telephone calls received by the telephone operator.
2. The number of printing mistakes in a book.

Probability Mass Function

Let X be a discrete random variable which takes values $x_1, x_2, \cdots$. Each value is associated with probability $p_i = P(X = x_i)$, then p_i is called the **Probability Mass Function** of the random variable X, provided $p_i (i = 1,\ 2,\ \cdots)$ satisfies the following two conditions:

$$1.\ p_i \geq 0,\ \forall i$$

$$2.\ \sum_i p_i = 1$$

Note: The pair $\{x_i,\ p_i\}$ is called **probability distribution of the discrete random variable** X.

Cumulative Distribution Function of D.R.V.

Cumulative Distribution Function (C.D.F.)

The function $F(x)$ or $F_X(x)$ is called the **Cumulative Distribution function** of the discrete random variable X and is defined as:

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} p(x_i)$$

Properties of C.D.F.

1. $0 \leq F(x) \leq 1$
2. $F(x)$ is a non- decreasing function of x,
   i.e. $F(x_1) \leq F(x_2)$ if $x_1 < x_2$
3. $F(-\infty) = \lim_{x \to -\infty} F(x) = 0 = P(X \leq -\infty)$
   $F(\infty) = \lim_{x \to \infty} F(x) = 1 = P(X \leq \infty)$
4. If X is a discrete R.V. taking values $x_1, x_2, ...$, where $x_1 < x_2 < ....$, Then $P(X = x_i) = F(x_i) - F(x_{i-1})$.

Mathematical Expectation of Discrete Random Variable

Mean value of X

The value $E(X)$ is called the **Expectation of *X*** (or) **Expected value of *X*** (or) **Mean value of *X*** and is defined as:

Mean of $X = E(X) = \overline{X} = \sum_i x_i p(x_i)$, **if** X **is a discrete random variable**.

Properties of Mean value of X

1. $E(aX) = aE(X)$, where a is a constant.
2. $E(aX + b) = aE(X) + b$, where a and b are constants.
3. $E(k_1 X_1 + k_2 X_2 + \cdots + k_n X_n) = k_1 E(X_1) + k_2 E(X_2) + \cdots + k_n E(X_n)$, where $k_1, \ k_2, \ \cdots, \ k_n$ are constants.
4. If $X_1, \ X_2, \ \cdots, \ X_n$, are independent random variables, then

   $E(X_1 \ X_2 \cdots X_n) = E(X_1) E(X_2) \cdots E(X_n)$, Multiplication theorem on expectatio
5. $E\left[X - \overline{X}\right] = E[X - E(X)] = E(X) - E(X) = 0. \left[\overline{X} = E(X) = \text{Mean of } X\right]$

Variance and Properties of Discrete Random Variable

Variance

Variance of X:

$$\text{Var}\,(X) = \text{V}(X) \quad i.e.\ \sigma_X^2 = E\left(X - \overline{X}\right)^2, \text{ where } \overline{X} = E\,(X)$$

$$= \sum_x \left(x - \overline{X}\right)^2 p\,(x), \quad \text{if } X \text{ is a D.R.V. and } p(x) \text{ is P.M.F. of} X.$$

$$\text{(or)}\sigma_X^2 = E\left(X^2\right) - [E(X)]^2$$

$$= \sum_x x^2 p(x) - \left[\sum_x xp(x)\right]^2, \text{if } X \text{ is a D.R.V. and } p(x) \text{ is P.M.F. of } X.$$

Properties of Variance

1. $\text{Var}(X) \geq 0$
2. $\text{Var}(a) = 0$, where a is a constant.
3. $\text{Var}\,(a \pm bX) = \text{Var}\,(a) + b^2\,\text{Var}\,(X) = b^2\,\text{Var}\,(X)$, where a and b are constants.

Standard Deviation and Moments of Discrete Random Variable

Standard Deviation

Standard Deviation of

$$X = \sqrt{\mathrm{Var}\,(X)} = \sigma_X.$$

Moments

The expected value of an integral power of a random variable is called its moments.
Moments are classified as two types.

1. Moments about mean (μ).
2. Moments about any point (a).

Examples of Discrete R.V.

Example 1

From a lot containing 25 items, 5 of which are defective, 4 items are chosen at random. If X is the number of defective found, obtain the probability distribution of X, when the items are chosen
(i) **without replacement and**
(ii) **with replacement.**

Solution: Since only 4 items are chosen, X can take the values $0, 1, 2, 3$ and 4. The lot contains 20 non- defective and 5 defective items.

Case (i): When the items are chosen without replacement, we can assume that all the 4 items are chosen simultaneously.

$$
\begin{aligned}
P(X=r) &= P \text{ (choosing exactly } r \text{ defective items),} \\
&= P \text{ (choosing } r \text{ defective and } (4-r) \text{ good items),} \\
&= \frac{{}^{5}C_r \times {}^{20}C_{4-r}}{{}^{20}C_4} (r = 0,\ 1,\ ..,\ 4)
\end{aligned}
$$

Examples of Discrete R.V. Continued...

Case (ii): When the items are chosen with replacement, we note that the probability of an item being defective remains the same in each draw.

$$p = \frac{5}{25} = \frac{1}{5},\ q = \frac{4}{5} \text{ and } n = 4.$$

The problem is one of performing 4 Bernoulli's trials and finding the probability of exactly r successes.

$$P(X = r) = {}^4C_r \left(\frac{1}{5}\right)^4 \left(\frac{4}{5}\right)^{4-r}, \quad (r = 0,\ 1,\ ..,\ 4).$$

Example 2

A shipment of 6 television sets contains 2 defective sets. A hotel makes a random purchase of 3 of the sets. If X is the number of defective sets purchased by the hotel, find the probability distribution of X. Solution:

All the 3 sets are purchased simultaneously. Since there are only 2 defective sets in the lot, X can take the values $0,\ 1$ and 2.

Examples of Discrete R.V. Continued...

$$
\begin{aligned}
P(X=r) &= P\,(\text{choosing exactly } r \text{ defective items}),\\
&= P\,(\text{choosing } r \text{ defective and } (3-r) \text{ good items}),\\
&= \frac{{}^{2}C_r \times {}^{4}C_{3-r}}{{}^{6}C_3}, \quad (r = 0,\ 1,\ 2)
\end{aligned}
$$

The required probability distribution is represented in the form of the following table.

$X = 4$	p_r
0	$\frac{1}{5}$
1	$\frac{3}{5}$
2	$\frac{1}{5}$
Total	1

Examples of Discrete R.V. Continued...

Example 3

Let X denotes the number of heads in an experiment of tossing two coins. Find
(a) probability distribution, (b) Cumulative distribution
(c) Mean of X (d) Variance of X
(e) $P(X \leq 1)$ (f) $P(|X| \leq 1)$
(g) $P(X \geq 1)$ (h) find minimum value of c such that $P(X \leq c) > 1/2$.

Solution

We know that, by tossing two coins, the sample space is

$$\begin{aligned} S &= \{HH, HT, TH, TT\} \\ n(S) &= |S| = 4 \end{aligned}$$

Given that X is a random variable which denotes the number of heads.

$$X(HH) = 2, \; X(HT) = 1, \; X(TH) = 1, \; X(TT) = 0$$

The range space of X is $R_X = 0, 1, 2. \Rightarrow X$ is a discrete random variable.

$$X(HH) = 2,\ X(HT) = 1,\ X(TH) = 1,\ X(TT) = 0 \qquad (1)$$

The range space of X is $R_X = \{0, 1, 2\}$.
$\Rightarrow X$ is a discrete random variable.
(a) Probability distribution of X is $\{x, p(x)\}$:

$$P(X = 0) = \frac{1}{4} \qquad \text{[number of times } X = 0 \text{ is 1 by (1)]}$$

$$P(X = 1) = \frac{2}{4} \qquad \text{[number of times } X = 1 \text{ is 2 by (1)]}$$

$$P(X = 2) = \frac{1}{4} \qquad \text{[number of times } X = 2 \text{ is 1 by (1)]}$$

Probability distribution of X is

R.V.	X	0	1	2
P.M.F.	$P(X = x)$	$\frac{1}{4}$	$\frac{2}{4} = \frac{1}{2}$	$\frac{1}{4}$

Examples of Discrete R.V. Continued...

(b) Cumulative distribution $F(X = x) = P(X \leq x)$:

R.V.	X	0	1	2
C.D.F.	$F(X = x)$	$F(x=0) = P(X \leq 0) = \frac{1}{4}$	$F(x=1) = P(X \leq 1) = \frac{3}{4}$	$F(x=2) = P(X \leq 2) = \frac{4}{4} = 1$

(c) Mean of X :

$$\begin{aligned}
E(X) &= \sum_x xP(X = x) \\
&= \sum_{x=0}^{x=2} x \cdot P(X = x) \\
&= 0 \cdot P(X = 0) + 1 \cdot P(X = 1) + 2 \cdot P(X = 2) \\
&= 0 \cdot \frac{1}{4} + 1 \cdot \frac{2}{4} + 2 \cdot \frac{1}{4} \qquad \text{[refer (2)]} \\
&= 0 + \frac{1}{2} + \frac{1}{2} \\
E(X) &= 1
\end{aligned}$$

Examples of Discrete R.V. Continued...

(d) Variance of X : $V(X) = E\left(X^2\right) - [E(X)]^2$ then

$$
\begin{aligned}
E\left(X^2\right) &= \sum_x x^2 \cdot P(X = x) \\
&= \sum_{x=0}^{x=2} x^2 \cdot P(X = x) \\
&= 0^2 \cdot P(X = 0) + 1^2 \cdot P(X = 1) + 2^2 \cdot P(X = 2) \\
&= 0 + 1 \cdot \frac{2}{4} + 4 \cdot \frac{1}{4} \qquad \text{[refer (2)]} \\
&= \frac{1}{2} + 1 = \frac{3}{2} \\
V(X) &= \frac{3}{2} - 1 = \frac{1}{2} \qquad [E(X) = 1]
\end{aligned}
$$

Examples of Discrete R.V. Continued...

$$\textbf{(e)}\ P(X \le 1) = P(X=0) + P(X=1)$$
$$= \frac{1}{4} + \frac{2}{4} = \frac{3}{4} \qquad \text{[refer (2)]}$$

(f) $P(|X| \le 1) = P(-1 \le X \le 1)$ by the definition of $|X| \le 1$

$$= P(X=-1) + P(X=0) + P(X=1)$$
$$= 0 + \frac{1}{4} + \frac{2}{4} = \frac{3}{4} \qquad \text{[refer (2)]}$$

(g)) $P(X \ge 1) = 1 - P(X < 1)$

$$P(X=0) = \frac{1}{4} \qquad \text{[refer (2)]}$$

Examples of Discrete R.V. Continued...

(h) Minimum value of c such that $P(X \leq c) > 1/2$.

X	$P(X \leq c) > \frac{1}{2}$	Remark
0	$P(X \leq 0) = P(X = 0) = \frac{1}{4} \not> \frac{1}{2}$	$c \neq 0$
1	$P(X \leq 1) = P(X = 0) + P(X = 1) = \frac{1}{4} + \frac{2}{4} = \frac{3}{4} > \frac{1}{2}$	$c = 1$
2	$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{1}{4} + \frac{2}{4} + \frac{1}{4} = 1 > \frac{1}{2}$	$c = 2$

$P(X \leq c) > \frac{1}{2}$ satisfies for $c = 1, 2$.
Minimum value of c is 1.

Example 4

Let X takes values 1, 2, 3, 4 such that

$$2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4).$$

Find the distributions of X.

Solution

Here X is a discrete random variable.
Let $2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4) = k$

Examples of Discrete R.V. Continued...

$$\begin{aligned} P(X=1) &= \frac{k}{2} \\ P(X=2) &= \frac{k}{3} \\ P(X=3) &= k \\ P(X=4) &= \frac{k}{5} \end{aligned}$$

We know that, by the property of probability

$$\begin{aligned} \sum_{x} P(X=x) &= 1 \\ \sum_{x=1}^{x=4} P(X=x) &= 1 \\ P(X=1)+P(X=2)+P(X=3)+P(X=4) &= 1 \\ \frac{k}{2}+\frac{k}{3}+k+\frac{k}{5} &= 1 \end{aligned}$$

Examples of Discrete R.V. Continued...

$$\frac{15k + 10k + 30k + 6k}{30} = 1$$

$$61k = 30 \Rightarrow k = \frac{30}{61}$$

The probability and cumulative distributions table of X is

R.V.	x	1	2	3	4
P.M.F.	$P(X = x)$	15/61	10/61	30/61	6/61

When $x < 1,\ F(x) = 0$
When $1 \leq x < 2,\ F(x) = P(X = 1) = \frac{15}{61}$
When $2 \leq x < 3,\ F(x) = P(X = 1) + P(X = 2) = \frac{25}{61}$
When $3 \leq x < 4,\ F(x) = P(X = 1) + P(X = 2) + P(X = 3) = \frac{55}{61}$
When $x \geq 4,\ F(x) = P(x = 1) + P(x = 2) + P(x = 3) + P(x = 4) = 1.$

Examples of Discrete R.V. Continued...

Example 5

A discrete random variable X has the probability function given below:

X	0	1	2	3	4	5	6	7
$P(X)$	0	a	$2a$	$2a$	$3a$	a^2	$2a^2$	$7a^2 + a$

Find

(a) a

(b) cumulative distribution function

(c) $P(X < 6)$

(d) find maximum value of c such that $P(X \leq c) < \frac{3}{4}$.

(e) $P(3 \leq X < 6)$

(f) $P(2X + 3 < 7)$

(g) $P(X > 1 / X < 3)$

Solution

Given X is discrete R.V. with P.M.F. $P(X)$ with an unknown a.

(a) We know that, by the definition P.M.F. is $\sum_{x} P(X = x) = 1$

Examples of Discrete R.V. Continued...

$$
\begin{aligned}
\sum_{x=0}^{x=7} P(X=x) &= 1 \\
0+a+2a+3a+a^2+2a^2+7a^2+a &= 1 \\
10a^2+9a &= 1 \\
10a^2+9a-1 &= 0 \\
10a^2+10a-a-1 &= 0 \\
10a(a+1)-(a+1) &= 0 \\
(10a-1)(a+1) &= 0 \\
a &= \frac{1}{10} \text{ or } -1 \\
\text{But } a &\neq -1[\text{ Probability } \in [0,\ 1]] \\
a &= \frac{1}{10}
\end{aligned}
$$

Probability distribution table becomes

R.V.	x	0	1	2	3	4	5	6	7
P.M.F.	$p(x)$	0	1/10	2/10	2/10	3/10	1/100	2/100	17/100

(b) Cumulative distribution function $F(X=x)=P(X\leq x)$:

R.V.	x:	0	1	2	3	4	5	6	7
C.D.F.	$F(x)$:	0	1/10	3/10	5/10	8/10	81/100	83/100	1

Examples of Discrete R.V. Continued...

$$\begin{aligned}
\textbf{(c)}\ P(X<6) &= P(X=0)+P(X=1)+P(X=2)+P(X=3) \\
&\quad + P(X=4)+P(X=5) \\
&= 1-[P(X\geq 6)] \qquad\qquad (P(\overline{A})=1-P(A)) \\
&= 1-[P(X=6)+P(X=7)] = 1-[2/100+17/100] \\
&= 1-[19/100] = \frac{100-19}{100} = \frac{81}{100} = 0.81
\end{aligned}$$

(d) To find the maximum value of c such that $P(X\leq c)<\frac{3}{4}$.

X	$P(X\leq c)>\frac{3}{4}=0.75$	Remarks
0	$P(X\leq 0)=F(0)=0<0.75$	$c=0$
1	$P(X\leq 1)=F(1)=\frac{1}{10}=0.1<0.75$	$c=1$
2	$P(X\leq 2)=F(2)=\frac{3}{10}=0.3<0.75$	$c=2$
3	$P(X\leq 3)=F(3)=\frac{5}{10}=0.5<0.75$	$c=3$
4	$P(X\leq 4)=F(4)=\frac{8}{10}=0.8\not<0.75$	$c\neq 4$
5	$P(X\leq 5)=F(5)=\frac{81}{100}=0.81\not<0.75$	$c\neq 5$

X	$P(X \le c) > \frac{3}{4} = 0.75$	Remarks
6	$P(X \le 6) = F(6) = \frac{83}{100} = 0.83 \not> 0.75$	$c \ne 6$
7	$P(X \le 7) = F(7) = \frac{100}{100} = 1 \not> 0.75$	$c \ne 7$

$P(X \le c) < \frac{3}{4}$ satisfies for $c = 0,\ 1,\ 2,\ 3.$
Maximum value of c is 3.

$$\textbf{(e)}\ P(3 \le X < 6) = P(X = 3) + P(X = 4) + P(X = 5)$$
$$= 2/10 + 3/10 + 1/100 = \frac{51}{100} = 0.51$$

$$\begin{aligned}\textbf{(f)}\ P(2X + 3 < 7) &= P(2X + < 7 - 3)\\ &= P(2X < 4)\\ &= P(X < 2)\\ &= P(X = 0) + P(X = 1)\\ &= 0 + 1/10 = \frac{1}{10} = 0.1\end{aligned}$$

Examples of Discrete R.V. Continued...

$$\textbf{(g)}\ P(X>7/X<3)=\frac{P[(X>1)\cap(X<3)]}{P(X<3)} \qquad \left[P(A/B)=\frac{P(A\cap B)}{P(B)}\right]$$

$$=\frac{P[(X=2,3,4,5,6,7)\cap(X=0,1,2)]}{P(X=0,1,2)}$$

$$=\frac{P[(X=2)]}{P(X=0)+P(X=1)+P(X=2)}$$

$$=\frac{\frac{2}{10}}{0+\frac{1}{10}+\frac{2}{10}}=\frac{\frac{2}{10}}{\frac{3}{10}}$$

$$=\frac{2}{3}=0.66666\bar{6}$$

$$\cong 0.667$$

Example

1. The duration of telephonic conversation.
2. The path of the aeroplane from Chennai to Hydarabad.

Probability Density Function

Let X be a **continuous random variable** such that

$$P\left\{x - \frac{1}{2}dx \leq X \leq x + \frac{1}{2}dx\right\} = f(x)dx$$

then $f(x)$ is called the **probability density function of** X, provided $f(x)$ satisfies the following conditions:

$$\begin{aligned} &1. \quad f(x) \geq 0, \ \text{for all } x \in R_x, \ \text{and} \\ &2. \quad \int_{-\infty}^{\infty} f(x)\, dx = 1. \end{aligned}$$

Cumulative Distribution Function of C.R.V.

Cumulative Distribution Function

The function $F(x)$ or $F_X(x)$ is called the **Cumulative Distribution function** of the continuous random variable X and is defined as:

$$F(x) = P(-\infty < X < x) = \int_{-\infty}^{x} f(x)\, dx$$

Properties of C.D.F.

1. $0 \leq F(x) \leq 1$.
2. $F(x)$ is a non-decreasing function of X *i.e.* $F(x_1) \leq F(x_2) \quad (\text{ if } x_1 < x_2)$
3. $F(-\infty) = \lim_{x \to -\infty} F(x) = 0 = P(X \leq -\infty), \; F(\infty) = \lim_{x \to \infty} F(x) = 1 = P(X \leq \infty)$
4. Moreover, $P(a \leq X \leq b)$ or $P(a < X < b)$ or *etc.* of a C.R.V. X (for curve $f(x)$, the probability curve of the R.V. X) is defined as:
$$P(a \leq X \leq b) = P(a < X < b) = P(a \leq X < b) = P(a < X \leq b) = \int_a^b f(x)dx$$
5. The relation between C.D.F. $F(x)$ and P.D.F. $f(x)$:
$$\frac{d}{dx}[F(x)] = f(x)$$

Mathematical Expectation of Continuous R.V.

Mean value of X

The value $E(X)$ is called the **Expectation of *X*** (or) **Expected value of *X*** (or) and is defined as:
Mean of X,

$$E(X) = \overline{X} = \int_{-\infty}^{\infty} x\, f(x)\, dx,$$

if X is a continuous random variable.
Note: 1. $E(X)$ is the first moment of X.
2. In general, $E[X^r]$ is the r^{th} moment of the random variable X.

Properties of Mean value of X

1. $E(a) = a$, where a is a constant.
2. $E(aX) = aE(X)$, where a is a constant.
3. $E(aX + b) = aE(X) + b$, where a and b are constants.

Mathematical Expectation of Continuous R.V.

Properties of Mean value of X

1. $E\left(k_1 X_1 + k_2 X_2 + \cdots + k_n X_n\right) = k_1 E\left(X_1\right) + k_2 E\left(X_2\right) + \cdots + k_n E\left(X_n\right)$, where $k_1, k_2, \cdots, k_n$ are constants.
2. If $X_1, X_2, \cdots, X_n$, are independent random variables, then

$$E\left(X_1\ X_2 \cdots X_n\right) = E\left(X_1\right) E\left(X_2\right) \cdots E\left(X_n\right)$$

(Multiplication theorem on expectation)

3. $E\left[X - \overline{X}\right] = E[X - E(X)] = E(X) - E(X) = 0.$
   $\left[\because \overline{X} = E(X) = \text{Mean of } X\right]$

Variance

Variance of X: Var(X)=$V(X)$ *i.e.*

$$\sigma_X^2 = E\left(X - \overline{X}\right)^2, \text{ where } \overline{X} = E\left(X\right)$$

$$= \int_{R_x} \left(x - \overline{X}\right)^2 f\left(x\right) dx, \quad \text{if } X \text{ is a C.R.V. and } f(x) \text{ is P.D.F. of } X.$$

(or)

Mathematical Expectation of Continuous R.V.

$$\sigma_X^2 = E\left(X^2\right) - [E(X)]^2$$

$$\sigma_X^2 = \int\limits_{-\infty}^{\infty} x^2 f(x) - \left[\int\limits_{-\infty}^{\infty} x f(x)\right]^2, \quad \text{if } X \text{ is a C.R.V. and } f(x) \text{ is P.D.F. of } X.$$

Standard Deviation of X

$$X = \sqrt{\text{Var }(X)} = \sigma_X$$

Properties of Variance

1. $\text{Var}(X) \geq 0$
2. $\text{Var}(a) = 0$, where a is a constant.
3. $\text{Var}(a \pm bX) = \text{Var}(a) + b^2\text{Var}(X) = b^2\ \text{Var}(X)$, where a and b are constants.
4. $\text{Var}(aX \pm bY) = a^2\text{Var}(X) + b^2\text{Var}(Y)$. If X and Y are independent.

Examples of Continuous R.V.

Example 1

Check whether the following functions are probability density function (P.D.F.):

(a) $f(x) = 6x(1-x), 0 \leq X \leq 1$

(b) $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}, -\infty < X < \infty$

(c) $f(x) = \begin{cases} \frac{100}{x^2}, & X > 100 \\ 0, & X < 100 \end{cases}$

(d) $f(x) = \sin x, 0 < x < \pi$

(e) $f(x) = \begin{cases} 0, & x < 2 \\ \frac{1}{18}(3+2x), & 2 \leq x \leq 4 \\ 0, & x > 4 \end{cases}$

Solution

Here $f(x)$ is defined in the interval (a, b) which contains uncountably infinite values.
X is a continuous random variable. If X is a continuous R.V. and $f(x)$ is a function defined in an interval of the form (a, b), then

$\int_a^b f(x)dx = 1,$ given $f(x)$ is **pdf** in (a, b), $\int_a^b f(x)dx \neq 1,$ given $f(x)$ is **not pdf**.

Examples of Continuous R.V. Continued...

(a) Given $f(x) = 6x(1-x) = 6\left(x - x^2\right), 0 \le X \le 1$.

We have to prove $\int\limits_0^1 f(x)\, dx = 1$ (1)

$$\begin{aligned}
\text{LHS of (1)} &= \int\limits_0^1 f(x)dx = \int\limits_0^1 6\left(x - x^2\right) dx \\
&= \left[\frac{6x^2}{2} - \frac{6x^3}{3}\right]_0^1 = \left[3x^2 - 2x^3\right]_0^1 \\
&= [(3-2) - (0-0)] = 1 \\
&= \text{RHS of (1)}
\end{aligned}$$

The given $f(x)$ is P.D.F.

(b) Given $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}, -\infty < X < \infty$.

We have to prove $\int\limits_{-\infty}^{\infty} f(x)dx = 1$ (2)

$$\text{LHS of (2)} = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\pi}\left(\frac{1}{1+x^2}\right)\,dx$$
$$= \frac{1}{\pi}\left[\tan^{-1} x\right]_{-\infty}^{\infty} = \frac{1}{\pi}\left[\tan^{-1}(\infty) - \tan^{-1}(-\infty)\right]$$
$$= \frac{1}{\pi}\left[\left(\frac{\pi}{2}\right) - \left(\frac{-\pi}{2}\right)\right]$$
$$= \frac{1}{\pi}[\pi] = 1 = \text{RHS of (2)}$$

The given $f(x)$ is P.D.F.

(c) Given $f(x) = \begin{cases} \frac{100}{x^2}, & x > 100 \\ 0, & x < 100 \end{cases}$

We have to prove $\int_{-\infty}^{\infty} f(x)dx = 1$ (3)

Examples of Continuous R.V. Continued...

$$\text{LHS of (3)} = \int_{-\infty}^{100} f(x)dx + \int_{100}^{\infty} f(x)dx = 0 + \int_{100}^{\infty} \frac{100}{x^2} dx$$
$$= 100 \left[\frac{-1}{x}\right]_{100}^{\infty} = 100 \left[0 - \frac{-1}{100}\right]$$
$$= 1 = \text{RHS of (3)}$$

The given $f(x)$ is P.D.F.

(d) Given $f(x) = \sin x, 0 < x < \pi$

We have to prove $\int_{0}^{\pi} f(x)\, dx = 1$ (4)

$$\text{LHS of (4)} = \int_{0}^{\pi} f(x)dx = \int_{0}^{\pi} \sin x dx = [-\cos x]_0^{\pi} = -[\cos x]_0^{\pi}$$
$$= -[(\cos 0) - (\cos \pi)] = -[1 - (-1)] = 2 \neq 1 \neq \text{RHS of (4)}$$

The given $f(x)$ is **not P.D.F.**

(e) Given $f(x) = \begin{cases} 0, & x < 2 \\ \frac{1}{18}(3+2x), & 2 \leq x \leq 4 \\ 0, & x > 4 \end{cases}$

We have to prove $\int\limits_{-\infty}^{\infty} f(x)dx = 1 \qquad (5)$

$$\begin{aligned} \text{LHS of (5)} &= \int_{-\infty}^{2} f(x)\,dx + \int_{2}^{4} f(x)\,dx + \int_{4}^{\infty} f(x)\,dx \\ &= 0 + \frac{1}{18}\left[3x + x^2\right]_2^4 + 0 \\ &= \frac{1}{18}\left[(12+16) - (6+4)\right] = 1 = \text{RHS of (5)} \end{aligned}$$

The given $f(x)$ is P.D.F.

Example 2

A continuous random variable X has a p.d.f. $f(x) = 3x^2, 0 \leq X \leq 1$. Find k & α such that

(a) $P(X \leq k) = P(X > k)$ [(b)] $P(X > \alpha) = 0.1$

(c) $P(|X| \leq 1)$ [(d)] $P(X > \beta) = 0.05$

Solution : Here X is a continuous random variable.

Given, pdf $f(x) = 3x^2, 0 \leq x \leq 1$. (1)

(a). Find k from the given equation $P(X \leq k) = P(X > k)$ (2)

$$\text{We know that, } P(X \leq k) + P(X > k) = 1$$

$$2P(X \leq k) = 1 \qquad [P(X \leq k) = P(X > k)]$$

$$P(X \leq k) = \frac{1}{2}$$

$$(2) \Rightarrow P(X \leq k) = P(X > k) = \frac{1}{2} \qquad (3)$$

Examples of Continuous R.V. Continued...

From (3), we have $P(X \leq k) = \frac{1}{2}$ or $P(X > k) = \frac{1}{2}$.

Method 1	Method 2
$P(X \leq k) = \frac{1}{2}$	$P(X > k) = \frac{1}{2}$
i.e. $\int_{-\infty}^{k} f(x)dx = \frac{1}{2}$	*i.e.* $\int_{k}^{\infty} f(x)dx = \frac{1}{2}$
$\int_{0}^{k} 3x^2 dx = \frac{1}{2}$	$\int_{k}^{1} 3x^2 dx = \frac{1}{2}$
$[x^3]_0^k = \frac{1}{2}$ $k^3 = \frac{1}{2}$	$\left[x^3\right]_k^1 = \frac{1}{2}$ $1 - k^3 = \frac{1}{2}$
$k = \left(\frac{1}{2}\right)^{\frac{1}{3}}$	$k = \left(\frac{1}{2}\right)^{\frac{1}{3}}$

(b). Find α from given equation $P(X > \alpha) = 0.1$

$$i.e. \int_{\alpha}^{\infty} f(x)dx = 0.1 \Rightarrow \int_{\alpha}^{1} 3x^2 dx = 0.1 \qquad [\text{ by } (1)]$$

Examples of Continuous R.V. Continued...

$$\begin{aligned}
\left[x^3\right]_\alpha^1 &= 0.1 = \frac{1}{10} \\
1 - \alpha^3 &= \frac{1}{10} \\
-\alpha^3 &= \frac{1}{10} - 1 \\
\alpha^3 &= \frac{9}{10} \\
\alpha &= \left(\frac{9}{10}\right)^{\frac{1}{3}}
\end{aligned}$$

(c). $P(|X| \leq 1) = P(-1 \leq X \leq 1)$

$$= \int_{-1}^{1} f(x)dx = \int_{-1}^{0} (0)dx + \int_{0}^{1} 3x^2 dx \qquad \text{[by (1)]}$$

$$= \left[x^3\right]_0^1 = 1$$

Examples of Continuous R.V. Continued...

(d). Find β from given equation $P(X > \beta) = 0.05$:

$$
\begin{aligned}
\text{i.e., } P(X > \beta) &= 0.05 \\
\int_{\beta}^{1} f(x)\; dx &= 0.05 \\
\int_{\beta}^{1} 3x^2 dx &= 0.05 \\
3\left(\frac{x^3}{3}\right)_{\beta}^{1} &= 0.05 \\
1 - \beta^3 &= 0.05 \\
\beta^3 &= 1 - 0.05 \\
\beta^3 &= 0.95 \\
\beta &= 0.9830476
\end{aligned}
$$

Examples of Continuous R.V. Continued...

Example 3

For the triangular distribution
$$f(x) = \begin{cases} x, & X \in [0,1] \\ 2-x, & X \in [1,2] \\ 0, & \textbf{otherwise} \end{cases}.$$
Find mean and variance.
Solution : Here X is a continuous random variable with pdf

$$f(x) = \begin{cases} x, & X \in [0,1] \\ 2-x, & X \in [1,2] \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

Mean of X:

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

$$= \int_{-\infty}^{0} (0)dx + \int_{0}^{1} x \cdot xdx + \int_{1}^{2} x \cdot (2-x)dx + \int_{0}^{\infty} (0)dx \qquad [\text{ by } (1)]$$

Examples of Continuous R.V. Continued...

$$
\begin{aligned}
&= \int_0^1 x^2 dx + \int_1^2 (2x - x^2) dx \\
&= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_1^2 \\
&= \left[\frac{1}{3} - 0\right] + \left[\left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\right] \\
&= \frac{1}{3} + \left[3 - \frac{7}{3}\right] \\
&= \frac{1}{3} + \frac{2}{3}
\end{aligned}
$$

$$\therefore E[X] = 1 \tag{2}$$

Examples of Continuous R.V. Continued...

Variance of X : $\text{Var}(X) = E\left[X^2\right] - [E(X)]^2$
Now,

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{-\infty}^{0} x^2(0)\, dx + \int_{0}^{1} x^2 \cdot xdx + \int_{1}^{2} x^2 \cdot (2-x)dx$$

$$+ \int_{0}^{\infty} x^2(0)\, dx$$

$$= \int_{0}^{1} x^3 dx + \int_{1}^{2} \left(2x^2 - x^3\right) dx = \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2$$

$$= \frac{1}{4}[1-0] + \left[\left(\frac{16}{3} - \frac{16}{4}\right) - \left(\frac{2}{3} - \frac{1}{4}\right)\right]$$

$$= \frac{1}{4} + \left[\frac{14}{3} - \frac{15}{4}\right] = \frac{1}{4} + \left[\frac{56-45}{12}\right] = \frac{1}{4} + \left[\frac{11}{12}\right] = \frac{14}{12}$$

$$E\left[X^2\right] = \frac{7}{6} \Rightarrow \text{Var}(X) = E\left[X^2\right] - [E(X)]^2 = \frac{7}{6} - 1 = \frac{1}{6}$$

1. Moments about Mean (Central Moments)

The r^{th} moment about mean (μ) for a random variable X is defined as

$$\mu_r = E\left(X - \mu\right)^r$$

The first four moments about the mean are

$$\mu_1 = E\left(X - \mu\right)^1 = E(X) - E(\mu) = \mu - \mu = 0$$
$$\mu_2 = E\left(X - \mu\right)^2 = \text{Variance of } X = E\left(X^2\right) - \left[E(X)\right]^2$$
$$\mu_3 = E\left(X - \mu\right)^3$$
$$\mu_4 = E\left(X - \mu\right)^4$$

Central Moment Continued...

2. Moments about Any Point

The r^{th} moment about any point(a) for a random variable X is defined as

$$\mu'_r = E\left(X - a\right)^r$$

Moments about Origin (Raw Moments)

The r^{th} moment about origin for a random variable X is defined as

$$\mu'_r = E\left(X^r\right)$$

The first four moments about the origin are

$$\mu'_1 = E\left(X^1\right),\ \mu'_2 = E\left(X^2\right),\ \mu'_3 = E\left(X^3\right),\ \mu'_4 = E\left(X^4\right)$$

Central Moment Continued...

Central moments in terms of Raw Moments

$$\mu_1 = 0$$

$$\mu_2 = \text{Var}(X) = \mu_2' - (\mu_1')^2$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

In general, $\mu_r = \mu_r' - {}^rC_1\mu_{r-1}'(\mu_1') + {}^rC_2\mu_{r-2}'(\mu_1')^2 - {}^rC_3\mu_{r-3}'(\mu_1')^3 + \cdots + (-1)^r(\mu_1')^r$

Note: $\mu_0' = 1,\ \mu_1' = \text{Mean of } X = E(X),\ \mu_2' = E(X^2)$

Example of Continuous R.V. for Central Moments

Example 4

The density function of a random variable X is given by $f(x) = Kx(2-x), 0 < x < 2$.

(i) **Find K.**

(ii) r^{th} **moment**

(i). To find K: **Solution :** Here X is continuous random variable with pdf has unknown K. To find K:

$$\int_0^2 Kx\,(2-x)\,dx = 1, \quad K\int_0^2 \left(2x - x^2\right)dx = 1$$

$$K\left[2\frac{x^2}{2} - \frac{x^3}{3}\right]_0^2 = 1, K\left[\left(4 - \frac{8}{3}\right) - (0-0)\right] = 1$$

$$K\left[\frac{4}{3}\right] = 1$$

$$K = \frac{3}{4}$$

(ii). r^{th} moment:

$$E\left[X^r\right] = \int_0^2 x^r f(x)\,dx = \int_0^2 x^r \frac{3}{4} x(2-x)\,dx$$

$$= \frac{3}{4}\int_0^2 \left(2x^{r+1} - x^{r+2}\right)dx = \frac{3}{4}\left[\frac{2x^{r+2}}{r+2} - \frac{x^{r+3}}{r+3}\right]_0^2$$

$$= \frac{3}{4}\left[\left(2\cdot\frac{2^{r+2}}{r+2} - \frac{2^{r+3}}{r+3}\right) - (0-0)\right] = \frac{3}{4}\left[\frac{2^{r+3}}{r+2} - \frac{2^{r+3}}{r+3}\right]$$

$$= \frac{3\left(2^{r+3}\right)}{4}\left[\frac{1}{r+2} - \frac{1}{r+3}\right]$$

$$= \frac{3\left(2^{r+3}\right)}{2^2}\left[\frac{r+3-r-2}{(r+2)(r+3)}\right]$$

$$= \frac{3\left(2^{r+1}\right)}{(r+2)(r+3)}$$

Characteristic function

Characteristic function

Although higher order moments of a R.V. X may be obtained directly by using the definition of $E(X^n)$, it will be easier in many problems to compute them through the characteristic function or equivalently through the moment generating function of the R.V. X. While the characteristic function always exists, the moment generating function need not.

Characteristic function of a R.V. X (discrete or continuous) is defined as $E(e^{iX\omega})$ and denoted as $f(\omega)$. If X is a discrete R.V. that can take the values $x_1,\ x_2,\ \ldots$ such that $P(X = x_r) = p_r$, then

$$\phi(\omega) = \sum_r e^{ix_r\omega} p(x_r).$$

If X is a continuous R.V. with density function $f(x)$, then

$$\phi(\omega) = \int_{-\infty}^{\infty} e^{ix\omega} f(x) dx.$$

Properties of Characteristic function

Properties

1. $\mu'_n = E(X^n)$= the coefficient of $\frac{i^n\omega^n}{n}$ in the expansion of $\phi(\omega)$ in series of ascending powers of $i\omega$.
2. $\mu'_n = \frac{1}{i^n}\left[\frac{d^n}{d\omega^n}\phi(\omega)\right]_{\omega=0}$.
3. If the characteristic function of a R.V. X is $\phi_x(\omega)$ and if $Y = aX + b$, then $\phi_Y(\omega) = e^{ib\omega}\phi_Y(a\omega)$.
4. If X and Y are independent R.Vs., then
$$\phi_{X+Y}(\omega) = \phi_X(\omega)\dot{\phi}_Y(\omega)$$
5. If the characteristic function of a continuous R.V. X with density function $f(x)$ is $\phi(\omega)$, then the Fourier inversion formula $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(\omega)e^{-ix\omega}d\omega$
$$\phi(\omega) = \int_{-\infty}^{\infty} f(x)e^{ix\omega}dx$$
6. If the density function of X is known, the density function of $Y = g(X)$ can be found from the C.F. of Y, provided $Y = g(X)$ is one-to-one.

Binomial Distribution

Bernoulli trial

A Bernoulli trial (or binomial trial) is a random experiment in which there are only two possible outcomes namely success (s) and failure (f).
The sample space of a Bernoulli trial is $S = \{s, f\}$.

Bernoulli experiment

The experiment consists of 'n' independent repeated Bernoulli trials.

Bernoulli distribution

Let us consider an experiment consists of $'n'$ independent trials which results successes(S) and failures (F) of the random form

$$S\ S\ F\ S\ \cdots\ F\ F\ S$$

Let X be a random variable which denotes the number of success, specifically $x \in X$ be the number of successes and hence we have $'n - x'$ number of failures.

Bernoulli distribution Continued...

Let p be the corresponding probability to get a success and q be the corresponding probability to get a failure.
The probability to get this form $S\ S\ F\ S\ \cdots\ F\ F\ S$ is

$$\begin{aligned}
P(S\ S\ F\ S\ \cdots\ F\ F\ S) &= P(S)\cdot P(S)\cdot P(F)\cdot P(S)\cdots P(F)\cdot P(F)\cdot P(S) \\
&\qquad\qquad\qquad\qquad\qquad [\because \text{ independent trials}] \\
&= p\cdot p\cdot q\cdot p\cdots q\cdot q\cdot p \\
&= \Big[p\cdot p\cdots p\Big]_{(x \text{ times})}\Big[q\cdot q\cdots q\Big]_{(n-x \text{ times})} \\
&= p^x q^{n-x}
\end{aligned}$$

This probability value is for 'x' successes in sequence above form $SSFS\cdots FFS$ only.
But $'x'$ successes in $'n'$ trials can occur in ${}^nC_x$ ways.
$\therefore$ Probability of $'x'$ successes in $'n'$ trials is ${}^nC_x p^x q^{n-x}$, *i.e.*

$$P(X = x \text{ successes}) = {}^nC_x p^x q^{n-x}$$

where $x = 0, 1, 2, \cdots, n$ with $p + q = 1$.

Note

1. $P(X = x) = {}^nC_x p^x q^{n-x}$ is the $(x+1)^{\text{th}}$ term in the binomial expansion of $(p+q)^n$.
$$\left[(p+q)^n = {}^nC_0 p^0 q^n + {}^nC_1 p^1 q^{n-1} + {}^nC_2 p^2 q^{n-2} + \cdots + {}^nC_n p^n q^0\right]$$
2. $\sum_{x=0}^{n} P(x) = \sum_{x=0}^{n} {}^nC_x p^x q^{n-x} = (p+q)^n = 1^n = 1.$

Definition

The random variable X that counts the number of successes, in the 'n' Bernoulli trials is said to follow a **Binomial distribution with parameters** n **and** p, written as $B(n,p)$. Symbolically, $X \sim B(n,\ p)$, $n \in N, p \in [0,1]$.
The Probability mass function of the Binomial distributed discrete random variable X is

$$P\,(X = x) = P(x) = {}^nC_x p^x q^{n-x},\ x = 0, 1, 2, 3, ...n,\ \text{with } p + q = 1.$$

Assumptions of the Binomial distribution

(1) The random experiments corresponds to two possible outcomes (success or failure).

(2) Number of trials is finite.

(3) The trials are independent.

(4) The probability of success is constant in any trial.

Formula

(1) Probability mass function of $X \sim B(n,\ p)$ is

$$P(X = x) = \ {}^nC_x p^x q^{n-x}$$

where $x = 0,\ 1,\ 2,\ 3,\ ...,\ n$ with $p + q = 1$.

Continued...

where

$n =$ number of trials.

$X =$ Random variable (Discrete), represents number of successes, which follows Binomial distribution.

$x =$ value of random variable X.

$p =$ probability of success in single trial.

$q =$ probability of failure in single trial $= 1 - p$.

$N =$ Number of times 'n' trials are repeated (or) Total number of sets

Mathematical Expectation of Binomial distribution

The Probability mass function of the Binomial distributed discrete random variable X is

$$P(X = x) = {}^nC_x p^x q^{n-x}, \; x = 0, \; 1, \cdots, n \text{ where } p + q = 1$$

Characteristic function of X

$$\phi(\omega) = (1 - p + pe^{i\omega})^n = (q + pe^{i\omega})^n$$

Mean of X in terms of Characteristic Function

As we know that

$$\begin{aligned} P(X = x) &= {}^nC_x p^x q^{n-x}, \quad x = 0, \; 1, \; 2, \; ...., \; n \\ \text{Characteristic Function } \phi(\omega) &= E(e^{i\omega x}) \\ &= \sum_{x=0}^{n} e^{i\omega x} \cdot {}^nC_x p^x q^{n-x} \end{aligned}$$

Mathematical Expectation of Binomial distribution Continued...

$$
\begin{aligned}
\phi(\omega) &= \sum_{x=0}^{n} {}^nC_x (pe^{i\omega})^x q^{n-x} \\
&= q^n + {}^nC_1 (pe^{i\omega}) q^{n-1} + \ldots\ldots + (pe^{i\omega})^n \\
&= (q + pe^{i\omega})^n
\end{aligned}
$$

From the properties of characteristic function, we know that

$$
\mu_n' = \frac{1}{i^n} \left[\frac{d^n}{d\omega^n} \phi(\omega) \right]_{\omega=0}
$$

Variance of X in terms of Characteristic Function

We have $Var(X) = \mu_2' - {\mu_1'}^2$, we find

$$
\frac{d^n}{d\omega^n} \phi(\omega) = n(q + pe^{i\omega})^{n-1} \cdot pie^{i\omega}
$$

Mathematical Expectation of Binomial distribution Continued...

$$\begin{aligned}
\frac{d^n}{d\omega^n}\phi(\omega) &= n(q+pe^{i\omega})^{n-1}\cdot pie^{i\omega} \\
\frac{d^n}{d\omega^n}\phi(\omega)|_{w=0} &= n(q+p)^{n-1}pi \\
\mu_1' &= \frac{1}{i}\left[\frac{d}{d\omega}\phi(\omega)\right]_{\omega=0} = \frac{1}{i^n}npi = np
\end{aligned}$$

Differentiating $\phi(\omega)$ w.r.to ω,

$$\begin{aligned}
\frac{d}{d\omega}\phi(\omega) &= npi\left[(q+pe^{i\omega})^{n-1}e^{i\omega}\right] \\
\frac{d^2}{d\omega^2}\phi(\omega) &= npi\left[(n-1)(q+pe^{i\omega})^{n-2}e^{i\omega}\cdot pe^{i\omega}+(q+pe^{i\omega})^{n-1}ie^{i\omega}\right] \\
\left[\frac{d^2}{d\omega^2}\phi(\omega)\right]_{\omega=0} &= npi[(n-1)pi+i] \\
&= npi^2[(n-1)p+1]
\end{aligned}$$

Mathematical Expectation of Binomial distribution Continued...

$$\begin{aligned}\left[\frac{d^2}{d\omega^2}\phi(\omega)\right]_{\omega=0} &= npi[(n-1)pi+i] \\ &= npi^2[(n-1)p+1]\end{aligned}$$

Now Variance of X is

$$\begin{aligned}Var(X)=\mu_2 &= \mu_2' - {\mu_1'}^2 \\ &= np[(n-1)p+1]-n^2p^2 \\ &= np[np-p+1-np] \\ &= npq.\end{aligned}$$

Examples of Binomial Distribution

Example 1

Consider an example of tossing 2 coins. Then the way of finding the same values of probability using,
(1) the ideas probability,
(2) random variable probability,
(3) and Binomial distribution probability.

Solution:

(1)	(2)	(3)
S.S. $= \{HH, HT, TH, TT\}$ $P(\text{2 heads}) = \frac{1}{4}$ $P(\text{1 head}) = \frac{2}{4}$ $P(\text{0 head}) = \frac{1}{4}$	S.S. $= \{HH, HT, TH, TT\}$ $P(X=2) = \frac{1}{4}$ $P(X=1) = \frac{2}{4}$ $P(X=0) = \frac{1}{4}$	$P(X=2) = \frac{1}{4}$ $P(X=1) = \frac{2}{4}$ $P(X=0) = \frac{1}{4}$

Examples of Binomial Distribution Continued...

Importance of the table is to find probability of 'x' heads
We can write the sample space for 2 coins in limited time. But we **cannot write** the sample space for 10 or more coins in limited time. That's why, we use **Binomial distribution** for any number of finite coins.

Example 2

Let X denotes the number of heads in an experiment of tossing two coins. Find probability distribution by using Binomial distribution.

Solution:
$X =$ a discrete R.V. which denotes the **number of heads**.
$p =$ **Probability of getting a head from a single coin** $= \dfrac{1}{2}$
$q = 1 - p = 1 - \dfrac{1}{2} = \dfrac{1}{2}$
$n =$ number of coins $= 2$.
$X \sim$ B.D. $(n,\ p)$.

Examples of Binomial Distribution Continued...

We know that, the P.M.F. of the Binomial Distribution is
$P(X = x) = {}^nC_x p^x q^{n-x}$

$$\text{When } X = 0, P(X = 0) = {}^2C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{2-0} = 1 \cdot 1 \cdot \frac{1}{4} = \frac{1}{4}$$

$$\text{When } X = 1, P(X = 1) = {}^2C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{2-1} = 2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{2}{4}$$

$$\text{When } X = 2, P(X = 2) = {}^2C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{2-2} = 1 \cdot \frac{1}{4} \cdot 1 = \frac{1}{4}$$

The probability distribution is

X:	0	1	2
$P(x)$:	$1/4$	$2/4$	$1/4$

Examples of Binomial Distribution Continued...

Example 3

Find Binomial distribution and $P(X = 4)$, for
(a) mean 4, variance = 3 (b) mean 4, variance = 5.
Solution : Given distribution is Binomial. We know that the P.M.F. of the Binomial distribution is

$$P(X = x) = {}^{n}C_{x} p^{x} q^{n-x}, x = 0, 1, 2, \cdots, n \quad (1)$$

$$\text{Mean of B.D.} = np \quad (2)$$

$$\text{Variance of B.D.} = npq \quad (3)$$

Relation between Mean & Variance is of B.D is Mean $>$ Variance. (4)

(a)

$$\text{Mean} = np = 4 \quad (5)$$

$$\text{Variance} = npq = 3 \quad (6)$$

$$4q = 3 \quad [\text{ by } (5)]$$

$$q = \frac{3}{4}$$

Examples of Binomial Distribution Continued...

$$\Rightarrow np = 4 \Rightarrow n\frac{1}{4} = 4 \qquad \left[p = \frac{1}{4}\right] \Rightarrow n = 16$$

$$\Rightarrow P(X = x) = {}^{16}C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{16-x}$$

$$i.e.\ P(X = 4) = {}^{16}C_4 \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^{12} = 0.22$$

(b) Here Mean $<$ Variance, which is not possible for Binomial distribution.
Explanation:

$$\text{Mean} = np = 4$$

$$\text{Variance} = npq = 5 \Rightarrow 4q = 5 \Rightarrow q = \frac{5}{4} > 1$$

which is not possible. [$0 \leq$ Probability (p (or) q) ≤ 1.]
$\Rightarrow$ Given data in (b) is not applicable for Binomial distribution.

Examples of Binomial Distribution Continued...

Example 4

An irregular 6**-faced dice is such that the probability that it gives** 3 **even numbers in** 5 **throws is twice the probability that it gives** 2 **even numbers in** 5 **throws. How many sets of exactly** 5 **trials can be expected to give no even number out of** 2500 **sets?**

Solution: Let the probability of getting an even number with the unfair dice be p. Let X denote the number of even numbers obtained in 5 trials (throws).

$$\begin{aligned} P(X=3) &= 2 \times P(X=2) \\ {}^5C_3 p^3 q^2 &= 2 \times {}^5C_2 p^2 q^3 \\ p &= 2q = 2(1-p) \\ 3p &= 2 \text{ or } p = \frac{2}{3} \text{ and } q = \frac{1}{3} \end{aligned}$$

Now P (getting no even number)

$$= P(X=0) = {}^5C_0 p^0 q^5 = (\frac{1}{5})^5 = \frac{1}{243}$$

Examples of Binomial Distribution Continued...

Number of sets having no success (even number) out of N sets
$= N \times P(X = 0)$

$$\begin{aligned} \text{Required number of sets} &= 2500 \times \frac{1}{243} \\ &= 10, \quad \text{nearly} \end{aligned}$$