Problems on Z-transform Properties

1. If $x(n) = x_1(n) * x_2(n)$ where $x_1(n) = (1/3)^n u(n)$ and $x_2(n) = (1/5)^n u(n)$, find $X(z)$ using convolution property of Z transform

Sol:

$$x_1(n) = (1/3)^n u(n) \Rightarrow X_1(z) = \frac{1}{1-\frac{1}{3}z^{-1}} \quad |z| > 1/3$$

$$x_2(n) = (1/5)^n u(n) \Rightarrow X_2(z) = \frac{1}{1-\frac{1}{5}z^{-1}} \quad |z| > 1/5$$

Using convolution property

$$x_1(n) * x_2(n) \xleftrightarrow{Z} X_1(z) \cdot X_2(z)$$

$$\therefore X(z) = \frac{1}{1-\frac{1}{3}z^{-1}} \cdot \frac{1}{1-\frac{1}{5}z^{-1}} \quad \text{ROC} : |z| > 1/3$$

2. Determine the signal $x(n)$ whose z-transform is given by: $X(z) = \log(1-az^{-1}) \quad |z|>|a|$

Sol: Given $X(z) = \log(1-az^{-1})$

$$\Rightarrow \frac{d}{dz}X(z) = \frac{1}{1-az^{-1}}(az^{-2}) = \frac{az^{-2}}{1-az^{-1}}$$

Multiplying $-z$ on both sides:

$$-z\frac{d}{dz}X(z) = \frac{-az^{-1}}{1-az^{-1}}$$

$$= -az^{-1}\left[\frac{1}{1-az^{-1}}\right]$$

$$= -az^{-1}\left[Z\left[a^n u(n)\right]\right]$$

$$= -a\,Z\left[a^{n-1}u(n-1)\right] \quad \text{— (1)} \quad \left[\text{Time shifting property}\right]$$

Using differentiation property

$$Z\left[n x(n)\right] = -z\frac{d}{dz}X(z) \quad \text{— (2)}$$

Comparing (1) & (2)

$$n x(n) = -a\left[a^{n-1}u(n-1)\right]$$

$$\therefore x(n) = \frac{-a^n u(n-1)}{n}.$$

3. Find z-transform of $na^n u(n)$

Sol: W.K.T $Z\left[a^n u(n)\right] = \dfrac{1}{1-az^{-1}} \quad |z| > a$

Using the property - multiplication by n:

$$Z[n\,x(n)] = -z \cdot \frac{d}{dz} X(z)$$

$$\therefore Z[n a^n u(n)] = -z \frac{d}{dz}\left[\frac{1}{1-az^{-1}}\right]$$

$$= -z \left[\frac{-1}{(1-az^{-1})^2}(+az^{-2})\right]$$

$$= \frac{az^{-1}}{(1-az^{-1})^2} = \frac{a/z}{\frac{(z-a)^2}{z^2}} = \frac{az}{(z-a)^2}$$

$$\therefore X(z) = \frac{az}{(z-a)^2} \quad |z| > a$$

4. Determine the z-transform of following signal:

$$x(n) = \frac{1}{2}(n^2+n)\left(\frac{1}{3}\right)^{n-1} u(n-1)$$

Sol:

$$x(n) = \frac{1}{2}(n^2+n)\left(\frac{1}{3}\right)^{n-1} u(n-1)$$

$$= \frac{1}{2} n^2 \left(\frac{1}{3}\right)^{n-1} u(n-1) + \frac{1}{2} n \left(\frac{1}{3}\right)^{n-1} u(n-1)$$

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W.K.T $\quad \mathcal{Z}\left[\left(\frac{1}{3}\right)^n u(n)\right] = \dfrac{z}{z-\frac{1}{3}} \quad |z| > \frac{1}{3}$

Using time shifting property:

$$\mathcal{Z}\left[\left(\frac{1}{3}\right)^{n-1} u(n-1)\right] = z^{-1} \times \frac{z}{z-\frac{1}{3}} = \frac{1}{z-\frac{1}{3}}.$$

Using multiplication property:

$$\mathcal{Z}\left[n \cdot \left(\frac{1}{3}\right)^{n-1} u(n-1)\right] = -z \cdot \frac{d}{dz}\left[\frac{1}{z-\frac{1}{3}}\right]$$

$$= -z \times \frac{-1}{\left(z-\frac{1}{3}\right)^2} = \frac{z}{\left(z-\frac{1}{3}\right)^2}$$

and $\mathcal{Z}\left[n^2 \cdot \left(\frac{1}{3}\right)^{n-1} u(n-1)\right] = -z \dfrac{d}{dz}\left(\dfrac{z}{\left(z-\frac{1}{3}\right)^2}\right)$

$$= \frac{z\left(z+\frac{1}{3}\right)}{\left(z-\frac{1}{3}\right)^3}$$

$$\therefore X(z) = \frac{1}{2}\left[\frac{z\left(z+\frac{1}{3}\right)}{\left(z-\frac{1}{3}\right)^3} + \frac{z}{\left(z-\frac{1}{3}\right)^2}\right]$$

$$= \frac{z^2}{\left(z-\frac{1}{3}\right)^3} \quad \text{ROC } |z| > \frac{1}{3}$$

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5. Find $x(\infty)$ if $X(z)$ is given by $\dfrac{3z+4}{(z-1)(z+4)}$

Sol: Using final value theorem:

$$x(\infty) = \lim_{z \to 1} (z-1)X(z)$$

$$= \lim_{z \to 1} (z-1) \cdot \frac{3z+4}{(z-1)(z+4)}$$

$$\therefore \lim_{z \to 1} \frac{3z+4}{z+4} = 7/5$$