Unit- III. Applications of Partial Differential Equations.

Classification of Partial differential equations of the second Order :

The most general linear partial differential equation of second order can be written as

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = 0 \longrightarrow (1)$$

ie, $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$

where A, B, C, D, E, F are in general functions of x & y.

The equation (1) of second order (linear) is said to

(i) elliptic if $B^2 - 4AC < 0$

(ii) hyperbolic if $B^2 - 4AC > 0$

(iii) parabolic if $B^2 - 4AC = 0$.

Examples :-

(1) Elliptic Type : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Laplace equation in two dimension)

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$ (Poisson's equation)

(2) Parabolic type : $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ (one- dimensional heat equation)

(3) Hyperbolic type : $\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ (one- dimensional wave equation).

Problems :- Classify the following equations :

① $x u_{xx} + u_{yy} = 0$

$A = x \quad B = 0 \quad C = 1$

$B^2 - 4AC = 0 - 4(1)(x)$

$= -4x$

The equation is elliptic if $x > 0$

The equation is hyperbolic if $x < 0$

The equation is parabolic if $x = 0$.

Note :- The same differential equation may be elliptic in one region, parabolic in another and hyperbolic in some other region.

---

② $\frac{\partial u}{\partial x^2} + 2 \frac{\partial u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$

$A = 1, \quad B = 2, \quad C = 1$

$B^2 - 4AC = 4 - 4(1)(1) = 0$.

Hence, the equation is parabolic at all points.

---

③ $x f_{xx} + y f_{yy} = 0, \quad x > 0, \; y > 0$

$A = x \quad B = 0 \quad C = y$

$B^2 - 4AC = 0 - 4xy$

$= -4xy$

$\therefore$ It is elliptic for all $x > 0, \; y > 0$.

---

④ $x^2 f_{xx} + (1-y^2) f_{yy} = 0$

here, $A = x^2 \quad B = 0 \quad C = 1-y^2$

$$B^2 - 4AC = 0 - 4(x^2)(1-y^2)$$
$$= -4x^2(1-y^2)$$
$$= 4x^2(y^2-1)$$

For all x, except $x=0$ (x^2 is positive)

If $-1<y<1$, y^2-1 is negative.

$\therefore$ For $-\infty<x<\infty$ $(x\neq 0)$, $-1<y<1$, the equation is elliptic.

For $-\infty<x<\infty$, $x\neq 0$, $y<-1$ or $y>1$, the equation is hyperbolic.

For $x=0$ for all y or for all x, $y=\pm 1$, the equation is parabolic.

---

⑤ $u_{xx} + 4u_{xy} + (x^2+4y^2)u_{yy} = \sin(x+y)$.

here, $A=1$; $B=4$; $C = x^2+4y^2$

$$B^2 - 4AC = 16 - 4(1)(x^2+4y^2)$$
$$= 4(4-(x^2+4y^2))$$
$$= 4(4-x^2-4y^2).$$

The equation is elliptic if
$$4-x^2-4y^2<0$$
$$-x^2-4y^2<-4.$$
$$x^2+4y^2>4.$$
$$\frac{x^2}{4}+\frac{y^2}{1}>1$$

$\therefore$ It is elliptic in the region outside the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$.

The equation is hyperbolic if
$$4-x^2-4y^2>0$$
$$-x^2-4y^2>-4$$
$$x^2+4y^2<4$$
$$\frac{x^2}{4}+\frac{y^2}{1}<1$$

$\therefore$ It is hyperbolic in the region inside the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$

The equation is parabolic if $4 - x^2 - 4y^2 = 0$

$$-x^2 - 4y^2 = -4$$

$$x^2 + 4y^2 = 4$$

$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

It is parabolic on the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$

---

(6) The Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

here, $A = 1, \quad B = 0, \quad C = 1$

$\therefore \; B^2 - 4AC = 0 - 4 < 0$.

Hence the equation is elliptic

The poisson's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$

here also, $B^2 - 4AC = -4 < 0$.

Hence the equation is elliptic.

One dimensional heat equation $\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

here, $A = \alpha^2, \quad B = 0, \quad C = 0$

$\therefore \; B^2 - 4AC = 0 - 4(\alpha^2)(0) = 0$

Hence the equation is parabolic.

One dimensional wave equation $\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$

here, $A = \alpha^2, \quad B = 0, \quad C = -1$

$B^2 - 4AC = 0 - 4(\alpha^2)(-1) = 4\alpha^2 > 0$

$\therefore$ The equation is hyperbolic.

Exercise problems:-

① $(x+1)u_{xx} - 2(x+2)u_{xy} + (x+3)u_{yy} = 0$

② $\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} - 12\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + 7u = x^2+y^2$

③ $(1+x^2) f_{xx} + (5+2x^2) f_{xy} + (4+x^2) f_{yy} = 2\sin(x+y)$

④ $(1-x^2) f_{xx} - 2xy f_{xy} + (1-y^2) f_{yy} = 0$.

⑤ prove $f_{xx} + 2f_{xy} + 4f_{yy} = 0$ is elliptic

⑥ prove $f_{xx} - 2f_{xy} + f_{yy} = 0$ and $u_{xx} = u_t$ are parabolic