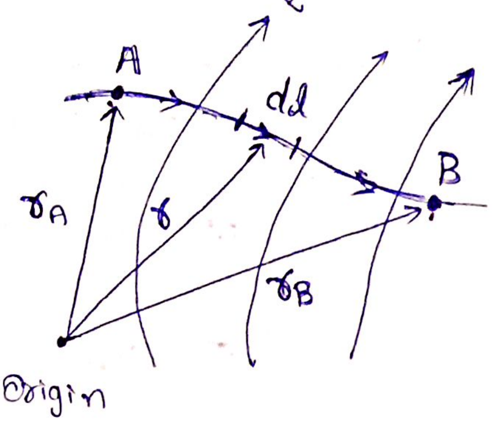
*. ELECTRIC POTENTIAL:-

Fig: Displacement of point charge in an electrostatic field E.



E can be obtained from coulombs law in general or Gauss's law when the charge distribution is symmetric.

'E' can be obtained from the electric scalar potential 'V'. It is easy to handle scalar when compared to vector.

Point charge Q from A to B in E, from Coulomb's law, the force on Q is $F = QE$, so the work done in displacing charge by dl is

$$dW = -F \cdot dl = -QE \cdot dl \longrightarrow ①$$

The -ve sign indicates the work is being done by an external agent.

total work done or potential energy required, in moving Q from A to B is

$$\boxed{W = -Q\int_A^B E \cdot dl} \longrightarrow ②$$

dividing W by Q gives the potential energy per unit charge. This quantity is denoted by V_{AB}, is known as potential difference b/w points A & B. Thus

$$\boxed{V_{AB} = \frac{W}{Q} = -\int_A^B E \cdot dl}$$

Note:- • In V_{AB}, A is initial point and B is the final point.

- If V_{AB} is -ve, there is a loss in P.E in moving Q from A to B, this implies that the work is being done by the field. However, if V_{AB} is +ve, there is a gain in P.E in the movement; an external agent performs the work.
- V_{AB} is independent of the path taken.
- V_{AB} is measured in jouls per Coulomb, commonly referred to as volts (V).

E due to a point charge Q located at the origin can be written as

$$E = \frac{Q}{4\pi\epsilon_0 r^2} a_r$$

$$\therefore V_{AB} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon_0 r^2} a_r \cdot dr\, a_r = \frac{-Q}{4\pi\epsilon_0}\int_{r_A}^{r_B} \frac{1}{r^2} dr = \frac{Q}{4\pi\epsilon_0} \left.\frac{1}{r}\right]_{r_A}^{r_B}$$

$$\Rightarrow V_{AB} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right] \longrightarrow (3)$$

$$\Rightarrow V_{AB} = \frac{Q}{4\pi\epsilon_0 r_B} - \frac{Q}{4\pi\epsilon_0 r_A}$$

$$\boxed{\therefore V_{AB} = V_B - V_A}$$

V_B & V_A are potentials at B & A, respectively. Thus the potential difference V_{AB} regarded as the potential at B with reference to A. In problems involving point charges, it is customary to choose infinity as reference; that is we assume the potential at infinity is zero. $V_A = \frac{Q}{4\pi\epsilon_0 r_A}$, $r_A \to \infty$ $V_A = 0$

The potential at any point ($r_B \to r$) due to a point charge Q located at origin is

$$\boxed{V = \frac{Q}{4\pi\epsilon_0 r}}.$$

def:- The potential at any point is the potential difference between that point and a chosen point at which the potential is zero.

$$V = -\int_{\infty}^{r} E \cdot dl$$

If the point charge 'Q' is not located at origin but at a point whose position vector is r', the potential $V(x,y,z)$ or simply $V(r)$ at r becomes

$$V(r) = \frac{Q}{4\pi\epsilon_0 |r - r'|}.$$

$Q_1, Q_2 \ldots Q_n$ at $r_1, r_2, \ldots r_n$ the potential at r is

$$V(r) = \frac{Q_1}{4\pi\epsilon_0 |r - r_1|} + \frac{Q_2}{4\pi\epsilon_0 |r - r_2|} + \cdots + \frac{Q_n}{4\pi\epsilon_0 |r - r_n|}.$$

$$V(r) = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{n} \frac{Q_k}{|r - r_k|} \quad \text{(point charges)}.$$

for continuous charges $V(\bar{r}) = \frac{1}{4\pi\epsilon_0} \int_L \frac{\rho_L(\bar{r}')\,dl'}{|\bar{r}-\bar{r}'|}$ (line charge)

$V(\bar{r}) = \frac{1}{4\pi\epsilon_0} \int_S \frac{\rho_S(\bar{r}')\,ds'}{|\bar{r}-\bar{r}'|}$ (sur ch)

$V(\bar{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho_V(\bar{r}')\,dv'}{|\bar{r}-\bar{r}'|}$ (volu ch)

$V_{AB} = V_B - V_A = -\int E \cdot dl = \frac{W}{Q}$. $\quad V = \frac{Q}{4\pi\epsilon_0 r} + C$

→ Two point charges $-4\mu C$ & $5\mu C$ are located at $(2,-1,3)$ and $(0,4,-2)$, Find the potential at $(1,0,1)$ assuming zero potential at infinity.

Sol:- $Q_1 = -4\mu C$, $Q_2 = 5\mu C$

$$V(\bar{r}) = \frac{Q_1}{4\pi\epsilon_0 |\bar{r}-\bar{r}_1|} + \frac{Q_2}{4\pi\epsilon_0 |\bar{r}-\bar{r}_2|} + C_0 \qquad C_0 = 0, \; V(\infty) = 0$$

$|\bar{r}-\bar{r}_1| = |(1,0,1)-(2,-1,3)| = |(-1,1,-2)| = \sqrt{6}$.

$|\bar{r}-\bar{r}_2| = \sqrt{26}$

$$V(1,0,1) = \frac{10^{-6}}{4\pi \times \frac{10^{-9}}{36\pi}} \left[-\frac{4}{\sqrt{6}} + \frac{5}{\sqrt{26}}\right] = -5.872 \text{ kV}.$$

Relation b/w E & V:-

$V_{AB} = V_B - V_A$

$V_{BA} = V_A - V_B = -(V_B - V_A) = -V_{AB}$

$V_{BA} + V_{AB} = \oint E \cdot dl = 0$

or $\boxed{\oint E \cdot dl = 0.}$ ⟶ (1)

Fig:- Conservative nature of electrostatic field.

physically, this implies that no net work is done in moving a charge along a closed path in an electrostatic field. Applying Stokes th

$$\oint E \cdot dl = \int (\nabla \times E) \cdot ds = 0$$

(or) $\nabla \times E = 0$. ⟶ (2)

any vector that satisfies (1) & (2) are called conservative or irrotational.

It is referred to the second Maxwell's equation for static electric fields.

Potential, $V = -\int E \cdot dl$

$$dV = -E \cdot dl = -(E_x a_x + E_y a_y + E_z a_z) \cdot (dx\, a_x + dy\, a_y + dz\, a_z)$$

$$= -E_x dx - E_y dy - E_z dz . \longrightarrow (3)$$

but

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz . \longrightarrow (4)$$

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}$$

$$E = -\nabla V$$

$\downarrow$

E is opposite to the direction in which V increases.

$$\nabla \times E = 0 \Rightarrow \nabla \times \nabla V = 0 .$$

*. AN ELECTRIC DIPOLE AND FLUX LINES :-

An electric dipole is formed when two point charges of equal magnitude but opposite sign are ~~sp~~ separated by a small distance.

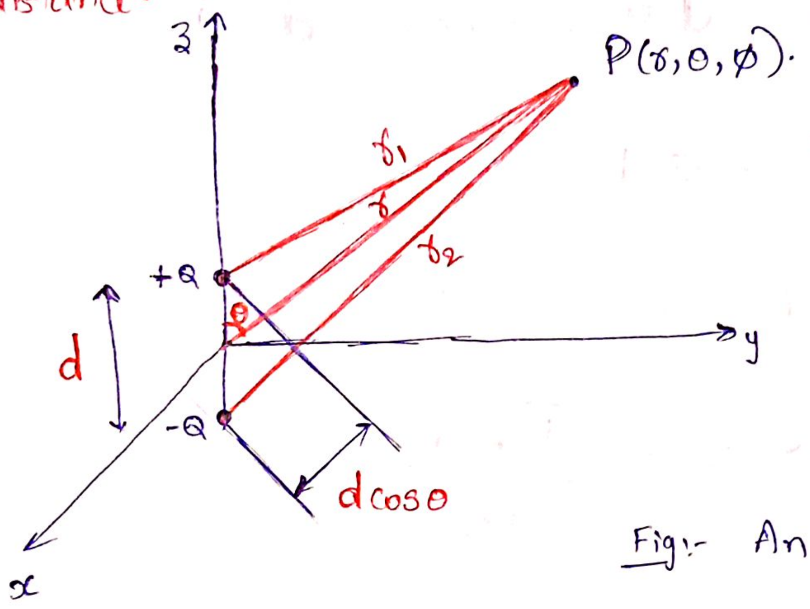


Fig:- An electric dipole.

Consider the dipole shown in Fig. The potential at point $P(r, \theta, \phi)$ is given by

$$V = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

$$\Rightarrow V = \frac{Q}{4\pi\varepsilon_0}\left[\frac{r_2 - r_1}{r_1 r_2}\right] \longrightarrow (1)$$

where r_1 and r_2 are the distances between P and $+Q$ & P and $-Q$, respectively.

~~If~~ $r >> d$, $\quad r_2 = r_1 + d\cos\theta$

$$\therefore r_2 - r_1 = d\cos\theta.$$

and $\quad r_1 r_2 \simeq r^2$,

then eq. (1) becomes

$$\Rightarrow \quad V = \frac{Q}{4\pi\varepsilon_0} \frac{d\cos\theta}{r^2} \longrightarrow (2)$$

since $d\cos\theta = \vec{\mathbf{d}}\cdot\vec{a}_r$, where $\vec{\mathbf{d}} = d\vec{a}_z$, if we define

$\vec{\mathbf{P}} = Q\vec{d}$ as the dipole moment,

eq (2) can be written as

$$\Rightarrow \quad V = \frac{\vec{P}\cdot\vec{a}_r}{4\pi\varepsilon_0 r^2} \longrightarrow (3)$$

<u>Note</u>:- The dipole moment $\vec{P}$ is directed from $-Q$ to $+Q$.

If the dipole center is not at the origin but at r', eq (3) becomes

$$\cancel{V(\vec{r}) = \frac{\vec{P}\cdot(\vec{r}-\vec{r}')}{4\pi\varepsilon_0 |\vec{r}-\vec{r}'|^3}}$$

$$\boxed{V(r) = \frac{\vec{P}\cdot(r-r')}{4\pi\varepsilon_0 |r-r'|^3}} \longrightarrow (4).$$

The electric field due to the dipole with center at the origin, can be obtained readily from

$$\vec{E} = -\nabla V$$

$$\Rightarrow \vec{E} = -\left[\frac{\partial V}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a}_\theta\right]$$

$$\Rightarrow \vec{E} = -\left[\frac{\partial}{\partial r}\left(\frac{Q}{4\pi\epsilon_0}\frac{d\cos\theta}{r^2}\right)\vec{a}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{Q}{4\pi\epsilon_0}\frac{d\cos\theta}{r^2}\right)\vec{a}_\theta\right]$$

$$\Rightarrow \vec{E} = \frac{Qd\cos\theta}{2\pi\epsilon_0 r^3}\vec{a}_r + \frac{Qd\sin\theta}{4\pi\epsilon_0 r^3}\vec{a}_\theta$$

or

$$\vec{E} = \frac{Qd}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\vec{a}_r + \sin\theta\,\vec{a}_\theta\right)$$

$$\therefore \boxed{\vec{E} = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\vec{a}_r + \sin\theta\,\vec{a}_\theta\right)}$$

where $p = |\vec{p}| = Qd$.

Note:-

- A point charge is monopole and its electric field

$$E \propto \frac{1}{r^2} \text{ and } V \propto \frac{1}{r}$$

- For dipole., $E \propto \frac{1}{r^3}$ and $V \propto \frac{1}{r^2}$

An electric flux line is an imaginary path or line drawn in such a way that its direction at any point is the direction of the electric field at that point.

In other words, they are the lines to which the electric flux density $\vec{D}$ is tangential at every point.

Any surface on which the potential is the same throughout is known as an equipotential surface. The intersection of an equipotential surface and a plane results in a path or line known as an equipotential line. No work is done in moving a charge from one point to another along an equipotential line or surface $(V_A - V_B) = 0$ and hence

$$\int_L E \cdot dl = 0.$$

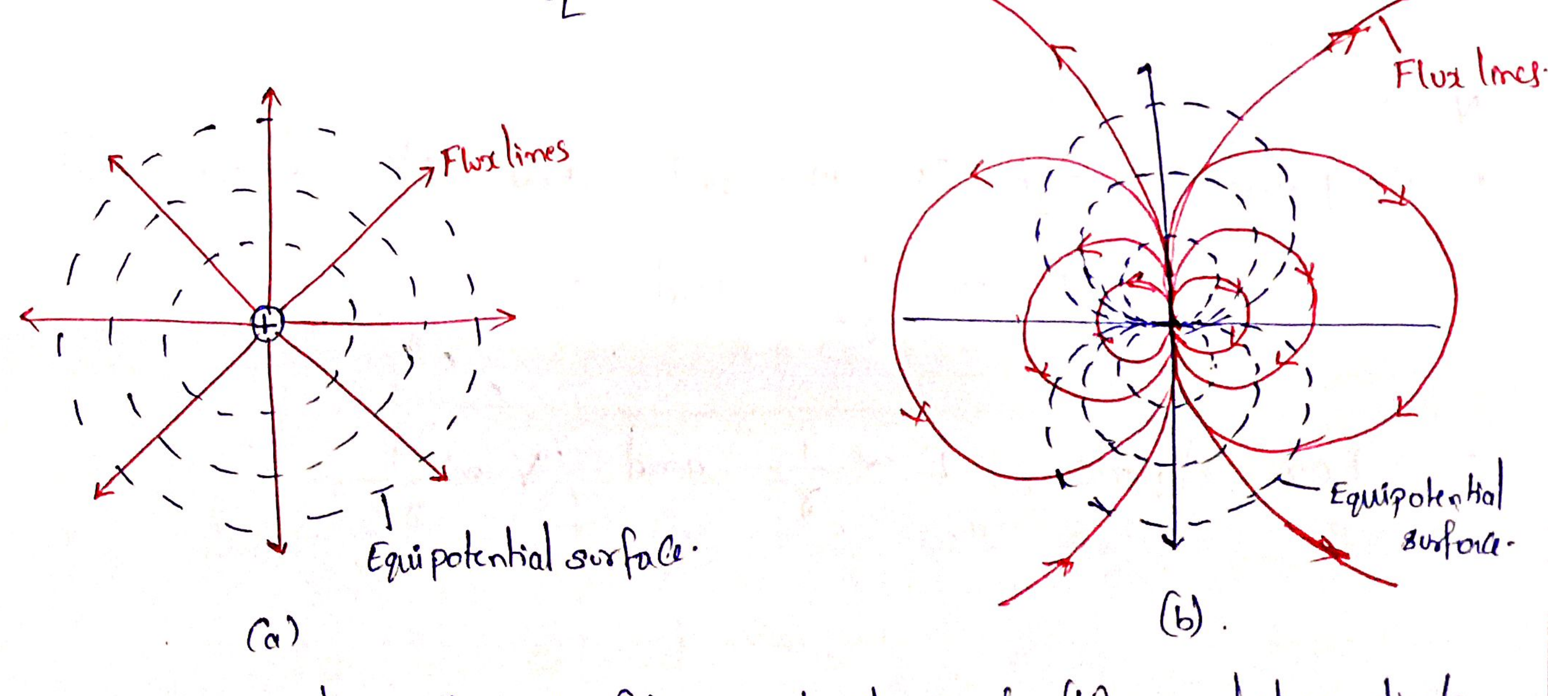


Fig:- Equipotential surfaces for (a) a point charge & (b) an electric dipole.

*. ENERGY DENSITY IN ELECTROSTATIC FIELDS :-

To determine the energy present in an assembly of charges, we must ~~And~~ first determine the amount of work necessary to assemble them. Suppose we wish to position three point charges Q_1, Q_2, and Q_3 in an initially empty space shown in Fig.

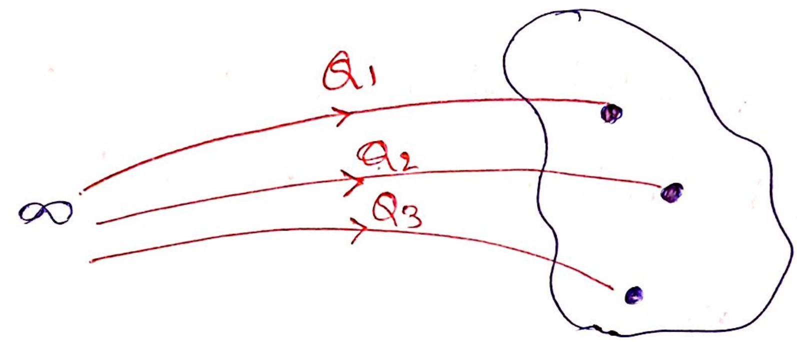


Fig :- Assembling of charges.

No work is required to transfer Q_1 from infinity to P_1 because the space is initially charge free and there is no electric field. The work done in transferring Q_2 from infinity to P_2 is equal to the product of Q_2 and the potential V_{21} at P_2 due to Q_1. Similarly, the work done in positioning Q_3 at P_3 is equal to $Q_3(V_{32} + V_{31})$, where V_{32} & V_{31} are the potentials at P_3 due to Q_2 and Q_1, respectively.

Hence the total work done in positioning the three charges is $W_E = W_1 + W_2 + W_3$.

$$\Rightarrow W_E = 0 + Q_2 V_{21} + Q_3 (V_{31} + V_{32}). \longrightarrow (1)$$

If the charges were positioned in reverse order,

$$\Rightarrow W_E = W_3 + W_2 + W_1$$

$$\Rightarrow W_E = 0 + Q_2 V_{23} + Q_1 (V_{12} + V_{13}). \longrightarrow (2)$$

where V_{23} is the potential at P_2 due to Q_3, V_{12} and V_{13} are, respectively, the potentials at P_1 due to Q_2 & Q_3.

adding (1) + (2) gives..

$$\Rightarrow 2W_E = Q_1 (V_{12} + V_{13}) + Q_2 (V_{21} + V_{23}) + Q_3 (V_{31} + V_{32}).$$

$$\Rightarrow 2W_E = Q_1 V_1 + Q_2 V_2 + Q_3 V_3.$$

$$\boxed{\therefore W_E = \frac{1}{2}(Q_1 V_1 + Q_2 V_2 + Q_3 V_3)} \longrightarrow (3)$$

where V_1, V_2, and V_3 are total potentials at P_1, P_2 and P_3 respectively.

In general, if there are 'n' point charges, eq(3) becomes

$$\boxed{W_E = \frac{1}{2} \sum_{k=1}^{n} Q_k V_k} \quad \text{(in joules)}. \longrightarrow (4)$$

If, instead of point charges, the region has a continuous charge distribution, the summation in eq. (4) becomes integration; that is,

$$W_E = \frac{1}{2}\int_L \rho_L V\, dl \quad \text{(line charge)} \rightarrow (5)$$

$$W_E = \frac{1}{2}\int_S \rho_S V\, ds \quad \text{(surface charge)} \rightarrow (6)$$

$$W_E = \frac{1}{2}\int_V \rho_v V\, dv \quad \text{(volume charge)} \rightarrow (7)$$

Since $\rho_v = \nabla\cdot\vec{D}$, eq. (7) can be further developed to yield

$$W_E = \frac{1}{2}\int_V (\nabla\cdot\vec{D})\, V\, dv \longrightarrow (8)$$

But for any vector $\vec{A}$ and scalar V, the identity

$$\nabla\cdot V\vec{A} = \vec{A}\cdot\nabla V + V(\nabla\cdot\vec{A})$$

or $$(\nabla\cdot\vec{A})V = \nabla\cdot V\vec{A} - \vec{A}\cdot\nabla V \longrightarrow (9)$$

Applying the identity in eq. (9) to (8), we get

$$\Rightarrow W_E = \frac{1}{2}\int_V (\nabla\cdot V\vec{D})\, dv - \frac{1}{2}\int_V (\vec{D}\cdot\nabla V)\, dv$$

By applying divergence theorem to the first term on the RHS of this equation, we have

$$\Rightarrow W_E = \frac{1}{2}\oint_S (V\vec{D})\cdot ds - \frac{1}{2}\int_V (\vec{D}\cdot\nabla V)\, dv \rightarrow (10)$$

We know that V varies as $\frac{1}{r}$ and $\vec{D}$ as $\frac{1}{r^2}$ for point charges; V varies as $\frac{1}{r^2}$ and $\vec{D}$ as $\frac{1}{r^3}$ for dipoles; and so on. Hence, $V\vec{D}$ in the first term on the R.H.S must vary at least as $\frac{1}{r^3}$ while dS varies as r^2. Consequently, the first integral in eq. (10) must tend to ~~zero~~ ~~as~~ zero as the surface S becomes large.

$$\Rightarrow W_E = 0 - \frac{1}{2}\int_v (\vec{D}\cdot\nabla V)\,dv$$

$$\Rightarrow W_E = -\frac{1}{2}\int_v (\vec{D}\cdot -\vec{E})\,dv$$

$(\because \vec{E} = -\nabla V$
$+ \vec{D} = \epsilon_0\vec{E})$

$$\Rightarrow W_E = \frac{1}{2}\int_v (\vec{D}\cdot\vec{E})\,dv$$

$$\Rightarrow W_E = \frac{1}{2}\int_v \epsilon_0\vec{E}\cdot\vec{E}\,dv$$

$$\Rightarrow W_E = \frac{1}{2}\int_v \epsilon_0 |\vec{E}|^2\,dv$$

$$\boxed{\therefore W_E = \frac{1}{2}\int_v \vec{D}\cdot\vec{E}\,dv = \frac{1}{2}\int_v \epsilon_0 E^2\,dv}$$

From this, we can define electrostatic energy density w_E (in J/m^3) as

$$w_E = \frac{dW_E}{dv} = \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{1}{2}\varepsilon_0 E^2 = \frac{D^2}{2\varepsilon_0}$$

$$\boxed{\therefore \quad W_E = \int_v w_E \, dv}$$