S. Kunal Keshan

RA2011004010051

ECE – A

18MAB101J

RA2011004010051
S. KUNAL KESHAN

13-01-2021 18MAB101T - CALCULUS AND LINEAR ALGEBRA

UNIT-IV- APPLICATION OF DIFFERENTIAL CALCULUS

ASSIGNMENT-II

1. Find the radius of curvature at any point on the curve
$x = a(\cos t + t\sin t)$
$y = a(\sin t - t\cos t)$

Solution: Given,

$x = a(\cos t + t\sin t)$ — ①

$y = a(\sin t - t\cos t)$ — ②

Differentiating ① and ② w.r.t parameter t.

$$\frac{dx}{dt} = a(\cancel{-\sin t} + t\cos t \cancel{+ \sin t})$$
$$= at\cos t \quad - ③$$

$$\frac{dy}{dt} = a(\cos t - (t(-\sin t) + \cos t)$$
$$= a(\cos t + t\sin t - \cos t)$$
$$= at\sin t \quad - ④$$

Equating ④ ÷ ③, we get.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{at\sin t}{at\cos t} = \tan t - ⑤$$

$y_1 = \tan t$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan t)$$
$$= \frac{d}{dt}(\tan t)\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = Sec^2 t \frac{1}{at\,Cos t}$$

$$y_2 = \frac{Sec^3 t}{at}$$

Radius of Curvature $= \frac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{(1+\tan^2 t)^{3/2}}{\frac{Sec^3 t}{at}}$$

$1 + \tan^2 t = Sec^2 t$

$$= \frac{(Sec^2 t)^{3/2}}{Sec^3 t} \times at$$

$$= \frac{Sec^3 t}{Sec^3 t} \times at$$

$$= at$$

Radius of Curvature, $\rho = at$

Q2. Find the Co-ordinates of Center of Curvature of the Curve $y = x^3 - 6x^2 + 3x + 1$ at $(1, -1)$.

Solution: Given,

$$y = x^3 - 6x^2 + 3x + 1$$

To find Centre of Curvature at $(1, -1)$.

We know that the Co-ordinates of the Centre of Curvature is given by

$$\bar{x} = x - \left[\frac{y_1(1+y_1^2)}{y_2}\right] \text{ and } \bar{y} = y + \left[\frac{1+y_1^2}{y_2}\right]$$

$$\frac{dy}{dx} = 3x^2 - 12x + 3$$

$$y_1 = \left.\frac{dy}{dx}\right|_{(1,-1)} = 3 - 12 + 3$$

$$= -6$$

$$\frac{d^2y}{dx^2} = 6x - 12$$

$$y_2 = \left.\frac{d^2y}{dx^2}\right|_{(1,-1)} = -6$$

Substitution y_1 and y_2 in $\bar{x}$ and $\bar{y}$, we get,

$$\bar{x}\Big|_{(1,-1)} = 1 - \left[\frac{-6(1 + 36)}{-6}\right]$$

$$= 1 - [37] = -36.$$

$$\bar{y}\Big|_{(1,-1)} = -1 + \left[\frac{1 + 36}{-6}\right] = -1 - \frac{37}{6}$$

$$= \frac{-6 - 37}{6} = -\frac{43}{6}.$$

The Coordinates of the Centre of Curvature of the Curve $y = x^3 - 6x^2 + 3x + 1$ is $(-36, -\frac{43}{6})$

Q 3. Find the equation of Circle of Curvature of the Curve $x^3 + y^3 = 3axy$ at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

Solution: Given,

$$x^3 + y^3 = 3axy \quad -①$$

To find Circle of Curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$.

we know that,

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2 \text{ is the Circle of Curvature.} \quad -②$$

Diff ① wrt x.

$$3x^2 + 3y^2 \frac{dy}{dx} = 3ax\frac{dy}{dx} + 3ay \quad -③$$

Diff again wrt x.

$$6x + 3y^2 \cdot \frac{d^2y}{dx^2} + 6y\left(\frac{dy}{dx}\right)^2 = 3ax\frac{d^2y}{dx^2} + 3a\frac{dy}{dx} + 3a\frac{dy}{dx} \quad - ④$$

Eqn ③ can be written as.

$$3y^2\frac{dy}{dx} - 3ax\frac{dy}{dx} = 3ay - 3x^2$$

$$\frac{dy}{dx} = \frac{3ay - 3x^2}{3y^2 - 3ax}$$

Then $y_1 = \left.\frac{dy}{dx}\right|_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} = \dfrac{3a \times \frac{3a}{2} - 3\times\frac{9a^2}{4}}{3\times\frac{9a^2}{4} - 3a\times\frac{3a}{2}}$

$$= \frac{\frac{9a^2}{2} - \frac{27a^2}{4}}{27a^2/4 - 9a^2/2}$$

④

$\times \frac{2}{2}$ to $\frac{9a^2}{2}$

$$y_1 = \frac{\frac{18a^2 - 27a^2}{4}}{\frac{27a^2 - 18a^2}{4}} = \frac{-9a^2}{9a^2} = -1 = y_1$$

Eqs (4) can be written as.

$$3y^2 \frac{d^2y}{dx^2} - 3ax\frac{d^2y}{dx^2} + 6y\left(\frac{dy}{dx}\right)^2 = 6a\frac{dy}{dx} - 6x$$

$$\frac{d^2y}{dx^2} = \frac{6a\,y_1 - 6x - 6y \cdot y_1^2}{3y^2 - 3ax}$$

Then $y_2 = \left.\frac{d^2y}{dx^2}\right|_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} = \frac{6a(-1) - 6 \times \frac{3a}{2} - 6 \times \frac{3a}{2} \times (-1)^2}{3 \times \frac{9a^2}{2} - 3a \times \frac{3a}{2}}$

$$= \frac{-6a - 9a - 9a}{\frac{27a^2}{2} - \frac{9a^2}{2}}$$

$$= \frac{-24a}{\frac{\cancel{18}9a^2}{\cancel{2}}} = \frac{-24}{9a} = -\frac{8}{3a} = y_2$$

Radius of Curvature $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{(1+(-1)^2)^{3/2}}{-\frac{8}{3a}} = \frac{\cancel{2}\sqrt{2} \times 3a}{-\cancel{8}4}$$

$$|\rho| = \frac{3\sqrt{2}a}{4}$$

Coordinates of the Centre of Curvature is,

$$\bar{x} = x - \left[\frac{y_1 (1 + y_1^2)}{y_2} \right]$$

$$\bar{y} = y + \left[\frac{1 + y_1^2}{y_2} \right]$$

Substituting the Values of y_1 and y_2 we get.

$$\bar{x} \Big|_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} = \frac{3a}{2} - \left[\frac{(-1)(1 + (-1)^2)}{-\frac{8}{3a}} \right]$$

$$= \frac{3a}{2} + \left(-\frac{3a}{8} \right)$$

$$= \frac{3a}{2} - \frac{3a}{4} = \frac{6a - 3a}{4} = \frac{3a}{4} = \bar{x}$$

$$\bar{y} \Big|_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} = \frac{3a}{2} + \left[\frac{1 + (-1)^2}{\frac{-8}{3a}} \right]$$

$$= \frac{3a}{2} - \left[\frac{3a}{4} \right]$$

$$= \frac{3a}{4} = \bar{y}$$

Then the Circle of Curvature is

$$(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$$

$$\left(x - \frac{3a}{4}\right)^2 + \left(y - \frac{3a}{4}\right)^2 = \left(\frac{3\sqrt{2}\,a}{4}\right)^2$$

$$\left(x - \frac{3a}{4}\right)^2 + \left(y - \frac{3a}{4}\right)^2 = \frac{9a^2}{8} \;//$$

4. Find the evolute of the parabola $x^2 = 4ay$.

Solution: the given Curve is a parabola, therefore its parametic form is,

$$x = at^2$$
$$y = 2at.$$

diff w.r.t t, we get

$$\frac{dx}{dt} = 2at$$

$$\frac{dy}{dt} = 2a$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t} = y_1$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}(1/t) \times \frac{dt}{dx} = -\frac{1}{t^2} \times \frac{1}{2at}$$

$$= -\frac{1}{2at^3} = y_2$$

The Coordinates for the Centres of Curvature is given by,

$$\bar{x} = x - \left[\frac{y_1(1 + (y_1)^2)}{y_2}\right]$$

$$\bar{y} = y + \left[\frac{1 + y_1^2}{y_2}\right]$$

$$\bar{x} = at^2 - \left[\frac{\frac{1}{t}\left(1 + \frac{1}{t^2}\right)}{\frac{-1}{2at^3}}\right]$$

$$= at^2 + \left[\frac{1}{t}\left(\frac{t^2+1}{t^2}\right) / \frac{1}{2at^3}\right]$$

$$\bar{x} = at^2 + 2a(t^2+1)$$

$$\bar{x} = 3at^2 + 2a$$

$$\bar{y} = 2at + \left[\frac{1 + \frac{1}{t^2}}{\frac{-1}{2at^3}} \right]$$

$$= 2at - \left[\frac{t^2+1}{\cancel{t^2}} \times 2at^{\cancel{3}} \right]$$

$$= 2at - \left[2at^3 + 2at \right]$$

$$\bar{y} = -2at^3$$

Eliminating t from $\bar{x}$ and $\bar{y}$

$$\bar{x} = 3at^2 + 2a \implies t^2 = \frac{\bar{x} - 2a}{3a} \quad -①$$

$$\bar{y} = -2at^3 \implies t^3 = -\frac{\bar{y}}{2a} \quad -②$$

Taking $①^3$ and $②^2$ we get.

$$t^6 = \left(\frac{\bar{x} - 2a}{3a}\right)^3 \quad \text{and} \quad t^6 = \left(\frac{-\bar{y}}{2a}\right)^2.$$

Comparing the two eqs, we get.

$$\left(\frac{\bar{x} - 2a}{3a}\right)^3 = \left(\frac{-\bar{y}}{2a}\right)^2$$

$$\frac{(\bar{x} - 2a)^3}{27a^{\cancel{3}}} = \frac{\bar{y}^2}{4\cancel{a^2}}$$

$$4(\bar{x} - 2a)^3 = 27a\bar{y}^2$$

$\bar{x} = x$ and $\bar{y} = y$

$4(x - 2a)^3 = 27ay^2$ is the required evolute.

Q 5. Find the envelope of the family of lines.

i) $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$, θ - parameter

ii) $\frac{x}{t} + yt = 2c$; c is constant.

Solution:

i) $$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1 \quad -(1)$$

Given θ is the parameter.

$\div$ Eqs (1) by $\cos\theta$.

$$\frac{x}{a} + \frac{y}{b}\tan\theta = \sec\theta$$

Squaring on both sides.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2}\tan^2\theta + \frac{2xy}{ab}\tan\theta = \sec^2\theta.$$

$(\sec^2\theta = 1+\tan^2\theta)$

$$\frac{y^2}{b^2}\tan^2\theta + \frac{2xy}{ab}\tan\theta + \frac{x^2}{a^2} = 1+\tan^2\theta.$$

$$\left(\frac{y^2}{b^2}-1\right)\tan^2\theta + \frac{2xy}{ab}\tan\theta + \frac{x^2}{a^2} - 1 = 0$$

It is of the form,

$$A\alpha^2 + B\alpha + C = 0.$$

The envelope of the family of lines is given by

$B^2 - 4AC = 0$; $A = \frac{y^2}{b^2} - 1$; $B = \frac{2xy}{ab}$, $C = \frac{x^2}{a^2} - 1$.

$$\frac{4x^2y^2}{a^2b^2} - 4\left[\frac{y^2-b^2}{b^2}\right] \times \left[\frac{x^2-a^2}{a^2}\right] = 0$$

$$\frac{4x^2y^2}{a^2b^2} - 4\left[\frac{x^2y^2 - y^2a^2 - b^2x^2 + a^2b^2}{a^2b^2}\right] = 0$$

$$\frac{\cancel{4x^2y^2} - \cancel{4x^2y^2} + 4y^2a^2 + 4b^2x^2 - 4a^2b^2}{a^2b^2} = 0$$

$$\frac{4y^2a^2 + 4b^2x^2 - 4a^2b^2}{a^2b^2} = 0$$

$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1$ is the envelope and is the equation of an ellipse with major axis on the y-axis.

ii) $\frac{x}{t} + yt = 2c$; c - constant. - (1)

(1) can be written as

$x + yt^2 = 2ct$

$yt^2 - 2ct + x = 0$

It is of the form $A\alpha^2 + B\alpha + C = 0$

Envelope is $B^2 - 4AC = 0$ $\quad A = y \quad B = -2c \quad C = x$

$4c^2 - 4yx$

$yx = c^2$

$y = \frac{c^2}{x}$ is the envelope.