SRM UNIVERSITY
(Under section 3 of UGC Act 1956)

18MAB102T- Advanced Calculus and Complex Analysis

UNIT V
COMPLEX INTEGRATION

TOPICS DISCUSED

- *Line integral*
- *Cauchy's integral theorem(without proof)*
- *Cauchy's integral formula (with proof)*
- *Application of Cauchy's integral formula*
- *Taylor's and Laurent's expansion(statements only)*
- *Singularities*
- *Poles and Residues*
- *Cauchy's residue theorem (with proof)*
- *Evaluation of line integrals*

LINE INTEGRAL

Definition :

Let $w = f(z)$ be a continuous function of the complex variable $z = x + iy$ along a curve c with end points A and B

$$\int_c f(z)dz = \int_c (udx - vdy) + i(vdx + udy)$$

EXAMPLE 1

$$Evaluate \int_C \bar{z}dz \quad from\ A(0,0)\ to\ B(4,2)\ along$$

$$the\ curve\ C\ and\ z = t^2 + it$$

Solution:

$$Let\ \bar{z} = x - iy\ , \quad z = x + iy = t^2 + it$$

$$\Rightarrow x = t^2, y = t$$

$$dx = 2tdt \;,dy = dt \;\; and \;\; dz = dx + idy$$

$$= 2tdt + idt$$

$$= (2t + i)dt$$

$$Also \;\; x = 0,4 \;\Rightarrow t = 0,2$$

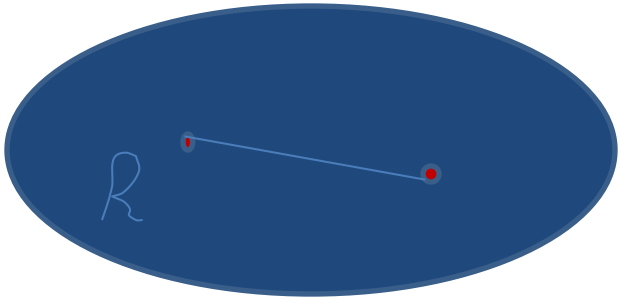
$$y = 0,2 \Rightarrow t = 0,2$$

$$I = \int_c \bar{z}dz = \int_0^2 (t^2 - it)(2t + i)dt = 10 - \frac{8}{3}i$$

DEFINITIONS

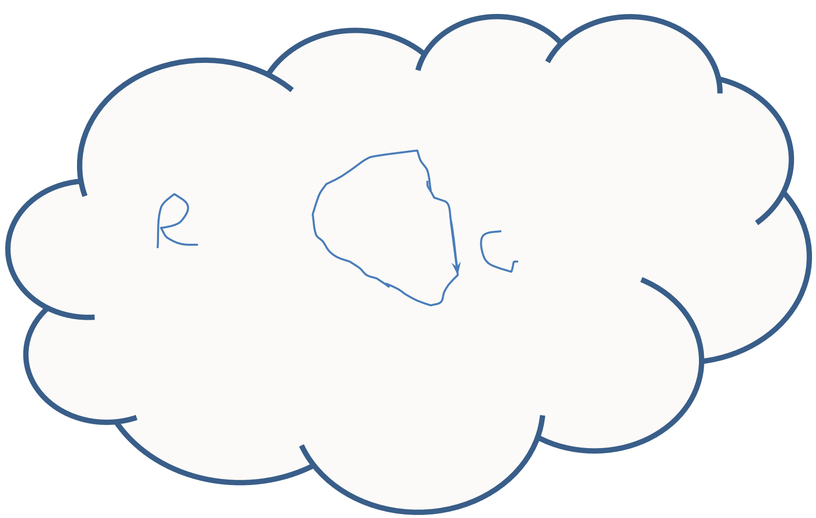
Connected region :

A region R is said to be connected when two points of it are connected by a curve; the curve should lie inside the region.



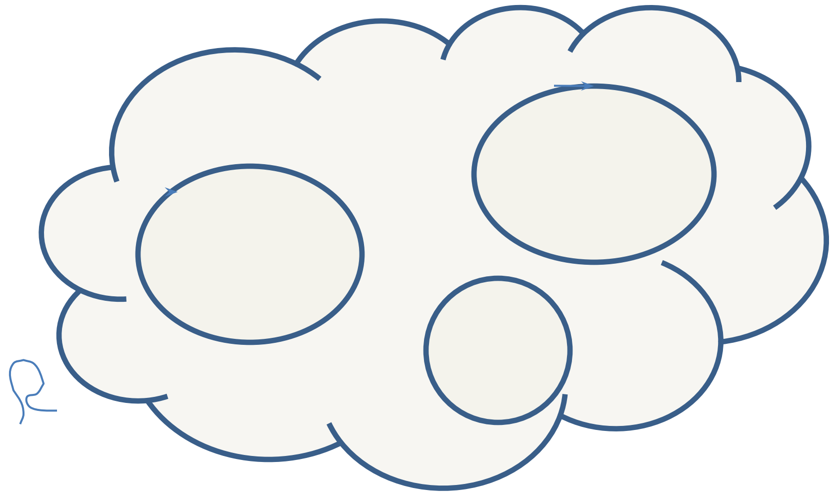
Simply Connected region :

A region R is said to be simply connected if any closed curve which lies in R can be shrunk to a point without leaving R



Multiply Connected region:

A region which is not simply connected.



NOTE

Multiply connected regions can be converted into a simply connected region by strip cuts

CAUCHY'S INTEGRAL THEOREM (or) CAUCHY'S FUNDAMENTAL THEOREM

If $f(z)$ is analytic and its derivatives $f'(z)$ is continuous at all points on and inside a simple closed curve C, then

$$\int_c f(z)\,dz = 0$$

CAUCHY'S INTEGRAL THEOREM FOR MULTIPLY CONNECTED REGION

If $f(z)$ *is analytic and its derivatives* $f'(z)$ *is continuous at all points in the region bounded by the simple closed curve* C_1 & C_2 *then*

$$\int_{c_1} f(z)\,dz = \int_{c_2} f(z)\,dz$$

CAUCHY'S INTEGRAL FORMULA

If $f(z)$ *is analytic inside and on a simple closed curve C that encloses a simple connected region R and if ' a ' is any point in R then*

$$f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}\,dz$$

Where C is described in the anticlockwise direction

CAUCHY'S INTEGRAL FORMULA FOR THE DERIVATIVES OF AN ANALYTIC FUNCTION

If a function $f(z)$ is analytic within and on a simple closed curve C and ' a ' is any point lying in it, then

$$f^n(a) = \frac{n!}{2\pi i}\oint_c \frac{f(z)}{(z-a)^{n+1}}dz$$

In general, $f^n(a) = \dfrac{n!}{2\pi i}\displaystyle\oint_c \frac{f(z)}{(z-a)^{n+1}}dz$

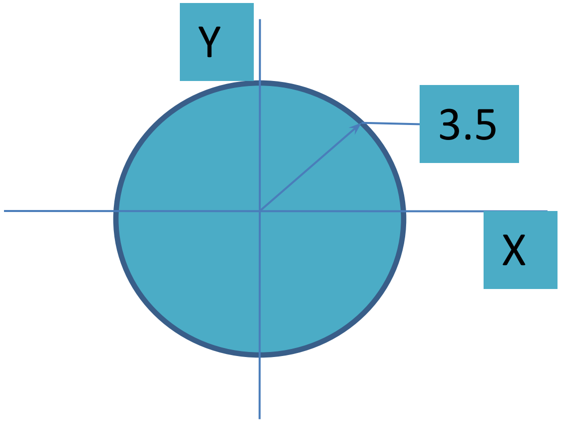
EXAMPLES 1

$$Evaluate \int_c \frac{dz}{z^2 - 7z + 12} \; where\, C \; is \; the$$

$$circle \; |z| = 3.5$$

$$Solution : Singularpoints : z^2 - 7z + 12 = 0 \Rightarrow z = 4, 3$$

$$z = 4 \; lies \; outside \; the \; circle \; |z| = 3.5$$



$$z = 3 \; lies \; inside \; the \; circle \; |z| = 3.5$$

$$\oint_c \frac{dz}{(z-4)(z-3)} = \oint_c \frac{\left(\frac{1}{z-4}\right)}{z-3} dz$$

$$Here \;\; f(z) = \frac{1}{z-4} \;\; is \; analytic \; inside \; C$$

$$\oint \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$= 2\pi i f(3)$$

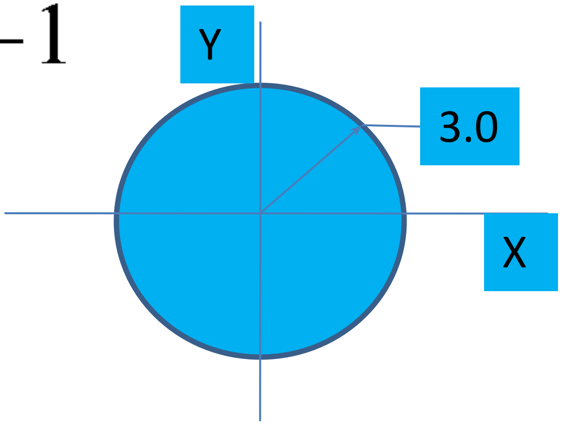
$$= -2\pi i$$

EXAMPLES 2

$$Evaluate \oint_c \frac{z-2}{z(z-1)}dz \quad where\ C\ is\ a\ circle\ |z|=3$$

$$Solution: \text{Singular points } z=0,1 \ \ lies\ \ inside\ \ C$$

$$Now\,Consider\ \frac{1}{z(z-1)}=\frac{A}{z}+\frac{B}{z-1}$$



$$A=-1 \ \ and\ B=1$$

$$\therefore \quad \frac{1}{z(z-1)}=\frac{-1}{z}+\frac{1}{z-1}$$

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WKT

$$\oint \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$$\therefore \oint_c \frac{(z-2)}{z(z-1)}\,dz = \oint_c \left(\frac{1}{z-1} - \frac{1}{z}\right)(z-2)\,dz$$

$$= 2\pi i f(0) - 2\pi i f(1)$$

$$= 2\pi i(-2) - 2\pi i(-1)$$

$$= -2\pi i$$

EXAMPLES 3

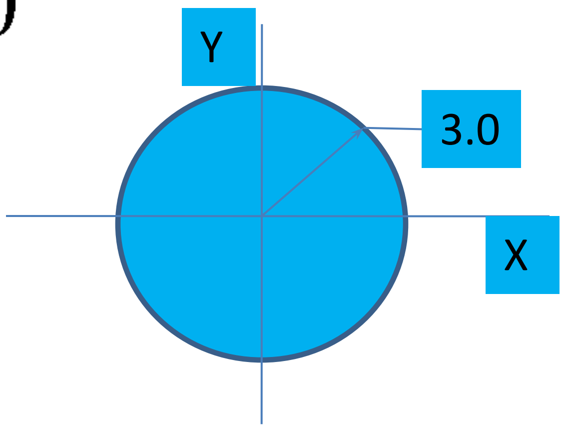
$$Evaluate \oint_c \frac{\cos \pi z^2}{(z-1)(z-2)} dz \quad where \ |z| = 3$$

using cauchy residues theorem

$so \ln :$

$Singular$ points: (z-1)(z-2)=0

$\Rightarrow z = 1, 2 \quad lies\ inside\ |z| = 3$



$$Now\,Consider\ \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$A = -1 \ \ and\ B = 1$$

$$\therefore \quad \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$\oint_c \frac{\cos \pi z^2}{(z-1)(z-2)} dz = -\oint_c \frac{\cos \pi z^2}{(z-1)} dz + \oint_c \frac{\cos \pi z^2}{(z-2)} dz$$

$$= -2\pi i f(1) + 2\pi i f(2)$$

$$= 4\pi i$$

TAYLORS SERIES

A function $f(z)$ be *analytic at all points inside a circle ' C ' with its center at ' a ' and radius* r *, we can expand as*

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots\ldots\ldots$$

$$\ldots + \frac{f^n(a)}{n!}(z-a)^n + \ldots.\infty$$

$$= \sum_{n=1}^{\infty} \frac{(z-a)^n}{n!} f^n(a)$$

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EXAMPLE 1

$$Expand\ \ \frac{1}{z-2}\ \ at\ z=1\ is\ aTaylor's series.$$

$$Solution: Let$$

$$f(z)=\frac{1}{z-2} \qquad \Rightarrow \quad f(1)=-1$$

$$f^{'}(z)=\frac{-1}{(z-2)^2} \qquad \Rightarrow \quad f^{'}(1)=-1$$

$$f^{''}(z)=\frac{2}{(z-2)^3} \qquad \Rightarrow \quad f^{''}(1)=2$$

$$f'''(z) = \frac{-6}{(z-2)^4} \quad \Rightarrow f'''(1) = -6$$

Taylor's series of $f(z)$ *about the point* $z = 1$ *is*

$$f(z) = -1 + \frac{(-1)}{1!}(z-1) + \frac{(2)}{2!}(z-1)^2 + \frac{(-6)}{3!}(z-1)^3 +$$

$$f(z) = -1 - (z-1) + (z-1)^2 + (z-1)^3 + \ldots$$

EXAMPLE 2

Expand $\cos z$ *at* $z = 0$ *is a Taylor's series.*

Solution : *Let*

$$f(z) = \cos z \qquad \Rightarrow \quad f(0) = 1$$

$$f'(z) = -\sin z \qquad \Rightarrow \quad f'(1) = 0$$

$$f''(z) = -\cos z \qquad \Rightarrow \quad f''(1) = -1$$

$$f'''(z) = \sin z \qquad \Rightarrow \quad f'''(1) = 0$$

$$Taylor's\ series\ of\ f(z)\ about\ the\ point\ z=0\ is$$

$$f(z)=1+\frac{(0)}{1!}(z-0)+\frac{(-1)}{2!}(z-0)^2+\frac{(0)}{3!}(z-0)^3+$$

$$f(z)=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\ldots\ldots\ldots$$

NOTE

$$If\ a=0\ then\ the\ taylor's\ series\ become\ Maclaurin's\ series$$

$$f(z)=f(0)+\frac{f'(0)}{1!}(z)+\frac{f''(0)}{2!}(z)^2+\ldots\ldots\ldots$$

$$\ldots+\frac{f^n(0)}{n!}(z)^n+\ldots.\infty$$

LAURENTS SERIES:

If $f(z)$ *is analytic on two concentric circle* C_1 *and* C_2 *of radii* r_1 *and* r_2 *with center at 'a' and also on the annular region R bounded by* C_1 *and* C_2 *then for all Z in R*

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n} \quad \textit{where}$$

$$a_n=\frac{1}{2\pi i}\oint_{c_1}\frac{f(z)}{(z-a)^{n+1}}dz \ ; \ b_n=\frac{1}{2\pi i}\oint_{c_2}\frac{f(z)}{(z-a)^{1-n}}dz$$

Both the integral being taken anticlockwise direction

EXAMPLE 1

Find the Laurent's series for $f(z) = \dfrac{z-1}{(z+2)(z+3)}$

in the region $2 < |z| < 3$

Soln: *Let* $\dfrac{z-1}{(z+2)(z+3)} = \dfrac{A}{z+2} + \dfrac{B}{z+3}$

$$\Rightarrow z-1 = A(z+3) + B(z+2)$$

$$\Rightarrow A = -3 \quad , \quad B = 4$$

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$$\therefore \; f(z) = \frac{-3}{z+2} + \frac{4}{z+3}$$

$$Let \;\; 2 < |z| < 3 \;\Rightarrow |z| > 2 \quad and \quad |z| < 3$$

$$\Rightarrow \frac{2}{|z|} < 1 \quad and \quad \frac{|z|}{3} < 1$$

$$f(z) = \frac{-3}{z\left(1 + \frac{2}{z}\right)} + \frac{4}{3\left(1 + \frac{3}{z}\right)}$$

$$f(z) \;\; = \;\; \frac{-3}{z}\left(1 + \frac{2}{z}\right)^{-1} + \frac{4}{3}\left(1 + \frac{3}{z}\right)^{-1}$$

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EXAMPLE 2

Find the Laurent's series for $f(z) = \dfrac{1}{z^2 - 3z + 2}$

in the region $(i)\ 1 < |z| < 2 \quad (ii)\ |z| > 2 \quad (iii)\ |z-1| < 1$

Soln : *Let* $f(z) = \dfrac{1}{z^2 - 3z + 2}$

Consider $\dfrac{1}{z^2 - 3z + 2} = \dfrac{A}{z-1} + \dfrac{B}{z-2}$

$$\Rightarrow A = -1 \ , \ B = 1$$

$$\therefore \ f(z) = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$(i)\ 1 < |z| < 2 \ \Rightarrow |z| > 1 \quad and \quad |z| < 2$$

$$\Rightarrow \frac{1}{|z|} < 1 \quad and \quad \frac{|z|}{2} < 1$$

$$f(z) = \frac{-1}{z\left(1 - \frac{1}{z}\right)} + \frac{1}{2\left(\frac{z}{2} - 1\right)}$$

$$f(z) \quad = \quad \frac{-1}{z}\left(1 - \frac{1}{z}\right)^{-1} - \frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1}$$

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$$(ii)\ |z| > 2 \Rightarrow \frac{2}{|z|} < 1$$

$$f(z) = \frac{-1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$f(z) = \frac{-}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$(iii)\ |z-1| < 1 \Rightarrow Put\ z-1 = u \Rightarrow z = u+1\ \&\ |u| < 1$$

$$f(z) = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$f(z) = \frac{-1}{u+1-1} + \frac{1}{u+1-2}$$

$$= \frac{-1}{u} + \frac{1}{u-1}$$

$$= \frac{-1}{u} - (1-u)^{-1}$$

$$f(z) = \frac{-1}{z-1} - \left[1 + (z-1) + (z-1)^2 + \ldots\ldots\right]$$

SINGULAR POINTS

A point $z = z_0$ *at which a function* $f(z)$ *fails to be analytic is called a singular point or singularity of* $f(z)$

Example

$f(z) = \dfrac{1}{z-3}$ *, here* $z = 3$ *is a* singular point of f(z)

TYPES OF SINGULAR POINTS

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ISOLATED SINGULARITY

$A\ point\ z = z_0\ is\ said\ to\ be\ an\ isolated\ singularity$

$of\ f(z)\ if\ (i)\ f(z)\ is\ not\ analytic\ at\ z = z_0$

$(ii)\ There\ exist\ a\ neighbourhood\ of\ z = z_0$

$containing\ no\ other\ singularity$

$Example :- f(z) = \dfrac{1}{z}\ \ is\ an\ analytic$

$every\ where\ except\ at\ z = 0$

$\therefore\ z = 0\ is\ an\ isolated\ singularity$

NOTE

If $z = z_0$ *is an isolated singular point of a function* $f(z)$ *then the singularity is called*

(i) *Removable singularity*

(ii) *A pole*

(iii) *An essential singularity.*

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REMOVABLE SINGULARITY

A singular point $z = z_0$ *is called a removable singularity of* $f(z)$ *if* $\lim_{z \to z_0} f(z)$ *exist and is finite*

Example : $f(z) = \dfrac{\sin z}{z}$

$$= \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots\ldots \right)$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \ldots$$

There is no negative power of Z.

Therefore $z = 0$ *is removable singularity*

POLES

An analytic function $f(z)$ *with a singularity at* $z = a$ *if* $\lim\limits_{z\to a} f(z) = \infty$ *then* $z = a$ *is a pole of* $f(z)$.

SIMPLE POLES

A pole of order one is called a simple pole

ESSENTIAL SINGULARITY

If the principal part contains an infinite no of non-zero terms then $z = z_0$ *is known as an essential singularity*

$$Example : f(z) = e^{\frac{1}{z}}$$

$z = 0$ *is a* singular points

$$But\, e^{\frac{1}{z}} = 1 + \frac{\frac{1}{z}}{1!} + \frac{\frac{1}{z^2}}{2!} + \ldots\ldots$$

$$= 1 + \frac{1}{z} + \frac{1}{2z^2} + \ldots\ldots$$

Here $f(z)$ *has* infinite number of -ve powers of z

$\therefore$ z = 0 is a essential singularity.

EVALUATION OF RESIDUES OF f(z)

(i) *Residue of* $f(z)$ *at its simple pole* $z = z_0$ *is given by*

$$R = \operatorname{Re}(z = z_0) = \lim_{z \to z_0} (z = z_0) f(z)$$

(ii) *Residue of* $f(z)$ *at its pole* $z = z_0$ *of order* n *is given by*

$$R = \operatorname{Re}(z = z_0) = \lim_{z \to z_0.} \left[\frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (z = z_0)^n f(z) \right]$$

CAUCHY RESIDUES THEOREM

If $f(z)$ be analytic at all points inside and on a simple closed curve C except for a finite no of isolated singularity $z_1, z_2, \ldots\ldots z_n$ inside C, then

$$\int_c f(z)\,dz = 2\pi i(\text{sum of the residue of } f(z) \text{ at } z_1, z_2, \ldots\ldots z_n)$$

$$= 2\pi i \sum_{i=1}^{n} R_i \text{, where } R_i \text{ is the residue of } f(z) \text{ at } z = z_i$$

EXAMPLE-1

$$Evaluate \oint_c \frac{\cos \pi z^2}{(z-1)(z-2)} dz \quad where \ |z| = 3$$

using cauchy residues theorem

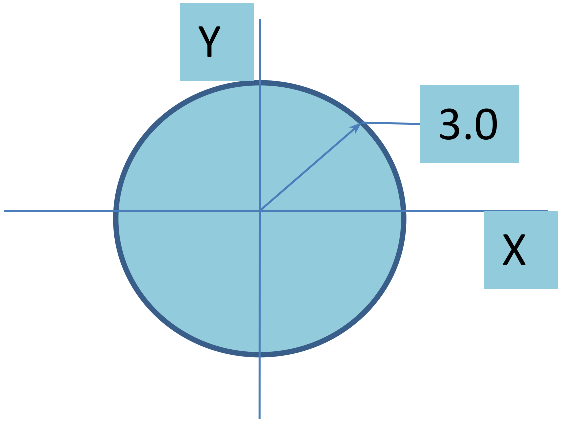
$soln:$

$Singular$ points: (z-1)(z-2)=0

$\Rightarrow$ z=1, 2 $is \ a \ pole \ of \ order \ one.$

$\therefore \ Its \ a \ simple \ pole$

$z = 1\ ,\ 2\ \ both\ lies\ inside\ the\ circle\ |z| = 3$



Now

$$\mathrm{Res}_1(z=1) = \lim_{z\to 1}(z-1)f(z)$$

$$= \lim_{z\to 1}(z-1)\frac{\cos \pi z^2}{(z-1)(z-2)}$$

$$= 1$$

$$\mathrm{Res}_2(z=2) = \lim_{z\to 2}(z-2)f(z)$$

$$= \lim_{z\to 2}(z-2)\frac{\cos\pi z^2}{(z-1)(z-2)}$$

$$= 1$$

$$\oint_c \frac{\cos\pi z^2}{(z-1)(z-2)} = \oint_c f(z)dz$$

$$= 2\pi i(sum\ of\ \ residues)$$

$$= 2\pi i(R_1 + R_2)$$

$$= 4\pi i$$

EXAMPLE-2

$$(ii)\quad Evaluate\ \oint_c \frac{\sin \pi z + \cos \pi z^2}{z+z^2}dz\ \ where\ C\ \ is\ a\ circle\ |z| = 2$$

$$so\ln : (\text{Hint})$$

$$z = 0, 1\ \ are\ \ simple\ \ pole\ \ \&\ \ both\ \ lies\ inside\ \ the\ \ circle\ |z| = 2$$

$$R_1(z=0) = 1\ \ \&\ \ R_2(z=1) = 1$$

$$\therefore\ \oint_c f(z)dz = 4\pi i$$

EXAMPLE-3

$$Find\ the\ residues\ at\ their\ poles\ of\ f(z)=\frac{z}{(z-1)^2}$$

$$so\ln: The\ poles\ are\ given\ by\ (z-1)^2=0$$

$$So\ \ z=1\ \ is\ \ a\ \ pole\ \ of\ \ order\ 2$$

$$R=\operatorname{Re}(z=z_0)=\lim_{z\to z_0}\left[\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}(z=z_0)^n f(z)\right]$$

$$\operatorname{Re}(z=1)=\lim_{z\to 1}\left[\frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}(z=1)^2\frac{z}{(z-1)^2}\right]$$

$$=\lim_{z\to 1}\frac{d}{dz}(z)=1$$

APPLICATION OF RESIDUES TO EVALUATE REAL INTEGRALS CONTOUR INTEGRATION (UNIT CIRCLE)

$$Type\ 1:\ \int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$$

$$Here\ \ z = e^{i\theta} \ \Rightarrow dz = ie^{i\theta}\, d\theta = iz\, d\theta$$

$$\Rightarrow d\theta = \frac{1}{iz}\, dz$$

$$Now\ let\ \ z = e^{i\theta} \ = \cos\theta + i\sin\theta$$

$$\frac{1}{z} = e^{-i\theta} \ = \cos\theta - i\sin\theta$$

$$\therefore \quad \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \ \&\ \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\therefore \int_0^{2\pi} f\left(\frac{1}{2}\left[z + \frac{1}{z}\right], \frac{1}{2i}\left[z - \frac{1}{z}\right]\right)\frac{dz}{zi}$$

EXAMPLE

$$Evaluate \int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$$

$$So\ln:$$

$$Let \quad z = e^{i\theta} \quad \Rightarrow dz = ie^{i\theta}\, d\theta = iz\, d\theta$$

$$\Rightarrow d\theta = \frac{1}{iz} dz$$

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) \;\&\; \sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$$

Now

$$I = \int_0^{2\pi} \frac{d\theta}{5 + 3\cos\theta} = \oint_c \frac{1}{5 + \frac{3}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz}$$

$$= \frac{2}{i} \oint_c \frac{dz}{3z^2 + 10z + 3} = \frac{2}{i} \oint_c f(z)\,dz$$

$$= \frac{2}{i}\left[2\pi i (sum\ of\ the\ residues\ of\ f(z)\right]$$

$$= 4\pi \left[sum\ of\ the\ residues\ of\ f(z) \right]$$

Hence

$$\operatorname{Re}\left(z = -\frac{1}{3} \right) = \lim_{z \to -\frac{1}{3}} \left(z + \frac{1}{3} \right) \frac{1}{(3z+1)(z+3)}$$

$$= \frac{1}{8}$$

$$\int_0^{2\pi} \frac{d\theta}{5 + 3\cos\theta} = 4\pi \left(\frac{1}{8} \right) = \frac{\pi}{2}$$

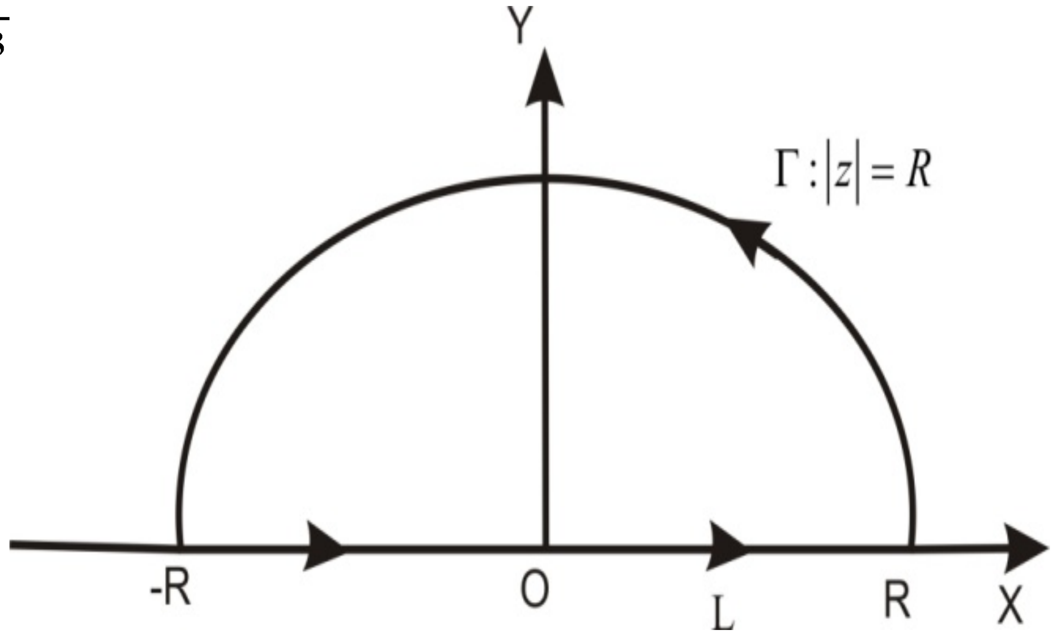
$$\int_0^{\infty} \frac{dx}{(x^2+a^2)^3}$$

$$I = \int_C \frac{dz}{(z^2+a^2)^3}$$

$$|z| = R$$

$$(z^2+a^2)^3 = 0$$

$$z^2 = -a^2$$



$$z = \pm ia$$

$$I = \int_C f(z)dz = 2\pi i\, R_1$$

$$\int_\Gamma f(z)dz + \int_L f(z)dz = 2\pi i\, R_1 \qquad \ldots(1)$$

$$R_1 = \frac{1}{\angle(n-1)} \lim_{z \to ai} \frac{d^{n-1}}{dz^{n-1}} (z-ai)^n f(z)$$

$$= \frac{1}{\angle 2} \lim_{z \to ai} \frac{d^2}{dz^2} (z-ai)^3 \frac{1}{(z-ai)^3 (z+ai)^3}$$

$$= \frac{1}{2} \lim_{z \to ai} \frac{d^2}{dz^2} (z + ai)^3$$

$$= \frac{1}{2} \lim_{z \to ai} \frac{d}{dz} [-3(z + ai)^{-4}](1)$$

$$= \frac{1}{2} \lim_{z \to ai} [12(z + ai)^{-5}]$$

$$= 6(2ai)^{-5} = \frac{6}{2^5 (ai)^5}$$

$$= \frac{3}{16a^5 (i^2)^2 i} = \frac{3}{16a^5 i}$$

$$\int_{\Gamma} f(z)dz + \int_{L} f(z)dz = 2\pi i \frac{3}{16a^5 i}$$

$$\int_{\Gamma} f(z)dz + \int_{-R}^{R} f(x)dx = \frac{3\pi}{8a^5}$$

$$R \to \infty$$

$$\lim_{R\to\infty} \int_{\Gamma} f(z)dz + \int_{-\infty}^{\infty} f(x)dx = \frac{3\pi}{8a^5} \qquad ...(2)$$

$$\lim_{R\to\infty} \int_{\Gamma} f(z)dz \to 0$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^3} = \frac{3\pi}{8a^5}$$

$$2\int_{0}^{\infty} \frac{dx}{(x^2+a^2)^3} = \frac{3\pi}{8a^5} = \frac{3\pi}{16a^5}$$

$$\int_{0}^{\infty} \frac{x \sin x dx}{x^2+a^2}$$

$$\int_{C} \frac{ze^{iz} dz}{z^2+a^2}$$

Γ

$|z| = R$

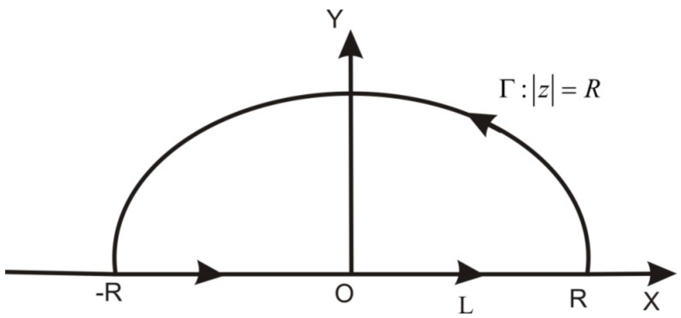
$$z^2 + a^2 = 0$$

$$z^2 = -a^2$$

$$\int_C f(z)dz = 2\pi i\, R_1$$

$$R_1 = \lim_{z \to ai}(z - ai)\frac{ze^{iz}}{(z - ai)(z + ai)}$$

$$= ai\,\frac{d^{i(ai)}}{2ai} = \frac{e^{-a}}{2}$$



$$\int_{\Gamma} f(z)dz + \int_{L} f(z)dz = 2\pi i \frac{e^{-a}}{2}$$

$$\int_{\Gamma} f(z)dz + \int_{-R}^{R} f(x)dx = \pi i e^{-a}$$

$$R \to \infty$$

$$\lim_{R\to\infty} \int_{\Gamma} f(z)dz + \int_{-\infty}^{\infty} f(x)dx = \pi i e^{-a}$$

$$\lim_{R\to\infty} \int_{\Gamma} f(z)dz \to 0$$

$$\int_{-\infty}^{\infty} f(x)dx = \pi i e^{-a}$$

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+a^2}\,dx = \pi i e^{-a}$$

$$\int_{-\infty}^{\infty} \frac{x(\cos x + i\sin x)}{x^2+a^2}\,dx = \pi i e^{-a}$$

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2}\,dx = \pi e^{-a}$$

$$\int_{0}^{\infty} \frac{x\sin x}{x^2+a^2}\,dx = \frac{\pi}{2} e^{-a}$$

Unit V - Completed

*** THANK YOU ***