Steady state conditions and non-zero boundary conditions.

(1) A bar, 10cm long, with insulated sides, has its ends A and B kept at 20° and 40°C., respectively, until steady-state conditions prevail, that is, until the temperature at any interior point no longer changes with time. The temperature at A is then suddenly raised to 50°C and at the same instant that at B is lowered to 10°C. Find the subsequent temperature function $u(x,t)$ at any time.

soln:- The partial differential equation of one dimensional heat flow is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ $\longrightarrow$ (a)

In steady state conditions, the temperature at any particular point does not vary with time ie, u depends only on x and not on time t.

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \left[\because \frac{\partial u}{\partial t} = 0 \text{ since } u \text{ is a function of } x \text{ only} \right]$$

Since u is a function of x only, the above equation can be written as $\frac{d^2u}{dx^2} = 0$ $(\alpha \neq 0)$

Hence when steady state conditions prevails the heat flow equation becomes

$$\frac{d^2u}{dx^2} = 0 \longrightarrow (b)$$

integrating eqn (b) w.r to 'x' twice, we get

$$\boxed{u(x) = ax+b} \longrightarrow (*)$$

The boundary conditions are

(i) $u(0) = 20$

(ii) $u(10) = 40$.

Applying b.c (i) in eqn: (*), we get

$$u(0) = a(0) + b = 20$$

$$\therefore \boxed{b=20}$$

Sub. in (*), we get $u(x) = ax + 20 \rightarrow$ (*)(*).

Applying b.c (ii) in eqn (*)(*), we get

$$u(10) = a(10) + 20 = 40.$$

$$10a = 20 \Rightarrow \boxed{a=2}$$

Sub $\boxed{a=2}$ in eqn (*)(*), we get

$$\boxed{u(x) = 2x+20}$$

When the temperatures at A and B are changed, the state is no longer steady. Then the temperature function $u(x,t)$ satisfies (a).

The boundary conditions in the second state are

$$u(0,t) = 50 \quad \forall t > 0$$

$$u(10,t) = 10 \quad \forall t > 0$$

The initial temperature of this state is the temperature in the previous steady-state. Hence the initial condition is

$$u(x,0) = 2x+20 \quad \text{for} \quad 0 < x < 10.$$

[Since non-zero boundary conditions have infinite number of values for A & B].

Therefore, in this case, we split the solution

$u(x,t)$ into two parts.

$$\text{ie, } u(x,t) = u_s(x) + u_t(x,t) \longrightarrow \text{(I)}$$

where $u_s(x)$ is a steady state solution of (a)

and $u_t(x,t)$ is a transient solution which decreases

with increase of t.

To find steady state temperature $u_s(x)$:

The Boundary conditions are

(i) $u_s(0) = 50$

(ii) $u_s(10,t) = 10$.

The steady state temperature is given by

$$u_s(x) = a_1 x + b_1 \longrightarrow \text{(II)}$$

Applying boundary condition (i) in eqn (II), we get

$$u_s(0) = a_1(0) + b_1 = 50$$

$$\boxed{b_1 = 50}$$

Sub. b_1 in eqn (II), we get $u_s(x) = a_1 x + 50 \longrightarrow \text{(III)}$

Applying b.c (ii) in eqn (III), we get

$$u_s(10) = a_1(10) + 50 = 10$$

$$10a_1 = -40$$

$$\boxed{a_1 = -4}$$

Sub a_1 in (III), we get $\boxed{u_s(x) = -4x + 50.}$

To find $u_t(x,t)$:

we assume that $u_t(x,t)$ is a transient solution of

$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ and satisfying the equation

$$u(x,t) = u_s(x) + u_t(x,t)$$

$$\therefore \; u_t(x,t) = u(x,t) - u_s(x) \longrightarrow \text{(IV)}$$

we have to find the boundary conditions for $u_t(x,t)$

Putting $x=0$ in IV, we get

$$u_t(0,t) = u(0,t) - u_s(0) = 50 - 50 = 0.$$

$$\therefore \; u_t(0,t) = 0.$$

Putting $x=l$ in IV, we get

$$u_t(l,t) = u(l,t) - u_s(l) = 10 - 10 = 0.$$

$$\therefore \; u_t(l,t) = 0.$$

Putting $t=0$ in (IV), we get

$$u_t(x,0) = u(x,0) - u_s(x)$$

$$= 2x + 20 - (-4x + 50)$$

$$= 2x + 20 + 4x - 50$$

$$= 6x - 30$$

$$\therefore \; u_t(x,0) = 6x - 30$$

Now for the function $u_t(x,t)$ we have the following boundary conditions.

(i) $u_t(0,t) = 0$ $\forall t$

(ii) $u_t(l,t) = 0$ $\forall t$

(iii) $u_t(x,0) = 6x - 30.$ for $0 < x < 10$

The suitable solution is

$$u_t(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\alpha^2\lambda^2 t} \longrightarrow \text{(i)}$$

Applying b.c (i) in eqn (1), we get

$$u_t(0,t) = A e^{-\alpha^2\lambda^2 t} = 0.$$

either $A=0$ or $e^{-\alpha^2\lambda^2 t} = 0$.

$e^{-\alpha^2\lambda^2 t} \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore \boxed{A=0}$

Sub $A=0$ in eqn (1), we get

$$u(x,t) = B \sin \lambda x \, e^{-\alpha^2\lambda^2 t} \longrightarrow (2)$$

Applying b.c (ii) in eqn (2), we get

$$u(10,t) = B \sin 10\lambda \, e^{-\alpha^2\lambda^2 t} = 0.$$

here, $B \neq 0$ ($\because$ If $B=0$, we get a trivial solution)

$e^{-\alpha^2\lambda^2 t} \neq 0$ ($\because$ it is defined $\forall t$).

$$\sin 10\lambda = 0.$$
$$\sin 10\lambda = \sin n\pi$$
$$10\lambda = n\pi$$
$$\Rightarrow \boxed{\lambda = \frac{n\pi}{10}}$$

Sub. $\lambda = \frac{n\pi}{10}$ in eqn (2), we get

$$u(x,t) = B \sin \frac{n\pi x}{10} \, e^{-\frac{\alpha^2 \cdot \pi^2 t}{100}}$$

$$u(x,t) = B_n \sin \frac{n\pi x}{10} \, e^{-\frac{\alpha^2 n^2 \pi^2 t}{100}}$$ where $B = B_n$, & B_n is any constant.

The most general soln. is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{10} \, e^{\frac{-\alpha^2 n^2 \pi^2 t}{100}} \longrightarrow (3).$$

Applying b.c (iii) in eqn (3), we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{10} (1) = 6x-30 \longrightarrow (4)$$

To find B_n expand $6x-30$ in half-range sine series in $(0,10)$

$$6x-30 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} \quad \text{where} \longrightarrow (5)$$

$$b_n = \frac{2}{10}\int_0^{10} f(x) \sin\frac{n\pi x}{10}\, dx$$

From (4) & (5), we get $\boxed{B_n = b_n}$

$$\therefore B_n = \frac{2}{10}\int_0^{10} (6x-30) \sin\frac{n\pi x}{10}\, dx$$

$$= \frac{1}{5}\left[(6x-30)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - 6\left(\frac{-\sin\frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)^2}\right)\right]_0^{10}$$

$$= \frac{1}{5}\left[(30)\left(\frac{10}{n\pi}\right)(-(-1)^n) - (-30)\left(\frac{10}{n\pi}\right)(-1)\right]$$

$$= \frac{1}{5}\left[\frac{-300}{n\pi}(-1)^n - \frac{300}{n\pi}\right]$$

$$= \frac{1}{5}\left[\frac{-300}{n\pi}\right]\left[(-1)^n + 1\right]$$

$$= \frac{-60}{n\pi}\left(1+(-1)^n\right)$$

$\therefore$ sub in eqn (3), we get

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{-60}{n\pi}\left(1+(-1)^n\right) \sin\frac{n\pi x}{10}\, e^{\frac{-\alpha^2 n^2 \pi^2 t}{100}}.$$

$$\therefore u(x,t) = u_s(x) + u_t(x,t)$$

$$u(x,t) = 50-4x + \sum_{n=1}^{\infty} \frac{-60}{n\pi}\left(1+(-1)^n\right) \sin\frac{n\pi x}{10}\, e^{\frac{-\alpha^2 n^2 \pi^2 t}{100}}$$

which is temperature distribution.

Exercise Problem

1. A rod AB of length 10 cm. has its ends A and B kept at temperature 30°C and 100°C respectively until the steady-state conditions prevail. At sometime later, the temperature at A is lowered to 20°C and that at B to 40°C, and then these temperatures are maintained. Find the subsequent temperature distribution.