Root Mean square value (RMS) value

Let $f(x)$ be defined in (a,b)

Then
$$\bar{y}_{(RMS)} = \sqrt{\frac{1}{(b-a)}\int_a^b [f(x)]^2\,dx}$$

RMS is always $\bar{y}$ only.

$$\bar{y}^2 = \frac{1}{b-a}\int_a^b [f(x)]^2\,dx$$

Parseval's Formula:

$$\bar{y}^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Where a_0, a_n, and b_n are Fourier co-efficients.

① Find the RMS value of $f(x) = (x - x^2)$ in $(-1, 1)$

Sol:

$$\text{RMS} \Rightarrow \bar{y} = \sqrt{\frac{1}{b-a}\int_a^b [f(x)]^2\,dx}$$

$$= \sqrt{\frac{1}{2}\int_{-1}^{1}(x-x^2)^2\,dx}$$

$$= \sqrt{\frac{1}{2}\int_{-1}^{1}[x^2 + x^4 - 2x^3]\,dx}$$

$$= \sqrt{\frac{8}{15}}$$

② Find the fourier series of $f(x) = x^2$ in $(-\pi, \pi)$. Deduce $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$

Sol: Given $f(x) = x^2$, Period $= 2\pi$

$\boxed{l = \pi}$

$$\therefore \boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx} \longrightarrow ①$$

Here $f(x) = x^2$ is even function in $(-\pi, \pi)$.

$$\therefore a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = 0.$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$\boxed{a_0 = \frac{2}{3}\pi^2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos(nx)\,dx$$

$$= \frac{2}{\pi}\left\{ (x^2)\left(\cancel{\frac{\sin nx}{n}}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\cancel{\frac{-\sin nx}{n^3}}\right)\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{ 2x\frac{\cos(nx)}{n^2}\right\}_0^{\pi}$$

$$= \frac{2}{\pi n^2}\left\{ 2\pi \cos n\pi - 0\right\}$$

$$a_n = \frac{4(-1)^n}{n^2}$$

$b_n = 0$ [since $f(x)$ is even in $(-\pi, \pi)$.]

$$\therefore \boxed{f(x) = \pi^2/3 + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx} \longrightarrow (2)$$

Deduction:

[By substituting the value of x in the series (2), we won't get the required deduction]

By parseval's formula,

$$\bar{y}^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \longrightarrow (3)$$

To find: $\bar{y}^2$

$$\bar{y}^2 = \frac{1}{b-a}\int_a^b [f(x)]^2 dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x^2)^2 dx = \frac{2}{2\pi}\int_0^{\pi} x^4 dx =$$

$$= \frac{1}{\pi}\left[\frac{\pi^5}{5}\right] = \frac{\pi^4}{5}$$

From (3),

$$\frac{\pi^4}{5} = \left(\frac{4\pi^4}{9}\right)\Big/4 + \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{16\left((-1)^n\right)^2}{n^4} + 0\right]$$

$$\frac{\pi^4}{5} = \frac{\pi^4}{9} + \frac{16}{2}\sum_{n=1}^{\infty}\frac{1}{n^4} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\frac{\pi^4}{5} - \frac{\pi^4}{9}}{8}$$

$$\boxed{= \frac{\pi^4}{90}}$$

Scanned with CamScanner

③ Obtain the Fourier series for $f(x) = \begin{cases} \pi x & ; 0 \le x \le 1 \\ \pi(2-x) & ; 1 \le x \le 2 \end{cases}$

Deduce that

(i) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

(ii) $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}$

Sol:

$$f(x) = \begin{cases} \pi x & ; 0 \le x \le 1 \\ \pi(2-x) & ; 1 \le x \le 2 \end{cases} \qquad \text{Period} = 2L = 2 \quad \boxed{L=1}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

where $a_0 = \frac{1}{L}\int_0^2 f(x)\,dx = \frac{1}{1}\int_0^2 f(x)\,dx$

$$= \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx$$

$$= \pi\left(\frac{x^2}{2}\right)_0^1 + \pi\left(2x - \frac{x^2}{2}\right)_1^2$$

$$= \frac{\pi}{2} + \pi\left[\left(4 - \frac{4}{2}\right) - \left(2 - \frac{1}{2}\right)\right]$$

$$\boxed{a_0 = \pi}$$

$$a_n = \frac{1}{L}\int_0^2 f(x)\cos(n\pi x)\,dx$$

$$= \frac{1}{1}\left\{\int_0^1 \pi x\cos(n\pi x)\,dx + \int_1^2 \pi(2-x)\cos(n\pi x)\,dx\right\}$$

$$= \pi\left\{(x)\left(\cancel{\frac{\sin n\pi x}{n\pi}}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right\}_0^1 +$$

$$\pi\left\{(2-x)\left(\cancel{\frac{\sin n\pi x}{n\pi}}\right) - (-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right\}_1^2$$

$$= \pi\left\{\frac{\cos n\pi}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right\} + \pi\left\{-\frac{\cos 2n\pi}{n^2\pi^2} + \frac{\cos n\pi}{n^2\pi^2}\right\}$$

$$= \pi\left\{\frac{2\cos n\pi}{n^2\pi^2} - \frac{2}{n^2\pi^2}\right\} = \frac{2}{n^2\pi}\left\{(-1)^n - 1\right\}$$

$$b_n = \frac{1}{l}\int_0^2 f(x)\sin(n\pi x)\,dx$$

$$= \frac{1}{1}\left\{\int_0^1 \pi x \sin(n\pi x)\,dx + \int_1^2 \pi(2-x)\sin(n\pi x)\,dx\right\}$$

$$= \pi\left\{(x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\cancel{\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)}\right\}_0^1 +$$

$$\pi\left\{\cancel{(2-x)}\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\cancel{\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)}\right\}_1^2$$

$$= \pi\left\{\left(-\frac{\cos n\pi}{n\pi} - 0\right)\right\} + \pi\left\{0 - \left[-1\,\frac{\cos n\pi}{n\pi}\right]\right\}$$

$$b_n = 0.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(n\pi x) + b_n\sin(n\pi x)\right]$$

$$f(x) = \pi/2 + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}\left((-1)^n - 1\right)\cos(n\pi x)$$

$$f(x) = \pi/2 + \frac{2}{\pi}\left[\frac{-2}{1^2}\cos\pi x + 0 - \frac{2}{3^2}\cos 3\pi x + \cdots\right] \longrightarrow ①$$

Deduction:

(i) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$ in the range $(0, 2)$

At $x = 1$, (discontinuous) $\Rightarrow f(1) = \dfrac{f(1^-) + f(1^+)}{2}$

$$= \frac{\pi + \pi}{2} = \pi$$

① $\Rightarrow$

$$f(1) = \pi = \pi/2 + 2/\pi\left[\frac{-2}{1^2}(-1) - \frac{2}{3^2}(-1) + \cdots\right]$$

$$\pi = \pi/2 + 2/\pi\left[\frac{2}{1^2} + \frac{2}{3^2} + \cdots\right]$$

$$\pi - \pi/2 = \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{\pi - \pi/2}{4/\pi} = \frac{1}{1^2} + \frac{1}{3^2} + \cdots$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \boxed{= \frac{\pi^2}{8}}$$

(ii) Deduction $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots$

Using PARSEVAL's Formula

$$\bar{y}^2 = \frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right) \longrightarrow (2)$$

<u>to find: $\bar{y}^2$</u>

$$\bar{y}^2 = \frac{1}{2}\int_0^2 [f(x)]^2 dx = \frac{1}{2}\left\{\int_0^1 (\pi x)^2 dx + \int_1^2 (\pi(2-x))^2 dx\right\}$$

$$= \frac{1}{2}\left\{\int_0^1 \pi^2 x^2 dx + \int_1^2 \pi^2 (2^2+x^2-4x)\,dx\right\}$$

$$= \frac{1}{2}\pi^2\left\{\left[\frac{x^3}{3}\right]_0^1 + \left[4x+\frac{x^3}{3}-4\frac{x^2}{2}\right]_1^2\right\}$$

$$= \frac{\pi^2}{2}\left\{\left[\left(\frac{1}{3}\right)\right]+\left[\left(\cancel{8}+\frac{8}{3}-\cancel{8}\right)-\left(4+\frac{1}{3}-2\right)\right]\right\}$$

$$= \frac{\pi^2}{2}\left\{\frac{2}{3}\right\} = \frac{\pi^2}{3}$$

<u>By the formula (2),</u>

$$\frac{\pi^2}{3} = \frac{\pi^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{2}{n^2\pi}\left((-1)^n-1\right)\right]^2$$

$$\frac{\pi^2}{3} = \frac{\pi^2}{4}+\frac{1}{2}\sum_{n=1,3,5\ldots}\left(\frac{-4}{n^2\pi}\right)^2$$

$$= \frac{\pi^2}{4}+\frac{16}{2}\sum_{n=1,3,5\ldots}\frac{1}{n^4\pi^2}$$

$$\frac{\pi^2}{3}-\frac{\pi^2}{4} = \frac{8}{\pi^2}\sum_{n=1,3,5,}\frac{1}{n^4}$$

$$\frac{\pi^2/12}{8/\pi^2} = \left[\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right]$$

$$\Rightarrow \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots = \frac{\pi^2}{12}\times\frac{\pi^2}{8}$$

$$\boxed{=\frac{\pi^4}{96}}$$