1 Power Spectral Density Function (PSDF)/ Power Density Spectrum (PDS)

Definition 1.0.1. *If $\{X(t)\}$ be a stationary process (either in strict sense or wide sense) with autocorrelation function $R_{XX}(\tau)$, then the power spectral density of $\{X(t)\}$ is the Fourier transform of $R_{XX}(\tau)$ is denoted by $S_{XX}(\omega)$ or $S_X(\omega)$ or $S(\omega)$ and is given by*

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}\, d\tau \tag{1.1}$$

if the variable ω is replaced by $2\pi f$ with f as the frequency variable, then the spectral density function is function of f, the corresponding effect is given below:

$$S_{XX}(f) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i2\pi f\tau}\, d\tau \tag{1.2}$$

Wiener-Khinchine Relation

Equations (1.1) and (1.2) sometimes called as the Wiener-Khinchine relation.

Note: $R_{XX}(\tau) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\omega\tau}\, d\omega$ is the Fourier inverse transform of $S_{XX}(\omega)$.

$$\text{Similarly } R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(f)e^{-i2\pi f\tau}\, d\omega$$

1.0.1 Properties of PSDF

Property 1.0.1. *The value of the power spectral density function at zero frequency is equal to the total area under the graph of autocorrelation function.* i.e. *By applying $\omega = 0$ or $f = 0$, we have*

$$S_{XX}(0) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, d\tau$$

Property 1.0.2. *The mean square value of a WSS process is equal to the total area under the graph of the spectral density.* i.e. *By applying $\omega = 0$ or $f = 0$, we have*

$$E\{X^2(t)\} = R_{XX}(0) = \int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega$$

Property 1.0.3. *The spectral density of a real process is an even function.* i.e. $S_{XX}(-\omega) = S_{XX}(\omega)$.

Proof.

$$\begin{aligned}
S(-\omega) &= \int_{-\infty}^{\infty} R(\tau)e^{i\omega\tau}\,d\tau \\
&= \int_{-\infty}^{\infty} R(-u)e^{-i\omega u}\,du \quad (\text{Substuting } \tau = -u) \\
&= \int_{-\infty}^{\infty} R(u)e^{-i\omega u}\,du \;=\; S(\omega), \quad (\text{Since } R(u) = R(-u))
\end{aligned}$$

□

Property 1.0.4. *The spectral density of any real or complex process $\{X(t)\}$ is a real function of ω.* i.e. $S^*_{XX}(\omega) = S_{XX}(\omega)$.

Proof.

$$\begin{aligned}
S^*(\omega) &= \int_{-\infty}^{\infty} R^*(\tau)e^{i\omega\tau}\,d\tau \\
&= \int_{-\infty}^{\infty} R(-\tau)e^{i\omega\tau}\,d\tau \\
&= \int_{-\infty}^{\infty} R(u)e^{-i\omega u}\,du \;=\; S(\omega) \geq 0, \quad (\text{Substuting } u = -\tau)
\end{aligned}$$

□

Property 1.0.5. *The spectral density and the autocorrelation function of a real WSS process forms a Fourier cosine transform pair.*

Property 1.0.6 (Wiener-Khinchine Theorem). *If $X_T(\omega)$ is the Fourier transform of the truncated random process $X_T(\omega) = \begin{cases} X(t), & \text{for } |t| \leq T \\ 0, & \text{for } |t| > T \end{cases}$ where $\{X(t)\}$ is a real WSS process with PSD function $S(\omega)$, then we have $S_{XX}(\omega) = \lim_{T\to\infty}\left[\frac{1}{2T}E\{|X_T(\omega)|^2\}\right]$.*

Example/Solved Problems

Example: 1. *The power spectral density of a random process $\{X\ (t)\}$ is given by $S_{XX}\ (\omega) = \begin{cases} \pi, & |\omega| < 1 \\ 0, & elsewhere \end{cases}$. Find its autocorrelation function.*

Hints/Solution:
We know that $R_{XX}(\tau) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\omega\tau}\,d\omega$
$R_{XX}(\tau) = \dfrac{1}{2\pi}\displaystyle\int_{-1}^{1} \pi e^{i\omega\tau}\,d\omega \;= \dfrac{1}{\tau}\sin\tau$

Example: 2. *If $R_{XX}\,(\tau) = e^{2\lambda|\tau|}$ is the autocorrelation of a random process $\{X(t)\}$, obtain its spectral density.*

Hints/Solution:
From $R_{XX}(\tau)$ we will get,

$$\begin{aligned} S_{XX}(w) &= \int\limits_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}\,d\tau = \int\limits_{-\infty}^{\infty} e^{-2\lambda|\tau|}e^{-i\omega\tau}\,d\tau \\ &= \int\limits_{-\infty}^{0} e^{2\lambda\tau}e^{-i\omega\tau}\,d\tau + \int\limits_{0}^{\infty} e^{-2\lambda\tau}e^{-i\omega\tau}\,d\tau \\ &= \frac{1}{2\lambda - iw} + \frac{1}{2\lambda + iw} = \frac{4\lambda}{4\lambda^2 + w^2} \end{aligned}$$

Example: 3. *If the power spectral density of a WSS process is given by $S_{XX}(w) = \begin{cases} \frac{b}{a}(a - |w|), & |\omega| \le a \\ 0, & otherwise \end{cases}$. Find the autocorrelation function of the process.*

Hints/Solution:
$S_{XX}(w) = \begin{cases} \dfrac{b}{a}(a - |w|), & -a < w < a \\ 0, \text{otherwise} \end{cases}$

$$\begin{aligned} R_{XX}(\tau) &= \frac{1}{2\pi}\int\limits_{-a}^{a} \frac{b}{a}(a - |w|)e^{iw\tau}\,dw \\ &= \frac{1}{\pi}\int\limits_{0}^{a} \frac{b}{a}(a - w)\cos w\tau\,dw \\ &= \frac{2b}{a\pi\tau^2}\sin^2\left(\frac{a\tau}{2}\right) \end{aligned}$$

Definition 1.0.2 (Average Power)**.** *The average power of a random process $\{X(t)\}$ is denoted by P_{XX} and is given by*

$$P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\,d\omega$$

In terms of the the time average it is given by

$$P_{XX} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E[X^2(t)]\, dt$$

If $X(t)$ is a WSS process, then $E[X^2(t)]$ is constant and $P_{XX} = E[X^2(t)]$.

Important Formulae/Results:

$$\begin{aligned}
F^{-1}\left[\frac{2a}{a^2+\omega^2}\right] &= e^{-a|\tau|}\\
F^{-1}\left[\frac{1}{(a^2-\omega^2)^2}\right] &= u(\tau)\tau e^{a|\tau|}\\
F^{-1}\left[\frac{1}{(a^2+\omega^2)^2}\right] &= u(\tau)\tau e^{-a|\tau|} \text{ where } u(\tau) \text{ is unit step function, } \alpha > 0
\end{aligned}$$

Example: 4. *The power spectrum of a WSS process $X(t)$ is given by $S_{XX}(\omega) = \dfrac{4}{(4+\omega^2)}$. Find the corresponding autocorrelation and average power.*

Hints/Solution:
We know that $R_{XX}(\tau) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\omega\tau}\, d\omega$

$$\begin{aligned}
F(e^{-a|\tau|}) &= 2\int_0^{\infty} e^{-a\tau}\cos\omega\tau\, d\tau\\
&= 2\left[\frac{e^{-a\tau}}{a^2+\omega^2}(-a\cos\omega\tau + \omega\sin\omega\tau)\right]_0^{\infty}\\
&= \frac{2a}{a^2+\omega^2}, a > 0\\
\implies e^{-a|\tau|} &= F^{-1}\left[\frac{2a}{a^2+\omega^2}\right]
\end{aligned}$$

Substituting $a = 2$, we get $F^{-1}[S_{XX}(\omega)] = F^{-1}\left[\dfrac{4}{4+\omega^2}\right] = e^{-2|\tau|} = R_{XX}(\tau)$

Since $X(t)$ is a WSS process, the average power $P_{XX} = E[X^2(t)] = R_{XX}(0) = e^0 = 1$.

Example: 5. *The power spectrum of a WSS process $X(t)$ is given by $S_{XX}(\omega) = \dfrac{1}{(1+\omega^2)^2}$. Find the corresponding autocorrelation and average power.*

Hints/Solution:

We know that

$$e^{-a|\tau|} = F^{-1}\left[\frac{2a}{a^2+\omega^2}\right],\ u(\tau)\tau e^{-a\tau} = F^{-1}\left[\frac{1}{(a+i\omega)^2}\right] \text{ and}$$
$$u(\tau)\tau e^{a\tau} = F^{-1}\left[\frac{1}{(a-i\omega)^2}\right]$$

$$\begin{aligned}
R_{XX}(\tau) &= F^{-1}\left[S_{XX}(\omega)\right] \\
&= F^{-1}\left[\frac{1}{(1+\omega^2)^2}\right] = \frac{1}{4}F^{-1}\left\{\left[\frac{(1+i\omega)+(1-i\omega)}{(1+\omega)(1-i\omega)}\right]^2\right\} \\
&= \frac{1}{4}\left\{F^{-1}\left[\frac{1}{(1-i\omega)^2}\right] + F^{-1}\left[\frac{1}{(1+i\omega)^2}\right] + F^{-1}\left[\frac{2}{1+\omega^2}\right]\right\} \\
&= \frac{1}{4}\left\{u(\tau)\tau\left[e^{a\tau}+e^{-a\tau}\right] + \frac{1}{2}e^{-a|\tau|}\right\}
\end{aligned}$$

Since $X(t)$ is a WSS process, the average power $P_{XX} = E[X^2(t)] = R_{XX}(0) = \frac{1}{4}$.

Example: 6. *Check whether the following functions are valid power density spectrums.*

(i) $S_{XX}(\omega) = \dfrac{4+\omega^2}{(4\omega^4+3\omega^2+5)}$

(ii) $S_{XX}(\omega) = \dfrac{\omega^2}{(\omega^6+3\omega^2+3)}$

(iii) $S_{XX}(\omega) = \dfrac{\omega+4}{(5+\omega^2)}$

Hints/Solution:

We know that $S_{XX}(\omega)$ is an even function and real. We need to verify whether $S_{XX}(-\omega) = S_{XX}(\omega)$, if it is, then the given function is a valid spectral density function.

(i) $S_{XX}(-\omega) = \dfrac{4+\omega^2}{(4\omega^4+3\omega^2+5)} = S_{XX}(\omega)$

$\therefore$ the given function is a valid power spectrum.

(ii) $S_{XX}(-\omega) = \dfrac{\omega^2}{(\omega^6+3\omega^2+3)} = S_{XX}(\omega)$

$\therefore$ the given function is a valid power spectrum.

(iii) $S_{XX}(\omega) = \dfrac{-\omega+4}{(5+\omega^2)} \neq S_{XX}(\omega)$

$\therefore$ the given function is not a valid power spectrum.

Example: 7. *Find the autcorrelation function of the process whose power density spectrum is given by* $\dfrac{9+\omega^2}{\omega^4+5\omega^2+4}$. *Also find its average power(mean square value).*

Hints/Solution:

We know that $R_{XX}(\tau) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\omega\tau}\, d\omega$

Using partial fractions, we will have

$\dfrac{9+\omega^2}{(1+\omega^2)(4+\omega^2)} = \dfrac{8/3}{(1+\omega^2)} + \dfrac{-5/3}{(4+\omega^2)}$

Also we know from the Fourier transform that

$$\begin{aligned}
F(e^{-a|\tau|}) &= 2\int_0^{\infty} e^{-a\tau}\cos\omega\tau\, d\tau \\
&= 2\left[\frac{e^{-a\tau}}{a^2+\omega^2}(-a\cos\omega\tau+\omega\sin\omega\tau)\right]_0^{\infty} \\
&= \frac{2a}{a^2+\omega^2}, a>0 \\
\implies e^{-a|\tau|} &= F^{-1}\left[\frac{2a}{a^2+\omega^2}\right]
\end{aligned}$$

$$\begin{aligned}
F^{-1}\left[S_{XX}(\omega)\right] &= \frac{4}{3}F^{-1}\left[\frac{2\cdot 1}{1+\omega^2}\right] - \frac{5}{12}F^{-1}\left[\frac{2\cdot 2}{4+\omega^2}\right] \\
&= \frac{4}{3}e^{-|\tau|} - \frac{5}{12}e^{-2|\tau|} = R_{XX}(\tau)
\end{aligned}$$

Now, the average power is given by $P_{XX} = R_{XX}(0) = E\{X^2(t)\} = \dfrac{4}{3} + \dfrac{5}{12} = \dfrac{11}{12}$

Example: 8. *Find the autocorrelation function corresponding to the PSD* $S_{XX}(\omega) = \dfrac{157+12\omega^2}{(16+\omega^2)(9+\omega^2)}$. *Also find the average power.*

Hints/Solution:

We know that $R_{XX}(\tau) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\omega\tau}\, d\omega$

Using partial fractions, we will have

$\dfrac{157+12\omega^2}{(16+\omega^2)(9+\omega^2)} = \dfrac{5}{(16+\omega^2)} + \dfrac{7}{(9+\omega^2)}$

Also we know from the Fourier transform that

$$\begin{aligned}
F(e^{-a|\tau|}) &= 2\int_0^{\infty} e^{-a\tau}\cos\omega\tau\, d\tau \\
&= 2\left[\frac{e^{-a\tau}}{a^2+\omega^2}(-a\cos\omega\tau+\omega\sin\omega\tau)\right]_0^{\infty} \\
&= \frac{2a}{a^2+\omega^2}, a>0 \\
\implies e^{-a|\tau|} &= F^{-1}\left[\frac{2a}{a^2+\omega^2}\right]
\end{aligned}$$

$$\begin{aligned} F^{-1}\left[S_{XX}(\omega)\right] &= \frac{5}{8}F^{-1}\left[\frac{2\cdot 4}{16+\omega^2}\right] + \frac{7}{6}F^{-1}\left[\frac{2\cdot 3}{9+\omega^2}\right] \\ &= \frac{5}{8}e^{-4|\tau|} + \frac{7}{6}e^{-3|\tau|} = R_{XX}(\tau) \end{aligned}$$

Now, the average power is given by $P_{XX} = R_{XX}(0) = E\{X^2(t)\} = \frac{5}{8} + \frac{7}{6} = \frac{43}{24}$

Example: 9. *The auto correlation function of the binary transmission is given by* $R_{XX}(\tau) = \begin{cases} 1-|\tau|, & \text{for } |\tau| \leq 1 \\ 0, & \text{elsewhere} \end{cases}$. *Find the PSD.*

Hints/Solution:

$$\begin{aligned} S_{XX}(\tau) &= \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-iw\tau}\,d\tau \\ &= \int_{-1}^{1} (1-|\tau|)[\cos(w\tau) - i\sin(w\tau)]\,d\tau \\ &= 2\int_{0}^{1} (1-\tau)\cos(w\tau)\,d\tau \\ &= \frac{2}{w^2}[1-\cos(w)] = \frac{4}{w^2}\left[\sin^2\left(\frac{w}{2}\right)\right] \end{aligned}$$
