# 18MAB102T- ADVANCED CALCULUS AND COMPLEX ANALYSIS

## Unit I - Double and Triple Integrals

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# Area using double Integration

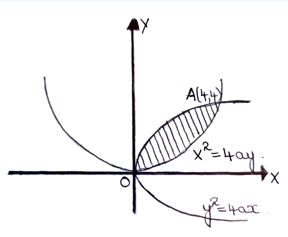
Area of the region $R$ by rectangular co-ordinates is $\int\int_R dxdy = \int\int_R dydx$

Area of the region $R$ by polar co-ordinates is $\int\int_R rdrd\theta$

**Problem: 1**

Show, by double integration, that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$

**Solution:** The region by integration is as shown.



Solving $y^2 = 4ax$ and $x^2 = 4ay$ are set $(0,0),(4a,4a)$. Take a strip parallel to y-axis implies limits for $y = \frac{x^2}{4a}$ and $y = 2\sqrt{ax}$ and then $x$ varies form 0 to 4a.

$$\therefore \text{Area}=\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dydx = \int_0^{4a} [y]_{\frac{x^2}{4a}}^{2\sqrt{ax}} dx$$

$$= \int_0^{4a} \left(2\sqrt{a}x^{1/2} - \frac{x^2}{4a}\right) dx = \left[2\sqrt{a}\frac{x^{1/2}+1}{\frac{1}{2}+1} - \frac{1}{4a}\frac{x^3}{3}\right]_0^{4a}$$

$$= \left[\frac{4}{3}\sqrt{a}x^{3/2} - \frac{1}{12a}x^3\right]_0^{4a} = \frac{4}{3}\sqrt{a}(4a^{3/2}) - \frac{1}{12a}(4a^3)$$

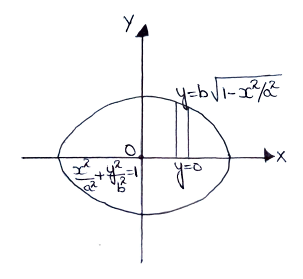
$$= \frac{4}{3}a^2(4\times 2) - \frac{1}{12a}\times 64a^3$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2 \text{ Sq.units.}$$

## Problem: 2

Find by double integration, the area enclosed by the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

**Solution:** The curve is symmetrical about both axes.




$\therefore$ Area =4 area in I quadrant

$$= 4\iint_A dydx = 4\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} dydx$$

$$= 4\int_0^a [y]_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx = \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

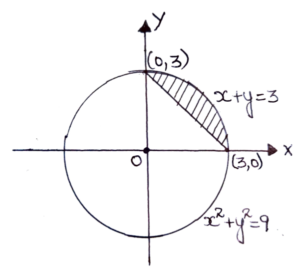
$$= \frac{4b}{a} \times \frac{a^2}{2} \times \frac{\pi}{2} = \pi ab \text{ Sq.units.}$$

**Note:** If $a = b$, the ellipse becomes circle. Then Area=$\pi r^2$

### Problem: 3

Find by double integration, smaller of the areas bounded by the circle $x^2 + y^2 = 9$ and $x + y = 3$.

**Solution:** The region of integration is as shown $y$ varies from $3 - x$ to $\sqrt{9-x^2}$ and $x$ varies from 0 to 3




$$\text{Area} = \int\int dydx$$

$$= \int_0^3 \int_{3-x}^{\sqrt{9-x^2}} dydx = \int_0^3 [\sqrt{9-x^2} - (3-x)]dx$$

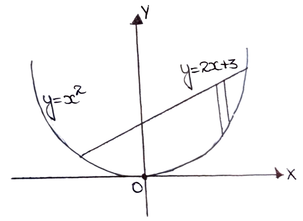
$$= \left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2} sin^{-1}\left(\frac{x}{3}\right)\right]_0^3 - 3[x]_0^3 + \left[\frac{x^2}{2}\right]_0^3$$

$$= \frac{9}{2}\frac{\pi}{2} - 9 + \frac{9}{2} = \frac{9\pi}{4} - \frac{9}{2} = \frac{9}{4}(\pi - 2) \text{ Sq.units.}$$

## Problem: 4

Find by double integration, the area bounded by the parabola $y = x^2$ and the line $y = 2x + 3$.

**Solution:** The region of integration is as shown.




Solving $y = 2x + 3$ we get $x^2 - 2x - 3 = 0$. (i.e.) $x = 3, -1$.
Required area $= \int\int dydx$ where $y$ varies from $y = x^2$ and $y = 2x + 3$.
Further $x$ varies from -1 to 3.

$$\therefore \text{Required area} = \int_{-1}^{3} \int_{x^2}^{2x+3} dydx = \int_{-1}^{3} [y]_{x^2}^{2x+3} dx$$

$$= \int_{-1}^{3} (2x + 3 - x^2)dx = \left[2\left(\frac{x^2}{2}\right) + 3x - \frac{x^3}{3}\right]_{-1}^{3}$$
$$= \frac{32}{3} = 10\frac{2}{3} \text{ Sq.units.}$$

### Problem: 5

Find by double integration, the area lying inside the circle $r = a\, sin\theta$ and outside the cordioid $r = a(1 - cos\theta)$.

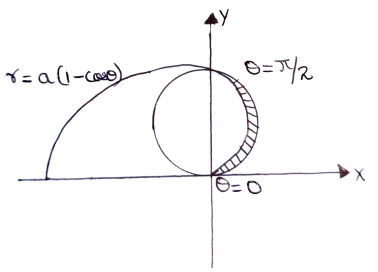
**Solution:** Eliminating $r$ between the equations of two curves
$sin\theta = 1 - cos\theta \ or \ sin\theta + cos\theta = 1$.
Squaring $1 + sin2\theta = 1 \ or \ sin2\theta = 0 \therefore 2\theta = 0$ or $\pi$
(i.e.) $\theta = 0$ or $\dfrac{\pi}{2}$ .
For the required area, $r$ varies from $a(1 - cos\theta)$ to $asin\theta$ and $\theta$ varies from $0 \ to \ \dfrac{\pi}{2}$.

$$\text{Required area} = \int_{0}^{\pi/2} \int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta$$

$$= \int_{0}^{\frac{\pi}{2}} \left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta}$$

$$= \frac{1}{2}\int_{0}^{\pi/2} a^2[\sin^2\theta - (1-\cos\theta)^2]d\theta$$

$$= \frac{a^2}{2}\int_{0}^{\pi/2} (2\cos\theta - 2\cos^2\theta)d\theta = a^2\left(1-\frac{\pi}{4}\right)$$

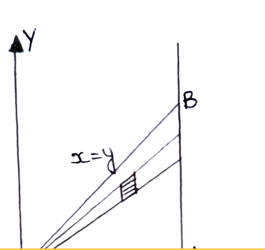


# Change of polar coordinates

## Problem: 1

Change to polar coordinates and evaluate $\int_0^a \int_y^a x\,dx\,dy$

**Solution:** The region of integration is $x = y, x = a, y = 0, y = a$.
(i.e) The triangle OAB putting $x = r\cos\theta, y = r\sin\theta$, the line $x = y$ becomes $r\cos\theta = r\sin\theta$
(i.e) $\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$



Hence in polar form $I = \int\limits_0^{\frac{\pi}{4}} \int\limits_0^{a/cos\theta} r^2 cos\theta drd\theta$

$= \int\limits_0^{\frac{\pi}{4}} \left(\frac{r^3}{3}\right)_0^{a/cos\theta} d\theta$

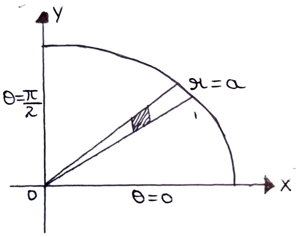
$= \frac{a^3}{3} \int\limits_0^{\frac{\pi}{4}} sec^2\theta d\theta = \frac{a^3}{3}[tan\theta]_0^{\pi/4}$

$= \frac{a^3}{3}$

## Problem: 2

Change to polar coordinates and evaluate $\int\limits_0^{a} \int\limits_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)} dxdy$

**Solution:** Putting $x = rcos\theta, y = rsin\theta$, the given limits $y^2 = a^2 - x^2$.
(i.e) The circle $x^2 + y^2 = a^2$ changes to $r = a$ and $y = 0$.




i.e. The x-axis changes to initial line $\theta = 0$. Hence, in the given region $r$ changes from 0 to a and $\theta$ changes from 0 to $\dfrac{\pi}{2}$.

$$I = \int_0^{\pi/2}\int_0^{a} e^{-r^2} r\,dr\,d\theta$$

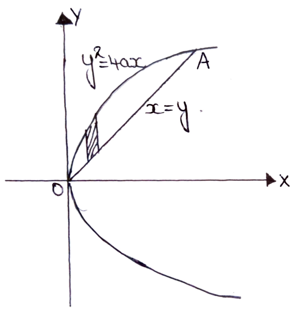
$$= \int_0^{\pi/2}\left(\frac{-1}{2}e^{-r^2}\right)_0^a d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}(e^{-a^2}-1)\,d\theta = \frac{\pi}{4}(1-e^{-a^2}).$$

## Problem: 3

Evaluate $\int_{0}^{4a} \int_{y^2/4a}^{y} dydx$ by changing to polar coordinates.

**Solution:** The region of integration is bounded by the parabola $x = y^2/4a$



(i.e) $y^2 = 4ax$ and the line $x = y$
By putting $x = rcos\theta, y = rsin\theta$, the parabola becomes
$r^2 sin^2\theta = 4arcos\theta$.

(i.e) $r = \dfrac{4acos\theta}{sin^2\theta}$ and the line becomes $x\ cos\theta = r\ sin\theta$
(i.e) $\theta = \dfrac{\pi}{4}$

Hence $r$ varies from 0 to $\dfrac{4acos\theta}{sin^2\theta}$ and $\theta$ varies from $\dfrac{\pi}{4}$ to $\dfrac{\pi}{2}$.

$$I = \int_{\pi/4}^{\pi/2} \int_{0}^{\frac{4acos\theta}{sin^2\theta}} r d\theta dr$$

$$= \int_{\pi/4}^{\pi/2} \left(\frac{r^2}{2}\right)_0^{\frac{4acos\theta}{sin^2\theta}} d\theta$$

$$= \frac{1}{2} \int_{\pi/4}^{\pi/2} \left[\frac{16a^2cos^2\theta}{sin^4\theta}\right] d\theta$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} cot^2\theta csc^2\theta d\theta$$

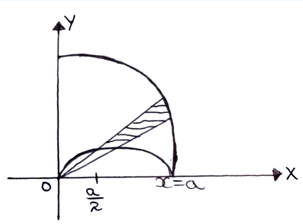
$$= 8a^2 \left[\frac{-cot^3\theta}{3}\right]_{\pi/4}^{\pi/2} = \frac{-8a^2}{3}[0-1] = \frac{8a^2}{3}$$

**Problem: 4**

Express the following integral in polar coordinates and evaluate

$$\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dxdy}{\sqrt{a^2-x^2-y^2}}$$

**Solution:** The limits of $y$ are $\sqrt{ax-x^2}$ and $\sqrt{a^2-x^2}$.



The equations of the circles now become

i) $r^2 - arcos\theta = 0$ $(i.e)$ $r = acos\theta$

ii) $r^2 = a^2$ $(i.e)$ $r = a$

Hence $r$ changes from $r = acos\theta$ $to$ $a$ and $\theta$ changes from 0 to $\dfrac{\pi}{2}$

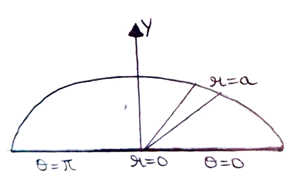
$$\therefore\ I = \int_0^{\pi/2} \int_{acos\theta}^{a} \frac{rdrd\theta}{\sqrt{a^2 - r^2}}$$

$$= \int_0^{\pi/2} [-\sqrt{a^2 - r^2}]_{acos\theta}^{a} d\theta$$

$$= \int_0^{\pi/2} asin\theta d\theta = [acos\theta]_0^{\pi/2} = a.$$

## Problem: 5

Evaluate $\int\limits_{-a}^{a}\int\limits_{0}^{\sqrt{a^2-x^2}}(x^2+y^2)dxdy$ by changing to polar coordinates.




**Solution:** Put $x = r\ cos\theta, y = rsin\theta \therefore dxdy = rdrd\theta$

$$I = \int\limits_{-a}^{a}\int\limits_{0}^{\sqrt{a^2-x^2}}(x^2+y^2)dxdy$$

$$= \int\limits_{0}^{\pi}\int\limits_{0}^{a} r^2.rdrd\theta = \left(\frac{r^4}{4}\right)_0^a [\theta]_0^{\pi} = \frac{\pi a^4}{4}$$

## Problem: 1

Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ by triple integrals.

**Solution:** Volume $=8\times$ volume in the first octant.

$$V=8\times\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dzdydx$$

$$=8\times\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}[z]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dydx$$

$$=8c\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\left[\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right]\mathrm{dydx}$$

Put $r^2 = \left(1 - \frac{x^2}{a^2}\right) b^2$

$$= \frac{8c}{b} \int_0^a \int_0^r \sqrt{r^2 - y^2}\, dy dx$$

$$= \frac{8c}{b} \int_0^a \left[ \frac{r^2}{2} sin^{-1} \frac{y}{r} + \frac{y}{2} \sqrt{r^2 - y^2} \right]_0^r dx$$

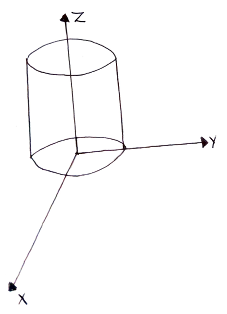
$$= \frac{2c\pi}{b} \int_0^a r^2 dx = \frac{2c\pi}{b} \int_0^a \left(1 - \frac{x^2}{a^2}\right) b^2 dx$$

$$= 2cb\pi \left(x - \frac{x^3}{3a^2}\right)_0^a = \frac{4\pi}{3} abc$$

## Problem: 2

Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

**Solution:** $z$ varies from $z = 0$ to $z = 4 - y$ and $x, y$ varies over all the points of the circle $x^2 + y^2 = 4$.




$$\text{Volume } V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{4-y} dzdydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [z]_0^{4-y} dydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y) dydx$$

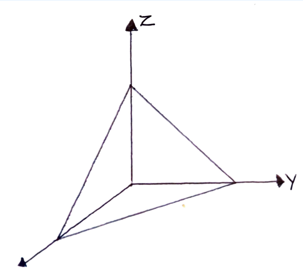
$$= \int_{-2}^{2} \left[ 4y - \frac{y^2}{2} \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx = 8 \times 2 \int_{-2}^{2} \sqrt{4 - x^2} dx$$

$$V = 16 \left[ \frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2} sin^{-1} \left( \frac{x}{2} \right) \right] = 16\pi$$

## Problem: 3

Find the volume of the tetrahedron bounded by the coordinate planes and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

**Solution:** Volume required=$\int \int \int dxdydz$ with limits.



$$= \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dzdydx$$

$$= c\int_0^a \int_0^{b(1-\frac{x}{a})} \left(1-\frac{x}{a}-\frac{y}{b}\right) dydx$$

$$= c\int_0^a \left[\left(1-\frac{x}{a}\right)y-\frac{y^2}{2b}\right]_0^{b(1-\frac{x}{a})} dx$$

$$= c\int_0^a \left[b\left(1-\frac{x}{a}\right)^2-\frac{b^2}{2b}\left(1-\frac{x}{a}\right)^2\right] dx$$

$$= \frac{bc}{2}\int_0^a (1-\tfrac{x}{a})^2 dx$$

$$= \frac{bc}{2}\left[\frac{\left(1-\frac{x}{a}\right)^2}{3}\times\left(\frac{-a}{1}\right)\right]$$

$$= \frac{abc}{6}$$