Problems :-

1. Find Fourier cosine and sine transforms of e^{-ax}, $a>0$ and hence deduce the inversion formula.

soln:-

$$F_c(f(x)) = \sqrt{2/\pi}\int_0^\infty f(x)\cos sx\,dx \quad \text{[Fourier cosine transform]}$$

$$F_c(e^{-ax}) = \sqrt{2/\pi}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$\left[\int_0^\infty e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

here $a=-a$, $b=s$

$$= \sqrt{2/\pi}\left[\frac{e^{-ax}}{(-a)^2+s^2}(-a\cos sx + s\sin sx)\right]_0^\infty$$

$$= \sqrt{2/\pi}\left[0-\left(\frac{1}{a^2+s^2}(-a)\right)\right]$$

$$= \sqrt{2/\pi}\left(\frac{a}{a^2+s^2}\right) \text{ if } a>0.$$

By inversion formula,

$$f(x) = \sqrt{2/\pi}\int_0^\infty F_c(s)\cos sx\,ds$$

$$e^{-ax} = \sqrt{2/\pi}\int_0^\infty \sqrt{2/\pi}\,\frac{a}{a^2+s^2}\cos sx\,ds$$

$$e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos sx}{a^2+s^2}\,ds.$$

$$\therefore \int_0^\infty \frac{\cos sx}{a^2+s^2}\,ds = \frac{\pi}{2a}\cdot e^{-ax},\ a>0.$$

Fourier sine transform:

$$F_s(f(x)) = \sqrt{2/\pi} \int_0^\infty f(x) \sin sx \, dx$$

$$F_s(e^{-ax}) = \sqrt{2/\pi} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$\left[\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx) \right]$$

here $a = -a$, $b = s$.

$$= \sqrt{2/\pi} \left[\frac{e^{-ax}}{(-a)^2+s^2} (-a \sin sx - s \cos sx) \right]_0^\infty$$

$$= \sqrt{2/\pi} \left[0 - \left(\frac{1}{a^2+s^2} (-s) \right) \right]$$

$$= \sqrt{2/\pi} \left[\frac{s}{a^2+s^2} \right].$$

By inversion formula,

$$f(x) = \sqrt{2/\pi} \int_0^\infty F_s(s) \sin sx \, ds$$

$$e^{-ax} = \sqrt{2/\pi} \int_0^\infty \sqrt{2/\pi} \left(\frac{s}{a^2+s^2} \right) \sin sx \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{s}{a^2+s^2} \sin sx \, ds.$$

$$\therefore \int_0^\infty \frac{s}{a^2+s^2} \sin sx \, dx = \frac{\pi}{2} e^{-ax}, \quad a > 0.$$

2) Using Parseval's identity, evaluate

(i) $\int_0^\infty \frac{dx}{(a^2+x^2)^2}$ and (ii) $\int_0^\infty \frac{x^2}{(a^2+x^2)^2}\,dx$ if $a>0$.

Soln:-

we know that, If $f(x) = e^{-ax}$ then $F_s(s) = \sqrt{2/\pi}\,\frac{s}{a^2+s^2}$

and $F_c(s) = \sqrt{2/\pi}\,\frac{a}{a^2+s^2}$

(i) Using Parseval's identity,

$$\int_0^\infty |f(x)|^2\,dx = \int_0^\infty |F_c(s)|^2\,ds.$$

$$\int_0^\infty (e^{-ax})^2\,dx = \int_0^\infty \left(\sqrt{2/\pi}\,\frac{a}{a^2+s^2}\right)^2 ds.$$

$$\int_0^\infty e^{-2ax}\,dx = \int_0^\infty \frac{2}{\pi}\,\frac{a^2}{(a^2+s^2)^2}\,ds.$$

$$\left.\frac{e^{-2ax}}{-2a}\right|_0^\infty = \frac{2a^2}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)^2}$$

$$0-\left(-\tfrac{1}{2a}\right) = \frac{2a^2}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)^2}$$

$$\int_0^\infty \frac{ds}{(a^2+s^2)^2} = \frac{\pi}{2a^2}\left(\frac{1}{2a}\right)$$

$$= \frac{\pi}{4a^3},\ a>0.$$

$$\therefore \int_0^\infty \frac{dx}{(a^2+x^2)^2} = \frac{\pi}{4a^3},\ a>0.$$

(ii) using Parseval's identity,

$$\int_0^\infty |f(x)|^2\,dx = \int_0^\infty |F_s(s)|^2\,ds$$

$$\int_0^\infty (e^{-ax})^2\,dx = \int_0^\infty \left(\sqrt{2/\pi}\,\frac{s}{a^2+s^2}\right)^2 ds$$

$$\Rightarrow \int_0^\infty e^{-2ax}\,dx = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(a^2+s^2)^2}\,ds.$$

$$\frac{1}{2a} = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(a^2+s^2)^2}\,ds.$$

$$\therefore \int_0^\infty \frac{s^2}{(a^2+s^2)^2}\,ds = \frac{1}{2a}\times\frac{\pi}{2}$$

$$= \frac{\pi}{4a}$$

$$\therefore \int_0^\infty \frac{x^2}{(a^2+x^2)^2}\,dx = \frac{\pi}{4a},\quad a>0.$$

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(3) Evaluate $\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)}$ using transform methods.

Soln:- Let $f(x) = e^{-ax}$, $g(x) = e^{-bx}$ then

$$F_c(s) = \sqrt{2/\pi}\,\frac{a}{a^2+s^2} \quad \& \quad G_c(s) = \sqrt{2/\pi}\,\frac{b}{b^2+s^2}$$

$\therefore$ Using, $\int_0^\infty F_c(s)\cdot G_c(s)\,ds = \int_0^\infty f(x)\,g(x)\,dx.$

$$\int_0^\infty \sqrt{2/\pi}\,\frac{a}{a^2+s^2}\cdot\sqrt{2/\pi}\,\frac{b}{b^2+s^2}\,ds = \int_0^\infty e^{-ax}\cdot e^{-bx}\,dx$$

$$\int_0^\infty \frac{2}{\pi}\,\frac{ab}{(a^2+s^2)(b^2+s^2)}\,ds = \int_0^\infty e^{-(a+b)x}\,dx$$

$$\frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)} = \left.\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= 0 - \left(\frac{1}{-(a+b)}\right) = \frac{1}{a+b}$$

$$\therefore \int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)} = \frac{\pi}{2ab}\cdot\frac{1}{a+b}, \quad a,b>0.$$

$$\therefore \int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2ab(a+b)} \text{ if } a,b>0.$$

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(4) Find Fourier sine transform of $1/x$.

Soln:-

$$F_s(f(x)) = \sqrt{2/\pi}\int_0^\infty f(x)\sin sx\, dx$$

$$F_s(1/x) = \sqrt{2/\pi}\int_0^\infty \frac{1}{x}\cdot \sin sx\, dx.$$

Put $sx = \theta. \Rightarrow x = \theta/s.$ when $x=0,\ \theta=0$

$s\,dx = d\theta. \Rightarrow dx = \frac{d\theta}{s}.$ $x=\infty,\ \theta=\infty.$

$$= \sqrt{2/\pi}\int_0^\infty \frac{\sin\theta}{\theta/s}\cdot\frac{d\theta}{s}.$$

$$= \sqrt{2/\pi}\int_0^\infty \frac{\cancel{s}\sin\theta}{\theta}\,\frac{d\theta}{\cancel{s}}$$

$$= \sqrt{2/\pi}\int_0^\infty \frac{\sin\theta}{\theta}\,d\theta \quad \left(\because \int_0^\infty \frac{\sin\theta}{\theta}\,d\theta = \frac{\pi}{2}\right)$$

$$= \sqrt{2/\pi}\times\frac{\pi}{2} = \sqrt{\frac{\pi}{2}}.$$

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(5) Show that

(i) $F_s(x.f(x)) = -\frac{d}{ds} F_C(s)$ (ii) $F_c[x.f(x)] = \frac{d}{ds} F_s(s)$ and

hence find Fourier cosine and sine transform of xe^{-ax}.

soln:-

(i) $F_c(s) = \sqrt{2/\pi} \int_0^\infty f(x) \cos sx \, dx$.

diff. b.s with respect to 's', we get

$$\frac{d}{ds} F_c(s) = \frac{d}{ds}\left[\sqrt{2/\pi} \int_0^\infty f(x) \cos sx \, dx\right]$$

$$= \sqrt{2/\pi} \int_0^\infty f(x) \frac{\partial}{\partial s}(\cos sx) \, dx$$

$$= \sqrt{2/\pi} \int_0^\infty f(x) (-\sin sx \cdot x) \, dx.$$

$$= -\sqrt{2/\pi} \int_0^\infty x f(x) \sin sx \, dx$$

$$= - F_s[x.f(x)]$$

$$\therefore F_s[x f(x)] = -\frac{d}{ds} F_c(s).$$

Fourier sine transform of xe^{-ax}.

$(\because F_c(s) = F_c(f(x)))$

ie, $F_s[x.f(x)] = -\frac{d}{ds} F_c(f(x))$

$$F_s[xe^{-ax}] = -\frac{d}{ds} F_c(e^{-ax}).$$

$$= -\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}\right]$$

$$= -\sqrt{2/\pi}\left[\frac{(a^2+s^2)(0)-a(2s)}{(a^2+s^2)^2}\right]$$

$$= \sqrt{2/\pi}\ \frac{2as}{(a^2+s^2)^2}.$$

$$\therefore\ F_s(xe^{-ax}) = \sqrt{2/\pi}\ \frac{2as}{(a^2+s^2)^2}.$$

(ii)

$$F_s(s) = \sqrt{2/\pi}\int_0^\infty f(x)\sin sx\,dx \quad (\text{definition}).$$

diff. both sides w.r.to 's', we get

$$\frac{d}{ds}F_s(s) = \frac{d}{ds}\left[\sqrt{2/\pi}\int_0^\infty f(x)\sin sx\,dx\right]$$

$$= \sqrt{2/\pi}\int_0^\infty f(x)\,\frac{\partial}{\partial s}(\sin sx)\,dx$$

$$= \sqrt{2/\pi}\int_0^\infty f(x)\,(\cos sx\cdot x)\,dx$$

$$= \sqrt{2/\pi}\int_0^\infty x\,f(x)\cos sx\,dx.$$

$$= F_c\left[x.f(x)\right]$$

$$\therefore\ F_c(xf(x)) = \frac{d}{ds}F_s(s).$$

<u>Fourier cosine transform of xe^{-ax}:</u>

ie, $F_c(xf(x)) = \frac{d}{ds}F_s(f(x))$.

$$\downarrow$$

$$F_c(xe^{-ax}) = \frac{d}{ds}F_s(e^{-ax}).$$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\ \frac{s}{a^2+s^2}\right]$$

$$= \sqrt{2/\pi}\ \frac{d}{ds}\left[\frac{s}{a^2+s^2}\right]$$

$$= \sqrt{2/\pi}\left[\frac{(a^2+s^2)(1) - s(2s)}{(a^2+s^2)^2}\right]$$

$$= \sqrt{2/\pi}\left[\frac{a^2+s^2-2s^2}{(a^2+s^2)^2}\right]$$

$$= \sqrt{2/\pi}\left[\frac{a^2-s^2}{(a^2+s^2)^2}\right]$$

$$\therefore\ F_c(xe^{-ax}) = \sqrt{2/\pi}\left(\frac{a^2-s^2}{(a^2+s^2)^2}\right).$$

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6) Find Fourier cosine transform of $e^{-a^2x^2}$ and hence evaluate Fourier sine transform of $xe^{-a^2x^2}$.

Soln:-

$$F_c(f(x)) = \sqrt{2/\pi}\int_0^\infty f(x)\cos sx\,dx.$$

$$F_c(e^{-a^2x^2}) = \sqrt{2/\pi}\int_0^\infty e^{-a^2x^2}\cos sx\,dx$$

$$= \sqrt{2/\pi}\times\frac{1}{2}\int_{-\infty}^\infty e^{-a^2x^2}\cos sx\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2x^2}\cos sx\,dx.$$

$$= \text{Real part of } \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2x^2}\cdot e^{isx}\,dx.$$

$$= \text{R.P of } \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2x^2+isx}\,dx.$$

we know that,

$$\int_{-\infty}^\infty f(x)\,dx = 2\int_0^\infty f(x)\,dx.$$

$$e^{isx} = \cos sx + i\sin sx.$$

Real part of $e^{isx} = \cos sx$.

$$= \text{R.P. of } \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)}\, dx}$$

[Refer: (9) Show that the transform of $e^{-s^2/2}$ is $e^{-s^2/2}$ by finding the fourier transform of $e^{-a^2x^2}$, $a>0$]

$$= \text{R.P of } \frac{1}{a\sqrt{2}} e^{-s^2/4a^2}$$

$$\therefore F_c(e^{-a^2x^2}) = \frac{1}{a\sqrt{2}} e^{-s^2/4a^2}$$

Fourier sine transform of $xe^{-a^2x^2}$

WKT, $F_s[xf(x)] = -\frac{d}{ds} F_c(s)$ $\quad \because F_c(s) = F_c(f(x))$

$$F_s[xe^{-a^2x^2}] = \frac{-d}{ds} F_c(e^{-a^2x^2})$$

$$= \frac{-d}{ds}\left[\frac{1}{a\sqrt{2}} e^{-s^2/4a^2}\right]$$

$$= -\left[\frac{1}{a\sqrt{2}} e^{-s^2/4a^2} \cdot \frac{-2s}{4a^2}\right]$$

$$= \frac{s}{2a^3\sqrt{2}} e^{-s^2/4a^2}$$

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(7) Solve for $f(x)$ from the integral equation

$$\int_0^{\infty} f(x) \sin sx\, dx = \begin{cases} 1 & \text{for } 0 \le s < 1 \\ 2 & \text{for } 1 \le s < 2 \\ 0 & \text{for } s > 2 \end{cases} \longrightarrow (*)$$

Proof:- Multiplying (*) by $\sqrt{2/\pi}$, both sides

$$\underbrace{\sqrt{2/\pi} \int_0^{\infty} f(x) \sin sx\, dx}_{\downarrow \text{ (definition)}} = \sqrt{2/\pi} \begin{cases} 1 & \text{for } 0 \le s < 1 \\ 2 & \text{for } 1 \le s < 2 \\ 0 & \text{for } s > 2 \end{cases}$$

$$F_s(f(x)) = \begin{cases} \sqrt{2/\pi} & \text{for } 0 \le s < 1 \\ 2\sqrt{2/\pi} & \text{for } 1 \le s < 2 \\ 0 & \text{for } s > 2 \end{cases}$$

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$\therefore\ f(x) = F_s^{-1}\left(\begin{cases} \sqrt{2/\pi} & \text{for } 0 \le s < 1 \\ 2\sqrt{2/\pi} & \text{for } 1 \le s < 2 \\ 0 & \text{for } s \geq 2 \end{cases}\right)$ $\quad f(x) = \sqrt{2/\pi}\int_0^\infty F_s(s)\sin sx\, ds$

$= \sqrt{2/\pi}\left[\int_0^1 \sqrt{2/\pi}\sin sx\, ds + \int_1^2 2\sqrt{2/\pi}\sin sx\, ds + \cancel{\int_2^\infty 0 \sin sx\, ds}\right]$

$= \frac{2}{\pi}\int_0^1 \sin sx\, ds + \frac{4}{\pi}\int_1^2 \sin sx\, ds$

$= \frac{2}{\pi}\left[\left(-\frac{\cos sx}{x}\right)_0^1 + \frac{4}{\pi}\left(-\frac{\cos sx}{x}\right)_1^2\right.$

$= \frac{2}{\pi}\left[\left(-\frac{\cos x}{x} + \frac{1}{x}\right) + \frac{4}{\pi}\left(-\frac{\cos 2x}{x} + \frac{\cos x}{x}\right)\right.$

$= \frac{2}{\pi}\left[\frac{1}{x} - \frac{\cos x}{x} - \frac{2\cos 2x}{x} + \frac{2\cos x}{x}\right)$

$f(x) = \frac{2}{\pi x}\left(1 + \cos x - 2\cos 2x\right)$

8) Find the function if its sine transform is $\frac{e^{-as}}{s}$.

Soln:

Let $F_s(f(x)) = \frac{e^{-as}}{s}$.

$f(x) = \sqrt{2/\pi}\int_0^\infty F_s(0)\sin sx\, ds.$

$f(x) = \sqrt{2/\pi}\int_0^\infty \frac{e^{-as}}{s}\sin sx\, ds \longrightarrow (1)$

$\therefore\ \frac{df}{dx} = \frac{d}{dx}\left[\sqrt{2/\pi}\int_0^\infty \frac{e^{-as}}{s}\sin sx\, ds\right]$

$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s}\frac{\partial}{\partial x}(\sin sx)\, ds$

$$= \sqrt{2/\pi} \int_0^{\infty} \frac{e^{-as}}{\cancel{s}} \cdot \cos sx \cdot \cancel{s}\, ds$$

$$= \sqrt{2/\pi} \int_0^{\infty} e^{-as} \cos sx \, ds. \qquad a=-a \quad b=x$$

$$= \sqrt{2/\pi} \left[\frac{e^{-as}}{(-a)^2+x^2} \left(-a\cos xs + x\sin xs\right)\right]_0^{\infty}$$

$$= \sqrt{2/\pi} \left(0 - \left(\frac{-a}{a^2+x^2}\right)\right)$$

$$= \sqrt{2/\pi} \left(\frac{a}{a^2+x^2}\right)$$

integrating w.r to 'x' on both sides,

$$\therefore \int \frac{df}{dx}\, dx = \sqrt{2/\pi} \int \frac{a}{a^2+x^2}\, dx$$

$$f(x) = \sqrt{2/\pi}\, \cancel{a} \cdot \frac{1}{\cancel{a}} \tan^{-1}\left(\frac{x}{a}\right) + c.$$

$$f(x) = \sqrt{2/\pi} \tan^{-1}(x/a) + c \longrightarrow (2).$$

At $x=0$, $f(0)=0$ using (1)

using this in equation (2), $f(0) = 0 + c$.

ie, $\boxed{0 = c}$

$$\therefore f(x) = \sqrt{2/\pi} \tan^{-1}(x/a).$$

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9) Find Fourier sine and cosine transform of x^{n-1}.

soln:-

we know that, $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}\, dx, \; n>0$

Replace x by ax, $a>0$.

$$= \int_0^{\infty} e^{-ax} (-ax)^{n-1}\, a\,dx$$

$$= \int_0^\infty e^{-ax} \cdot a^{n-1} x^{n-1} a\,dx$$

$$= a^n \int_0^\infty e^{-ax} x^{n-1}\,dx$$

$$\therefore \int_0^\infty e^{-ax} x^{n-1}\,dx = \frac{\Gamma(n)}{a^n}, \quad n>0,\ a>0.$$

we can prove the result even if a is complex.

Setting $a = is$

$$\int_0^\infty e^{-isx} x^{n-1}\,dx = \frac{\Gamma(n)}{(is)^n}$$

$$\frac{1}{i^n} = \left(\frac{1}{i}\right)^n = \left(\frac{1}{i} \times \frac{i}{i}\right)^n = \left(\frac{i}{i^2}\right)^n = (-i)^n$$

$$= \frac{(-i)^n\,\Gamma(n)}{s^n}$$

$$= \frac{e^{-\pi/2\, ni}\,\Gamma(n)}{s^n}$$

$$-i = e^{-\pi/2\, i}$$

Equating real and imaginary parts on both sides, we get

ie, $$\int_0^\infty x^{n-1} \cos sx\,dx = \frac{\Gamma(n)}{s^n} \cos\frac{n\pi}{2}$$

$$\int_0^\infty x^{n-1} \sin sx\,dx = \frac{\Gamma(n)}{s^n} \sin\frac{n\pi}{2}$$

Multiply $\sqrt{2/\pi}$ on both sides, we get

$$\sqrt{2/\pi} \int_0^\infty x^{n-1} \cos sx\,dx = \sqrt{2/\pi}\,\frac{\Gamma(n)}{s^n} \cos\frac{n\pi}{2}$$

$$\sqrt{2/\pi} \int_0^\infty x^{n-1} \sin sx\,dx = \sqrt{2/\pi}\,\frac{\Gamma(n)}{s^n} \sin\frac{n\pi}{2}$$

$$\therefore \quad F_c(x^{n-1}) = \sqrt{2/\pi}\ \frac{\Gamma(n)}{s^n} \cos\frac{n\pi}{2} \qquad (1)$$

$$F_s(x^{n-1}) = \sqrt{2/\pi}\ \frac{\Gamma(n)}{s^n} \sin\frac{n\pi}{2} . \qquad (2)$$

Taking $n = 1/2$, in (1)

$$F_c\left(x^{1/2-1}\right) = \sqrt{2/\pi}\ \frac{\Gamma 1/2}{s^{1/2}} \cos\frac{\pi}{4}$$

$$F_c\left(x^{-1/2}\right) = \sqrt{\cancel{2}/\cancel{\pi}}\ \frac{\cancel{\sqrt{\pi}}}{s^{1/2}} \cdot \frac{1}{\cancel{\sqrt{2}}} \qquad \left[\because \Gamma 1/2 = \sqrt{\pi}\right]$$

$$F_c\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{s^{1/2}}$$

ie, $F_c\left(1/\sqrt{x}\right) = \frac{1}{\sqrt{s}}$

III^{ly} $F_s\left(1/\sqrt{x}\right) = \frac{1}{\sqrt{s}}$

Note :-

$\frac{1}{\sqrt{x}}$ is self- reciprocal under Fourier sine and cosine transform.