Problems :-

① Solve : $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$. subject to

(i) $u(0,t) = 0$ for $t \geq 0$

(ii) $u(l,t) = 0$ for $t \geq 0$

(iii) $u(x,0) = \begin{cases} x & \text{for } 0 \leq x \leq l/2 \\ l-x & \text{for } l/2 \leq x \leq l . \end{cases}$

soln :-

here the given equation is a one dimensional heat flow equation and therefore the correct solution is

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\alpha^2\lambda^2 t} \longrightarrow ①$$

The boundary conditions are

(i) $u(0,t) = 0$ for $t \geq 0$

(ii) $u(l,t) = 0$ for $t \geq 0$

(iii) $u(x,0) = \begin{cases} x & \text{for } 0 \leq x \leq l/2 \\ l-x & \text{for } l/2 \leq x \leq l \end{cases}$

Applying b.c (i) in eqn ①, we get

$$u(0,t) = A e^{-\alpha^2\lambda^2 t} = 0 .$$

either $A = 0$ or $e^{-\alpha^2\lambda^2 t} = 0$.

$e^{-\alpha^2\lambda^2 t} \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore \boxed{A = 0}$

Substituting $\boxed{A=0}$ in eqn ①, we get

$$u(x,t) = B\sin\lambda x\, e^{-\alpha^2\lambda^2 t} \longrightarrow ②$$

Applying b.c (ii) in eqn ②, we get

$$u(l,t) = B\sin\lambda l\, e^{-\alpha^2\lambda^2 t} = 0$$

here, $B \neq 0$ since if $B=0$ we get trivial solution.

$e^{-\alpha^2\lambda^2 t} \neq 0$ ($\because$ it is defined $\forall t$)

hence, $\sin\lambda l = 0$

$\sin\lambda l = \sin n\pi$, where n is an integer

$\lambda l = n\pi$

$\therefore \lambda = \frac{n\pi}{l}$

Substituting $\lambda = \frac{n\pi}{l}$ in equation ②, we get

$$u(x,t) = B\sin\frac{n\pi x}{l}\, e^{\frac{-\alpha^2 n^2\pi^2 t}{l^2}}$$

$$u(x,t) = B_n \sin\frac{n\pi x}{l}\, e^{\frac{-\alpha^2 n^2\pi^2 t}{l^2}} \longrightarrow (*)$$ where B_n is any constant.

Since the partial differential equation (heat equation) is linear any linear combinations of solutions (or sum of the solutions) of the form $(*)$ with $n = 1,2,3,\ldots$ is also a solution of the equation.

$\therefore$ The most general solution of $(*)$ can be written as

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\, e^{\frac{-\alpha^2 n^2\pi^2 t}{l^2}} \longrightarrow ③$$

Applying boundary condition (iii) in eqn ③, we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\,(1) = f(x) \longrightarrow ④$$

where $f(x) = \begin{cases} x & \text{for } 0 < x < l/2 \\ l-x & \text{for } l/2 < x < l \end{cases}$

To find B_n expand $f(x)$ in $(0,l)$ in a half-range Fourier sine series we get

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{where} \quad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \longrightarrow (5)$$

From equation (4) & (5), we get

$$\boxed{B_n = b_n}$$

$$\therefore B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} x \sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} (l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[(x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^{l/2} +$$

$$(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\Bigg]_{l/2}^{l}$$

$$= \frac{2}{l}\left[\left[\frac{l}{2}\left(\frac{l}{n\pi}\right)\left(-\cos\frac{n\pi}{2}\right) + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] + 0 - \left(\left(\frac{l}{2}\right)\left(\frac{l}{n\pi}\right)\left(-\cos\frac{n\pi}{2}\right) - \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{2}{l}\left[\cancel{-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \cancel{\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2}{l}\left[\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

Substitute the value of B_n in equation (3), we get

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}}.$$

2) Find the temperature $u(x,t)$ in a silver bar of length 10cm, constant cross-section of $1\ cm^2$ area, density $10.6\ gm/cm^3$, thermal conductivity $1.04\ cal/cm\ deg.\ sec$; specific heat $0.056\ cal/gm.\ deg.$) which is perfectly insulated laterally, if the ends are kept at $0°c$, and if initially the temperature is $5°c$ at the centre of the bar and falls uniformly to zero at its ends.

soln:- In this problem, $\alpha^2 = k/\rho c = \dfrac{1.04}{(10.6)(0.056)} = 1.75$

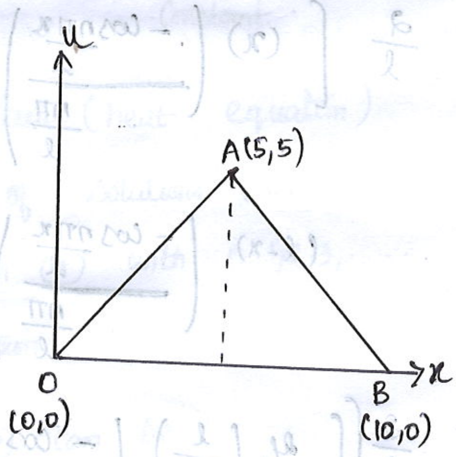
The one dimensional heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

The boundary conditions are

(i) $u(0,t) = 0$ for all $t \geq 0$

(ii) $u(10,t) = 0$ for all $t \geq 0$.



The equation of line along OA is $\underset{x_1\ u_1}{(0,0)}$ $\underset{x_2\ u_2}{(5,5)}$

$$\frac{u-u_1}{u_2-u_1} = \frac{x-x_1}{x_2-x_1}$$

$$\Rightarrow \frac{u-0}{5-0} = \frac{x-0}{5-0} \Rightarrow \frac{u}{5} = \frac{x}{5}$$

$$\Rightarrow \boxed{u = x}, \quad 0 \leq x \leq 5.$$

The equation of line along AB is $\underset{x_1\ u_1}{(5,5)}$ $\underset{x_2\ u_2}{(10,0)}$

$$\Rightarrow \frac{u-5}{0-5} = \frac{x-5}{10-5}$$

$\Rightarrow$

$\Rightarrow \dfrac{u-5}{-5} = \dfrac{x-5}{5} \Rightarrow u-5 = -x+5$

$\Rightarrow u = -x+10, \quad 5 \le x \le 10.$

$\Rightarrow \cancel{-u+5} = \cancel{x-5}$

$\therefore$ (ii) $u(x,0) = \begin{cases} x & \text{in } 0 \le x \le 5 \\ -x+10 & \text{in } 5 \le x \le 10. \end{cases}$

The suitable solution is

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\alpha^2\lambda^2 t} \longrightarrow (1)$$

Applying b.c (i) in eqn (1), we get

$$u(0,t) = A(1)\, e^{-\alpha^2\lambda^2 t} = 0.$$

either $A=0$ or $e^{-\alpha^2\lambda^2 t} = 0.$

$e^{-\alpha^2\lambda^2 t} \ne 0$ ($\because$ it is defined for all t)

$\boxed{\therefore A=0}$

Sub $A=0$ in eqn (1), we get

$$u(x,t) = B\sin\lambda x\, e^{-\alpha^2\lambda^2 t} \longrightarrow (2)$$

Applying b.c (ii) in eqn (2), we get

$$u(10,t) = B\sin 10\lambda\, e^{-\alpha^2\lambda^2 t} = 0.$$

here, $B \ne 0$ since if $B=0$ we get trivial solution.

$e^{-\alpha^2\lambda^2 t} \ne 0$ ($\because$ it is defined for all t)

hence, $\sin 10\lambda = 0$

$\sin 10\lambda = \sin n\pi$, n is an integer

$10\lambda = n\pi$

$\boxed{\lambda = \dfrac{n\pi}{10}}$

Sub. $\lambda = \dfrac{n\pi}{10}$ in eqn (2), we get

$$u(x,t) = B \sin\frac{n\pi x}{10} e^{-\frac{\alpha^2 n^2 \pi^2 t}{100}}$$

$$u(x,t) = B_n \sin\frac{n\pi x}{10} e^{-\frac{\alpha^2 n^2 \pi^2 t}{100}}$$ where $B = B_n$, B_n is any constant.

The most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} e^{-\frac{\alpha^2 n^2 \pi^2 t}{100}} \longrightarrow ③$$

Applying b.c (iii) in eqn ③, we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} = f(x) \longrightarrow ④$$ where

$$f(x) = \begin{cases} x & \text{in } 0 \le x \le 5 \\ 10-x & \text{in } 5 \le x \le 10 \end{cases}$$

To find B_n expand $f(x)$ in half-range sine series in the interval $(0, 10)$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} \longrightarrow ⑤$$ where $b_n = \frac{2}{10}\int_0^{10} f(x) \sin\frac{n\pi x}{10}\,dx$

From ④ & ⑤, we get $\boxed{B_n = b_n}$

$$\therefore B_n = \frac{2}{10}\left[\int_0^5 x \sin\frac{n\pi x}{10}\,dx + \int_5^{10} (10-x)\sin\frac{n\pi x}{10}\,dx\right]$$

$$= \frac{1}{5}\left[\left[(x)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)^2}\right)\right]_0^5 + \right.$$

$$\left.\left[(10-x)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{10}}{\left(\frac{n\pi}{10}\right)^2}\right)\right]_5^{10}\right]$$

$$= \frac{1}{5}\left[\left[(5)\left(\frac{10}{n\pi}\right)\left(-\cos\frac{n\pi}{2}\right) + \frac{100}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)\right] +\right.$$

$$\left[0-\left((5)\left(\frac{10}{n\pi}\right)\left(-\cos\frac{n\pi}{2}\right)-\frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]$$

$$\Rightarrow \frac{1}{5}\left[\cancel{-\frac{50}{n\pi}\cos\frac{n\pi}{2}}+\frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}+\cancel{\frac{50}{n\pi}\cos\frac{n\pi}{2}}+\frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$=\frac{1}{5}\left[\frac{200}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$\Rightarrow \frac{40}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

$\therefore$ sub. the value of B_n in eqn (3), we get

$$u(x,t)=\sum_{n=1}^{\infty}\frac{40}{n^2\pi^2}\sin\frac{n\pi}{2}\cdot\sin\frac{n\pi x}{10}\,e^{-\frac{\alpha^2 n^2\pi^2 t}{100}}$$

Exercise problems.

(1) Solve the equation $\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}$ subject to the boundary conditions $u(0,t)=0$, $u(l,t)=0$, $u(x,0)=x$.

(2) A homogeneous rod of conducting material of length 'l' units has ends kept at zero temperature and the temperature at the centre is T and falls uniformly to zero at the two ends. Find $u(x,t)$.