Steady state Conditions and zero boundary Conditions.

① A rod 30cm long has its ends A and B kept at 20°c and 80°c respectively, until steady state Conditions prevail. The temperature at each end is then suddenly reduced to 0°c. and kept so. Find the resulting temperature function $u(x,t)$ taking $x=0$ at A.

Soln:- The P.D.E of one dimensional heat flow is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow ⓐ$$

In steady state Conditions, the temperature at any particular point does not vary with time. ie, u depends only on x and not on time t.

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \left[\because \frac{\partial u}{\partial t} = 0 \text{ since } u \text{ is a function of } x \text{ only} \right]$$

Since u is a function of x only, the above equation Can be written as $\frac{d^2 u}{dx^2} = 0$. $(\alpha \neq 0)$

Hence when ~~steady~~ steady state Conditions prevails the heat flow equation becomes

$$\frac{d^2 u}{dx^2} = 0. \longrightarrow ②$$

int. eqn ② w.r.to 'x' twice, we get

$$\boxed{u(x) = ax+b} \longrightarrow (*)$$

The boundary conditions in steady state are

(i) $u(0) = 20$

(ii) $u(30) = 80$

Applying b.c (i) in eqn ⊛, we get

$$u(0) = a(0) + b = 20$$

$$\therefore \boxed{b = 20}$$

Sub. $b = 20$ in eqn ⊛ we get

$$u(x) = ax + 20 \longrightarrow ⊛⊛.$$

Applying b.c (ii) in eqn ⊛⊛, we get

$$u(30) = a(30) + 20 = 80$$

$$30a = 60$$

$$\boxed{a = 2}.$$

Sub $a = 2$ in eqn ⊛⊛, we get

$$\therefore \boxed{u(x) = 2x + 20}$$

When the temperature at A and B are reduced to zero, the temperature distribution changes and the state is no more steady state. For this transient state, the boundary conditions are

(i) $u(0,t) = 0 \quad \forall\, t \geq 0$

(ii) $u(30,t) = 0 \quad \forall\, t \geq 0.$

The initial temperature of this state is the temperature in the previous steady-sate. Hence the initial condition is

(iii) $u(x,0) = 2x + 20$, for $0 < x < 30$.

Now, we have to find $u(x,t)$ satisfying the conditions (i), (ii) and (iii) and the P.D.E. (a). The correct solution of ① is of the form

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\alpha^2\lambda^2 t} \longrightarrow ①$$

Applying b.c (i) in eqn ①, we get

$$u(0,t) = A(1)\, e^{-\alpha^2\lambda^2 t} = 0.$$

either $A=0$ or $e^{-\alpha^2\lambda^2 t} = 0$.

but $e^{-\alpha^2\lambda^2 t} \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore \boxed{A=0}$.

Sub. $A=0$ in eqn ①, we get

$$u(x,t) = B\sin\lambda x\, e^{-\alpha^2\lambda^2 t} \longrightarrow ②$$

Applying b.c (ii) in eqn ②, we get

$$u(30,t) = B\sin 30\lambda\, e^{-\alpha^2\lambda^2 t} = 0.$$

$B \neq 0$ ($\because$ If $B=0$, we get trivial solution).

$e^{-\alpha^2\lambda^2 t} \neq 0$ ($\because$ it is defined $\forall t$)

$\sin 30\lambda = 0$

$\sin 30\lambda = \sin n\pi$, where n is an integer

$30\lambda = n\pi$

$$\boxed{\lambda = \frac{n\pi}{30}}$$

Sub. $\lambda = \frac{n\pi}{30}$ in eqn ②, we get

$$u(x,t) = B\sin\frac{n\pi x}{30}\, e^{-\frac{\alpha^2 n^2\pi^2 t}{900}}$$

$$u(x,t) = B_n \sin\frac{n\pi x}{30}\, e^{-\frac{\alpha^2 n^2\pi^2 t}{900}}$$ where $B = B_n$, B_n is any constant.

The most general solution can be written as

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{30} e^{-\frac{\alpha^2 n^2\pi^2 t}{900}} \longrightarrow (3)$$

Applying b.c (iii) in eqn (3), we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{30} (1) = 2x+20. \longrightarrow (4)$$

To find B_n, expand $2x+20$ in half-range sine series in the interval $(0,30)$.

$$2x+20 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{30} \text{ where} \longrightarrow (5)$$

$$b_n = \frac{2}{30}\int_0^{30} f(x) \sin\frac{n\pi x}{30} dx$$

From (4) & (5), we get

$$\therefore \boxed{B_n = b_n}$$

$$B_n = \frac{2}{30}\int_0^{30} (2x+20) \sin\frac{n\pi x}{30} dx$$

$$= \frac{1}{15}\left[(2x+20)\left(\frac{-\cos\frac{n\pi x}{30}}{\frac{n\pi}{30}}\right) - (2)\left(\frac{-\sin\frac{n\pi x}{30}}{\left(\frac{n\pi}{30}\right)^2}\right)\right]_0^{30}$$

$$= \frac{1}{15}\left[(80)\left(\frac{30}{n\pi}\right)(-(-1)^n) + \frac{2(900)}{n^2\pi^2}(0) - \left(20\right)\left(\frac{30}{n\pi}\right)(-1) - 2\left(\frac{900}{n^2\pi^2}\right)(0)\right].$$

$$= \frac{1}{15}\left[\frac{2400}{n\pi}(-1)^{n+1} + \frac{600}{n\pi}\right]$$

$$= \frac{1}{15} \times \frac{600}{n\pi}\left[4(-1)^{n+1} + 1\right]$$

$$= \frac{40}{n\pi}\left[1 + 4(-1)^{n+1}\right].$$

Sub. in eqn 'sub. the value of B_n in eqn ⑤,

we get

The temperature distribution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{40}{n\pi}\left(1+4(-1)^{n+1}\right) \sin\frac{n\pi x}{30}\, e^{\frac{-\alpha^2 n^2 \pi^2 t}{900}} \text{ degrees.}$$

---

2) A rod of length 'l' has its ends A and B kept at 0°C and 100°C until steady state condition prevail. If the temperature at B is reduced suddenly to 0°C and kept so while that of A is maintained, find the temperature $u(x,t)$ at a distance x from A and at time t.

Soln:-

The partial differential equation of one dimensional heat flow is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow ⓐ$$

In steady state conditions, the temperature at any particular point does not vary with time. ie, u depends only on x and not on time t.

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \left[\because \frac{\partial u}{\partial t} = 0 \text{ since } u \text{ is a function of } x \text{ only}\right]$$

Since u is a function of x only, the above equation can be written as $\frac{d^2u}{dx^2} = 0$ $(\alpha \neq 0)$

Hence when steady state conditions prevails the heat flow equation becomes

$$\frac{d^2u}{dx^2} = 0 \longrightarrow (2)$$

integrating eqn (2). w.r to 'x' twice, we get

$$u(x) = ax+b \longrightarrow (*)$$

The boundary conditions are

(i) $u(0) = 0$

(ii) $u(l) = 100$

Applying b.c (i) in eqn (*), we get

$$u(0) = a(0) + b = 0$$

$$\therefore \boxed{b=0}$$

Sub $b=0$ in eqn (*), $u(x) = ax+0 \longrightarrow (*)(*)$

Applying b.c (ii) in eqn (*)(*), we get

$$u(l) = a(l) = 100$$

$$a = \frac{100}{l}.$$

Sub $a = \frac{100}{l}$ in eqn (*)(*), we get

$$\boxed{u(x) = \frac{100x}{l}}$$

When the temperature at B is reduced to zero, the temperature distribution changes and the state is no more steady state. For this transient state, the boundary conditions are

(i) $u(0,t) = 0 \quad \forall t \geq 0$

(ii) $u(l,t) = 0 \quad \forall t \geq 0$

The initial temperature of this state is the temperature in the previous steady-state. Hence the initial condition is

(iii) $u(x,0) = \frac{100x}{l}$ for $0 \le x \le l$.

Now, we have to find $u(x,t)$ satisfying the conditions (i), (ii) and (iii) and the P.D.E (a).

The suitable solution of ① is of the form

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\alpha^2\lambda^2 t} \longrightarrow ①$$

Applying b.c (i) in eqn ①, we get

$$u(0,t) = A(1)\, e^{-\alpha^2\lambda^2 t} = 0.$$

either $A=0$ or $e^{-\alpha^2\lambda^2 t} = 0$.

$e^{-\alpha^2\lambda^2 t} \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore \boxed{A=0}$

sub $A=0$ in eqn ①, we get

$$u(x,t) = B\sin\lambda x\, e^{-\alpha^2\lambda^2 t} \longrightarrow ②$$

Applying b.c (ii) in eqn ②, we get

$$u(l,t) = B\sin\lambda l\, e^{-\alpha^2\lambda^2 t} = 0.$$

here, $B \neq 0$ ($\because$ If $B=0$, we get trivial solution)

$e^{-\alpha^2\lambda^2 t} \neq 0$ ($\because$ it is defined for all t)

$\sin\lambda l = 0$

$\therefore \sin\lambda l = \sin n\pi$, where n is an integer

$\lambda l = n\pi$

$\therefore \boxed{\lambda = \frac{n\pi}{l}}$

Sub $\lambda = \frac{n\pi}{l}$ in eqn ②, we get

$$u(x,t) = B \sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$$

$$u(x,t) = B_n \sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$$ where $B = B_n$, B_n is any constant.

The most general solution can be written as

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}} \longrightarrow ③$$

Applying b.c (iii) in eqn ③, we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} (1) = \frac{100x}{l}. \longrightarrow ④$$

To find B_n, expand $\frac{100x}{l}$ in half-range sine series in $(0,l)$.

$$\frac{100x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \longrightarrow ⑤$$ where

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx.$$

From ④ & ⑤, we get $\boxed{B_n = b_n}$

$$B_n = \frac{2}{l}\int_0^l \frac{100x}{l} \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2(100)}{l^2}\int_0^l x \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{200}{l^2}\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^l$$

$$= \frac{200}{l^2}\left[l\left(\frac{l}{n\pi}\right)(-\cos n\pi)\right]$$

$$= \frac{200}{l^2}\left[-\frac{l^2}{n\pi}(-1)^n\right] = \frac{200}{n\pi}(-1)^{n+1}$$

Sub. the value of B_n in eqn ③, we get

The temperature distribution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{200}{n\pi}(-1)^{n+1} \sin\frac{n\pi x}{l}\, e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}}.$$

<u>Exercise problems</u>:

① A rod of length l has its ends A and B kept at 0°C and 120°C respectively, until steady state conditions prevail. If the temperature at B is reduced to 0°C and kept so while that of A is maintained. Find the temperature distribution in the rod.

Ans: $u(x,t) = \frac{240}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi x}{l}\, e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}}.$