Unit- 4

Fourier Transforms.

Complex Fourier Transform (Infinite)

Let $f(x)$ be a function defined in $(-\infty, \infty)$ and be piece-wise continuous in each finite partial interval and absolutely integrable in $(-\infty, \infty)$. Then the complex Fourier Transform of $f(x)$ is defined by

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

Inversion Theorem for Complex Fourier Tranform :-

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

Fourier Integral theorem:

If $f(x)$ is piece-wise continuously differentiable and absolutely integrable in $(-\infty, \infty)$, then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{i(x-t)s} dt\, ds.$$

Properties of Fourier Tranforms :

Thm: 1 Fourier Tranform is linear.

ie, $F[af(x) + bg(x)] = aF(f(x)) + bF[g(x)]$

where F stands for Fourier transform.

Proof:- $F[af(x) + bg(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (af(x) + bg(x))\, e^{isx} dx$ (by definition)

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} a f(x) e^{isx} dx + \int_{-\infty}^{\infty} b g(x) e^{isx} dx\right]$$

$$= a \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx + b \cdot \int_{-\infty}^{\infty} g(x) e^{isx} dx$$

$$= a F[f(x)] + b F[g(x)].$$

Theorem: 2 Shifting theorem

If $F[f(x)] = F(s)$, then $F(f(x-a)) = e^{isa} F(s)$.

Proof:

$$F(f(x-a)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{isx} dx \quad [\text{by definition}]$$

Put $x - a = t \Rightarrow x = a+t$

$dx = dt$

when $x = -\infty$, $t = \infty$

$x = \infty$, $t = \infty$.

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i(a+t)s} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot e^{ias} \cdot e^{its} ds.$$

$$= e^{ias} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{its} ds}. \quad [\because \text{'t' is a dummy variable}]$$

$$= e^{ias} F(s).$$

Theorem : 3 Change of scale property.

If $F\{f(x)\} = F(s)$, then $F\{f(ax)\} = \frac{1}{|a|} F\left(\frac{s}{a}\right)$, where $a \neq 0$.

Proof:

$$F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{isx} dx \quad [\text{by definition}]$$

Case (i) Put $ax = t \Rightarrow x = t/a$ when $x = \infty,\ t = \infty$

and $a > 0$, $a\,dx = dt$ $\quad x = -\infty,\ t = -\infty$

$$dx = \frac{dt}{a}$$

$$\therefore \quad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i(s/a)t} \frac{dt}{a}$$

$$= \frac{1}{a} \cdot \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i(s/a)t}\, dt}$$ (by the definition)

$$= \frac{1}{a} \cdot F(s/a), \quad \text{where } a > 0.$$

Case (ii).

Put $ax = t$ & $a < 0$.

$x = t/a$ & $dx = \frac{dt}{a}$

when $x = \infty$, $t = -\infty$

$x = -\infty$, $t = \infty$.

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t)\, e^{i(s/a)t} \frac{dt}{a}.$$

$$= -\frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i(s/a)t}\, dt$$

$$= \frac{-1}{a} F(s/a), \quad \text{where } a < 0.$$

$$\therefore \quad F\{f(ax)\} = \frac{1}{|a|} F(s/a).$$

Theorem: 4

$$F\{e^{iax} f(x)\} = F(s+a)$$

Proof:

$$F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x)\, e^{isx}\, dx$$ (by the definition)

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s+a)x} f(x)\, dx$$

$$= F(s+a).$$

Theorem: 5 **Modulation theorem.**

If $F\{f(x)\} = F(s)$, then $F\{f(x)\cos ax\} = \frac{1}{2}\left[F(s-a) + F(s+a)\right]$

Proof:-

$$F[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos ax \, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left(\frac{e^{iax} + e^{-iax}}{2}\right) e^{isx}\, dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} f(x)\, e^{iax} . e^{isx}\, dx\right] + \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} f(x)\, e^{-iax} e^{isx} dx\right]\right]$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x}\, dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i(s-a)x} dx\right]$$

$$= \frac{1}{2}\left[F(s+a) + F(s-a)\right]$$

Theorem: 6 If $F(f(x)) = F(s)$, then $F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n} F(s)$.

Proof:- By the definition, $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$.

differentiating w.r to 's' both sides, n times

$$\frac{d^n F(s)}{ds^n} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (ix)^n f(x) e^{isx}\, dx$$

$$= (i)^n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n f(x) e^{isx}\, dx$$

$$= (i)^n F\{x^n f(x)\}$$

$$F\{x^n f(x)\} = \frac{1}{(i)^n} \frac{d^n F(s)}{ds^n}$$

$$F\{x^n f(x)\} = (-i)^n \frac{d^n}{ds^n} F(s). \quad \left[\because \frac{1}{(i)^n} = \left(\frac{1}{i}\right)^n = \left(\frac{1}{i} \times \frac{i}{i}\right)^n = \left(\frac{i}{-1}\right)^n = (-i)^n\right]$$

Theorem : 7

$$F\{f'(x)\} = -isF(s) \text{ if } f(x) \to 0 \text{ as } x \to \pm\infty.$$

Proof :-

$$F\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} d(f(x))$$

$u = e^{isx}$ $\quad v = f(x)$ $\quad \int u\, dv = uv - \int v\, du.$

$\frac{du}{dx} = (is)e^{isx}$

$$= \frac{1}{\sqrt{2\pi}} \left[\left\{ e^{isx} f(x) \right\}_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\, (is) e^{isx} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 - is \int_{-\infty}^{\infty} f(x) e^{isx}\, dx \right]$$

$[\because \text{if } f(x) \to 0 \text{ as } x \to \pm\infty]$

$$= -is \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx}$$

$$= -is\, F(s)$$

Theorem : 8

$$F\left\{ \int_a^x f(x)\, dx \right\} = \frac{F(s)}{-is}$$

Let $\quad \phi(x) = \int_a^x f(x)\, dx$

Then $\phi'(x) = f(x)$

$$F(\phi'(x)) = (-is) F[\phi(x)] \quad (\text{by Thm. 7})$$

$$F[f(x)] = -is\, F[\phi(x)]$$

$$F[\phi(x)] = \frac{F(s)}{-is} \quad //$$

$$F\{x^n f(x)\} = \frac{1}{(i)^n} \frac{d^n}{ds^n} F(s) \qquad \frac{1}{(i)^n} = \left(\frac{1}{i} \times \frac{i}{i}\right)^n = \left(\frac{i}{-1}\right)^n = (-i)^n$$

$$= (-i)^n \frac{d^n}{ds^n} F(s).$$

Problems :-

(1) Find the Fourier transform of $f(x) = \begin{cases} 1-x^2 & \text{if } |x|<1 \\ 0 & \text{if } |x|>1 \end{cases}$

hence evaluate $\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right) \cos\frac{x}{2}\, dx$.

Solution :-

$$F(s) = F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \Big[\int_{-\infty}^{-1} \underbrace{f(x)}_{(0)} e^{isx}\, dx + \int_{-1}^{1} f(x)\, e^{isx}\, dx + \int_{1}^{\infty} \underbrace{f(x)}_{(0)} e^{isx}\, dx \Big]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)\,(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \Big[\int_{-1}^{1} \underbrace{\underbrace{(1-x^2)}_{\text{even}}\, \underbrace{\cos sx}_{\text{even}}}_{\text{even}}\, dx + i \int_{-1}^{1} \underbrace{\underbrace{(1-x^2)}_{\text{even}}\, \underbrace{\sin sx}_{\text{odd}}}_{\text{odd}}\, dx \Big]$$

$$= \frac{1}{\sqrt{2\pi}} \Big[2\int_0^1 (1-x^2) \cos sx\, dx + 0 \Big]$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1-x^2) \cos sx\, dx$$

$$= \sqrt{2/\pi} \Big[(1-x^2)\frac{\sin sx}{s} - (-2x)\left(\frac{-\cos sx}{s^2}\right) - 2\left(\frac{-\sin sx}{s^3}\right) \Big]_0^1$$

$$= \sqrt{2/\pi} \Big[\frac{-2\cos s}{s^2} + 2\frac{\sin s}{s^3} \Big]$$

$$= \sqrt{2/\pi} \Big[\frac{-2s\cos s + 2\sin s}{s^3} \Big]$$

$$= \frac{-2\sqrt{2}}{\sqrt{\pi}} \left[\frac{s\cos s - \sin s}{s^3} \right]$$

Using inversion formula:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{-2\sqrt{2}}{\sqrt{\pi}} \left[\frac{s\cos s - \sin s}{s^3} \right] e^{-isx}\, ds$$

$$= \frac{-2}{\pi} \int_{-\infty}^{\infty} \left[\frac{s\cos s - \sin s}{s^3} \right] e^{-isx}\, ds.$$

$$\int_{-\infty}^{\infty} \left(\frac{s\cos s - \sin s}{s^3} \right) (\cos sx - i \sin sx)\, ds = -\pi/2\, f(x)$$

$$= -\pi/2 \begin{cases} (1-x^2) & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1. \end{cases}$$

$$= \begin{cases} -\pi/2\,(1-x^2) & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

Equating real parts,

$$\int_{-\infty}^{\infty} \left(\frac{s\cos s - \sin s}{s^3} \right) \cos sx\, ds = \begin{cases} -\pi/2\,(1-x^2) & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

Put $x = 1/2$

$$\int_{-\infty}^{\infty} \left(\frac{s\cos s - \sin s}{s^3} \right) \cos s/2\, ds = -\frac{\pi}{2} \left(1 - 1/4\right)$$

$$\underbrace{\int_{-\infty}^{\infty} \underbrace{\left(\frac{s\cos s - \sin s}{s^3} \right)}_{\text{even}} \underbrace{\cos s/2}_{\text{even}}\, ds}_{\text{even}} = \frac{-3\pi}{8}$$

$$2\int_0^\infty \left(\frac{s\cos s - \sin s}{s^3}\right)\cos s/_2\, ds = -\frac{3\pi}{8}.$$

$$\therefore \int_0^\infty \left(\frac{s\cos s - \sin s}{s^3}\right)\cos s/_2\, ds = -\frac{3\pi}{16}.$$

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2) Find the Fourier transform of $f(x)$ given by $f(x) = \begin{cases} 1 & \text{for } |x| < a \\ 0 & \text{for } |x| > a > 0 \end{cases}$

and hence evaluate $\int_0^\infty \frac{\sin x}{x}\, dx$ and $\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\, ds$.

Solution:-

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\underset{(0)}{\cancel{\int_{-\infty}^{-a} f(x) e^{isx}\, dx}} + \int_{-a}^{a} f(x) e^{isx}\, dx + \underset{(0)}{\cancel{\int_{a}^{\infty} f(x) e^{isx}\, dx}} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (1)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\underbrace{\int_{-a}^{a} \cos sx\, dx}_{\text{even function}} + i \underbrace{\int_{-a}^{a} \sin sx\, dx}_{\text{odd function (0)}} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[2 \int_0^a \cos sx\, dx \right]$$

$$= \sqrt{2/\pi} \left[\frac{\sin sx}{s} \right]_0^a$$

$$= \sqrt{2/\pi}\, \frac{\sin as}{s}$$

Using inversion formula, we get

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} \, ds.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin as}{s} e^{-isx} \, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} e^{-isx} \, ds.$$

$$\int_{-\infty}^{\infty} \frac{\sin as}{s} e^{-isx} \, ds = \pi f(x)$$

$$\int_{-\infty}^{\infty} \frac{\sin as}{s} (\cos sx - i \sin sx) \, ds = \pi \begin{cases} 1 & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}$$

$$\int_{-\infty}^{\infty} \frac{\sin as}{s} (\cos sx - i \sin sx) \, ds = \begin{cases} \pi & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}$$

Equating real parts,

$$\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s} \, ds = \begin{cases} \pi & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}$$

Put $x = 0$.

$$\int_{-\infty}^{\infty} \frac{\sin as}{s} \, ds = \pi$$

even function

$$2 \int_{0}^{\infty} \frac{\sin as}{s} \, ds = \pi$$

$$\int_{0}^{\infty} \frac{\sin as}{s} \, ds = \pi/2$$

Put $as = \theta \Rightarrow s = \theta/a$

$a \, ds = d\theta \Rightarrow ds = \frac{d\theta}{a}$

when $s = 0, \ \theta = 0$

$s = \infty, \ \theta = \infty$

$$\int_0^\infty \frac{\sin\theta}{\theta/a}\frac{d\theta}{a} = \pi/2$$

$$\int_0^\infty \frac{\alpha\sin\theta}{\theta}\frac{d\theta}{\alpha} = \pi/2$$

$$\int_0^\infty \frac{\sin\theta}{\theta}d\theta = \pi/2 .$$

Convolution Theorem or Faltung Theorem:

Definition: The Convolution of two functions $f(x)$ and $g(x)$ is defined as $f*g = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,g(x-t)\,dt.$

Theorem: The Fourier transform of the Convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

ie, $F\{f(x)*g(x)\} = F(s)\cdot G(s) = F\{f(x)\}\cdot F\{g(x)\}$.

Parseval's identity:

If $F(s)$ is the Fourier transform of $f(x)$

then, $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds.$

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(3) Find the Fourier transform of $f(x)$ given by

$$f(x) = \begin{cases} 1 & \text{for } |x| < a \\ 0 & \text{for } |x| > a > 0 \end{cases}$$

and prove that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}.$

Solution:-

Refer Previous problem, we know that

$$F(s) = \sqrt{\frac{2}{\pi}}\ \frac{\sin as}{s}.$$

<u>Using Parseval's identity,</u>

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$\int_{-a}^{a} (1)\, dx = \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}}\ \frac{\sin as}{s} \right)^2 ds$$

$$(a-(-a)) = \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin as}{s} \right)^2 ds$$

even function.

$$\cancel{2}a = \frac{\cancel{2}}{\pi} \left[2 \int_{0}^{\infty} \left(\frac{\sin as}{s} \right)^2 ds \right]$$

$$\frac{a \times \pi}{2} = \int_{0}^{\infty} \left(\frac{\sin as}{s} \right)^2 ds$$

Put $as = \theta \Rightarrow s = \theta/a$ when $s = 0,\ \theta = 0$

$a\, ds = d\theta$ $\qquad s = \infty,\ \theta = \infty.$

$ds = \frac{d\theta}{a}.$

$$\int_{0}^{\infty} \left(\frac{\sin \theta}{\theta/a} \right)^2 \frac{d\theta}{a} = \frac{\pi a}{2}$$

$$\int_{0}^{\infty} a^{\cancel{2}} \left(\frac{\sin \theta}{\theta} \right)^2 \frac{d\theta}{\cancel{a}} = \frac{\pi \cancel{a}}{2}$$

$$\therefore \int_{0}^{\infty} \left(\frac{\sin \theta}{\theta} \right)^2 d\theta = \frac{\pi}{2}.$$

4) Find the Fourier transform of $f(x) = \begin{cases} 1-|x| & \text{if } |x|<1 \\ 0 & \text{for } |x|>1 \end{cases}$ (7)

and hence find the value $\int_0^\infty \frac{\sin^4 t}{t^4}\, dt$.

Soln:-

$$F(f(x)) = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx .$$

$$= \frac{1}{\sqrt{2\pi}} \left[\underbrace{\int_{-\infty}^{-1} f(x) e^{isx}\, dx}_{(0)} + \int_{-1}^{1} f(x) e^{isx} dx + \underbrace{\int_{1}^{\infty} f(x) e^{isx}\, dx}_{(0)} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|) e^{isx}\, dx .$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{1} (1-|x|) (\cos sx + i \sin sx)\, dx \right.$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{1} \underbrace{\underbrace{(1-|x|)}_{\text{even}} \underbrace{\cos sx}_{\text{even}}}_{\text{even}}\, dx + \underbrace{i \int_{-1}^{1} \underbrace{(1-|x|)}_{\text{e.f}} \underbrace{\sin sx}_{\text{o.f}}\, dx}_{\text{o.f} \quad (0)} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[2 \int_0^1 (1-|x|) \cos sx\, dx \right]$$

$$= \sqrt{2/\pi} \int_0^1 (1-x) \cos sx\, dx \qquad \left[\because 1-|x| = 1-x \text{ in interval } (0,1) \right]$$

$$= \sqrt{2/\pi} \left\{ \left[(1-x) \left(\frac{\sin sx}{s} \right) - (-1) \left(-\frac{\cos sx}{s^2} \right) \right]_0^1 \right.$$

$$= \sqrt{2/\pi} \left(\frac{-\cos s}{s^2} + \frac{1}{s^2} \right) = \sqrt{2/\pi} \left(\frac{1-\cos s}{s^2} \right)$$

Using Parseval's identity,

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds$$

$$\int_{-1}^{1} (1-|x|)^2\,dx = \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s^2}\right)\right)^2 ds$$

$$2\int_{0}^{1} (1-x)^2\,dx = \frac{2}{\pi}\int_{-\infty}^{\infty} \left(\frac{1-\cos s}{s^2}\right)^2 ds$$ even function

$$\cancel{2}\left(\frac{(1-x)^3}{-3}\right)_0^1 = \frac{\cancel{2}}{\pi}\left[2\int_{0}^{\infty} \left(\frac{1-\cos s}{s^2}\right)^2 ds\right]$$

$$1/3 = \frac{2}{\pi}\int_{0}^{\infty} \left(\frac{1-\cos s}{s^2}\right)^2 ds.$$

$$\frac{\pi}{6} = \int_{0}^{\infty} \left(\frac{2\sin^2 s/2}{s^2}\right)^2 ds.$$

$$\frac{\pi}{6} = 4\int_{0}^{\infty} \left(\frac{\sin^2 s/2}{s^2}\right)^2 ds.$$

[$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\sin^2\theta/2 = \frac{1-\cos\theta}{2}$

$2\sin^2\theta/2 = 1-\cos\theta.$

$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a.n+1}$]

Put $t = s/2 \Rightarrow s = 2t$

$ds = 2dt$

When $s=0$, $t=0$

$s=\infty$, $t=\infty$.

$$\frac{\pi}{24} = \int_{0}^{\infty} \left(\frac{\sin^2 t}{(2t)^2}\right)^2 2dt$$

$$\int_{0}^{\infty} \frac{\sin^4 t}{16t^4}\,2dt = \frac{\pi}{24}$$

$$\int_{0}^{\infty} \frac{\sin^4 t}{8t^4}\,dt = \frac{\pi}{24} \Rightarrow \int_{0}^{\infty} \frac{\sin^4 t}{t^4}\,dt = \frac{\pi}{3}.$$