# Properties of Z-Transform.

1. **Linearity**:

Weighted sum of 2 signals is equal to the weighted sum of individual transforms.

$$X_1(z) = Z[x_1(n)]$$
$$X_2(z) = Z[x_2(n)]$$

then

$$Z[a x_1(n) + b x_2(n)] = a X_1(z) + b X_2(z)$$

Proof:-
$$Z[a x_1(n) + b x_2(n)] = \sum_{n=-\infty}^{\infty} [a x_1(n) + b x_2(n)] z^{-n}$$

$$= a \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + b \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}.$$

ROC for transform of a sum of sequences of ROC of the individual transform.

2. **Time Shifting**:-

If $X(z) = Z[x(n)]$ then
$$Z[x(n-m)] = z^{-m} X(z).$$

Proof:-
$$Z[x(n-m)] = \sum_{n=-\infty}^{\infty} x(n-m) z^{-n}$$

Let $n-m = p$.
$$= \sum_{p=-\infty}^{\infty} x(p) z^{-(p+m)}$$

$$= \sum_{n=-\infty}^{\infty} x(p) z^{-p} z^{-m}$$

$$= \underbrace{\left[ \sum_{n=-\infty}^{\infty} x(p) z^{-p} \right]}_{X(z)} z^{-m}.$$

$$= z^{-m} X(z)$$

If $m > 0$, ROC of $z^{-m} X(z)$ is same as that of $X(z)$ except for $n = 0$.

3. <u>Multiplication by exponential sequence.</u>

If $X(z) = Z[x(n)]$ then

$$Z[a^n x(n)] = X[a^{-1} z].$$

Proof:- $$Z[a^n x(n)] = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}.$$

$$= \sum_{n=-\infty}^{\infty} x(n) (a^{-1} z)^{-n}$$

$$= X(a^{-1} z).$$

ROC $R_1 < |a^{-1} z| < R_2$

$$R_1 < \left| \frac{z}{a} \right| < R_2$$

$$|a| R_1 < |z| < |a| R_2$$

## 4. Time Reversal

If $X(z) = Z[x(n)]$ then

$$Z[x(-n)] = X z^{-1}.$$

Proof:

$$Z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

Let $l = -n$ then

$$Z[x(-n)] = \sum_{n=-\infty}^{\infty} x(l) z^{l}.$$

$$= \sum_{n=-\infty}^{\infty} x(l) (z^{-1})^{l}$$

$$= X(z^{-1})$$

ROC: $|R_1| < z^{-1} < R_2$

that is $\frac{1}{R_2} < z < \frac{1}{R_1}$

## 5) Multiplication by n

If $Z[x(n)] = X(z)$ then

$$Z[n\,x(n)] = -z \frac{d}{dz} X(z).$$

Proof:-

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} n\, x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} n\, x(n) z^{-n}$$

$$= z \sum_{n=-\infty}^{\infty} n\, x(n)\, z^{-n-1}$$

$$= z \sum_{n=-\infty}^{\infty} n\, x(n)\, z^{-(n+1)}$$

$$= z \sum_{n=-\infty}^{\infty} x(n)\, n z^{-(n+1)}$$

$$= z \sum_{n=-\infty}^{\infty} x(n) \left[ \frac{-d\left(z^{-n}\right)}{dz} \right]$$

$$= -z \sum_{n=-\infty}^{\infty} x(n) \left[ \frac{d}{dz}\left(z^{-n}\right) \right]$$

$$= -z \frac{d}{dz} \left[ x(n)\, z^{-n} \right]$$

$$Z[n\, x(n)] = -z \frac{d}{dz} X(z).$$

$$Z[n^k x(n)] = \left(-z \frac{d}{dz}\right)^k X(z)$$

6. Convolution.

$$Z[x(n) * h(n)] = X(z) \cdot H(z)$$

$\downarrow$ Convolutional operator.

Proof :-

Let $y(n) = x(n) * h(n)$

$$y(n) = x(n) * h(n) = X(z) \cdot H(z).$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)$$

Take z transform for both sides.

$$Y(z) = \sum_{n=-\infty}^{\infty} \left[ \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \right] z^{-n}$$

Interchanging order of summation.

$$Y(z) = \sum_{k=-\infty}^{\infty} x(k)\, z^{-k} \cdot \sum_{n=-\infty}^{\infty} h(n-k)\, z^{-n+k}$$

$$\left[ z^{-k} \cdot z^{k} = 1 \right].$$

$$Y(z) = \sum_{k=-\infty}^{\infty} x(k) z^{-k} \sum_{n=-\infty}^{\infty} h(n-k)\, z^{-(n-k)}$$

$$Y(z) = X(z) \cdot H(z)$$

6. Conjugation.

If $Z[x(n)] = X(z)$ then

$$Z[x^*(n)] = X^*(z^*)$$

7. Initial Value Theorem:

If $X(z) = Z[x(n)]$ then

$$x(0) = \lim_{z \to \infty} X(z)$$

Proof:-

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} = x(0) + x(1) z^{-1} + x(2) z^{-2} + \cdots$$

We know that when $z \to \infty$. all the terms will vanish except $x(0)$.

So,

$$\lim_{z \to \infty} X(z) = \lim_{z \to \infty} \sum_{n=0}^{\infty} x(n) z^{-n} = x(0)$$

$$\lim_{z \to \infty} X(z) = x(0).$$

8) Final Value Theorem.

$$x(\infty) = \lim_{z \to 1} (z-1) X(z).$$

9) Parseval's Relation

$$\sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) = \frac{1}{2\pi j} \oint_c X_1(z) X_2^*\left(z^{-1}\right) dz$$