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ECE – A

Advanced Calculus and Complex Analysis-

18MAB102T

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18MAB102T - ADVANCED CALCULUS AND COMPLEX ANALYSIS.

ASSIGNMENT- I

1\. Evaluate $\int_0^{\log a}\int_0^{x}\int_0^{x+y} (e^{x+y+z})\, dz\, dy\, dx$

Soln:

Let $I = \int_0^{\log a}\int_0^{x}\int_0^{x+y} (e^{x+y+z})\, dz\, dy\, dx$

Order of Integration is as follows

$I = \int_0^{\log a}\left[\int_0^{x}\left[\int_0^{x+y} (e^{x+y+z})\, dz\right] dy\right] dx$ $\quad (e^{x+y+z} = e^x e^y e^z)$ $\quad (\int e^x e^y e^z \, dz = e^x e^y e^z)$

$= \int_0^{\log a}\left[\int_0^{x}\left[e^{x+y+z}\right]_0^{x+y} dy\right] dx$

$= \int_0^{\log a}\left[\int_0^{x}(e^{2x+2y} - e^{x+y})\cdot dy\right] dx$ $\quad (\int e^{2x+2y}\, dy = \frac{e^{2x+2y}}{2})$ $\quad (\int e^{x+y}\, dy = e^{x+y})$

$= \int_0^{\log a}\left[\frac{e^{2x+2y}}{2} - e^{x+y}\right]_0^{x} dx$

$= \int_0^{\log a}\left(\frac{e^{4x}}{2} - e^{2x} - \left(\frac{e^{2x}}{2} - e^{x}\right)\right) dx$

$= \int_0^{\log a}\left(\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^{x}\right) dx$

$= \left[\frac{e^{4x}}{8} - \frac{e^{2x}}{2} - \frac{e^{2x}}{4} + e^{x}\right]_0^{\log a} = \left[\frac{e^{4x}}{8} - \frac{3e^{2x}}{4} + e^{x}\right]_0^{\log a}$

$I = \frac{e^{4\log a}}{8} - \frac{e^{2\log a}}{4} + e^{\log a} - \left(\frac{e^0}{8} - \frac{3e^0}{4} + e^0\right)$

$= \frac{a^4}{8} - \frac{a^2}{4} + a - \frac{1}{8} + \frac{3}{4} - 1 \quad \times 8$

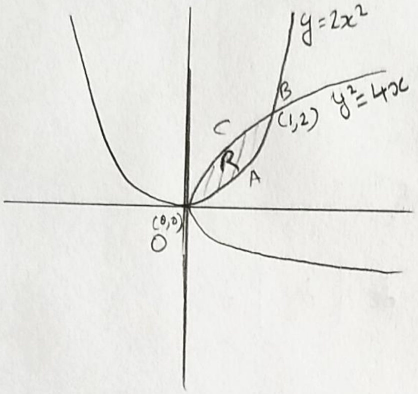
$= a^4 - 2a^2 + 8a - 1 + 6 - 8$

$= a^4 - 2a^2 + 8a - 3$ //

Using Double Integration find the area enclosed by the Curves $y = 2x^2$ and $y^2 = 4x$.

(2)

Soln.



$y = 2x^2$ — ①

y	0	2	8	18	32
x	0	1	2	3	4

$y^2 = 4x$ — ②

y	0	2	$2\sqrt{2}$	$2\sqrt{3}$	4
x	0	1	2	3	4

Solving ① ant ②

$y = 2x^2$ and $y^2 = 4x$

then

$(2x^2)^2 = 4x$

$4x^4 = 4x$

$x = 1$

when $x = 1$

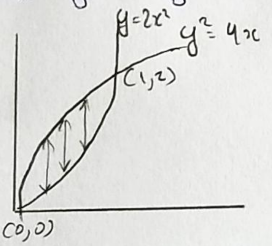
$y = 2$ — ①

$y = \pm 2$ — ②

Area enclosed $=$ Area OABC

$= \int_x \int_y dx\, dy.$

y- limits given by vertical line drawn across the area

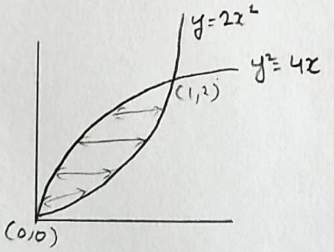


Lower limit $=$ $y = 2x^2$

upper limit $=$ $y^2 = 4x$

$y = 2\sqrt{x}$

x- limits given by horizontal line drawn across the area



Lower limit $x = 0$

upper limit $x = 1$

$\therefore$ Area Enclosed $= \displaystyle\int_{x=0}^{x=1} \int_{y=2x^2}^{y=2\sqrt{x}} dy\, dx$

$I = \displaystyle\int_0^1 \left[y \right]_{2x^2}^{2\sqrt{x}} \cdot dx$

$= \displaystyle\int_0^1 (2\sqrt{x} - 2x^2) \cdot dx$

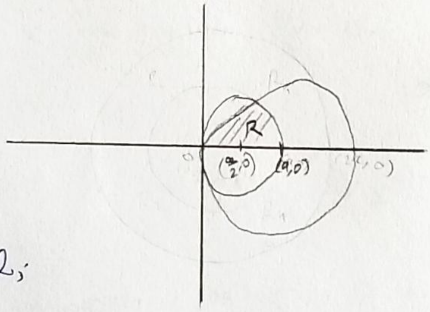
$$I = \left[\frac{2x^{3/2}}{3/2} - \frac{2x^3}{3}\right]_0^1$$

$$= \left[\frac{4}{3} - \frac{2}{3}\right] - [0]$$

$$I = \frac{2}{3}$$

2. Evaluate $\iint r^4 \, dr\, d\theta$ area between the Circles $r = a\cos\theta$ and $r = 2a\cos\theta$.

Soln



$r = a\cos\theta$

$\sqrt{x^2+y^2} = a \cdot \frac{x}{\sqrt{x^2+y^2}}$

$x^2+y^2 = a.x.$

$x^2 - ax + y^2 = 0$

$x^2 - ax + \frac{a^2}{4} + y^2 = \frac{a^2}{4}$

$\left(x - \frac{a}{2}\right)^2 + (y-0)^2 = \left(\frac{a}{2}\right)^2$.

Since it is symmetrical;

$$\text{Area} = 2 \cancel{4} \times \iint_{\theta\ r} r^4 \, dr\, d\theta.$$

Limits of r =

L.L = $r = a\cos\theta$

UL = $r = 2a\cos\theta$.

Limits of θ

$LL = \theta = 0$

$UL = \theta = \frac{\pi}{2}$

$r = 2a\cos\theta$

$\sqrt{x^2+y^2} = \frac{2a x}{\sqrt{x^2+y^2}}$

$x^2+y^2 - 2ax = 0$

$x^2 - 2ax + a^2 + y^2 = a^2$

$(x-a)^2 + (y-0)^2 = a^2$

$$I = 2\int_{\theta=0}^{\pi/2}\int_{r=a\cos\theta}^{2a\cos\theta} r \cdot dr\, d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a\cos\theta}^{2a\cos\theta} d\theta.$$

$\left[\cos^2\theta = \frac{\cos 2\theta + 1}{2}\right]$

$$= \int_0^{\pi/2} (4a^2\cos^2\theta - a^2\cos^2\theta) \cdot d\theta$$

$$= a^2 \int_0^{\pi/2} \cancel{4}^{2}\left(\frac{\cos 2\theta + 1}{\cancel{2}}\right) - \left(\frac{\cos 2\theta + 1}{2}\right) \cdot d\theta$$

$$= a^2\left[\cancel{\cos}\left[\frac{\sin 2\theta}{2} + \theta\right] \times 2 - \left(\frac{\sin 2\theta}{4} + \frac{\theta}{2}\right)\right]_0^1$$

$$I = a^2\left[\sin 2\theta + 2\theta - \frac{\sin 2\theta}{4} - \frac{\theta}{2}\right]_0^{\pi/2}$$ (4)

$$= a^2\left[\frac{3\sin 2\theta}{4} + \frac{3\theta}{2}\right]_0^{\pi/2}$$

$$= a^2\left[\frac{3\sin\pi}{4} + \frac{3\pi}{4} - 0\right]$$

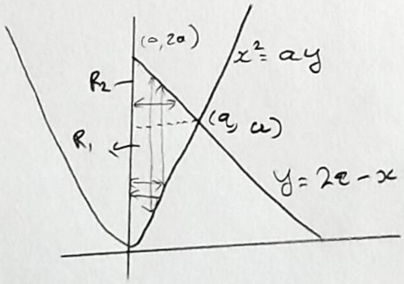
$$= \frac{3a^2\pi}{4}$$ //

3. ~~Evaluate~~ Change the order of integration in $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\,dy\,dx$ and hence evaluate.

Soln

In given double integral, integration is wrt y first and then wrt x.

On changing the order of integration, integration wrt x first and then wrt y.



$y = \frac{x^2}{a}$; $y = 2a - x$

$(2a-y)^2 = ay$

$4a^2 + y^2 - 4ay = ay$

$y^2 - 5ay + 4a^2 = 0$

$y = a, 4a$

then

$x = a, -2a$

only first quadrant

In Region R_1

x Lts -

$0 \to \sqrt{ay}$

y Lts

$0 \to a$

In Region R_2

x Lts -

$0 \to 2a - y$

y Lts -

$a \to 2a$.

$\therefore$ Area $= I = \iint_{R_1} xy\,dx\,dy + \iint_{R_2} xy\,dx\,dy$

$$= \int_0^a \int_0^{\sqrt{ay}} xy\,dx\,dy + \int_a^{2a}\int_0^{2a-y} xy\,dy\,dx$$

Integrating over R_1,

$$I_1 = \int_0^a \int_0^{\sqrt{ay}} xy \, dx dy$$

$$= \int_0^a \left[\int_0^{\sqrt{ay}} x \cdot dx \right] y \, dy \Big]$$

$$= \int_0^a \left[\frac{x^2}{2} \right]_0^{\sqrt{ay}} y \cdot dy \Big]$$

$$= \frac{1}{2} \int_0^a (ay \cdot y \, dy)$$

$$= \frac{a}{2} \left[\frac{y^3}{3} \right]_0^a = \frac{a^4}{6} = I_1$$

Integrating over R_2,

$$I_2 = \int_a^{2a} \int_0^{2a-y} xy \, dy dx$$

$$= \int_a^{2a} \left[\int_0^{2a-y} x \cdot dx \right] y \cdot dy \Big]$$

$$\int_0^{2a} \left[\frac{x^2}{2} \right]_0^{2a-y} y \cdot dy = \frac{1}{2} \int_0^{2a} (2a-y)^2 \, dy.$$

$$= \frac{1}{2} \int_0^{2a} (4a^2 + y^2 - 4ay) \, dy$$

$$= \frac{1}{2} \left[4a^2 y + \frac{y^3}{3} - \frac{4ay^2}{2} \right]_0^{2a}$$

$$= \frac{1}{2} \left[4a^2 \cdot 2a + \frac{8a^3}{3} - 2a \cdot 4a^2 \right]$$

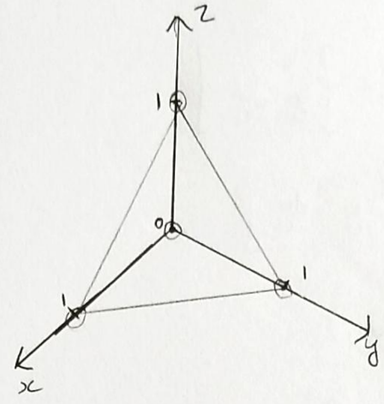
$$= \frac{1}{2} \left[8a^3 + \frac{8a^3}{3} - 8a^3 \right] = \frac{4a^3}{3}$$

$$I = I_1 + I_2$$

$$= \frac{a^4}{6} + \frac{4a^3}{3} = a^4 + 8a^3 \, //.$$

4. Evaluate $\iiint dx\,dy\,dz$, Where V is the Volume of the tetrahedron whose vertices are $(0,0,0)$ $(0,1,0)$ $(1,0,0)$ $(0,0,1)$ (6)

Soln



Formula of tetrahedron is given by,

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad a,b,c=1 \text{ unit length}$$

$$x+y+z=1$$

Limits of $z=0 \rightarrow z=1-x-y$

Limits of $y=0 \rightarrow y=1-x$

Limits of $x=0 \rightarrow x=1$.

$$I=\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} dx\,dy\,dz.$$

$$=\int_0^1\left[\int_0^{1-x}\left(\int_0^{1-x-y}dz\right)dy\right]dx.$$

$$=\int_0^1\left[\int_0^{1-x}[z]_0^{1-x-y}dy\right]dx$$

$$=\int_0^1\left[\int_0^{1-x}(1-x-y)\,dy\right]dx$$

$$=\int_0^1\left[y-xy-\frac{y^2}{2}\right]_0^{1-x}dx = \int_0^1\left[(1-x)-x(1-x)-\frac{(1-x)^2}{2}\right]dx$$

$$= \int_0^1 \left[1 - x - x + x^2 - \frac{(1 + x^2 - 2x)}{2}\right] dx$$

$$= \int_0^1 \left[1 - 2x + x^2 - \frac{1}{2} - \frac{x^2}{2} + x\right] dx$$

$$= \int_0^1 \left(\frac{1}{2} - x + \frac{x^2}{2}\right) dx$$

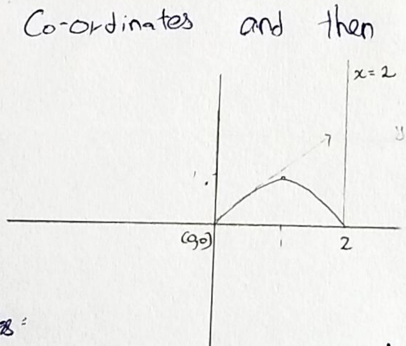
$$= \left[\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{6}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \frac{1}{6} \;//$$

$$V = \frac{1}{6}$$

5. Change into polar Co-ordinates and then evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dx\,dy$.

Soln



$y = \sqrt{2x - x^2}$

x	0	1	2
y	0	1	0

Transformation formulas:

$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

$\iint dx\,dy = \iint r\,.\,dr\,.\,d\theta$.

Limits

$x = r\cos\theta$

$x = 2$; $x = 0$

$r = 0 \rightarrow r = \frac{2}{\cos\theta}$

$\theta = 0 \rightarrow \theta = \frac{\pi}{4}$

$$\text{Area} = I = \int_{\theta=0}^{\pi/4} \int_0^{2/\cos\theta} \frac{\cancel{r}\cos\theta}{\cancel{r}}\,.\,r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \left[\int_0^{2/\cos\theta} r\,dr\right] \cos\theta\, d\theta = \int_0^{\pi/4} \left[\frac{r^2}{2}\right]_0^{2/\cos\theta} \cos\theta\,.\,d\theta$$

8

$$I = \frac{1}{2}\int_0^{\pi/4} \frac{4}{\cos^2\theta} \cos\theta \, . d\theta.$$

$$= 2 \int_0^{\pi/4} \sec\theta \quad = 2 \left[\ln (\sec\theta + \tan\theta) \right]_0^{\pi/4}$$

$$= 2 \left[\ln \left| \sec\frac{\pi}{4} + \tan\frac{\pi}{4} \right| + \ln \left| \sec 0 + \tan 0 \right| \right]$$

$$= 2 \left[\ln |\sqrt{2}+1| + \ln |1.| \right]$$

$$= 2 \ln (\sqrt{2}+1) \,\|.$$