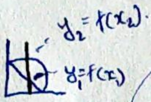
# UNIT-I

# DOUBLE AND TRIPLE INTEGRALS

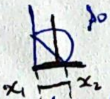
## Steps to find Limits of Integration:

1. Sketch the region and label the bounding curves.

2. Imagine a vertical line (L) cutting through R. Mark the y-values where L enters and leaves. Those are Y Limits.



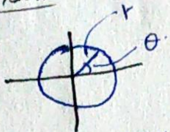

3. Choose x-limits that include all the vertical lines through R.




So $\int_{x_1}^{x_2}\int_{f(x_1)}^{f(x_2)} f(x,y)\,dA$

↳ includes all ⊥ lines.

## Double Integral on Polar Coordinates and Conversion.




Area is given by

$$\iint_{\theta\ r} f(r,\theta)\, dr\, d\theta$$

**Note:**

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2$

$dx\,dy = r\,dr\,d\theta$

## DOUBLE INTEGRAL

$$\iint_R f(x,y)\,dA = \int_{dy}\int_{dx} f(x,y)\,dx\,dy$$

( Integration here is similar to partial diff. First it is integrated wrt $x$, keeping $y$ constant, then integrated wrt $y$ keeping $x$ const. )

### Properties:

1. $\iint_R k f(x,y)\,dA = k\iint_R f(x,y)\,dA$
2. $\iint_R (f \pm g)\,dA = \iint_R f\,dA \pm \iint_R g\,dA$
3. $\iint_R f(x,y)\,dA \geq 0$ if, $f(x,y) \geq 0$.
4. $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$ if, $f(x,y) \geq g(x,y)$.
5. $R_1, R_2 \to$ non overlapping Regions. Then,
$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

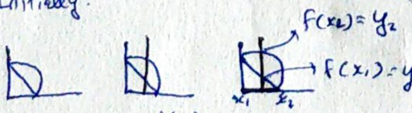
## DOUBLE AND TRIPLE INTEGRAL

## Change in Order of Integration:

1. Identify the region of Integration.
2. Identify the Constant and Variable limits.
3. Switch Variable → Constant, Constant → Variable.
4. Use "Steps to find limits of integration" to find new limits.

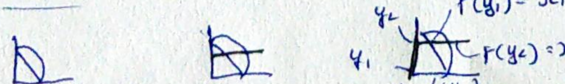
Eg:

Initially.




$\iint f(x,y)$    $\int\int_{f(x_1)}^{f(x_2)} f(x,y)$    $\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)$

In CIO



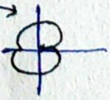

$\iint f(x,y)$    $\iint_{f(y_1)}^{f(y_2)} f(x,y)$    $\int_{y_1}^{y_2}\int_{f(y_1)}^{f(y_2)} f(x,y)$

⚠ **Note:** Double and Triple Integration more or less are the same process.

| Area by Double Int. | Area by Triple Int. |
|---|---|
| $dA = \int_{y_1}^{y_2}\int_{x_1}^{x_2} dx\,dy$ | $dV = \int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} dx\,dy\,dz$ |

## CARTOID:

$r = a(1-\cos\theta)$



## TETRAHEDRON:

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$$

$$V = \frac{abc}{6}$$

$$\int\sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

# UNIT - 2
# VECTOR CALCULUS

## VECTOR CALCULUS

- **Gradient of Scalar function**

$\phi(x, y, z) \rightarrow$ Scalar fun.

$$\text{grad } \phi = \nabla \phi = \hat{i}\frac{d\phi}{dx} + \hat{j}\frac{d\phi}{dy} + \hat{k}\frac{d\phi}{dz}$$

- **Vector Differential Operator (VDO) ($\nabla$)**

$$\nabla = \hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}$$

(partial derivatives)

- **Laplace operator ($\nabla^2$)**

$$\nabla^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2} = \nabla \cdot \nabla$$

- Convergence is basically ($\text{div}\vec{F} < 0$)

- **Divergence of Vector function**

$\vec{F}(x, y, z) \rightarrow$ ~~Scalar~~ Vector fn.

$$\text{div}\vec{F} = \nabla \cdot \vec{F} = \frac{dF_x}{dx} + \frac{dF_y}{dy} + \frac{dF_z}{dz}$$

Also if $\text{div}\vec{F} = 0$ then it is Solenoidal!

- **Curl of Vector function**

$\vec{F}(x, y, z) \rightarrow$ Vector fun.

$$\text{Curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ F_x & F_y & F_z \end{vmatrix}$$

if $\text{Curl}\vec{F} = 0$ then it is Irrotational!

- **Additional Identities**

If Vector is Conservative

$\rightarrow \nabla \times \vec{F} = 0$

$\rightarrow \nabla \times (\nabla \phi) = 0$

$\rightarrow \nabla \cdot (\nabla \times \vec{F}) = 0$

- **Directional Derivation**

$$\frac{\nabla \phi \cdot \vec{a}}{|\vec{a}|}$$

$\vec{a} \rightarrow$ direction vector

$\nabla\phi$: grad$\phi$

- **Line Integrals**

$$\int_C \vec{F} \cdot \vec{dr}$$

Curve parametric: $x = x(t)$, $z = z(t)$, $y = y(t)$

where

$$\vec{dr} = \vec{dx} + \vec{dy} + \vec{dz}$$

1. Substitute $dr$ and find dot product
2. Substitute the parametric form
3. Integrate

- **Unit normal Vector**

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|}$$

$\nabla\phi \rightarrow$ grad$\phi$

- **Greens Theorem**

$$\oint_C P\,dx + Q\,dy = \iint_R \left[\frac{dQ}{dx} - \frac{dP}{dy}\right] dx\,dy$$

- **Angle btwn two Surfaces**

$$\cos\theta = \frac{\nabla \phi_1 \cdot \nabla \phi_2}{|\nabla \phi_1|\,|\nabla \phi_2|}$$

- **Stokes theorem** (Cauchys)

$$\oint_C \vec{F} \cdot \vec{dr} = \iint_S \text{Curl}\vec{F} \cdot \hat{n}\,ds$$

- **Surface Integrals**

$$\iint_S \vec{F} \cdot \hat{n}\,ds = \iint_R \frac{\vec{F} \cdot \hat{n}}{|\vec{n} \cdot \vec{k}|}\,dx\,dy$$

$$\iint_S F \cdot ds = \iint_D F(r(u,v))\,\|\vec{r}_u \times r_v\|\,d$$

- **Gauss Divergence Theorem**

$$\iint_S \vec{F} \cdot \hat{n}\,ds = \iiint_V \nabla \cdot \vec{F}\,dV$$

$\nabla \cdot \vec{F} \rightarrow$ div V

- **Area by Greens Theorem**

$$\frac{1}{2}\oint_C x\,dy - y\,dx$$

# UNIT - 3

# LAPLACE TRANSFORMATION

## LAPLACE TRANSFORMATION

- **Laplace Transformation**

$F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt, \ t > 0$

- **Important Results**

$\rightarrow L[e^{at}] = \frac{1}{s-a}, \ s > a$

$\rightarrow L[e^{-at}] = \frac{1}{s+a}, \ s > -a$

$\rightarrow L[\cos at] = \frac{s}{s^2+a^2}$

$\rightarrow L[\sin at] = \frac{a}{s^2+a^2}$

$\rightarrow L[\cosh at] = \frac{s}{s^2-a^2}$

$\rightarrow L[\sinh at] = \frac{a}{s^2-a^2}$

$\rightarrow L[1] = \frac{1}{s}$

$\rightarrow L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$ or $\frac{n!}{s^{n+1}}$ (gamma function)

$\Gamma(n+1) = n\Gamma(n)$

- **Laplace Transform of Derivatives**

1. $L[f'(t)] = sL[f(t)] - f(0)$
2. $L[f''(t)] = s^2L[f(t)] - sf(0) - f'(0)$
3. $L[f'''(t)] = s^3L[f(t)] - s^2f(0) - sf'(0) - f''(0)$

- **First Shifting Theorem**

If $L[f(t)] = F(s)$

then,

$L[e^{at}f(t)] = F(s-a)$

$L[e^{-at}f(t)] = F(s+a)$

- **Second Shifting Theorem**

if $L[f(t)] = F(s)$

and $G(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$

then,

$L[G(t)] = e^{-as}F(s)$

(or)

$L[f(t-a)] = e^{-as}F(s)$

- **For Piecewise Functions**

L of Piecewise func.

$f(T+t) = f(t)$

Period $T > 0$

Eg: $f(t+2\pi) = \sin t = f(t)$

$L[f(t)] = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\, dt$

- **Change of Scale of Property**

If $L[f(t)] = F(s)$

then,

$L[f(at)] = \frac{1}{a} F\left[\frac{s}{a}\right]$

- **Initial Value Theorem**

If $L[f(t)] = F(s)$

$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$

- **Final Value Theorem**

$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

- **When t is multiplied to f(t)**

$L[tf(t)] = -\frac{d}{ds}F(s)$

$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$

- **When t is divided to f(t)**

given that $\lim_{t \to 0} \frac{f(t)}{t}$ exists

then

$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\, ds$

- **Inverse Laplace**

$\rightarrow L^{-1}\left[\frac{1}{(s^2-a^2)}\right] = te^{at}$

$\rightarrow L^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n$

- **First Shifting Property:**

$$L^{-1}[F(s+a)] = e^{-at} L^{-1}[F(s)]$$

- **Inverse Laplace of Derivatives and Integrals**

$$L^{-1}[F'(s)] = -t(L^{-1}[F(s)])$$

$$L^{-1}[sF(s)] = \frac{d}{dt} L^{-1}[F(s)]$$

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)] \cdot dt$$

$$L^{-1}[e^{-as}F(s)] = L^{-1}[F(s)]_{t \to t-a} \; H(t-a)$$

$$= f(t-a)\,H(t-a)$$

- **Method of Partial Fractions:**

1. $$\frac{f_n}{s(s+a)} = \frac{A}{s} + \frac{B}{s+a}$$

2. $$\frac{f_n}{s(s^2+2as+a^2)} = \frac{A}{s} + \frac{Bs+C}{(s^2+2as+a^2)}$$

3. $$\frac{f_n}{(s+a)^3} = \frac{A}{(s+a)} + \frac{B}{(s+a)^2} + \frac{C}{(s+a)^3}$$

4. $$\frac{f_n}{(s^2+a^2)^2(s^2+b^2)} = \frac{As+b}{(s^2+a^2)} + \frac{Cs+D}{(s^2+a^2)^2} + \frac{Es+F}{(s^2+b^2)}$$

- **Convolution:**

Convolution of $f(t)$ and $g(t)$ is defined as

$$\int_0^t f(u)\,g(t-u) \cdot du$$

$$f(t) * g(t) = \int_0^t f(u)\,g(t-u) \cdot du$$

**Convolution Theorem:**

→ $L[f(t) * g(t)]$
$= L[f(t)] \cdot L[g(t)]$

→ $f(t) * g(t)$
$= L^{-1}[F(s)] \cdot L^{-1}[G(s)]$

→ $L^{-1}[F(s)] * L^{-1}[G(s)]$
$= L^{-1}[F(s)] \cdot L^{-1}[G(s)]$

$$L^{-1}[1] = \delta(t)$$