Homogeneous linear equation with constant co-efficients

A homogeneous linear partial differential equation of n^{th} order with constant co-efficients is of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \dots + a_n \frac{\partial^n z}{\partial y^n} = F(x,y)$$

put $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$

$$\left(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \dots + a_n D'^n\right) z = F(x,y)$$

(or)

$$f(D,D') z = F(x,y) \quad \text{--- (1)}$$

The complete solution of (1) contains two parts (i) Complementary function and (ii) Particular integral.

Complementary function:

$$f(D,D') z = F(x,y) \quad \text{--- (2)}$$

Now the complementary function of (2) is the solution of

$$f(D,D') = 0 \quad \text{--- (3)}$$

If we factorise $f(D,D')$ into linear factors

$$\left[(D - m_1 D')(D - m_2 D') \dots (D - m_n D')\right] z = 0$$

where $m_1, m_2, \dots m_n$ are roots of

$$f(m,1) = 0 \quad \left[\text{Replace } D = m,\ D' = 1\right]$$

Case (i) If $m_1 \neq m_2 \neq m_3 \dots \neq m_n$ then

$$C.F = \phi_1(y+m_1x) + \phi_2(y+m_2x) + \dots + \phi_n(y+m_nx)$$

Case (ii) If $m_1 = m_2,\ m_3 \neq m_4 \neq \dots \neq m_n$ then

$$CF = \phi_1(y+m_1x) + x\phi_2(y+m_1x) + \phi_3(y+m_3x) + \dots \phi_n(y+m_nx)$$

Case (iii) If $m_1 = m_2 = m_3 = \dots = m_r$ then

$$CF = \phi_1(y+m_1x) + x\phi_2(y+m_1x) + x^2\phi_3(y+m_1x) + \dots + x^{r-1}\phi_r(y+m_rx) + \dots$$

Particular Integral: $f(D, D')z = F(x,y)$, then

$$PI = \frac{F(x,y)}{f(D,D')}$$

$\boxed{\text{Type I}}$ $F(x,y) = e^{ax+by}$

$$PI = \frac{e^{ax+by}}{f(D,D')}$$

Replace $D = $ co.eff $x = a$

$D' = $ co.eff $y = b$

then $PI = \dfrac{e^{ax+by}}{f(a,b)}$ Provided $f(a,b) \neq 0$.

If $f(a,b) = 0$ then

$$PI = \frac{x\,e^{ax+by}}{f'(D,D')}$$

[Differentiate denominator partially with respect to D]

Replace $D = a$

$D' = b$

then

$PI = \dfrac{x\,e^{ax+by}}{f'(a,b)}$ Provided $f'(a,b) \neq 0$.

If $f'(a,b)=0$

then $PI = \dfrac{x^2 e^{ax+by}}{f''(D,D')}$ Replace $D=a$, $D'=b$

$$PI = \frac{x^2 e^{ax+by}}{f''(a,b)} \quad \text{Provided } f''(a,b)\neq 0.$$

Continue the above process. till get the correct answer.

Type II $F(x,y) = \cos(ax+by)$ or $\sin(ax+by)$

$$PI = \frac{\cos(ax+by)}{f(D,D')} \text{ (or) } \frac{\sin(ax+by)}{f(D,D')}$$

$$= R.P \frac{e^{i(ax+by)}}{f(D,D')} \text{ (or) } IP \frac{e^{i(ax+by)}}{f(D,D')}$$

$\boxed{\begin{array}{l} e^{i\theta} = \cos\theta + i\sin\theta \\ \cos\theta = R.P\, e^{i\theta} \\ \sin\theta = I.P\, e^{i\theta} \end{array}}$

now same as type I.

$$= RP \frac{e^{iax+iby}}{f(ia,ib)} \text{ (or) } IP \frac{e^{iax+iby}}{f(ia,ib)}$$

finally collect Real part	finally collect imaginary part

another method

$$PI = \frac{\cos(ax+by)}{f(D,D')} \text{ (or) } \frac{\sin(ax+by)}{f(D,D')}$$

Replace $D^2 = -a^2$, $D'^2 = -b^2$, $DD' = -(ab)$

Type - III

$$F(x,y) = x^m y^n.$$

$$PI = \frac{x^m y^n}{f(D, D')}$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \dots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$= \frac{x^m y^m}{D \text{ (or) } D' \left[1 \pm \phi(D \cdot D')\right]}$$

function of D or D'

$$= \frac{1}{D \text{ or } D'} \left[1 \pm \phi(D, D')\right]^{-1} (x^m y^n)$$

$$\boxed{\begin{aligned} \frac{1}{D^2} &= \iint dx\,dx \\ \frac{1}{D'^2} &= \iint dy\,dy \end{aligned}}$$

expand using $(1 \pm x)^{-1}$ then multiply $x^m y^n$ inside and simplification we get the answer

$$\boxed{D = \frac{\partial}{\partial x}, \quad D' = \frac{\partial}{\partial y}}$$

1. Solve $(D^3 - 3D^2D' + 2DD'^2)z = 0$

$D = m$
$D' = 1$

$m^3 - 3m^2 + 2m = 0$

$m(m^2 - 3m + 2) = 0$

$m(m-1)(m-2) = 0$

$m = 0, 1, 2.$

$\therefore \; CF = \phi_1(y) + \phi_2(y+x) + \phi_3(y+2x)$

$PI = 0 \; [R.H.S = 0]$

$\therefore \; z = CF + PI.$

$$\boxed{z = \phi_1(y) + \phi_2(y+x) + \phi_3(y+2x)}$$

2. $(D^3 + DD'^2 - D^2D' - D'^3)z = 0$

$D = m, \; D' = 1$

A. equation: $m^3 + m - m^2 - 1 = 0$

$m = 1$ is one root.

co-effient

m^3	m^2	m	cons
1	-1	1	-1
0	1	0	1
1	0	1	0

one root

$\uparrow m^2 \quad \uparrow m \quad \uparrow$ cons

$m^2 + 1 = 0$

$m^2 = -1$

$m = \pm i$

$m = 1, -i, +i$

$CF = \phi_1(y+x) + \phi_2(y-ix) + \phi_3(y+ix)$

$PI = 0.$

$$\boxed{z = CF + PI = \phi_1(y+x) + \phi_2(y-ix) + \phi_3(y+ix)}$$

③ Solve $(D^2 - 4DD' + 4D'^2) z = e^{2x+y}$.

$D = m \quad D' = 1$

$m^2 - 4m + 4 = 0$

$(m-2)(m-2) = 0$

$m = 2, 2$ (repeat roots)

~~$CF = \phi_1(y+2x) + \phi_2(y+2x)$~~

$CF = \phi_1(y+2x) + x\phi_2(y+2x)$

$PI = \dfrac{e^{2x+y}}{D^2 - 4DD' + 4D'^2}$

Replace.

$D = $ co. eff. $x = 2$

$D' = $ co. eff $y = 1$

$= \dfrac{e^{2x+y}}{4 - 8 + 4} = \dfrac{e^{2x+y}}{0}$ X. not possible

$PI = \dfrac{x e^{2x+y}}{2D - 4D'}$ [denomitor P.D. w.r to D]

$= \dfrac{x e^{2x+y}}{4 - 4}$ X. $\quad D = 2, \; D' = 1$

$PI = \dfrac{x^2 e^{2x+y}}{2}$ (again denominator P.D w.r to D)

$\therefore z = CF + PI$.

$$\boxed{z = \phi_1(y+2x) + x\,\phi_2(y+2x) + \frac{x^2 e^{2x+y}}{2}}$$

(4) (7)

Solve $(D^3 - 2D^2D')z = \sin(x+2y) + 3x^2y$

L.HS Replace $D = m$. $D' = 1$

$$m^3 - 2m^2 = 0$$
$$m^2(m-2) = 0$$
$$m = 0, 0, 2.$$

$$CF = \phi_1(y) + x\phi_2(y) + \phi_3(y+2x)$$

$$PI_1 = \frac{\sin(x+2y)}{D^3 - 2D^2D'}$$

$e^{i\theta} = \cos\theta + i\sin\theta$
$IP\, e^{i\theta} = \sin\theta$

$$= IP\, \frac{e^{ix+2iy}}{D^3 - 2D^2D'}$$

Replace $D = $ co-eff $x = i$
$D' = $ co-eff $y = 2i$

$$= IP\, \frac{e^{i(x+2y)}}{-i - 2(-1)2i}$$

$$= IP\, \frac{e^{i(x+2y)}}{3i} \times \frac{i}{i}$$

$$= IP\, \frac{\{\cos(x+2y) + i\sin(x+2y)\} \times i}{-3}$$

collecting Imaginary parts

$$\boxed{PI_1 = -\frac{\cos(x+2y)}{3}}$$

$$PI_2 = \frac{3x^2y}{D^3 - 2D^2D'}$$

$$= \frac{3x^2y}{D^3\left[1 - \frac{2D'}{D}\right]}$$

$$(1-x)^{-1} = 1+x+x^2+x^3+\dots$$

$$= \frac{1}{D^3}\left[1 - \frac{2D'}{D}\right]^{-1}(3x^2y)$$

$$= \frac{1}{D^3}\left[1 + \frac{2D'}{D} + \frac{4D'^2}{D^2} + \frac{8D'^3}{D^3} + \dots\right](3x^2y)$$

$$= \frac{1}{D^3}\left[3x^2y + \frac{2}{D}D'(3x^2y) + \frac{4}{D^2}D'^2(3x^2y) + \dots\right]$$

$$\boxed{D' = \frac{\partial}{\partial y}}$$

$$D'(3x^2y) = 3x^2$$

$$(D')^2(3x^2y) = D'(3x^2) = 0$$

$\therefore$ (all higher order derivative vanish) like D'''

$$= \frac{1}{D^3}\left[3x^2y + \frac{2}{D}3x^2 + 0 + 0\right]$$

$$\frac{1}{D^3} = \iiint dx\,dx\,dx$$

$$= \frac{\cancel{3}y\,x^5}{\cancel{3}\cdot 4\cdot 5} + \frac{6}{D^4}(x^2)$$

$$\frac{1}{D^4} = \iiiint dx\,dx\,dx\,dx$$

$$= \frac{x^5y}{20} + 6\frac{x^6}{3\cdot 4\cdot 5\cdot 6}$$

$$PI_2 = \frac{x^5y}{20} + \frac{x^6}{60}$$

$$z = CF + PI_1 + PI_2.$$

$$\boxed{z = \phi_1(y) + x\phi_2(y) + \phi_3(y+2x) - \frac{1}{3}\cos(x+2y) + \frac{x^5y}{20} + \frac{x^6}{60}}$$

(5) $(D^2 - 2DD' + D'^2)z = \cos(x-3y)$

$D = m, \; D' = 1$ on (LHS)

A equation $m^2 - 2m + 1 = 0$

$m = 1, 1$. (Repeat roots)

$CF = \phi_1(y+x) + x\phi_2(y+x)$

$PI_1 = \dfrac{\cos(x-3y)}{D^2 - 2DD' + D'^2}$ (method - I)

$= RP \dfrac{e^{ix-3iy}}{D^2 - 2DD' + D'^2}$ $\quad D = i$, $D' = -3i$

$= RP \dfrac{e^{i(x-3y)}}{-1 - 2(3) + (-9)}$

$= RP \dfrac{[\cos(x-3y) + i\sin(x-3y)]}{-16}$

$$\boxed{PI = \frac{\cos(x-3y)}{-16}}$$

OR

$PI = \dfrac{\cos(x-3y)}{D^2 - 2DD' + D'^2}$ $\quad$ Replace $D^2 = -a^2 = -1$

$D'^2 = -b^2 = -9$

$DD' = -ab = 3$

$= \dfrac{\cos(x-3y)}{-1 - 2(3) - 9}$

$= \dfrac{-1}{16}\cos(x-3y)$

$$\boxed{z = CF + PI = \phi_1(y+x) + x\phi_2(y+x) - \frac{1}{16}\cos(x-3y)}$$

(6) solve $(D^2 - DD')z = \sin x \sin 2y$.

$D = m$
$D' = 1$

$m^2 - m = 0$

$m(m-1) = 0$

$m = 0, 1$

$CF = \phi_1(y) + \phi_2(y+x)$.

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$\frac{\cos(A-B) - \cos(A+B)}{2} = \sin A\sin B$

$$PI = \frac{\sin x \sin 2y}{D^2 - DD'} = \frac{\cos(x-2y)}{2(D^2-DD')} - \frac{1}{2}\frac{\cos(x+2y)}{D^2-DD'}$$

$$= RP\frac{e^{ix-2iy}}{2(D^2-DD')} - \frac{1}{2}R.P\frac{e^{ix+2iy}}{D^2-DP'}$$

$D = i$
$D' = 2i$

$D = i$
$D' = -2i$

$$= RP\frac{e^{i(x-2y)}}{2(-1-2)} - \frac{1}{2}RP\frac{e^{i(x+2y)}}{-1+2}$$

$$= \frac{\cos(x-2y)}{-6} - \frac{1}{2}\frac{\cos(x+2y)}{1}$$

$$z = \phi_1(y) + \phi_2(y+x) - \frac{\cos(x-2y)}{6} - \frac{\cos(x+2y)}{2} \; //$$

(7) Solve $(D^2 + 2DD' + D'^2)z = \sinh(x+y)$

$D = m$ / $D' = 1$

$m^2 + 2m + 1 = 0$

$m = -1, -1$ (roots repeat)

$CF = \phi_1(y-x) + x\,\phi_2[y-x)$

$$\boxed{\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}}$$

$$PI = \frac{\sinh(x+y)}{D^2+2DD'+D'^2}$$

$$= \frac{e^{x+y} - e^{-x-y}}{2(D^2+2DD'+D'^2)}$$

$$= \frac{1}{2}\left[\frac{e^{x+y}}{1+2+1} - \frac{e^{-x-y}}{1+2+1}\right]$$

$$= \frac{1}{2}\left[\frac{e^{x+y}}{4} - \frac{e^{-x-y}}{4}\right]$$

$$= \frac{1}{4}\sinh(x+y)$$

$\therefore\ z = CF + PI = \phi_1(y-x) + x\,\phi_2(y-x) + \frac{1}{4}\sinh(x+y).$

Practice problems:

① solve $(D^3 - 7DD'^2 - 6D'^3)\,z = x^2y + \sin(x+2y)$

Ans. $\phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x) + \frac{x^5y}{60} - \frac{1}{75}\cos(x+2y)$

② solve $(D^2 - 2DD')\,z = x^3y + e^{2x}$.

Ans. $\phi_1(y) + \phi_2(y+2x) + \frac{e^{2x}}{4} + \frac{x^5y}{20} + \frac{x^6}{60}$

③ solve $(D^2 - 4D'^2)\,z = \cos 2x \cos 3y$

④ solve $(D^2 + 4DD' - 5D'^2)\,z = x + y^2 + \pi$.

$z = \phi_1(x+y) + \phi_2(y-5x) + \frac{x^3}{6} + \frac{x^2}{2}(y^2+\pi) - \frac{4}{3}x^3y + \frac{7}{4}x^4$

If $F(x,y) = e^{ax+by} f(x,y)$ where $f(x,y) = \cos ax$ (or) $\sin ax$ (or) $x^m y^n$.

then $PI = \dfrac{e^{ax+by} f(x,y)}{\phi(D, D')}$

$PI = e^{ax+by} \left[\dfrac{f(x,y)}{\phi(D+a, D'+b)} \right]$ $D \to D +$ co.eff 'x'
$D' \to D' +$ co.eff 'y'

then as usual earlier method we can solve.

① Solve $(D^3 + D^2D' - DD'^2 - D'^3) z = e^x \cos 2y$

$D = m$. $D' = 1$

$m^3 + m^2 - m - 1 = 0$

$(m+1)^2 (m-1) = 0$

$m = 1, -1, -1$

$C.F = \phi_1(y+x) + \phi_2(y-x) + x\,\phi_3(y-x)$

$PI = \dfrac{e^x \cos 2y}{D^3 + D^2D' - DD'^2 - D'^3}$ $D \to D+1$ (co.eff x)
$D' \to D' + 0$ (no co eff y)

$= e^x \left[\dfrac{\cos 2y}{(D+1)^3 + (D+1)^2 D' - (D+1) D'^2 - D'^3} \right]$

$= e^x \left[\dfrac{RP\ e^{i2y}}{1 + 2i + 4 + 8i} \right]$ $D \Rightarrow 0$ (because co effi of x is 0)
$D' = 2i$

$= e^x \left[\dfrac{RP\ (\cos 2y + i \sin 2y)}{5 + 10i} \dfrac{(5-10i)}{(5-10i)} \right]$

$$= e^{x}\left[\frac{5\cos 2y + 10\sin 2y}{(5)^2 - (10i)^2}\right]$$

$$= \frac{5e^{x}[\cos 2y + 2\sin 2y]}{25+100} = \frac{e^{x}[\cos 2y + 2\sin 2y]}{25}.$$

$\therefore z = PI + CF$

$$z = \phi_1(y+x) + \phi_2(y-x) + x\phi_3(y-x) + \frac{e^{x}}{25}[\cos 2y + 2\sin 2y]$$

② solve $(D^2 + DD' - 6D'^2)z = y\cos x.$

$D = m \quad D' = 1$

$m^2 + m - 6 = 0$

$(m-2)(m+3) = 0$

$m = +2, -3$

$CF = \phi_1(y+2x) + \phi_2(y-3x)$

$$PI = \frac{y\cos x}{D^2 + DD' - 6D'^2}$$

$$= \frac{1}{(D-2D')(D+3D')} y\cos x$$

(factorise)

$$= \frac{1}{D-2D'}\left[\int (a+3x)\cos x\,dx\right] \quad \text{where } y = a+3x$$

$$= \frac{1}{D-2D'}\left[(a+3x)\sin x - 3\int \sin x\,dx\right]$$

$$= \frac{1}{D-2D'}\left[y\sin x + 3\cos x\right]$$

$$= \int \left[(a-2x)\sin x + 3\cos x\right] dx \quad \text{where } y = a - 2x$$

$$= \left[(a-2x)(-\cos x) - (-2)(-\sin x) + 3\sin x\right]$$

$$PI = -y\cos x + \sin x$$

$$\therefore \boxed{z = \phi_1(y+2x) + \phi_2(y-3x) + \sin x - y\cos x.}$$

x