1.4.3 Quantization of Continuous-Amplitude Signals

As we have seen, a digital signal is a sequence of numbers (samples) in which each number is represented by a finite number of digits (finite precision).

The process of converting a discrete-time continuous-amplitude signal into a digital signal by expressing each sample value as a finite (instead of an infinite) number of digits is called *quantization*. The error introduced in representing the continuous-valued signal by a finite set of discrete value levels is called *quantization error* or *quantization noise*.

We denote the quantizer operation on the samples $x(n)$ as $Q[x(n)]$ and let $x_q(n)$ denote the sequence of quantized samples at the output of the quantizer. Hence

$$x_q(n) = Q[x(n)]$$

Then the quantization error is a sequence $e_q(n)$ defined as the difference between the quantized value and the actual sample value. Thus

$$e_q(n) = x_q(n) - x(n) \tag{1.4.25}$$

We illustrate the quantization process with an example. Let us consider the discrete-time signal

$$x(n) = \begin{cases} 0.9^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

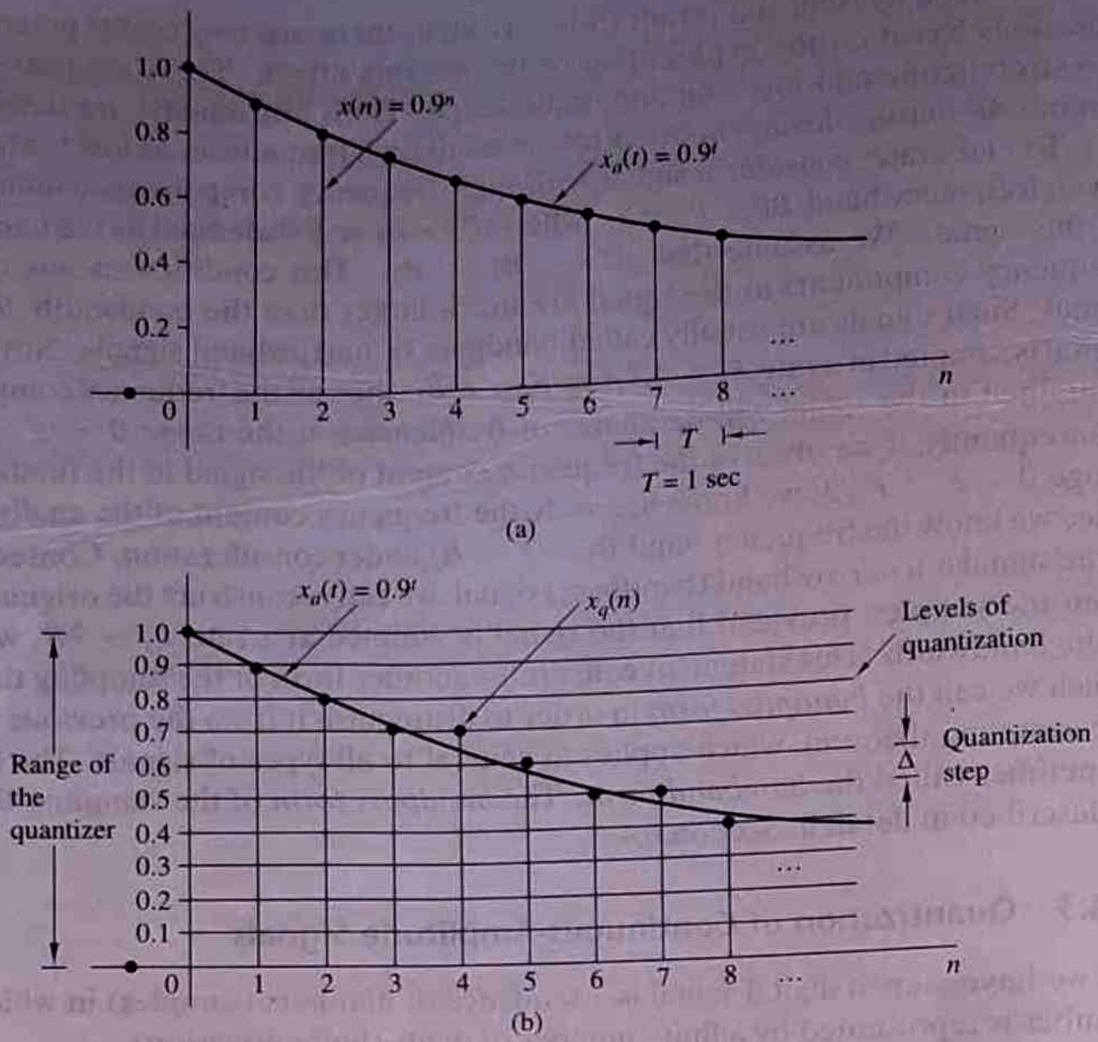

Figure 1.4.7 Illustration of quantization.

obtained by sampling the analog exponential signal $x_a(t) = 0.9^t$, $t \geq 0$ with a sampling frequency $F_s = 1$ Hz (see Fig. 1.4.7(a)). Observation of Table 1.2, which shows the values of the first 10 samples of $x(n)$, reveals that the description of the sample value $x(n)$ requires n significant digits. It is obvious that this signal cannot be processed by using a calculator or a digital computer since only the first few samples can be stored and manipulated. For example, most calculators process numbers with only eight significant digits.

However, let us assume that we want to use only one significant digit. To eliminate the excess digits, we can either simply discard them (*truncation*) or discard them by rounding the resulting number (*rounding*). The resulting quantized signals $x_q(n)$ are shown in Table 1.2. We discuss only quantization by rounding, although it is just as easy to treat truncation. The rounding process is graphically illustrated in Fig. 1.4.7(b). The values allowed in the digital signal are called the *quantization levels*, whereas the distance Δ between two successive quantization levels is called the *quantization step size* or *resolution*. The rounding quantizer assigns each sample of $x(n)$ to the nearest quantization level. In contrast, a quantizer that performs truncation would have assigned each sample of $x(n)$ to the quantization level below

TABLE 1.2 Numerical Illustration of Quantization with One Significant Digit Using Truncation or Rounding

n	$x(n)$ Discrete-time signal	$x_q(n)$ (Truncation)	$x_q(n)$ (Rounding)	$e_q(n) = x_q(n) - x(n)$ (Rounding)
0	1	1.0	1.0	0.0
1	0.9	0.9	0.9	0.0
2	0.81	0.8	0.8	−0.01
3	0.729	0.7	0.7	−0.029
4	0.6561	0.6	0.7	0.0439
5	0.59049	0.5	0.6	0.00951
6	0.531441	0.5	0.5	−0.031441
7	0.4782969	0.4	0.5	0.0217031
8	0.43046721	0.4	0.4	−0.03046721
9	0.387420489	0.3	0.4	0.012579511

it. The quantization error $e_q(n)$ in rounding is limited to the range of $-\Delta/2$ to $\Delta/2$, that is,

$$-\frac{\Delta}{2} \leq e_q(n) \leq \frac{\Delta}{2} \tag{1.4.26}$$

In other words, the instantaneous quantization error cannot exceed half of the quantization step (see Table 1.2).

If $x_{\min}$ and $x_{\max}$ represent the minimum and maximum values of $x(n)$ and L is the number of quantization levels, then

$$\Delta = \frac{x_{\max} - x_{\min}}{L - 1} \tag{1.4.27}$$

We define the *dynamic range* of the signal as $x_{\max} - x_{\min}$. In our example we have $x_{\max} = 1$, $x_{\min} = 0$, and $L = 11$, which leads to $\Delta = 0.1$. Note that if the dynamic range is fixed, increasing the number of quantization levels L results in a decrease of the quantization step size. Thus the quantization error decreases and the accuracy of the quantizer increases. In practice we can reduce the quantization error to an insignificant amount by choosing a sufficient number of quantization levels.

Theoretically, quantization of analog signals always results in a loss of information. This is a result of the ambiguity introduced by quantization. Indeed, quantization is an irreversible or noninvertible process (i.e., a many-to-one mapping) since all samples in a distance $\Delta/2$ about a certain quantization level are assigned the same value. This ambiguity makes the exact quantitative analysis of quantization extremely difficult. This subject is discussed further in Chapter 6, where we use statistical analysis.

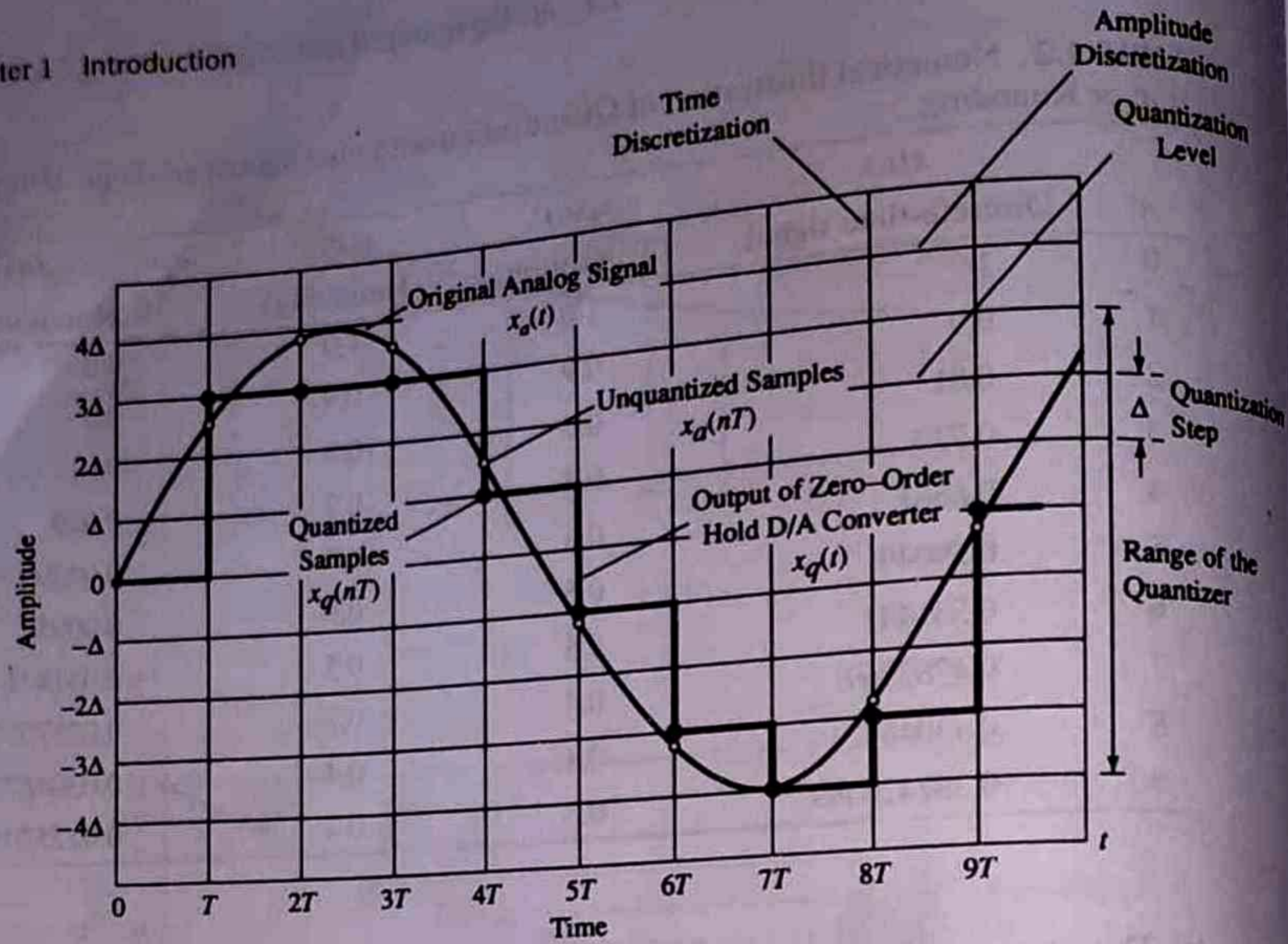

Figure 1.4.8 Sampling and quantization of a sinusoidal signal.