3.4 The Discrete Fourier Transform

The DFT of a finite duration sequence $x(n)$ is obtained by sampling the Fourier transform $X(e^{j\omega})$ at N equally spaced points over the interval $0 \le \omega \le 2\pi$ with a spacing of $\frac{2\pi}{N}$. The DFT, denoted by $X(k)$ is defined as

$$X(k) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} \qquad 0 \le k \le N-1 \qquad \text{... (3.14)}$$

The Fourier transform $X(e^{j\omega})$ is periodic in ω, with period 2π and it's inverse Fourier transform is equal to discrete-time sequence $x(n)$. In section (1.20) it was shown that when a continuous-time signal is sampled with sampling time T, the spectrum of the resulting discrete-time sequence becomes a periodic function of frequency with period $\frac{2\pi}{T}$. Similarly, when $X(e^{j\omega})$ is sampled with sampling period $\frac{2\pi}{N}$, the corresponding discrete-time sequence $x_p(n)$ becomes periodic in time with period N (Fig. 3.1) where

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN) \quad \text{... (3.15)}$$

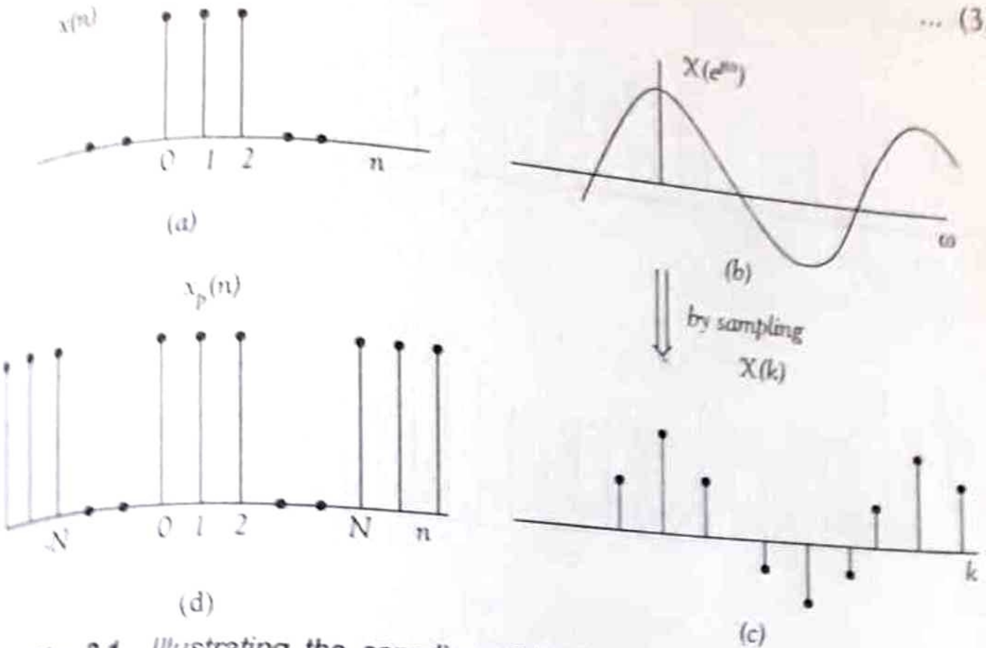

Fig. 3.1 Illustrating the sampling of Fourier transform of a sequence (a) Finite duration sequence, (b) Fourier transform of the sequence, (c) Sampled version of Fourier transform, (d) Periodic sequence

Thus the periodic sequence $x_p(n)$, corresponding to $X(k)$ for $k = 0$ to $N-1$ obtained by sampling $X(e^{j\omega})$ in the interval 0 to 2π, is formed from $x(n)$ by adding together an infinite number of shifted replicas of $x(n)$. Let us consider an example in which the sequence $x(n)$ is of length $L = 9$ and the value of $N = 10$ illustrated in Fig. 3.2a. When we sample the frequency spectrum of $x(n)$ taking 10 sampling points (i.e., $N > L$) over the interval 2π, we obtain a periodic sequence $x_p(n)$ as shown in Fig. 3.2b in which the delayed replicas of $x(n)$ do not overlap, and the period of the periodic sequence $x_p(n)$ is recognizable as $x(n)$. In Fig. 3.2c, the same sequence $x(n)$ is used but the value of N is equal to 7 (i.e., $N < L$). In this case the replicas of $x(n)$ overlap and one period of $x_p(n)$ is not identical to $x(n)$. This is a form of time-domain aliasing which is due to undersampling of the Fourier transform of $x(n)$. From above discussion we find that the sequence $x(n)$ cannot be recovered from $x_p(n)$ when the number of sampling points N is less than the length of the sequence L. If $N > L$, $x(n)$ can be recovered through the relation

$$x(n) = x_p(n) \qquad 0 \le n \le N-1$$

$$= 0 \qquad \text{otherwise}$$

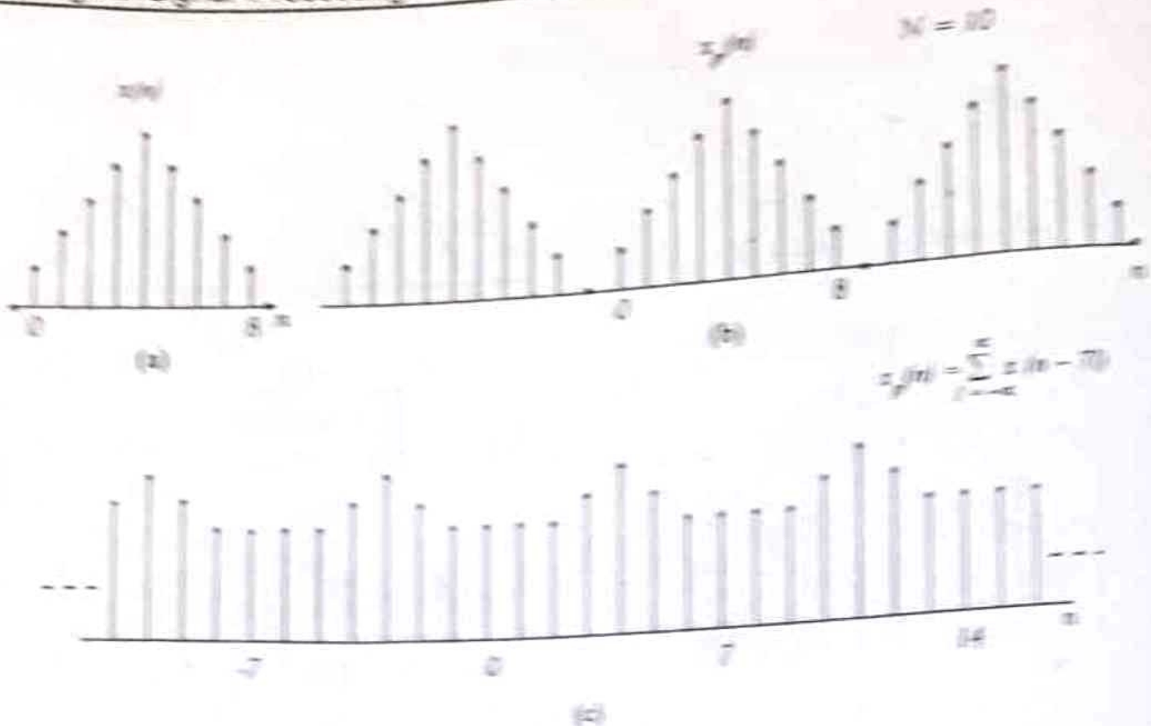

Fig.3.2 (a) Finite duration sequence x(n) of length L = 9
(b) Periodic sequence $x_p(n)$ corresponding to sampling the Fourier transform of x(n) with N = 10
(c) Periodic sequence $x_p(n)$ corresponding to sampling the Fourier transform of x(n) with N = 7

Let $x(n)$ is a causal, finite duration sequence containing L samples. Then it's Fourier transform is given by

$$X(e^{j\omega}) = \sum_{n=0}^{L-1} x(n)\, e^{-j\omega n} \qquad \text{... (3.17)}$$

If we sample $X(e^{j\omega})$ at N equally spaced points over $0 \le \omega \le 2\pi$, we obtain

$$X(k) = X(e^{j\omega})\Big|_{\omega = 2\pi k/N} = \sum_{n=0}^{L-1} x(n) e^{-j2\pi kn/N} \qquad \text{... (3.18)}$$

Since time domain aliasing occurs if $N < L$, to prevent it, we increase the duration of $x(n)$ from L to N samples by appending appropriate number of zeros, which is known as zero padding.

Since zero valued elements contribute nothing to sum the Eq. (3.18) can be written as

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad 0 \le k \le N-1 \qquad \text{... (3.18a)}$$

which is called an N-point DFT.

Since $x_p(n)$ is periodic extension of $x(n)$ with period N, it can be expressed in Fourier series expansion

$$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_p(k)\, e^{j2\pi kn/N} \qquad \text{... (3.19)}$$

We have already seen in section (3.2) that the discrete-Fourier series coefficients $X_p(k)$ of the periodic sequence $x_p(n)$. is itself is a periodic sequence with period N. The DFT $X(k)$ is related to the DFS coefficients $X_p(k)$ by

$$\begin{aligned} X(k) &= X_p(k) && 0 \le k \le N-1 \\ &= 0 && \text{otherwise} \end{aligned} \qquad \text{... (3.20)}$$

Substituting Eq. (3.16) and Eq. (3.20) in the Eq. (3.19) we get

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}$$

which is called as inverse discrete Fourier transform.

The formulas for DFT and IDFT are

DFT

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \qquad 0 \le k \le N-1 \qquad \text{... (3.21)}$$

IDFT

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j2\pi kn/N} \qquad 0 \le n \le N-1 \qquad \text{... (3.22)}$$

For notation purpose discrete Fourier transform and inverse discrete Fourier transform given in Eq. (3.21) and Eq. (3.22) can be represented by

$$X(k) = DFT\left[\, x(n) \,\right] \qquad \text{... (3.23a)}$$

and $$x(n) = IDFT\left[\, X(k) \,\right] \qquad \text{... (3.23b)}$$

Example 3.1: *Find the DFT of a sequence $x(n) = \{1, 1, 0, 0\}$ and find the IDFT of $Y(k) = \{1, 0, 1, 0\}$.*

Solution

Let us assume $N = L = 4$

we have $X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N} \quad k = 0, 1, \ldots N-1$

$$X(0) = \sum_{n=0}^{3} x(n) = x(0) + x(1) + x(2) + x(3)$$

$$= 1 + 1 + 0 + 0 = 2$$

$$X(1) = \sum_{n=0}^{3} x(n)e^{-j\pi n/2} = x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi} + x(3)e^{-j3\pi/2}$$

$$= 1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2}$$

$$= 1 - j$$

$$X(2) = \sum_{n=0}^{3} x(n)e^{-j\pi n} = x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$

$$= 1 + \cos\pi - j\sin\pi$$

$$= 1 - 1 = 0$$

$$X(3) = \sum_{n=0}^{3} x(n)e^{-j3n\pi/2} = x(0) + x(1)e^{-j3\pi/2} + x(2)e^{-j3\pi} + x(3)e^{-j9\pi/2}$$

$$= 1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}$$

$$= 1 + j$$

$$X(k) = \left\{2, 1 - j, 0, 1 + j\right\}$$

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\,e^{j2\pi nk/N} \qquad n = 0, 1 \ldots N-1$$

$$y(0) = \frac{1}{4} \sum_{k=0}^{3} Y(k) \qquad n = 0, 1, 2, 3$$

$$= \frac{1}{4} \left[y(0) + y(1) + y(2) + y(3) \right]$$

$$= \frac{1}{4} [1 + 0 + 1 + 0]$$

$$= 0.5$$

$$y(1) = \frac{1}{4} \sum_{k=0}^{3} Y(k)\, e^{j\pi k/2}$$

$$y(1) = \frac{1}{4} \left[Y(0) + Y(1)e^{j\pi/2} + Y(2)e^{j\pi} + Y(3)e^{j3\pi/2} \right]$$

$$= \frac{1}{4} [1 + 0 + \cos\pi + j\sin\pi + 0]$$

$$= \frac{1}{4} \left[1 + 0 - 1 + 0 \right]$$

$$= 0$$

$$y(2) = \frac{1}{4} \left[Y(0) + Y(1)e^{j\pi} + Y(2)e^{j2\pi} + Y(3)e^{j3\pi} \right]$$

$$= \frac{1}{4} [1 + 0 + \cos 2\pi + j\sin 2\pi + 0]$$

$$= \frac{1}{4} \left[1 + 0 + 1 + 0 \right] = 0.5$$

$$y(3) = \frac{1}{4}\left[Y(0) + Y(1)e^{j3\pi/2} + Y(2)e^{j3\pi} + Y(3)e^{j9\pi/2} \right]$$

$$= \frac{1}{4}\,[1 + 0 + \cos 3\pi + j \sin 3\pi + 0]$$

$$= \frac{1}{4}\left[1 + 0 + (-1) + 0\right]$$

$$= 0$$

$$y(n) = \{0.5, 0, 0.5, 0\}$$

Example 3.2: *Find the DFT of a sequence* $x(n) = 1$ *for* $0 \le n \le 2$
$= 0$ otherwise
for (i) $N = 4$ *(ii)* $N = 8$. *Plot* $|H(k)|$ *and* $\underline{|H(k)}$. *comment on the result.*

Solution:

Given $L = 3$

For $N = 4$, the periodic extension of $x(n)$ shown in Fig. 3.3 can be obtained by adding one zero (i.e., $N - L$ zeros).

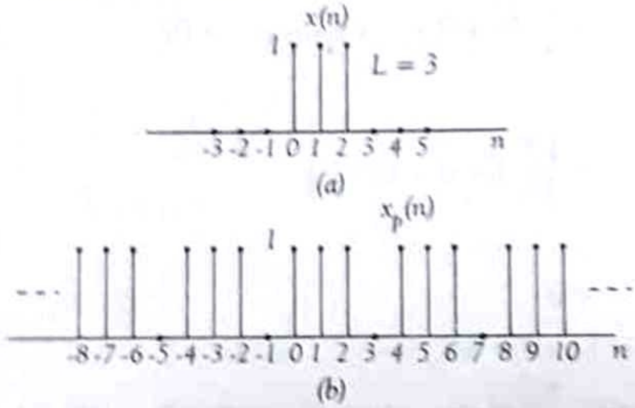

Fig. 3.3 (a) The sequence given in example 3.2
(b) Periodic extension of the sequence for N = 4

We have

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$$

$$k = 0, 1 \ldots N - 1$$

From Fig. 3.3b we find