UNIT -5

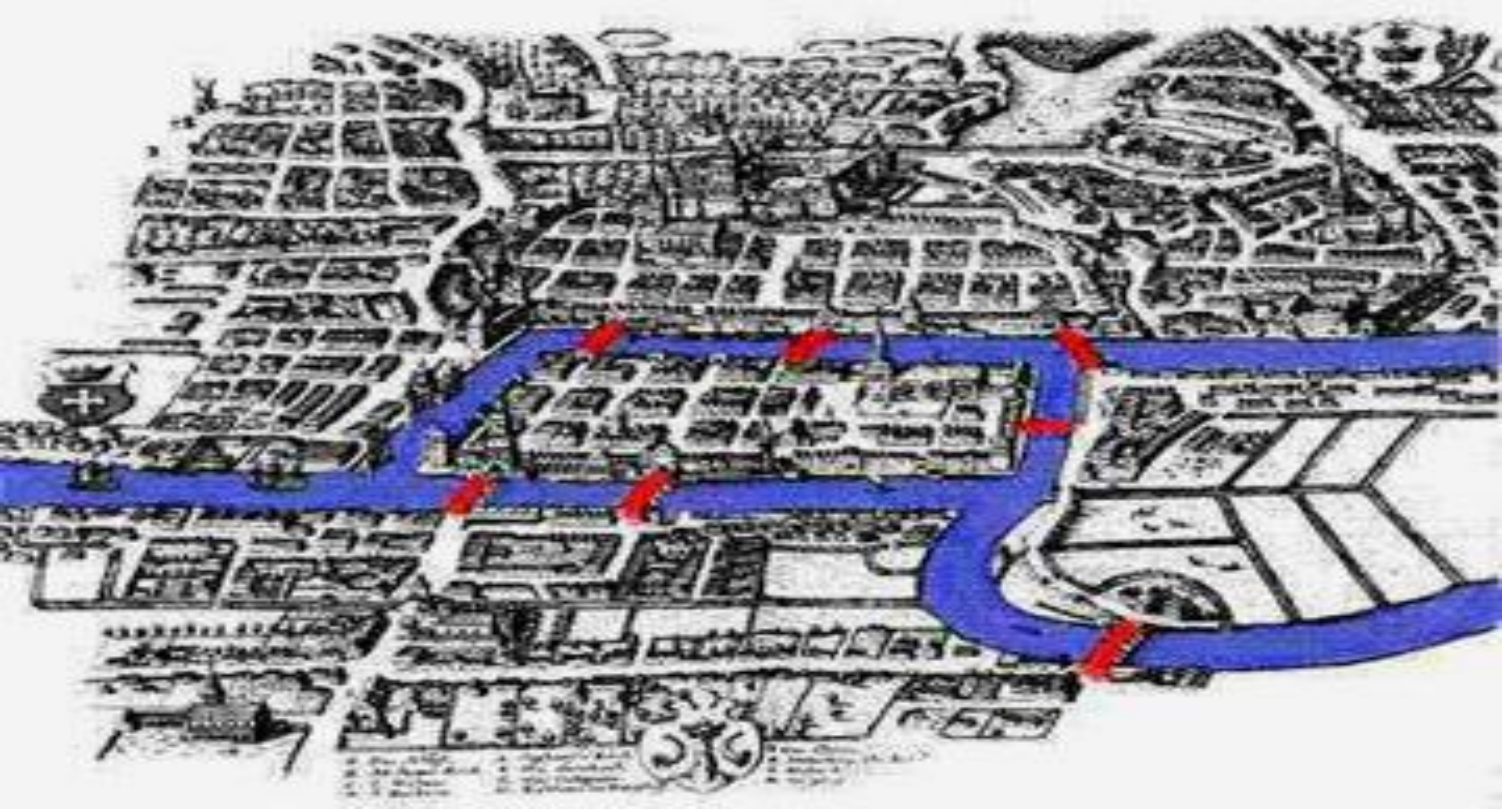
Graph Theory

Birth of Graph Theory

The Konigsberg Bridge Problem

- The city of Konigsberg was located on the Pregel river in Prussia.
- The city occupied 2 islands plus areas on both banks.
- These regions were linked by 7 bridges.
- The citizens wondered whether they could leave home, cross every bridge exactly once, and return home.

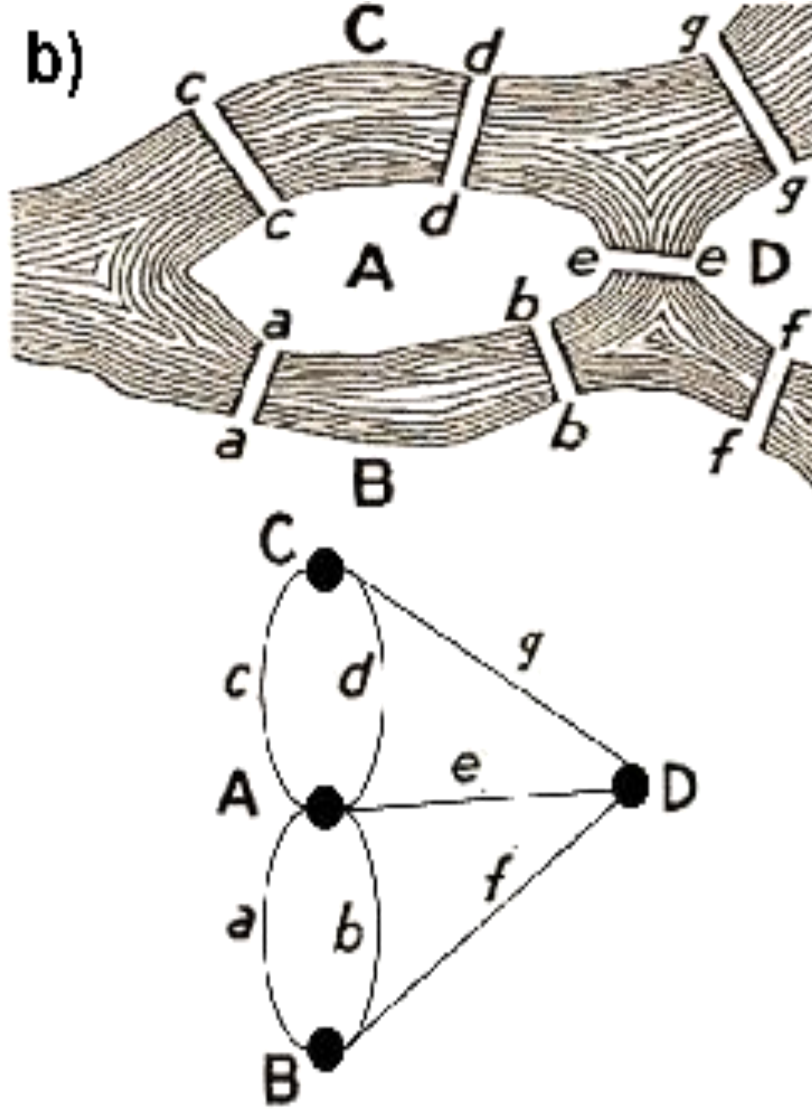
Konigsberg Bridge Problem



a)

Leonhard Euler [1707-1783]

b)

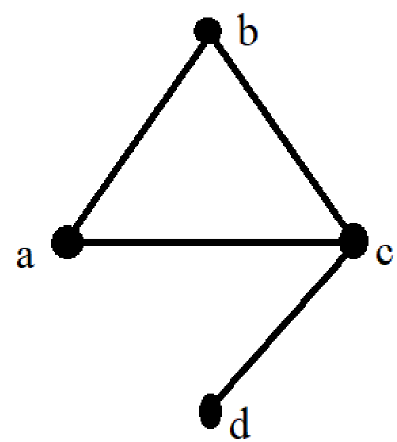


Euler's Graph

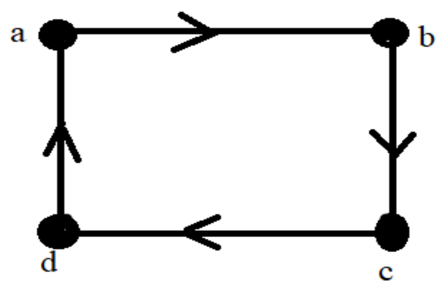
- Euler represented the seven bridges as seven edges and the four places as four vertices.
- He concluded that it is not possible to traverse all the bridges exactly once from start to end of any town which made Euler to introduce the concept of Eulerian path and Eulerian circuit which we will discuss in the later section.
- The following are the definitions and theorems in graph theory.

Graph: A graph G=(V, E) consists of a nonempty set V called the set of vertices (or nodes or points) and a set E of ordered or unordered pairs of elements of V, called edges (or lines) such that there is a mapping from E to V.

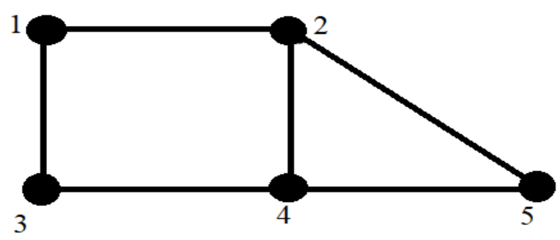
Example: Draw a graph G=(V, E) given by set with V={a,b,c,d} and E={(a, b), (a, c), (b, c), (c, d)}.



Directed Graph and undirected Graph: If is graph G=(V, E), each edge e∈E is associated with an ordered pair of vertices, then G is called a directed graph or digraph.

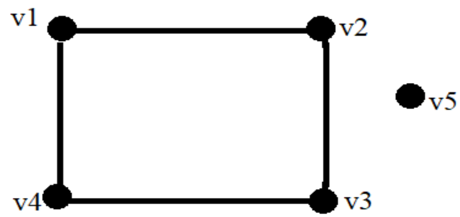


If each edge is associated with an unordered pair of vertices then G is called an undirected graph.



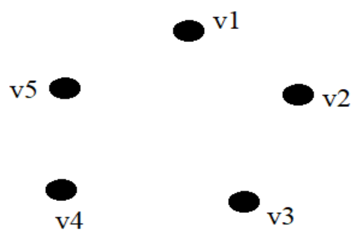
Isolated vertex: A vertex of a graph which is not adjacent to any other vertex.

Example: Let G= (V, E) with V={v_1, v_2, v_3, v_4, v_5} and E={(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1)}. Here v_5 is an isolated vertex.



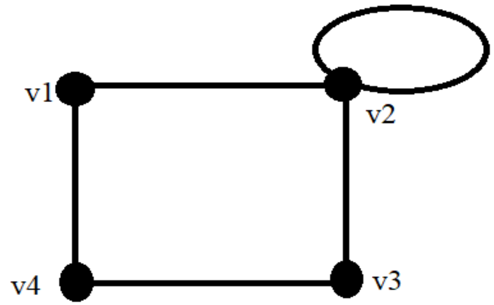
Null graph: A graph containing only isolated vertices is called null graph.

Example:



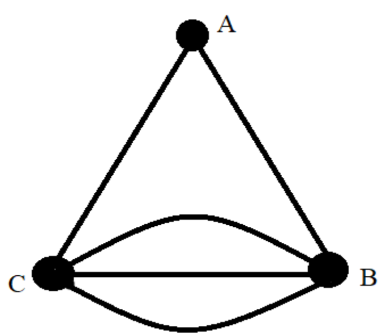
Here v_1, v_2, v_3, v_4 and v_5 are all isolated vertex.

Loop: An edge of a graph that joins a vertex to itself is called a loop.



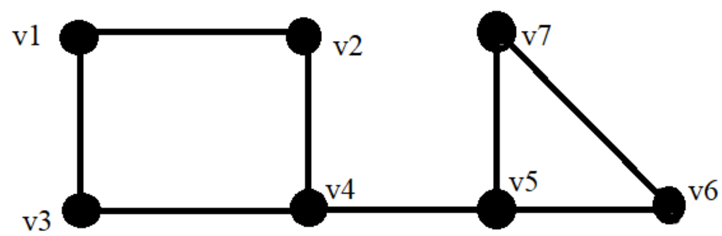
Here v2 is having a loop

Parallel edges: If in a directed graph or undirected graph certain pairs of vertices are joined by more than one edge, such edges are called paralled edge. A graph which contains some parallel edges is called a multigraph.

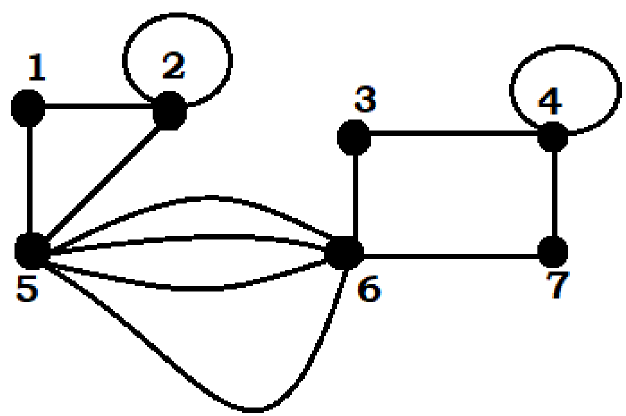


Here B and C are having parallel edges

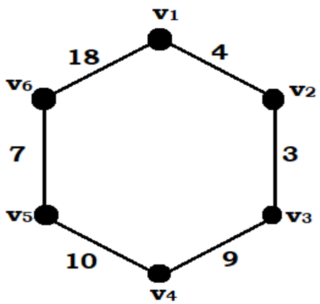
Simple Graph: A graph in which there is only one edge between a pair of vertices.



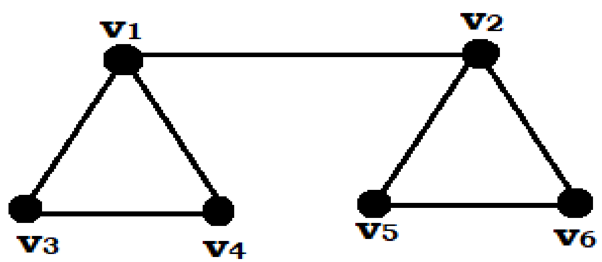
Pseudo graph: A graph in which loops and parallel edges are allowed.



Weighted graph: Graphs in which a number (weight) is assigned to each edge.

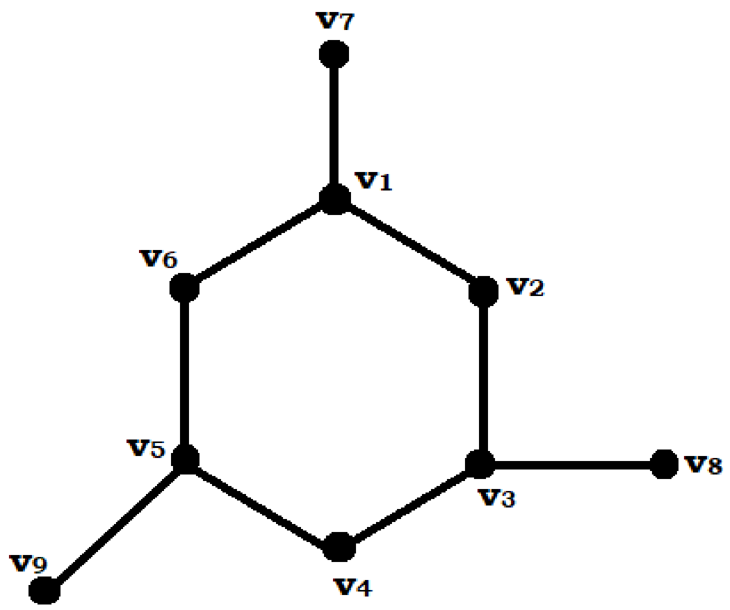


Degree of a Vertex: The degree of a vertex in an undirected graph is the number of edges incident with it, with the exception that a loop at a vertex contributes twice to the degree of that vertex. The degree of a vertex v is denoted by deg (v).Clearly the degree of an isolated vertex is zero.



Here deg (v_1)=3, deg (v_2)=3, deg (v_3)=2, deg (v_4)=2, deg (v_5)=2 and deg (v_6)=2.

Pendant vertex: If the degree of a vertex is one, it is called a pendant vertex.



Here v_7,v_8 and v_9 are called pendant vertices.

Theorem: (The Handshaking Theorem)

If G= (V, E) is an undirected graph with e edges, then $\sum_i \deg(v_i) = 2e$. [The sum of the degrees of all vertices of an undirected graph is twice the number of edges of the graph and hence even].

Proof: Since every edge is incident with exactly two vertices, every edge contributes 2 to the sum of the degree of the vertices.

Therefore all the e edges contribute (2e) to the sum of the degrees of the vertices.

Hence the proof.

Theorem: The number of vertices of odd degree in an undirected graph is even.

Proof:

Let G=(V,E) be the undirected graph. Let V_1 and V_2 be the set of vertices of G of even and odd degrees respectively.

Then by handshaking theorem,

$$2e = \sum_{v_i \in V_1} \deg(v_i) + \sum_{v_j \in V_2} \deg(v_j) \quad - \ (1)$$

Since each $\deg(v_i)$ is even, $\sum_{v_i \in V_1} \deg(v_i)$ is even

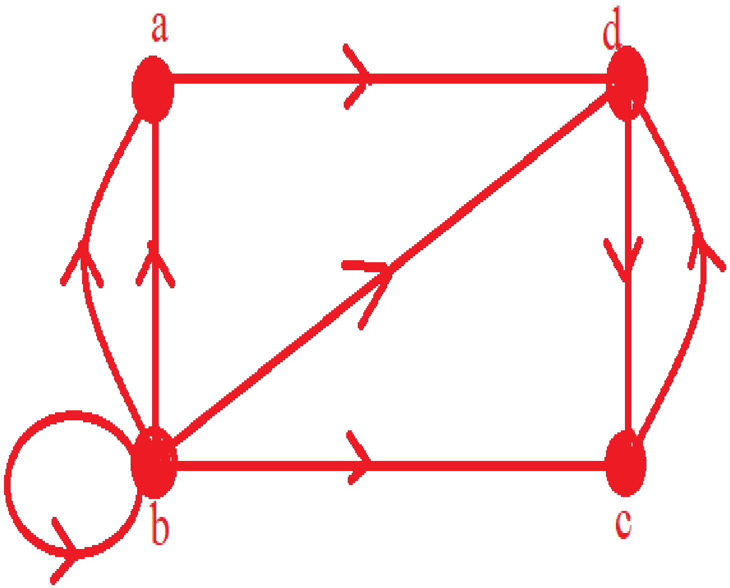
As left hand side of the equation – (1) is even,

we get $\sum_{v_j \in V_2} \deg(v_j)$ is even.

$\Rightarrow$ the number of vertices of odd degree is even.

The out degree and in degree of a vertex:

In a directed graph G, the out degree of a vertex *v* of G, denoted by $deg_G^+(v)$is the number of edges beginning at v and the in degree of *v*, denoted by $deg_G^-(v)$ is the number of edges ending at *v*.



$deg_G^+(a)$=1

$deg_G^+(b)$=5

$deg_G^+(c)$=1

$deg_G^+(d)$=1

$deg_G^-(a)$=2

$deg_G^-(b)$=1

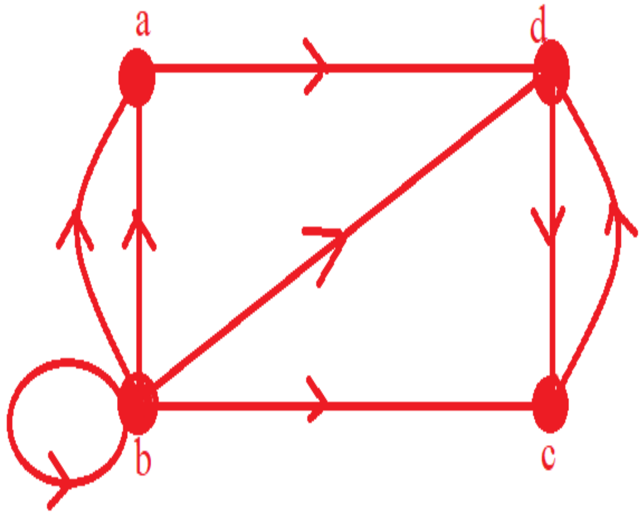
$deg_G^-(c)$=2

$deg_G^-(d)$=3

Theorem :

If G=(V,E) be a directed graph with e edges then $\sum_{v \in V} deg_G^+(v) = \sum_{v \in V} deg_G^-(v)$ =e.

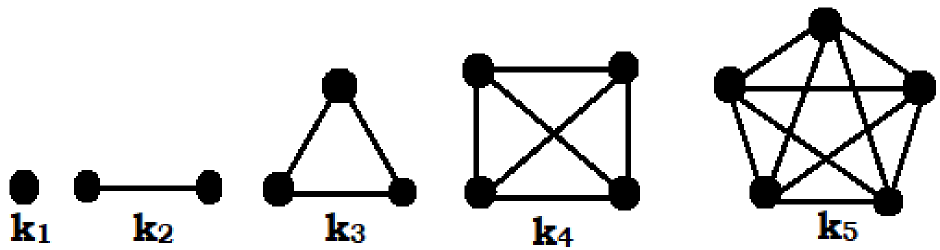
Verify handshaking theorem for the following digraph



Sum of the in-degrees + sum of the out-degrees= 2e
Sum of the in-degrees=8
Sum of the out-degrees=8
The number of edges (e) =8

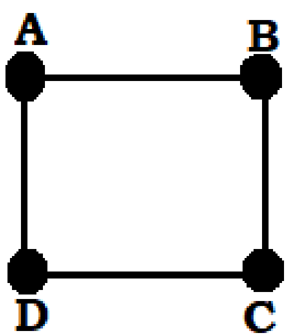
5.1.2 Special Simple Graphs:

Complete Graph: A simple graph, in which there is exactly one edge between each pair of distinct vertices, is called a complete graph.The complete graph on n vertices is denoted by K_n.

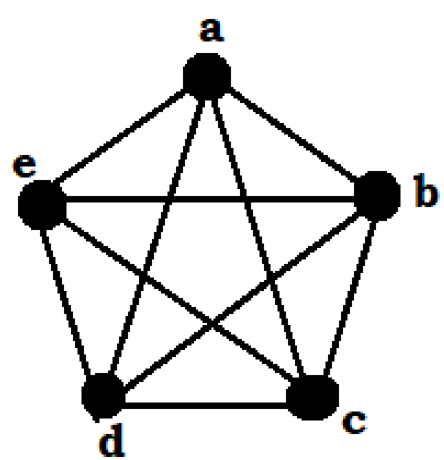


Note: The number of edges in K_n is nC_2 or $\frac{n(n-1)}{2}$. Hence, the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

Regular graph: If every vertex of a simple graph has the same degree, then the graph is called a regular graph. If every vertex in a regular graph has degree n, then the graph is called n-regular.

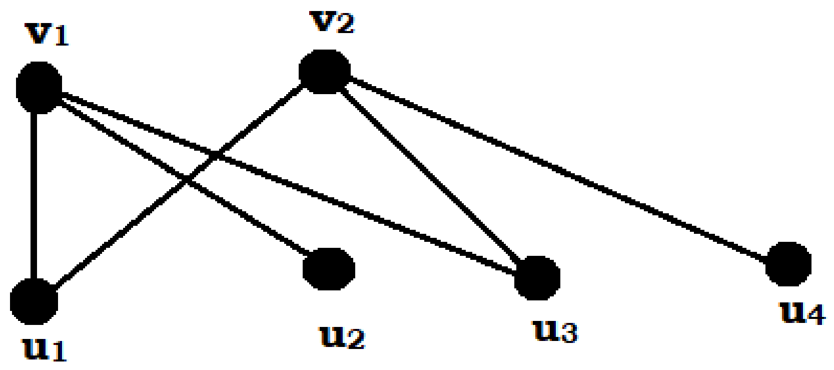


2 regular graph

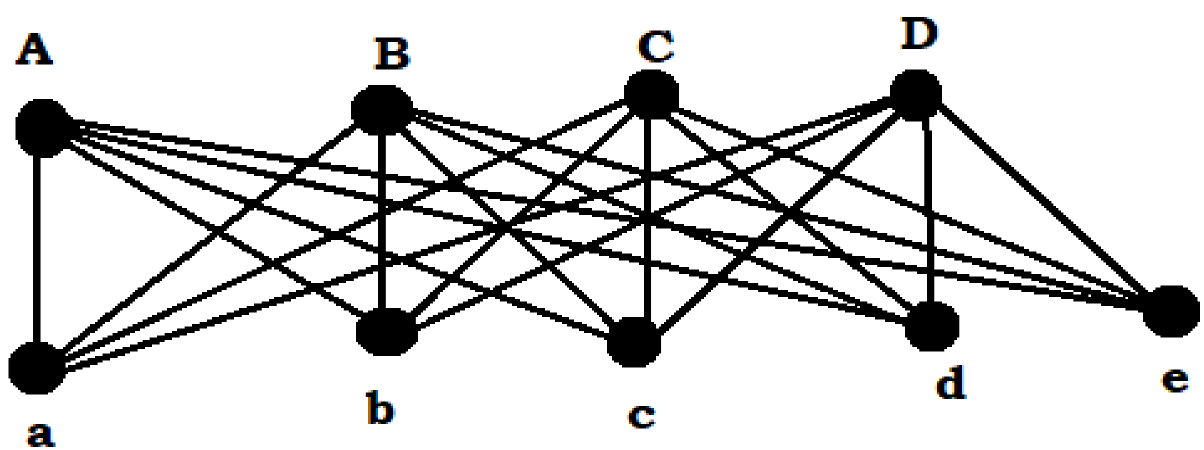


4 regular graph

Bipartite graph: If the vertex set V of a simple graph G= (V, E) can be partitioned into two subsets V_1 and V_2 such that every edge of G connects a vertex in V_1 and a vertex in V_2, then G is called bipartite graph.



Complete bipartite graph: If each vertex of V_1 is connected with every vertex of V_2 by an edge, then G is called a complete bipartite graph. If V_1 contains m vertices and V_2 contains n vertices, then the complete bipartite graph is denoted by $K_{m,n}$.



Theorem: The number of edges in a bipartite graph with n vertices is at most $\left(\frac{n^2}{4}\right)$.

Proof: Let the vertex set be partitioned into the two subsets V_1 and V_2. Let V_1 contain x vertices. Then V_2 contains (n-x) vertices. The largest number of edges of the graph can be obtained, when each of the x vertices in V_1 is connected to each of the (n-x) vertices in V_2. Therefore, the largest number of edges f(x)= x(n-x), is a function of x. Now we have to find the value of x for which f(x) is maximum. By calculus, f'(x)=n-2x and f''(x)=-2. f'(x)=0, when x=(n/2) and $f''(\frac{n}{2})$ <0.

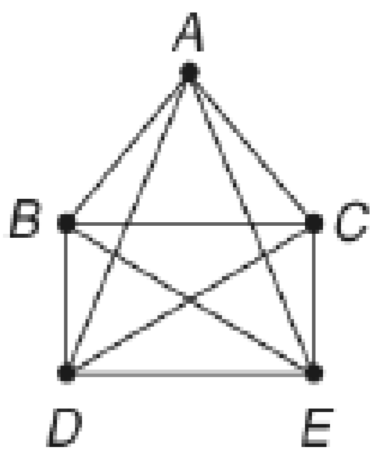
Hence, f(x) is maximum, when x=$\frac{n}{2}$.

Therefore the maximum number of edges required= $f(\frac{n}{2})$ =$\frac{n^2}{4}$.

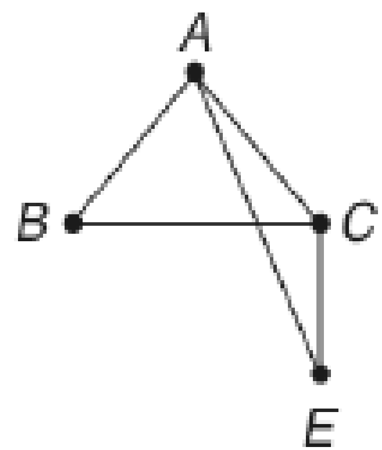
A graph $H = (V', E')$ is called a *subgraph* of $G = (V, E)$, if $V' \subseteq V$ and $E' \subseteq E$.

If $V' \subset V$ and $E' \subset E$, then H is called a *proper subgraph* of G.

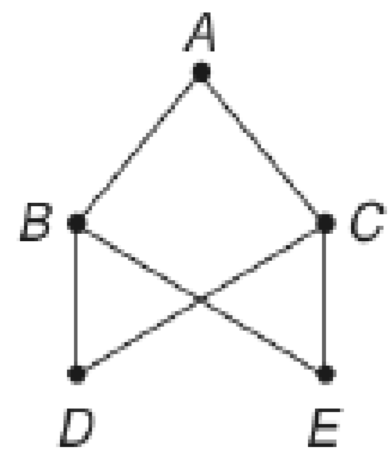
If $V' = V$, then H is called a *spanning subgraph of* G. A spanning subgraph of G need not contain all its edges.



Graph G

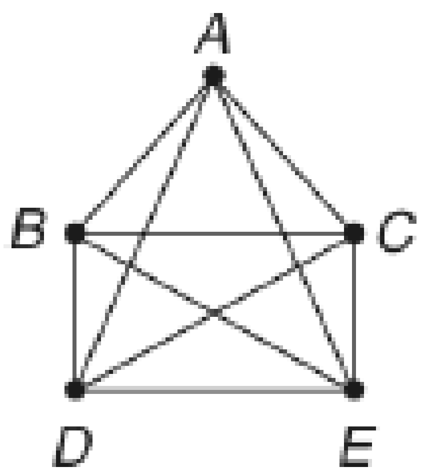


A subgraph of G
(A vertex deleted subgraph of G)

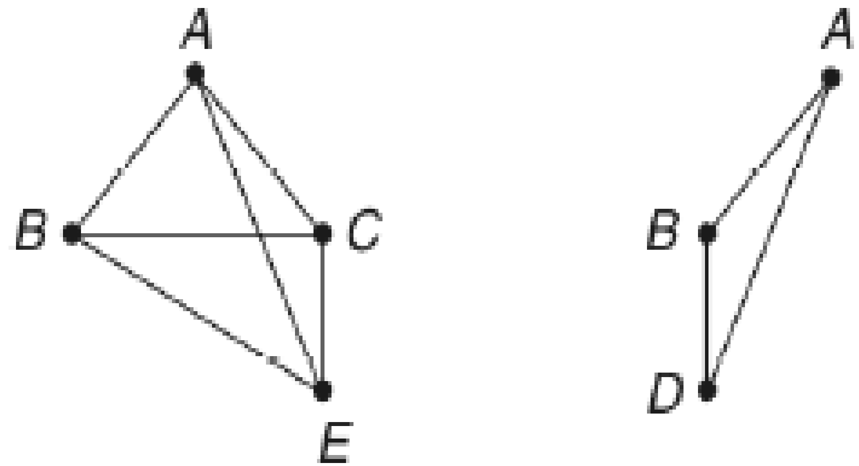


A spanning subgraph of G
(An edge deleted subgraph of G)

A subgraph $H = (V', E')$ of $G = (V, E)$, where $V' \subseteq V$ and E' consists of only those edges that are incident on the elements of V', is called an *induced subgraph of G*.



Graph G

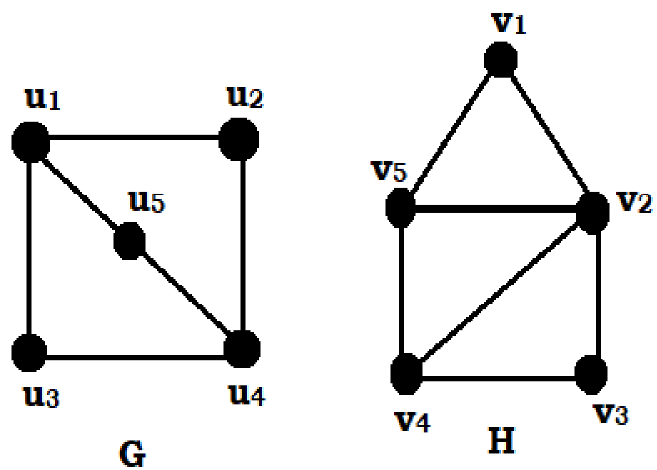


Induced subgraphs of G

Isomorphism: The simple graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are isomorphic if there exists a one to one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1, if and only if f(a) and f(b) are adjacent in G_2 for all a, b $\in V_1$, such a function f is called an isomorphism.

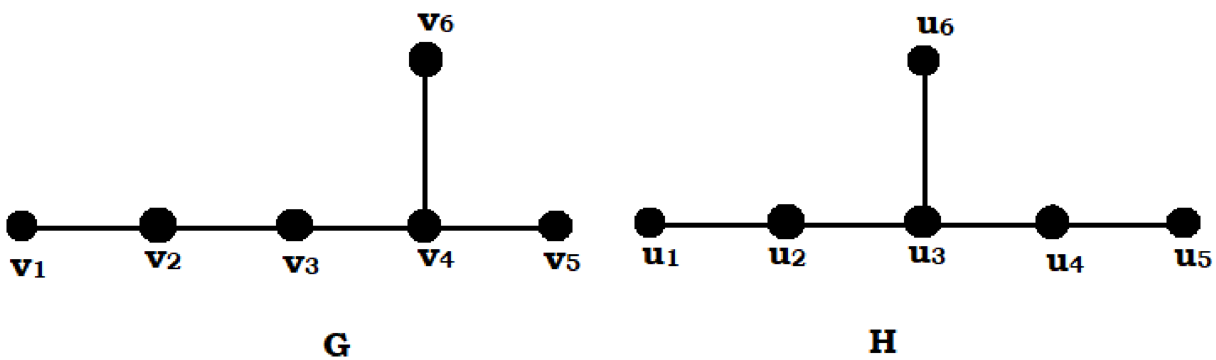
Two simple graphs that are not isomorphic are called nonisomorphic

Example: Show that the graphs G and H are isomorphic.

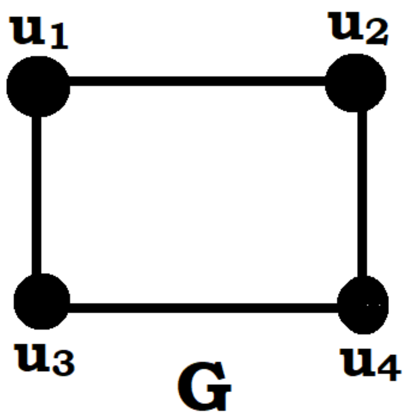


In G, |V|=5 and |E|=6. In H, |V|=5 and |E|=7. The number of edges in both the graphs G and H are not equal. Therefore the graphs G and H are not isomorphic.

Problem: Show that the graphs G and H are not isomorphic.



Problem: Show that the following graphs G and H are isomorphic.



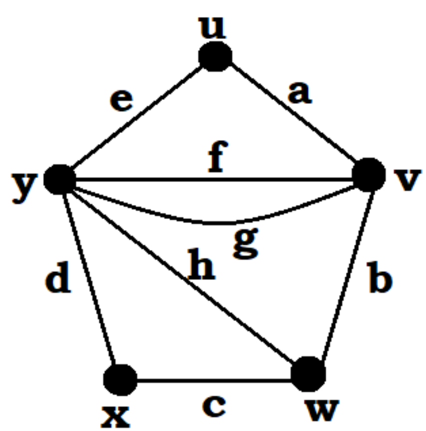
The graphs G and H are isomorphic

5.3.1 Paths and Cycles:

Walk: A walk in G is a finite non-null sequence $W=v_0e_1v_1e_2v_2\ldots e_kv_k$, whose terms are alternatively vertices and edges, such that for $1\leq i\leq k$, the ends of e_i are v_{i-1} and v_i. W is a walk from v_0 to v_k or a (v_o,v_k)-walk.The vertices v_0 and v_k are called the origin and terminus of W respectively and $v_1,v_2,\ldots,v_{k-1}$ its internal vertices. The integer k is the length of W.

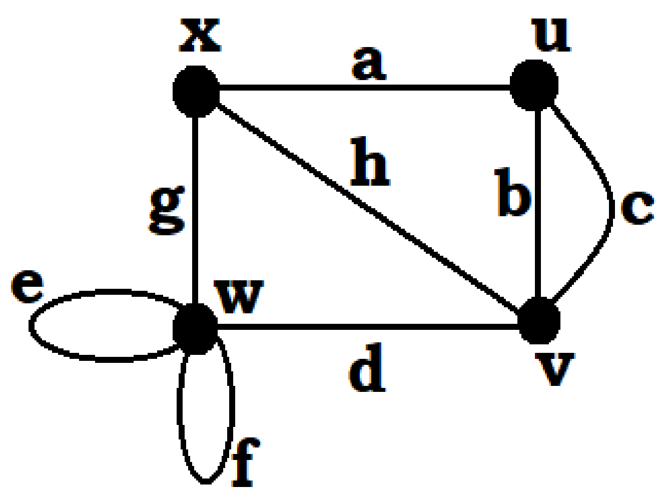
If the edges $e_1,e_2,e_3,\ldots,e_k$ of a walk W are distinct, W is called a trail.

If the vertices $v_0,v_1,v_2,\ldots,v_k$ are distinct, W is called a path.



Walk –uavfyfvgyhwbv; Trail-wcxdyhwbvgy and Path-xcwhyeuav

Cycle: A walk is closed if it has positive length and its origin and terminus are same. A closed trail whose origin and internal vertices are distinct is a cycle.

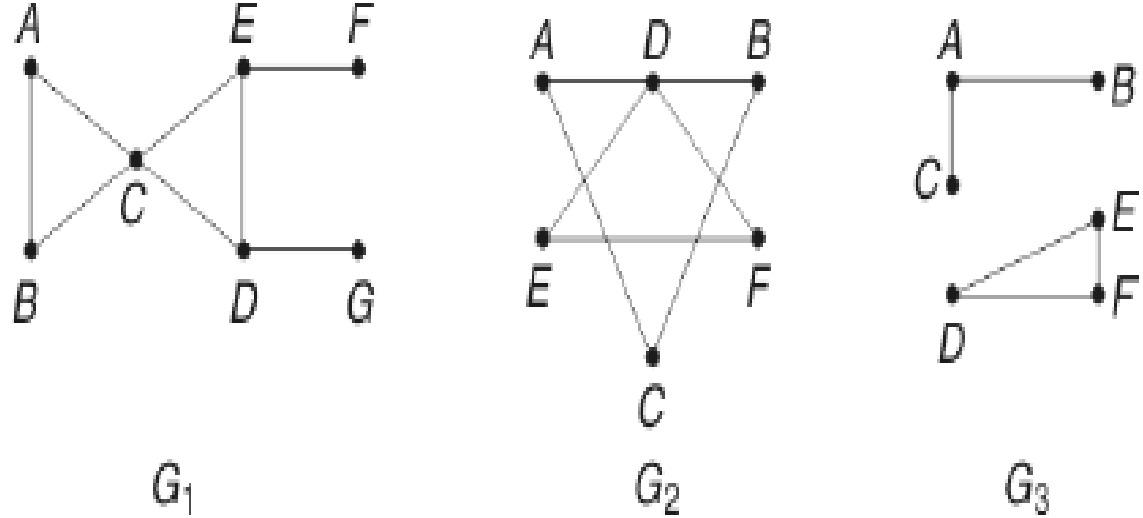


Closed trail: ucvhxgwfwdvbu; Cycle: xaubvhx

An undirected graph is said to be connected if a path between every pair of distinct vertices of the graph.

A graph that is not connected is called *disconnected*.

G_1 and G_2 are connected, while G_3 is not connected.



Theorem: If a graph G (either connected or not) has exactly two vertices of odd degree then there is a path joining these two vertices.

Proof:

Case (i) Let G be connected.

Let v_1 and v_2 be the only vertices of G which are of odd degree. But we have already proved that the number of odd vertices is even. Clearly there is a path connecting v_1 and v_2, since G is connected.

Case (ii) Let G be disconnected.

Then the components of G are connected. Hence, v_1 and v_2 should belong to the same component of G. Hence, there is a path between v_1 and v_2.

Theorem: The maximum number of edges in a simple connected graph G with n vertices and k components is $\frac{(n-k)(n-k+1)}{2}$.

Proof:

Let the number of vertices in the i^{th} component of G be $n_i \ (i \geq 1)$.

Then $n_1+n_2+\ldots+n_k= n$ or $\sum_{i=1}^{k} n_i = n$ - (1)

Hence

$$\sum_{i=1}^{k}(n_i - 1) = (n_1 - 1) + (n_2 - 1) + \ldots + (n_k - 1)$$

$$= \sum_{i=1}^{k} n_i - k$$

$$= n - k$$

Squaring on both sides we get,

$$(n_1 - 1)^2 + (n_2 - 1)^2 + \ldots + (n_k - 1)^2 \le (n-k)^2$$

$$n_1{}^2 + n_2{}^2 + \ldots + n_k{}^2 - 2n_1 - 2n_2 \ldots - 2n_k + k \le (n-k)^2$$

$$\Rightarrow \sum_{i=1}^{k} n_i{}^2 - 2n + k \le (n-k)^2 \quad \therefore$$

$$\Rightarrow \sum_{i=1}^{k} n_i{}^2 \le (n-k)^2 + 2n - k \qquad \text{- (2)}$$

Now the maximum number of edges in the i^th^ component of $G = \frac{1}{2} n_i (n_i - 1)$

$\therefore$ maximum number of edges of G,

$$= \frac{1}{2} \sum_{i=1}^{k} n_i (n_i - 1)$$

$$= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^{\ 2} - n\right]$$

$$\leq \frac{1}{2}\left[(n-k)^2 + 2n - k - n\right]$$

$$\leq \frac{1}{2}\left[(n-k)^2 + (n-k)\right]$$

$$\leq \frac{(n-k)(n-k+1)}{2}$$

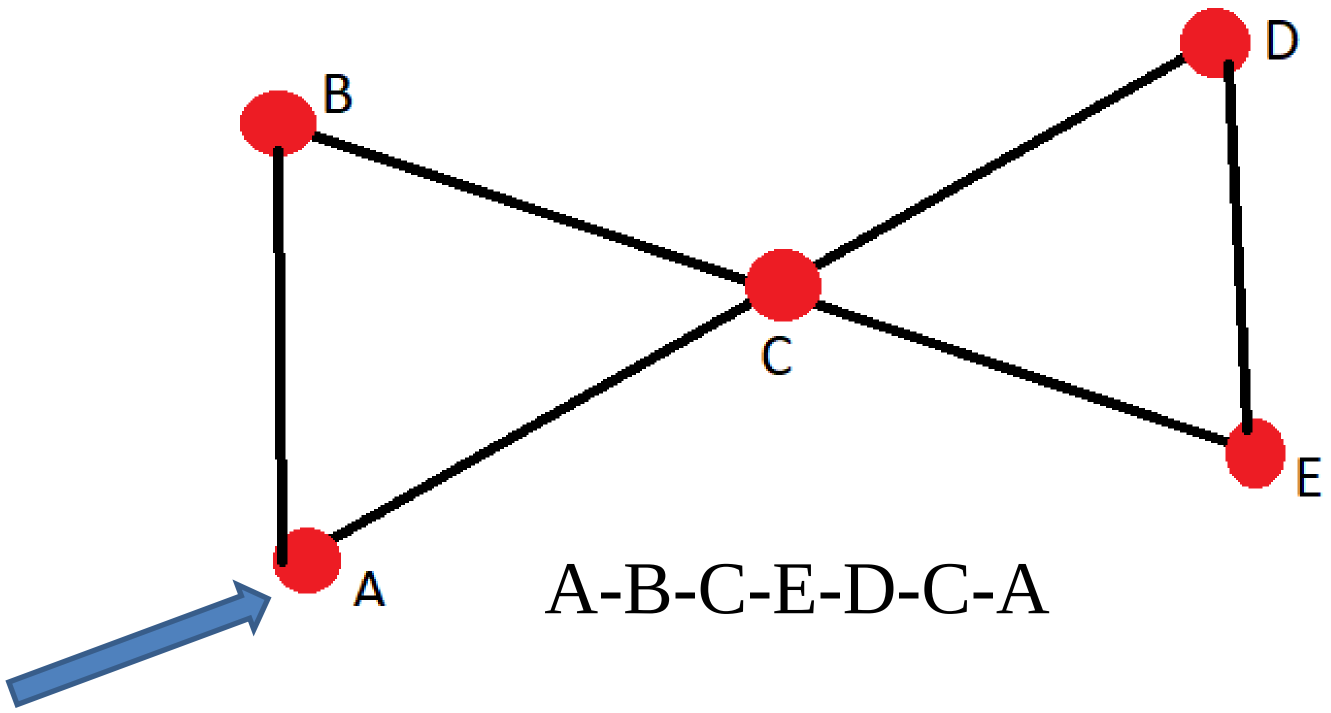
$\therefore$ The maximum number of edges of $G \leq \dfrac{(n-k)(n-k+1)}{2}$

Eulerian Path: A path of graph G is called an Eulerian path, if it includes each edge of G exactly once

Eulerian Circuit: A circuit of a graph G is called an Eulerian circuit, if it include each edge of G exactly once.

Eulerian Graph: A graph containing an Eulerian circuit is called an Euler graph

EULERIAN CIRCUIT



deg(A)=2

deg(B)=2

deg(C) =4

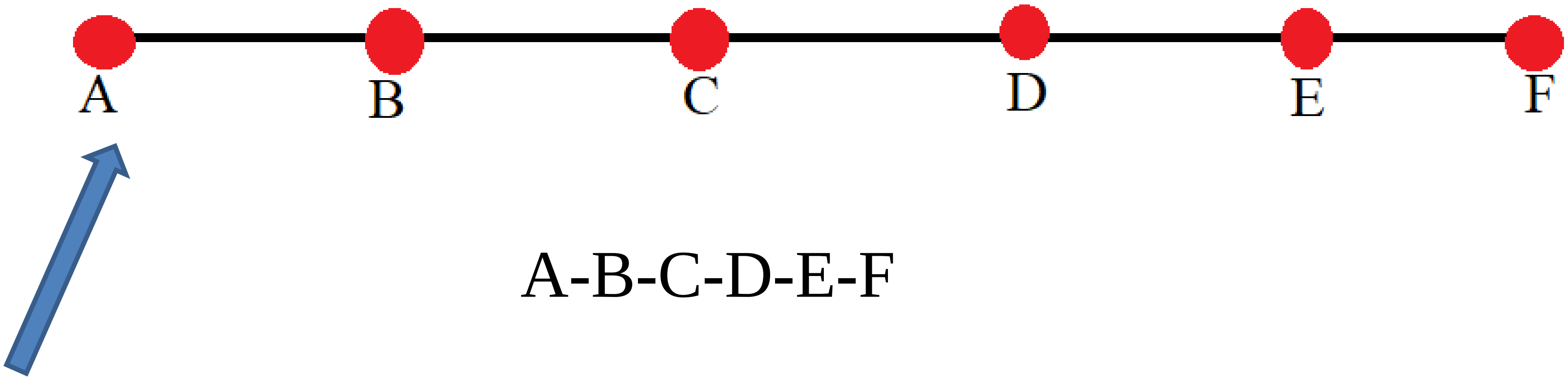
deg(D)=2

deg(E)=2

A-B-C-E-D-C-A

Theorem: A connected graph contains an Eulerian circuit if and only if each of its vertices is of even degree.

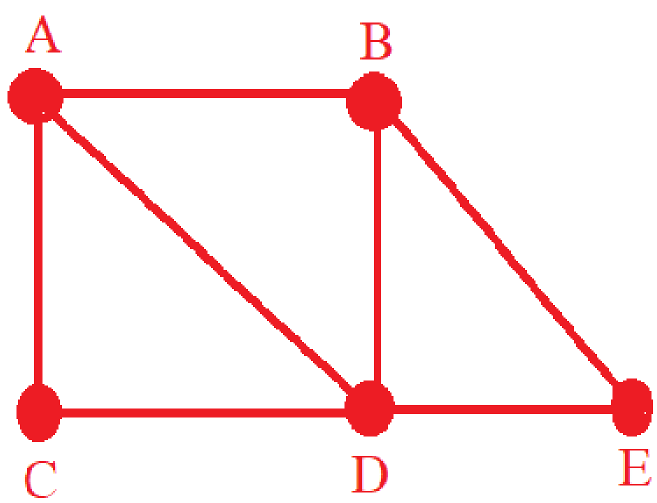
EULERIAN PATH



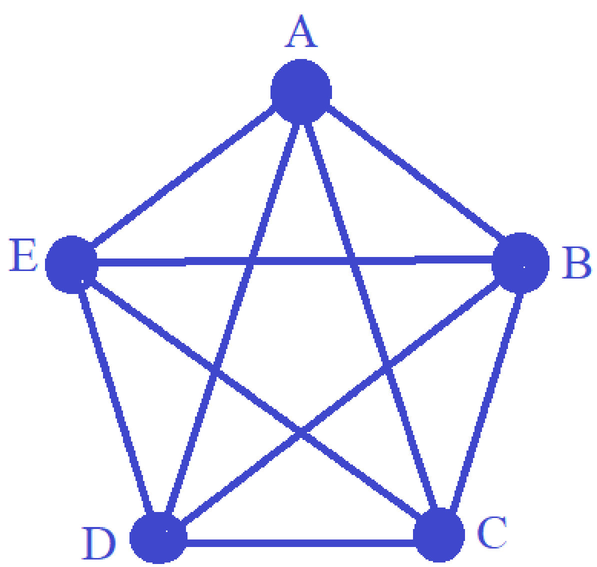
A-B-C-D-E-F

Theorem: A connected graph contains an Euler path, if and only if it has exactly two vertices of odd degree.

Find an Euler path or an Euler circuit, if it exists for the following graphs:



G1



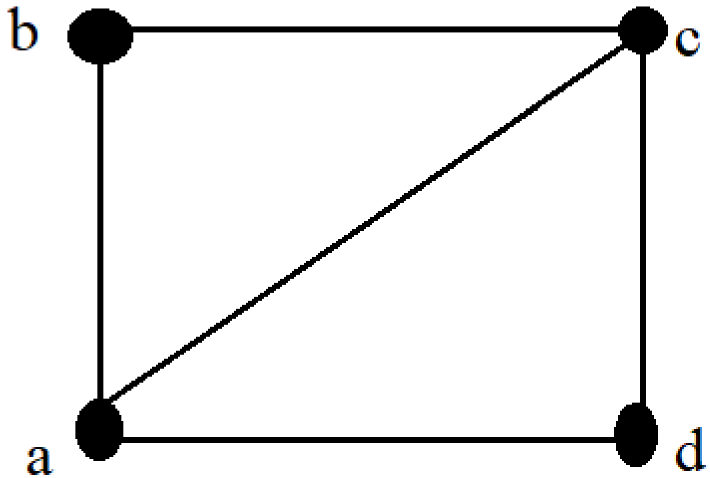
G2

Hamiltonian Path: A path of a graph G is called a Hamiltonian path, if it includes each vertex of G exactly once.

Hamiltonian Circuit: A circuit of a graph G is called a Hamiltonian circuit, if it includes each vertex of G exactly once, except the starting and end vertices which appear twice.

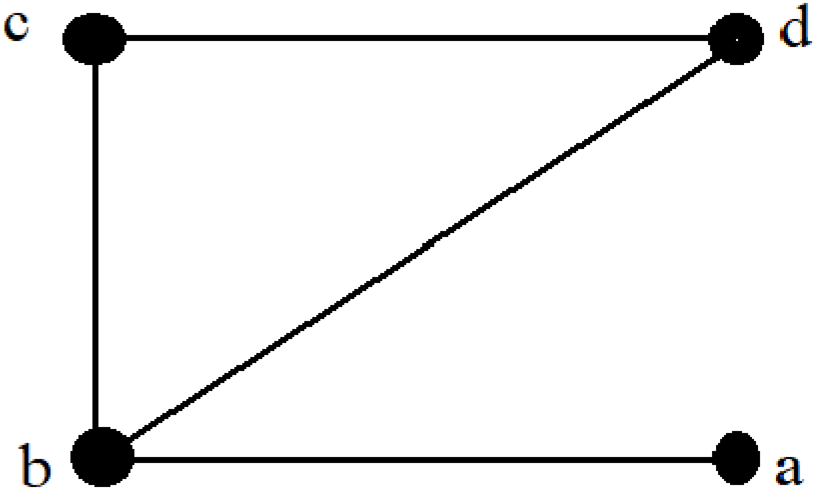
Hamiltonian Graph: A graph containing a Hamiltonian circuit is called a Hamiltonian graph.

Hamiltonian Circuit



a-b-c-d-a

Hamiltonian Path



a-b-c-d

Problem:

Give an example of a graph which contains:

i) an Eulerian circuit and a Hamiltonian circuit.

ii) an Eulerian circuit but not a Hamiltonian circuit.

iii) a Hamiltonian circuit and but not an Eulerian circuit.

iv) neither an Eulerian circuit nor a Hamiltonian circuit.

Matrix Representation of Graphs

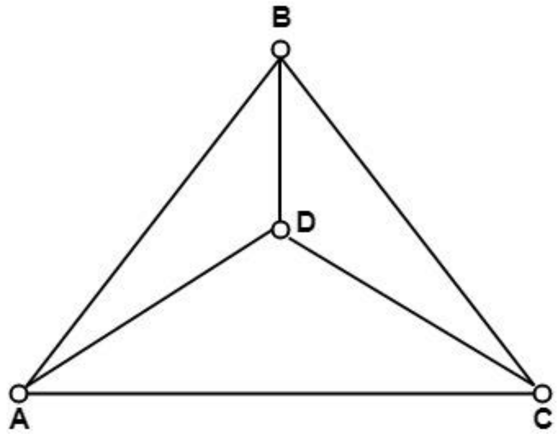
Adjacency Matrix Representation: If an Undirected Graph G consists of n vertices then the adjacency matrix of a graph is an n x n matrix A = $[a_{ij}]$ and defined by

$$a_{ij} = \begin{bmatrix} 1, \text{if } \{V_i, V_j\} \text{is an edge i.e., } v_i \text{is adjacent to } v_j \\ 0, \text{if there is no edge between } v_i \text{ and } v_j \end{bmatrix}$$

If there exists an edge between vertex v_i and v_j, where i is a row and j is a column then the value of a_{ij}=1.

If there is no edge between vertex v_i and v_j, then value of a_{ij}=0.

Find the adjacency matrix M_A of graph G shown in Fig:



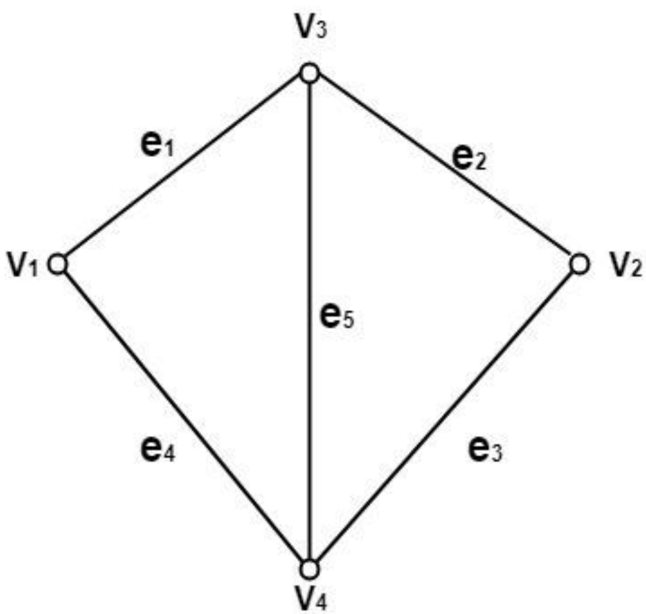
Since graph G consist of four vertices. Therefore, the adjacency matrix will be 4 x 4 matrix.

$$M_A = \begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Incidence Matrix Representation: If an Undirected Graph G consists of n vertices and m edges, then the incidence matrix is an n x m matrix $C = [c_{ij}]$ and defined by

$$c_{ij} = \begin{bmatrix} 1, \text{if the vertex } V_i \text{ incident by edge } e_j \\ 0, \text{otherwise} \end{bmatrix}$$

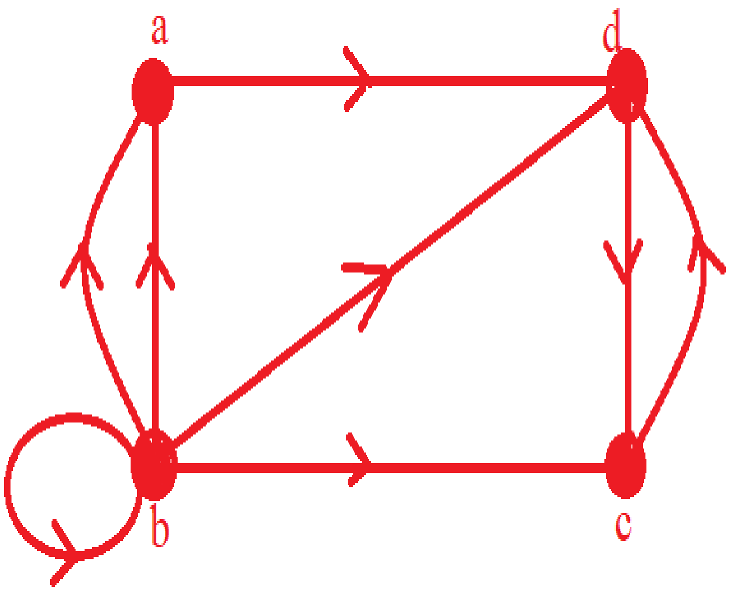
Consider the undirected graph G as shown in fig. Find its incidence matrix M_I.



$$M_I = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

The out degree and in degree of a vertex:

In a directed graph G, the out degree of a vertex *v* of G, denoted by $deg_G^+(v)$is the number of edges beginning at v and the in degree of *v*, denoted by $deg_G^-(v)$ is the number of edges ending at *v*.



$deg_G^+(a)$=1

$deg_G^+(b)$=5

$deg_G^+(c)$=1

$deg_G^+(d)$=1

$deg_G^-(a)$=2

$deg_G^-(b)$=1

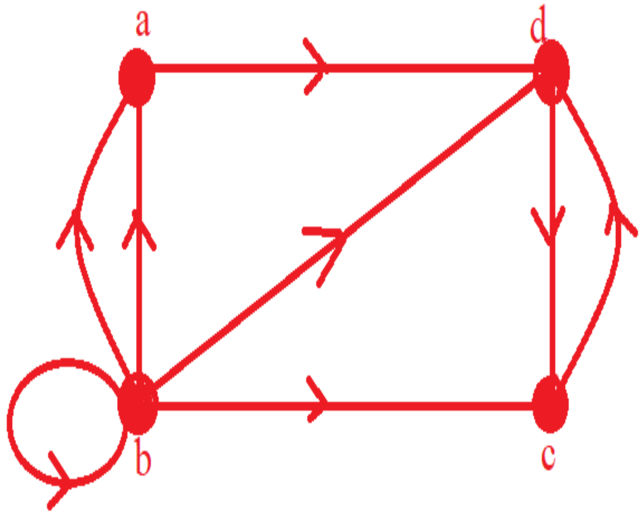
$deg_G^-(c)$=2

$deg_G^-(d)$=3

Theorem :

If G=(V,E) be a directed graph with e edges then $\sum_{v\in V} deg_G^+(v) = \sum_{v\in V} deg_G^-(v)$ =e.

Verify handshaking theorem for the following digraph



Sum of the in-degrees + sum of the out-degrees= 2e

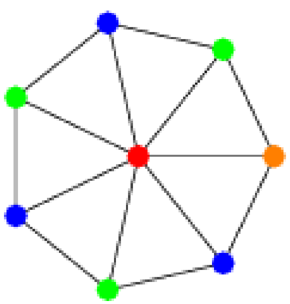
Sum of the in-degrees=8

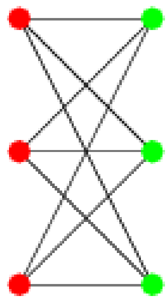
Sum of the out-degrees=8

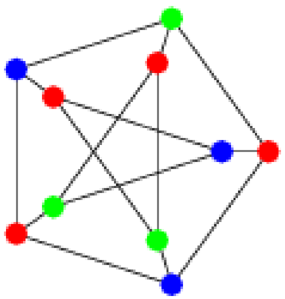
The number of edges (e) =8

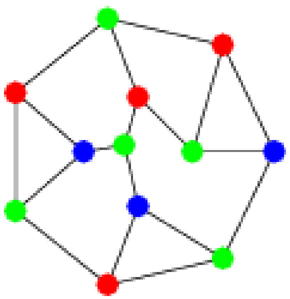
Graph colouring

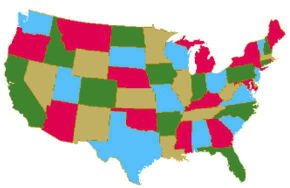
➢ A graph has been coloured if colour is assigned to each vertex in such a way that adjacent vertices have different colours.

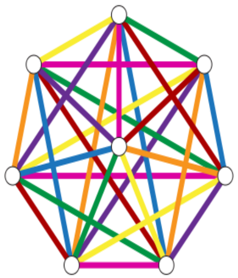


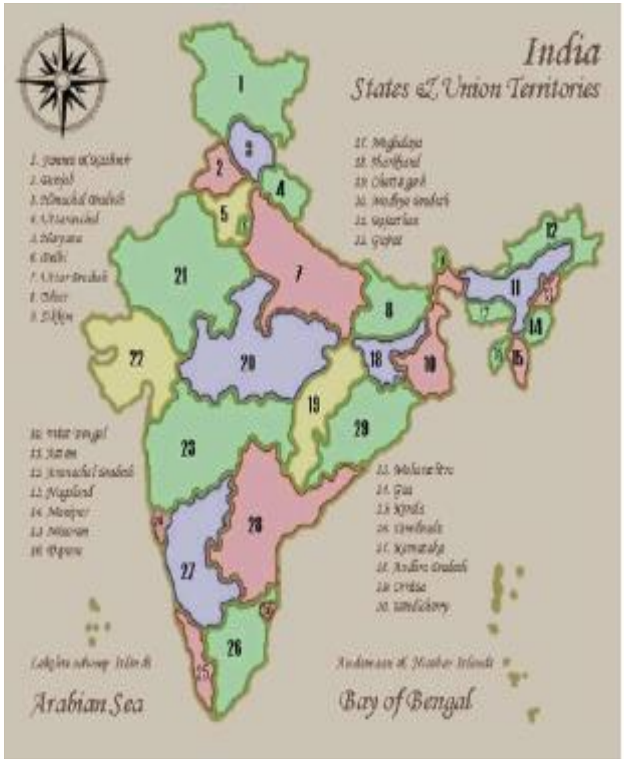












- Vertex colouring of a graph G is a mapping from f:V(G) to S.
- The elements of S are called colors.
- The vertices of one color form a color class.
- If |S| = k then it is k-coloring.
- A graph is k colorable if it has proper k-coloring.
- In proper coloring each color class is a stable set.
- Therefore, one may consider k-coloring as a partition of the vertex set of G into k stable sets which are disjoint.

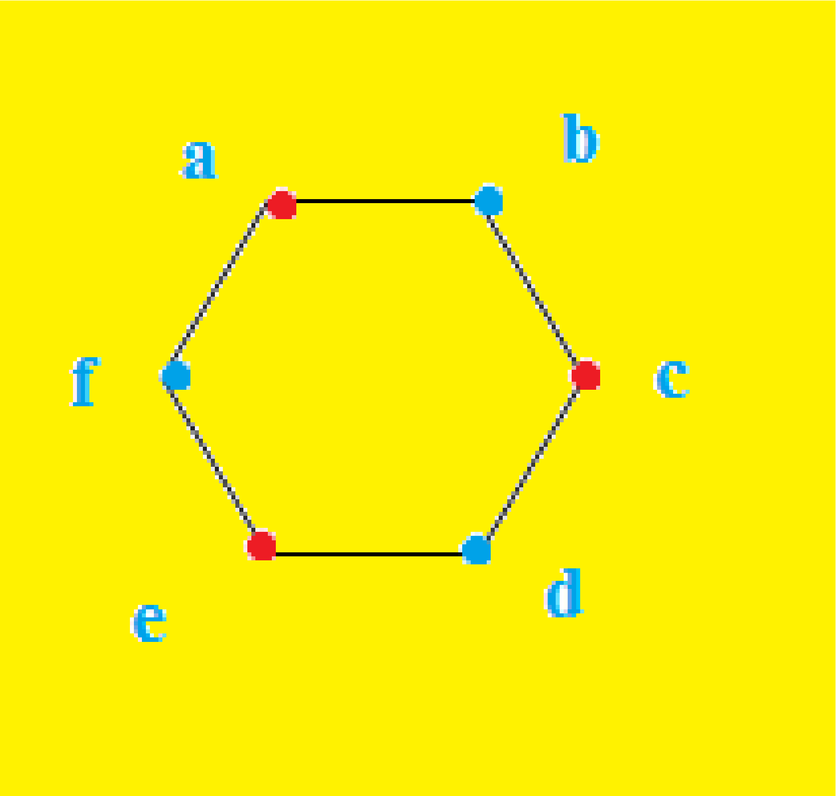
Chromatic number

The chromatic number of a graph is the smallest number of colours with which it can be coloured.

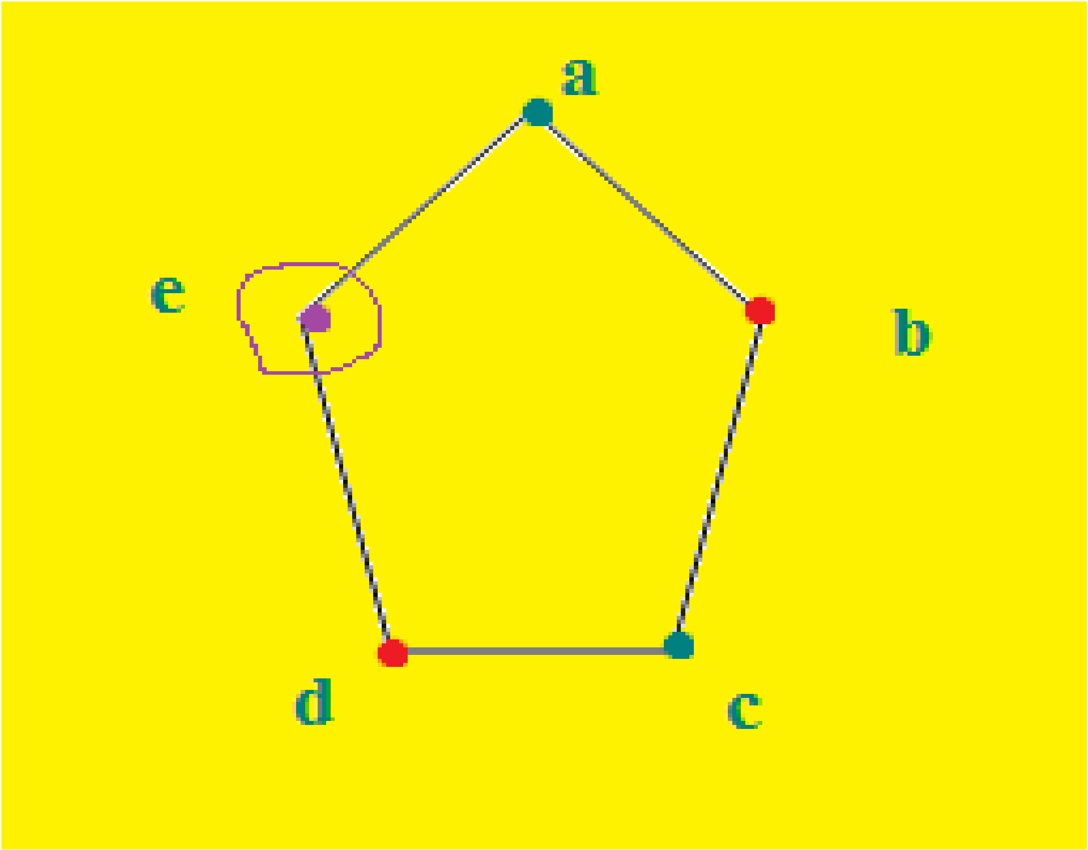
What is the chromatic number of the graph C_n $n \geq 3$. C_n denotes a cycle with n vertices.

In general 2 colors are needed when n is even.

Example: C_6 cycle with six vertices only two colors needed red and blue.



Example: C_5 cycle with five vertices three colors needed.



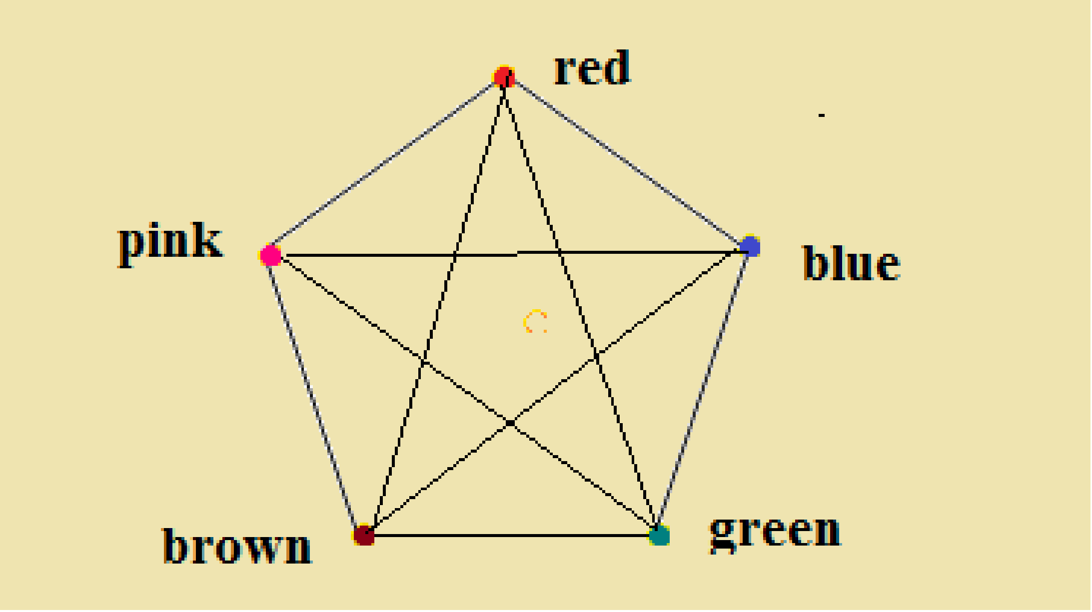
Here pick a vertex color it green color the next vertex with red, proceed the same way till you reach the fourth vertex. The fifth vertex cannot be red or green so we need a third color say purple as shown above.

What is the chromatic number of K_n?

No two vertices can be assigned the same color because every two vertices of this graph are adjacent. Therefore the chromatic number of K_n is n.

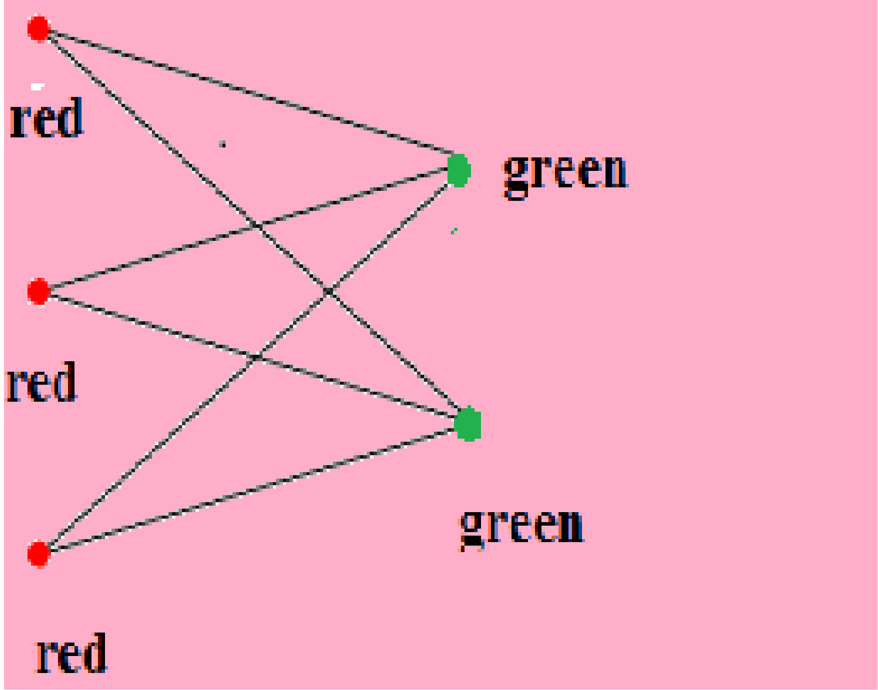
Example: K_5

Five distinct colors are needed so chromatic number is 5.



Example: Coloring of $K_{3,2}$ bipartite graph.

Only two colors are needed to color a complete bipartite graph.



Applications of graph coloring

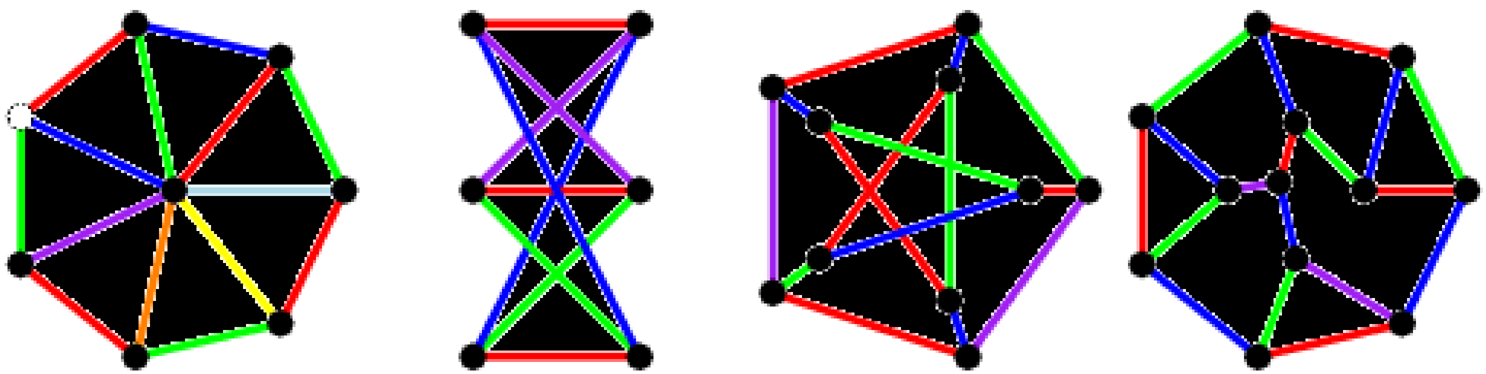
1. Scheduling final exams:

- Vertices represent the courses
- There is an edge between the courses if there is a common student in the courses the vertices represent.
- Each time slot for the exams may be represented by different color.
- Scheduling of exams corresponds to coloring of the associated graph.

Edge colouring:

An edge colouring of G is a mapping from the edge set E(G) to the elements of S. The edges of one colour form a colour class.

If |S|=k then it is called k-edge colouring (objective is to use minimum number of colours).



Weisstein, Eric W. "Edge Coloring." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/EdgeColoring.html

Four color theorem:

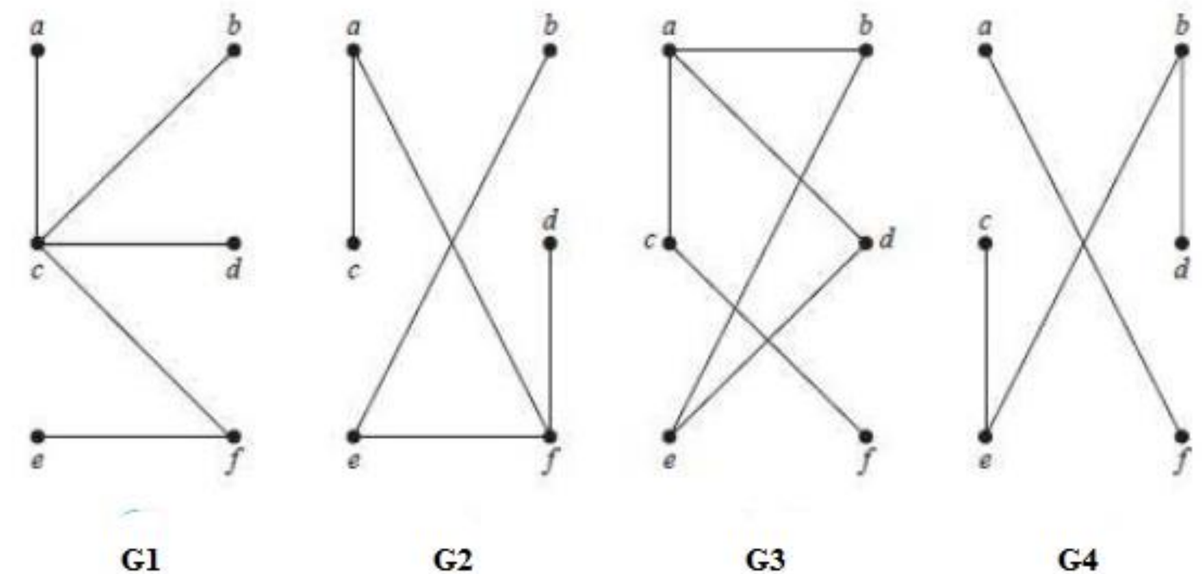
The **four color theorem**, or the **four color map theorem**, states that, given any separation of a plane into contigious regions, producing a figure called a *map*, no more than four colors are required to color the regions of the map so that no two adjacent regions have the same color.

Adjacent means that two regions share a common boundary curve segment, not merely a corner where three or more regions meet.

TREES

A connected graph without any circuits is called a tree. It is a simple graph and has no loops and no parallel edges.

Find which of them are not trees and why?



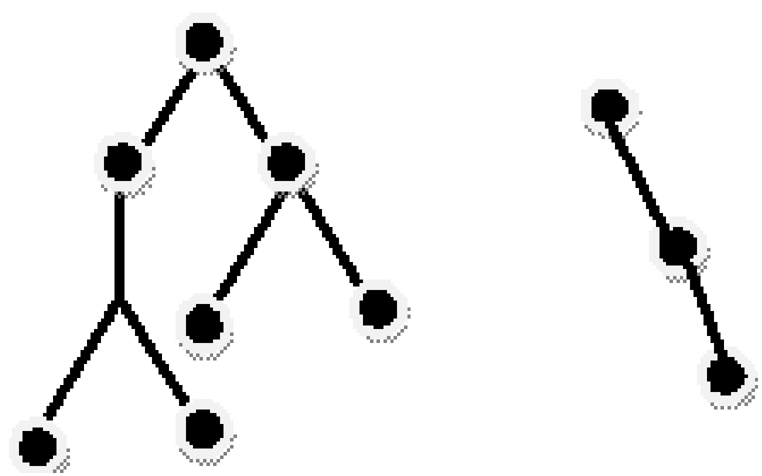
G1 and G4 are trees other two graphs contain circuit.

Definition of tree:

A connected graph without any circuits is called a tree. It is a simple graph and has no loops and no parallel edges.

Definition of forest:

A forest is a graph whose components are all trees.



PROPERTIES OF TREES

- An undirected graph is a tree, if and only if, there is a unique simple path between every pair of vertices.
- A tree with n vertices has n-1 edges.
- Any connected graph with n vertices and n-1 edges is called a tree.
- Any circuitless graph with n vertices and n-1 edges is a tree.

Property 1:

An undirected graph is a tree if and only if , there is a unique simple path between every pair of vertices.

Proof:

(i) Consider a undirected graph T to be the tree under consideration.

Then by the definition of tree it is connected. Hence there is a simple path between any pair of vertices say, v_i and v_j.

If possible, let there be two paths between v_i and v_j and the other from v_j to v_i. Union of these two paths would contain a circuit.

But T cannot have a circuit as it is a tree.

Hence there is a unique simple path between every pair of vertices in T.

(ii) Converse:

Given that there is a unique simple path between every pair of vertices in T. We need to prove that T is a tree.

By the assumption T is connected. If possible let T contain a circuit. That is there is a pair of vertices v_i and v_j between which two paths exist which is against our assumption. Hence T cannot have a circuit and so T is a tree.

Property 2:

If G is a tree with n vertices then G has n-1 edges.

Proof: Proof is by induction.

Basis step: If n=1, the number of vertices is one and the number of edges is zero. Hence the theorem is true for n=1.

Induction step: For n=2, the number of vertices is 2 and edges is 1. Hence also true for n=2 Assume that the theorem is true for .

Consider a tree with k+1 vertices. Let e_{ij} be an edge of the tree connecting v_i and v_j. Since G is a tree e_{ij} is the only edge of the tree connecting v_i and v_j.

Hence deleting e_{ij} will disconnect G into two components.

Each such component has fewer than k+1 vertices and each of them are trees. Let the two components have k_1 and k_2 vertices respectively.

Then the total number of edges in G will be

$$k_1-1+k_2-1+eij = k_1-1+k_2-1+1$$

$$=k_1-1+k_2$$

$$=k+1-1=k.$$

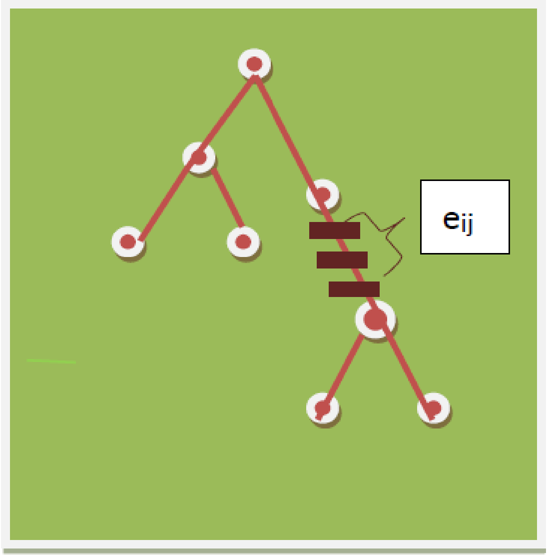
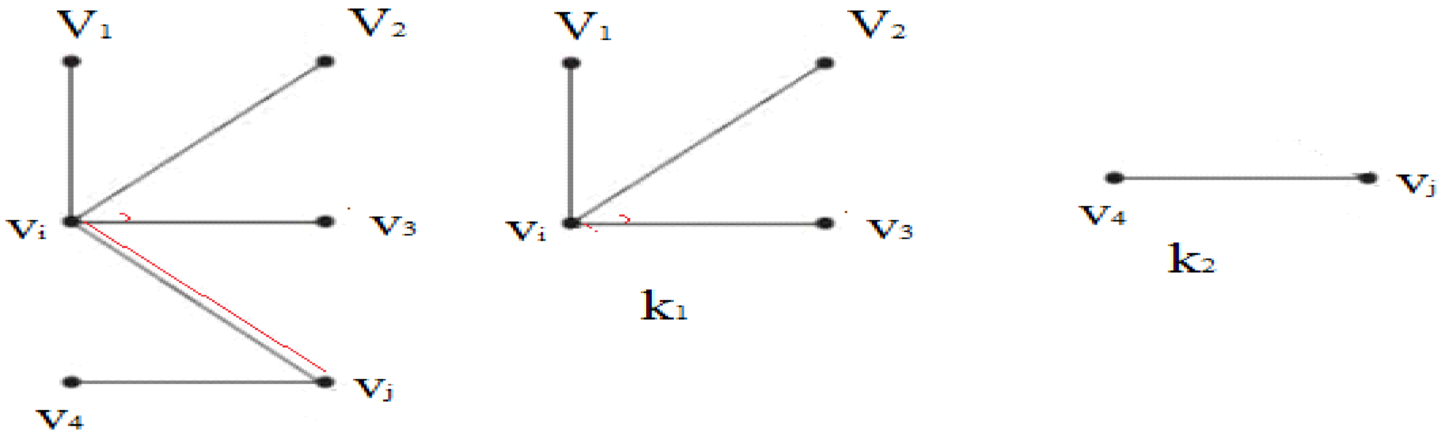


Illustration using a graph of 5 vertices.



Assume that the theorem is true for k≤5

K_1 has 4 vertices and hence no of edges is 3.

K_2 has 2 vertices and hence the number of edges is one.

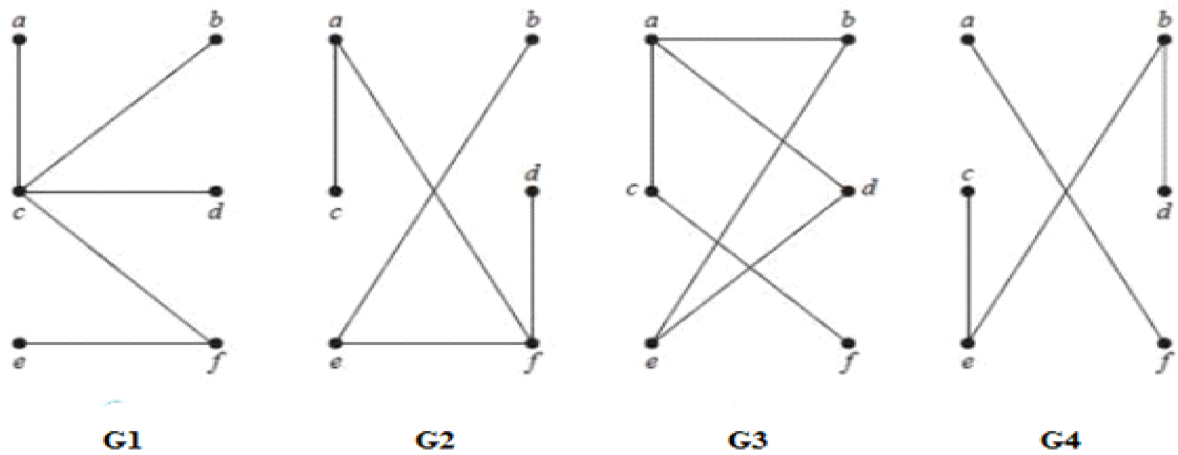
The number of edges in k= $k_1+k_2-1-1+e_{ij}$= 3+1= 4 that is nothing but number of vertices 5 minus 1 hence the proof.

Property3:

Any connected graph with n vertices and n-1 edges is a tree.

Proof: Assume that G has a cycle of length p. Then there are p points and p lines on the cycle and for each of the n-p points not on the cycle there is an incident lineon a geodesic to a point of the cycle. Since each such line is different, the number of edges n-p+p=n>n-1 which is a contradiction.

Problem: Which of them are not trees and why?



G1 and G4 are trees, the other two graphs contain circuit and hence are not trees.

Property 4:

Any circuitless graph with n vertices and n-1 edges is a tree.

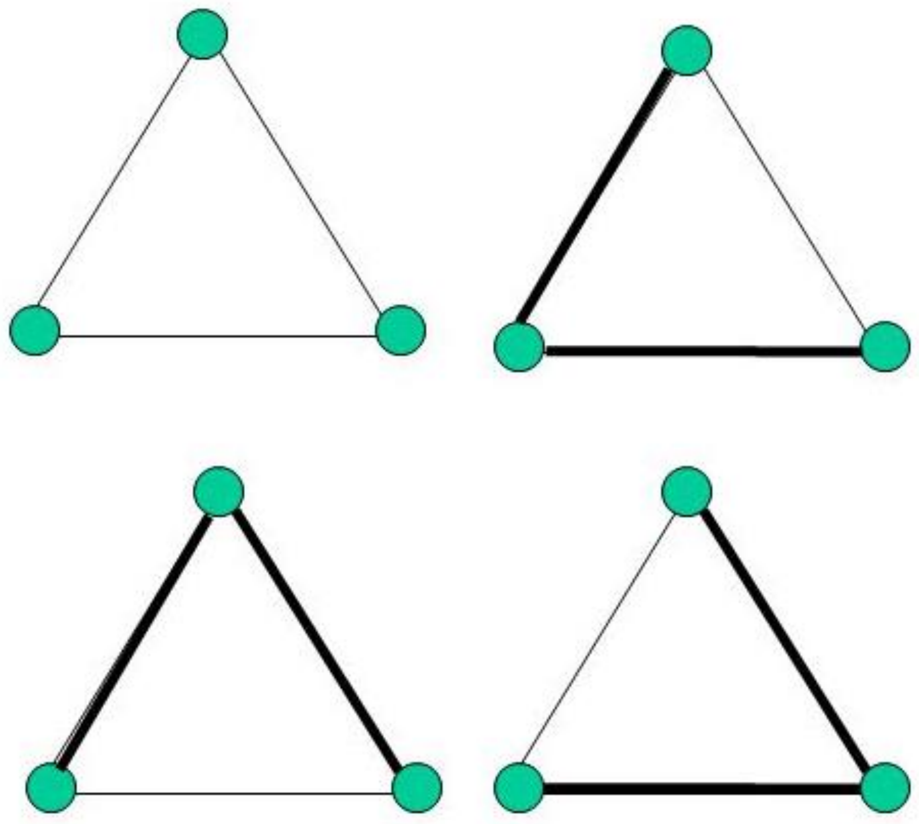
SPANNING TREES

If the subgraph T of a connected graph G is a tree containing all the vertices of G then T is called the spanning tree of G.

Example:

Since G has three vertices any spanning tree of G will also have three vertices and hence two edges.

This can be done in $3C_2$ ways that is 3 ways as shown in the figure.



Definition of spanning set: Let G=(V, E) be a connected undirected graph. A spanning set for G is a subset F of E such that (V, F) is connected.

Definition of spanning tree: If the subgraph T of a connected graph G is a tree containing all the vertices of G then T is called the spanning tree of G.

Definition of branch of tree: Any edge in a spanning tree T of G is called a branch of T.

Definition of chord: an edge in G that is not in a specific given spanning tree T is called a chord.

Facts:

1. A graph can have more than one spanning tree.
2. Number of F=number of V-1, obviously.
3. Every connected graph has atleast one spanning tree.
4. With respect to any of the spanning trees, a connected graph of n vertices and e edges has n-1 tree branches and e-n+1 chords.
5. Rank of G = number of branches in any spanning tree of G.
6. Nulllity of G=number of chords in G.

MINIMUM SPANNING TREE

Definition:

If G is a connected weighted graph, the spanning tree of G with the smallest total weight is called minimum spanning tree of G.

Two popular algorithms for constructing minimum spanning trees is

- Prim's Algorithm
- Kruskal's Algorithm

KRUSKAL'S ALGORITHM

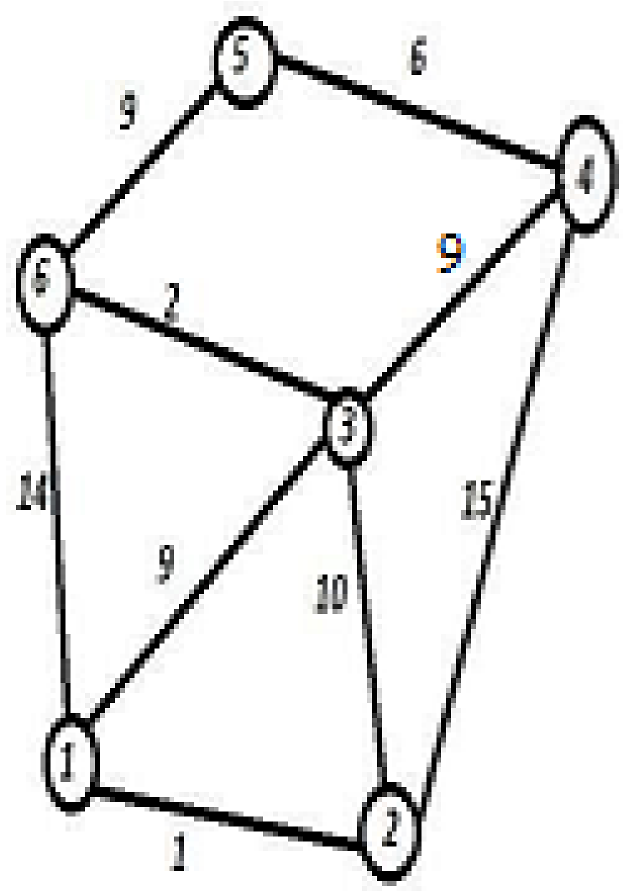
1. The edges of the graph are arranged in the order of increasing weights
2. An edge with minimum weight is selected as an edge of the required spanning tree.
3. Those edges with minimum weight that do not form a circuit with already selected edges are successively added.
4. The procedure terminates once n-1 edges have been selected.

NOTE:

The weight of a minimum spanning tree is unique, different minimum spanning trees are possible if 2 or more edges have same weight

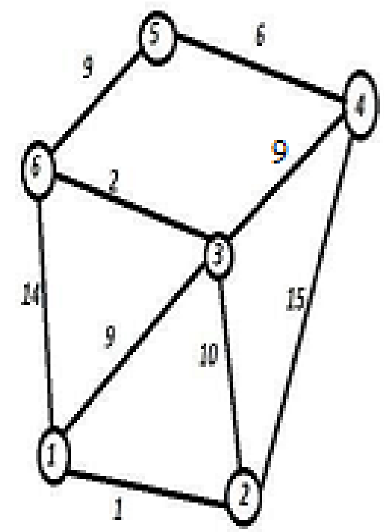
Problem:

Find the minimum spanning tree for the weighted graph shown in the figure by using Kruskal's alg



Solution:

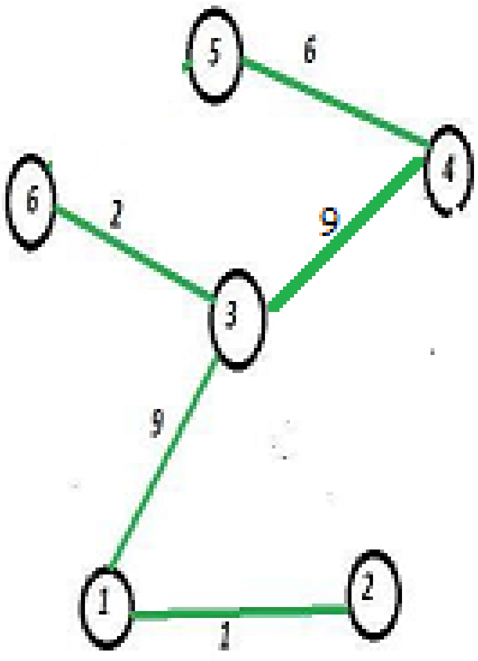
To find the minimum spanning tree for the weighted graph shown in the figure by using Kruskal’s algorithm.

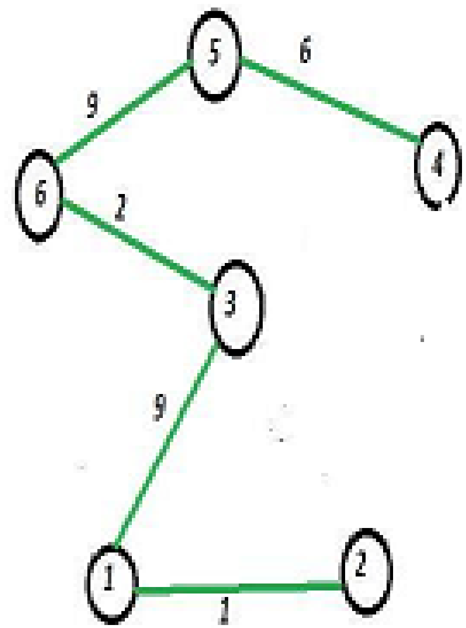


Number of vertices is 6 .
Hence the number of edges to be considered is 5.

Edge	Weight	Included or not	If not included circuit formed
1-2	1	Yes	-
6-3	2	Yes	-
5-4	6	Yes	-
1-3	9	Yes	-
5-6	9	Yes	-

The other spanning trees possible for the same graph are





There are two spanning trees for the above problem.
Their minimum spanning weight is 27.(unique)

Concepts based on minimum spanning trees (MST):

- The number of edges in MST with n nodes is (n-1).
- The weight of MST of a graph is always unique. However there may be different ways to get this weight (if there edges with same weights).
- The weight of MST is sum of weights of edges in MST.
- Maximum path length between two vertices is (n-1) for MST with n vertices.
- There exists only one path from one vertex to another in MST.
- Removal of any edge from MST disconnects the graph.
- For a graph having edges with distinct weights, MST is unique.

Problem: How many minimum spanning trees are possible using Kruskal's algorithm for a given graph?

Solution: If all edges weight are distinct, minimum spanning tree is unique.
If two edges have same weight, then we have to consider both possibilities and find possible minimum spanning trees.


Thank you