Unit-3
Analysis of LTI CT system

By

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Contents

(modeling of LTI-CT system
Classical method
zero state response
zero input response
Total response)

Modeling of LTI-CT system

- The 2 basic methods are used to analyse the response of linear time invariant (LTI) continuous time system.
- Method-1 is differential equation method, method-2 is convolutional integral method.

Method-1: In first method, we develop a mathematical description for continuous time system. It is an ordinary linear differential equation with constant coefficient of the form

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \ldots\ldots\ldots\ldots + a_1 \frac{dy(t)}{dt} + a_0 y(t) =$$

$$b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \ldots\ldots\ldots\ldots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

Where $x(t) =$ input of the system

$y(t) =$ output of the system

Modeling of LTI-CT system (Cont.)

- For a given input, the output of the system can be obtained by solving the differential equation.
- The solution of the differential equation consists of 2 parts:

(i) Zero state response(Forced response):-

The zero state response of the system is response due to the input when the initial state of the system is zero.

(ii) Zero input response (natural response)

It is the response of the system due to the initial state of the system, and making input zero.

Method-2: The output of the LTI-CT system can be obtained using convolution integrals.

Solution of differential equation (Classic method)

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Natural response (zero input response)

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Natural response (zero input response) (Cont.)

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Natural response (zero input response) (Cont.)

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Forced response (zero state response)

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Forced response (zero state response) (Cont.)

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Problem-1

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Problem-1 (Cont.)

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Problem-1 (Cont.)

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Problem-1 (Cont.)

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Problem-1 (Cont.)

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Problem-1 (Cont.)

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Problem-1 (Cont.)

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Problem-1 (Cont.)

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Problem 2

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Problem-2 (Cont.)

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Problem-2 (Cont.)

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Problem-2 (Cont.)

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Problem-2 (Cont.)

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(Convolution integral Properties of convolution, Step response)

Representation of a continuous time signal

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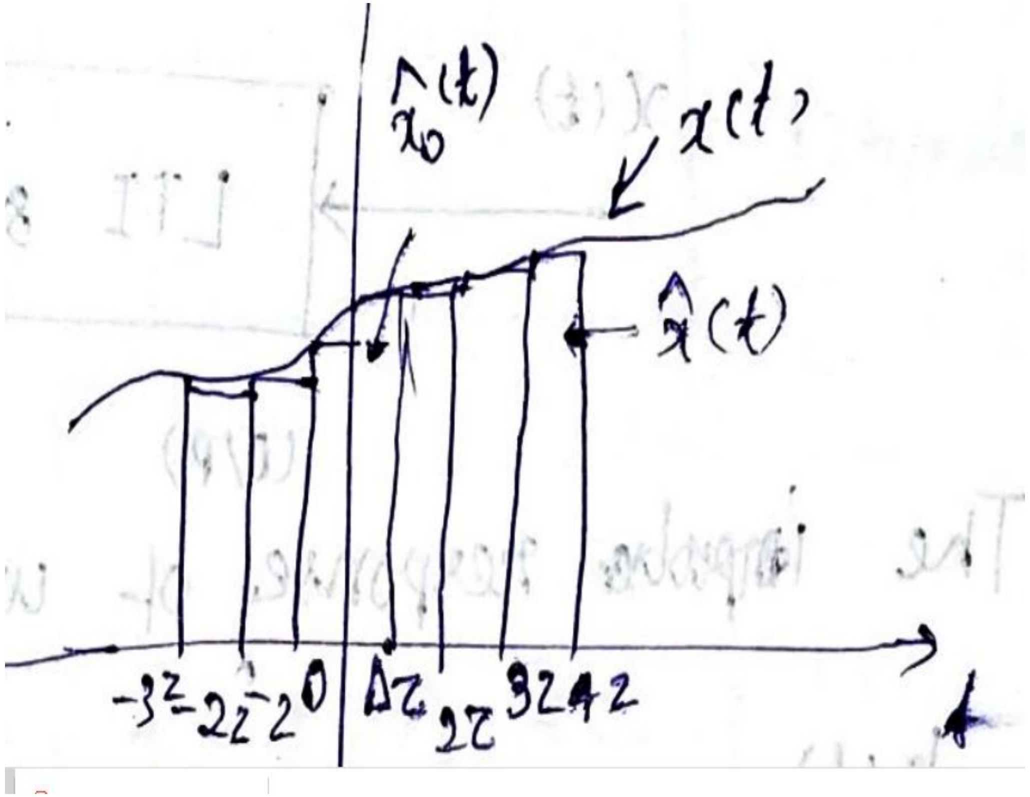

Representation of a continuous time signal (Cont.)

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Convolution integral

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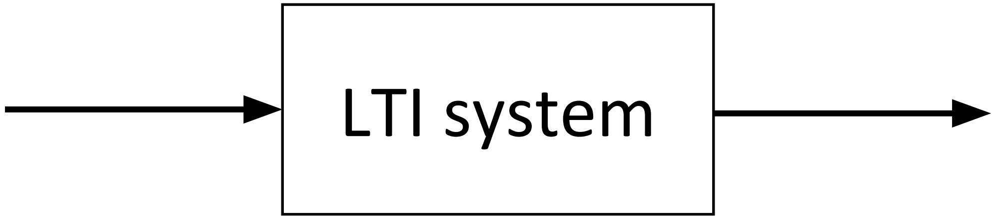

The response (output) of unit impulse function is denoted as $h(t)$

$$h(t) = T[\delta(t)]$$

Where $\delta\ (t)$ is unit impulse function

As we know $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\ d\tau$

As $y(t) = T[x(t)]$

So, $y(t) = T[\ \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\ d\tau]$

$= \int_{-\infty}^{\infty} x(\tau)T[\delta(t-\tau)]\ d\tau$

As we know $T[\delta\big((t-\tau)\big)] = h(t-\tau)$ (due to time invariant system)

Convolution integral (Cont.)

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Properties of Convolution

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Properties of Convolution (Cont.)

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Properties of Convolution (Cont.)

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Properties of Convolution (Cont.)

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Properties of Convolution (Cont.)

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Properties of Convolution (Cont.)

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Properties of Convolution (Cont.)

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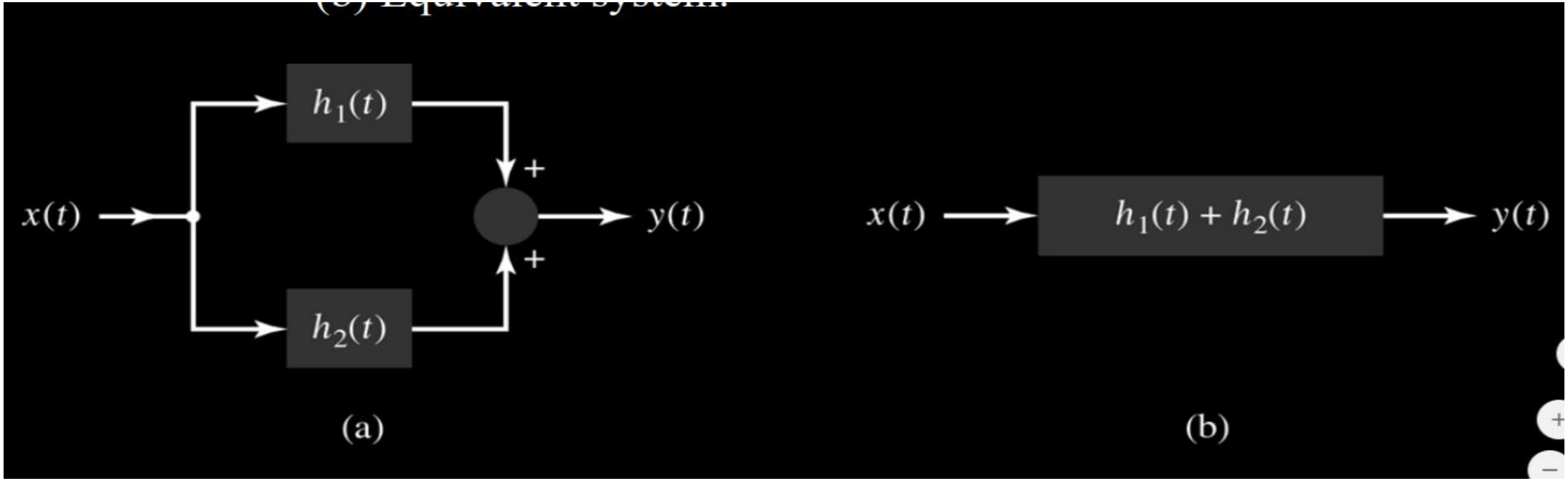
Impulse response of interconnected systems

System in parallel



Impulse response of interconnected systems (Cont.)

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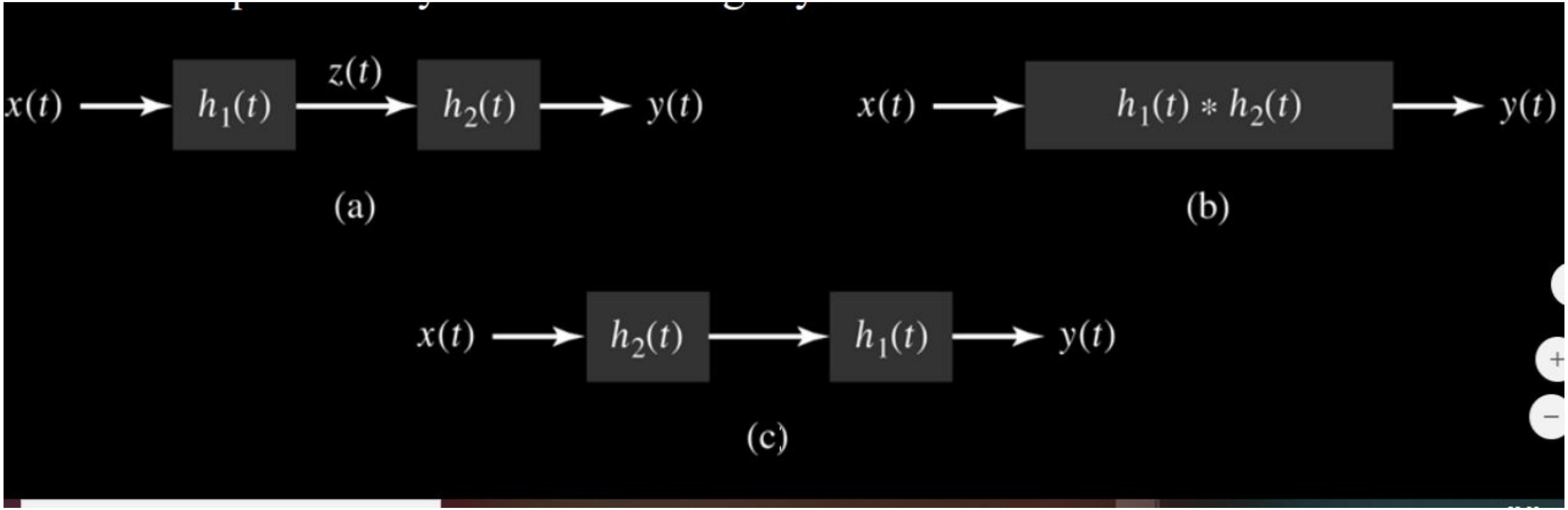

Step response

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Problems

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Problems (Cont.)

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Problems (Cont.)

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Problems (Cont.)

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Problems (Cont.)

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Problems (Cont.)

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Problems (Cont.)

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Problems (Cont.)

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Problems (Cont.)

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Stability

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Problems (Cont.)

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Problems (Cont.)

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(Laplace transform, Properties of Laplace transform, Problem solving)

Signal and system using the Laplace transform

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Converting Fourier transform to Laplace transform

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Converting Fourier transform to Laplace transform(Cont.)

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Summary of Laplace Transform

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Convergence of the Laplace transform

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Convergence of the Laplace transform (Cont.)

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S-Plane

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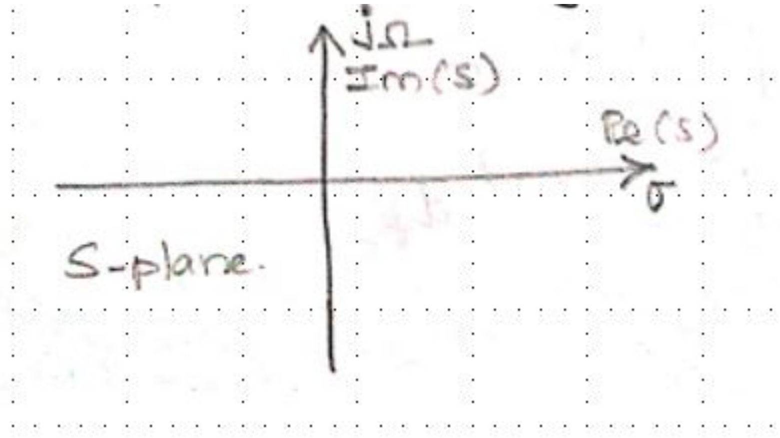

Properties of ROC

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Problems

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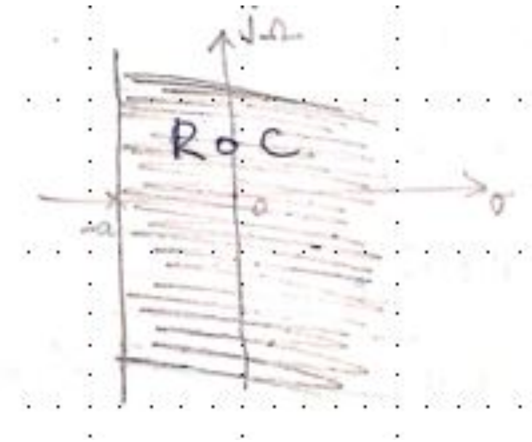

Problems (Cont.)

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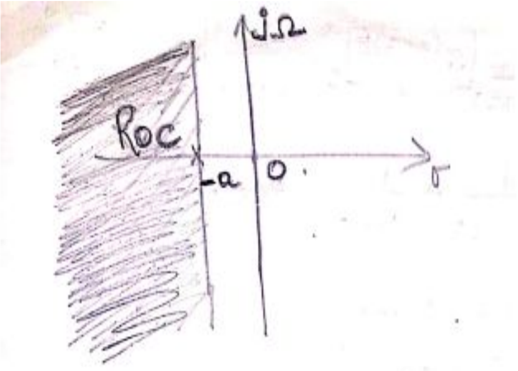

Problems (Cont.)

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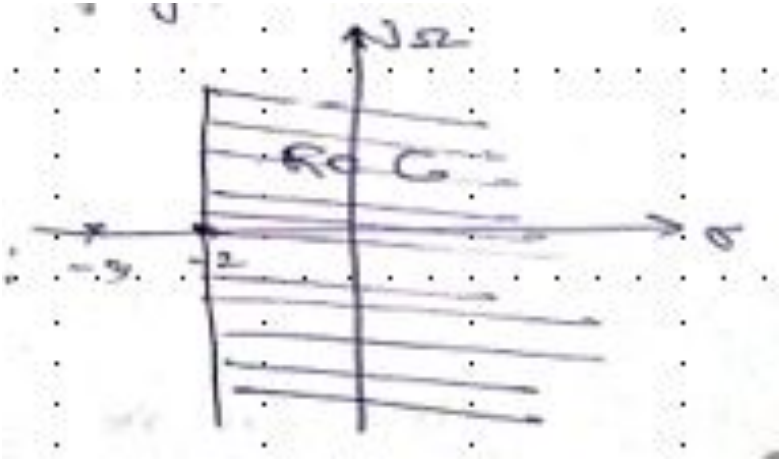

Problems (Cont.)

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Problems (Cont.)

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The unilateral Laplace transform

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Problems

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Problems (Cont.)

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Problems (Cont.)

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Properties of Laplace transform

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Properties of Laplace transform (Cont.)

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Properties of Laplace transform (Cont.)

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Properties of Laplace transform (Cont.)

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Properties of Laplace transform (Cont.)

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Properties of Laplace transform (Cont.)

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Properties of Unilateral Laplace transform (Cont.)

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Properties of Unilateral Laplace transform (Cont.)

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Proof. From the definition of Laplace transform, we can write

$$L\left[\frac{d}{dt}x(t)\right] = \int_0^\infty \frac{d}{dt}x(t)e^{-st}dt$$

This equation may be integrated by parts by letting

$$u = e^{-st} \quad \text{and} \quad dv = dx(t)$$
$$du = -se^{-st}dt \quad \text{and} \quad v = x(t)$$

$$\boxed{\int udv = vu - \int vdu}$$

$$L\left[\frac{d}{dt}x(t)\right] = e^{-st}x(t)\Big|_0^\infty - \int_0^\infty x(t)(-se^{-st})dt$$
$$= e^{-st}x(t)\Big|_0^\infty + s\int_0 x(t)e^{-st}dt$$
$$= -x(0^-) + sX(s)$$
$$\Rightarrow \quad L\left[\frac{d}{dt}x(t)\right] = sX(s) - x(0^-)$$

since $\lim_{t\to\infty} x(t)e^{-st} = 0$.

To find the transform of the second derivative, let us write

$$\frac{d^2x(t)}{dt^2} = \frac{d}{dt}\left[\frac{d}{dt}x(t)\right]$$
$$L\left[\frac{d^2x(t)}{dt^2}\right] = sL\left[\frac{d}{dt}x(t)\right] - \frac{dx}{dt}(0^-)$$
$$= s[sX(s) - x(0^-)] - \frac{dx}{dt}(0^-)$$
$$= s^2X(s) - sx(0^-) - \frac{dx}{dt}(0^-)$$

In the above expression the quantity $\frac{dx}{dt}(0^-)$ is the derivative of $x(t)$ evaluated at $t = 0^-$.

Similarly

$$L\left[\frac{d^nx(t)}{dt^n}\right] = s^nX(s) - s^{n-1}x(0^-) - s^{n-2}\frac{dx}{dt}(0^-)\ldots - \frac{d^{n-1}x(0^-)}{dt^{n-1}}$$

Properties of Unilateral Laplace transform (Cont.)

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Properties of Unilateral Laplace transform (Cont.)

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Properties of Unilateral Laplace transform (Cont.)

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Problems

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Problems (Cont.)

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Problems (Cont.)

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Problems (Cont.)

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Initialvalue

$$x(0) = \lim_{s\to\infty} sX(s)$$

$$= \lim_{s\to\infty} \frac{s(s+5)}{s^2+3s+2} = \lim_{s\to\infty} \frac{s^2+5s}{s^2+3s+2}$$

$$= \lim_{x\to 0} \frac{1+5x}{1+3x+2x^2} \quad \left(s = \frac{1}{x}\right)$$

$$= 1$$

Final value

$$x(\infty) = \lim_{s\to 0} sX(s)$$

$$= \lim_{s\to 0} \frac{s(s+5)}{s^2+3s+2} = 0$$

(Inverse of Unilateral Laplace transform, Inverse of Bilateral Laplace transform)

Inverse of Unilateral Laplace Transforms

$$X(s) = \frac{N(s)}{D(s)}$$

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Inverse of Unilateral Laplace transform (Cont.)

Inverse of Unilateral Laplace transform (Cont.)

Inverse of Unilateral Laplace transform (Cont.)

Inverse of Unilateral Laplace transform (Cont.)

Inverse of Unilateral Laplace transform (Cont.)

Problems

Problems (Cont.)

Problems (Cont.)

$$\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] = e^{-2t}u(t)$$

$$\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-t}u(t)$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+2)^2}\right] = te^{-t}u(t)$$

Problems (Cont.)

3.Q. Find the Inverse Laplace Transform of $X(s) = \dfrac{2s+1}{(s+1)(s^2+2s+2)}$

$$= \frac{A}{s+1} + \frac{B}{s-(-1+j1)} + \frac{B^*}{s-(-1-j1)}$$

Roots of $s^2+2s+2=0$ are $\dfrac{-2\pm\sqrt{4-8}}{2} = \dfrac{-2\pm j2}{2} = -1\pm j1$

$$A = (s+1)\frac{(2s+1)}{(s+1)(s^2+2s+2)}\bigg|_{s=-1}$$

$$= \frac{2(-1)+1}{(-1)^2+2(-1)+2} = \frac{-1}{1} = -1$$

$$B = (s+1-j)\frac{2s+1}{(s+1)(s+1+j)(s+1-j)}\bigg|_{s=-1+j}$$

$$= \frac{2(-1+j)+1}{(-1+j+1)(-1+j+1+j)}$$

$$= \frac{-1+2j}{j(2j)} = \frac{-1}{2}(-1+2j) = 0.5-j$$

$$\Rightarrow X(s) = \frac{-1}{s+1} + \frac{0.5-j}{s-(-1+j1)} + \frac{0.5+j}{s-(-1-j1)}$$

Problem (Cont.)

Taking inverse Laplace-transform we get

$$
\begin{aligned}
x(t) &= -e^{-t}u(t) + (0.5-j)e^{(-1+j1)t}u(t) + (0.5+j)e^{(-1-j1)t}u(t) \\
&= -e^{-t}u(t) + (0.5-j)e^{-t}e^{jt}u(t) + (0.5+j)e^{-t}e^{-jt}u(t) \\
&= -e^{-t}u(t) + 0.5e^{-t}e^{jt}u(t) - je^{-t}e^{jt}u(t) + 0.5e^{-t}e^{-jt}u(t) \\
&\quad + je^{-t}e^{-jt}u(t) \\
&= -e^{-t}u(t) + 0.5e^{-t}(e^{jt}+e^{-jt})u(t) - je^{-t}(e^{jt}-e^{-jt})u(t) \\
&= -e^{-t}u(t) + e^{-t}\cos t\,u(t) + 2e^{-t}\sin t\,u(t) \\
&= -e^{-t}u(t) + e^{-t}(\cos t + 2\sin t)u(t)
\end{aligned}
$$

$$\boxed{\begin{aligned} \frac{e^{jt}+e^{-jt}}{2} &= \cos t \\ e^{jt}-e^{-jt} &= 2j\sin t \end{aligned}}$$

(or)

Problem (Cont.)

Inverse of Bilateral Laplace transform

Example

Problems

Solved Problem 7.31 Find the inverse Laplace transform of

$$X(s) = \frac{2}{(s+4)(s-1)} \quad \text{if the region of convergence is}$$

(b) $\mathrm{Re}(s) > 1$
(c) $\mathrm{Re}(s) < -4$

Solution:

Given

$$X(s) = \frac{2}{(s+4)(s-1)}$$

$$= \frac{A}{s+4} + \frac{B}{s-1}$$

$$A = \cancel{(s+4)} \frac{2}{\cancel{(s+4)}(s-1)}\Bigg|_{s=-4} = \frac{-2}{5}$$

$$B = \cancel{(s-1)} \frac{2}{(s+4)\cancel{(s-1)}}\Bigg|_{s=1} = \frac{2}{5}$$

$$\Rightarrow X(s) = \frac{-2}{5}\frac{1}{s+4} + \frac{2}{5(s-1)}$$

Problems (Cont.)

(a) The $X(s)$ has poles at -4 and 1. The strip of ROC is $-4 < \text{Re}(s) < 1$ as shown in Fig. 7.14. The pole at -4, which is at the left of the strip of ROC, corresponds to the causal signal and the pole at 1 to the right of the strip of ROC corresponds to anticausal signal. Therefore

$$x(t) = \frac{-2}{5}e^{-4t}u(t) - \frac{2}{5}e^{t}u(-t)$$

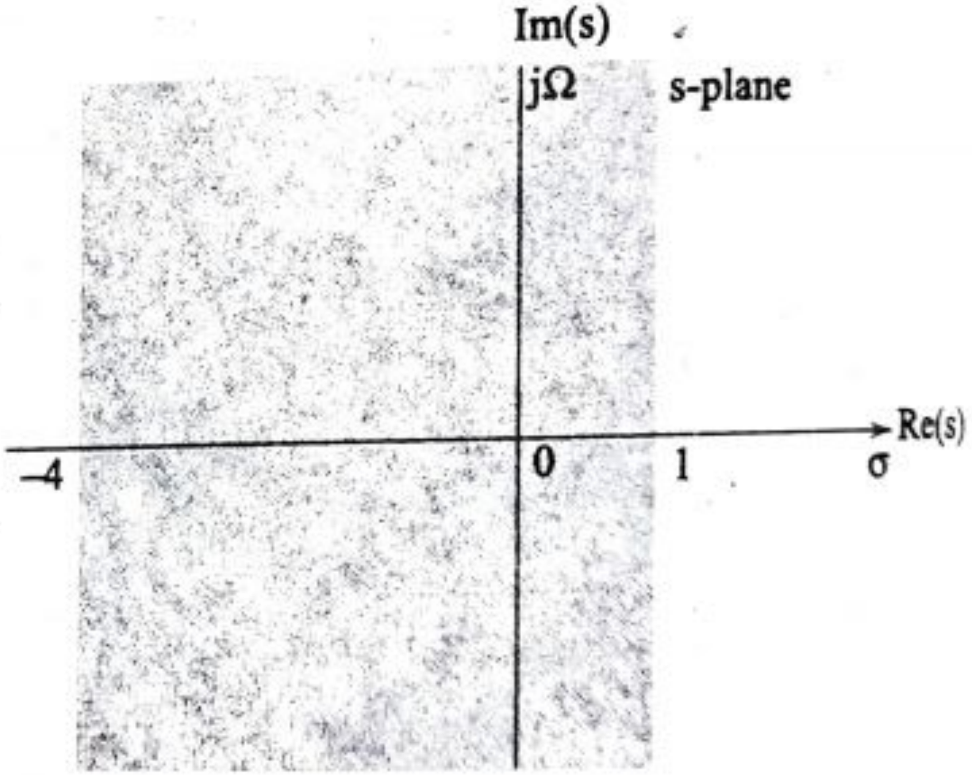


Fig. 7.14

Problems (Cont.)

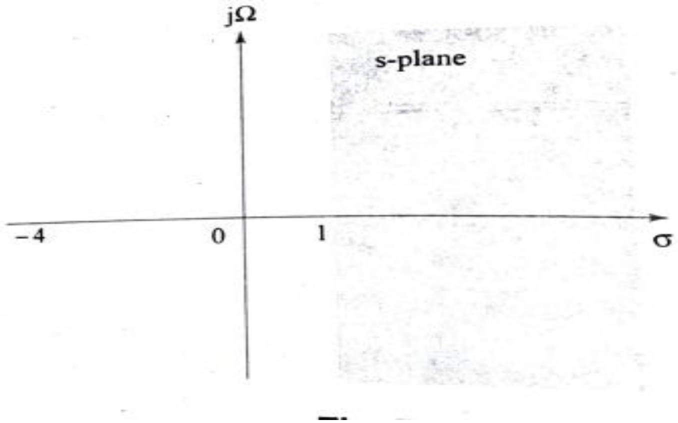
(b) The ROC is $\mathrm{Re}(s) > 1$.

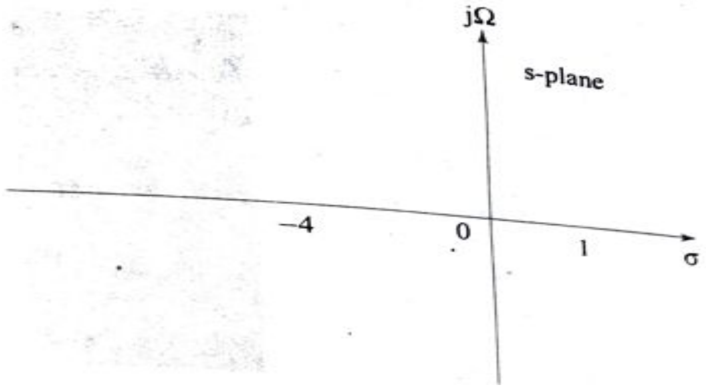
Both poles lies to the left of the ROC, so both poles correspond to causal signal. Therefore

$$x(t) = \frac{-2}{5}e^{-4t}u(t) + \frac{2}{5}e^{t}u(t)$$

(c) The ROC is $\mathrm{Re}(s) < -4$

Both poles lie to the right $x(t) = \frac{2}{5}e^{-4t}u(-t) - \frac{-2}{5}e^{t}u(-t)$ o anti-causal signals. Therefore





Problems (Cont.)

2Q. Find the signal whose bilateral transform is

$$X(s) = \frac{1}{(s+5)(s+1)} \qquad -5 < \mathrm{Re}(s) < -1$$

Solution:

$$X(s) = \frac{1}{(s+5)(s+1)}$$
$$= \frac{A}{s+5} + \frac{B}{s+1}$$
$$A = (s+5)\frac{1}{(s+5)(s+1)}\bigg|_{s=-5} = \frac{-1}{4}$$
$$B = (s+1)\frac{1}{(s+5)(s+1)}\bigg|_{s=-1} = \frac{1}{4}$$
$$X(s) = \frac{-1}{4}\cdot\frac{1}{s+5} + \frac{1}{4}\cdot\frac{1}{s+1}$$

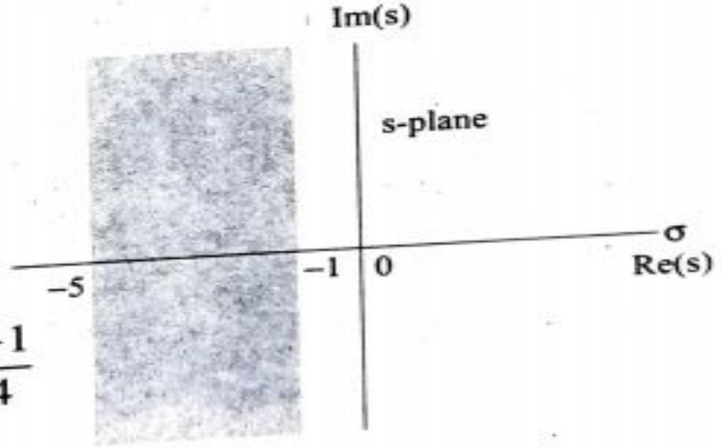


Fig. 7.17

ROC is $-5 < \mathrm{Re}(s) < -1$ shown in Fig. 7.17

The pole -5 is left to the region of convergence so this correspond to causal signal and the pole -1 is right of the ROC so this corresponds to anticausal signal.

$$x(t) = \frac{-1}{4}e^{-5t}u(t) - \frac{1}{4}e^{-t}u(-t)$$

Unit-2
Last Part
(System analysis with Fourier Transform)

Response of the System

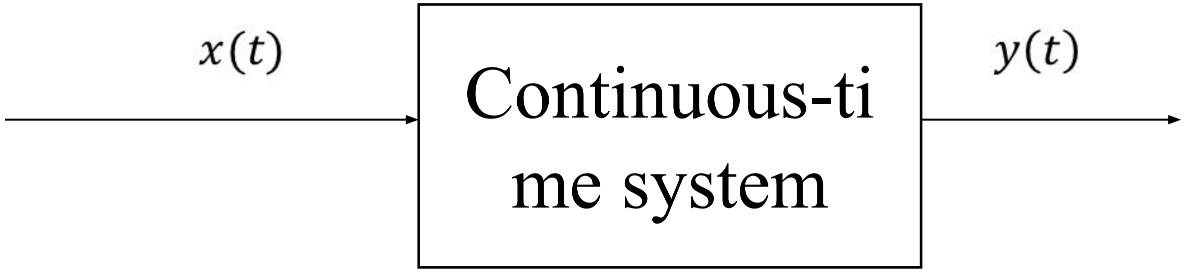
- According to the convolution property of the Fourier transform

<u>12. Convolution</u> $x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

If $F[x(t)] = X(j\Omega)$, $F[h(t)] = H(j\Omega)$,

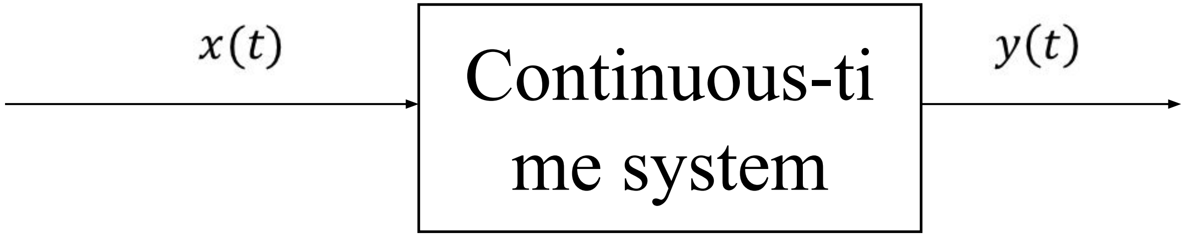
Then $F[x(t) * h(t)] = X(j\Omega)\ H(j\Omega)$

Let $x(t)$= *input of the system,* $h(t)$= *impulse response of the system or transfer function of the system,* $y(t)$ = *output of the system*



Response of the System

Fourier transform in LTI system analysis is used if, initial condition is zero or not given



- From the figure we can see that, $y(t) = x(t) * h(t)$

Applying Fourier transform both sides

$F[y(t)] = F[x(t) * h(t)]$ As we know, $F[x(t) * h(t)] = X(j\Omega)\, H(j\Omega)$

So, $Y(j\Omega) = X(j\Omega)\, H(j\Omega)$

So Fourier transform of impulse response of the system $= H(j\Omega) = \dfrac{Y(j\Omega)}{X(j\Omega)}$

Method to solve the system analysis using Fourier Transform

- If in a differential the relationship between input and output is mentioned **then, calculate the Fourier transform of transfer function or impulse response of the system.**
- *Fourier transform of impulse response of the system*$=H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)}$
- If question is asked for impulse response of the system $(h(t))$, **then compute inverse Fourier transform from the $\boldsymbol{H(j\Omega)}$.**
- If question is asked for output (response) of the system for given input $x(t)$. Then, compute $X(j\Omega)$. After this, compute $Y(j\Omega)$
- $Y(j\Omega) = X(j\Omega)\, H(j\Omega)$
- Then compute inverse Fourier transform of $Y(j\Omega)$, which is output or response of the system $(y(t))$.

Property of Fourier transform related to differential equation

<u>7. Differentiation in time:-</u> If $F[x(t)] = X(j\Omega)$,

Then $F\left[\frac{d}{dt}x(t)\right] = j\Omega X(j\Omega)$,

Similarly, $F\left[\frac{d^n}{dt^n}x(t)\right] = (j\Omega)^n X(j\Omega)$

System analysis with Fourier transform

Consider an LTI system described by constant - coefficient differential equation of the form

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \qquad (6.96)$$

Taking, Fourier transform both sides

$$F\left[\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right] = F\left[\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right] \qquad (6.101)$$

Using the linear property Eq. (6.101) can be written as

$$\sum_{k=0}^{N} a_k F\left[\frac{d^k y(t)}{dt^k}\right] = \sum_{k=0}^{M} b_k F\left[\frac{d^k x(t)}{dt^k}\right] \tag{6.102}$$

Using differentiation property we can write

$$\sum_{k=0}^{N} a_k (j\Omega)^k Y(j\Omega) = \sum_{k=0}^{M} b_k (j\Omega)^k X(j\Omega)$$

$$\boxed{F\left[\frac{d^k x(t)}{dt}\right] = (j\Omega)^k X(j\Omega)}$$

$$Y(j\Omega)\left[\sum_{k=0}^{N} a_k (j\Omega)^k\right] = X(j\Omega)\left[\sum_{k=0}^{M} b_k (j\Omega)^k\right]$$

Thus the system transfer function

$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)} = \frac{\sum_{k=0}^{M} b_k (j\Omega)^k}{\sum_{k=0}^{N} a_k (j\Omega)^k} \tag{6.103}$$

Solved Problem 6.29 The input and the output of a causal LTI system are related by the differential equation

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)$$

(a) Find the impulse response of the system.
(b) What is the response of this system if $x(t) = te^{-2t}u(t)$.

Solution:

Given

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)$$

Taking Fourier transform on both sides we get

$$(j\Omega)^2 Y(j\Omega) + 6j\Omega\, Y(j\Omega) + 8Y(j\Omega) = 2X(j\Omega)$$

$$Y(j\Omega)\left[(j\Omega)^2 + 6j\Omega + 8\right] = 2X(j\Omega)$$

$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)} = \frac{2}{(j\Omega)^2 + 6j\Omega + 8}$$

The impulse response is inverse Fourier transform of $H(j\Omega)$

$$h(t) = F^{-1}\left[H(j\Omega)\right]$$

$$H(j\Omega) = \frac{2}{(j\Omega)^2 + 6j\Omega + 8}$$

$$= \frac{2}{(j\Omega+4)(j\Omega+2)}$$

$$\boxed{\begin{aligned} A+B&=0 \\ 2A+4B&=2 \\ \Rightarrow \quad B&=1 \\ A&=-1 \end{aligned}}$$

$$= \frac{A}{(j\Omega+4)} + \frac{B}{j\Omega+2}$$

$$= \frac{-1}{(j\Omega+4)} + \frac{1}{j\Omega+2}$$

$$c_i = (s-p_i)\mathrm{X(s)}|_{s=p_i}$$

$$h(t) = F^{-1}\left[\frac{-1}{j\Omega+4}\right] + F^{-1}\left[\frac{1}{j\Omega+2}\right]$$

$$\Rightarrow \quad h(t) = -e^{-4t}u(t) + e^{-2t}u(t) \qquad \boxed{F\left[e^{-at}u(t)\right] = \frac{1}{a+j\Omega}}$$

If $x(t) = t\,e^{-2t}u(t)$

$$X(j\Omega) = \frac{1}{(2+j\Omega)^2} \qquad \boxed{F\left[te^{-at}u(t)\right] = \frac{1}{(a+j\Omega)^2}}$$

$$Y(j\Omega) = \frac{1}{(2+j\Omega)^2}\frac{2}{(j\Omega+4)(j\Omega+2)}$$

$$= \frac{2}{(j\Omega+2)^3(j\Omega+4)}$$

$$= \frac{A}{(j\Omega+2)} + \frac{B}{(j\Omega+2)^2} + \frac{C}{(j\Omega+2)^3} + \frac{D}{(j\Omega+4)}$$

$$= \frac{1}{4(j\Omega+2)} - \frac{1}{2(j\Omega+2)^2} + \frac{1}{(j\Omega+2)^3} - \frac{1}{4(j\Omega+4)}$$

$$y(t) = \frac{1}{4}e^{-2t}u(t) - \frac{1}{2}t\,e^{-2t}u(t) + \frac{t^2}{2}e^{-2t}u(t) - \frac{1}{4}\,e^{-4t}u(t)$$

Solved Problem 6.30 Find the frequency response of an LTI system described by the difference equation

$$\frac{d^2y(t)}{dt^2}+5\frac{dy(t)}{dt}+6y(t)=2x(t)$$

Solution:

Given

$$\frac{d^2y(t)}{dt^2}+5\frac{dy(t)}{dt}+6y(t)=2x(t)$$

Apply Fourier transform on both sides

$$(j\Omega)^2Y(j\Omega)+5j\Omega Y(j\Omega)+6Y(j\Omega)=2X(j\Omega)$$

$$Y(j\Omega)\left[(j\Omega)^2+5j\Omega+6\right]=2X(j\Omega)$$

The frequency response

$$H(j\Omega)=\frac{Y(j\Omega)}{X(j\Omega)}=\frac{2}{(j\Omega)^2+5j\Omega+6}$$

Solved Problem 6.37 Consider a causal LTI system with frequency response

AU APR. 2008

$$H(j\Omega) = \frac{1}{j\Omega + 3}$$

For a particular input $x(t)$ this system is observed to produce the output

$$y(t) = e^{-3t}u(t) - e^{-4t}u(t).$$

Determine $x(t)$.

Solution:

Given

$$H(j\Omega) = \frac{1}{j\Omega + 3}$$

$$y(t) = e^{-3t}\,u(t) - e^{-4t}\,u(t)$$

Apply Fourier transform on both sides to obtain

$$Y(j\Omega) = \frac{1}{j\Omega + 3} - \frac{1}{j\Omega + 4}$$

$$= \frac{1}{(j\Omega+3)(j\Omega+4)}$$

we know

$$Y(j\Omega) = H(j\Omega)\,X(j\Omega)$$

$$\Rightarrow \quad X(j\Omega) = \frac{Y(j\Omega)}{H(j\Omega)}$$

$$= \left[\frac{1}{(j\Omega+3)(j\Omega+4)}\right] \Big/ \left(\frac{1}{j\Omega+3}\right)$$

$$= \frac{1}{j\Omega+4}$$

Therefore the input

$$x(t) = e^{-4t}\,u(t)$$

Solved Problem 6.40 Consider the continuous - time LTI system with frequency response

$$H(j\Omega) = \frac{a - j\Omega}{a + j\Omega}$$

where $a > 0$. What is the magnitude of $H(j\Omega)$? What is $\underline{/H(j\Omega)}$? What is the impulse response of this system?

$$H(j\Omega) = \frac{a - j\Omega}{a + j\Omega}$$

$$|H(j\Omega)| = \frac{|a - j\Omega|}{|a + j\Omega|} = \frac{\sqrt{a^2 + \Omega^2}}{\sqrt{a^2 + \Omega^2}} = 1$$

$$H(j\Omega) = \frac{(a - j\Omega)}{(a + j\Omega)} \frac{(a - j\Omega)}{(a - j\Omega)}$$

$$= \frac{(a - j\Omega)^2}{a^2 + \Omega^2} = \frac{a^2 - \Omega^2 - j2a\Omega}{a^2 + \Omega^2}$$

$$= \frac{a^2 - \Omega^2}{a^2 + \Omega^2} - j \quad \frac{2a\Omega}{a^2 + \Omega^2}$$

$$\angle H(j\Omega) = \tan^{-1}\left[\frac{-2a\Omega}{a^2 - \Omega^2}\right]$$

$$H(j\Omega) = \frac{a - j\Omega}{a + j\Omega}$$

$$= \frac{2a}{a + j\Omega} - 1$$

$$\Rightarrow h(t) = 2a\, e^{-at} u(t) - \delta(t)$$

Unit-3
Last Part
(Analysis of LTI system using Laplace transform)

Response of the System

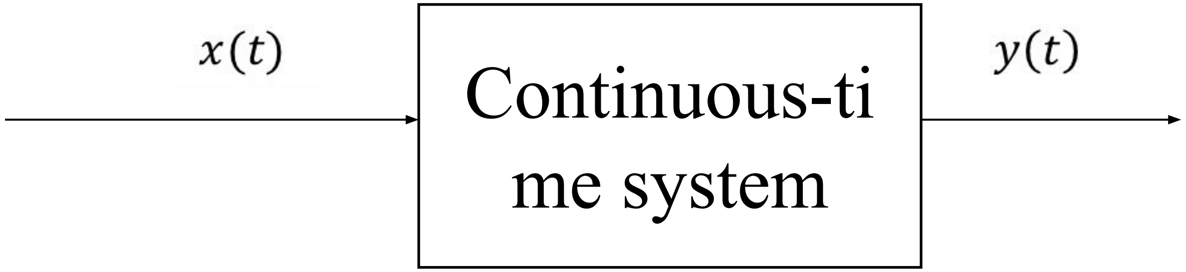
- According to the convolution property of the Laplace transform

<u>12. Convolution</u> $x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

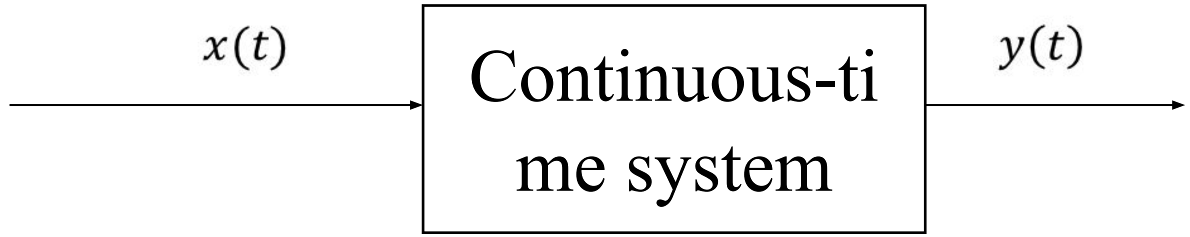
If $\mathrm{L}[x(t)] = X(s)$, $L[h(t)] = H(s)$,

Then $L[x(t) * h(t)] = X(s)\, H(s)$

Let $x(t)$= input of the system, $h(t)$= impulse response of the system or transfer function of the system, $y(t)$ *= output of the system*



Response of the System



- From the figure we can see that, y(t) $= x(t) * h(t)$

Applying Laplace transform both sides

L[y(t)] $= L[x(t) * h(t)]$ As we know, $L[x(t) * h(t)] = X(s)\ H(s)$

So, Y$(s) = X(s)\ H(s)$

So Fourier transform of impulse response of the system$=H(s) = \dfrac{Y(s)}{X(s)}$

Method to solve the system analysis using Laplace Transform

- If in a differential the relationship between input and output is mentioned **then, calculate the Laplace transform of transfer function or impulse response of the system.**
- *Laplace transform of impulse response of the system*$=H(s) = \frac{Y(s)}{X(s)}$
- Put the initial condition if it is given, for zero state response: initial condition will be zero. If initial condition is not given, consider it as 0.
- If question is asked for impulse response of the system $(h(t))$, **then compute inverse Laplace transform from the $H(s)$.**
- If question is asked for output (response) of the system for given input $x(t)$. Then, compute $X(s)$. After this, compute $Y(s)$
- $\mathrm{Y}(s) = X(s)\,H(s)$
- Then compute inverse Laplace transform of $\mathrm{Y}(s)$, which is output or response of the system $(y(t))$.

Properties of Laplace used for analysis of LTI system

8.Transform of derivatives:- If $L[x(t)] = X(s)$,

Then $L\left[\frac{d}{dt}x(t)\right] = SX(S)$ - $x(0^-)$

In the above expression the quantity $\frac{dx}{dt}(0^-)$ is the derivative of $x(t)$ evaluated at $t = 0^-$.
Similarly

$$L\left[\frac{d^n x(t)}{dt^n}\right] = s^n X(s) - s^{n-1}x(0^-) - s^{n-2}\frac{dx}{dt}(0^-)\ldots - \frac{d^{n-1}x(0^-)}{dt^{n-1}}$$

Problem-1

Q. Using Laplace transform, solve the following differential equations

(i) $\dfrac{d^2y(t)}{dt^2} + 3\dfrac{dy(t)}{dt} + 2y(t) = \dfrac{d}{dt}x(t)$ if $y(0^-) = 2; \dfrac{dy(0^-)}{dt} = 1$ and $x(t) = e^{-t}u(t)$

Taking Laplace transform on both sides

$$\left[s^2Y(s) - sy(0^-) - \frac{dy}{dt}(0^-)\right] + 3[sY(s) - y(0^-)]$$

$$+2Y(s) = sX(s) - x(0^-)$$

$$y(0^-) = 2; \frac{dy(0^-)}{dt} = 1$$

$$(s^2Y(s) - 2s - 1) + 3[sY(s) - 2] + 2Y(s) = sX(s)$$

$$Y(s)[s^2 + 3s + 2] = 2s + 7 + s\underline{X(s)}$$

Problem-1 (Cont.)

Given $x(t) = e^{-t}u(t)$

$$X(s) = \frac{1}{s+1}$$

$$\Rightarrow \quad Y(s)[s^2+3s+2] = 2s+7+\frac{s}{s+1}$$

$$Y(s) = \frac{2s+7}{s^2+3s+2} + \frac{s}{(s+1)(s^2+3s+2)}$$

$$= \frac{(2s+7)(s+1)+s}{(s+1)(s^2+3s+2)}$$

$$= \frac{2s^2+10s+7}{(s+1)(s^2+3s+2)}$$

$$= \frac{2s^2+10s+7}{(s+1)^2(s+2)}$$

$$= \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2}$$

Problem-1 (Cont.)

$$A = \frac{1}{1!}\frac{d}{ds}\left[\cancel{(s+1)^2}\frac{2s^2+10s+7}{\cancel{(s+1)^2}(s+2)}\right]\Bigg|_{s=-1}$$

$$= \frac{(s+2)(4s+10)-(2s^2+10s+7)}{(s+2)^2}\Bigg|_{s=-1}$$

$$=7$$

$$B = (s+1)^2\frac{2s^2+10s+7}{(s+1)^2(s+2)}\Bigg|_{s=-1} = -1$$

$$C = (s+2)\frac{2s^2+10s+7}{(s+1)^2(s+2)}\Bigg|_{s=-2} = -5$$

$$Y(s) = \frac{7}{s+1} - \frac{1}{(s+1)^2} - \frac{5}{s+2}$$

$$\frac{d(u/v)}{dx} = \frac{\frac{Vd(u)}{dx} - \frac{ud(v)}{dx}}{v^2}$$

$$c_i = (s-p_i)\mathrm{X(s)}|_{s=p_i}$$

$$c_{lk} = (s-p_k)^l X(s)|_{s=p_k}$$

$$c_{ik} = \frac{1}{(l-i)!}\frac{d^{l-i}[(s-p_k)^l X(s)]}{ds^{l-i}}|_{s=p_k}$$

Taking inverse Laplace transform on both sides, we get

$$y(t) = 7e^{-t}u(t) - te^{-t}u(t) - 5e^{-2t}u(t)$$

Problem-2

2. Q. Using Laplace transform, solve the following differential equations

$$\frac{d^3y(t)}{dt^3}+\frac{7d^2y(t)}{dt}+\frac{16dy(t)}{dt}+12y(t)=x(t)\text{ if }\frac{dy(0^-)}{dt}=0;\frac{d^2y(0^-)}{dt^2}=0$$

$y(0^-)=0$ and $x(t)=\delta(t)$

(ii) $$\frac{d^3y(t)}{dt^3}+7\frac{d^2y(t)}{dt^2}+16\frac{dy(t)}{dt}+12y(t)=x(t)$$

Applying Laplace transform on both sides we get

$$\left[s^3Y(s)-s^2y(0^-)-s\frac{dy(0^-)}{dt}-\frac{d^2y(0^-)}{dt^2}\right]+7\left[s^2Y(s)-sy(0^-)-\frac{dy(0^-)}{dt}\right]$$
$$+16[sY(s)-y(0^-)]+12Y(s)=X(s) \tag{7.73}$$

Given

$$y(0^-)=0$$
$$\frac{dy(0^-)}{dt}=0 \quad\text{and}\quad \frac{d^2y(0^-)}{dt^2}=0$$

Problem-2 (Cont.)

$$c_i = (s - p_i)X(s)|_{s=p_i}$$

$$c_{lk} = (s - p_k)^l X(s)|_{s=p_k}$$

$$c_{ik} = \frac{1}{(l-i)!}\frac{d^{l-i}[(s-p_k)^l X(s)]}{ds^{l-i}}|_{s=p_k}$$

Substituting these values in Eq. (7.73), we get

$$s^3Y(s) + 7s^2Y(s) + 16sY(s) + 12Y(s) = X(s)$$

$$Y(s)[s^3 + 7s^2 + 16s + 12] = X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{1}{s^3 + 7s^2 + 16s + 12}$$

For $x(t) = \delta(t); X(s) = 1$. Therefore

$$\begin{aligned} Y(s) &= \frac{1}{s^3 + 7s^2 + 16s + 12} \\ &= \frac{1}{(s+3)(s+2)^2} \\ &= \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s+3} \end{aligned}$$

$$A = \frac{1}{1!}\frac{d}{ds}\left[(s+2)^2\frac{1}{(s+3)(s+2)^2}\right]\Bigg|_{s=-2}$$

$$\begin{array}{r|rrrr} -3 & 1 & 7 & 16 & 12 \\ & 0 & -3 & -12 & -12 \\ \hline & 1 & 4 & 4 & 0 \end{array}$$

$$\begin{aligned} \Rightarrow s^3 + 7s^2 + 6s + 12 &= (s+3)(s^2+4s+4) \\ &= (s+3)(s+2)^2 \end{aligned}$$

Problem-2 (Cont.)

$$= \frac{d}{ds}\left[\frac{1}{s+3}\right]\Bigg|_{s=-2}$$

$$= \frac{-1}{(s+3)^2}\Bigg|_{s=-2} = -1$$

$$B = (s+2)^2 \frac{1}{(s+3)(s+2)^2}\Bigg|_{s=-2} = 1$$

$$C = (s+3)\frac{1}{(s+3)(s+2)^2}\Bigg|_{s=-3} = 1$$

$$Y(s) = \frac{-1}{s+2} + \frac{1}{(s+2)^2} + \frac{1}{s+3}$$

$$\Rightarrow \quad y(t) = -e^{-2t}u(t) + te^{-2t}u(t) + e^{-3t}u(t)$$

Problem-3

3. Q. A system is described by the following differential equation

$$\frac{d^2y(t)}{dt^2}+7\frac{dy(t)}{dt}+12y(t)=x(t)$$

Determine the response of the system to a unit step applied at $t=0$. The initial conditions are $y(0^-)=-2;\quad \frac{dy}{dt}(0^-)=0$

Solution:

$$\frac{d^2y(t)}{dt^2}+7\frac{dy(t)}{dt}+12y(t)=x(t)$$

Applying Laplace-transform on both sides, we have

$$\left[s^2Y(s)-sy(0^-)-\frac{dy}{dt}(0^-)\right]+7[sY(s)-y(0^-)]+12Y(s)=X(s)$$

Problem-3 (Cont.)

Substituting $y(0^-) = -2$ and $\frac{dy}{dt}(0^-) = 0$, we obtain

$$s^2Y(s) + 2s + 7sY(s) + 14 + 12Y(s) = X(s)$$

$$(s^2 + 7s + 12)Y(s) + 2s + 14 = X(s)$$

$$t) = 7e^{-t}u(t) - te^{-t}u(t) - 5e^{-2t}u(t)$$

a) For a unit step input

$$x(t) = u(t)$$

$$X(s) = \frac{1}{s}$$

$$\Rightarrow Y(s) = \frac{1}{s(s^2 + 7s + 12)} - \frac{(2s + 14)}{s^2 + 7s + 12}$$

$$= \frac{1 - 2s^2 - 14s}{s(s+3)(s+4)}$$

$$= \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+4}$$

$$= \frac{1}{12s} - \frac{25}{3(s+3)} + \frac{25}{4(s+4)}$$

$$A = s\frac{1 - 2s^2 - 14s}{s(s+3)(s+4)}\bigg|_{s=0} = \frac{1}{12}$$

$$B = (s+3)\frac{(1 - 2s^2 - 14s)}{s(s+3)(s+4)}\bigg|_{s=-3} = \frac{1 - 18 + 42}{(-3)(1)} = -\frac{25}{3}$$

$$C = \frac{(s+4)(1 - 2s^2 - 14s)}{s(s+3)(s+4)}\bigg|_{s=-4} = \frac{1 - 32 + 56}{(-4)(-1)} = \frac{25}{4}$$

$$= \frac{1}{12}u(t) - \frac{25}{3}e^{-3t}u(t) + \frac{25}{4}e^{-4t}u(t)$$

Problem-4

4. Q. For a system with transfer function

$$H(s) = \frac{s+5}{s^2+5s+6}$$

find the zero-state response if the input $x(t)$ is $e^{-3t}u(t)$

Solution: Given

$$H(s) = \frac{s+5}{s^2+5s+6}$$

$$\frac{Y(s)}{X(s)} = \frac{s+5}{(s+2)(s+3)}$$

Problem-4 (Cont.)

For

$$x(t) = e^{-3t}u(t) \quad X(s) = \frac{1}{s+3}.$$

$$\Rightarrow \quad Y(s) = \frac{s+5}{(s+2)(s+3)^2}$$

$$= \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{(s+3)^2}$$

$$A = (s+2)\frac{s+5}{(s+2)(s+3)^2}\bigg|_{s=-2} = \frac{-2+5}{(-2+3)^2} = 3$$

$$B = \frac{1}{1!}\frac{d}{ds}\left[(s+3)^2\frac{s+5}{(s+2)(s+3)^2}\right]\bigg|_{s=-3}$$

$$= \frac{(s+2)-(s+5)}{(s+2)^2}\bigg|_{s=-3}$$

$$= \frac{-3}{(-3+2)^2} = -3$$

Problem-4 (Cont.)

$$L[e^{-at}u(t)] = \frac{1}{s+a}$$

$$C = (s+3)^2 \frac{s+5}{(s+2)(s+3)^2}\bigg|_{s=-3} = -2$$

$$Y(s) = \frac{3}{s+2} - \frac{3}{s+3} - \frac{2}{(s+3)^2}$$

Taking inverse Laplace transform, we have

$$y(t) = 3e^{-2t}u(t) - 3e^{-3t}u(t) - 2te^{-3t}u(t)$$

Problem 5

5.Q. Find the impulse and the step response of the following systems.

$$H(s) = \frac{10}{s^2 + 6s + 10}$$

$$H(s) = \frac{10}{s^2 + 6s + 10}$$

$$\frac{Y(s)}{X(s)} = \frac{10}{s^2 + 6s + 10}$$

For an impulse $x(t) = \delta(t); X(s) = 1$

$$L[e^{-at} \sin(\Omega_0 t)u(t)] = \frac{\Omega_0}{(s+a)^2 + {\Omega_0}^2}$$

$$\Rightarrow \quad Y(s) = \frac{10}{s^2 + 6s + 10}$$

$$= \frac{10}{(s+3)^2 + 1^2}$$

$$L[e^{-at} \cos(\Omega_0 t)u(t)] = \frac{(s+a)}{(s+a)^2 + {\Omega_0}^2}$$

Taking inverse Laplace transform we get

$$y(t) = 10e^{-3t} \sin t\, u(t)$$

Problem 5 (cont.)

(ii) For a unit step input

$$x(t) = u(t)$$

$$X(s) = \frac{1}{s}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{10}{s^2 + 6s + 10}$$

$$Y(s) = \frac{10}{s(s^2 + 6s + 10)}$$

$$= \frac{A}{s} + \frac{Bs + C}{s^2 + 6s + 10}$$

$$\Rightarrow \quad \frac{10}{s(s^2 + 6s + 10)} = \frac{A(s^2 + 6s + 10) + s(Bs + C)}{s(s^2 + 6s + 10)}$$

$$(A + B)s^2 + (6A + C)s + 10A = 10$$

Comparing the coefficient of s^2, s and constant we get

$$A + B = 0 \; ; \; 6A + C = 0$$

$$10A = 10$$

$$A = 1$$

Problem 5 (cont.)

$$B = -1$$

$$C = -6$$

$$L[e^{-at}\sin(\Omega_0 t)u(t)] = \frac{\Omega_0}{(s+a)^2 + {\Omega_0}^2}$$

$$L[e^{-at}\cos(\Omega_0 t)u(t)] = \frac{(s+a)}{(s+a)^2 + {\Omega_0}^2}$$

$$\begin{aligned} Y(s) &= \frac{1}{s} + \frac{-s-6}{s^2+6s+10} \\ &= \frac{1}{s} - \frac{(s+6)}{s^2+6s+10} \\ &= \frac{1}{s} - \frac{s+6}{(s+3)^2+1} \\ &= \frac{1}{s} - \left\{ \frac{s+3}{(s+3)^2+1} + \frac{3}{(s+3)^2+1} \right\} \end{aligned}$$

Taking inverse Laplace transform we get

$$\begin{aligned} y(t) &= u(t) - \left\{ e^{-3t}\cos t\,u(t) + 3e^{-3t}\sin t\,u(t) \right\} \\ &= \left[1 - e^{-3t}\{\cos t + 3\sin t\}\right] u(t) \end{aligned}$$

Problem 6

6 Q. Find the impulse and the step response of the following systems.

$$H(s) = \frac{s+2}{s^2+5s+4}$$

$$\frac{Y(s)}{X(s)} = \frac{s+2}{(s+1)(s+4)}$$

For an impulse $x(t) = \delta(t)$
$X(s) = 1$

$$\Rightarrow \quad Y(s) = \frac{s+2}{(s+1)(s+4)}$$

$$= \frac{A}{s+1} + \frac{B}{s+4}$$

$$A = (s+1)\left.\frac{s+2}{(s+1)(s+4)}\right|_{s=-1} = \frac{1}{3}$$

$$B = (s+4)\left.\frac{(s+2)}{(s+1)(s+4)}\right|_{s=-4} = \frac{2}{3}$$

$$Y(s) = \frac{1}{3(s+1)} + \frac{2}{3(s+4)}$$

Problem 6 (Cont.)

Taking inverse Laplace transform, we have

$$y(t) = \frac{1}{3}e^{-t}u(t) + \frac{2}{3}e^{-4t}u(t)$$

For a unit stepinput

$$x(t) = u(t)$$

$$X(s) = \frac{1}{s}$$

Then

$$Y(s) = \frac{s+2}{s(s+1)(s+4)}$$

$$= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$A = s\,\frac{s+2}{s(s+1)(s+4)}\bigg|_{s=0} = \frac{1}{2}$$

$$B = (s+1)\frac{s+2}{s(s+1)(s+4)}\bigg|_{s=-1} = \frac{-1}{3}$$

$$C = (s+4)\,\frac{s+2}{s(s+1)(s+4)}\bigg|_{s=-4} = \frac{-1}{6}$$

$$Y(s) = \frac{1}{2s} - \frac{1}{3(s+1)} - \frac{1}{6(s+4)}$$

Taking inverse Laplace transform

$$y(t) = \frac{1}{2}u(t) - \frac{1}{3}e^{-t}u(t) - \frac{1}{6}e^{-4t}u(t)$$

Problem 6 (Cont.)

Solution:

Given

$$X(s) = \frac{2s^2 + 9s - 47}{(s+1)(s^2 + 6s + 25)}$$

$$\frac{2s^2 + 9s - 47}{(s+1)(s^2 + 6s + 25)} = \frac{A}{s+1} + \frac{Bs + C}{s^2 + 6s + 25}$$

$$= \frac{A(s^2 + 6s + 25) + (Bs + C)(s+1)}{(s+1)(s^2 + 6s + 25)}$$

$$= \frac{As^2 + 6As + 25A + Bs^2 + Cs + Bs + C}{(s+1)(s^2 + 6s + 25)}$$

$$= \frac{(A+B)s^2 + (6A + B + C)s + 25A + C}{(s+1)(s^2 + 6s + 25)}$$

Problem 6 (Cont.)

Comparing L.H.S and R.H.S. we get

$$A+B=2$$
$$6A+B+C=9$$
$$25A+C=-47$$

Solving for A, B and C we get

$$A=\frac{-27}{10}; \quad B=\frac{47}{10}; \quad C=\frac{41}{2}$$

$$X(s)=\frac{-27}{10(s+1)}+\frac{\frac{47}{10}s+\frac{41}{2}}{s^2+6s+25}$$
$$=\frac{-27}{10}\frac{1}{s+1}+\frac{47}{10}\left[\frac{s+\frac{205}{47}}{(s+3)^2+4^2}\right]$$
$$=\frac{-27}{10}\cdot\frac{1}{s+1}+\frac{47}{10}\left[\frac{s+3}{(s+3)^2+4^2}+\frac{16}{47}\cdot\frac{4}{(s+3)^2+4^2}\right]$$

Problem 6 (Cont.)

Taking inverse Laplace transform on both sides we get

$$x(t) = \frac{-27}{10}e^{-t} + \frac{47}{10}e^{-3t}\cos 4t + \frac{8}{5}e^{-3t}\sin 4t$$

$$= \frac{-27}{10}e^{-t} + e^{-3t}\left[\frac{47}{10}\cos 4t + \frac{8}{5}\sin 4t\right]$$