Solution of Partial differential Equations

A solution (or) integral of a partial differential equation is a relation between the independent and depent variables which satisfies the given partial differential equation.

Complete solution (or) Integral:

A solution in which the number of arbitrary Constants is equal to the number of independent variables.

~~Particular~~ Particular Integral (or) solution:

In Complete integral if we give particular values to the arbitrary Constants.

Singular integral (or) solution:

Singular solution does not contain any arbitrary Constants and arbitrary functions.

General solution (or) Integral:

A solution of a partial differential equation which contains as many arbitrary functions as the order of the equation is called the General solution.

First order non-linear PDE of standard types:

1. Type I : $F(p,q)=0$
2. Type II - $z=px+qy+f(p,q)$ (Clairaut's form)
3. Type III : $F(z,p,q)=0$
   $F(x,p,q)=0$
   $F(y,p,q)=0$
4. Type IV : $F(x,y,p,q)=0$ [Separable equations].

Type-I $F(p,q)=0$

Working rule:

Given $F(p,q)=0$ —— ①

$\Rightarrow$ Consider $z=ax+by+c$ be the soln of ① —— ②

$\Rightarrow$ If we diff eqn ② p. D. w.r to x & y
we get $p=a$, $q=b$

$\Rightarrow$ $F(a,b)=0$, solve this for b, we get $b=f(a)$

$$\boxed{z=ax+f(a)y+c} \longrightarrow \text{Complete solution} \quad —— ③$$

To find singular soln:

eqn ③ P. D. w. r to 'c' we get

$\boxed{0\neq 1}$ $\therefore$ there is no singular soln.

General soln:

put $c=\phi(a)$ in eqn ③

and p. Diff w.r to 'a' we get eqn ④

Eliminating a from eqn ③ & ④ we get general solution.

1. Solve $\sqrt{p}+\sqrt{q}=1$ — (1) [Form $F(p,q)=0$ type - I]

Soln

Assume that $z=ax+by+c$ — (2) be the Comple solution (or) integral of eqn (1).

where.

$$\sqrt{a}+\sqrt{b}=1$$

$$\sqrt{b}=1-\sqrt{a}$$

$$b=[1-\sqrt{a}]^2$$

$$z=ax+(1-\sqrt{a})^2y+c$$ be the complete soln: — (3)

Singular soln:

If eqn (3) w.r to 'c' means

$0\neq 1$

$\therefore$ no singular solution

general solution:

put $c=\phi(a)$ in eqn (3)

$$z=ax+(1-\sqrt{a})^2y+\phi(a) \quad — (4)$$

eqn (4) P.D. w.r.to 'a'

$$0=x+2(1-\sqrt{a})\left(\frac{-1}{2\sqrt{a}}\right)y+\phi'(a) \quad — (5)$$

Eliminating a from eqn (4) & (5) we get general solution.

② **Example:** Solve $p^2+q^2 = npq$ —— ①

Let us assume $z = ax+by+c$ be the soln

$$a^2+b^2-nab = 0$$

$A = 1$
$B = -na$
$C = b^2$

solving for 'b'

$$b = \frac{na \pm \sqrt{a^2n^2-4a^2}}{2}$$

$$b = \frac{a}{2}\left[n \pm \sqrt{n^2-4}\right]$$

Hence the complete solution is

$$z = ax + \frac{a}{2}\left[n \pm \sqrt{n^2-4}\right]y + c \quad \text{—— ②.}$$

singular: eqn ② P.D w.r to 'c'

$0 \neq 1$ no singular solution.

general soln: put $c = \phi(a)$ in eqn ②

$$z = ax + \frac{a}{2}\left[n \pm \sqrt{n^2-4}\right]y + \phi(a) \quad \text{—— ③}$$

eqn ③ P.D. w.r to 'a'

$$0 = x + \frac{\left[n \pm \sqrt{n^2-4}\right]y}{2} + \phi'(a) \quad \text{—— ④}$$

Eliminating a from ③ & ④ we get general solution.

Practice problems:

① Solve $pq = 1$

② $p^2 + q^2 = m^2$

Type -2 $z = px + qy + f(p,q)$ —— ①

[Clairauts' Form]

Procedure:

⇒ Complete integral: put $p = a$ & $q = b$

$$z = ax + by + f(a,b) \quad \text{—— ②}$$

⇒ singular soln.
eqn no ② P.D. w. r to a, & b then try to eliminate a, b from the eqn.

⇒ general soln put = ~~b~~ $b = \phi(a)$ in ② and Diff P. D. w. r to 'a'. eliminate 'a' we get general soln.

Example 1 Solve: $z = px + qy + \sqrt{1+p^2+q^2}$ —— ①

∴ The Complete solution ~~are~~ $(p = a,\ q = b)$

$$z = ax + by + \sqrt{1+a^2+b^2} \quad \text{—— ②}$$

Singular soln:
eqn no ② P D. w. r to 'a'

$$0 = x + \frac{1}{\cancel{2}\sqrt{1+a^2+b^2}} \cancel{2}a$$

$$\frac{a}{\sqrt{1+a^2+b^2}} = -x. \quad \text{—— ③}$$

eqn 2 P.D w.r to 'b'

$$0 = y + \frac{1}{\cancel{2}\sqrt{1+a^2+b^2}} \cancel{2} b$$

$$\frac{b}{\sqrt{1+a^2+b^2}} = -y \quad \text{———} \quad (4)$$

now calculate $1-(-x)^2 - (-y)^2$

$$= 1 - \frac{a^2}{1+a^2+b^2} - \frac{b^2}{1+a^2+b^2}$$

$$1-x^2-y^2 = \frac{1+\cancel{a^2}+\cancel{b^2}-\cancel{a^2}-\cancel{b^2}}{1+a^2+b^2} = \frac{1}{1+a^2+b^2}$$

$$\sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}}$$

eqn 3 & 4 becomes

$$x = -a\sqrt{1-x^2-y^2}$$

$$a = -\frac{x}{\sqrt{1-x^2-y^2}}$$

$$y = -b\sqrt{1-x^2-y^2}$$

$$b = \frac{-y}{\sqrt{1-x^2-y^2}}$$

substitute in eqn 2

$$z = \frac{-x\,(x)}{\sqrt{1-x^2-y^2}} - y\frac{y}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$z = \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}} = \frac{\sqrt{1-x^2-y^2}\,\cancel{\sqrt{1-x^2-y^2}}}{\cancel{\sqrt{1-x^2-y^2}}}$$

squari both sid $\boxed{x^2+y^2+z^2=1}$ is singuler soln.

Scanned with CamScanner

general soln

put $b=\phi(a)$ in eqn ②

$$z = ax + \phi(a)y + \sqrt{1+a^2+\phi(a)^2} \quad —③$$

eqn ③ P.D. w.r to 'a'

$$0 = x + \phi'(a)y + \frac{1}{2\sqrt{1+a^2+[\phi(a)]^2}}\left[2a + 2\phi(a)\phi'(a)\right] \quad —④$$

Eliminating a from ③ & ④ we get general soln.

Example: $z = px + qy + p^2 - q^2$ —① put $\begin{bmatrix} p=a \\ q=b \end{bmatrix}$

The complete soln: $z = ax + by + a^2 - b^2$ —②

singular soln:

eqn ② P.D. w.r to 'a'

$$0 = x + 2a$$

$$a = -x/2 \quad —③$$

eqn ② P.D. w.r to 'b'

$$0 = y - 2b$$

$$\boxed{\frac{y}{2} = b} \quad —④$$

Substitute ③ & ④ in eqn ②

$$z = -\frac{x}{2}x + \frac{y}{2}y + \frac{x^2}{4} - \frac{y^2}{4}$$

$$z = \frac{x^2}{4} - \frac{x^2}{2} + \frac{y^2}{2} - \frac{y^2}{4}$$

$\boxed{4z = y^2 - x^2}$ is the singular solution.

general solution:

eqn ② put $b = \phi(a)$

$$z = ax + \phi(a)y + a^2 - [\phi(a)]^2 \quad —⑤$$

eqn ⑤ P.D. w.r to 'a'

$$0 = x + \phi'(a)y + 2a - 2[\phi(a)][\phi'(a)] \quad —⑥$$

Eliminating a from ⑤ & ⑥ we get general soln.

Practice problems:

① solve $z = px + qy + p^2q^2$

② $z = px + qy + 3pq$.

③ $z = px + qy + p^2 + q^2 + pq$.

Type 3:

case (i) $F(x, p, q) = 0$

Since z is function of x and y.

then $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

$$\boxed{dz = p\,dx + q\,dy} \quad — ①$$

Procedure:

Assume: $q = a$ (some constant)

$F(x, p, a) = 0$

Solving for p, we get $p = \phi(x, a)$

$\therefore dz = p\,dx + q\,dy$

$dz = \phi(x, a)\,dx + a\,dy$.

Integrating

$z = f(x, a) + ay + c$ — Complete soln. — ②

Singular solution:

eqn number ② P.D. w.r to c

$0 \neq 1$. so no singular soln.

General solution.

put $c = \phi(a)$ in eqn ②

$z = f(x, a) + ay + \phi(a)$ — ③

eqn ③ P.D w.r to 'a'

$0 = f'(x, a) + y + \phi'(a)$ — ④

Eliminating 'a' from ③ & ④ we get general solution.

1. solve $p = 2qx$. $[F(x, p, q) = 0]$

put $q = a$

$p = 2ax$.

w.k.T $dz = p\,dx + q\,dy$

$dz = 2ax\,dx + a\,dy$

Integrating

$z = ax^2 + ay + c$ — (1) — Complete solution.

Singular solution: eqn (1) P.D w.r to c.

$0 \neq 1$ so no singular solution.

general soln: put $c = \phi(a)$ in eqn (1)

$z = ax^2 + ay + \phi(a)$ — (2)

eqn (2) P.D w.r to 'a'

$0 = x^2 + y + \phi'(a)$ — (3)

Eliminating a from (2) & (3) we get general solution.

2. solve $q = px + p^2$ $[F(x, p, q) = 0]$

put $q = a$.

$p^2 + px - a = 0$ solving for p

$$p = -\frac{x \pm \sqrt{x^2+4a}}{2}$$

w.k.T $dz = p\,dx + q\,dy$

$$dz = \left(\frac{-x \pm \sqrt{x^2+4a}}{2}\right)dx + a\,dy$$

Integrating

$$z = \frac{-x^2}{4} \pm \frac{1}{2}\int \sqrt{x^2+4a}\,dx + ay + b$$

$$z = -\frac{x^2}{4} \pm \frac{1}{2}\left\{2a\sinh^{-1}\left(\frac{x}{2\sqrt{a}}\right) + \frac{x}{2}\sqrt{x^2+4a}\right\} + ay + b$$

Complete soln.

$\Rightarrow$ no singular solution.

$\Rightarrow$ general soln. [try yourself].

$q = a$

Practice problems:

① Solve $q = 2px$ Complete Soln: $z = \frac{a}{2}\log x + ay + c$.

② solve $\sqrt{p} + \sqrt{q} = \sqrt{x}$.

Type - 3 (ii)

$F(y, p, q) = 0$

Procedure: Assume $p = a$.

then $F(y, a, q) = 0$

solving for q, we obtain $q = \phi(y, a)$

$dz = a\,dx + \phi(y, a)\,dy$

$z = ax + f(y, a) + c$ — Complete solution. — ①

Singular solution: eqn ① P D w.r to c we get

$0 \neq 1$

so no singular soln.

General solution: put $c = \phi(a)$ in eqn ①

$z = ax + f(y, a) + \phi(a)$ — ②

eqn ② P.D. w.r 'a'

$0 = x + f'(y, a) + \phi'(a)$ — ③

Eliminating a from eqn ② and ③ we get general solution.

Example 1: solve: $pq = y$. $f(p, q, y) = 0$

Assume $p = a$ = constant

then $aq = y$ $q = \frac{y}{a}$

W.K.T $dz = p\,dx + q\,dy$

$dz = a\,dx + \frac{y}{a}\,dy$

Integrating $z = ax + \frac{y^2}{2a} + c$ — Complete solution — ①

<u>singular soln:</u>

eqn ① P.D. w.r to 'c'

$0 \neq 1$ So no singular solution.

<u>general Solution:</u>

put $c = \phi(a)$ in eqn ①

$$z = ax + \frac{y^2}{2a} + \phi(a) \quad — ②$$

eqn ② P.D. w.r to 'a'

$$0 = x + \frac{y^2}{2}\left(\frac{-1}{a^2}\right) + \phi'(a) \quad — ③$$

Eliminating 'a' from eqn ② & ③ we get general solution.

<u>Type 3 (iii)</u>

$$F(z, p, q) = 0.$$

As a trail solution, assume that z is a function of $u = x + ay$, where a is an arbitrary constant.

Now $z = f(u) = f(x+ay)$ —①

eqn ① P.D w.r to 'x'

$$p = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du} \cdot 1 \quad \therefore \boxed{p = \frac{dz}{du}}$$

eqn ① P.D w.r to 'y'

$$q = \frac{dz}{du}\frac{du}{dy} = \frac{dz}{du}(a) \qquad q = a\left(\frac{dz}{du}\right)$$

Substituting p&q value in $F(z, p, q) = 0$ we get.

$F\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0$ which is ordinary differential equation of first order. solving for $\frac{dz}{du}$. we obtain $\frac{dz}{du} = \phi(z, a)$ (say)

$$\frac{dz}{\phi(z,a)} = du \quad \text{integrating}$$

$f(z, a) = x + ay + c$ —② is complete integral.

<u>Singular integral:</u> eqn ② P D w.r to 'c' we get $0 \neq 1$ [So no singular solution]

<u>General Solution:</u> put $c = \phi(a)$ in eqn ② we get eqn ③

then eqn ③ P.D w.r to ③

Elimination 'a' from ② & ③ we get general solution.

1 Solve $q(p^2 z + q^2) = 4 \qquad F(z, p, q) = 0$

Assume $z = f(x + ay)$ and $u = x + ay$

then: $p = \dfrac{dz}{du}$

$$q = a\frac{dz}{du}.$$

$$a\left[z\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2\right] = 4$$

$$\left(\frac{dz}{du}\right)^2 = \frac{4}{9[z + a^2]}$$

$$\frac{dz}{du} = \frac{2}{3}\frac{1}{\sqrt{z + a^2}}$$

$$3\sqrt{z + a^2}\, dz = 2\, du$$

Integrating $\dfrac{3\,(z + a^2)^{3/2}}{3/2} = 2u + 2b$

$$(z + a^2)^{3/2} = x + ay + b$$

$(z + a^2)^{3/2} = (x + ay + b)$. This is Complete inte — (1)

<u>singular</u> <u>soln</u>: ~~put b =~~

eqn (1) P. D. w.r to 'b'

$0 = \cancel{\quad}(1)$

$\therefore$ so no singular solution.

<u>general</u> <u>solution</u>:

put $b = \phi(a)$

$$(z + a^2)^{3/2} = x + ay + \phi(a) \quad \text{— (2)}$$

eqn (2) P. D. w.r to 'a'

$$\tfrac{3}{2}(z + a^2)^{1/2}(2a) = y + \phi'(a) \quad \text{— (3)}$$

Eliminating 'a' from eqn (2) & (3) we get general solution.

2 **Solve:**

$$p(1+q) = qz.$$

Assume $z = f(u)$ where $u = x + ay$

$$p = \frac{dz}{du} \qquad q = a\frac{dz}{du}$$

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = a\frac{dz}{du}z$$

$$a\frac{dz}{du} = az - 1$$

$$\frac{a\,dz}{az-1} = du$$

Integrating $\log(az-1) = u + c.$

$$\log(az-1) = x + ay + c \longrightarrow \text{Complete integral} \longrightarrow ①$$

singular soln eqn ① P.D. w.r.to 'c' we get

$0 \neq 1$ so no singular solution.

General solution: put $c = \phi(a)$

$$\log(az-1) = x + ay + \phi(a) \quad —②$$

eqn ② P.D w.r to 'a'

$$\frac{1}{az-1}z = y + \phi'(a) \quad — ③$$

Eliminating a from eqn ② & ③ we get general solution.

Practice Problems:

① solve $z = p^2 + q^2$ Complete soln: $2\sqrt{z} = \frac{1}{\sqrt{1+a^2}}(x+ay) + b$

② solve $p(1-q^2) = q(1-z)$ " $2\sqrt{1-a+az} = x + ay + c$

Scanned with CamScanner

Type-IV $f_1(x,p) = f_2(y,q)$

Method of separable equations

Step 1: put $f_1(x,p) = f_2(y,q) = a$

step 2: write $p = g_1(x,a)$, and $q = g_2(y,a)$

step 3: putting $dz = pdx + qdy$ and integrating, we get the complete integral as

$$z = \int g_1(x,a)\,dx + \int g_2(y,a)\,dy + b. \quad \text{--- (1)}$$

Singular solution:

eqn number (1) P.d. w.r to 'b' we get $0 \neq 1$ so no singular solution.

General solution:

put $b = \phi(a)$ in eqn (1) we get eqn (2)

then eqn (2) P.D. w. r to 'a' we get eqn (3)

eliminating 'a' from eqn (2) & (3) we get general solution.

1. solve $p^2 y(1+x^2) = qx^2$.

$$\frac{p^2(1+x^2)}{x^2} = \frac{q}{y} = a \text{ (assume)}$$

Comparing.

$$p^2 = \frac{ax^2}{(1+x^2)} \qquad p = \frac{\sqrt{a}\,x}{\sqrt{1+x^2}}$$

$$q = ay$$

w. K.T: $dz = p\,dx + q\,dy$

$$dz = \frac{\sqrt{a}\,x}{\sqrt{1+x^2}}\,dx + ay\,dy$$

Integrating

$$z = \sqrt{a}\sqrt{1+x^2} + \frac{ay^2}{2} + b \quad \text{—— (1)}$$

∴ complete integral.

Singular integral:

eqn (1) P D w.r to 'b' we get

$0 \neq 1$. so no singular integr

General integral: put $b = \phi(a)$

$$z = \sqrt{a}\sqrt{1+x^2} + \frac{ay^2}{2} + \phi(a) \quad \text{—— (2)}$$

eqn (2) P.D w.r to 'a' we get

$$0 = \frac{1}{2\sqrt{a}}\sqrt{1+x^2} + \frac{y^2}{2} + \phi'(a) \quad \text{—— (3)}$$

eliminating 'a' from (2) & (3) we get general solution.

2) Solve $p^2+q^2 = x+y$

$$p^2 - x = y - q^2 = a \text{ (say)}$$

$$p^2 = a+x$$

$$p = \sqrt{a+x}$$

$$y = q^2 + a$$

$$y - a = q^2$$

$$q = \sqrt{y-a}$$

W.K.T $dz = p\,dx + q\,dy$

$$dz = \sqrt{a+x}\,dx + \sqrt{y-a}\,dy$$

$$z = \frac{(x+a)^{3/2}}{3/2} + \frac{(y-a)^{3/2}}{3/2} + b \quad \text{--- (1)}$$

This is Complete integral.

Singular solution:

eqn (1) P.D w.r.to 'b' we get

$0 \neq 1$ so no singular solution.

General solution: put $b = \phi(a)$ in eqn (1)

$$z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y-a)^{3/2} + \phi(a) \quad \text{--- (2)}$$

eqn (2) P.D w.r.to 'a'

$$0 = (x+a)^{1/2} + (y-a)^{1/2}(-1) + \phi'(a) \quad \text{--- (3)}$$

Eliminating a from (2) & (3) we get general solution.

Practice Problems:

(1) Solve $p^2+q^2 = x^2+y^2$ Complete soln: $z = \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2+x^2} + \frac{y}{2}\sqrt{y^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right) + b$.

(2) Solve $p - x^2 = q + y^2$ Soln: $z = ax + \frac{x^3}{3} + ay - \frac{y^3}{3} + b$

(3) $p+q = \sin x + \sin y$