change of order of integration

In double integral with limits, the change of order of integration changes the limits of integration.

1) Change the order of integration in the integral $\int_{-a}^{a} \int_{0}^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$

Sol:- Given $\int_{-a}^{a} \int_{0}^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$

clearly y is independent, x is dependent.

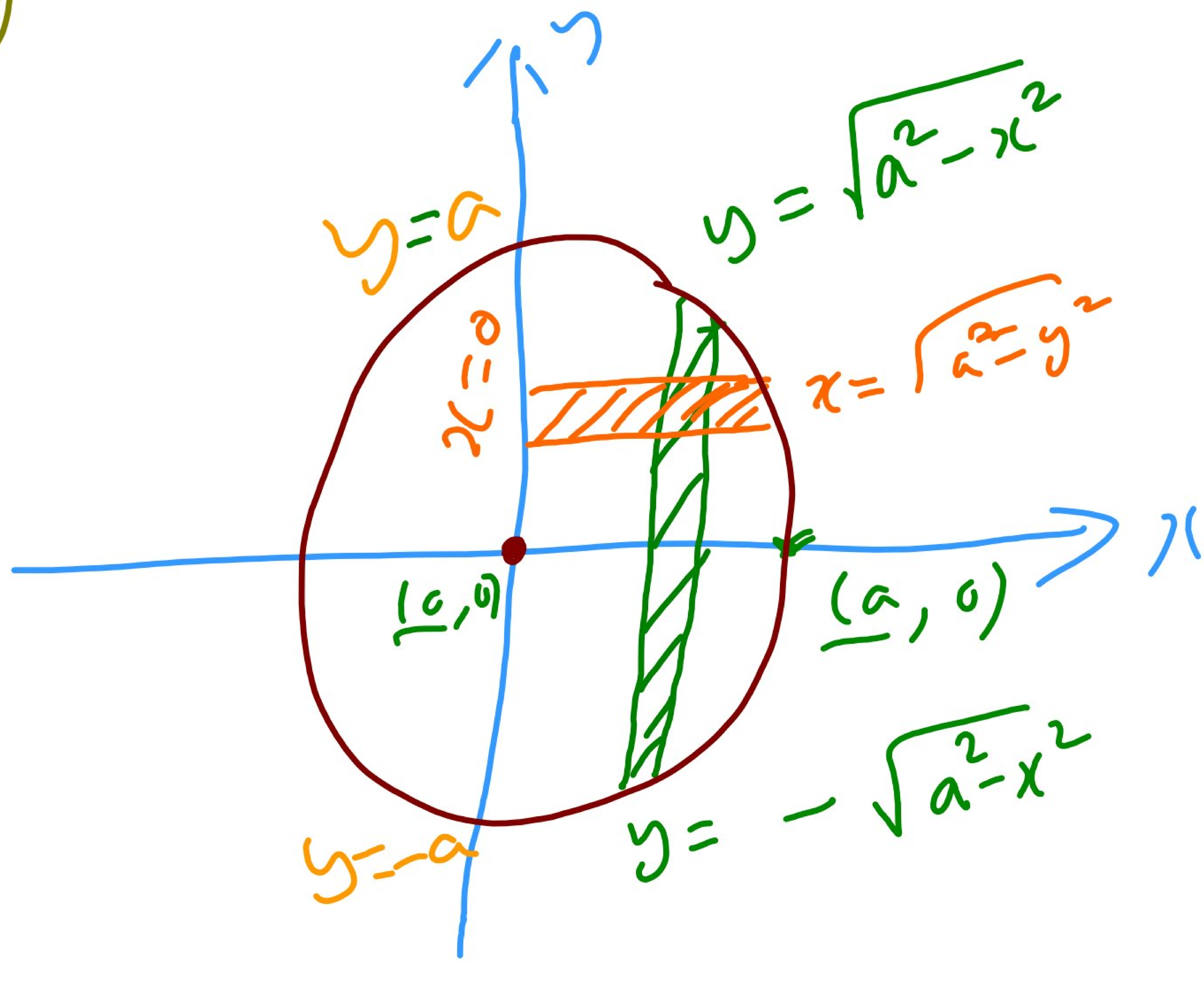
$\therefore$ The path is parallel to x-axis

x is varying from 0 to $\sqrt{a^2-y^2}$

y is varies from $-a$ to a

To change the order of integration.

first take the path is parallel to y-axis



$$\left[\because x=\sqrt{a^2-y^2} \quad x^2+y^2=a^2\right]$$

$\therefore$ y is varies from

$-\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$

and x is varies from

0 to a

Thus $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy = \int_{x=0}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\,dx\,dy$

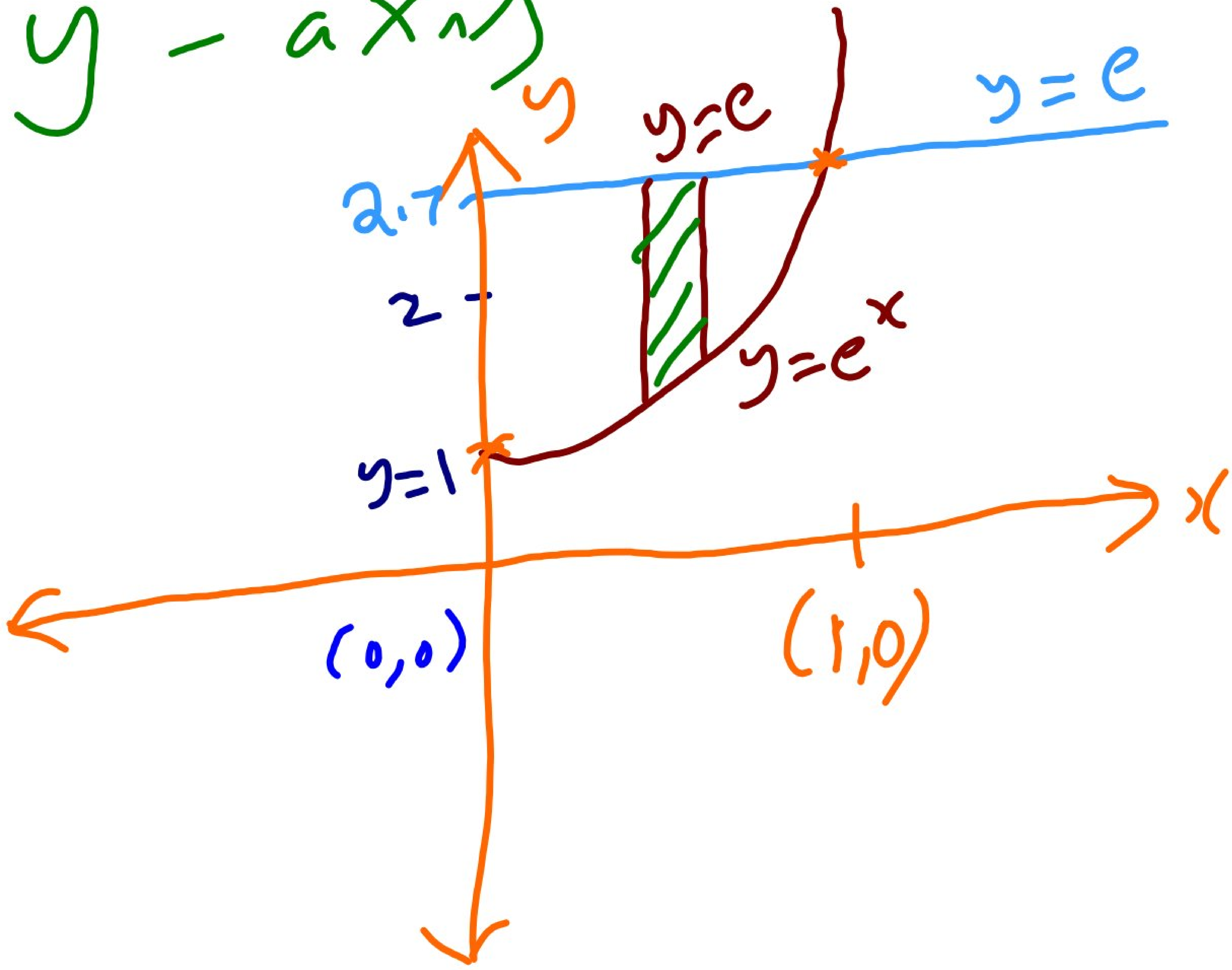
2) Evaluate $\int_0^1\int_{e^x}^{e} \frac{dy\,dx}{\log y}$ by changing the order of integration

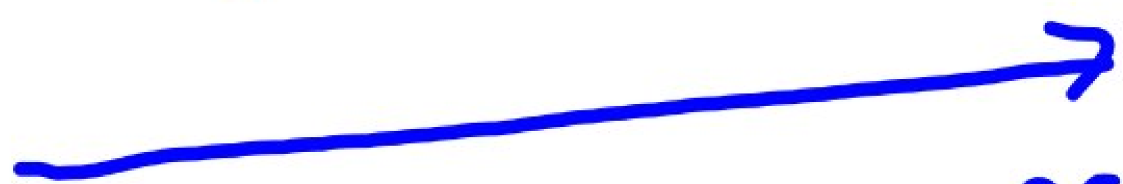
Sol:- From the given bouble integration $\int_0^1\int_{e^x}^{e} \frac{1}{\log y}\,dx\,dy$

Clearly x is independent and y is dependent

$\therefore$ The path is parallel to y-axis

$y = e^x$





$x = 0$

$y = e^0$

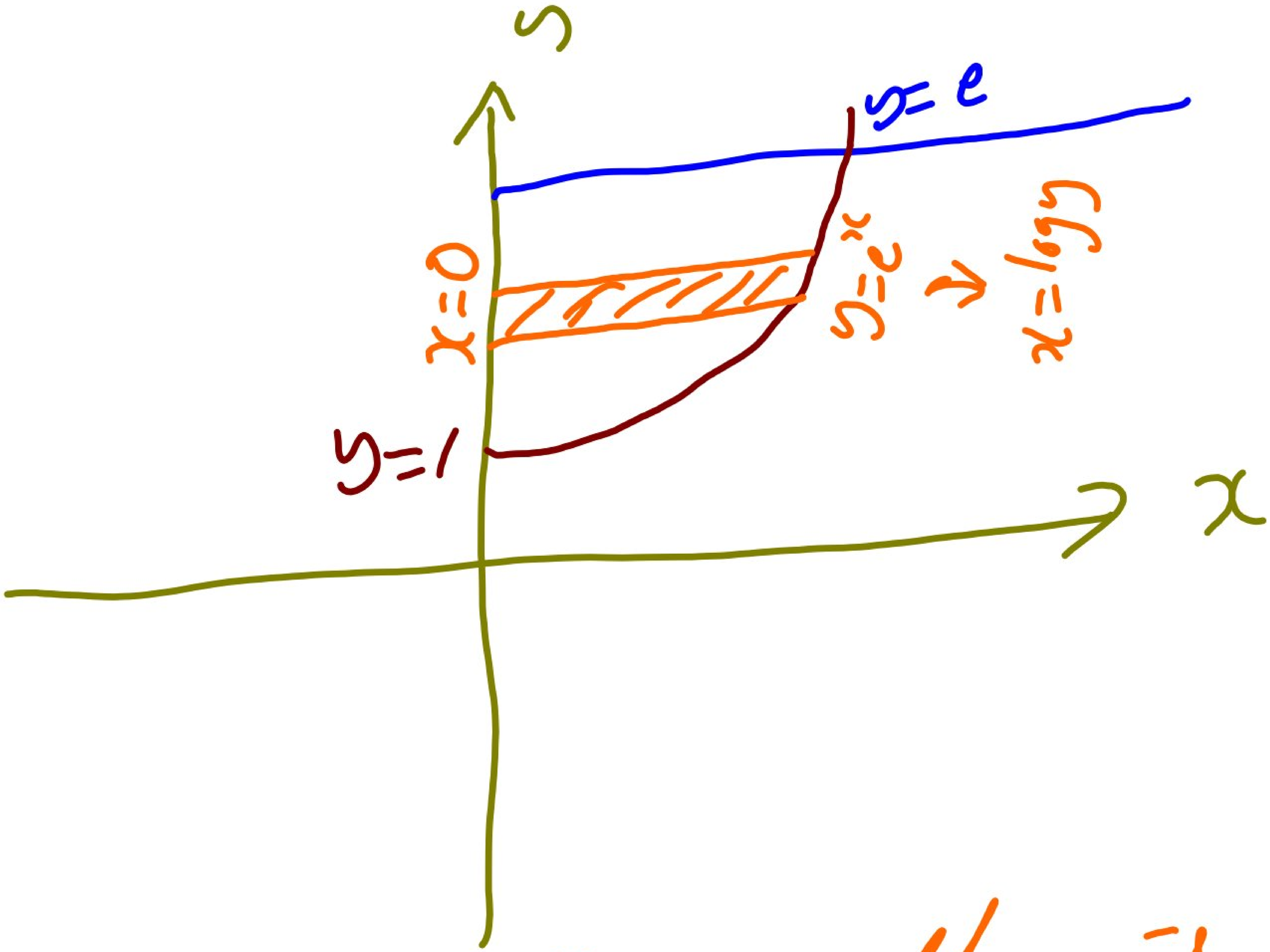
$y = 1$

$x = 1$

$y = e^1$

$y = 2.718$

To change the order of integration



∴ Taken the path is parallel to x-axis

x is varies from 0 to log y

and

y is varies from 1 to e

$$\therefore \int_{x=0}^{1}\int_{y=e^{x}}^{e}\frac{1}{\log y}\,dx\,dy=\int_{y=1}^{e}\int_{x=0}^{\log y}\frac{1}{\log y}\,dx\,dy$$

$$= \int_{y=1}^{e} \frac{1}{\log y} \left\{ \int_{x=0}^{\log y} 1 \, dx \right\} dy$$

$$= \int_{y=1}^{e} \frac{1}{\log y} \left(x \right)_{0}^{\log y} dy$$

$$= \int_{y=1}^{e} \frac{1}{\log y} \left(\log y - 0 \right) dy$$

$$= \int_{y=1}^{e} \frac{1}{\cancel{\log y}} \cancel{\log y} \, dy$$

$$= \int_{y=1}^{e} 1 \, dy$$

$$= \left(y \right)_{1}^{e}$$

$$= e - 1$$

3) Evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2} \, dx\, dy$ by changing the order of integration.

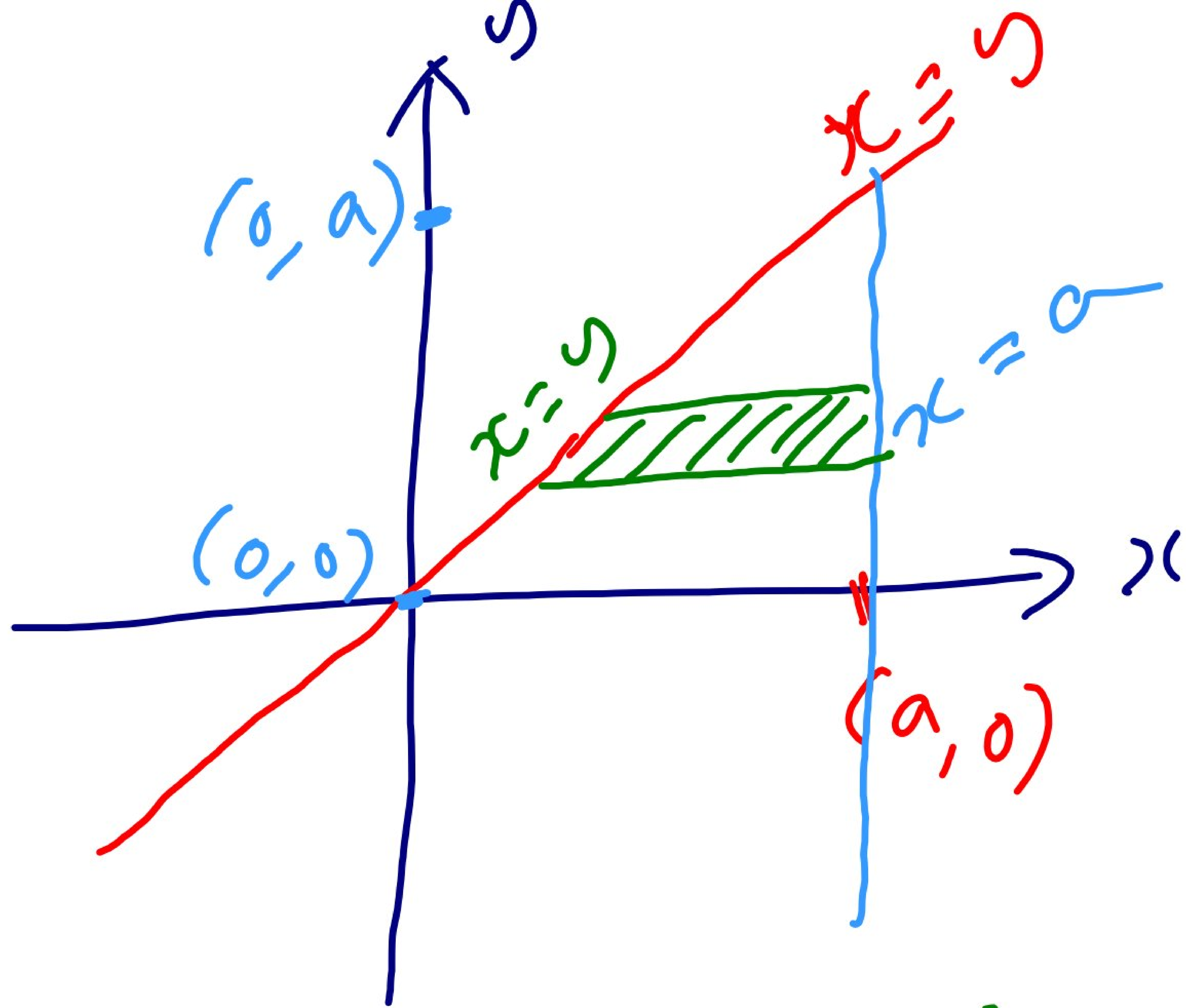
Sol:- Given $\int_{y=0}^{a} \int_{x=y}^{a} \frac{x}{x^2+y^2} \, dx\, dy$

Clearly y is independent and x is dependent

∴ the path is parall to x-axis

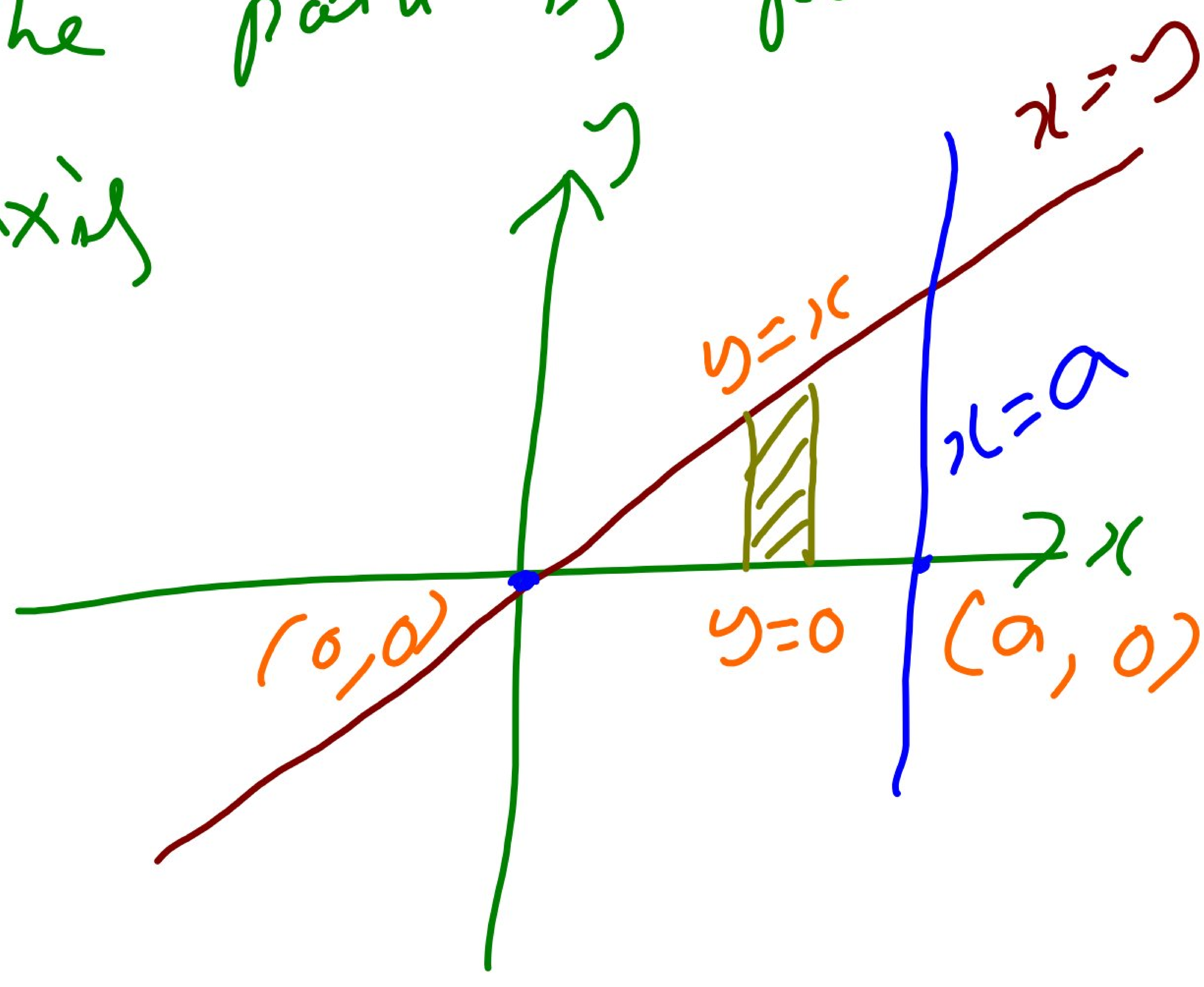
∴ x is varies from y to a

and y is varies from 0 to a



To change the order of integration as,

the path is parallel to y_axis



$\therefore$ y is varies from 0 to x

and x is vary from 0 to a

$$\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy = \int_{x=0}^{a} \int_{y=0}^{x} \frac{x}{x^2+y^2}\,dx\,dy$$

$$= \int_{x=0}^{a} x \left\{ \int_{y=0}^{x} \frac{1}{y^2+x^2}\,dy \right\} dx$$

$$\boxed{\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}}$$

$$= \int_{x=0}^{a} x \left[\frac{1}{x} \tan^{-1}\left(\frac{y}{x}\right) \right]_0^x dx$$

$$= \int_{x=0}^{a} x \left(\frac{1}{x} \tan^{-1}\left(\frac{x}{x}\right) - \frac{1}{x}\tan^{-1}(0) \right) dx$$

$$= \int_{x=0}^{a} x\left(\frac{1}{x}\,\frac{\pi}{4} - \frac{1}{x}\,(0)\right)dx$$

$$= \frac{\pi}{4}\int_{x=0}^{a} \cancel{x}\,\frac{1}{\cancel{x}}\,dx$$

$$= \frac{\pi}{4}\left(x\right)_0^a$$

$$= \frac{\pi}{4}\left(a-0\right)$$

$$= \frac{a\pi}{4}$$

4) Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dx\,dy$ by changing the order of integration

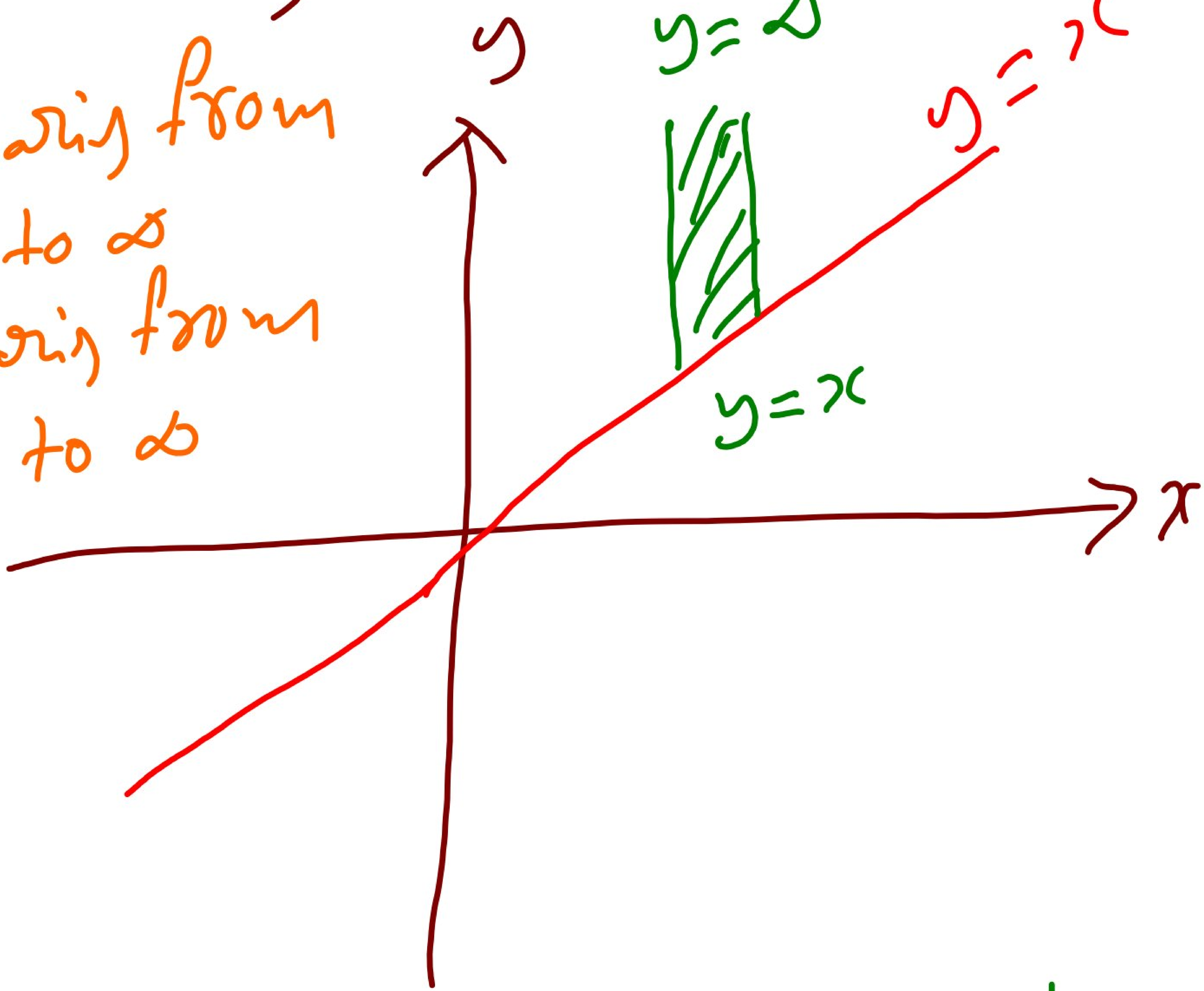
Sol:- Given $\int_{x=0}^{\infty} \int_{y=x}^{\infty} \frac{e^{-y}}{y}\,dx\,dy$

Clearly x is independent and y is dependent

$\therefore$ The path is parallel y-axis

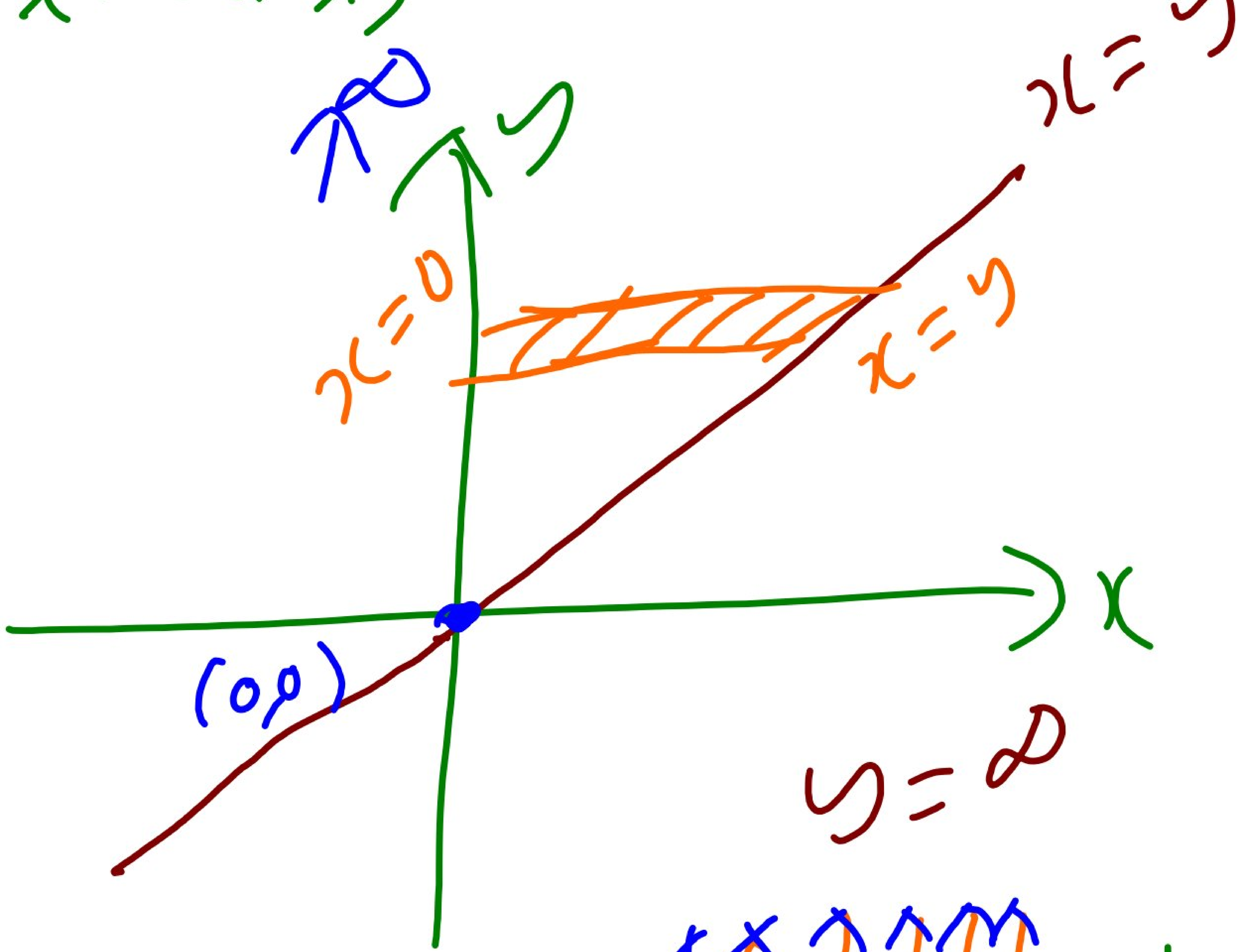
x is varies from 0 to ∞

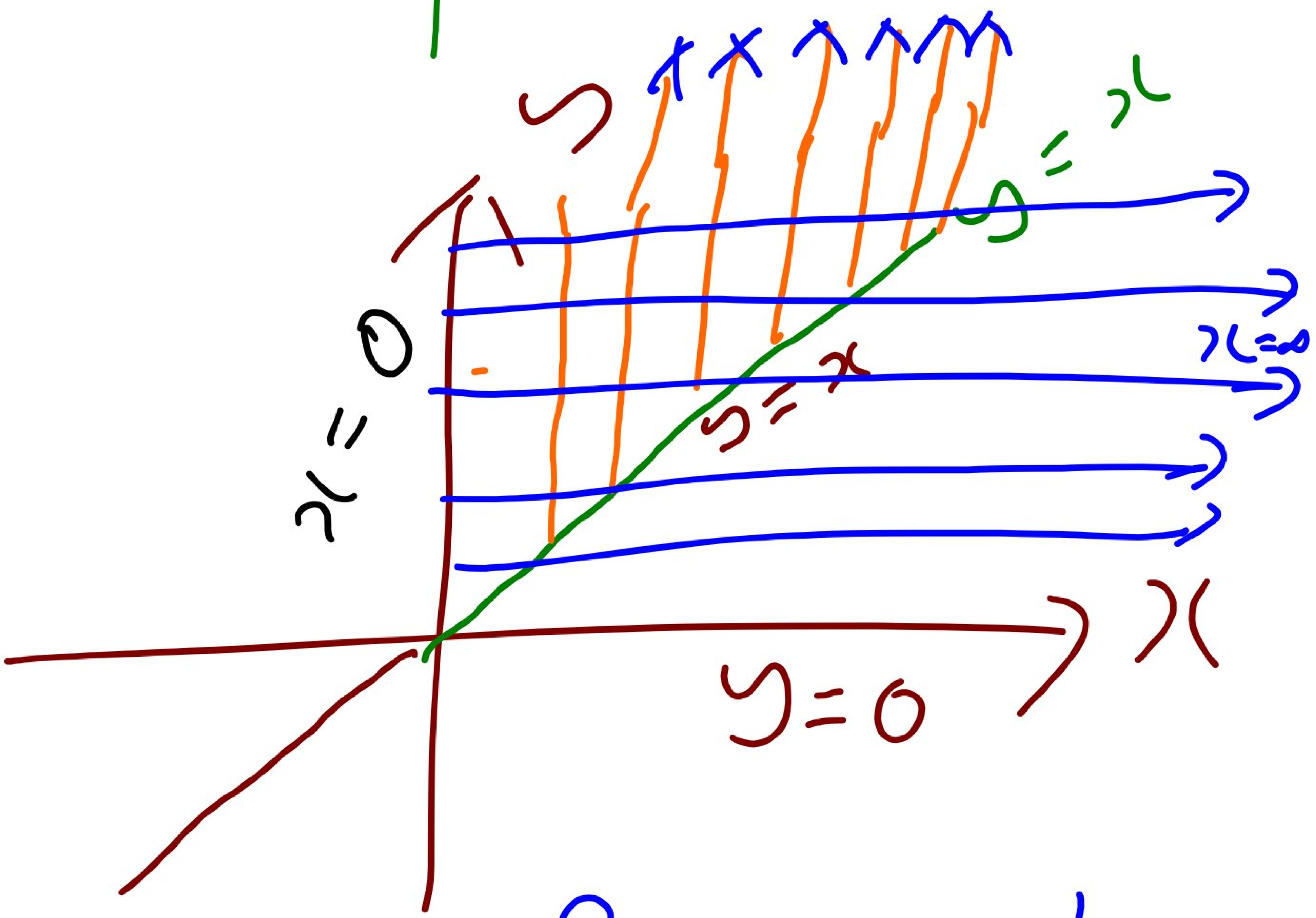
y is varies from x to ∞



To change the order of the integration as

The path is parallel to x-axis





x is varies from 0 to y

y is varis from 0 to ∞

$$\int_{x=0}^{\infty}\int_{y=x}^{\infty} \frac{e^{-y}}{y}\,dx\,dy$$

$$= \int_{y=0}^{\infty}\int_{x=0}^{y} \frac{e^{-y}}{y}\,dx\,dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{y}\left[\int_{x=0}^{y} 1\,dx\right]dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{y}\,(x)_0^{y}\,dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{\cancel{y}}\,\cancel{(y-0)}\,dy$$

$$= \left(\frac{e^{-y}}{-1}\right)_0^{\infty}$$

$$= \frac{e^{-\infty}}{-1} - \frac{e^{0}}{-1}$$

$$= 0 + 1$$

$$= 1$$

H.W page 276 : Ex 7.4

277 : Ex 7.7

278 : Ex 7.8

Ex 7.9

Ex 7.10