Applications to Solving Finite Difference Equations.

$$Z[y(n)] = Y(z)$$

$$Z[y(n+1)] = zY(z) - zy(0)$$

$$Z[y(n+2)] = z^2Y(z) - z^2y(0) - zy(1)$$

$$\vdots$$

$$Z[y(n+k)] = z^k\left[Y(z) - y(0) - \frac{y(1)}{z} - \frac{y(2)}{z^2} - \cdots - \frac{y(k-1)}{z^{k-1}}\right]$$

(1) Solve : $y_{n+2} - 4y_{n+1} + 4y_n = 0$ given $y_0 = 1$ and $y_1 = 0$.

Soln:-

Taking Z-transform on both sides of the difference equation, we get

$$Z[y_{n+2} - 4y_{n+1} + 4y_n] = Z[0]$$

$$Z[y_{n+2}] - 4Z[y_{n+1}] + 4Z[y_n] = 0.$$

$$z^2Y(z) - z^2y(0) - zy(1) - 4[zY(z) - zy(0)] + 4Y(z) = 0.$$

$$z^2Y(z) - z^2(1) - \cancel{z(0)} - 4[zY(z) - z(1)] + 4Y(z) = 0 \quad (\because y_0 = 1 \text{ \& } y_1 = 0)$$

$$z^2Y(z) - z^2 - 4zY(z) + 4z + 4Y(z) = 0.$$

$$[z^2 - 4z + 4]Y(z) = z^2 - 4z$$

$$Y(z) = \frac{z^2 - 4z}{z^2 - 4z + 4}$$

$$Y(z)\cdot z^{n-1} = \frac{(z^2-4z)\,z^{n-1}}{(z-2)^2}$$

$$Y(z)\cdot z^{n-1} = \frac{z^{n+1}-4z^n}{(z-2)^2}$$

$z=2$ is a pole of order 2.

Residue of $Y(z)\cdot z^{n-1}$ at $z=2$ of order 2 $(n=1)$ $\Big\} = \frac{1}{1!}\lim \frac{d}{dz}\cancel{(z-2)^2}\frac{z^{n+1}-4z^n}{\cancel{(z-2)^2}}$

$$= \lim_{\to 2} (n+1)z^n - 4(nz^{n-1}).$$

$$\Rightarrow (n+1)2^n - 4n(2^{n-1}) \qquad 2^{2+n-1}$$

$$= n2^n + 2^n - n(2\cdot 2^n)$$

$$\Rightarrow 2^n(n+1-2n)$$

$$= 2^n(1-n).$$

$$\therefore\ y(n) = 2^n(1-n),\quad n=0,1,2,\ldots$$

2) Solve: $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ given $y_0 = y_1 = 0$.

Soln:-

$$y_{n+2} + 6y_{n+1} + 9y_n = 2^n$$

Taking Z-trans. on b.s of the difference eqn,

$$Z[y_{n+2} + 6y_{n+1} + 9y_n] = Z(2^n)$$

$$Z[y_{n+2}] + 6Z[y_{n+1}] + 9Z(y_n) = z/z-2$$

$$[z^2Y(z) - z^2y(0) - zy(1)] + 6[zY(z) - y_0] + 9Y(z) = z/z-2$$

$$z^2Y(z) - \cancel{z^2(0)} - \cancel{z(0)} + 6zY(z) - \cancel{6(0)} + 9Y(z) = z/z-2$$

$$Y(z)[z^2 + 6z + 9] = z/z-2$$

$$Y(z) = \frac{z}{(z^2+6z+9)(z-2)}$$

$$Y(z) = \frac{z}{(z-2)(z+3)^2}$$

$$\frac{Y(z)}{z} = \frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2}$$

Solve, $A = \frac{1}{25}$ $B = -\frac{1}{25}$ $C = -\frac{1}{5}$

$$\therefore Y(z) = \frac{1/25}{z-2} - \frac{1/25}{z+3} - \frac{1/5}{(z+3)^2}$$

Taking inverse Z-transform on both sides, we get

$$Z^{-1}[Y(z)] = Z^{-1}[RHS]$$

$$y(n) = \frac{1}{25} Z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25} Z^{-1}\left[\frac{z}{z+3}\right] - \frac{1}{5}\left[\frac{z}{(z+3)^2}\right]$$

$$= \frac{1}{25}(2^n) - \frac{1}{25}(-3)^n - \frac{1}{5}\left[\frac{1}{-3}\underbrace{\frac{-3z}{(z+3)^2}}_{n(-3)^n}\right]$$

$$= \frac{2^n}{25} - \frac{(-3)^n}{25} - \frac{1}{5} n(-3)^{n-1}.$$

$$\left[\because \; Z[a^n] = \frac{z}{z-a} \quad \text{and} \quad Z[na^n] = \frac{az}{(z-a)^2}\right]$$

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3) Solve : $x(n+2) - 3x(n+1) + 2x(n) = 0$ given $x(0) = 0$, $x(1) = 1$.

Soln:

Given $x(n+2) - 3x(n+1) + 2x(n) = 0$.

Taking Z-trans. on both sides, we get

$$Z[x(n+2) - 3x(n+1) + 2x(n)] = Z(0)$$

$$Z[x(n+2)] - 3Z[x(n+1)] + 2Z(x(n)] = 0.$$

$$z^2X(z) - z^2\underbrace{\cancel{x(0)}}_{(0)} - zx(1) - 3[zX(z) - \underbrace{\cancel{x(0)}}_{(0)}] + 2X(z) = 0$$

$$z^2X(z) - zx(1) - 3zX(z) + 2X(z) = 0.$$

$$[z^2 - 3z + 2]X(z) - z(1) = 0.$$

$$X(z) = \frac{z}{z^2 - 3z + 2}$$

$$\frac{X(z)}{z} = \frac{1}{(z-1)(z-2)} = \frac{A}{(z-2)} + \frac{B}{(z-1)}$$

Solve, $A=1$, $B=-1$.

$$\therefore \frac{X(z)}{z} = \frac{1}{z-2} + \frac{-1}{z-1}$$

$$X(z) = \frac{z}{z-2} - \frac{1}{z-1}$$

Taking inverse z-transform on b.s, we get

$$x(n) = 2^n - 1, \quad n = 0,1,2,\dots$$

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(4) Solve: $y(n) - y(n-1) = u(n) + u(n-1)$.

we know that,

$$Z[x(n-m)] = z^{-m} X(z) \qquad Z(u(n)) = \frac{z}{z-1}$$

$$u(n-m) = z^{-m} \frac{z}{z-1}$$

Taking Z-trans. on both sides, we get

$$Z[y(n) - y(n-1)] = Z[u(n) + u(n-1)]$$

$$= Z[y(n)] - Z[y(n-1)] = Z(u(n)) + Z[u(n-1)]$$

$$Y(z) - z^{-1} Y(z) = \frac{z}{z-1} + z^{-1} \cdot \frac{z}{z-1}$$

$$Y(z)\left[1 - \frac{1}{z}\right] = \frac{z}{z-1} + \frac{1}{\cancel{z}} \cdot \frac{\cancel{z}}{z-1}$$

$$Y(z)\left[\frac{z-1}{z}\right] = \frac{z}{z-1} + \frac{1}{z-1}$$

$$Y(z) = \frac{z+1}{z-1} \times \frac{z}{z-1}$$

$$Y(z) = \frac{z^2+z}{(z-1)^2}$$

$$Y(z) \cdot z^{n-1} = \frac{z^2+z}{(z-1)^2} \cdot z^{n-1} =$$

$$Y(z) \cdot z^{n-1} = \frac{z^{n+1}+z^n}{(z-1)^2}$$

$z=1$ is pole of order 2.

Residue of $Y(z) \cdot z^{n-1}$ at $z=1$ of order 2 $\Big\} = \frac{1}{1!} \lim_{z \to 1} \frac{d}{dz} \cancel{(z-1)^2} \cdot \frac{z^{n+1}+z^n}{\cancel{(z-1)^2}}$

$$= \lim_{z \to 1} \frac{d}{dz}\left(z^{n+1}+z^n\right)$$

$$= \lim_{z \to 1} (n+1) z^{n+1-\cancel{1}} + n z^{n-1}$$

$$\Rightarrow n+1+n = 1+2n.$$

$\therefore$ $x(n) = R$ where R is residue of $Y(z) \cdot z^{n-1}$

$\therefore$ $x(n) = 1+2n, \quad n = 0, 1, 2, \ldots$

Exercise problems:

1. Solve: $y(n) - ay(n-1) = u(n)$
2. Solve: $y_{n+2} + y_n = 2$ given $y_0 = y_1 = 0$.
3. Solve: $y_{n+2} - 4y_n = 0$ using Z-transform.