Infinite Fourier Cosine and Sine transform:

Infinite Fourier Cosine transform:

$$F_c(s) = F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$f(x) = \sqrt{2/\pi} \int_0^\infty F_c(s) \cos sx \, ds \quad (\text{inversion formula})$$

Infinite Fourier sine transform:

$$F_s(s) = F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds \quad (\text{inversion formula})$$

Properties of cosine and Sine transform:

① $F_c[a f(x) + b g(x)] = a F_c(f(x)) + b G_c(g(x))$

② $F_c[f(x) \sin ax] = \frac{1}{2}[F_s(a+s) + F_s(a-s)]$

③ $F_c[f(x) \cos ax] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$

④ $F_c[f(ax)] = \frac{1}{a} F_c(s/a)$

⑤ $F_s[a f(x) + b g(x)] = a F_s(f(x)) + b F_s(g(x))$

⑥ $F_s[f(x) \sin ax] = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$

⑦ $F_s[f(x) \cos ax] = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$

⑧ $F_s[f(ax)] = \frac{1}{a} F_s\left(\frac{s}{a}\right)$.

① $F_c\,[a f(x) + b g(x)] = a F_c\,(f(x)) + b F_c\,(g(x))$

Proof :-

$$F_c\,[a f(x) + b g(x)] = \sqrt{2/\pi}\int_0^\infty (a f(x) + b g(x)) \cos sx\, dx \quad (\text{by definition})$$

$$= \underbrace{a\sqrt{2/\pi}\int_0^\infty f(x)\cos sx\, dx}_{} + b\sqrt{2/\pi}\int_0^\infty g(x) \cos sx\, dx$$

$$= a F_c\,(f(x)) + b F_c\,(g(x))$$

$$\therefore\ F_c\,(a f(x) + b g(x)) = a F_c\,(f(x)) + b F_c\,(g(x)).$$

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③ $F_c\,(f(x) \sin ax) = \frac{1}{2}\,[F_s(a+s) + F_s(a-s)]$

Proof :-

$$F_c\,(f(x)\sin ax) = \sqrt{2/\pi}\int_0^\infty f(x) \sin ax \cos sx\, dx \quad (\text{by definition})$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\left(\frac{\sin(a+s)x + \sin(a-s)x}{2}\right) dx$$

$$= \frac{1}{2}\left[\sqrt{2/\pi}\int_0^\infty f(x) \sin(a+s)x\, dx + \sqrt{2/\pi}\int_0^\infty f(x) \sin(a-s)x\, dx\right]$$

$$= \frac{1}{2}\,[F_s(a+s) + F_s(a-s)].$$

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⑥ $F_s\,(f(x)\sin ax) = \frac{1}{2}\,[F_c(s-a) - F_c(s+a)]$

Proof :-

$$F_s\,(f(x)\sin ax) = \sqrt{2/\pi}\int_0^\infty f(x) \sin ax \sin sx\, dx \quad (\text{by definition})$$

$$= \sqrt{2/\pi}\int_0^\infty f(x) \sin sx \sin ax\, dx$$

$$= \sqrt{2/\pi} \int_0^\infty f(x) \left(\frac{\cos(s-a)x - \cos(s+a)x}{2} \right] dx$$

$$= \frac{1}{2} \left[\sqrt{2/\pi} \int_0^\infty f(x) \cos(s-a)x \, dx - \sqrt{2/\pi} \int_0^\infty f(x) \cos(s+a) \, dx \right]$$

$$= \frac{1}{2} \left[F_c(s-a) - F_c(s+a) \right]$$

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(8) $F_s(f(ax)) = \frac{1}{a} F_s(s/a)$

Proof :-

$$F_s(f(ax)) = \sqrt{2/\pi} \int_0^\infty f(ax) \sin sx \, dx \quad (\text{by defn.})$$

Put $ax = t \Rightarrow x = t/a$ when $x = 0, \; t = 0$

$a\,dx = dt$ $\qquad x = \infty, \; t = \infty.$

$$= \sqrt{2/\pi} \int_0^\infty f(t) \sin\left(\frac{st}{a}\right) \frac{dt}{a}.$$

$$= \frac{1}{a} \sqrt{2/\pi} \int_0^\infty f(t) \sin\left(s/a \, t\right) dt$$

$$= \frac{1}{a} F_s(s/a)$$

Note: Remaining properties (try).

Identities :

If $F_c(s)$, $G_c(s)$ are the Fourier cosine transforms and $F_s(s)$, $G_s(s)$ are the Fourier sine transforms of $f(x)$ and $g(x)$ respectively, then

① $$\int_0^\infty f(x)g(x)\,dx = \int_0^\infty F_c(s)\,G_c(s)\,ds$$

② $$\int_0^\infty f(x)\,g(x)\,dx = \int_0^\infty F_s(s)\,.\,G_s(s)\,ds$$

③ $$\int_0^\infty |f(x)|^2\,dx = \int_0^\infty |F_c(s)|^2\,ds = \int_0^\infty |F_s(s)|^2\,ds.$$