*. GAUSS'S LAW :- (MAXWELL'S EQUATION)

Gauss's law states that the total electric flux 'ψ' through any closed surface is equal to the total charge enclosed by that surface.

Thus $$\psi = Q_{enc}$$

that is, $$\psi = \oint_S d\psi = \oint_S \vec{D} \cdot \vec{ds}$$

$$= \text{total charge enclosed} \quad Q = \int_V \rho_v \, dv$$

or

INTEGRAL FORM

$$\boxed{Q = \oint_S \vec{D} \cdot \vec{ds} = \int_V \rho_v \, dv} \longrightarrow (1)$$

By applying divergence theorem to $\oint_S D \cdot ds$, we get

$$\oint_S \vec{D} \cdot \vec{ds} = \int_V \nabla \cdot \vec{D} \, dv$$

from (1)

$$\int_V \nabla \cdot \vec{D} \, dv = \int_V \rho_v \, dv.$$

POINT FORM

$$\boxed{\therefore \rho_v = \nabla \cdot \vec{D}} \longrightarrow (2)$$

which is the first of the four Maxwell's equations.

Eq. (2) states that the volume charge density is the same as the divergence of the electric flux density.

NOTE :-

- Gauss's law is an alternative statement of Coulomb's law; proper application of the divergence theorem to Coulomb's law resuts in Gauss's law

- Gauss's law provides an easy means of finding $\vec{E}$ or $\vec{D}$ for symmetrical charge distributions such as a point charge, an infinite line charge etc.

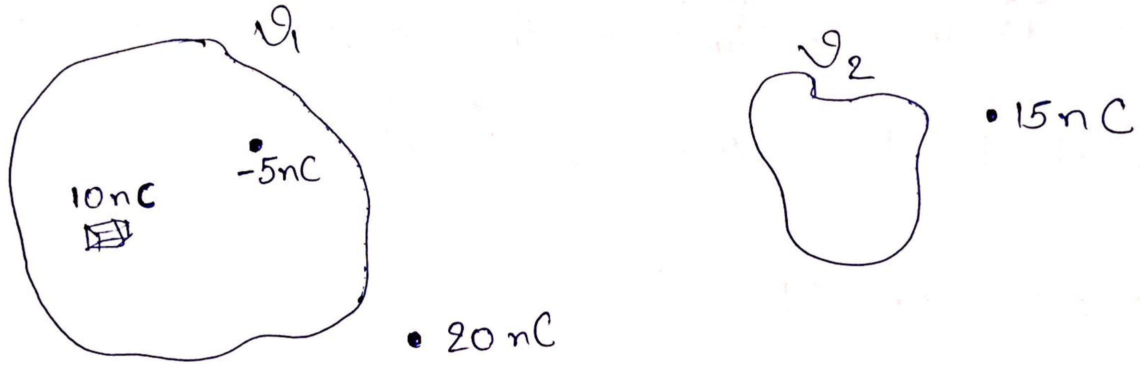

Fig :- Illustration of Gauss's law ; flux leaving v_1 is 5 nC and that leaving v_2 is 0 C.

- We cannot use the law to determine $\vec{E}$ or $\vec{D}$ when the charge distribution is not symmetric; we must resort to Coulomb's law to determine $\vec{E}$ or $\vec{D}$ in that case.

*. APPLICATIONS OF GAUSS'S LAW:—

The procedure for applying Gauss's law to calculate the electric field involves

(i). Knowing whether symmetry exists.

(ii). Construct a mathematical closed surface (known as a Gaussian surface).

(iii). The surface is chosen such that $\vec{D}$ is normal or tangential to the Gaussian surface.

When $\vec{D}$ is normal to the surface, $\vec{D}\cdot\vec{ds} = D\,\vec{ds}$ because $\vec{D}$ is constant on the surface: When $\vec{D}$ is tangential to the surface, $\vec{D}\cdot\vec{ds} = 0$.

A. POINT CHARGE:—

Suppose a point charge Q is located at the origin. To determine $\vec{D}$ at a point P, it is easy to see that choosing a spherical surface containing P will satisfy symmetry conditions.

Thus a spherical surface centered at the origin is the Gaussian surface in this case.

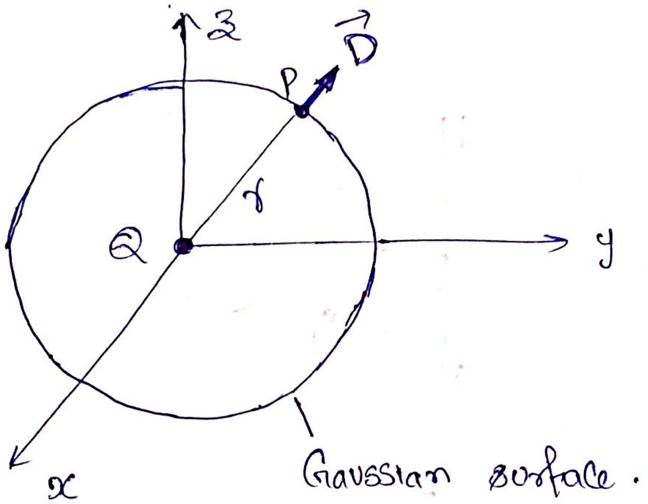

Fig:- Gaussian surface about a point charge.

Since $\vec{D}$ is everywhere normal to the Gaussian surface, that is, $\vec{D} = D_r \vec{a}_r$, applying Gauss's law (Ψ = Qenclosed) gives

$$Q = \oint \vec{D} \cdot \vec{ds} = D_r \oint ds$$

$$\Rightarrow Q = D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$\Rightarrow Q = D_r \, 4\pi r^2$$

$4\pi r^2$ is the surface area of the Gaussian surface.

$$\boxed{\therefore \vec{D} = \frac{Q}{4\pi r^2} \vec{a_r}}$$

B. INFINITE LINE CHARGE :-

Suppose the infinite line of uniform charge ρ_L C/m lies along the z-axis. To determine $\vec{D}$ at a point P, we choose a cylindrical surface containing P to satisfy symmetry condition.

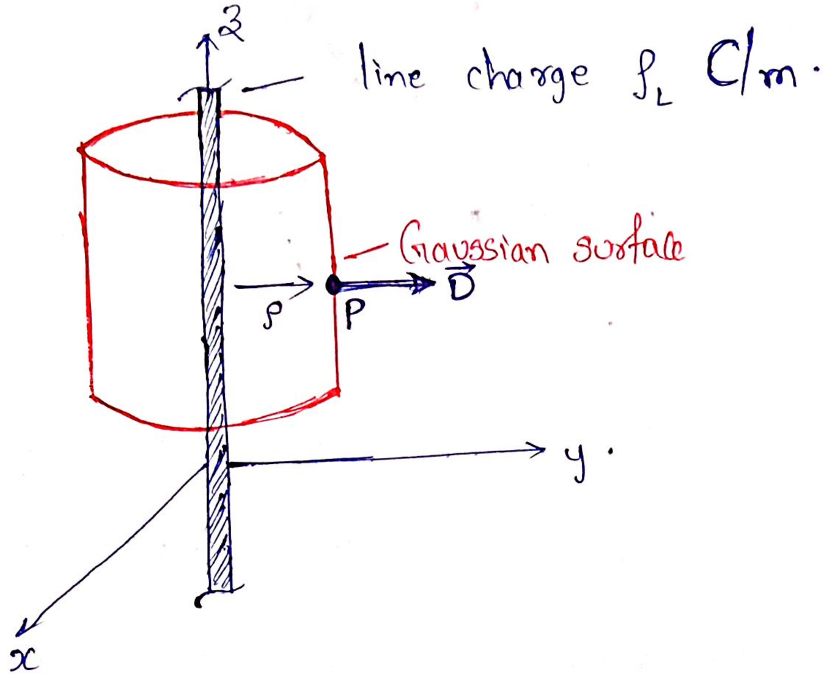

Fig:- Gaussian surface about an infinite line charge.

$\vec{D}$ is constant on and normal to the cylindrical Gaussian surface; that is $\vec{D} = D_\rho \vec{a}_\rho$.

If we apply Gauss's law to an arbitrary length 'l' of the line

$$\rho_L l = Q = \oint \vec{D} \cdot \vec{ds} = D_\rho \oint ds = D_\rho \, 2\pi \rho l.$$

where $\oint ds = 2\pi \rho l$ is the surface area of the Gaussian surface.

Note:- $\int \vec{D} \cdot ds$ evaluated on the top and bottom surfaces of the cylinder is zero since $\vec{D}$ has no z-component; that means that $\vec{D}$ is tangential to those surfaces.

$$\boxed{\therefore \vec{D} = \frac{\rho_L}{2\pi \rho} \vec{a}_\rho}$$

C. INFINITE SHEET OF CHARGE:-

Consider the infinite sheet of uniform charge ρ_s C/m^2 lying on the $z=0$ plane. To determine $\vec{D}$ at point P, we choose a rectangular box that is cut symmetrically by the sheet of charge and has two of its faces parallel to the sheet.

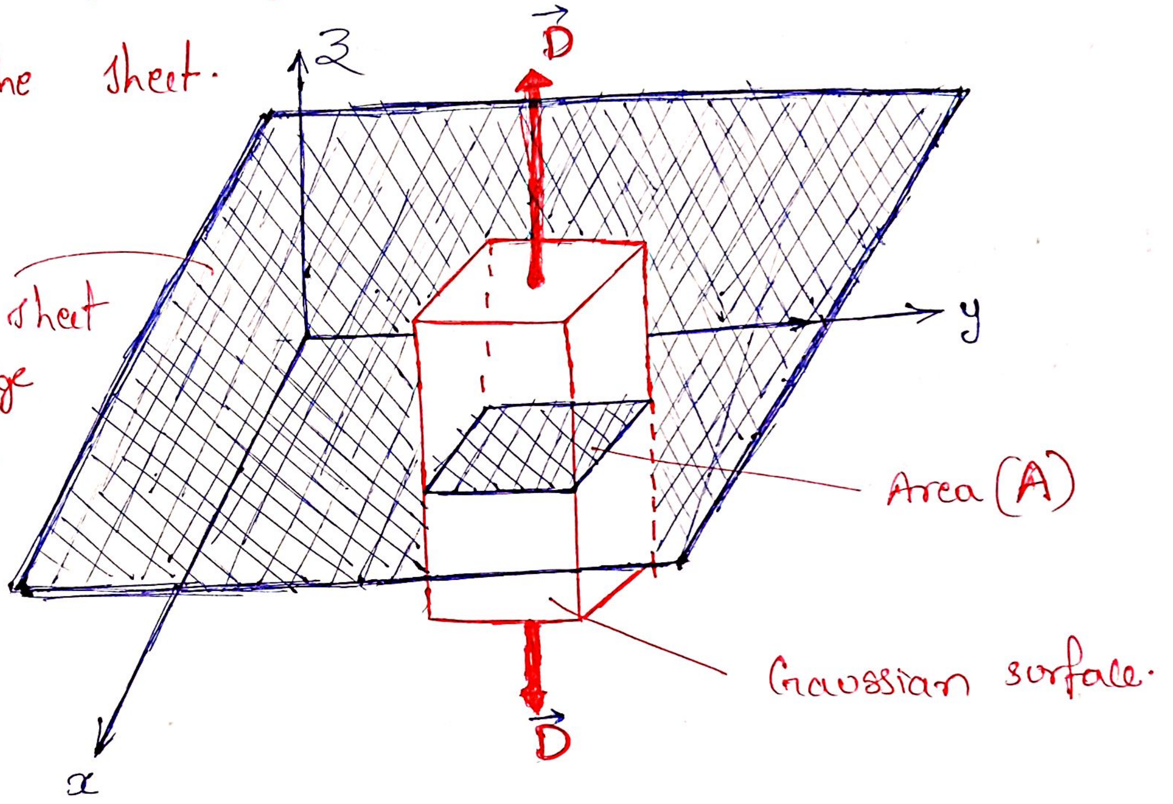

Fig:- Gaussian surface about an infinite sheet of charge.

$\vec{D}$ is normal to the sheet, $\vec{D} = D_z \vec{a}_z$, and applying Gauss's law gives.

$$Q = \rho_s \int ds$$

and

$$Q = \oint \vec{D} \cdot \vec{ds}$$

$$\Rightarrow \quad \rho_s \int ds = D_z \left[\int_{top} ds + \int_{bottom} ds \right] \rightarrow (1)$$

NOTE:- $\vec{D}\cdot d\vec{S}$ evaluated on the sides of the box is zero because $\vec{D}$ has no components along $\vec{a}_x$ and $\vec{a}_y$. If the top and bottom area of the box each has area A, eq. (1) becomes

$$\rho_S A = D_z (A + A).$$

and thus $$\boxed{\vec{D} = \frac{\rho_S}{2}\vec{a}_z}$$

D. UNIFORMLY CHARGED SPHERE :-

Consider a sphere of radius 'a' with a uniform charge ρ_v C/m³. To determine $\vec{D}$ every where, we construct Gaussian surfaces for cases $r \le a$ and $r \ge a$ separately. Since the charge has spherical symmetry, it is obvious that a spherical surface is an appropriate Gaussian surface.

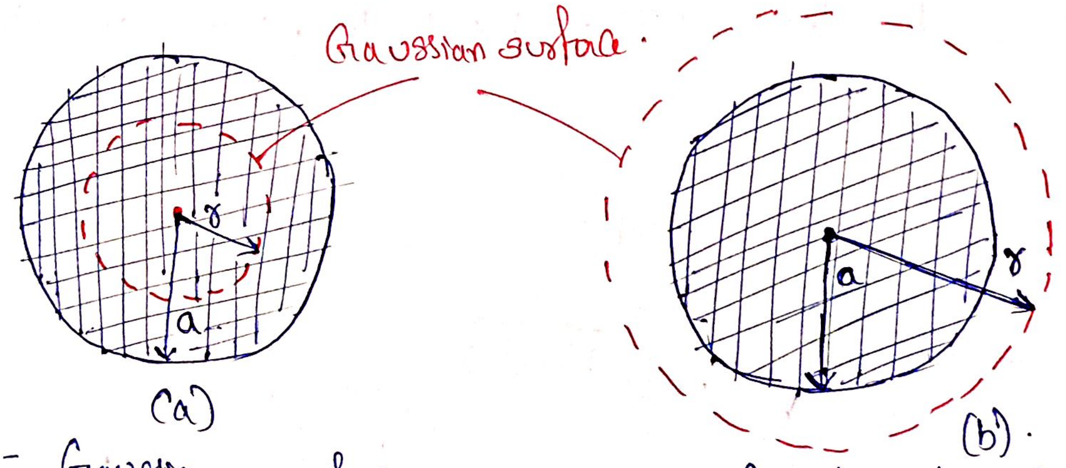

Fig:- Gaussian surface for a uniformly charged sphere when (a) $r \le a$ & (b) $r \ge a$

For $r \le a$, the total charge enclosed by the spherical surface of radius r, is

$$Q_{enc} = \int \rho_v \, dv = \rho_v \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$\therefore Q_{enc} = \rho_v \frac{4}{3} \pi r^3 \longrightarrow (1)$$

and

$$\psi = \oint \vec{D} \cdot \vec{ds} = D_r \oint ds = D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$\therefore \psi = D_r \, 4\pi r^2 \longrightarrow (2)$$

Hence, $\psi = Q_{enc}$ gives

$$\Rightarrow D_r \, 4\pi r^2 = \frac{4\pi r^3}{3} \rho_v$$

$$\Rightarrow D_r = \frac{r}{3} \rho_v$$

and $\vec{D} = D_r \vec{a}_r + D_\theta \vec{a}_\theta + D_\phi \vec{a}_\phi$

$$\boxed{\therefore \vec{D} = \frac{r}{3} \rho_v \vec{a}_r} \qquad 0 < r \le a. \rightarrow (3)$$

For $r \ge a$, the charge enclosed by the surface is the entire charge in this case, that is

$$Q_{enc} = \int \rho_v dv = \rho_v \int dv = \rho_v \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$\therefore Q_{enc} = \rho_v \frac{4}{3} \pi a^3$$

while

$$\psi = \oint \vec{D} \cdot \vec{ds} = D_r 4\pi r^2$$

$$\psi = Q_{enc} \Rightarrow D_r 4\pi r^2 = \frac{4}{3} \pi a^3 \rho_v$$

$$\boxed{\therefore \vec{D} = \frac{a^3}{3r^2} \rho_v \vec{a_r}} \quad r \geq a \longrightarrow (4)$$

Thus from (3) & (4)

$$\vec{D} = \begin{cases} \frac{r}{3} \rho_v \vec{a_r} & 0 < r \leq a \\ \frac{a^3}{3r^2} \rho_v \vec{a_r} & r \geq a \end{cases}$$

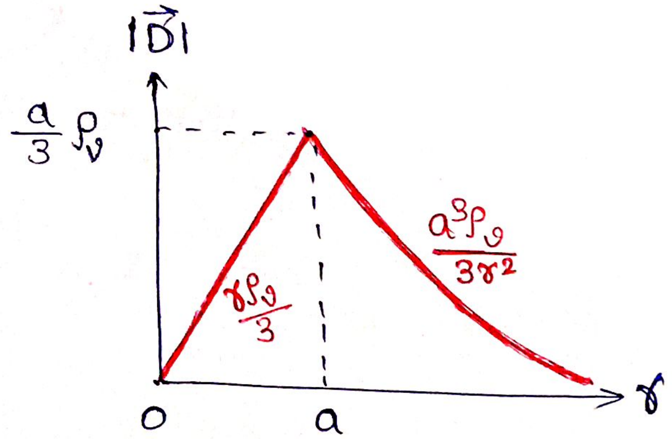

Fig :- Sketch of $|\vec{D}|$ against 'r' for a uniformly charged sphere.