1.3.1 Continuous-Time Sinusoidal Signals

A simple harmonic oscillation is mathematically described by the following continuous-time sinusoidal signal:

$$x_a(t) = A\cos(\Omega t + \theta), \quad -\infty < t < \infty \tag{1.3.1}$$

shown in Fig. 1.3.1. The subscript a used with $x(t)$ denotes an analog signal. This signal is completely characterized by three parameters: A is the *amplitude* of the sinusoid, Ω is the *frequency* in radians per second (rad/s), and θ is the *phase* in radians. Instead of Ω, we often use the frequency F in cycles per second or hertz (Hz), where

$$\Omega = 2\pi F \tag{1.3.2}$$

In terms of F, (1.3.1) can be written as

$$x_a(t) = A\cos(2\pi F t + \theta), \quad -\infty < t < \infty \tag{1.3.3}$$

We will use both forms, (1.3.1) and (1.3.3), in representing sinusoidal signals.

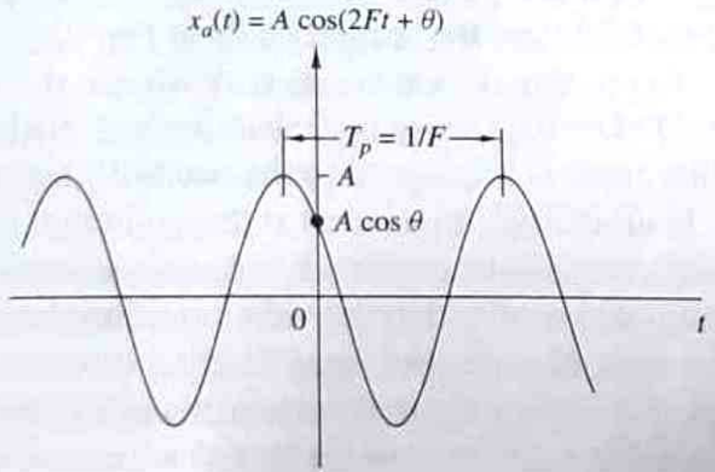

Figure 1.3.1
Example of an analog sinusoidal signal.

1.3.2 Discrete-Time Sinusoidal Signals

A discrete-time sinusoidal signal may be expressed as

$$x(n) = A\cos(\omega n + \theta), \quad -\infty < n < \infty \tag{1.3.7}$$

where n is an integer variable, called the sample number, A is the *amplitude* of the sinusoid, ω is the *frequency* in radians per sample, and θ is the *phase* in radians.

If instead of ω we use the frequency variable f defined by

$$\omega \equiv 2\pi f \tag{1.3.8}$$

the relation (1.3.7) becomes

$$x(n) = A\cos(2\pi f n + \theta), \quad -\infty < n < \infty \tag{1.3.9}$$

The frequency f has dimensions of cycles per sample. In Section 1.4, where we consider the sampling of analog sinusoids, we relate the frequency variable f of a discrete-time sinusoid to the frequency F in cycles per second for the analog sinusoid. For the moment we consider the discrete-time sinusoid in (1.3.7) independently of the continuous-time sinusoid given in (1.3.1). Figure 1.3.3 shows a sinusoid with frequency $\omega = \pi/6$ radians per sample ($f = \frac{1}{12}$ cycles per sample) and phase $\theta = \pi/3$.

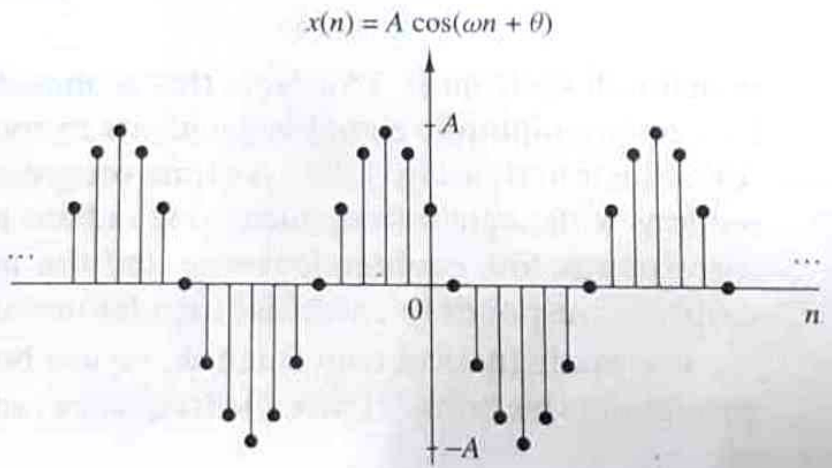

Figure 1.3.3 Example of a discrete-time sinusoidal signal ($\omega = \pi/6$ and $\theta = \pi/3$).

Analog-to-Digital and Digital-to-Analog Conversion

Most signals of practical interest, such as speech, biological signals, seismic signals, radar signals, sonar signals, and various communications signals such as audio and video signals, are analog. To process analog signals by digital means, it is first necessary to convert them into digital form, that is, to convert them to a sequence of numbers having finite precision. This procedure is called *analog-to-digital (A/D) conversion*, and the corresponding devices are called *A/D converters* (*ADCs*).

Conceptually, we view A/D conversion as a three-step process. This process is illustrated in Fig. 1.4.1.

1. *Sampling*. This is the conversion of a continuous-time signal into a discrete-time signal obtained by taking "samples" of the continuous-time signal at discrete-time instants. Thus, if $x_a(t)$ is the input to the sampler, the output is $x_a(nT) \equiv x(n)$, where T is called the *sampling interval*.

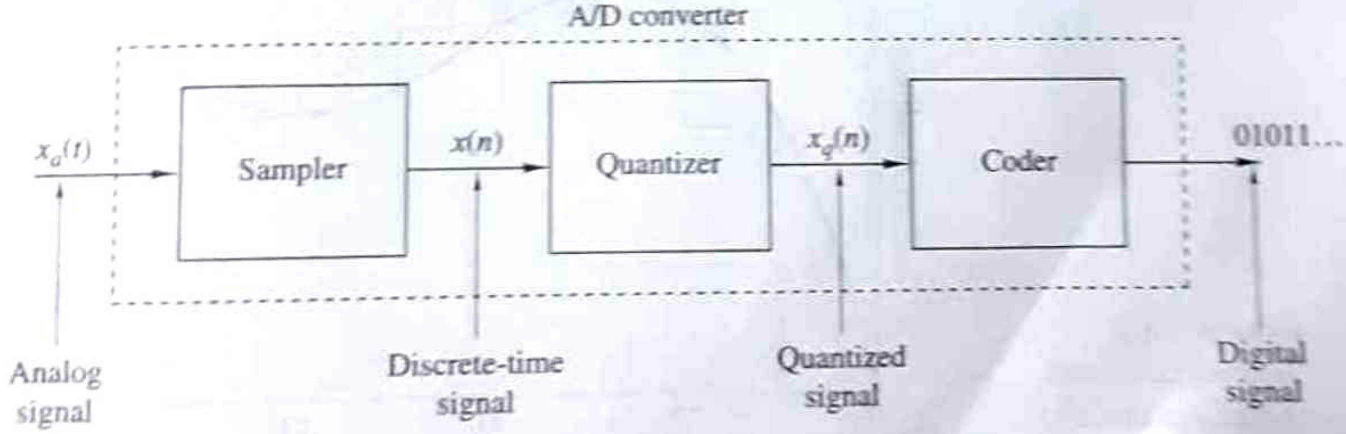

Figure 1.4.1 Basic parts of an analog-to-digital (A/D) converter.

2. *Quantization*. This is the conversion of a discrete-time continuous-valued signal into a discrete-time, discrete-valued (digital) signal. The value of each signal sample is represented by a value selected from a finite set of possible values. The difference between the unquantized sample $x(n)$ and the quantized output $x_q(n)$ is called the quantization error.
3. *Coding*. In the coding process, each discrete value $x_q(n)$ is represented by a b-bit binary sequence.

1.4.1 Sampling of Analog Signals

There are many ways to sample an analog signal. We limit our discussion to *periodic* or *uniform sampling*, which is the type of sampling used most often in practice. This is described by the relation

$$x(n) = x_a(nT), \qquad -\infty < n < \infty \tag{1.4.1}$$

where $x(n)$ is the discrete-time signal obtained by "taking samples" of the analog signal $x_a(t)$ every T seconds. This procedure is illustrated in Fig. 1.4.3. The time interval T between successive samples is called the *sampling period* or *sample interval* and its reciprocal $1/T = F_s$ is called the *sampling rate* (samples per second) or the *sampling frequency* (hertz).

Periodic sampling establishes a relationship between the time variables t and n of continuous-time and discrete-time signals, respectively. Indeed, these variables are linearly related through the sampling period T or, equivalently, through the sampling rate $F_s = 1/T$, as

$$t = nT = \frac{n}{F_s} \tag{1.4.2}$$

As a consequence of (1.4.2), there exists a relationship between the frequency variable F (or Ω) for analog signals and the frequency variable f (or ω) for discrete-time signals. To establish this relationship, consider an analog sinusoidal signal of the form

$$x_a(t) = A\cos(2\pi Ft + \theta) \tag{1.4.3}$$

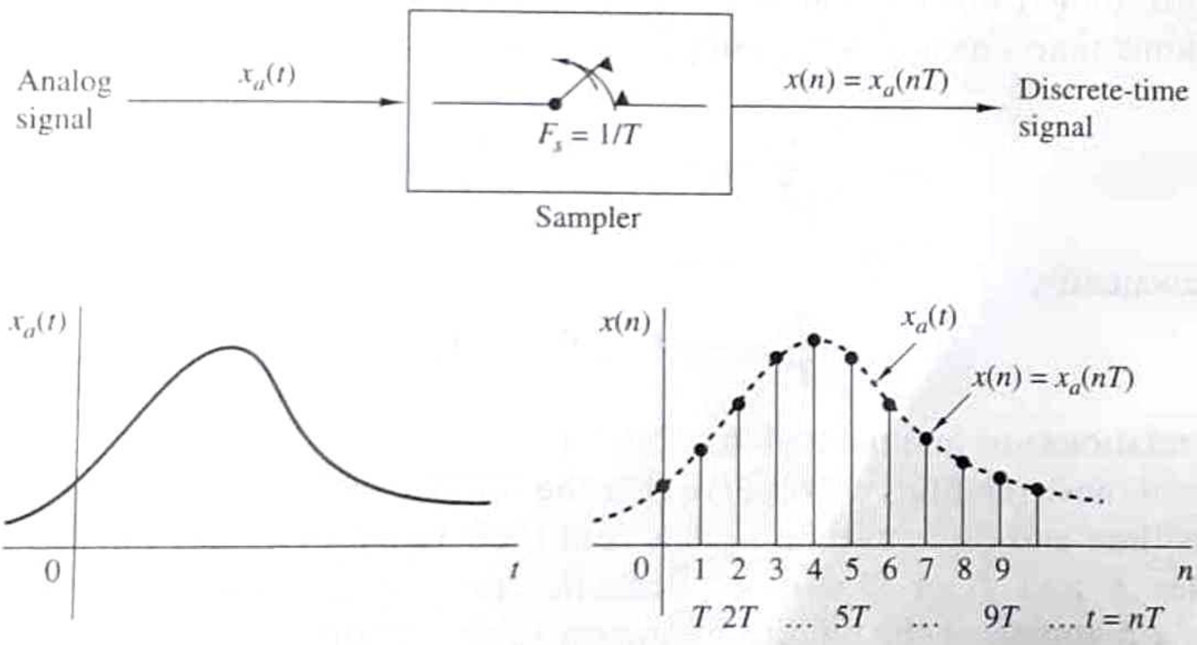

Figure 1.4.3 Periodic sampling of an analog signal.

which, when sampled periodically at a rate $F_s = 1/T$ samples per second, yields

$$\begin{aligned} x_a(nT) \equiv x(n) &= A\cos(2\pi F nT + \theta) \\ &= A\cos\left(\frac{2\pi nF}{F_s} + \theta\right) \end{aligned} \tag{1.4.4}$$

If we compare (1.4.4) with (1.3.9), we note that the frequency variables F and f are linearly related as

$$f = \frac{F}{F_s} \tag{1.4.5}$$

or, equivalently, as

$$\omega = \Omega T \tag{1.4.6}$$

The relation in (1.4.5) justifies the name *relative ornormalized frequency*, which is sometimes used to describe the frequency variable f. As (1.4.5) implies, we can use f to determine the frequency F in hertz only if the sampling frequency F_s is known.

We recall from Section 1.3.1 that the ranges of the frequency variables F or Ω for continuous-time sinusoids are

$$\begin{aligned} -\infty < F < \infty \\ -\infty < \Omega < \infty \end{aligned} \tag{1.4.7}$$

However, the situation is different for discrete-time sinusoids. From Section 1.3.2 we recall that

$$\begin{aligned} -\frac{1}{2} < f < \frac{1}{2} \\ -\pi < \omega < \pi \end{aligned} \tag{1.4.8}$$

By substituting from (1.4.5) and (1.4.6) into (1.4.8), we find that the frequency of the continuous-time sinusoid when sampled at a rate $F_s = 1/T$ must fall in the range

$$-\frac{1}{2T} = -\frac{F_s}{2} \leq F \leq \frac{F_s}{2} = \frac{1}{2T} \tag{1.4.9}$$

or, equivalently,

$$-\frac{\pi}{T} = -\pi F_s \leq \Omega \leq \pi F_s = \frac{\pi}{T} \tag{1.4.10}$$

These relations are summarized in Table 1.1.

From these relations we observe that the fundamental difference between continuous-time and discrete-time signals is in their range of values of the frequency variables F and f, or Ω and ω. Periodic sampling of a continuous-time signal implies a mapping of the infinite frequency range for the variable F (or Ω) into a finite frequency range for the variable f (or ω). Since the highest frequency in a

TABLE 1.1 Relations Among Frequency Variables

Continuous-time signals		Discrete-time signals
$\Omega = 2\pi F$		$\omega = 2\pi f$
$\frac{\text{radians}}{\text{sec}}$ Hz		$\frac{\text{radians}}{\text{sample}}$ $\frac{\text{cycles}}{\text{sample}}$
	$\xrightarrow{\omega = \Omega T,\ f = F/F_s}$	$-\pi \le \omega \le \pi$
	$\xleftarrow{\Omega = \omega/T,\ F = f \cdot F_s}$	$-\frac{1}{2} \le f \le \frac{1}{2}$
$-\infty < \Omega < \infty$		$-\pi/T \le \Omega \le \pi/T$
$-\infty < F < \infty$		$-F_s/2 \le F \le F_s/2$

discrete-time signal is $\omega = \pi$ or $f = \frac{1}{2}$, it follows that, with a sampling rate F_s, the corresponding highest values of F and Ω are

$$F_{\max} = \frac{F_s}{2} = \frac{1}{2T}$$
$$\Omega_{\max} = \pi F_s = \frac{\pi}{T} \tag{1.4.11}$$

Therefore, sampling introduces an ambiguity, since the highest frequency in a continuous-time signal that can be uniquely distinguished when such a signal is sampled at a rate $F_s = 1/T$ is $F_{\max} = F_s/2$, or $\Omega_{\max} = \pi F_s$. To see what happens to frequencies above $F_s/2$, let us consider the following example.

EXAMPLE 1.4.1

The implications of these frequency relations can be fully appreciated by considering the two analog sinusoidal signals

$$x_1(t) = \cos 2\pi(10)t$$
$$x_2(t) = \cos 2\pi(50)t \tag{1.4.12}$$

which are sampled at a rate $F_s = 40$ Hz. The corresponding discrete-time signals or sequences are

$$x_1(n) = \cos 2\pi\left(\frac{10}{40}\right)n = \cos\frac{\pi}{2}n$$
$$x_2(n) = \cos 2\pi\left(\frac{50}{40}\right)n = \cos\frac{5\pi}{2}n \tag{1.4.13}$$

However, $\cos 5\pi n/2 = \cos(2\pi n + \pi n/2) = \cos \pi n/2$. Hence $x_2(n) = x_1(n)$. Thus the sinusoidal signals are identical and, consequently, indistinguishable. If we are given the sampled values generated by $\cos(\pi/2)n$, there is some ambiguity as to whether these sampled values correspond to $x_1(t)$ or $x_2(t)$. Since $x_2(t)$ yields exactly the same values as $x_1(t)$ when the two are sampled at $F_s = 40$ samples per second, we say that the frequency $F_2 = 50$ Hz is an *alias* of the frequency $F_1 = 10$ Hz at the sampling rate of 40 samples per second.