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ECE – A

Advanced Calculus and Complex Analysis-

18MAB102T

ADVANCED CALCULUS AND COMPLEX ANALYSIS

18MAB102T ASSIGNMENT - II

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1. Verify Stokes Theorem for
$F = (y-z+2)\hat{i} + (yz+4)\hat{j} - xy\hat{k}$
where S is an open surface of a cube
$x=0, x=2, y=0, y=2$ and $z=0$ and $z=2$.

Soln: Given Cube is an open cube.
By Stokes Theorem;

$$\oint_c F \cdot dr = \iint_S \text{Curl } \vec{F} \cdot \hat{n} \, dS.$$

Assuming that the Cube is open in the z direction,
The normal vector is, $\hat{n} = \hat{k}$.

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (y-z+2) & (yz+4) & -xy \end{vmatrix}$$

$$= \vec{i}(-x-y) - \vec{j}(-y+1) + \vec{k}(0-1).$$

$$= -\vec{i}(x+y) + \vec{j}(y-1) - \vec{k}$$

$$\text{Curl } \vec{F} \cdot \hat{n} = [-\vec{i}(x+y) + \vec{j}(y-1) - \vec{k}] \cdot \vec{k}$$

$$= -1$$

$$\oint_c F \cdot dr = \int_{y=0}^{2}\int_{x=0}^{2} \text{Curl } \vec{F} \cdot \hat{n} \, dx \, dy$$

$$= \int_0^2\int_0^2 -1 \, dx dy = -\int_0^2 [x]_0^2 \, dy = -2[y]_0^2$$

$$= -4$$

2. Verify Gauss divergence theorem for the function $F = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ taken over the cube bounded by the planes $x=0$, $x=1$, $y=0$, $y=1$, $z=0$ and $z=1$.

Soln:

Given.

$F = 4xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k}$

By Gauss divergence theorem.

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dv.$$

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right)$$

$$= 4z - 2y + y.$$

$$= 4z - y.$$

then,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1\int_0^1 (4z-y)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 \left[4xz - xy\right]_0^1 dy\,dz$$

$$= \int_0^1 \left[4zy - \frac{y^2}{2}\right]_0^1 dz$$

$$= \left[4\cdot\frac{z^2}{2} - \frac{z}{2}\right]_0^1 = 2 - \frac{1}{2} = \frac{3}{2}$$

3. Find the Laplace Transform of the Periodic Function,

$$f(t) = \begin{cases} t & \text{if } 0 < t < 1 \\ 2-t & 1 < t < 2 \end{cases}$$

given that $f(t+2) = f(t)$.

Soln. The Laplace Transform for a Periodic function is given by,

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) \cdot dt \qquad \text{where } T = 2$$

$$= \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t) \cdot dt$$

$$= \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} \cdot t \cdot dt + \int_1^2 e^{-st} (2-t) \cdot dt \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\left(t\left(\frac{e^{-st}}{-s}\right) - \left(\frac{e^{-st}}{s^2}\right)\right)_0^1 + \left((2-t)\left[\frac{e^{-st}}{-s}\right] + 1\left[\frac{e^{-st}}{s^2}\right]\right)_1^2 \right]$$

$$= \frac{1}{1-e^{-2s}} \left[-\cancel{\frac{e^{-s}}{s}} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} + \frac{e^{-2s}}{s^2} + \cancel{\frac{e^{-s}}{s}} - \frac{e^{-s}}{s^2} \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{1}{s^2} \left[1 - 2e^{-s} + e^{-2s} \right] \right]$$

$$= \frac{(e^{-s}-1)^2}{s^2(1-e^{-2s})} = \frac{(1-e^{-s})^{\cancel{2}}}{s^2(1+e^{-s})\cancel{(1-e^{-s})}}$$

$$L\{f(t)\} = \frac{(1-e^{-s})}{s^2(1+e^{-s})} = \frac{1}{s^2}\left[\frac{1 - \dfrac{e^{-s}\cdot e^{+s/2}}{e^{+s/2}}}{1 + \dfrac{e^{-s}\cdot e^{+s/2}}{e^{+s/2}}}\right]$$

$$= \frac{1}{s^2}\left[\frac{e^{+s/2} - e^{-s/2}}{e^{s/2} + e^{-s/2}}\right]$$

$$= \frac{1}{s^2}\tanh\left[\frac{s}{2}\right] \qquad \left[\tanh x = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right].$$

4. Using Convolution theorem evaluate $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$

Soln. wkt.

$$L[f(t) * g(t)] = F(s)\,G(s) \quad -① $$

Then;

$$L^{-1}[F(s)] \cdot L^{-1}[G(s)] = f(t) * g(t) \quad - ②$$

Also;

$$f(t) * g(t) = \int_0^t f(t)\, g(t-u)\cdot du \quad - ③$$

Given

$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$ can be written as.

$$L^{-1}\left[\frac{s}{s^2+a^2}\right] L^{-1}\left[\frac{s}{s^2+b^2}\right] = \cos at * \cos bt$$

$f(t) = \cos at$; $g(t) = \cos bt$

On substituting in eqn 3.

$$\cos at * \cos bt = \int_0^t \cos at \cos b(t-u) \cdot du.$$

$$[2\cos A \cos B = \cos(A+B) + \cos(A-B)].$$

$$= \frac{1}{2}\int_0^t [\cos(at+bt-bu) + \cos(at-bt+bu)] \cdot du.$$

$$= \frac{1}{2}\left[\frac{\sin(at+bt-bu)}{-b} + \frac{\sin(at-bt+bu)}{b}\right]_0^t.$$

$$= \frac{1}{-2b}[\sin(at+bt-bt) - \sin(at+bt) - (\sin at - bt + bt) - \sin(at-bt)]$$

$$= \frac{1}{-2b}[\cancel{\sin at} - \sin(at+bt) - \cancel{\sin at} + \sin(at-bt)]$$

$$= \frac{\sin(at+bt) - \sin(at-bt)}{2b}$$

$\therefore$

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \frac{\sin(at+bt) - \sin(at-bt)}{2b}.$$

5. Using Laplace Transform method solve $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + xt = e^{-t}$,
Given $x(0) = 2$ and $x'(0) = 1$.

Soln. wkt.

$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$.

$L[f'(t)] = sL[f(t)] - f(0)$.

Rewriting the eqn we get.

$x''(t) - 2x'(t) + t\,x(t) = e^{-t}$.

Taking Laplace transform on both sides we get.

$L[x''(t)] - 2L[x'(t)] + L[t\,x(t)] = L[e^{-t}]$.

$\Rightarrow s^2 L[x(t)] - s\,x(0) - x'(0) - 2sL[x(t)] + 2x(0)$

$- \frac{d}{ds} L[x(t)] = \frac{1}{s+1}$

(Substituting)

$\Rightarrow s^2 L[x(t)] - 2s - 1 - 2sL[x(t)] + 4 - \frac{d}{ds} L[x(t)] = \frac{1}{s+1}$.

$L[x(t)]\left(s^2 - 2s - \frac{d}{ds}\right) + 3 = \frac{1}{s+1}$

$L[x(t)](s^2 - 2s) = \frac{1 - 3s - 3}{s+1}$

$L(x(t)) = \frac{-3s-2}{(s+1)(s^2-2s)}$.

$x(t) = L^{-1}\left[\frac{-3s-2}{(s+1)(s^2-2s)}\right]$ → F(S)

F(S) can be rewritten as:

$$\frac{-3s-3}{(s+1)(s^2-2s)} = \frac{A}{s+1} + \frac{Bs+C}{s^2-2s} \quad -①$$

(or)

$$-3s-\cancel{3}\,2 = A(s^2-2s) + (Bs+C)(s+1).$$

When $s=-1$	$s=0$	$s=2$.
$3-2 = A(1+2)$	$-2 = C$	$(2B+C)(3) = -8.$
$1 = 3A$		$6B+3C = -8$
$A = \frac{1}{3}$		$6B-6 = -8.$
		$6B = -2$
		$B = -\frac{1}{3}$

Substituting A, B and C in ① and finding laplace we get,

$$L^{-1}\left[\frac{1}{3(s+1)} + \frac{s}{3(s^2-2s)} - \frac{2}{(s^2-2s)}\right]$$

$\because s^2-2s$ can be rewritten as. $(s-1)^2-1$

$$L^{-1}\left[\frac{1}{3(s+1)}\right] - \frac{1}{3}L^{-1}\left[\frac{s}{(s-1)^2-1}\right] - 2L^{-1}\left[\frac{1}{(s-1)^2-1}\right].$$

$$= \frac{1}{3}e^{-t} - \frac{e^t}{3}L^{-1}\left[\frac{s}{s^2-1}\right] - 2e^t\left[\frac{1}{s^2-1}\right].$$

$$\boxed{= \frac{e^{-t}}{3} - \frac{e^t}{3}\cosh t - 2e^t\sinh t = x(t).}$$