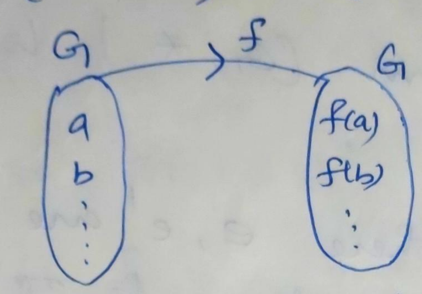
Group Homomorphism :-

If $(G, *)$ & (G', Δ) are two groups, then a mapping $f : G \to G'$ is called a group homomorphism if

$$f(a * b) = f(a) \,\Delta\, f(b) \quad \forall\, a, b \in G \;\&\; f(a), f(b) \in G'$$



$(G, *)$ $\qquad$ (G', Δ)

Eg :- $(G, +)$ – G - set of all real no.

$(G', \times)$ – G' – set of non zero real numbers.

$f(x) = 2^x \quad \forall\, x \in G.$

Let $a, b \in G$ then $f(a) = 2^a$, $f(b) = 2^b$.

$$f(a+b) = 2^{a+b}$$
$$= 2^a \cdot 2^b \quad \forall\, a, b \in G.$$
$$= f(a) \cdot f(b)$$

Note :- A group homomorphism f is called group isomorphism if f is 1-1 & onto.

properties :-

If $f : G \to G'$ is a group homomorphism from $(G, *)$ to (G', Δ) then

(i) $f(e) = e'$ where e, e' are identity elements of G & G'

(ii) for any $a \in G$, $f(a^{-1}) = [f(a)]^{-1}$

(iii) If H is a subgroup of G, then $f(H)$ is a subgroup of G'

i.e. $f(H) = \{f(h) \mid h \in H\}$

Proof :-

Let $x \in G$, i.e. $f(x) \in G'$

Since f is a homomorphism.

$\because x * e = x$

$$e' \Delta f(x) = f(x)$$
$$= f(e * x)$$

$= f(e) \triangle f(x)$ $\because f$ is a homomorphism

By using Right cancellation law

$e' = f(e)$.

(ii) To prove $f(x^{-1}) = [f(x)]^{-1}$ $\forall x \in G$.

Let $x \in G \Rightarrow x^{-1} \in G$.

ie $f(x) \in G'$ & $f(x^{-1}) \in G'$.

Now f is a homomorphism

$f(x) \triangle f(x^{-1}) = f(x * x^{-1})$ $\because a * a^{-1} = e$

$= f(e)$

$= e'$

ie $f(x^{-1}) = [f(x)]^{-1}$

(iii) Let $h_1, h_2 \in H$ & $h_1' \ \& \ h_2' \in f(H)$

ie $h_1' = f(h_1)$ & $h_2' = f(h_2)$.

$h_1' \triangle (h_2')^{-1} = f(h_1) \triangle [f(h_2)]^{-1}$

$= f(h_1) \triangle f(h_2^{-1})$ by property (ii)

$= f(h_1 * h_2^{-1})$ $\because f$ is a homomorphism

$\in f(H)$ $\because$ H is a subgroup $a, b \in H \Rightarrow a * b^{-1} \in H$.

ie $h_1' \circ (h_2')^{-1} \in f(H)$

for $h_1', h_2' \in f(H)$.

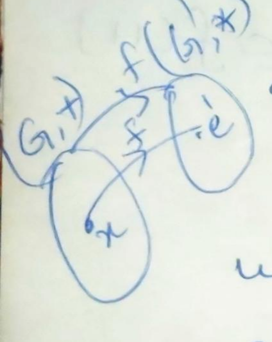
Thus $h_1', h_2' \in f(H)$

$\Rightarrow h_1' \circ (h_2')^{-1} \in f(H)$

$\therefore f(H)$ is a subgroup of G'.

Kernel of a homomorphism :-

If $f : G \to G'$ is a homomorphism then the kernel of homomorphism f is denoted by $\ker f$ or K and defined as.



The set of those elements of G which are mapped to the identity element of G' under mapping f.

$$\ker f \text{ or } K = \{ x : x \in G \mid f(x) = e' \}$$

$e' \to$ identity element of G'

↓ contains all the elements of G which are mapped by f onto e' of G'

1. $(G, +)$ — Additive group — set of Real numbers

$(G', \times) \to$ set of non zero Real numbers.

$f(x) = 2^x \quad \forall x \in G.$

$a, b \in G, \quad f(a) = 2^a$

$f(b) = 2^b.$

$$f(a+b) = 2^{a+b}$$
$$= 2^a \cdot 2^b$$
$$= f(a) \cdot f(b)$$

$\therefore f$ is a homomorphism.

$f(x) = e' = 1$

$2^x = 1 \qquad \mathrm{Ker} f = \{0\}$

$\boxed{x=0} \qquad \mathrm{Ker} f \in G.$

2. $(G, +)$ — set of integer.

$G' = \{1, -1\} \quad (G', \times).$

$f(x) = (-1)^x \quad \forall x \in G.$

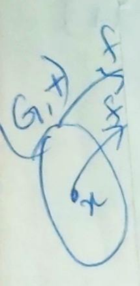
Let $a, b \in G$.

$$f(a+b) = (-1)^{a+b}$$
$$= (-1)^{a} \cdot (-1)^{b}$$
$$= f(a) \cdot f(b)$$

$\therefore$ f is a homomorphism.

$$f(x) = e' = 1$$
$$(-1)^{x} = 1$$
$$x = 0, \pm 2, \pm 4, \pm 8 \, \cdots$$
$$\ker f = \{0, \pm 2, \pm 4, \pm 8 \cdots\}$$

(3). $f : G \to G'$. $G' = \mathbb{Z}_4$ (Addition modulo 4.



$+$ $\quad G = \mathbb{Z}$

$a = 2, \; b = 5$ $\quad$ 2, 5 $\quad$ $f(2) = \bar{2} = 2$, $f(5) = \bar{5} = 1$

$$f(2+5) = f(7) = \bar{7} = 3$$

$f(2) +_4 f(5)$
$2 +_4 1$
$= 3$

$\therefore$ f is a group homomorphism

$$\ker f = \{x \in G \mid f(x) = e' \; e' \in G'\}$$
i.e. $m \in \mathbb{Z} \mid f(m) = 0 = e'$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$f(0) = \bar{0} = 0$

$f(1) = \bar{1} = 1 \neq 0.$

$f(-1) = (\overline{-1}) = -1+4 = 3 \neq 0$

$f(4) = \bar{4} = 0$

$f(-4) = -\bar{4} = -4+4 = 0.$

$= \{0, \pm 4, \pm 8 \cdots\}$

$k \in Z$

$= k \cdot 4$

(4) consider the groups. $(R^+, \cdot)$ & $(R, +)$

let $f : R^+ \to R$ be defined by

$f(x) = \log_{10} x$ check f is a homomorphism

or not.

soln For $x, y \in R^+$

$f(x \cdot y) = \log_{10}(xy)$

$= \log_{10} x + \log_{10} y$

$= f(x) + f(y)$

ie $f(x \cdot y) = f(x) + f(y)$

f preserves the operation.

$\therefore f$ is a homomorphism.

(5) If R & C are additive groups of real & complex no. resp & if the mapping $f : C \to R$ is defined by $f(x+iy) = x$. Show that f is a homomorphism & find $\ker f$.

Soln Let $z_1, z_2 \in C$, Let $z_1 = a+ib$, $z_2 = c+id$

$a+ib,\ c+id \in C$.

$$f(a+ib+(c+id)) \quad f(z_1+z_2)$$

$$= f(a+c+i(b+d))$$

$$= a+c.$$

$$= f(a+ib) + f(c+id)$$

$$= f(z_1) + f(z_2)$$

$\therefore$ f preserves operation.

$\therefore$ f is a homomorphism.

$$\ker f = \{ x+iy \in C \mid f(x) = 0 \}$$

$\because 0$ is the identity element of $(R, +)$

$$f(x) = f(a+ib) = 0$$

ie $a = 0$.

$\therefore$ $\ker(f) = \{$ All purely imaginary no $\}$.

(6) If G is the set of all ordered pairs (a,b) of real no & $*$ is the binary operation defined by

$$(a,b) * (c,d) = (a+c, b+d)$$

G' is the additive group of real no & $f : G \to G'$ is defined by $f(a,b) = a$ $\forall$ $(a,b) \in G$. check f is homomorphism.

Soln

$(a,b), (c,d) \in G$.

$$f((a,b) * (c,d)) = f(a+c, b+d)$$

$$= a+c$$

$$= f(a,b) + f(c,d)$$

$\therefore$ f preserves operation.

f is homomorphism.