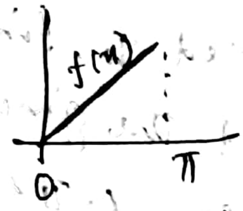
Half-Range Series

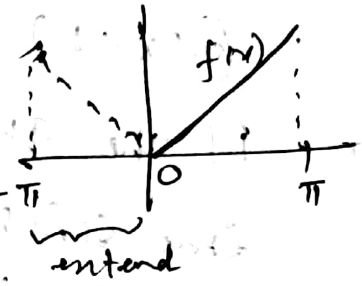
Half-Range Cosine series in the interval $0 < x < \pi$

* If we need a cosine series for $f(x)$ defined in $(0 < x < \pi)$ then we will make the given $f(x)$ as even function in the interval $(-\pi < x < \pi)$

* Graphically, For (eg)



to Convert → even fn



(Given problem)

Then the Fourier series will be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

[Period = 2π
Half period = π]

where

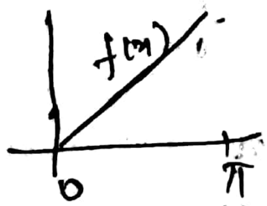
$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Half-Range Sine series in the interval $(0 < x < \pi)$

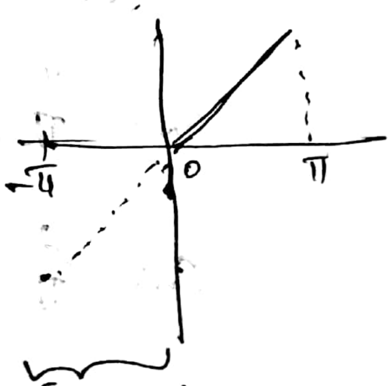
* we will make a given function $f(x)$ in the interval $0 < x < \pi$ to a odd function in $(-\pi < x < \pi)$

* Graphically



Given problem

to convert → odd fn



(extend)

Note:

(1) Don't check whether the given function is odd/even for finding half Range problems.

(2) If the question is asked to find half Range cosine series, then extend the given function is even in $(-L<x<L)$ or $(-\pi<x<\pi)$ from $(0<x<L)$ or $(0<x<\pi)$.

(3) If the question is asked to find half Range sine series, then extend the given function is odd in $(-L<x<L)$ (or) $(-\pi<x<\pi)$ from $(0<x<L)$ (or) $(0<x<\pi)$.

(4) For half Range problems, the given period is $f(x)$ in $(0<x<L)$ itself half period. ie, then half period $= L$.

① Find the half Range Fourier cosine / sine series for $f(x)=x$ in $0<x<\pi$.

Sol: Here $f(x)=x$ in $0<x<\pi$

Half period $= \pi$, Total period $= 2\pi$.

To find: Half Range cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \longrightarrow ①$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{ (x)\left(\cancel{\frac{\sin nx}{n}}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{ \frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right\}$$

$$= \frac{2}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right)$$

$$= \begin{cases} -\frac{4}{\pi n^2} & \text{if } n\text{-odd} \\ 0 & \text{if } n\text{-even}\end{cases}$$

$$\therefore f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n\text{-odd}} \frac{1}{n^2}\cos nx$$

<u>To find</u>: Half Range Sine Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (2)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left\{ (x)\left(-\frac{\cos nx}{n}\right) - (1)\cdot\left[\cancel{-\frac{\sin nx}{n^2}}\right]\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left(-\pi\frac{\cos n\pi}{n}\right) + 0\right\}$$

$$= \frac{2}{\pi}\left(-\pi\frac{(-1)^n}{n}\right)$$

$$= \frac{2(-1)^{n-1}}{n}$$

$$f(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin nx$$

(2) Express $f(x) = x(\pi - x)$, $0 < x < \pi$ as a Fourier series of periodicity 2π containining (i) sine terms (ii) cosine terms.

Deduce $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

Sol:

Given $f(x) = x(\pi - x)$, $0 < x < \pi$ [For the half range problems always the given interval is half-interval]

(i) Cosine series:

Here, Half Period $= L = \pi$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \ldots$

Where

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}(x\pi - x^2)\,dx = \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^{\pi}$

$\boxed{a_0 = \frac{\pi^2}{3}}$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\cos nx\,dx$

$= \frac{2}{\pi}\left\{(\pi x - x^2)\cancel{\left(\frac{\sin nx}{n}\right)} - (\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + \cancel{(-2)\left(\frac{-\sin nx}{n^3}\right)}\right\}_0^{\pi}$

$= \frac{2}{\pi}\left\{(\pi - 2x)\left(\frac{\cos nx}{n^2}\right)\right\}_0^{\pi}$

$= \frac{2}{\pi}\left\{\left(\frac{-\pi\,(-1)^n}{n^2}\right) - \left(\pi/n^2\right)\right\}$

$a_n = \frac{-2}{n^2}\left[(-1)^n + 1\right]$

$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{-4}{n^2} & \text{if } n \text{ is even} \end{cases}$

$\therefore$ The cosine series is.

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$= \frac{\pi^2/3}{2} + (-2)\sum_{n=1}^{\infty}\frac{(-1)^n+1}{n^2}\cos nx$

$f(x) = \pi^2/6 + (-2)\sum_{n=1}^{\infty}\frac{(-1)^n+1}{n^2}\cos nx$

$f(x) = \frac{\pi^2}{6} - 2\left\{0 + \frac{2\cos 2x}{2^2} + 0 + \frac{2}{4^2}\cos 4x + \cdots\right\}$

Deduction:

$x = \pi/2 \Rightarrow$ $f(\pi/2) = \frac{\pi^2}{6} - 2\left(\frac{2}{2^2}(-1) + \frac{2}{4^2}(1) + \frac{2}{6^2}(-1) + \cdots\right)$

$\pi/2(\pi - \pi/2) = \frac{\pi^2}{6} + 4\left(\frac{1}{2^2} - \frac{1}{4^2} + \cdots\right)$

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$$\frac{\pi^2}{4} - \frac{\pi^2}{6} = \frac{4}{4}\left(\frac{1}{1^2} - \frac{1}{2^2} + \cdots\right)$$

$$\boxed{\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \cdots}$$

(ii) Sine series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2) \sin nx \, dx$$

$$= \frac{2}{\pi}\left\{ \cancel{(\pi x - x^2)\left(\frac{-\cos nx}{n}\right)} - (\pi - 2x)\cancel{\left(\frac{-\sin nx}{n^2}\right)} + (-2)\left(\frac{+\cos nx}{n^3}\right)\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{ \left(0 - \frac{2(-1)^n}{n^3} + 2/n^3\right\}\right.$$

$$= \frac{4}{\pi n^3}\left[1 - (-1)^n\right]$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8}{\pi n^3} & \text{if } n \text{ is odd} \end{cases}$$

The sine series is

$$f(x) = \frac{8}{\pi}\sum_{n=1,3,5,\ldots} \frac{1}{n^3} \sin(nx)$$

(3) Find the half range cosine series for $f(x) = (x-1)^2$ in the interval $0 < x < 1$.

S.T. $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

Sol: Here the half Period $\boxed{L = 1}$, since question is half range

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x).$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$= \frac{2}{1}\int_0^1 (x-1)^2\,dx$$

$$= 2\left[\frac{(x-1)^3}{3}\right]_0^1$$

$$= \frac{2}{3}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{1}\int_0^1 (x-1)^2 \cos(n\pi x)\,dx$$

$$= 2\left\{ (x-1)^2 \cancel{\left(\frac{\sin n\pi x}{n\pi}\right)} - (2(x-1))\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (2)\cancel{\left(+\frac{\sin n\pi x}{n^3}\right)} \right\}_0^1$$

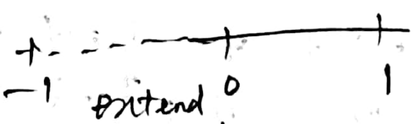
$$= 2\left\{ 2(x-1)\frac{\cos n\pi x}{n^2\pi^2}\right\}_0^1$$

$$= 4\left\{ 0 - (-1)\frac{(-1)^n}{n^2\pi^2}\right\}$$

$$a_n = \frac{4}{n^2\pi^2}$$

$$\therefore f(x) = \frac{2/3}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$$

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\left\{\frac{\cos \pi x}{1^2} + \frac{\cos 2\pi x}{2^2} + \cdots\right\}$$



Deduction:-

At $x=0$, treat as discontinuous

$$f(0) = \frac{f(0^-)+f(0^+)}{2} = \frac{(1)^2+(1)^2}{2} = 1$$

Hence, $1 = \frac{1}{3} + \frac{4}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$

$$\Rightarrow \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots \boxed{= \frac{\pi^2}{6}}$$