Problems on Functions :-

(1). If $f : Z \to N \cup \{0\}$ defined by

$$f(x) = \begin{cases} 2x-1 & x > 0 \to \text{odd} \\ -2x & x \le 0 \to \text{even} \end{cases}$$

(i) Prove that f is 1-1 & onto

(ii) find f^{-1}

Solution :- Let $x_1, x_2 \in Z$ & let $f(x_1) = f(x_2)$

Then either both $f(x_1)$ & $f(x_2)$ are both odd or both even.

[$\because$ From the definition of f, an odd no. cannot be equal to an even number]

If they are both odd, then

$$2x_1 - 1 = 2x_2 - 1$$
$$x_1 = x_2$$

If they are both even, then

$$-2x_1 = -2x_2$$
$$x_1 = x_2$$

Thus whenever $f(x_1) = f(x_2)$ we get $x_1 = x_2$

Hence, $f(x)$ is one-one

Let $y \in N$, If y is odd its prein...

is $y = 2x - 1$

$x = \frac{y+1}{2} > 0$.

& $f\left(\frac{y+1}{2}\right) = 2\left(\frac{y+1}{2}\right) - 1$

$= y + 1 - 1 = y$ $\quad (y \in N, \frac{y+1}{2} \in Z)$

If y is even, then its preimage

is $-y/2$.

$f(-y/2) = -2\left(-\frac{y}{2}\right) = y$ as $-\frac{y}{2} < 0$.

If $y = 0$ its preimage is zero

Thus for every $y \in N \cup \{0\}$, there

exists $x \in Z$ s.t. $f(x) = y$.

Hence f is onto.

Hence f is invertible.

b. $y = f(x) = \begin{cases} 2x - 1 & x > 0 \\ -2x & x \leq 0. \end{cases}$

$f^{-1} : N \cup \{0\} \longrightarrow Z$ defined by

$$f^{-1}(y) = x = \begin{cases} \dfrac{y+1}{2} & y = 1, 3, 5 \cdots \\ -\dfrac{y}{2} & y = 0, 2, 4, 6 \cdots \end{cases}$$

or

$$f^{-1}(x) = \begin{cases} \dfrac{x+1}{2}, & x = 1, 3, 5 \cdots \\ -x/2, & x = 0, 2, 4, 6 \cdots \end{cases}$$

(2). If $A = \{ x \in R \mid x \neq 1/2 \}$ & $f : A \to R - \{2\}$ is defined by $f(x) = \dfrac{4x}{2x-1}$

(i) Find the range of f

(ii) S.T f is invertible.

(iii) find $\mathrm{dom}(f^{-1})$, $\mathrm{range}(f^{-1})$ & formula for f^{-1}.

Solution:- If $y \in \mathrm{range}(f)$, then there exists $x \in \mathrm{dom}(f) = A$ such that $y = \dfrac{4x}{2x-1} = \dfrac{4}{2 - 1/x}$

$y = \infty$, if $2x - 1 = 0$ i.e. $x = 1/2$

$\therefore$ y is defined when $x \neq 1/2$ i.e. $x \in A$.

when $x = \pm\infty$, $y = \dfrac{4}{2 - 1/x} = 2$

$\therefore$ when $x \neq \pm\infty$ (i.e. $x \in A$), $y \neq 2$

$\therefore$ $y \in$ range (f), provided $y \neq 2$.

$\therefore$ range$(f) = \{y \in R \mid y \neq 2\}$

(ii) To prove f is invertible, we have to prove f is 1-1 + onto.

<u>f is 1-1 :-</u>

Let $f(a_1) = f(a_2)$

$$\frac{4a_1}{2a_1-1} = \frac{4a_2}{2a_2-1}$$

ie $8a_1a_2 - 4a_1 = 8a_1a_2 - 4a_2$

$-4a_1 = -4a_2$

$a_1 = a_2$

$\therefore$ f is 1-1.

<u>f is onto :</u>

Let $y \in$ range$(f) = \{y \in R \mid y \neq 2\}$.

If there is $x \in A$ s.t. $y = \frac{4x}{2x-1} = f(x)$.

ie $y(2x-1) = 4x$

or $x(2y-4) = y$

$$x = \frac{y}{2y-4} \in A = \text{dom}(f) \quad (x \neq 1/2)$$

Thus, for any $y \in$ range(f), there is

$\frac{y}{2y-4} \in A$ s.t $f\left(\frac{y}{2y-4}\right) = y$.

$$f\left(\frac{y}{2y-4}\right) = \frac{\frac{4y}{2y-4}}{2\cdot\frac{y}{2y-4} - 1}$$

Hence for any real $y (\neq 2) \in$ range f there exists an $x \in A$ s.t $f(x) = y$

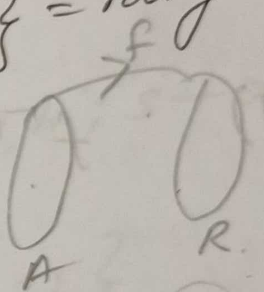
$$= \frac{4y}{(\cancel{2y-4})\left[\frac{\cancel{2y}-2y+4}{\cancel{2y-4}}\right]}$$

$$= y.$$

Hence f is onto.

As f is both one-one & onto, f is invertible.

(iii) $\text{dom}(f^{-1}) = \{y \in R \mid y \neq 2\} = \text{range}(f)$



range$(f^{-1}) = \text{dom}(f) = A$.

$f^{-1} : R \to A$

For any $y \in \text{dom}(f^{-1})$

$$f^{-1}(y) = x = \frac{y}{2y-4}$$

$$\therefore \; f^{-1}(x) = \frac{x}{2x-4}, \; x \neq 2$$

③ If $f, g, h : R \to R$ are defined by $f(x) = x+2$, $g(x) = \frac{1}{x^2+1}$, $h(x) = 3$. find

(i) $g \circ h \circ f$ (ii) $h \circ g \circ f$ (iii) $g \circ f^{-1}$ (iv) $f^{-1} \circ g \circ f$

(i) $(g \circ h \circ f)(x) = (g \circ h) \circ f(x) = (g \circ h)(f(x))$

$= (g \circ h)(x+2) = g[h(x+2)]$

$= g(3) = \frac{1}{10}$.

(ii) $(h \circ g \circ f)(x) = (h \circ g)(f(x))$

$= (h \circ g)(x+2)$

$= h[g(x+2)] = h\left(\frac{1}{(x+2)^2+1}\right)$

$= 3$.

$f^{-1}(y) = x$.

(iii) $f^{-1} : R \to R$ $f(x) = y$

$\Rightarrow x+2 = y$ or

$x = y-2$

$\therefore f(y-2) = y$ $\therefore f^{-1}(y) = x = y-2$ $f(x) = x+2$

$\therefore f^{-1}(x) = x-2$ $f^{-1}(x) = y = x-2$ $f(y-2) = y-2+2$

$= y$

$(g \circ f^{-1})(x) = g[f^{-1}(x)] = g(x-2)$

$= \frac{1}{(x-2)^2+1}$

Note that $[(g \circ f^{-1}) \circ f](x) = (g \circ f^{-1})(f(x))$

$= (g \circ f^{-1})(x+2)$

$= g[f^{-1}(x+2)] = g(x)$

$$= \frac{1}{x^2+1}$$

(iv) $\left[f^{-1}\circ(g\circ f)\right](x) = (f^{-1}\circ g)(x+2)$

$$= f^{-1}\left[\frac{1}{(x+2)^2+1}\right]$$

$$= \frac{1}{(x+2)^2+1} - 2$$

$$= \frac{1-2(x+2)^2-2}{(x+2)^2+1} = -\left[\frac{2(x+2)^2+1}{(x+2)^2+1}\right]$$

④ If $S = \{1,2,3,4,5\}$ & if $f, g, h : S \longrightarrow S$ are defined by

$f = \{(1,2), (2,1), (3,4), (4,5), (5,3)\}$

$g = \{(1,3), (2,5), (3,1), (4,2), (5,4)\}$

$h = \{(1,2), (2,2), (3,4)\ (4,3), (5,1)\}$

(i) verify whether $f\circ g = g\circ f$

(ii) Explain why f & g have inverses but h does not.

(iii) find f^{-1} & g^{-1}

(iv) Show that $(f\circ g)^{-1} = g^{-1}\circ f^{-1} \neq f^{-1}\circ g^{-1}$

soln

$(f \circ g)(1) = f[g(1)] = f(3) = 4$

$(f \circ g)(2) = f[g(2)] = f(5) = 3$

$(f \circ g)(3) = f[g(3)] = f(1) = 2$

$(f \circ g)(4) = f[g(4)] = f(2) = 1$

$(f \circ g)(5) = f[g(5)] = f(4) = 5$

$f \circ g = \{(1,4), (2,3), (3,2), (4,1), (5,5)\}$ —— (1)

$(g \circ f)(1) = g[f(1)] = g(2) = 5$

$(g \circ f)(2) = g[f(2)] = g(1) = 3$

$(g \circ f)(3) = g[f(3)] = g(4) = 2$

$(g \circ f)(4) = g[f(4)] = g(5) = 4$

$(g \circ f)(5) = g[f(5)] = g(3) = 1$

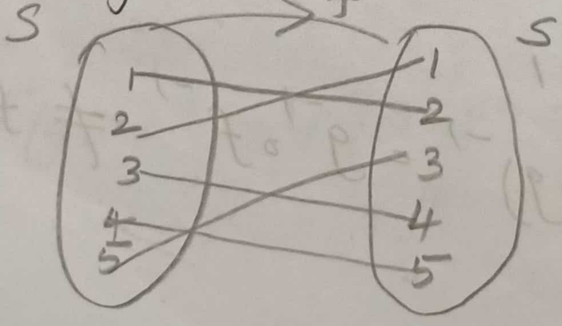
$\therefore g \circ f = \{(1,5), (2,3), (3,2), (4,4), (5,1)\}$ —— (2)

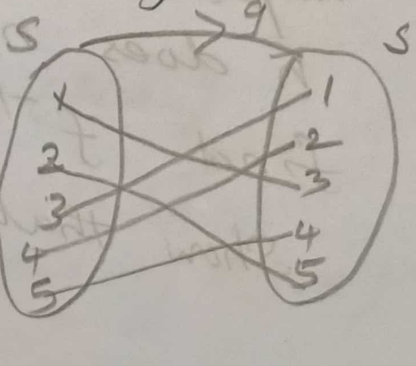
Hence, from (1) & (2)

$f \circ g \neq g \circ f$.

(ii) Both f & g are one to one & onto.

$\therefore$ They are invertible. i.e. f^{-1} & g^{-1} exist.

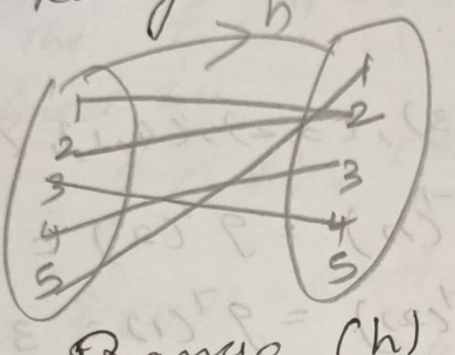




each & every element in the domain should have distinct images in the codomain.

$\therefore$ f & g is 1-1

& Range = codomain — onto.



$h(1) = h(2) = 2$

but $1 \neq 2$.

$\therefore$ h is not 1-1

Also Range (h) $= \{1, 2, 3, 4\} \neq S$.

$\therefore$ 5 does not have pre image in S.

$\therefore$ h is also not onto.

$\therefore$ the inverse of h does not exist.

Hence.

(ii) f^{-1} is obtained by reversing the elements in all the ordered pairs of f.

$f^{-1} = \{(2,1), (1,2), (4,3), (5,4), (3,5)\}$ (A)

It is easily verified that

$f \circ f^{-1} = f^{-1} \circ f = \{(1,1), (2,2), (3,3), (4,4), (5,5)\}$

$= I$.

$|||^{ly}$ $g^{-1} = \{(3,1), (5,2), (1,3), (2,4), (4,5)\}$ — (B)

from (1)

$(f \circ g)^{-1} = \{(4,1), (3,2), (2,3), (1,4), (5,5)\}$

from (A) & (B)

$g^{-1} \circ f^{-1} = \{(1,4), (2,3), (3,2), (4,1), (5,5)\}$

$(g^{-1} \circ f^{-1})(1) = g^{-1}(f^{-1}(1)) = g^{-1}(2) = 4.$

$(g^{-1} \circ f^{-1})(2) = g^{-1}(f^{-1}(2)) = g^{-1}(1) = 3$

$(g^{-1} \circ f^{-1})(3) = g^{-1}(f^{-1}(3)) = g^{-1}(5) = 2$

$(g^{-1} \circ f^{-1})(4) = g^{-1}(f^{-1}(4)) = g^{-1}(3) = 1$

$(g^{-1} \circ f^{-1})(5) = g^{-1}(f^{-1}(5)) = g^{-1}(4) = 5.$

Again from (A) & (B).

$f^{-1} \circ g^{-1} = \{(1,5), (2,3), (3,2), (4,4), (5,1)\}$ (III)

$(f^{-1} \circ g^{-1})(1) = f^{-1}(g^{-1}(1)) = f^{-1}(3) = 5$

$(f^{-1} \circ g^{-1})(2) = f^{-1}(g^{-1}(2)) = f^{-1}(4) = 3$

$(f^{-1} \circ g^{-1})(3) = f^{-1}(g^{-1}(3)) = f^{-1}(1) = 2$

$(f^{-1} \circ g^{-1})(4) = f^{-1}(g^{-1}(4)) = f^{-1}(5) = 4$

$(f^{-1} \circ g^{-1})(5) = f^{-1}(g^{-1}(5)) = f^{-1}(2) = 1$

$\therefore (f \circ g)^{-1} = g^{-1} \circ f^{-1} \neq f^{-1} \circ g^{-1}$

Q). If $A = \{1, 2, 3, 4, 5\}$ $B = \{1, 2, 3, 8, 9\}$
& the $f : A \to B$ & $g : A \to A$ are
defined by
$f = \{(1,8), (3,9), (4,3), (2,1), (5,2)\}$
& $g = \{(1,2), (3,1), (2,2), (4,3), (5,2)\}$ if they
Find $f \circ g$, $g \circ f$, $f \circ f$ & $g \circ g$
exists.

Soln. Let $f : A \to B$ & $g : A \to A$
$\therefore$ $f \circ g : A \to B$.

So

$(f \circ g)(1) = f(g(1)) = f(2) = 1$

$(f \circ g)(2) = f[g(2)] = f(2) = 1$

$(f \circ g)(3) = f(g(3)) = f(1) = 8$

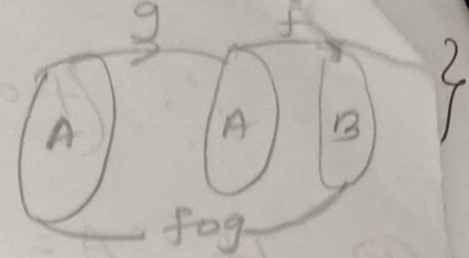
$(f \circ g)(4) = f(g(4)) = f(3) = 9$

$(f \circ g)(5) = f(g(5)) = f(2) = 1$

$\therefore f \circ g = \{(1,1), (2,1), (3,8), (4,9), (5,1)\}$

$Range(f) = \{1, 2, 3, 8, 9\}$

$dom(g) = \{1, 2, 3, 4, 5\}$



$\therefore range(f) \not\subseteq dom(g)$

Hence $(g \circ f)(a) = g[f(a)]$ is not defined.

i.e. $g \circ f$ is not defined.

$[f : A \to \underline{B},\ g : \underline{A} \to A$

Again $Range(f) = \{1, 2, 3, 8, 9\}$

$dom(f) = \{1, 2, 3, 4, 5\}$ $\quad f : A \to B$

$Range(f) \not\subseteq dom(f)$

Hence $f \circ f$ is not defined

$Range(g) = \{1, 2, 3\}$

$dom(g) = \{1, 2, 3, 4, 5\}$

Hence $Range\ g \subseteq dom\ g$

Hence $g \circ g$ is defined.

Now $(g \circ g)(1) = g[g(1)] = g(2) = 2$

$(g \circ g)(2) = g[g(2)] = g[2] = 2$

$(g \circ g)(3) = g[g(3)] = g(1) = 2$

$(g \circ g)(4) = g(g(4)) = g(3) = 1$

$(g \circ g)(5) = g(g(5)) = g(2) = 2$

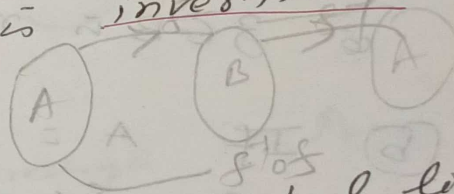
$\therefore g \circ g = \{(1,2),(2,2),(3,2),(4,1)\ (5,2)\}$

—— x —— x

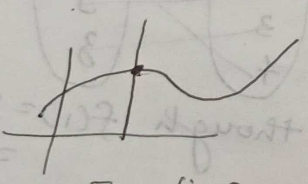
* If $f : A \to B$, then $f^{-1} : B \to A$ is said to be an inverse of f if for any $y \in B$
$f^{-1}(y) = x$ iff $f(x) = y$.

* A function $f : A \to B$ is said to be invertible if there exists a function $f^{-1} : B \to A$ s.t $f^{-1} \circ f$ is the identity map on A & $f \circ f^{-1}$ is the identity map on B. ie $f^{-1} \circ f = I_A$ & $f \circ f^{-1} = I_B$.

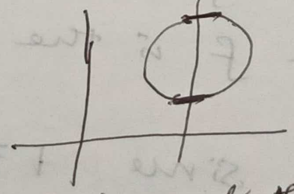
* If $f : A \to B$ then f is invertible iff f is bijective.



* vertical line Test

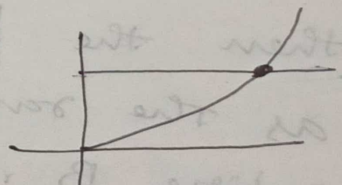


Function



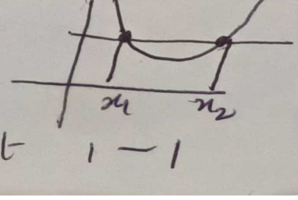
it cut the curve in two pts not a functions

Horizontal line



one line.

$f(x_1) = f(x_2)$
$x_1 \neq x_2$



not 1 — 1

(6) Give an example of a function $N \to N$ which is as a set of ordered pairs which is

(i) one to one but not onto.

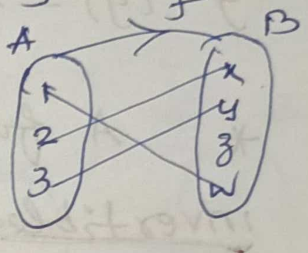
(ii) onto but not one to one

(c) both one-to one & onto.

d) neither one to one nor onto.

(i) If $A = \{1, 2, 3\}$, $B = \{x, y, z, w\}$ &

$f = \{(1, w), (2, x), (3, y)\}$



then the function f is $1-1$

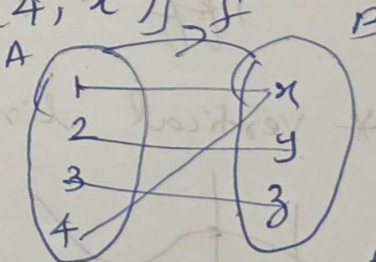
as distinct element of A are mapped into distinct element of B.

f is not onto, as the Range(f) is $\neq B$

as $z \in B$ is not the image of any element of A.

(b) If $A = \{1, 2, 3, 4\}$ $B = \{x, y, z\}$

& $f = \{(1, x), (2, y), (3, z), (4, x)\}$



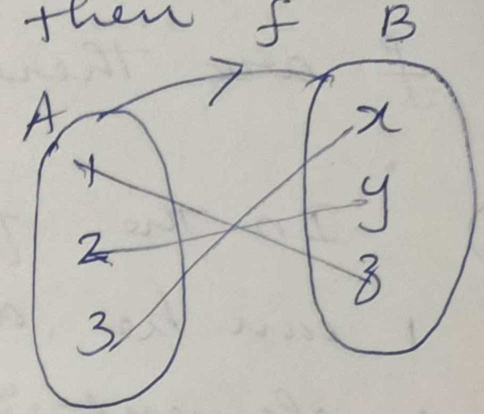
then the function f is onto.

as the range of f is the entire B.

f is not $1-1$. Since $1 \neq 4$ though $f(1) = f(4) = x$.

c) If $A = \{1, 2, 3\}$, $B = \{x, y, z\}$ &

$f = \{(1, z), (2, y), (3, x)\}$ then

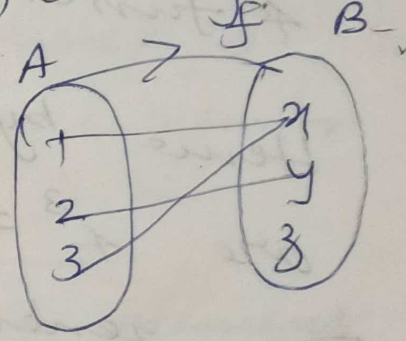


f is 1-1 & onto.

as $f(1)$, $f(2)$, $f(3)$ are all different & the range of f is B.

d) If $A = \{1, 2, 3\}$, $B = \{x, y, z\}$

then $f = \{(1, x), (2, y), (3, x)\}$



then f is neither 1-1 nor onto.

as $f(1) = f(3) = x$

but $1 \neq 3$. & as

range of f is not B.

ie $z \in B$ has no preimage in A.

1) If $A = \{1, 2, 3\}$ $B = \{w, x, y, z\}$ & $f : A \to B$ (i) how many functions f are there ?.

(a) In the function $f : A \to B$, the element 1 can be associated with any of the 4 elements of B. Thus, there are 4 functions with 1 as the argument. ||ly there are 4 functions with 2 & 4 functions with 3 as the argument.

Hence by the rule of product, there are $4^3 = 64$ functions from A to B.

In general if $|A| = m$ & $|B| = n$ there are n^m functions $f : A \to B$.