Since f is onto.

For every $b \in B$, there exist an element $a \in A$ such that
$$f(a) = b$$
Define $g: B \to A$ be a function, ~~where~~ $g(b) = a$ where $f(a) = b$

$(g \circ f)\, a = g(f(a))$

$= g(b) = a = I_A(a)$

Thus $g \circ f = I_A$.

For any $b \in B$

$(f \circ g)(b) = f(g(b))$

$= f(a) = b = I_B(b)$

$\therefore f \circ g = I_B$

Hence g is inverse of f

$\therefore f$ is invertible.

Theorem 5: Let $f: A \to B$ and $g: B \to C$ are invertible function, then $g \circ f: A \to C$ is also invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Proof: Since $f: A \to B$ and $g: B \to C$ are invertible.

i.e They are bijective.

Hence $g \circ f: A \to C$ is also bijective.

$\therefore g \circ f$ is also invertible

$\therefore (g \circ f)^{-1}$ exists and $(g \circ f)^{-1}: C \to A$.

Since $g^{-1}: C \to B$, $f^{-1}: B \to A$; $f^{-1} \circ g^{-1}: C \to A$

$(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ are functions from $C \to A$; thus domain and codomain of these two functions are equal.

To prove for all $c \in C$, $(g \circ f^{-1})(c) = (f^{-1} \circ g^{-1})(c)$

$\because g$ is onto, for every $c \in C$, there exists $b \in B$, such that $g(b) = c$

$\because f$ is onto, for every $b \in B$, there exists $a \in A$, such that $f(a) = b$

$\therefore (g \circ f)(a) = g[f(a)] = g(b) = c$

i.e $(g \circ f)^{-1}(c) = a$ — (1).

$g^{-1}(c) = b$, $f^{-1}(b) = a$

$\therefore (f^{-1} \circ g^{-1})(c) = f^{-1}(g^{-1}(c)) = f^{-1}(b) = a$ — (11)

Q) Determine whether or not each of the following relation is a function with domain $\{1,2,3,4\}$. If any relation is not a function, explain why?

① $R_1 = \{(1,1),(2,1),(3,1),(4,1),(3,3)\}$

R_1 is not a function, since there are two pairs $(3,1)$, $(3,3)$.
Hence image of the element 3 is not unique.

② $R_2 = \{(1,2),(2,3),(4,2)\}$

R_2 is not a function, since there is no image for 3 in domain.

③ $R_3 = \{(1,1),(2,1),(3,1),(4,1)\}$

It is a function even though the domain of 1, 2, 3, 4 is mapped to single element.

Q) If $f: R \to R$, $g: R \to R$; $f(x) = x^2 - 2$, $g(x) = x + 4$
Find gof and fog and state whether these functions are injective, bijective, surjective.

$$gof(x) = g[f(x)]$$
$$= g(x^2 - 2)$$
$$= x^2 - 2 + 4$$
$$= x^2 + 2$$

$$fog(x) = f[g(x)]$$
$$= f(x+4)$$
$$= (x+4)^2 - 2 = x^2 + 16 + 8x - 2$$
$$= x^2 + 8x + 14$$

Here $gof \neq fog$

$f: R \to R$; $f(x) = f(y)$
$$x^2 - 2 = y^2 - 2$$
$$x^2 = y^2$$
$$x = \pm y$$
f is not one → one, f is not injective.

For every $y \in \underset{(R)}{Y}$, there is an element $x \in \underset{(R)}{X}$, such that
$$f(x) = y.$$
$$\Rightarrow x^2 - 2 = y$$
$$\Rightarrow x = \pm\sqrt{y+2} \; . \notin R$$
Hence $f(x)$ is not onto, (surjective).

$g: R \to R$

$g(x) = g(y)$

$\therefore x + 4 = y + 4$.

$\Rightarrow x = y$.

$\therefore g$ is one to one.

For every $y \in Y$ there ~~element~~ exists $x \in X$, such that.

$g(x) = y$.

$\Rightarrow x + 4 = y$.

$\Rightarrow x = y - 4$

$\therefore$ It is onto.

$\therefore g$ is bijective.

Q) Show that the function $f: R \to R$ defined by $f(x) = \frac{x}{x+4}$ is one to one and onto and hence find the inverse.

$f(x) = f(y)$. [Given $f: R \to R$].

$\Rightarrow \frac{x}{x+4} = \frac{y}{y+4}$.

$\Rightarrow \cancel{xy} + 4x = \cancel{xy} + 4y$.

$\Rightarrow x = y$.

$\therefore f$ is one to one.

For every $y \in R$, there is $x \in R$, such that $f(x) = y$.

$\therefore \frac{x}{x+4} = y$.

$\Rightarrow x = (x+4)y$.

$\Rightarrow x = xy + 4y$

$\Rightarrow x(1-y) = 4y$

$\Rightarrow x = \frac{4y}{1-y}$.

$f\left(\frac{4y}{1-y}\right) = \frac{4y/1-y}{\frac{4y}{1-y} + 4} = y$

$\therefore$ It is onto.

$f^{-1}(x) = \frac{4y}{1-y}$.

Q) Show that $f: R-\{3\} \to R-\{1\}$ given by $f(x) = \frac{x-2}{x-3}$ is a bijective.

Let $f(x_1) = f(x_2)$

$\Rightarrow \frac{x-2}{x-3} = \frac{y-2}{y-3}$

$\Rightarrow \cancel{xy} - 3x - 2y + \cancel{6} = \cancel{xy} - 3y - 2x + \cancel{6}$

$\Rightarrow x = y \quad \therefore$ One to one

To prove onto

for every $y \in R-\{1\}$ ~~for~~ there exists $x \in R-\{3\}$ such that

$f(x) = y$

$\Rightarrow \frac{x-2}{x-3} = y$

$\Rightarrow x-2 = xy - 3y$.

$\Rightarrow x(1-y) = 2-3y$

$\Rightarrow x = \frac{2-3y}{1-y}$.

$\Rightarrow x = \frac{3y-2}{y-1}$

$f^{-1}(y) = x = \frac{3y-2}{y-1}$.

Q) Show that $f: R \to R$ defined as $f(x) = 3x^3 - 4$ is one to one function

let $f(x) = f(y)$

$\Rightarrow 3x^3 - 4 = 3y^3 - 4$.

$\Rightarrow x^3 = y^3$.

$\Rightarrow x^3 - y^3 = 0$

$\Rightarrow (x-y)(x^2 + xy + y^2) = 0$

$\Downarrow$ $\quad$ $\Downarrow$

$x-y=0$ $\quad$ Imaginary roots

But domain is real number

$\therefore x - y = 0$

$x = y$ $\therefore$ One to one.

Q) ~~If f: →~~ If $S = \{1,2,3,4,5\}$, if $f, g: S \to S$ are defined by.

$f = \{(1,2)(2,1)(3,4),(4,5),(5,3)\}$

$g = \{(1,3),(2,5),(3,1),(4,2),(5,4)\}$

$h = \{(1,2),(2,2),(3,4),(4,3),(5,1)\}$

(i) Verify $f \circ g = g \circ f$

(ii) Explain why f and g have inverse but h doesn't.

(iii) find f^{-1}, g^{-1}

(iv) Show that $(f \circ g)^{-1} = g^{-1} \circ f^{-1} \neq f^{-1} \circ g^{-1}$

~~(v) Verify whether $g \circ f = f \circ g$~~.

(i)

$f\circ g(1) = f[g(1)] = f(3) = 4$	$f\circ g(4) = f[g(4)] = f(2) = 1$
$f\circ g(2) = f[g(2)] = f(5) = 3$	$f\circ g(5) = f[g(5)] = f(4) = 5$.
$f\circ g(3) = f[g(3)] = f(1) = 2$	$f\circ g = \{(1,4),(2,3),(3,2),(4,1)$

(5,5)}

$g \circ f(1) = g[f(1)] = g(2) = 5.$

$g \circ f(2) = g(1) = 3$

$g \circ f(3) = g(4) = 2$

$g \circ f(4) = g(2) = 4$

$g \circ f(5) = 1$

$g \circ f = \{(1,5), (2,3), (3,2), (4,4), (5,1)\}$

$\therefore f \circ g \neq g \circ f.$

(iii) $f^{-1} = \{(2,1), (1,2), (4,3), (5,4), (3,5)\}$

$g^{-1} = \{(3,1), (5,2), (1,3), (2,4), (4,5)\}.$

Q) If $A = \{1,2,3,4,5\}$, $B = \{1,2,3,8,9\}$ are defined by

$f: A \to B$, $g: A \to A$

$f = \{(1,8), (3,9), (4,3), (2,1), (5,2)\}$

$g = \{(1,2), (3,1), (2,2), (4,3), (5,2)\}$

Find 1) $f \circ g$ 2) $g \circ f$ 3) $f \circ f$ 4) $g \circ g$. if they exist.

Q) $f: Z \to N \cup 0$ defined by $f(x) = \begin{cases} 2x - 1; & x > 0 \\ -2x; & x \leq 0 \end{cases}$

P.T (i) f is one to one & onto

(ii) find f^{-1}.