The Inverse Fourier Transform is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds = F^{-1}[F(s)]$$

NOTE *In this chapter, we shall use the symbols F for "the Fourier transform of" and F^{-1} for "the inverse Fourier transform of"*

UNIT 2

Properties of Fourier Transform

P1. Fourier Transform is linear

ie $F[af(x) + bg(x)] = aF(s) + bG(s)$

P2. Shifting Theorem

ie If $F[f(x)] = F(s)$, then $F[f(x-a)] = e^{ias} F(s)$.

P3. $F[e^{iax} f(x)] = F(s+a)$

P4. Change of scale property

ie If $F[f(x)] = F(s)$, then $F[f(ax)] = \frac{1}{|a|} F\left(\frac{s}{a}\right)$, $a \neq 0$

P5. If $F[f(x)] = F(s)$, then $F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n} F(s)$

P6. $F[f'(x)] = -is\, F(s)$ if $f(x) \to 0$ as $x \to \pm\infty$

P7. $F\left[\int_a^x f(x)\, dx\right] = \frac{F(x)}{-is}$.

P8. $F[f(-x)] = F(-s)$

P9. $F[\overline{f(x)}] = \overline{F(-s)}$ '–' denotes complex conjugate.

P10. $F[\overline{f(-x)}] = \overline{F(s)}$

$$= a\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx + b\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)\,e^{isx}\,dx$$

$$= aF(s) + bG(s)$$

Property 2 Shifting Theorem **(Anna Uni. Dec 2008)**

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)\,e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{is(a+t)}\,dt, \quad \text{by putting } x-a=t.$$

$$= e^{ias}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{ist}\,dt = e^{ias}\,F(s)$$

Property 3

$$F[e^{iax}f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iax}f(x)\,e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(s+a)x}f(x)\,dx = F(s+a).$$

Property 4 Change of scale property **(Anna Uni. Dec 2008)**

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)\,e^{isx}\,dx$$

put $ax = t,\ x = \frac{t}{a};\ t : -\infty \to \infty$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{i\frac{s}{a}t}\,\frac{dt}{a} \qquad \text{if } a>0$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right) \qquad \dots (1)$$

If $a < 0$ then t varies from $\infty \to -\infty$.

$$\therefore F[f(ax)] = \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} e^{i\frac{s}{a}t} f(t)\, dt$$

$$= -\frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\frac{s}{a}t} f(t)\, dt$$

$$= \frac{-1}{a} F\left(\frac{s}{a}\right),\ a < 0 \qquad \dots (2)$$

UNIT 2

From (1) and (2), we have

$$\boldsymbol{F[f(ax)] = \frac{1}{|a|} F\left(\frac{s}{a}\right)}$$

Property 5 **(Anna Uni. Dec 2008)**

$$\boldsymbol{F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx}$$

$$F'(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, (ix)\, dx$$

$$\frac{d^n}{ds^n}[F(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, (ix)^n\, dx$$

$$= (i)^n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n f(x)\, e^{isx}\, dx$$

$$\frac{d^n}{ds^n}[F(s)] = (i)^n F[x^n f(x)]$$

$$\boldsymbol{\therefore F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n} F(s)}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f'(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx}\, d\,[f(x)]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left[e^{isx} f(x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\,(is)\, e^{isx}\,dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 - is \int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx \right] \quad [\because f(x) \to 0 \text{ as } x \to \pm\infty]$$

$$...] = -is\,F(s)$$

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...rty 7

$$... = \int_{a}^{x} f(x)\,dx$$

...en $g' = f(x)$

$$F[g'(x)] = -is\,F[g(x)] \quad (\because \text{by } P6)$$

$$F[f(x)] = -is\,F\left[\int_{a}^{x} f(x)\,dx \right]$$

$$\left[\int_{a}^{x} f(x)\,dx \right] = \frac{F(s)}{-is}$$

...rty 8 **(A.U. May/June 2007)**

$$F[f(ax)] = \frac{1}{|a|} F\left(\frac{s}{a} \right), \quad a \neq 0$$

...$a = -1$

$$...[f(-x)] = F(-s)$$

UNIT 2

Property 9

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$F(-s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} f(x)\, dx$$

Taking complex conjugate on bothsides

$$\overline{F(-s)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} \overline{f(x)}\, dx$$

$$\therefore \overline{F(-s)} = F[\overline{f(x)}]$$

Property 10

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$\overline{F(s)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)}\, e^{-isx}\, dx$$

Put $x = -t;\ dx = -dt;\ t : \infty \to -\infty$

$$\therefore \overline{F(s)} = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} \overline{f(-t)}\, e^{ist}\, (-dt)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(-t)}\, e^{ist}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(-x)}\, e^{isx}\, dx = F[\overline{f(-x)}]$$

$$\therefore \overline{F(s)} = F[\overline{f(-x)}]$$

CONVOLUTION THEOREM AND PARSEVAL'S IDENTITY

The convolution of two functions $f(x)$ and $g(x)$ is defined by

$$f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

NVOLUTION THEOREM: (Anna Uni. Apr/May 2008)

tement: $F[f * g] = F(s) \cdot G(s)$

oof:

$$F[f * g] = F\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, F[g(x-t)]\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, G(s)\, dt \quad \text{by } P2$$

$$= G(s) \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt = F(s) \cdot G(s).$$

RSEVAL'S IDENTITY

atement:

A function $f(x)$ and its transform $F(s)$ satisfy the identity.

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds.$$

roof:

The convolution theorem can be written as

$F^{-1}[F(s)\, G(s)] = f * g$

ie $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, G(s)\, e^{-isx}\, ds = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$

put $x = 0$, we get

UNIT 2

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, G(s)\, ds = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(-t)\, dt \qquad \ldots (1)$$

Let $g(-t) = \overline{f(t)}$ or $g(t) = \overline{f(-t)}$

$\therefore G(s) = F[g(x)] = F[\overline{f(-x)}] = \overline{F(s)}$ by P10

$$(1) \Rightarrow \int_{-\infty}^{\infty} F(s)\, \overline{F(s)}\, ds = \int_{-\infty}^{\infty} f(t)\, \overline{f(t)}\, dt$$

ie $$\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(t)|^2\, dt$$

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SOLVED EXAMPLES

Example 1 (A.U. Oct/Nov 04, April/May 05, May/June 06, 07, Dec 2008)

Find the Fourier transform of the function $f(x)$ defined by

$$f(x) = \begin{bmatrix} 1 - x^2 \text{ if } |x| < 1 \\ 0 \text{ if } |x| > 1 \end{bmatrix}.$$ *Hence prove that*

(i) $$\int_0^{\infty} \left(\frac{\sin s - s \cos s}{s^3} \right) \cos \frac{s}{2}\, ds = \frac{3\pi}{16}$$

(ii) $$\int_0^{\infty} \left(\frac{\sin s - s \cos s}{s^3} \right)^2 ds = \frac{\pi}{15}$$

☞ **SOLUTION:** Fourier transform of $f(x)$ is given by

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2)(\cos sx + i \sin sx)\, dx$$