Gauss's Law:-

Gauss's law states that the total electric flux ψ through any closed surface is equal to the total charge enclosed by the surface.

$$\psi = Q_{enc}$$

i.e $$\psi = \oint_S d\psi = \oint_S D \cdot ds$$

$$= \text{total charge enclosed} \quad Q = \int_v \rho_v \, dv$$

(or)

$$Q = \oint_S D \cdot ds = \int_v \rho_v \, dv \quad -(1)$$

By applying divergence theorem to the middle term

$$\oint_S D \cdot ds = \int_v \nabla \cdot D \, dv \quad -(2)$$

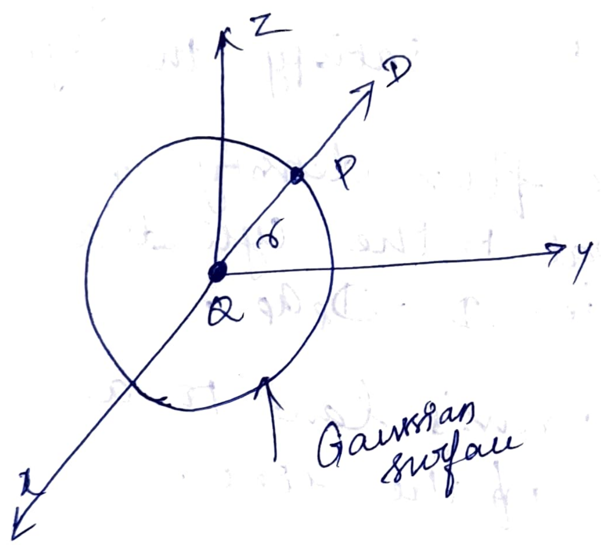
Comparing the two volume integrals

$$\boxed{\rho_v = \nabla \cdot D}$$

This is the 1st maxwell's eqn.

Point charge:-

* Suppose a point charge Q is located at the origin.
* Determine 'D' at a point P, choosing a spherical surface containing P will satisfy symmetry conditions.
* Spherical surface centered at the origin is the Gaussian surface.



Since D is everywhere normal to the Gaussian surface is $D = D_r a_r$ applying Gauss's law ($\psi = Q_{enclosed}$) gives

$$Q = \oint_S D \cdot ds = D_r \oint_S ds = D_r 4\pi r^2$$

where $\oint ds = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi$

$$= r^2 [-\cos\theta]_0^{\pi} [\phi]_0^{2\pi}$$

$$= r^2 (2)(2\pi) = 4\pi r^2$$

$$D_r = \frac{Q}{4\pi r^2}$$

$$D = \frac{Q}{4\pi r^2} \cdot a_r$$

Infinite line charge:-

→ Suppose the infinite line of uniform charge ρ_L C/m lies along the z-axis. To determine

→ D at a point P, we choose a cylindrical surface containing P to satisfy the symmetry condition

→ The electric flux density D is constant on & normal to the cylindrical Gaussian surface. ie) $D = D_\rho a_\rho$.

→ Apply Gauss's law to an arbitrary length 'l' of the line.

$$Q = \rho_L l = \oint_S D\, ds$$

$$= D_\rho \oint_S ds \rightarrow \int_0^{2\pi}\int_0^{l} \rho\, d\phi\, dz$$

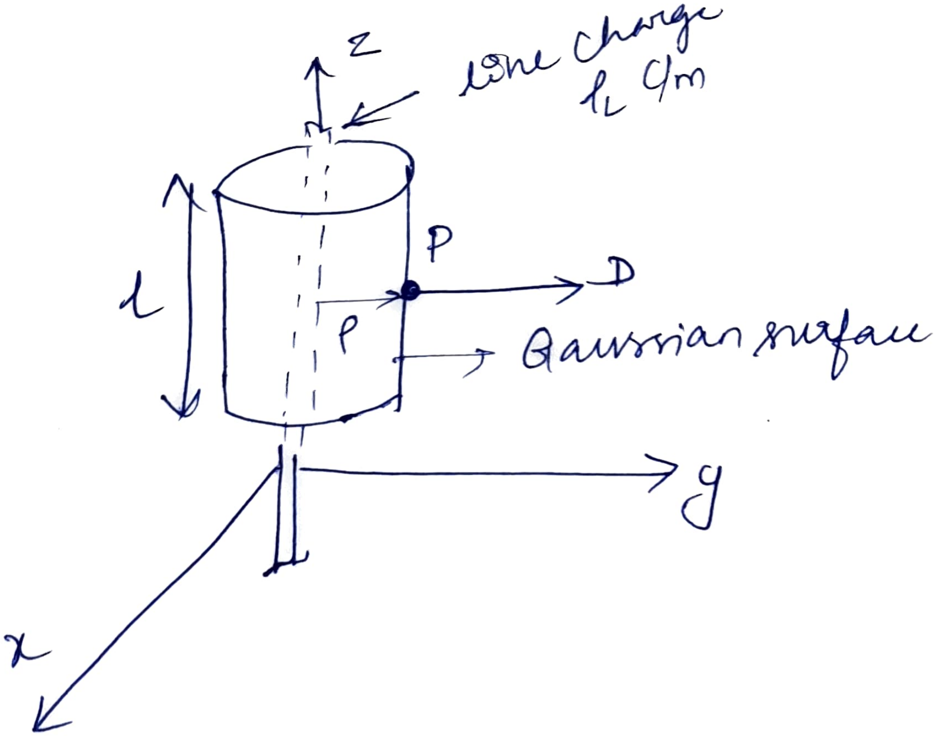
$$\rho_L l = D_\rho\, 2\pi\rho l \qquad D_\rho = \frac{\rho_L}{2\pi\rho}$$

* Where $\oint ds = 2\pi\rho l$ is the surface area of the Gaussian surface.

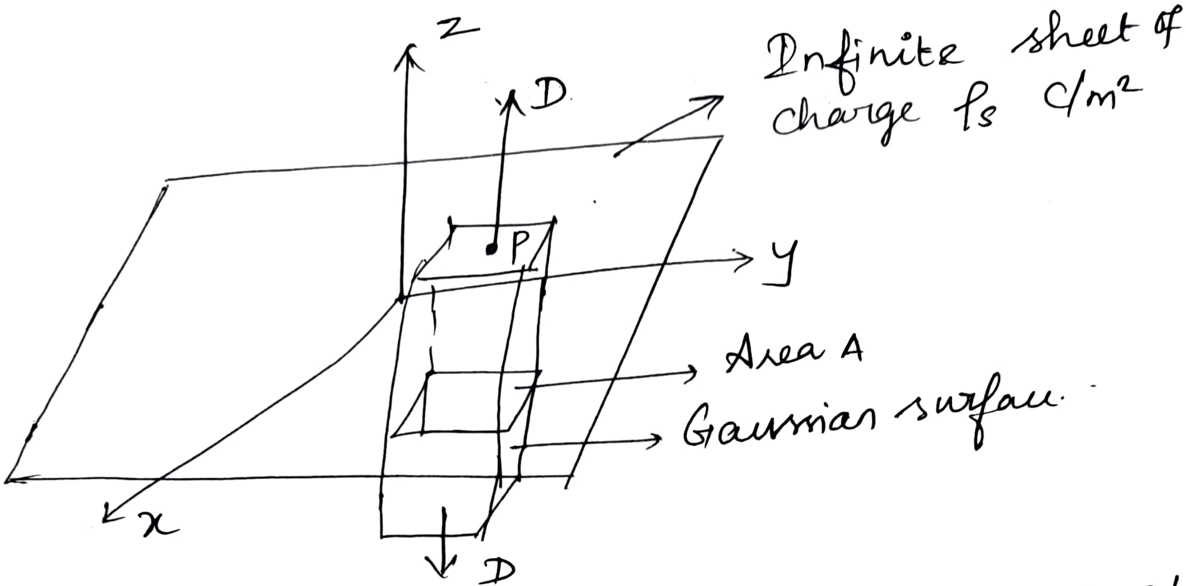
* Note that $\int D \cdot ds$ evaluated on the top & bottom surfaces of the cylinder is zero.

→ D has no z-component that means D is tangential to those surfaces.

$$D = \frac{\rho_L}{2\pi\rho} a_\rho.$$


z
line charge
ρ_L C/m
l
P
D
ρ
Gaussian surface
y
x

Infinite sheet of charge:-



Consider an infinite sheet of charge ρ_s C/m² lying on the $z=0$ plane.

To determine D at point P. We choose a rectangular box that is cut symmetrically by the sheet of charge and has two ~~side~~ of its faces parallel to the sheet as shown in fig. As D is normal to the sheet $D = D_z a_z$. apply gauss's law gives

$$\rho_s \int_s ds = Q = \oint_s D \cdot ds$$

$$= D_z \left[\int_{top} ds + \int_{bottom} ds \right]$$

$D \cdot ds \rightarrow$ on the sides of the box is zero because D has no component along a_x & a_y

If the top and bottom area of the box each has area A.

$$\rho_s A = D_z (A + A)$$

&

$$D_z = \frac{\rho_s A}{2A} = \frac{\rho_s}{2}$$

$$D = \frac{\rho_s}{2} a_z$$

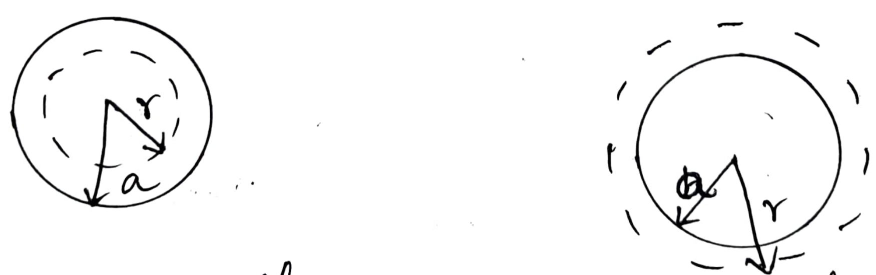
(or) $$E = \frac{D}{\varepsilon_0} = \frac{\rho_s}{2\varepsilon_0} a_z$$

Uniformly charged sphere

Consider a sphere of radius 'a' with a uniform charge ρ_0 C/m³.

To determine 'D' everywhere, construct Gaussian surfaces for cases $r \le a$ & $r \ge a$ separately.

spherical surface → appropriate Gaussian surface.



Gaussian surface for a uniformly charged sphere when $r \ge a$ & $r \le a$.

For $r \leq a$, The total charge enclosed by the spherical surface of radius 'r'.

$$Q_{enc} = \int_V \rho_v \, dv$$

$$= \rho_0 \int_V dv$$

$$= \rho_0 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \rho_0 \left[\left(\frac{r^3}{3} \right)_0^r (-\cos\theta)_0^\pi \, [\phi]_0^{2\pi} \right]$$

$$= \rho_0 \left[\frac{r^3}{3} - 0 \right] [-(-1-1)] [2\pi - 0]$$

$$= \rho_0 \, ((r^3/3)(2)(2\pi))$$

$$= \frac{\rho_0 4\pi r^3}{3}$$

&

$$\psi = \oint_S D \cdot ds = D_r \oint_S ds = D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$D_r [r^2 (-\cos\theta]_0^\pi \, [\phi]_0^{2\pi}]$$

$$= D_r r^2 (2)(2\pi)$$

$$= D_r 4\pi r^2$$

$\psi = Q_{enc}$ gives

$$D_r 4\pi r^2 = \frac{4\pi r^3}{3} \rho_0$$

$$D_r = \frac{r}{3} \rho_0$$

$$\boxed{D = \frac{r}{3} \rho_0 \, a_r \qquad 0 \leq r \leq a}$$

For $r \geq a$, the charge enclosed by the surface is the entire charge in this case,

$$Q_{enc} = \int_V \rho_v \, dv = \rho_0 \int_V dv$$

$$= \rho_0 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \rho_0 \cdot \tfrac{4}{3} \pi a^3$$

$$\psi = \oint_S D \cdot ds = D_r \oint_S ds$$

$$= D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$= D_r \, 4\pi r^2$$

$$\psi = Q_{enc}$$

$$\rho_0 \tfrac{4}{3} \pi a^3 = D_r \, 4\pi r^2$$

$$D_r = \frac{\rho_0 a^3}{3 r^2}$$

$$\boxed{D = \frac{a^3}{3r^2} \rho_0 \, a_r} \quad r \geq a$$

D everywhere is given by,

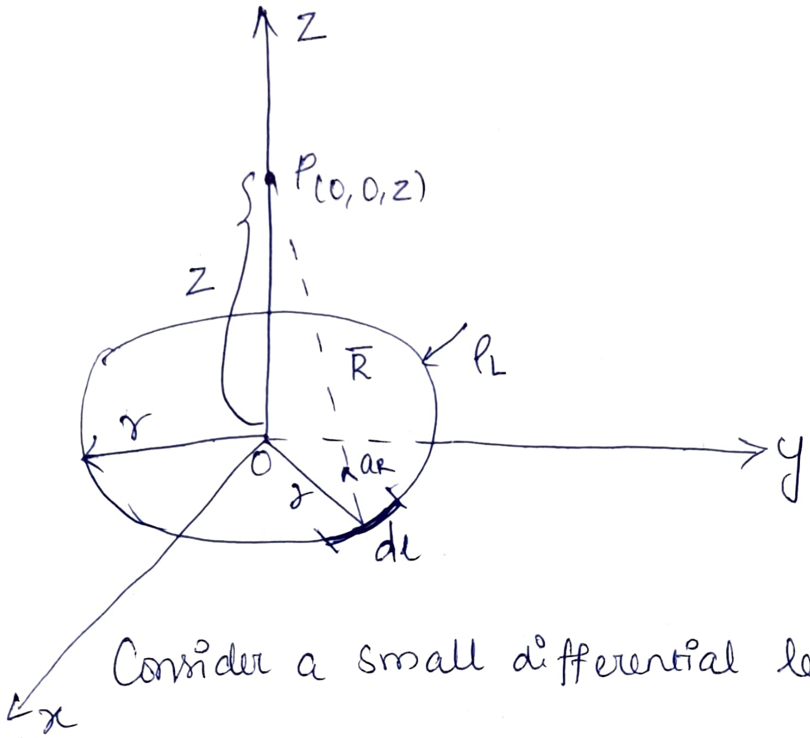
$$D = \begin{cases} \frac{r}{3} \rho_0 \, a_r & 0 \leq r \leq a \\ \frac{a^3}{3r^2} \rho_0 \, a_r & r \geq a \end{cases}$$

Electric Field Due to Charged Circular Ring

Consider a charged circular ring of radius 'r' placed in xy plane with centre at origin, carrying a charge uniformly along its circumference.

The charge density is ρ_L C/m

The point 'p' is at a perpendicular distance 'z' from the ring as shown in fig below.



Consider a small differential length 'dl' on this ring. The charge on it is dQ.

$$dQ = \rho_L \, dl \qquad E = \frac{Q}{4\pi\varepsilon_0 R^2}$$

$$d\bar{E} = \frac{\rho_L \, dl}{4\pi\varepsilon_0 R^2} a_R \qquad \text{--- (1)}$$

where

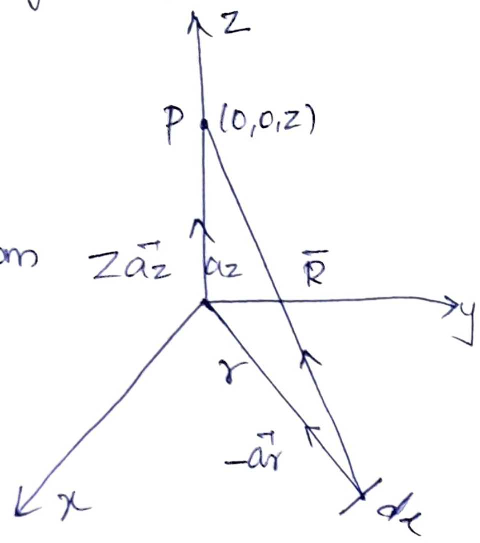
R = distance of point 'p' from dl.

Consider the cylindrical coordinate system.

For 'dl' we are moving in 'ϕ' direction

where $dl = r\,d\phi$

$R^2 = r^2 + z^2$



$\vec{R}$ can be obtained from its two components in cylindrical system.

1) distance 'r' in the direction of $-\vec{a_r}$ radially inwards ie $-r\vec{a_r}$

2) distance 'z' in the direction of $\vec{a_z}$ ie $z\vec{a_z}$

$$\vec{R} = -r\vec{a_r} + z\vec{a_z}$$

$$|\vec{R}| = \sqrt{(-r)^2 + (z)^2} = \sqrt{r^2+z^2}$$

$$\vec{a_R} = \frac{\vec{R}}{|R|} = \frac{-r\vec{a_r} + z\vec{a_z}}{\sqrt{r^2+z^2}}$$

$$d\vec{E} = \frac{\rho_L\, dL}{4\pi\varepsilon_0 \left(\sqrt{r^2+z^2}\right)^2} \times \frac{-r\vec{a_r} + z\vec{a_z}}{\sqrt{r^2+z^2}}$$

$$= \frac{\rho_L\,(r\,d\phi)}{4\pi\varepsilon_0\,(r^2+z^2)^{3/2}} \left(-r\vec{a_r} + z\vec{a_z}\right)$$

The radial components of $\vec{E}$ at point 'P' will be symmetrically placed in the plane parallel to 'xy' plane and are going to cancel each other. Hence neglecting $\vec{a_r}$ component from $d\vec{E}$ we get

$$d\vec{E} = \frac{\rho_L \, (r\,d\phi)}{4\pi \varepsilon_0 (r^2+z^2)^{3/2}} \; z\,\vec{a_z}$$

$$\vec{E} = \int_{\phi_0}^{2\pi} \frac{\rho_L \, r\,d\phi}{4\pi \varepsilon_0 (r^2+z^2)^{3/2}} \; z\,\vec{a_z}$$

$$= \frac{\rho_L \, r}{4\pi \varepsilon_0 \, (r^2+z^2)^{3/2}} \; z\,\vec{a_z} \, (\phi)_0^{2\pi}$$

$$= \frac{\rho_L \, r z \, (\cancel{2\pi}) \, \vec{a_z}}{\underset{2}{\cancel{4\pi}} \varepsilon_0 \, (r^2+z^2)^{3/2}}$$

$$= \frac{\rho_L \, r z}{2 \varepsilon_0 \, (r^2+z^2)^{3/2}} \; \vec{a_z}$$

where r = Radius of the ring

z = Perpendicular distance of point 'p' from the ring along the axis of the ring.