MODULE - 4

Random Process and Game Programming

Equip By:

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1. Use of random numbers in programs

2. Monte Carlo integration

3. Random walk

Use of random numbers in programs

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INTRODUCTION

Random numbers have many applications in science and computer programming, especially when there are significant uncertainties in a phenomenon of interest.

The key idea in computer simulations with random numbers is first to formulate an algorithmic description of the phenomenon we want to study.

This description frequently maps directly onto a quite simple and short Python program, where we use random numbers to mimic the uncertain features of the phenomenon.

Random numbers are used to simulate uncertain events

Deterministic problems

- Some problems in science and technology are desrcribed by exact mathematics, leading to precise results
- Example: throwing a ball up in the air $(y(t) = v_0 t - \frac{1}{2} g t^2)$

Stochastic problems

- Some problems appear physically uncertain
- Examples: rolling a die, molecular motion, games
- Use random numbers to mimic the uncertainty of the experiment.

Drawing Random Numbers:

Python has a module random for generating random numbers. The function call random.random() generates a random number in the half open interval1 [0, 1). We can try it out:

```
>>> import random
>>> random.random()
0.81550546885338104
>>> random.random()
 0.44913326809029852
>>> random.random()
 0.88320653116367454
```

All computations of random numbers are based on deterministic algorithms (see Exercise 8.16 for an example), so the sequence of numbers cannot be truly random. However, the sequence of numbers appears to lack any pattern, and we can therefore view the numbers as random2

Drawing random numbers

Python has a `random` module for drawing random numbers.
`random.random()` draws random numbers in $[0, 1)$:

```
>>> import random
>>> random.random()
0.81550546885338104
>>> random.random()
0.44913326809029852
>>> random.random()
0.88320653116367454
```

Notice

The sequence of random numbers is produced by a deterministic algorithm - the numbers just appear random.

Uniformly Distribution of random numbers

- `random.random()` generates random numbers that are uniformly distributed in the interval [0, 1)
- `random.uniform(a, b)` generates random numbers uniformly distributed in [a, b)
- Uniformly distributed means that if we generate a large set of numbers, no part of [a, b) gets more numbers than others

Uniformly Distribution of random numbers

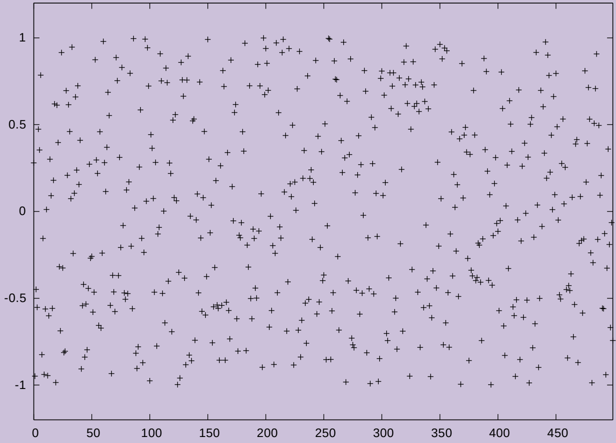
The numbers generated by random.random() tend to be equally distributed between 0 and 1, which means that there is no part of the interval [0, 1) with more random numbers than other parts. We say that the distribution of random numbers in this case is uniform.

The function random.uniform(a,b) generates uniform random numbers in the half open interval [a, b), where the user can specify a and b. With the following program (in file uniform_numbers0.py) we may generate lots of random numbers in the interval [−1, 1) and visualize how they are distributed :

Output figure shows the values of these 500 numbers, and as seen, the numbers appear to be random and uniformly distributed between −1 and 1

Distribution of random numbers visualized

```
N = 500 # no of samples
x = range(N)
y = [random.uniform(-1,1) for i in x]
from scitools.std import plot
plot(x, y, '+', axis=[0,N-1,-1.2,1.2])
```



Vectorized drawing of random numbers

- `random.random()` generates one number at a time
- `numpy` has a `random` module that e ciently generates a (large) number of random numbers at a time

```
from numpy import random
r = random.random()             # one no between 0 and 1
r = random.random(size=10000)   # array with 10000 numbers
r = random.uniform(-1, 10)      # one no between -1 and 10
r = random.uniform(-1, 10, size=10000) # array
```

- Vectorized drawing is important for speeding up programs!
- Possible problem: two `random` modules, one Python "built-in" and one in `numpy` (`np`)
- Convention: use `random` (Python) and `np.random`

```
random.uniform(-1, 1)              # scalar number
import numpy as np
np.random.uniform(-1, 1, 100000)   # vectorized
```

Computing the Mean and Standard Deviation

You probably know the formula for the mean or average of a set of n numbers $x_0, x_1, \ldots, x_{n-1}$:

$$x_m = \frac{1}{n} \sum_{j=0}^{n-1} x_j \,. \tag{1}$$

The amount of spreading of the x_i values around the mean x_m can be measured by the *variance*,

$$x_v = \frac{1}{n} \sum_{j=0}^{n-1} (x_j - x_m)^2 \,. \tag{2}$$

Textbooks in statistics teach you that it is more appropriate to divide by $n-1$ instead of n, but we are not going to worry about that fact in this document. A variant of (2) reads

$$x_v = \frac{1}{n} \left(\sum_{j=0}^{n-1} x_j^2 \right) - x_m^2 \,. \tag{3}$$

The good thing with this latter formula is that one can, as a statistical experiment progresses and n increases, record the sums

$$s_m = \sum_{j=0}^{q-1} x_j, \quad s_v = \sum_{j=0}^{q-1} x_j^2 \tag{4}$$

and then, when desired, efficiently compute the most recent estimate on the mean value and the variance after q samples by

$$x_m = s_m/q, \quad x_v = s_v/q - s_m^2/q^2 \,. \tag{5}$$

The *standard deviation*

$$x_s = \sqrt{x_v} \tag{6}$$

Computing the Mean and Standard Deviation

```
import sys N = int(sys.argv[1])
 import random as random_number
 from math import sqrt
sm = 0; sv = 0
for q in range(1, N+1):
x = random_number.uniform(-1, 1)
sm += x
sv += x**2
# write out mean and st.dev. 10 times in this loop:
if q % (N/10) == 0:
xm = sm/q
 xs = sqrt(sv/q - xm**2)
 print '%10d mean: %12.5e stdev: %12.5e' % (q, xm, xs)
```

OUTPUT:

100000 mean: 1.86276e-03 stdev: 5.77101e-01 200000 mean: 8.60276e-04 stdev: 5.77779e-01
300000 mean: 7.71621e-04 stdev: 5.77753e-01 400000 mean: 6.38626e-04 stdev: 5.77944e-01
500000 mean: -1.19830e-04 stdev: 5.77752e-01 600000 mean: 4.36091e-05 stdev: 5.77809e-01
700000 mean: -1.45486e-04 stdev: 5.77623e-01 800000 mean: 5.18499e-05 stdev: 5.77633e-01
900000 mean: 3.85897e-05 stdev: 5.77574e-01 1000000 mean: -1.44821e-05 stdev: 5.77616e-01

The Gaussian or Normal Distribution-

In some applications we want random numbers to cluster around a specific value m. This means that it is more probable to generate a number close to m than far away from m.

A widely used distribution with this qualitative property is the Gaussian or normal distribution4 . The normal distribution has two parameters: the mean value m and the standard deviation s.

The latter measures the width of the distribution, in the sense that a small s makes it less likely to draw a number far from the mean value, and a large s makes more likely to draw a number far from the mean value.

Single random numbers from the normal distribution can be generated by:

```
import random as random_number
 m=10
s=20
r = random_number.normalvariate(m, s)
print(r)
```

```
from numpy import random
 r = random.normal(m, s, size=N)
```

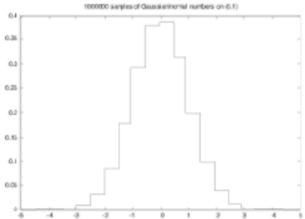
```
import sys
N = int(sys.argv[1])
m = float(sys.argv[2])
s = float(sys.argv[3])

import numpy as np
np.random.seed(12)
samples = np.random.normal(m, s, N)
print np.mean(samples), np.std(samples)

import scitools.std as st
x, y = st.compute_histogram(samples, 20, piecewise_constant=True)
st.plot(x, y, savefig='tmp.pdf',
        title ='%d samples of Gaussian/normal numbers on (0,1)' % N)
```

The corresponding program file is `normal_numbers1.py`, which gives a mean of -0.00253 and a standard deviation of 0.99970 when run with `N` as 1 million, `m` as 0, and `s` equal to 1. Figure 3 shows that the random numbers cluster around the mean $m = 0$ in a histogram. This normalized histogram will, as `N` goes to infinity, approach the famous, bell-shaped, normal distribution probability density function.

Figure 3: Normalized histogram of 1 million random numbers drawn from the normal distribution.



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Drawing integers

- Quite often we want to draw an integer from [a, b] and not a real number
- Python's `random` module and `numpy.random` have functions for drawing uniformly distributed integers:

```
import random
r = random.randint(a, b) # a, a+1, ..., b

import numpy as np
r = np.random.randint(a, b+1, N)              # b+1 is not included
r = np.random.random_integers(a, b, N)        # b is included
```

Random Integer Functions

Python’s random module has a built-in function randint(a,b) for drawing an integer in [a, b], i.e., the return value is among the numbers a, a+1, . . ., b-1, b

```
import random as random_number
r = random_number.randint(a, b)

from numpy inport random
 r = random.randint(a, b+1, N)

from numpy inport random
 r = random.random_integers(a, b, N)
```

Suppose we want to draw a random integer among the values 1, 2, 3, and 4, and that each of the four values is equally probable.

Onepossibility is to draw real numbers from the uniform distribution on, e.g., [0, 1) and divide this interval into four equal subintervals:

```
import random as random_number
 r = random_number.random()
 if 0 <= r < 0.25:
 r = 1
 elif 0.25 <= r < 0.5:
 r = 2
elif 0.5 <= r < 0.75:
 r = 3
else:
 r = 4
```

Example: Rolling a die

Problem

- Any no of eyes, 1-6, is equally probable when you roll a die
- What is the chance of getting a 6?

Solution by Monte Carlo simulation:

Rolling a die is the same as drawing integers in [1, 6].

```
import random
N = 10000
eyes = [random.randint(1, 6) for i in range(N)]
M = 0 # counter for successes: how many times we get 6 eyes
for outcome in eyes:
    if outcome == 6:
        M += 1
print 'Got six %d times out of %d' % (M, N)
print 'Probability:', float(M)/N
```

Probability: `M/N` (exact: 1/6)

Example: Rolling a die; vectorized version

```
import sys, numpy as np
N = int(sys.argv[1])
eyes = np.random.randint(1, 7, N)
success = eyes == 6          # True/False array
six = np.sum(success)        # treats True as 1, False as 0
print 'Got six %d times out of %d' % (six, N)
print 'Probability:', float(M)/N
```

Impoartant!

Use `sum` from `numpy` and not Python's built-in `sum` function! (The latter is slow, often making a vectorized version slower than the scalar version.)

Debugging programs with random numbers requires xing the seed of the random sequence

- Debugging programs with random numbers is di cult because the numbers produced vary each time we run the program
- For debugging it is important that a new run reproduces the sequence of random numbers in the last run
- This is possible by xing the seed of the `random` module: `random.seed(121)` (`int` argument)

```
>>> import random
>>> random.seed(2)
>>> ['%.2f' % random.random() for i in range(7)]
['0.96', '0.95', '0.06', '0.08', '0.84', '0.74', '0.67']
>>> ['%.2f' % random.random() for i in range(7)]
['0.31', '0.61', '0.61', '0.58', '0.16', '0.43', '0.39']

>>> random.seed(2)     # repeat the random sequence
>>> ['%.2f' % random.random() for i in range(7)]
['0.96', '0.95', '0.06', '0.08', '0.84', '0.74', '0.67']
```

By default, the seed is based on the current time

Drawing random elements from a list

There are di erent methods for picking an element from a list at random, but the main method applies `choice(list)`:

```
>>> awards = ['car', 'computer', 'ball', 'pen']
>>> import random
>>> random.choice(awards)
'car'
```

Alternatively, we can compute a random index:

```
>>> index = random.randint(0, len(awards)-1)
>>> awards[index]
'pen'
```

We can also shu e the list randomly, and then pick any element:

```
>>> random.shuffle(awards)
>>> awards[0]
'computer'
```

Example: Drawing cards from a deck; make deck and draw

Make a deck of cards:

```
# A: ace, J: jack, Q: queen, K: king
# C: clubs, D: diamonds, H: hearts, S: spades

def make_deck():
    ranks = ['A', '2', '3', '4', '5', '6', '7',
             '8', '9', '10', 'J', 'Q', 'K']
    suits = ['C', 'D', 'H', 'S']
    deck = []
    for s in suits:
        for r in ranks:
            deck.append(s + r)
    random.shuffle(deck)
    return deck

deck = make_deck()
```

Draw a card at random:

```
deck = make_deck()
card = deck[0]
del deck[0]

card = deck.pop(0) # return and remove element with index 0
```

Example: Drawing cards from a deck; draw a hand of cards

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Draw a hand of n cards:

```
def deal_hand(n, deck):
    hand = [deck[i] for i in range(n)]
    del deck[:n]
    return hand, deck
```

Note:

- `deck` is returned since the function changes the list
- `deck` is changed in-place so the change a ects the `deck` object in the calling code anyway, but returning changed arguments is a Python convention and good habit

Example: Drawing cards from a deck; deal

Deal hands for a set of players:

```
def deal(cards_per_hand, no_of_players):
    deck = make_deck()
    hands = []
    for i in range(no_of_players):
        hand, deck = deal_hand(cards_per_hand, deck)
        hands.append(hand)
    return hands

players = deal(5, 4)
import pprint; pprint.pprint(players)
```

Resulting output:

```
[['D4', 'CQ', 'H10', 'DK', 'CK'],
 ['D7', 'D6', 'SJ', 'S4', 'C5'],
 ['C3', 'DQ', 'S3', 'C9', 'DJ'],
 ['H6', 'H9', 'C6', 'D5', 'S6']]
```

Example: Drawing cards from a deck; analyze results (1)

Analyze the no of pairs or n-of-a-kind in a hand:

```
def same_rank(hand, n_of_a_kind):
    ranks = [card[1:] for card in hand]
    counter = 0
    already_counted = []
    for rank in ranks:
        if rank not in already_counted and \
                ranks.count(rank) == n_of_a_kind:
            counter += 1
            already_counted.append(rank)
    return counter
```

Example: Drawing cards from a deck; analyze results (2)

Analyze the no of combinations of the same suit:

```
def same_suit(hand):
    suits = [card[0] for card in hand]
    counter = {}   # counter[suit] = how many cards of suit
    for suit in suits:
        # attention only to count > 1:
        count = suits.count(suit)
        if count > 1:
            counter[suit] = count
    return counter
```

Example: Drawing cards from a deck; analyze results (3)

Analysis of how many cards we have of the same suit or the same rank, with some nicely formatted printout (see the book):

```
The hand D4, CQ, H10, DK, CK
    has 1 pairs, 0 3-of-a-kind and
    2+2 cards of the same suit.
The hand D7, D6, SJ, S4, C5
    has 0 pairs, 0 3-of-a-kind and
    2+2 cards of the same suit.
The hand C3, DQ, S3, C9, DJ
    has 1 pairs, 0 3-of-a-kind and
    2+2 cards of the same suit.
The hand H6, H9, C6, D5, S6
    has 0 pairs, 1 3-of-a-kind and
    2 cards of the same suit.
```

1 Use of random numbers in programs

2 Monte Carlo integration

3 Random walk

Monte Carlo integration

$$\int_a^b f(x)dx$$

$$\int_a^b f(x)dx$$

Principles of Monte Carlo Simulation

Assume that we perform N experiments where the outcome of each experiment is random. Suppose that some event takes place M times in these N experiments. An estimate of the probability of the event is then M/N.

The estimate becomes more accurate as N is increased, and the exact probability is assumed to be reached in the limit as $N \to \infty$.

(Note that in this limit, $M \to \infty$ too, so for rare events, where M may be small in a program, one must increase N such that M is sufficiently large for M/N to become a good approximation to the probability.)

There is a strong link between an integral and the average of the integrand

Recall a famous theorem from calculus: Let f_m be the mean value of $f(x)$ on $[a, b]$. Then

$$\int_a^b f(x)dx = f_m(b-a)$$

Idea: compute f_m by averaging N function values. To choose the N coordinates $x_0, \ldots, x_{N-1}$ we use random numbers in $[a, b]$. Then

$$f_m = N^{-1} \sum_{j=0}^{N-1} f(x_j)$$

This is called Monte Carlo integration.

Implementation of Monte Carlo integration; scalarversion

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```
def MCint(f, a, b, n):
    s = 0
    for i in range(n):
        x = random.uniform(a, b)
        s += f(x)
    I = (float(b-a)/n)*s
    return I
```

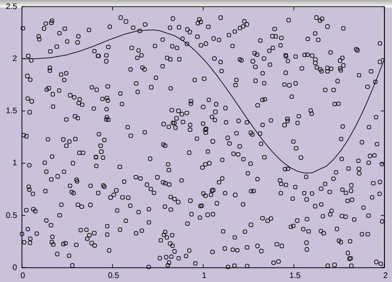
Implementation of Monte Carlo integration; vectorized version

```python
def MCint_vec(f, a, b, n):
    x = np.random.uniform(a, b, n)
    s = np.sum(f(x))
    I = (float(b-a)/n)*s
    return I
```

Remark:

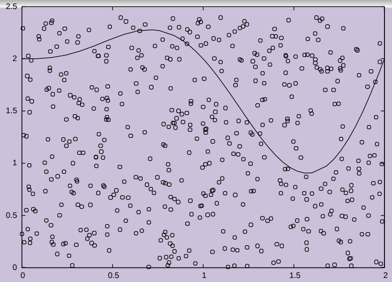
Monte Carlo integration is slow for $\int f(x)dx$ (slower than the Trapezoidal rule, e.g.), but very efficient for integrating functions of many variables $\int f(x_1, x_2, \ldots, x_n)dx_1 dx_2 \cdots dx_n$

Dart-inspired Monte Carlo integration



- Choose a box $B = [x_L, x_H] \times [y_L, y_H]$ with some geometric object G inside, what is the area of G?
- Method: draw N points at random inside B, count how many, M, that fall within G, G's area is then $M/N \times \text{area}(B)$
- Special case: G is the geometry between $y = f(x)$ and the x axis for $x \in [a, b]$, i.e., the area of G is $\int_a^b f(x)dx$, and our method gives $\int_a^b f(x)dx \approx \frac{M}{N} m(b - a)$ if B is the box $[a, b] \times [0, m]$

Dart-inspired Monte Carlo integration



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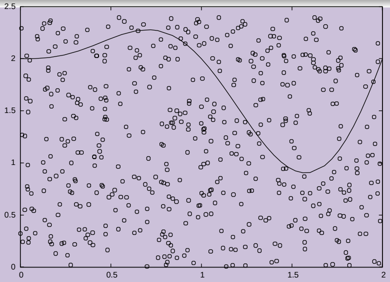
Dart-inspired Monte Carlo integration



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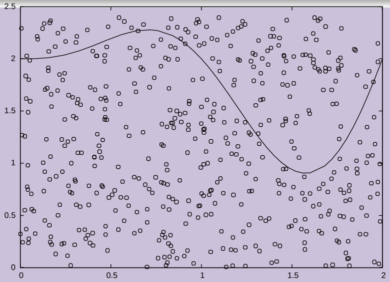
Dart-inspired Monte Carlo integration



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The code for the dart-inspired Monte Carlo integration

Scalar code:

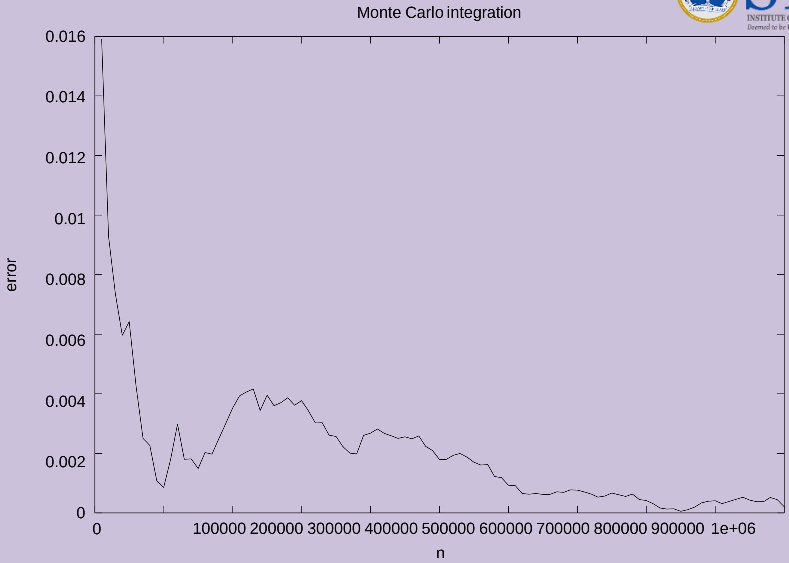
```
def MCint_area(f, a, b, n, fmax):
    below = 0 # counter for no of points below the curve
    for i in range(n):
        x = random.uniform(a, b)
        y = random.uniform(0, fmax)
        if y <= f(x):
            below += 1
    area = below/float(n)*(b-a)*fmax
    return area
```

Vectorized code:

```
from numpy import *

def MCint_area_vec(f, a, b, n, fmax):
    x = np.random.uniform(a, b, n)
    y = np.random.uniform(0, fmax, n)
    below = y[y < f(x)].size
    area = below/float(n)*(b-a)*fmax
    return area
```

The development of the error in Monte Carlo integration



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Computing Probabilities

With the mathematical rules from probability theory one may compute the probability that a certain event happens, say the probability that you get one black ball when drawing three balls from a hat with four black balls, six white balls, and three green balls.

Unfortunately, theoretical calculations of probabilities may soon become hard or impossible if the problem is slightly changed.

There is a simple “numerical way” of computing probabilities that is generally applicable to problems with uncertainty. The principal ideas of this approximate technique is explained below, followed by with three examples of increasing complexity.

Probabilities can be computed by Monte Carlo simulation

What is the probability that a certain event A happens?

Simulate N events and count how many times M the event A happens. The probability of the event A is then M/N (as $N \to \infty$).

Example:

You throw two dice, one black and one green. What is the probability that the number of eyes on the black is larger than that on the green?

```
import random
import sys
N = int(sys.argv[1])        # no of experiments
M = 0                       # no of successful events
for i in range(N):
    black = random.randint(1, 6)    # throw black
    green = random.randint(1, 6)    # throw green
    if black > green:               # success?
        M += 1
p = float(M)/N
print 'probability:', p
```

A vectorized version can speed up the simulations

```
import sys
N = int(sys.argv[1])        # no of experiments

import numpy as np
r = np.random.random_integers(1, 6, (2, N))

black = r[0,:]              # eyes for all throws with black
green = r[1,:]              # eyes for all throws with green
success = black > green     # success[i]==True if black[i]>green[i]
M = np.sum(success)         # sum up all successes

p = float(M)/N
print 'probability:', p
```

Run 10+ times faster than scalar code

The exact probability can be calculated in this (simple) example

All possible combinations of two dice:

```
combinations = [(black, green)
                for black in range(1, 7)
                for green in range(1, 7)]
```

How many of the `(black, green)` pairs that have the property `black > green`?

```
success = [black > green for black, green in combinations]
M = sum(success)
print 'probability:', float(M)/len(combinations)
```

How accurate and fast is Monte Carlo simulation?

Programs:

- `black_gt_green.py`: scalar version
- `black_gt_green_vec.py`: vectorized version
- `black_gt_green_exact.py`: exact version

```
Terminal> python black_gt_green_exact.py
probability: 0.416666666667

Terminal> time python black_gt_green.py 10000
probability: 0.4158

Terminal> time python black_gt_green.py 1000000
probability: 0.416516
real 0m1.725s

Terminal> time python black_gt_green.py 10000000
probability: 0.4164688
real 0m17.649s

Terminal> time python black_gt_green_vec.py 10000000
probability: 0.4170253
real 0m0.816s
```

Gami cation of this example

Suggested game:

Suppose a games is constructed such that you have to pay 1 euro to throw the two dice. You win 2 euros if there are more eyes on the black than on the green die. Should you play this game?

Code:

```
import sys
N = int(sys.argv[1])                    # no of experiments

import random
start_capital = 10
money = start_capital
for i in range(N):
    money -= 1                          # pay for the game
    black = random.randint(1, 6)        # throw black
    green = random.randint(1, 6)        # throw brown
    if black > green:                   # success?
        money += 2                      # get award

net_profit_total = money - start_capital
net_profit_per_game = net_profit_total/float(N)
print 'Net profit per game in the long run:', net_profit_per_game
```

Should we play the game?

```
Terminaldd> python black_gt_green_game.py 1000000
Net profit per game in the long run: -0.167804
```

No!

Example: Drawing balls from a hat

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We have 12 balls in a hat: four black, four red, and four blue

```
hat = []
for color in 'black', 'red', 'blue':
    for i in range(4):
        hat.append(color)
```

Choose two balls at random:

```
import random
index = random.randint(0, len(hat)-1)   # random index
ball1 = hat[index]; del hat[index]
index = random.randint(0, len(hat)-1)   # random index
ball2 = hat[index]; del hat[index]

# or:
random.shuffle(hat)    # random sequence of balls
ball1 = hat.pop(0)
ball2 = hat.pop(0)
```

What is the probability of getting two black balls or more?

```
def new_hat(): # make a new hat with 12 balls
    return [color for color in 'black', 'red', 'blue'
            for i in range(4)]

def draw_ball(hat):
    index = random.randint(0, len(hat)-1)
    color = hat[index]; del hat[index]
    return color, hat # (return hat since it is modified)

# run experiments:
n = input('How many balls are to be drawn? ')
N = input('How many experiments? ')
M = 0 # no of successes


for e in range(N):
    hat = new_hat()
    balls = []          # the n balls we draw
    for i in range(n):
        color, hat = draw_ball(hat)
        balls.append(color)
    if balls.count('black') >= 2: # two black balls or more?
        M += 1
print 'Probability:', float(M)/N
```

Examples on computing the probabilities

```
Terminal> python balls_in_hat.py
How many balls are to be drawn? 2
How many experiments? 10000
Probability: 0.0914

Terminal> python balls_in_hat.py
How many balls are to be drawn? 8
How many experiments? 10000
Probability: 0.9346

Terminal> python balls_in_hat.py
How many balls are to be drawn? 4
How many experiments? 10000
Probability: 0.4033
```

Guess a number game

Game:

Let the computer pick a number at random. You guess at the number, and the computer tells if the number is too high or too low.

Program:

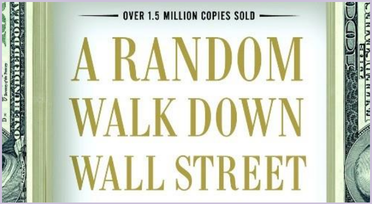
```
import random
number = random.randint(1, 100)   # the computer's secret number
attempts = 0    # no of attempts to guess the number
guess = 0       # user's guess at the number
while guess != number:
    guess = input('Guess a number: ')
    attempts += 1
    if guess == number:
        print 'Correct! You used', attempts, 'attempts!'
        break
    elif guess < number: print 'Go higher!'
    else:                 print 'Go lower!'
```

1 Use of random numbers in programs

2 Monte Carlo integration

3 Random walk

Random walk



Random walk in one space dimension

Basics of random walk in 1D:

- One particle moves to the left and right with equal probability
- n particles start at $x = 0$ at time $t = 0$ - how do the particles get distributed over time?

Applications:

- molecular motion
- heat transport
- quantum mechanics
- polymer chains
- population genetics
- brain research
- hazard games
- pricing of nancial instruments

Computing Statistics of the Particle Positions

Scientists interested in random walks are in general not interested in the graphics of our walk1D.py program, but more in the statistics of the positions of the particles at each step.

We may therefore, at each step, compute a histogram of the distribution of the particles along the x axis, plus estimate the mean position and the standard deviation.

These mathematical operations are easily accomplished by letting the SciTools function compute_histogram and the numpy functions mean and std operate on the positions array

```
mean_pos = mean(positions)
stdev_pos = std(positions)
pos, freq = compute_histogram(positions, nbins=int(xmax),
piecewise_constant=True)

xmean, ymean = [mean_pos, mean_pos], [yminv, ymaxv]
xstdv1, ystdv1 = [stdev_pos, stdev_pos], [yminv, ymaxv]
 xstdv2, ystdv2 = [-stdev_pos, -stdev_pos], [yminv, ymaxv]
```

Program for 1D random walk

```
from scitools.std import plot
import random

np = 4                       # no of particles
ns = 100                     # no of steps
positions = zeros(np) # all particles start at x=0
HEAD = 1;  TAIL = 2          # constants
xmax = sqrt(ns); xmin = -xmax   # extent of plot axis

for step in range(ns):
    for p in range(np):
        coin = random_.randint(1,2) # flip coin
        if coin == HEAD:
            positions[p] += 1   # step to the right
        elif coin == TAIL:
            positions[p] -= 1   # step to the left
    plot(positions, y, 'ko3',
         axis=[xmin, xmax, -0.2, 0.2])
    time.sleep(0.2)             # pause between moves
```

Random walk as a difference equation

Let x_n be the position of one particle at time n. Updating rule:

$$x_n = x_{n-1} + s$$

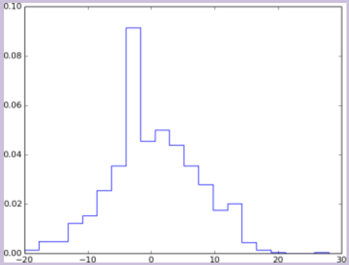
where $s = 1$ or $s = -1$, both with probability $1/2$.

Computing statistics of the random walk

Scientists are not interested in just looking at movies of random walks - they are interested in statistics (mean position, width of the cluster of particles, how particles are distributed)

```
mean_pos = mean(positions)
stdev_pos = std(positions)    # "width" of particle cluster

# shape of particle cluster:
from scitools.std import compute_histogram
pos, freq = compute_histogram(positions, nbins=int(xmax),
                              piecewise_constant=True)
plot(pos, freq, 'b-')
```



Vectorized implementation of 1D random walk

First we draw all moves at all times:

```
moves = numpy.random.random_integers(1, 2, size=np*ns)
moves = 2*moves - 3 # -1, 1 instead of 1, 2
moves.shape = (ns, np)
```

Evolution through time:

```
positions = numpy.zeros(np)
for step in range(ns):
    positions += moves[step, :]

    # can do some statistics:
    print numpy.mean(positions), numpy.std(positions)
```

Now to more exciting stu : 2D random walk

Let each particle move north, south, west, or east - each with probability 1/4

```
def random_walk_2D(np, ns, plot_step):
    xpositions = numpy.zeros(np)
    ypositions = numpy.zeros(np)
    NORTH = 1; SOUTH = 2; WEST = 3; EAST = 4

    for step in range(ns):
        for i in range(len(xpositions)):
            direction = random.randint(1, 4)
            if direction == NORTH:
                ypositions[i] += 1
            elif direction == SOUTH:
                ypositions[i] -= 1
            elif direction == EAST:
                xpositions[i] += 1
            elif direction == WEST:
                xpositions[i] -= 1
    return xpositions, ypositions
```

Vectorized implementation of 2D random walk

```
def random_walk_2D(np, ns, plot_step):
    xpositions = zeros(np)
    ypositions = zeros(np)
    moves = numpy.random.random_integers(1, 4, size=ns*np)
    moves.shape = (ns, np)
    NORTH = 1; SOUTH = 2; WEST = 3; EAST = 4

    for   step   in   range(ns):
        this_move = moves[step,:]
        ypositions += where(this_move == NORTH, 1, 0)
        ypositions -= where(this_move == SOUTH, 1, 0)
        xpositions += where(this_move == EAST, 1, 0)
        xpositions -= where(this_move == WEST,  1, 0)
    return xpositions, ypositions
```

Visualization of 2D random walk

- We plot every `plot_step` step
- One plot on the screen + one hardcopy for movie le
- Extent of axis: it can be shown that after n_s steps, the typical width of the cluster of particles (standard deviation) is of order $\sqrt{n_s}$, so we can set min/max axis extent as, e.g.,

```
xymax = 3*sqrt(ns); xymin = -xymax
```

Inside `for` loop over steps:

```
# just plot every plot_step steps:
if (step+1) % plot_step == 0:
    plot(xpositions, ypositions, 'ko',
         axis=[xymin, xymax, xymin, xymax],
         title='%d particles after %d steps' % \
               (np, step+1),
         savefig='tmp_%03d.png' % (step+1))
```

Class implementation of 2D random walk

- Can classes be used to implement a random walk?
- Yes, it sounds natural with class `Particle`, holding the position of a particle as attributes and with a method `move` for moving the particle one step
- Class `Particles` holds a list of `Particle` instances and has a method `move` for moving all particles one step and a method `moves` for moving all particles through all steps
- Additional methods in class `Particles` can plot and compute statistics
- Downside: with class `Particle` the code is scalar - a vectorized version `must` use arrays inside class `Particles` instead of a list of `Particle` instances
- The implementation is an exercise

Summary of drawing random numbers (scalar code)

Draw a uniformly distributed random number in [0, 1):

```
import random
r = random.random()
```

Draw a uniformly distributed random number in [a, b):

```
r = random.uniform(a, b)
```

Draw a uniformly distributed random integer in [a, b]:

```
i = random.randint(a, b)
```

Summary of drawing random numbers (vectorized code)

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Draw n uniformly distributed random numbers in [0, 1):

```
import numpy as np
r = np.random.random(n)
```

Draw n uniformly distributed random numbers in [a, b):

```
r = np.random.uniform(a, b, n)
```

Draw n uniformly distributed random integers in [a, b]:

```
i = np.random.randint(a, b+1, n)
i = np.random.random_integers(a, b, n)
```

Summary of probability computations

- Probability: perform N experiments, count M successes, then success has probability M/N (N must be large)
- Monte Carlo simulation: let a program do N experiments and count M (simple method for probability problems)

Thank you