① $$Z\{1/n\} = \sum_{n=1}^{\infty} 1/n \, z^{-n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n z^n} = \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \dots + \text{to } \infty$$

$$= \frac{1}{z} + \frac{(1/z)^2}{2} + \frac{(1/z)^3}{3} + \dots$$

$$= -\log(1 - 1/z) \quad \text{if } |1/z| < 1$$

$$= \log(1 - 1/z)^{-1} \quad \text{if } 1 < |z|$$

$$= \log\left(\frac{z-1}{z}\right)^{-1} \quad \text{if } |z| > 1$$

$$\log\left(\frac{z}{z-1}\right) \quad \text{if } |z| > 1$$

Note: $-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \quad |x| < 1.$

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② $$Z\left\{\cos\frac{n\pi}{2}\right\} = \sum_{n=0}^{\infty} \cos\frac{n\pi}{2} \cdot z^{-n}$$

$$= 1 - \frac{1}{z^2} + \frac{1}{z^4} - \dots \text{to } \infty$$

$$= (1 + 1/z^2)^{-1} = \frac{z^2}{z^2+1} \quad \text{if } |z| > 1.$$

Note:

$n=0$, $\cos 0 = 1$ $\qquad$ $n=3$, $\cos\frac{3\pi}{2} = 0$.

$n=1$ $\cos\pi/2 = 0$ $\qquad$ $n=4$, $\cos\frac{4\pi}{2} = 1$

$n=2$ $\cos\pi$ -1 $\qquad$ ⋮

3) Let $\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$.

$$\frac{1}{n\cancel{(n+1)}} = \frac{A(n+1)+Bn}{n\cancel{(n+1)}}$$

Put $n=-1$; $1 = -B$ $\therefore \boxed{B=-1}$ | Put $n=0$, $1 = A$ $\therefore \boxed{A=1}$

$$\frac{1}{n(n+1)} = \frac{+1}{n} - \frac{1}{n+1}.$$

$$Z\left\{\frac{1}{n(n+1)}\right\} = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n} - \sum_{n=1}^{\infty} \frac{1}{n+1} z^{-n}$$

$$= \log\left(\frac{z}{z-1}\right) - \left[\sum_{n=1}^{\infty} \frac{1}{(n+1)z^n}\right]$$

$$= \log\left(\frac{z}{z-1}\right) - \left[\frac{1}{2z^1} + \frac{1}{3z^2} + \frac{1}{4z^3} + \cdots\right]$$

$$= \log\left(\frac{z}{z-1}\right) - \left[\frac{1}{2}\left(\frac{1}{z}\right)^1 + \frac{1}{3}\left(\frac{1}{z}\right)^2 + \cdots\right]$$

$$= \log\left(\frac{z}{z-1}\right) - \frac{z}{z}\left[1-1 + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right) + \cdots\right]$$

$$= \log\left(\frac{z}{z-1}\right) - z\left[-\frac{1}{z} + \frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots\right)$$

$$= \log\left(\frac{z}{z-1}\right) + 1 - z\left[\underbrace{\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots}\right]$$

$$= \log\left(\frac{z}{z-1}\right) + 1 - z\left[-\log\left(1 - \frac{1}{z}\right)\right]$$

$$= \log\left(\frac{z}{z-1}\right) + 1 - z\left[-\log\left(\frac{z-1}{z}\right)\right]$$

$$= \log\left(\frac{z}{z-1}\right) + 1 - z\left(\log\left(\frac{z-1}{z}\right)^{-1}\right).$$

$$= \log\left(\frac{z}{z-1}\right) + 1 - z\left(\log\left(\frac{z}{z-1}\right)\right].$$

$$= (1-z)\log\left(\frac{z}{z-1}\right) + 1.$$

4) $Z\left[u(n-1)\right] = \sum_{n=1}^{\infty} u(n-1) \cdot z^{-n}$

we know that $\left[u(n-k) = \begin{cases} 1 & \text{for } (n-k) \geq 0 \\ 0 & \text{for } (n-k) < 0 \end{cases} \right]$

$$\Rightarrow \sum_{n=1}^{\infty} z^{-n} = \sum_{n=1}^{\infty} \frac{1}{z^n}$$

$$= \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

$$= \frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right)$$

$$= \frac{1}{z}\left(1 - 1/z\right)^{-1} \quad \text{if } |1/z| < 1.$$

$$= \frac{1}{z}\left[\frac{z-1}{z}\right)^{-1} \quad \text{if } 1 < |z|$$

$$= \frac{1}{z}\left[\frac{z}{z-1}\right] \quad \text{if } |z| > 1.$$

$$= \frac{1}{z-1} \quad \text{if } |z| > 1.$$

we know that

$$\left[\delta(n-k) = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}\right]$$

5) $$Z[3^n \delta(n-1)] = \sum_{n=0}^{\infty} 3^n \delta(n-1) z^{-n} =$$

here $k=1$

$$= 3^1 z^{-1} = 3/z.$$

6) $$Z\left[\cos\frac{n\pi}{2}\, u(n)\right] = \sum_{n=0}^{\infty} u(n)\cos\frac{n\pi}{2}\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \cos\frac{n\pi}{2}\, z^{-n}$$

[Refer 2nd problem in Page No: 15]

7) $$Z[\delta(n-k)] = \sum_{n=0}^{\infty} \delta(n-k) z^{-n}$$

$$= \frac{1}{z^k}, \text{ if } k \text{ is a positive integer.}$$

8) $$Z[ab^n] = \sum_{n=0}^{\infty} ab^n z^{-n}$$

$$= \sum_{n=0}^{\infty} a\left(\frac{b}{z}\right)^n = a\sum_{n=0}^{\infty} (b/z)^n$$

$$= a\left[1 + \frac{b}{z} + \left(\frac{b}{z}\right)^2 + \cdots \text{to } \infty\right)$$

$$= a\left[1 - b/z\right]^{-1} \text{ if } |b/z| < 1$$

$$= a\left[\frac{z-b}{z}\right)^{-1} \text{ if } |b| < |z|$$

$$= a\left[\frac{z}{z-b}\right] \text{ if } |z|>|b|.$$

$$= \frac{az}{z-b} \text{ if } |z|>|b|.$$

9) $$Z[x(n)] = \sum_{n=0}^{\infty} n \cdot z^{-n} = \sum_{n=0}^{\infty} \frac{n}{z^n}$$

Refer page NO: 9
problem NO: 4

Note: Answer: $\frac{z}{(z-1)^2}$ if $|z|>1$

10) $$Z\left[\frac{(n+1)(n+2)}{2}\right] = \frac{1}{2}Z\left[n^2+3n+2\right]$$

$Z(2) = 2Z(1)$

$$= \frac{1}{2}\left[Z[n^2] + 3Z(n) + Z(2)\right]$$

$$= \frac{1}{2}\left[\frac{z(z+1)}{(z-1)^3} + \frac{3z}{(z-1)^2} + \frac{2z}{z-1}\right] \text{ if } |z|>1.$$

11) $$Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) \cdot z^{-n}$$

$$= z^{-k} = \frac{1}{z^k}.$$

12) $$Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{0} z^{-n}$$

$$= \sum_{n=-\infty}^{0} z^{n}$$

$$= 1 + z + z^2 + \cdots$$

$$= (1-z)^{-1} \text{ if } |z|<1$$

$$= \frac{1}{1-z} \text{ if } |z|<1$$

(13) $$Z[x(n)] = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{a}{z}\right)^n$$

$$= 1 + \frac{1}{1!}\left(\frac{a}{z}\right) + \frac{1}{2!}\left(\frac{a}{z}\right)^2 + \frac{1}{3!}\left(\frac{a}{z}\right)^3 + \cdots$$

$$= 1 + \frac{(a/z)}{1!} + \frac{(a/z)^2}{2!} + \frac{(a/z)^3}{3!} + \cdots$$

$$= e^{a/z}$$

Note: $e^x = 1 + \frac{x}{\underline{1}} + \frac{x^2}{\underline{2}} + \frac{x^3}{\underline{3}} + \cdots$

Find the Z-transform of f(t) where f(t) is given by

① t	④ $\sin\omega t$ (HW)
② e^{-at}	⑤ $\cos\omega t$
③ e^{at} (HW)	⑥ t^K

$$Z[f(t)] = \sum_{n=0}^{\infty} f(nT)\,z^{-n}.$$

① $$Z[t] = \sum_{n=0}^{\infty} (nT)\,z^{-n}$$

$$= T\sum_{n=0}^{\infty} n z^{-n}$$

$$= T\underbrace{\sum_{n=0}^{\infty} n\cdot\frac{1}{z^n}}$$

$$= \frac{Tz}{(z-1)^2}$$ [Refer page No: 9 problem no: 4]

② $$Z[e^{-at}] = \sum_{n=0}^{\infty} e^{-anT}\cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} (e^{-aT})^n z^{-n} = \sum_{n=0}^{\infty}\left(\frac{e^{-aT}}{z}\right)^n$$

$$= 1+\left(\frac{e^{-aT}}{z}\right)^1 + \left(\frac{e^{-aT}}{z}\right)^2 + \dots$$

$$= \left(1+\frac{e^{-aT}}{z}+\left(\frac{e^{-aT}}{z}\right)^2+\dots\right)$$

$$= \left(1 - \frac{e^{-aT}}{z}\right)^{-1} \text{ if } \left|\frac{e^{-aT}}{z}\right| < 1$$

$$= \left(\frac{z - e^{-aT}}{z}\right)^{-1} \text{ if } |e^{-aT}| < |z|$$

$$= \left(\frac{z}{z-e^{-aT}}\right) \text{ if } |z| > |e^{-aT})$$

③ lllly $Z[e^{at}] = \frac{z}{z-e^{aT}}$ if $|z| > |e^{aT}|$. (Exercise).

⑤ $Z[\cos\omega t] = \sum_{n=0}^{\infty} \cos n\omega T \, z^{-n}$

$$= Z[\cos n\theta] \text{ where } \theta = \omega T$$

$$= \frac{z[z - \cos\omega T]}{z^2 - 2z\cos\omega T + 1}$$ [Refer page No: 11 & 12]

(or)

$$Z[\cos\omega t] = Z\left[\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right]$$

$$= \frac{1}{2}\left[Z(e^{i\omega t}) + Z(e^{-i\omega t})\right]$$

$$= \frac{1}{2}\left[\frac{z}{z - e^{i\omega T}} + \frac{z}{z - e^{-i\omega T}}\right]$$

$$= \frac{1}{2}\left[\frac{z(z-e^{-i\omega T}) + z(z-e^{i\omega T})}{(z-e^{i\omega T})(z-e^{-i\omega T})}\right]$$

$$= \frac{1}{2}\left[\frac{z^2 - ze^{-i\omega T} + z^2 - ze^{i\omega T}}{z^2 - ze^{-i\omega T} - ze^{i\omega T} + 1}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - z\left[e^{i\omega T} + e^{-i\omega T}\right]}{z^2 - z\left[e^{i\omega T} + e^{-i\omega T}\right] + 1}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - z\left[2\cos\omega T\right)}{z^2 - z(2\cos\omega T) + 1}\right]$$

$$\Rightarrow \frac{2}{2}\left[\frac{z^2 - z\cos\omega T}{z^2 - 2z\cos\omega T + 1}\right]$$

$$= \frac{z\left[z - \cos\omega T\right)}{z^2 - 2z\cos\omega T + 1}$$

④ $Z\left[\sin\omega t\right] = \left.\frac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1}\right\} \rightarrow$ Exercise.

b) $Z[t^k]$

$$Z[t^k] = -Tz \frac{d}{dz} Z[t^{k-1}] \quad (\text{To prove})$$

Pf:

$$Z[t^{k-1}] = \sum_{n=0}^{\infty} (nT)^{k-1} z^{-n}$$

$$\frac{d}{dz} Z[t^{k-1}] = \sum_{n=0}^{\infty} \frac{d}{dz} (nT)^{k-1} z^{-n}$$

$$= \sum (nT)^{k-1} (-n) z^{-(n+1)}$$

$$= -(Tz)^{-1} \sum_{n=0}^{\infty} (nT)^k z^{-n}$$

$$\therefore -Tz \frac{d}{dz} Z[t^{k-1}] = Z[t^k]$$

$$\therefore Z[t^k] = -Tz \frac{d}{dz} Z[t^{k-1}] \quad k = 1, 2, 3, \ldots$$

When $t = 1$ $\quad Z[t] = -Tz \frac{d}{dz} Z[1] = -Tz \frac{d}{dz}\left(\frac{z}{z-1}\right)$

$$= \frac{Tz}{(z-1)^2}$$

When $t = 2$ $\quad Z[t^2] = -Tz \frac{d}{dz} Z[t]$

$$= -Tz \frac{d}{dz}\left(\frac{Tz}{(z-1)^2}\right)$$

$$= \frac{T^2 z(z+1)}{(z-1)^3}$$