Unit V

Power Spectral Density Function

Problems on Spectral density

Example1: If $Y(t) = X(t+a) - X(t+a)$, prove that

$R_{YY}(\tau) = 2R_{XX}(\tau) - R_{XX}(\tau+2a) - R_{XX}(\tau-2a)$

Hence prove that $S_{YY}(\omega) = 4\sin^2 a\omega S_{XX}(\omega)$.

Solution:

$$
\begin{aligned}
R_{YY}(\tau) &= E\left(Y(t)Y(t+\tau)\right) \\
&= E\left[\left(X(t+a) - X(t-a)\right)\left(X(t+a+\tau) - X(t-a+\tau)\right)\right] \\
&= E\left[\left(X(t+a)X(t+a+\tau)\right)\right] - E\left(X(t-a)X(t-a+\tau)\right) \\
&\quad - E\left[\left(X(t+a)X(t-a+\tau)\right)\right] + E\left[\left(X(t-a)X(t-a+\tau)\right)\right] \\
&= R_{XX}(\tau) - R_{XX}(\tau+2a) - R_{XX}(\tau-2a) + R_{XX}(\tau) \\
&= 2R_{XX}(\tau) - R_{XX}(\tau+2a) - R_{XX}(\tau-2a)
\end{aligned}
$$

To prove

$S_{YY}(\omega) = 4\sin^2 a\omega\, S_{XX}(\omega)$

$$
\begin{aligned}
F^{-1}\left(R_{YY}(\tau)\right) &= 2S_{XX}(\tau) - \int_{-\infty}^{\infty} R_{XX}(\tau+2a)e^{-i\omega\tau}d\tau - \int_{-\infty}^{\infty} R_{XX}(\tau-2a)e^{-i\omega\tau}d\tau \\
&= 2S_{XX}(\omega) - \int_{-\infty}^{\infty} R_{XX}(u)e^{-i\omega(u-2a)}du - \int_{-\infty}^{\infty} R_{XX}(v)e^{-i\omega(v+2a)}dv
\end{aligned}
$$

where $\tau - 2a = v,\ \tau + 2a = u$

$$
\begin{aligned}
F^{-1}\left(R_{YY}(\tau)\right) &= 2S_{XX}(\omega) - e^{i\omega 2a}S_{XX}(\omega) - e^{-i\omega 2a}S_{XX}(\omega) \\
&= 2S_{XX}(\omega) - \left(e^{i\omega 2a} + e^{-i\omega 2a}\right)S_{XX}(\omega) \\
&= 2S_{XX}(\omega)\left[1 - \cos 2a\omega\right] \\
&= 4\sin^2(a\omega)S_{XX}(\omega).
\end{aligned}
$$

Example2:

If $R_{XX}(\tau) = ae^{-b|\tau|}$, find the spectral density function, where a and b are constants.

Solution: Power spectral density

$$S_{XX}(\omega)=\int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}d\tau$$

$$=\int_{-\infty}^{\infty} ae^{-b|\tau|}e^{-i\omega\tau}d\tau$$

$$=\int_{-\infty}^{\infty} ae^{-b|\tau|}\left(\cos\omega\tau - i\sin\omega\tau\right)d\tau$$

$$S_{XX}(\omega)=2a\int_{-\infty}^{\infty} e^{-b|\tau|}\cos\omega\tau d\tau$$

$$=2a\left[\frac{e^{-b\tau}}{b^2+\omega^2}\left(-b\cos\omega\tau+\omega\sin\omega\tau\right)\right]_0^{\infty}$$

$$=\frac{2ab}{b^2+\omega^2}.$$

Example 3:

A stationary random process x(t) has an autocorrelation function given by

$R_{XX}(\tau)=3e^{-|\tau|}+5e^{-4|\tau|}$ Find the power spectral density of the process.

Solution:

$$S_{XX}(\omega)=F\left[R_{XX}(\tau)\right]=\int_{-\infty}^{\infty}\left(3e^{-|\tau|}+5e^{-4|\tau|}\right)e^{-i\omega\tau}d\tau$$

$$=\int_{-\infty}^{\infty}\left(3e^{-|\tau|}+5e^{-4|\tau|}\right)e^{-i\omega\tau}d\tau$$

$$=2\int_{0}^{\infty}\left(3e^{-|\tau|}+5e^{-4|\tau|}\right)\cos\omega\tau d\tau$$

$$=2\left(\frac{3}{\left(1+\omega^2\right)}+\frac{5(4)}{\left(16+\omega^2\right)}\right)$$

$$=\frac{46\omega^2+136}{\left(1+\omega^2\right)\left(16+\omega^2\right)}$$

Example 4:

The power spectral of a stationary random process is given by $S_{XX}(\omega)=\begin{cases} A\ , -k<\omega<k, \\ 0, \text{ otherwise}\end{cases}$.

Find the autocorrelation function.

Solution:

$$R_{XX}(\tau) = F^{-1}\left(S_{XX}(\omega)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\tau\omega}d\omega$$

$$= \frac{1}{2\pi}\int_{-k}^{k} Ae^{i\tau\omega}d\omega$$

$$= \frac{A}{2\pi}\left(\frac{e^{i\,\omega\tau}}{i\tau}\right)_{-k}^{k} = \frac{A}{\pi\tau}\left\{\sin k\tau\right\}.$$

Example 5:

For a random process with power spectrum $S_{XX}(\omega) = \begin{cases} 1 - \dfrac{\omega^2}{4}, & |\omega| \leq 2 \\ 0, & \text{otherwise} \end{cases}$,

find the auto correlation function of the process.

Solution:

$$R_{XX}(\tau) = F^{-1}\left(S_{XX}(\omega)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\omega\tau}d\omega$$

$$= \frac{1}{2\pi}\int_{-2}^{2}\left(1-\frac{\omega^2}{4}\right)\left(\cos\omega\tau + i\sin\omega\tau\right)d\omega$$

$$= \frac{1}{\pi}\int_{0}^{2}\left(1-\frac{\omega^2}{4}\right)\cos\omega\tau d\omega$$

$$= \frac{1}{\pi}\left[\left(1-\frac{\omega^2}{4}\right)\frac{\sin\omega\tau}{\tau} - \left(-\frac{2\omega}{4}\right)\left(-\frac{\cos\omega\tau}{\tau^2}\right) + \left(-\frac{1}{2}\right)\left(-\frac{\sin\omega\tau}{\tau^3}\right)\right]_0^2$$

$$= \frac{1}{\pi}\left[\left(-\frac{\cos 2\tau}{\tau^2}\right) + \frac{1}{2\tau^3}\sin 2\tau\right]$$

Example 6:

Find the average power of the random process X(t) with power density spectrum

$$S_{XX}(\omega) = \frac{6\omega^2}{\left(1+\omega^2\right)^3}$$

Solution:

$$\text{Average power} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{6\omega^2}{\left(1+\omega^2\right)^3}\,d\omega \text{ put } \omega = \tan\theta; d\omega = \sec^2\theta d\theta$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{6\tan^2\theta}{\left(1+\tan^2\theta\right)^3}\sec^2\theta d\theta$$

$$= \frac{6}{\pi}\int_{0}^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta d\theta = \frac{6}{\pi}\int_{0}^{\frac{\pi}{2}}\left(\sin^2\theta - \sin^4\theta\right)d\theta = \frac{3}{8}.$$

Example 7:

A WSS random process X(t) has power spectral density $S_{XX}(\omega) = \frac{\omega^2}{\left(\omega^4+10\omega^2+9\right)}$ find the auto correlation function and mean square value of the process.

Solution:

$$R_{XX}(\tau) = F^{-1}\left(S_{XX}(\omega)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\omega\tau}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\omega^2}{\omega^4+10\omega^2+9}e^{i\omega\tau}d\omega$$

$$\frac{\omega^2}{\omega^4+10\omega^2+9} = \frac{u}{(u+1)(u+9)}, \; u = \omega^2,$$

$$= \frac{A}{(u+1)} + \frac{B}{(u+9)}$$

Solving $A = \frac{-1}{8}$, $B = \frac{9}{8}$

$$\frac{\omega^2}{\omega^4+10\omega^2+9} = \frac{\frac{-1}{8}}{\left(\omega^2+1\right)} + \frac{\frac{9}{8}}{\left(\omega^2+9\right)}$$

$$R_{XX}(\tau) = F^{-1}\left(S_{XX}(\omega)\right)$$

$$= \frac{-1}{2\times 8} F^{-1}\left(\frac{2}{\left(\omega^2+1\right)}\right) + \frac{1}{6}\times\frac{9}{8} F^{-1}\left(\frac{2\times 3}{\left(\omega^2+9\right)}\right)$$

$$= \frac{-1}{16} e^{-|\tau|} + \frac{3}{16} e^{-3|\tau|}$$

Mean Square value of the process =R(0)= $\frac{1}{8}$.

Example 8:

The power spectral density function of a zero mean wide sense stationary process X(t) is given by $S(\omega) = \begin{cases} 1 \text{ if } |\omega| < \omega_0 \\ 0 \text{ elsewhere} \end{cases}$. Find $R(\tau)$ and also show that $X(t) \; and \; X\left(t+\frac{\pi}{\omega_0}\right)$ are uncorrelated.

Solution:

$$E\left[X(t)\right] = 0$$

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega\tau} d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} 1.(\cos\omega\tau + i\sin\omega\tau) d\omega$$

$$= \frac{2}{2\pi}\int_{0}^{\omega_0} \cos\omega\tau d\omega = \frac{1}{\pi}(\sin\omega\tau)/_{0}^{\omega_0}$$

$$= \frac{1}{\pi\tau}\sin\omega_0\tau$$

To prove $X(t) \; and \; X\left(t+\frac{\pi}{\omega_0}\right)$ are uncorrelated we have to prove

$$Cov\left(X(t)\,,\, X\left(t+\frac{\pi}{\omega_0}\right)\right) = 0$$

$$Cov\left(X(t)\,,\, X\left(t+\frac{\pi}{\omega_0}\right)\right) = E\left(X(t)\; X\left(t+\frac{\pi}{\omega_0}\right)\right) - E\left(X(t)\right) E\left(X\left(t+\frac{\pi}{\omega_0}\right)\right)$$

$$= E\left(X(t)\; X\left(t+\frac{\pi}{\omega_0}\right)\right) + 0$$

X(t) is a WSS process.

$$E\left(X(t)\; X\left(t+\frac{\pi}{\omega_0} \right) \right) = R_{XX}\left(\frac{\pi}{\omega_0} \right) = \frac{1}{\pi\tau}\sin\left(\omega_0 \frac{\pi}{\omega_0} \right) = 0$$

$$Cov\left(X(t)\,,\; X\left(t+\frac{\pi}{\omega_0} \right) \right) = 0.$$

That is $X(t)$ *and* $X\left(t+\frac{\pi}{\omega_0} \right)$ are uncorrelated.

Cross Power Spectral Density Function

Definition: Let X(t) and Y(t) be two jointly stationary processes with cross correlation function $R_{XY}(\tau)$. Then the Fourier transform of $R_{XY}(\tau)$ is called the cross power density spectrum or cross power spectral density of X(t) and Y(t) and is denoted by $S_{XY}(\omega)$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-i\omega\tau} d\tau\,.$$

If given $S_{XY}(\omega)$ then $R_{XY}(\tau)$ is obtained by inverse Fourier transform given by

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} S_{XY}(\omega) e^{i\tau\omega} d\omega$$

similarly,

$$S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau) e^{-i\omega\tau} d\tau$$

$$R_{YX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YX}(\omega) e^{i\tau\omega} d\omega$$

Properties of cross power spectral density:

1. $S_{XY}(-\omega) = S_{YX}(\omega)$
2. $\operatorname{Re} S_{XY}(\omega)$ and $\operatorname{Re} S_{YX}(\omega)$ are even functions of ω.
3. $\operatorname{Im} S_{XY}(\omega)$ and $\operatorname{Im} S_{YX}(\omega)$ are odd functions of ω.
4. If X(t) and Y(t) are orthogonal, then $S_{XY}(\omega) = 0$ and $S_{YX}(\omega) = 0$.

Problems:

Example1:

If $S_{XY}(\omega) = \begin{cases} a + \dfrac{ib\omega}{\alpha} \text{ , } -\alpha < \omega < \alpha, \; \alpha > 0 \\ \qquad 0, \text{ otherwise} \end{cases}$ a and b are constants, find the cross correlation function.

Solution:

$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega) e^{i\tau\omega} d\omega$$

$$= \frac{1}{2\pi}\int_{-\alpha}^{\alpha}\left(a + \frac{ib\omega}{a}\right) e^{i\tau\omega} d\omega$$

$$= \frac{a}{2\pi}\int_{-\alpha}^{\alpha} e^{i\omega\tau} d\omega + \frac{ib}{2a\pi}\int_{-\alpha}^{\alpha} \omega e^{i\omega\tau} d\omega$$

$$= \frac{a}{2\pi}\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-\alpha}^{\alpha} + \frac{ib}{2a\pi}\left[\omega\frac{e^{i\omega\tau}}{i\tau} - 1.\frac{e^{i\omega\tau}}{(i\tau)^2}\right]_{-\alpha}^{\alpha}$$

$$= \frac{a}{\pi\tau}\sin\alpha\tau + \frac{b}{\pi\tau}\cos\alpha\tau - \frac{b}{\pi\alpha\tau^2}\sin\alpha\tau$$

$$R_{XY}(\tau) = \frac{1}{\pi\alpha\tau^2}\left[(a\alpha\tau - b)\sin\alpha\tau + b\alpha\tau\cos\alpha\tau\right].$$

Example2:

The cross power spectrum of real random processes X(t) and Y(t) is given by

$S_{XY}(\omega) = \begin{cases} a + ib\omega \text{ , if } |\omega| < 1 \\ \quad 0, \text{ otherwise} \end{cases}$ Find the cross correlation function.

Solution:

$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega) e^{i\tau\omega} d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1}(a + ib\omega) e^{i\tau\omega} d\omega$$

$$= \frac{a}{2\pi}\int_{-1}^{1} e^{i\omega\tau} d\omega + \frac{ib}{2\pi}\int_{-1}^{1} \omega e^{i\omega\tau} d\omega$$

$$= \frac{1}{\pi\tau^2}\left\{(a\tau - b)\sin\tau + b\tau\cos\tau\right\}.$$

Note: If $S_{XY}(\omega) = \begin{cases} a + jb\omega \text{ , if } |\omega| < 1 \\ 0, \text{ otherwise} \end{cases}$ then $R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega) e^{j\tau\omega} d\omega$

Linear Systems with Random Inputs

System:

It is a functional relationship between the input X(t) and the output Y(t). It is written as Y(X(t)) = f(X(t)), $-\infty < t < \infty$.

Linear System:

If $f(a_1 X_1(t) \pm a_2 X_2(t)) = a_1 f(X_1(t)) \pm a_2 f(X_2(t))$,then f is called a linear system.

Time Invariant System:

If Y (t + h) = f (X (t + h)), where Y(X(t)) = f(X(t)), then it is called a time invariant system.

Memoryless System:

If the output $Y(t_1)$ at a given time $t=t_1$ depends only on $X(t_1)$ and not on any other past or future values of X(t),then the system f is called a memoryless system.

Causal System:

If the value of the output Y(t) at time $t=t_1$ depends only on the past values of the input X(t) ,$t \le t_1$ .i.e $Y(t_1)= f(X(t) ,t \le t_1)$,then the system is called a causal system.

Unit Impulse Response Function

System in the form of Convolution:

$Y(t)= h(t)*X(t) = \int_{-\infty}^{\infty} h(u) X(t-u) du$, h(t) is called system weighting function or unit impulse response function.

Theorems:

1. If a system is such that its input X(t) and its output Y(t) are related by a convolution integral given by equation $Y(t)= h(t)*X(t) = \int_{-\infty}^{\infty} h(u) X(t-u) du$, the system is a linear time invariant system.
2. If the output to a time-invariant, stable linear system is a WSS process, the output will also be a WSS process.
3. If X(t) is a WSS process and if $Y(t)= h(t)*X(t) = \int_{-\infty}^{\infty} h(u) X(t-u) du$ then $S_{XY}(\omega)= S_{XX}(\omega) H^*(\omega)$ and $S_{YY}(\omega)= S_{XX}(\omega) |H(\omega)|^2$.
4. The power spectral densities of the input and output processes in the system are connected by the relations $S_{XY}(w)= S_{XX}(w) H^*(w)$ and $S_{YY}(\omega)= S_{XX}(\omega) |H(\omega)|^2$ where $H(\omega)$ is the Fourier transform of the unit impulse response function h(t).
   $H(\omega)= F(h(t)) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt$

$H(\omega)$ is called the system function or the power transfer function.

Note:

1. If $\int_{-\infty}^{\infty}|h(t)|\text{dt} < \infty$, the system is called stable.
2. In addition if h(t)=0 when t<0, the system is said to be casual.

Example1:

A circuit has unit impulse response given by h(t)=$\begin{cases} \frac{1}{T}, & 0 \le t \le T \\ 0, oherwise \end{cases}$

Evaluate $S_{YY}(\omega)$ in terms of $S_{XX}(\omega)$.

Solution:

We know $S_{YY}(\omega)$= $S_{XX}(\omega)$ $|\text{H}(\omega)|^2$, where H(ω) is the Fourier transform of h(t).

$$\text{H}(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t}\text{dt}$$

$$= \int_0^T \frac{1}{T}\, e^{-i\omega t}\text{dt}$$

$$= \frac{1}{T}\int_0^T (cos\ \omega t - \text{i sin } \omega t)\text{dt}$$

$$= \frac{1}{T}\left[\frac{sin\omega t}{\omega} + i\,\frac{cos\omega t}{\omega}\right]_0^T$$

$$=\frac{1}{\omega T}\ [\text{sin } \omega T\text{+icos } \omega T\text{-i}]$$

$$=\frac{1}{\omega T}\ [\text{sin } \omega T\text{-i(1-(cos } \omega T)]$$

$$|\text{H}(\omega)|^2=\frac{1}{\omega^2T^2}[(sin\omega T)^2 + (1 - cos\omega T)^2]$$

$$= \frac{1}{\omega^2T^2}[\ \text{2-2cos}\omega T\ ]$$

$$= \frac{4}{\omega^2T^2}\ \ sin^2\left(\frac{\omega T}{2}\right)$$

Example 2:

A WSS process X(t) is the input to linear system with impulse response h(t)=2e^{-7t}, t ≥ 0. If R_{XX}(τ)=$e^{-4|\tau|}$,find the power spectral density function of the output process Y(t).

Solution:

$S_{XX}(\omega)$ = Fourier transform of R_{XX}(τ).

$$= \int_{-\infty}^{\infty} R_{XX}(\tau)\ e^{-i\omega\tau}\text{d}\tau$$

$$=\int_{-\infty}^{\infty} e^{-4|\tau|}\ e^{-i\omega\tau}\text{d}\tau$$

$$=\int_{-\infty}^{\infty} e^{-4|\tau|}(\cos\omega\tau - i\ sin\ \omega\tau)\ d\tau$$

$$=\int_{-\infty}^{\infty} e^{-4|\tau|} \cos\omega\tau\ d\tau - i\int_{-\infty}^{\infty} e^{-4|\tau|} \sin\omega\tau\ d\tau$$

$$=2\int_0^{\infty} e^{-4\tau} \cos\omega\tau\ d\tau$$

$$= \frac{2e^{-4\tau}}{16+\omega^2}(-4\cos\omega\tau + \omega\sin\omega\tau)_0^\infty$$

$$= \frac{8}{16+\omega^2}$$

H(ω)= Fourier transform of h(t)

$$=\int_0^\infty h(t)\, e^{-i\omega t}\text{dt}$$

$$=\int_0^\infty 2e^{-7t}\, e^{-i\omega t}\text{dt}$$

$$=2 \quad \int_0^\infty 2e^{-(7+i\omega)t}\ \text{dt}$$

$$=2\left[\frac{e^{-t(7+i\omega)}}{-(7+i\omega)}\right]_0^\infty$$

$$=\frac{2}{7+i\omega}$$

$$|H(\omega)|=\frac{2}{\sqrt{49+\omega^2}}$$

$$|H(\omega)|^2=\frac{4}{49+\omega^2}$$

$$S_{YY}(\omega)= S_{XX}(\omega)\ |\text{H}(\omega)|^2=\frac{8}{16+\omega^2}\frac{4}{49+\omega^2} = \frac{32}{(16+\omega^2)(49+\omega^2)}$$

Example 3: A Wide Sense Stationary process X(t) is the input to a linear system with impulse response h(t)=$2e^{-t}$, t ≥ 0.If $R_{XX}(\tau)=e^{-2|\tau|}$,find the power spectral density function of the output process Y(t).

<u>Solution:</u>

$S_{XX}(\omega)$ = Fourier transform of $R_{XX}(\tau)$.

$$= \int_{-\infty}^\infty R_{XX}(\tau)\, e^{-i\omega\tau}\text{d}\tau$$

$$=\int_{-\infty}^\infty e^{-2|\tau|}\, e^{-i\omega\tau}\text{d}\tau$$

$$= \int_{-\infty}^\infty e^{-2|\tau|}(\cos\omega\tau - i\, sin\ \omega\tau)\ d\tau$$

$$=\int_{-\infty}^\infty e^{-2|\tau|}\cos\omega\tau\ d\tau - i\int_{-\infty}^\infty e^{-2|\tau|}\sin\omega\tau\ d\tau$$

$$=2\int_0^\infty e^{-2\tau}\cos\omega\tau\ d\tau$$

$$= \frac{2e^{-2\tau}}{4+\omega^2}(-2\cos\omega\tau + \omega\sin\omega\tau)_0^\infty$$

$$= \frac{4}{14+\omega^2}$$

H(ω)= Fourier transform of h(t)

$$=\int_0^\infty h(t)\, e^{-i\omega t}\text{dt}$$

$$=\int_0^\infty 2e^{-t}\, e^{-i\omega t}\text{dt}$$

$=2 \int_0^\infty 2e^{-(1+i\omega)t}\,\mathrm{dt}$

$=2\left[\frac{e^{-t(1+i\omega)}}{-(1+i\omega)}\right]_0^\infty$

$=\frac{2}{1+i\omega}$

$|H(\omega)|=\frac{2}{\sqrt{1+\omega^2}}$

$|H(\omega)|^2=\frac{4}{1+\omega^2}$

$S_{YY}(\omega)= S_{XX}(\omega)\ |\mathrm{H}(\omega)|^2=\frac{4}{4+\omega^2}\frac{4}{1+\omega^2} = \frac{16}{(4+\omega^2)(1+\omega^2)}$