<u>SEQUENCES IN ℝ</u>

<u>Sequences:</u>

Definition: A **sequence of real numbers** (or a **sequence in** ℝ) is a function defined on the set ℕ = {1, 2, . . . } of natural numbers whose range is contained in the set ℝ of real numbers.

Thus by defining a sequence on a set **A** we are defining a function **f:** ℕ →which set the it assigns every member of set ℕ an element of the set **A.**
Sequences are denoted by notation : **(x_n).**
In Sequences order of elements is important.
Consider, **A =** (1, 2, 3, 4, 5) and **B** = (1, 3, 2, 5, 4)
Sequence **A** ≠ **B** as the order in which the elements of sequences occur is different in both the Sequences A and B.

Important things to note about Sequences:
What makes a sequence different from an arbitrary set are the following important things:

1) **Rule:** Every Sequence in Mathematics is defined by a **function or a rule**; even arbitrary sequences are stated as being defined **'arbitrarily'** before writing.
   e.g.: D = $\{2^n: n \in \mathbb{N}\}$ here the function or rule being every term of the sequence is positive integral power of 2.
2) **Order:** Order in which the elements or terms of the sequences are arranged.
   e.g.: Consider the following arbitrarily defined sequences:
   A = {1,2,3,4,5} and B = {2,1,4,3,5} are both **different** when defined as **Sequences** but are equal when defined as **Sets.**
3) **Repetition:** In Sequences **repetition** of elements is **allowed**.
   e.g.: C = {1,1,1,...} is called a constant sequence of 1.

Examples of sequences:

1.) $(x_n) = \{(-1)^n: n \in \mathbb{N}\}$

2.) $(y_n) = \{\frac{1}{n}: n \in \mathbb{N}\}$

Example of not a sequence:

E = {2,5,3,4,6,7,...} is not a sequence as we have no rule for guessing or generating the next element in the sequence.

<u>Neighbourhood around a point:</u>

Definition: A neighbourhood around a point $x \in \mathbb{R}$ is any set **V** = $\{y \in \mathbb{R} \mid y \in (x-\varepsilon, x+\varepsilon)\}$ for some $\varepsilon > 0$, the neighbourhood is said to be an $\varepsilon-$ neighbourhood of x

So a neighbourhood around x is the set of all the points around x which are at a distance of arbitrary length $\varepsilon > 0$

Example: $= 1$, so a 1 – neighbourhood around the point 1 is illustrated by the red line in the below diagram

Limit of a Sequence:

Definition: A **sequence X** = $(\boldsymbol{x_n})$ in $\mathbb{R}$ is said to converge to a limit **L** in $\mathbb{R}$ if for every $\varepsilon > 0$ there exists a natural number **K(*ε*)** such that for all **n** $\geq$ **K(*ε*)** the terms $\boldsymbol{x_n}$ satisfy $|\, \boldsymbol{x_n} - \boldsymbol{L} \,| < \varepsilon$

The above definition basically says that if we have a limit **L** of a given sequence then we can find at least one element with index **K(*ε*)** of the sequence **X** at any distance of arbitrary length ***ε*** of our choice such that all the elements which succeed the element with **K(*ε*)** index have distance between them and **L** less than ***ε***

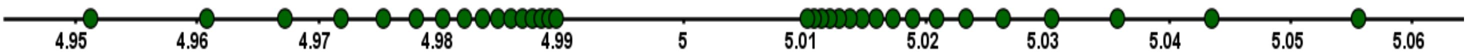
In terms of neighbourhood, the above definition can be understood as we can find the element with **K(*ε*)** for every $\varepsilon -$ neighbourhood of **L,** such that all the elements which succeed the element with **K(*ε*)** index have distance between them and **L** less than ***ε***

Sequences which have a limit are said to converge to the limit and are said as convergent sequences.

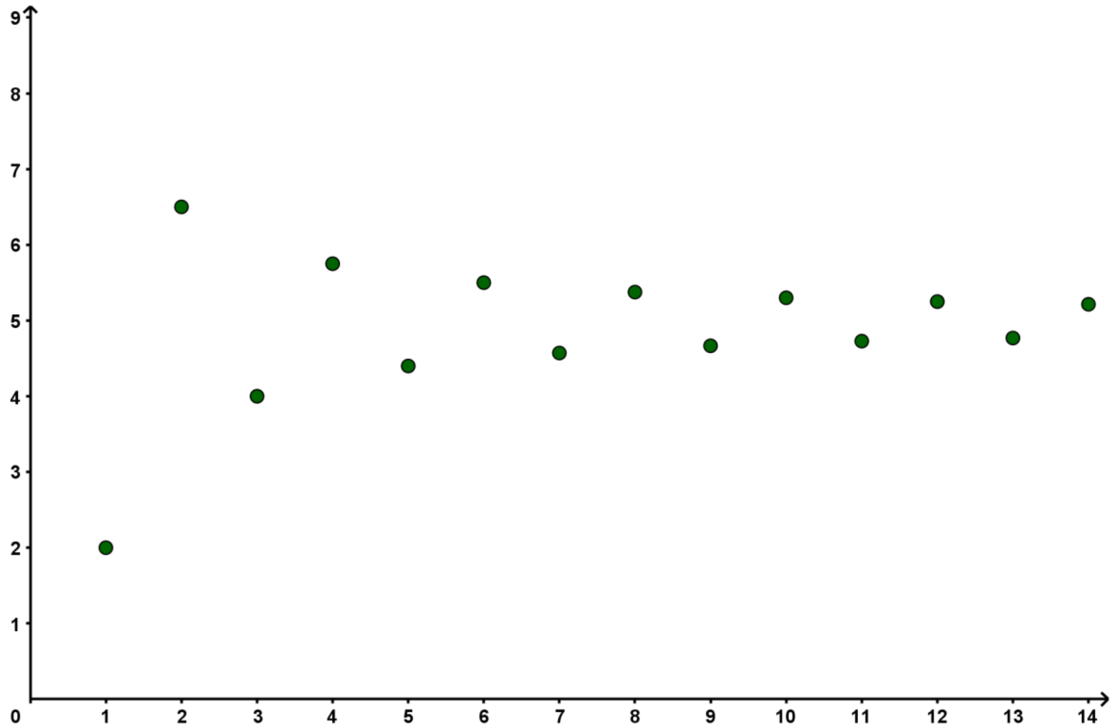
Sequences which do not have a limit are said to diverge or not convergent and are said as divergent sequences.

Consider a sequence defined as **X** = $\{x \mid x = 5 + 3\frac{(-1)^n}{n}, n \in \mathbb{N}\}$

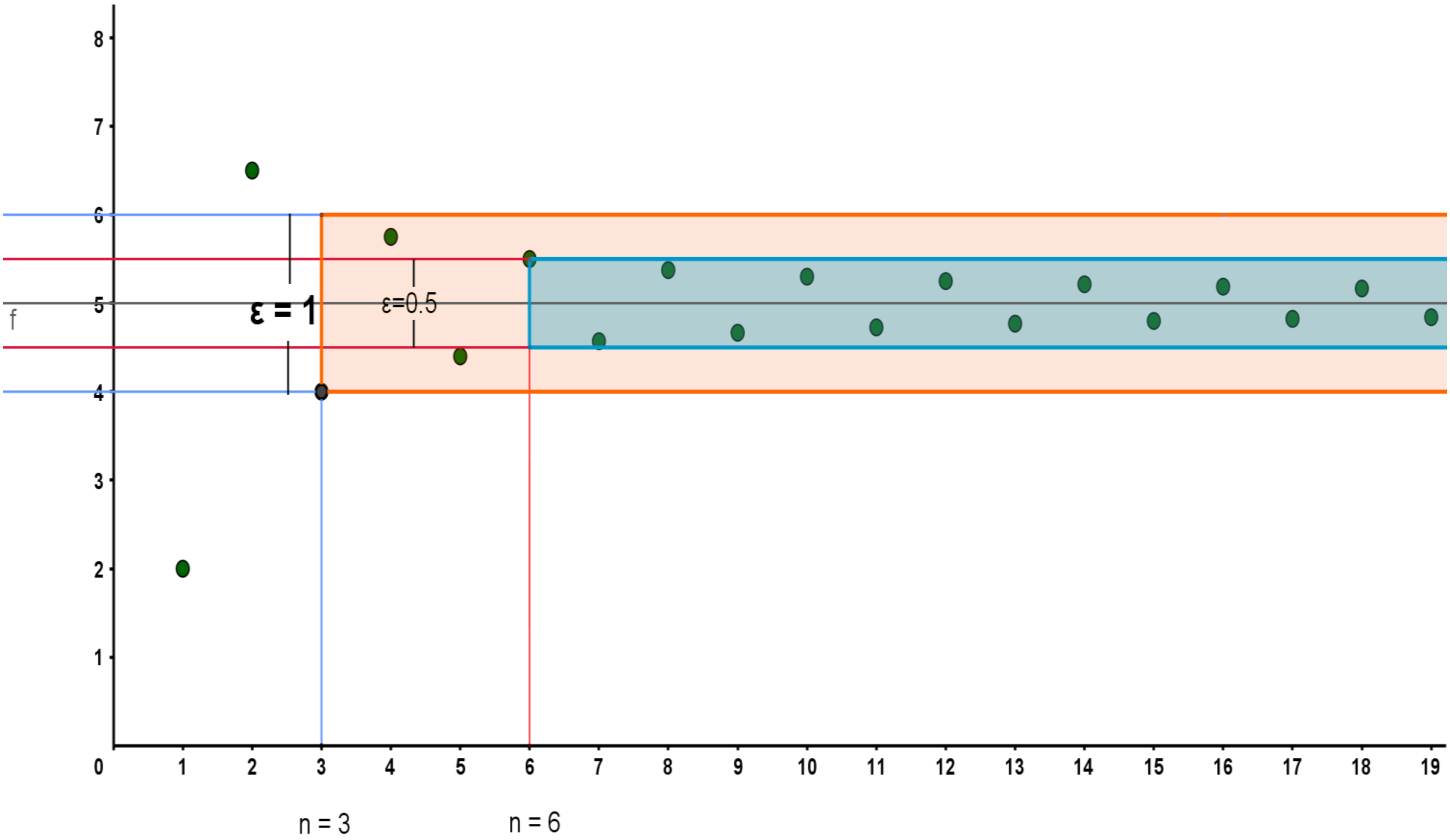
On X- axis the sequence is illustrated below:



It can be also seen in x-y plane with indices of terms plotted on x-axis and the sequence on y-axis below is the illustration:



The considered example is a convergent sequence and its convergence is illustrated below:



for $\boldsymbol{\varepsilon} = 1$ we can find $\mathbf{K}(\boldsymbol{\varepsilon}) = \mathbf{3}$ after which the succeeding terms stay at a distance of length at the most 1 .
or $\boldsymbol{\varepsilon} = 0.5$ we can find $\mathbf{K}(\boldsymbol{\varepsilon}) = \mathbf{6}$ after which the succeeding terms stay at a distance of length at the most 0.5 .
Continuing in this way we can find $\mathbf{K}(\boldsymbol{\varepsilon})$ for arbitrary $\boldsymbol{\varepsilon} > 0$

<u>Examples:</u>

1.) $(x_n) = \{(-1)^n : n \in \mathbb{N}\}$
2.) $(y_n) = \{\frac{1}{n} : n \in \mathbb{N}\}$
3.) $(z_n) = \{1, 1, 1, \ldots\}$

Consider the sequences in the example above

1.) The first sequence is an **oscillatory sequence** and is **divergent** we cannot find any limit for the sequence.

2.) In the second sequence is **convergent** and its **limit is 0**, $\left|\frac{1}{n} - \mathbf{0}\right| < \varepsilon$ since by **Archimedean property** we can always find a real number $\boldsymbol{\varepsilon}$, such that $\frac{1}{n} < \boldsymbol{\varepsilon}$, $n \in \mathbb{N}$

3.) The third sequence is said to be a **constant sequence** and is **convergent** with limit 1 as we can always find one element of the sequence namely 1 itself for every $\boldsymbol{\varepsilon} > 0$

References:

Introduction to real analysis – Robert Bartle and Donald Sherbert.