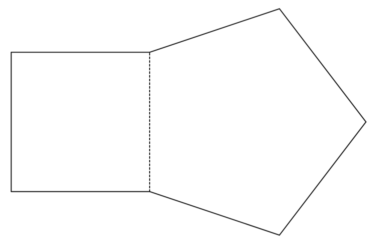
1. **LT1623** Amy, Beth, and Carlos want to stand in a line to take a group photo. However, Beth and Carlos refuse to stand directly next to each other. How many ways can Amy, Beth, and Carlos be ordered from left to right?

   *Written by: Linus Tang*

   **Answer:** $\boxed{2}$

   Since Beth and Carlos cannot stand directly next to each other, they must be on the two outside positions, and Amy must be in the middle. Beth and Carlos can take the left and right positions in either order, so the answer is $\boxed{2}$.

2. **LT1584** A figure is made of a square and a regular pentagon, which share an side of length 2, as shown in the figure below. What is the perimeter of the figure?

   *Written by: Linus Tang*

   **Answer:** $\boxed{14}$

   The figure has 7 sides, each of which has length 2, so the perimeter is $7 \times 2 = \boxed{14}$.

3. **LT1611** A birthday cake costs \$10.00, plus an additional \$0.50 for every decoration on it. Mr. Li orders two birthday cakes, the first of which has three decorations on it. If the subtotal was \$24.00, how many decorations were on the second cake?

   *Written by: Linus Tang*

   **Answer:** $\boxed{5}$

   The first cake cost $\$10.00 + 3 \times \$0.50 = \$11.50$, so the second cake must have cost the remaining \$12.50. The cake itself cost \$10.00, so the decorations added \$2.50 to the cost. This is the cost of $\boxed{5}$ decorations.

4. **LT1585** A square and a circle are drawn on a piece of paper. What is the maximum number of intersection points between the two shapes?

   *Written by: Linus Tang*

   **Answer:** $\boxed{8}$

Each edge of the square intersects the boundary of the circle at most twice. There are four edges, so the number of intersections is at most $4 \times 2 = \boxed{8}$.

(Insert diagram with construction of 8.)

5. **LT1561** Let $p$ be a prime number. The sum of the positive divisors of $2p$ is 42. What is $p$?

*Written by: Linus Tang*

**Answer:** $\boxed{13}$

Since the positive divisors of $2p$ are $1, 2, p$, and $2p$, we have that $1 + 2 + p + 2p = 42$, or $3p + 3 = 42$. Solving, $p = \boxed{13}$.

6. **LT1527** Tom's favorite number has four digits. The sum of the first three digits is 16, and the sum of the last three digits is 7. What is the first digit?

*Written by: Linus Tang*

**Answer:** $\boxed{9}$

Let $A$, $B$, $C$, and $D$ be the four digits in order. Then $A+B+C = 16$ and $B+C+D = 7$. Subtracting, $A - D = 9$. Since $A$ and $D$ are digits, the only way this can happen is if $A = \boxed{9}$ and $D = 0$.

7. **LT1587** The number $2024^2 = 4096576$ has 63 positive divisors. How many of these divisors are greater than 2024?

*Written by: Linus Tang*

**Answer:** $\boxed{31}$

The divisors form pairs that multiply to $2024^2$.

2024 pairs with itself, and the other 62 divisors form $62/2 = \boxed{31}$ pairs, each of which contains one number greater than 2024 and one number less than 2024.

8. **LT1589** There are initially 1000 bacteria in a petri dish. Every 20 minutes, each bacterium splits into two bacteria. How many bacteria are in the petri dish after 60 minutes?

*Written by: Linus Tang*

**Answer:** $\boxed{8000}$

The bacteria split $60/20 = 3$ times. After the first split, there are $1000 \times 2 = 2000$ bacteria. After the second, there are $2000 \times 2 = 4000$. After the third, there are $4000 \times 2 = \boxed{8000}$.

9. **LT1588** A palindrome is a sequence of letters that are in the same order when read from left to right or right to left. For example, *abcba* is a palindrome. How many ways can the seven letters in *pompoms* be rearranged to form a palindrome?

*Written by: Linus Tang*

**Answer:** $\boxed{6}$

Trivially $\boxed{6}$.

10. **BF1569** Find the least real number $N$ such that there exist no values of $x$ greater than or equal to $N$ that satisfy

$$\lfloor x^2 + \lfloor x^2 + \lfloor x^2 + \lfloor x^2 \rfloor \rfloor \rfloor \rfloor = 100,$$

where $\lfloor y \rfloor$ denotes the greatest integer less than or equal to $y$.

*Written by: Benjamin Fu*

**Answer:** $\boxed{\sqrt{26}}$

Observe that since $\lfloor x^2 \rfloor$ is always an integer and $\lfloor x^2 + n \rfloor = \lfloor x^2 \rfloor + n$ for all integers $n$, we have

$$\lfloor x^2 + \lfloor x^2 \rfloor \rfloor = \lfloor x^2 \rfloor + \lfloor x^2 \rfloor = 2 \lfloor x^2 \rfloor .$$

Similarly, we have

$$\lfloor x^2 + \lfloor x^2 + 2 \lfloor x^2 \rfloor \rfloor \rfloor = \lfloor x^2 + 3 \lfloor x^2 \rfloor \rfloor = 4 \lfloor x^2 \rfloor .$$

Thus, we have

$$4 \lfloor x^2 \rfloor = 100 \implies \lfloor x^2 \rfloor = 25.$$

It is easy to see that $N = \sqrt{26}$.

11. **BF1598** Ethan puts five slips of paper into a basket, labelled 1, 2, 3, 4, and 5. He then randomly draws out three slips of paper one by one, without replacement. Determine the probability that the last number Ethan drew was the largest of the three.

*Written by: Benjamin Fu*

**Answer:** $\boxed{\dfrac{1}{3}}$

By symmetry, it is equally likely for each of the three numbers Ethan drew to be the largest, so the answer is simply $\boxed{\frac{1}{3}}$.

12. **BF1591** Let $\omega$ be a circle of radius 1 and $A$ be its center. Let $B$ be a point on the circumference of $\omega$. If point $C$ is chosen uniformly and random from the interior of $\omega$, find the probability that $\triangle ABC$ is obtuse.

*Written by: Benjamin Fu*

**Answer:** $\boxed{\frac{3}{4}}$

We will consider the three disjoint cases. First, the probability that $\angle ABC > 90^\circ$ is clearly 0. Next, by the Inscribed Angle Theorem, we see that $\angle BCA > 90^\circ$ if and only if $C$ is within the circle with diameter $\overline{AB}$, which has a probability of $\frac{1}{4}$ to occur. Finally, it's easy to see that $\angle CAB > 90^\circ$ if and only if $C$ lies below the line passing through $A$ perpendicular to $AB$, which has a probability of $\frac{1}{2}$ to occur. Thus, our answer is $0 + \frac{1}{4} + \frac{1}{2} = \boxed{\frac{3}{4}}$

13. **LT1554** Suppose $a$ is a real number such that the equation $x^3 + ax^2 - 1000 = 0$ has three real solutions in $x$, one of which equals the sum of the other two. Determine the value of $a$.

*Written by: Linus Tang*

**Answer:** $\boxed{20}$

By Vieta's formulas, the sum of all three roots is $-a$. The root that equals the sum of the other two must then be $-\frac{a}{2}$. Plugging this in, $(-\frac{a}{2})^3 + a(-\frac{a}{2})^2 - 1000 = 0$. This simplifies to $\frac{a^3}{8} - 1000 = 0$, so $a = \boxed{20}$.

14. **LT1543** Find the greatest multiple of 11 whose digits are all distinct.

*Written by: Linus Tang*

**Answer:** $\boxed{9876524130}$

Since all digits are distinct, the best we can hope for is 10 digits. We want the largest digits in the largest place values, so we start constructing the number $987\ldots$.

By the divisibility by 11 rule, we want the digit sum of $0 + 1 + \cdots + 9 = 45$ to be split into alternating sums equaling 28 and 17.

Note that no number starting with $987654\ldots$ (with the digits $0, 1, 2, 3$ left to place) has this property, since the alternating digit sums are both greater than 17 already.

So, we backtrack a digit, writing 98765 with the digits $0, 1, 2, 3, 4$ left to place. The best way to achieve a sum of 18 in the even positions is to complete the number $\boxed{9876524130}$.

15. **BF1602** Let $ABCD$ be a rectangle with side lengths $AB = 10$ and $BC = 1$. A circle $\omega$ passes through $A$ and $B$ and is tangent to $\overline{CD}$. Find the radius of $\omega$.

*Written by: Benjamin Fu*

**Answer:** $\boxed{13}$

Let $P \neq A$ denote the second intersection of $AD$ with $\omega$. Note that since $\angle BAP = 90^\circ$,

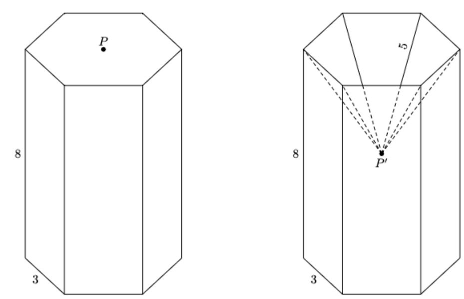
$\overline{BP}$ is a diameter of $\omega$. By Power of a Point, we have

$$5^2 = 1 \cdot DP \implies DP = 25 \implies AP = DP - 1 = 24.$$

Therefore, $BP = \sqrt{24^2 + 10^2} = 26$, so the radius of $\omega$ is $\boxed{13}$.

16. **DKG1184**

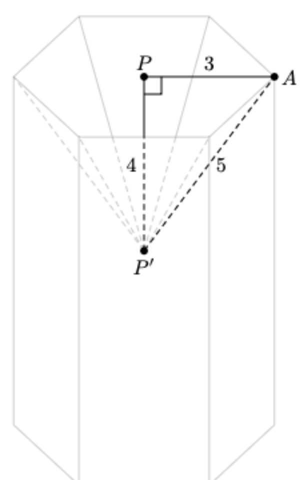
Consider a hexagonal prism whose height is 8 and base side length 3. Let point $P$ be the center of the top face. Let point $P$ be the center of the top face. This point $P$ is "pushed" down to a new point $P'$, which deforms the prism by creating a dent, making it lose volume. All the six new edges connecting $P'$ have length 5. What fraction of the prism's original volume did it lose in this process?




*Written by: Daksh Gupta*

**Answer:** $\boxed{\frac{1}{6}}$

The volume lost is a hexagonal pyramid, so to calculate its volume we need to figure out its height.




Let point $A$ be one of the vertexes connected to $P'$.

$\overrightarrow{PA}$ must be 3, since the distance from the center of a regular hexagon to one of its corners is the same as its side length (to see why this is true, split the hexagon into 6 equilateral triangles.

Also, since $P'$ is directly beneath $P$, $\angle APP'$ must be a right angle.

Now, we can solve for the height $PP'$ with Pythagoras, or just recognize that $\triangle APP'$ is a $3-4-5$ right triangle. This results in the height being 4.

Now to find the proportion of volume lost, let $B$ be the area of the hexagonal base, then we have:

$$\frac{\text{Volume Lost}}{\text{Total Volume}}$$
$$= \frac{\frac{1}{3}B(4)}{8B}$$
$$= \boxed{\frac{1}{6}}$$

17. **LT1562** Suppose that $a$, $b$, and $c$ are positive integers such that $\gcd(a,b,c) = 2024$ and $\operatorname{lcm}(a,b,c) = 2024000$. Let $M$ be the greatest possible value of $\gcd(a,b) \cdot \gcd(b,c) \cdot \gcd(c,a)$. How many positive divisors does $M$ have?

*Written by: Linus Tang*

**Answer:** $\boxed{832}$

First, write the prime factorizations $2024 = 2^3 \cdot 11 \cdot 23$ and $2024000 = 2^6 \cdot 5^3 \cdot 11 \cdot 23$.

We first compute the maximum power of 2 that can divide $\gcd(a,b)\gcd(b,c)\gcd(c,a)$. Of these three numbers, at most one can be divisible by $2^k$ for $k > 3$ (otherwise, $2^k$ would divide $\gcd(a,b,c)$). None of them can divide $2^k$ for $k > 6$, otherwise $2^k$ would divide $\operatorname{lcm}(a,b,c)$. Therefore, the maximum power of 2 that can divide $\gcd(a,b)\gcd(b,c)\gcd(c,a)$ is $2^3 \cdot 2^3 \cdot 2^6 = 2^{12}$.

By similar processes, the maximum powers of 5, 11, and 23 that can divide $\gcd(a,b)\gcd(b,c)\gcd(c,a)$ are $5^3$, $11^3$, and $23^3$.

These powers can be simultaneously achieved, for example, when $a = 2024$ and $b = c = 2024000$.

Therefore, $M = 2^12 \cdot 5^3 \cdot 11^3 \cdot 23^3$, which has $13 \cdot 4 \cdot 4 \cdot 4 = \boxed{832}$ positive divisors.

18. **BF1568** Let $ABCDEF$ be a regular hexagon with side length 1, and let $M$ and $N$ denote the midpoints of $\overline{BC}$ and $\overline{CD}$, respectively. Define $P$ to be the intersection of $\overline{AM}$ and $\overline{BN}$. Find the area of $\triangle BPM$.

*Written by: Benjamin Fu*

**Answer:** $\boxed{\frac{\sqrt{3}}{56}}$

Draw a line through $N$ parallel to $BC$ intersecting $AB$ and $AM$ at $X$ and $Y$, respectively. Since $N$ is the midpoint of $CD$, $X$ and $Y$ must be the midpoints of $AB$

and $AM$. Therefore, $XY = \frac{1}{2}BM = \frac{1}{4}$. Additionally $XN = \frac{1}{2}(AD + BC) = \frac{3}{2}$, so $YN = XN - XY = \frac{5}{4}$.

Next, because we have $\triangle BPM \sim \triangle NPY$, $BM = \frac{1}{2}$, and $YN = \frac{5}{4}$, the ratio between the distances from $P$ to $BM$ and $YN$ is $2:5$. Therefore, the distance from $P$ to $BM$ is $\frac{2}{2+5} = \frac{2}{7}$ the distance from $YN$ to $BN$. Once again, as $N$ is the midpoint of $CD$, the distance from $BM$ to $YN$ is a quarter of the distance from $BM$ to $EF$, which is $\sqrt{3}$, so the distance from $P$ to $BM$ is $\frac{2}{7} \cdot \frac{1}{4} \cdot \sqrt{3} = \frac{\sqrt{3}}{14}$. Thus, the area of $\triangle BPM$ is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{14} = \boxed{\frac{\sqrt{3}}{56}}$.

19. **BF1599** Determine the greatest positive integer $N$ such that for all integers $m$, we have that $N$ divides $m^7 - m$.

    *Written by: Benjamin Fu*

    **Answer:** $\boxed{42}$

    When $x = 2$, we have $x^7 - x = 126$, so we only have to check prime factors of $126 = 2 \cdot 3^2 \cdot 7$. Clearly, $x^7 - x$ is always divisible by 2, 3, and 7 by Fermat's Little Theorem. However, it is not always divisible by $3^2 = 9$ (for example, $3^7 - 3$ is 3 less than a multiple of 9). Therefore, $N = 2 \cdot 3 \cdot 7 = \boxed{42}$.

20. **BF1374** Bob has 9 toys, one of which is a toy car, and 3 boxes. Yesterday, he tossed each toy into a random box. What is the expected value of the number of toys in the box containing the toy car (including the car itself)?

    *Written by: Benjamin Fu*

    **Answer:** $\boxed{\dfrac{11}{3}}$

    Because the way the objects are distributed do not affect the likelihood of any of them being chosen, the scenario is mathematically equivalent to first choosing an object before distributing the objects. If we first pick an object, then distribute the 8 remaining objects among the boxes, there will be an average of $\frac{8}{3}$ objects per box. Whichever box our chosen object goes in will have one more object (the chosen object itself), for a final answer of $\boxed{\frac{11}{3}}$.

21. **BF1563** Find the number of integers $1 \leq n \leq 2024$ for which the remainder when $n^3$ is divided by 2025 is odd.

    *Written by: Benjamin Fu*

    **Answer:** $\boxed{990}$

    Notice that if $2025 \nmid n^3$, the remainders when $n^3$ and $(2025 - n)^3$ are divided by 2025 have opposite parity since

$$(2025-n)^3 \equiv -n^3 \equiv 2025-n^3 \pmod{2025}.$$

Therefore, there is a bijection between values of $n$ that leave an odd remainder and those that leave a nonzero even remainder, so our desired answer is simply half the number of values $n$ satisfying $2025 \nmid n^3$. Since $2025 = 3^4 \cdot 5^2$, we have $2025 \mid n^3$ if and only if $3^2 \cdot 5^1 \mid n$, which is satisfied by $\lfloor \frac{2024}{3^2 \cdot 5^1} \rfloor = 44$ values of $n$. Thus, there are $2024 - 44 = 1980$ values of $n$ satisfying $2025 \nmid n^3$, so our answer is $\frac{1980}{2} = \boxed{990}$.

22. **BF1564** Positive real numbers $a$, $b$, and $c$ satisfy the following equations:

$$ab + \frac{1}{c} = 1$$

$$bc + \frac{1}{a} = 2$$

$$ca + \frac{1}{b} = 4$$

Find the least possible value of $a+b+c$.

*Written by: Benjamin Fu*

**Answer:** $\boxed{\dfrac{7\sqrt{5}-7}{4}}$

We begin by rewriting the given equations as follows:

$$abc + 1 = c$$
$$abc + 1 = 2a$$
$$abc + 1 = 4b$$

Let $n = abc + 1$. Then multiplying the equations above together gives us a cubic polynomial in terms of $n$:

$$n^3 = 8n - 8 \Longrightarrow (n-2)(n^2+2n-4) = 0.$$

Therefore, $n \in \left\{2, -1+\sqrt{5}, -1-\sqrt{5}\right\}$. We wish to minimize $a+b+c = \frac{7}{4}(abc+1)$ while ensuring each of $a$, $b$, and $c$ are positive, which is achieved with $abc+1 = -1+\sqrt{5}$. Thus, our answer is $\boxed{\frac{7\sqrt{5}-7}{4}}$.

23. **LT1557** Harry randomly selects six distinct integers between 0 and 9, inclusive. What is the probability that the product of three of these integers equals the product of the other three?

*Written by: Linus Tang*

**Answer:** $\boxed{\dfrac{1}{105}}$

We first count the number of sets of six integers that have the desired property.

Note that 0 cannot be among the six integers, otherwise one product must be zero and the other product must be nonzero. Also, 5 cannot be among the six integers (since no other digits are divisible by 5), otherwise only one product will be divisible by 5. Similarly, 7 cannot be among the six integers.

Thus, all but one of the remaining choices $\{1, 2, 3, 4, 6, 8, 9\}$ must be chosen. Note that for any six integers, if they can be partitioned into two groups with the same product $n$, then the product of the six integers is a perfect square ($n^2$). We can check that the unchosen number must be 2 or 8, leaving a product of $1 \cdot 3 \cdot 4 \cdot 6 \cdot 8 \cdot 9 = 72^2$ or $1 \cdot 2 \cdot 3 \cdot 4 \cdot 6 \cdot 9 = 36^2$.

For these two cases, the partitions $1 \cdot 8 \cdot 9 = 3 \cdot 4 \cdot 6$ and $1 \cdot 4 \cdot 9 = 2 \cdot 3 \cdot 6$ work.

Going back to the original problem, there are $\binom{10}{6} = 210$ ways to select the six integers and 2 of them work, so the desired probability is $\frac{2}{210} = \boxed{\dfrac{1}{105}}$.

24. **BF1551** Let $ABCD$ be a rectangle with $AB = 120$ and $BC = 170$, and let $EFGH$ be a unit square within $ABCD$ such that $\overline{AB} \parallel \overline{EF}$ and $E$ is the closest vertex to $A$. Given that $\angle ABF = \angle BCG$ and $\angle CDH = \angle DAE$, find the least possible length of $\overline{AE}$.

*Written by: Benjamin Fu*

**Answer:** $\boxed{\dfrac{91\sqrt{2}}{2}}$

First, we can get rid of $EFGH$ entirely by collapsing the space between lines $EF$ and $GH$ as well as the space between lines $FG$ and $HE$. This shortens each side of the rectangle by 1 unit, so we now have $AB = 119$ and $BC = 169$. Additionally, all of $E$, $F$, $G$, and $H$ are consolidated into one single point, which we will name $P$. The given condition on the angles now become $\angle ABP = \angle BCP$ and $\angle CDP = \angle DAP$.

Consider triangle $\triangle BCP$. Since the sum of the angles in a triangle are equal to $180°$, we have

$$\angle BCP + \angle CPB + \angle PBC = 180°.$$

Furthermore, we also know $90° - \angle PBC = \angle ABP = \angle BCP$, and substituting this into the equation above gives us $\angle CPB = 90°$. Similar reasoning yields $\angle APD = 90°$. By the Inscribed Angle Theorem, $P$ must lie on both a circle with diameter $AD$ and a circle with diameter $BC$, so $P$ is equidistant from sides $AD$ and $BC$, with said distance being $\frac{AB}{2} = \frac{119}{2}$.

Define $a = AP$ and $b = DP$. We wish to minimize $AE = AP$, so we let $a < b$. The distance from $P$ to side $AD$ is $\frac{119}{2}$, so the area of $\triangle APD$ is

$$\frac{1}{2} \cdot 169 \cdot \frac{119}{2} = \frac{1}{2} \cdot ab \Longrightarrow 2ab = 169 \cdot 119.$$

Additionally, as $\angle ADP = 90°$, we also have $a^2 + b^2 = 169^2$. Hence, we obtain the two following equations:

$$a + b = \sqrt{a^2 + 2ab + b^2} = \sqrt{169^2 + 169 \cdot 119} = 156\sqrt{2}$$
$$-a + b = \sqrt{a^2 - 2ab + b^2} = \sqrt{169^2 - 169 \cdot 119} = 65\sqrt{2}.$$

Subtracting the second equation from the first, then dividing by 2 gives us our final answer of $a = \boxed{\frac{91\sqrt{2}}{2}}$.

25. **LT1622** Ten distinct cells are chosen randomly from a $100 \times 100$ grid. Let $p$ be the probability that there is a pair of chosen cells in the same row or the same column. Estimate the integer nearest $1000p$.

    Submit a positive integer $N$. If the correct answer is $A$, you will receive $\max(25(2 - \max(\frac{A}{N}, \frac{N}{A})), 0)$ points.

    *Written by: Linus Tang*

    **Answer:** $\boxed{396}$

    Will add solution and remarks later

26. **LT1620** Gilbert thinks of a number $n$, and writes down the equation

    $$\frac{1}{a} + \frac{1}{b} = \frac{1}{n}.$$

    This equation is satisfied by at least 200 ordered pairs of positive integers $(a, b)$. Estimate the smallest possible value of $n$.

    Submit a positive integer $N$. If the correct answer is $A$, you will receive $\max(25 - \sqrt{|A - N|}, 0)$ points.

    *Written by: Linus Tang*

    **Answer:** $\boxed{1260}$

    Clearing denominators, the equation is $an + bn = ab$. Using Simon's Favorite Factoring Trick, we can rewrite the equation as $ab - an - bn + n^2 = n^2$, which factors as

    $$(a - n)(b - n) = n^2.$$

    Now, we have that the number of positive integer solutions in $(a, b)$ is the number of positive divisors of $n^2$. We want to find small $n$ for which the number of positive divisors of $n^2$ is at least 200. Guessing and checking values of $n$ by their prime factorization, it turns out that $n = 2^2 \cdot 3^2 \cdot 5 \cdot 7 = 1260$ gives $(4+1)(4+1)(2+1)(2+1) = 225$ factors, and is the smallest $n$ that allows us to achieve at least 200.

27. **LT1621** Four points are chosen independently and uniformly at random from the interior of a unit square. Let $p$ be the probability that these points are the vertices of a convex quadrilateral. Estimate the integer nearest $1000p$.

Submit a positive integer $N$. If the correct answer is $A$, you will receive $\max(25 - \frac{|A-N|}{6}, 0)$ points.

*Written by: Linus Tang*

**Answer:** $\boxed{694}$

Found by computer simulation.

An approach to get an estimate by hand is to first estimate the average area, $A$, of a triangle determined by three random points in unit square, then use $p = 1 - 4A$.