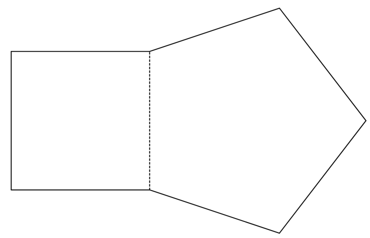
1. **LT1584** A figure is made of a square and a regular pentagon, which share an side of length 2, as shown in the figure below. What is the perimeter of the figure?



Written by: Linus Tang

Answer: $\boxed{14}$

The figure has 7 sides, each of which has length 2, so the perimeter is $7 \times 2 = \boxed{14}$.

2. **LT1623** Amy, Beth, and Carlos want to stand in a line to take a group photo. However, Beth and Carlos refuse to stand directly next to each other. How many ways can Amy, Beth, and Carlos be ordered from left to right?

Written by: Linus Tang

Answer: $\boxed{2}$

Since Beth and Carlos cannot stand directly next to each other, they must be on the two outside positions, and Amy must be in the middle. Beth and Carlos can take the left and right positions in either order, so the answer is $\boxed{2}$.

3. **LT1611** A birthday cake costs \$10.00, plus an additional \$0.50 for every decoration on it. Mr. Li orders two birthday cakes, the first of which has three decorations on it. If the subtotal was \$24.00, how many decorations were on the second cake?

Written by: Linus Tang

Answer: $\boxed{5}$

The first cake cost $\$10.00 + 3 \times \$0.50 = \$11.50$, so the second cake must have cost the remaining \$12.50. The cake itself cost \$10.00, so the decorations added \$2.50 to the cost. This is the cost of $\boxed{5}$ decorations.

4. **LT1589** There are initially 1000 bacteria in a petri dish. Every 20 minutes, each bacterium splits into two bacteria. How many bacteria are in the petri dish after 60 minutes?

Written by: Linus Tang

Answer: $\boxed{8000}$

The bacteria split $60/20 = 3$ times. After the first split, there are $1000 \times 2 = 2000$ bacteria. After the second, there are $2000 \times 2 = 4000$. After the third, there are $4000 \times 2 = \boxed{8000}$.

5. **LT1588** A palindrome is a sequence of letters that are in the same order when read from left to right or right to left. For example, *abcba* is a palindrome. How many ways can the seven letters in *pompoms* be rearranged to form a palindrome?

   Written by: Linus Tang

   Answer: $\boxed{6}$

   Trivially $\boxed{6}$.

6. **LT1587** The number $2024^2 = 4096576$ has 63 positive divisors. How many of these divisors are greater than 2024?

   Written by: Linus Tang

   Answer: $\boxed{31}$

   The divisors form pairs that multiply to 2024^2.

   2024 pairs with itself, and the other 62 divisors form $62/2 = \boxed{31}$ pairs, each of which contains one number greater than 2024 and one number less than 2024.

7. **LT1585** A square and a circle are drawn on a piece of paper. What is the maximum number of intersection points between the two shapes?

   Written by: Linus Tang

   Answer: $\boxed{8}$

   Each edge of the square intersects the boundary of the circle at most twice. There are four edges, so the number of intersections is at most $4 \times 2 = \boxed{8}$.

   (Insert diagram with construction of 8.)

8. **LT1527** Tom's favorite number has four digits. The sum of the first three digits is 16, and the sum of the last three digits is 7. What is the first digit?

   Written by: Linus Tang

   Answer: $\boxed{9}$

   Let A, B, C, and D be the four digits in order. Then $A+B+C = 16$ and $B+C+D = 7$. Subtracting, $A - D = 9$. Since A and D are digits, the only way this can happen is if $A = \boxed{9}$ and $D = 0$.

9. **BF1598** Ethan puts five slips of paper into a basket, labelled 1, 2, 3, 4, and 5. He then randomly draws out three slips of paper one by one, without replacement. Determine the probability that the last number Ethan drew was the largest of the three.

Written by: Benjamin Fu

Answer: $\boxed{\frac{1}{3}}$

By symmetry, it is equally likely for each of the three numbers Ethan drew to be the largest, so the answer is simply $\boxed{\frac{1}{3}}$.

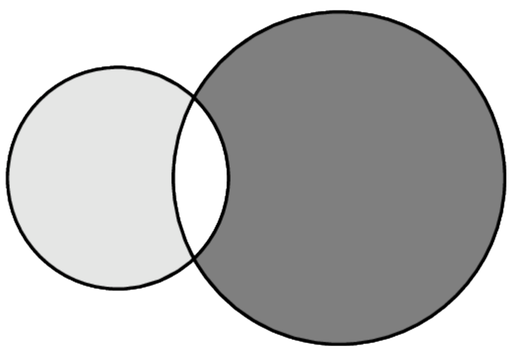
10. **LT1561** Let p be a prime number. The sum of the positive divisors of $2p$ is 42. What is p?

Written by: Linus Tang

Answer: $\boxed{13}$

Since the positive divisors of $2p$ are $1, 2, p$, and $2p$, we have that $1 + 2 + p + 2p = 42$, or $3p + 3 = 42$. Solving, $p = \boxed{13}$.

11. **LT1537** See the diagram below, consisting of two intersecting circles. The areas of the shaded regions are 30π and 45π. The difference between the radii of the circles is 1. What is the radius of the larger circle?



Written by: Linus Tang

Answer: $\boxed{8}$

If the intersection of the two circles has area x, then the circles have areas of $30\pi + x$ and $45\pi + x$. Thus, the difference between their areas is 15π, no matter what x is.

Letting r_1 and r_2 to be radii of the circles, we have $\pi r_1^2 - \pi r_2^2 = 15\pi$, or $r_1^2 - r_2^2 = 15$. We are also given $r_1 - r_2 = 1$. Dividing, we have $r_1 + r_2 = 15$, and solving, we have $r_1 = \boxed{8}$ and $r_2 = 7$.

12. **BZ1520** Al is assigned a set of 35 calculus problems for homework, and since the

problems are ordered by difficulty, the nth problem takes him n minutes to solve. To finish as quickly as possible, Al enlists the help of his older brother Gebra, who can solve each problem in only 1 minute. What is the minimum number of minutes it takes for Al and Gebra working simultaneously to finish the homework assignment?

Written by: Brian Zhou

Answer: $\boxed{28.}$

Since it doesn't matter which problems Gebra works on, it is optimal for Al to work from front to back and Gebra to work from back to front. If they meet in the middle perfectly and Al solved x problems, then the following equation is satisfied:

$$\frac{x(x+1)}{2} = 35 - x.$$

Solving the quadratic gives $x = 7$. This is conveniently an integer, so the answer is just $35 - 7 = \boxed{28}$ minutes.

13. **BF1569** Find the least real number N such that there exist no values of x greater than or equal to N that satisfy

$$\left\lfloor x^2 + \left\lfloor x^2 + \left\lfloor x^2 + \left\lfloor x^2 \right\rfloor \right\rfloor \right\rfloor \right\rfloor = 100,$$

where $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y.

Written by: Benjamin Fu

Answer: $\boxed{\sqrt{26}}$

Observe that since $\lfloor x^2 \rfloor$ is always an integer and $\lfloor x^2 + n \rfloor = \lfloor x^2 \rfloor + n$ for all integers n, we have

$$\left\lfloor x^2 + \left\lfloor x^2 \right\rfloor \right\rfloor = \left\lfloor x^2 \right\rfloor + \left\lfloor x^2 \right\rfloor = 2\left\lfloor x^2 \right\rfloor.$$

Similarly, we have

$$\left\lfloor x^2 + \left\lfloor x^2 + 2\left\lfloor x^2 \right\rfloor \right\rfloor \right\rfloor = \left\lfloor x^2 + 3\left\lfloor x^2 \right\rfloor \right\rfloor = 4\left\lfloor x^2 \right\rfloor.$$

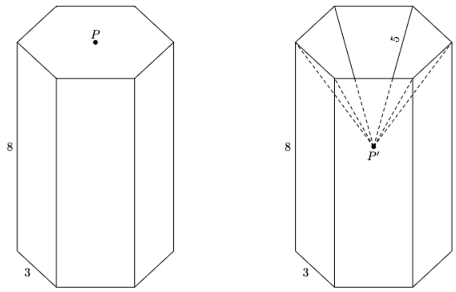
Thus, we have

$$4\left\lfloor x^2 \right\rfloor = 100 \Longrightarrow \left\lfloor x^2 \right\rfloor = 25.$$

It is easy to see that $N = \sqrt{26}$.

14. **DKG1184**

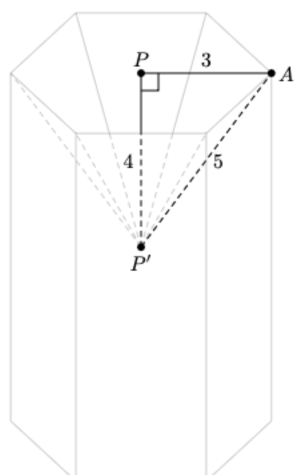
Consider a hexagonal prism whose height is 8 and base side length 3. Let point P be the center of the top face. Let point P be the center of the top face. This point P is "pushed" down to a new point P', which deforms the prism by creating a dent, making it lose volume. All the six new edges connecting P' have length 5. What fraction of the prism's original volume did it lose in this process?



Written by: Daksh Gupta

Answer: $\boxed{\frac{1}{6}}$

The volume lost is a hexagonal pyramid, so to calculate its volume we need to figure out its height.



Let point A be one of the vertexes connected to P'.

$\overrightarrow{PA}$ must be 3, since the distance from the center of a regular hexagon to one of its corners is the same as its side length (to see why this is true, split the hexagon into 6 equilateral triangles.

Also, since P' is directly beneath P, $\angle APP'$ must be a right angle.

Now, we can solve for the height PP' with Pythagoras, or just recognize that $\triangle APP'$ is a $3-4-5$ right triangle. This results in the height being 4.

Now to find the proportion of volume lost, let B be the area of the hexagonal base, then we have:

$$\frac{\text{Volume Lost}}{\text{Total Volume}} = \frac{\frac{1}{3}B(4)}{8B} = \boxed{\frac{1}{6}}$$

15. **BF1599** Determine the greatest positive integer N such that for all integers m, we have that N divides $m^7 - m$.

Written by: Benjamin Fu

Answer: $\boxed{42}$

When $x = 2$, we have $x^7 - x = 126$, so we only have to check prime factors of $126 = 2 \cdot 3^2 \cdot 7$. Clearly, $x^7 - x$ is always divisible by 2, 3, and 7 by Fermat's Little Theorem. However, it is not always divisible by $3^2 = 9$ (for example, $3^7 - 3$ is 3 less than a multiple of 9). Therefore, $N = 2 \cdot 3 \cdot 7 = \boxed{42}$.

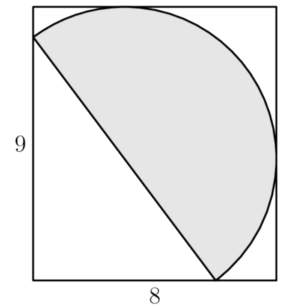
16. **BF1591** Let ω be a circle of radius 1 and A be its center. Let B be a point on the circumference of ω. If point C is chosen uniformly and random from the interior of ω, find the probability that $\triangle ABC$ is obtuse.

Written by: Benjamin Fu

Answer: $\boxed{\frac{3}{4}}$

We will consider the three disjoint cases. First, the probability that $\angle ABC > 90^\circ$ is clearly 0. Next, by the Inscribed Angle Theorem, we see that $\angle BCA > 90^\circ$ if and only if C is within the circle with diameter $\overline{AB}$, which has a probability of $\frac{1}{4}$ to occur. Finally, it's easy to see that $\angle CAB > 90^\circ$ if and only if C lies below the line passing through A perpendicular to AB, which has a probability of $\frac{1}{2}$ to occur. Thus, our answer is $0 + \frac{1}{4} + \frac{1}{2} = \boxed{\frac{3}{4}}$

17. **LT1538** A semicircle is inscribed within an 8×9 rectangle, such that the two endpoints of its diameter lie on two sides of the rectangle, and its arc is tangent to the other two sides. Find the length of the diameter of the semicircle.



Written by: Linus Tang

Answer: $\boxed{10}$

Let $a < b$ be real numbers such that $2a$ and $2b$ are the lengths of the legs of the right triangle between the diameter of the semicircle and the sides of the rectangle, and let r denote the radius of the semicircle. Using the tangency condition, we obtain the two following equations:

$$a + r = 8 \Longrightarrow a = 8 - r$$
$$b + r = 9 \Longrightarrow b = 9 - r.$$

Additionally, the Pythagorean Theorem gives us

$$(2a)^2 + (2b)^2 = (2r)^2 \Longrightarrow (8 - r)^2 + (9 - r)^2 = r^2.$$

Solving the quadratic yields $r = 5$ and $r = 29$, with the latter clearly being impossible. Therefore, our answer is $2r = \boxed{10}$.

18. **LT1533** There are 11 balls in a bag, labeled with distinct integers from 1 to 11. Every minute, Ann takes two of the balls from the bag at random, throws away the one with the smaller label, and puts the other back into the bag. After nine minutes, there are two balls left in the bag. What is the probability that one of these balls is the one with the label 10?

Written by: Linus Tang

Answer: $\boxed{\frac{2}{5}}$

This is the same as the probability that the pair $\{10, 11\}$ is never selected.

The probability that $\{10, 11\}$ is not selected during the first draw is $\frac{54}{55}$, since it is one of 55 possible pairs.

If it wasn't selected during the first draw, then the probability that it wasn't selected during the second draw is $\frac{44}{45}$, since there are now 10 balls and therefore 45 possible pairs.

The successive probabilities of not drawing $\{10, 11\}$ are $\frac{35}{36}, \frac{27}{28}, \ldots, \frac{2}{3}$.

Multiplying, the overall probability that $\{10, 11\}$ is not drawn during the first nine minutes is $\frac{54}{55} \cdot \frac{44}{45} \cdot \frac{35}{36} \cdot \frac{27}{28} \cdot \frac{20}{21} \cdot \frac{14}{15} \cdot \frac{9}{10} \cdot \frac{5}{6} \cdot \frac{2}{3}$. Many factors cancel (and in fact, the product telescopes in general), leaving the answer $\boxed{\dfrac{2}{5}}$.

19. **BF1602** Let $ABCD$ be a rectangle with side lengths $AB = 10$ and $BC = 1$. A circle ω passes through A and B and is tangent to $\overline{CD}$. Find the radius of ω.

Written by: Benjamin Fu

Answer: $\boxed{13}$

Let $P \neq A$ denote the second intersection of AD with ω. Note that since $\angle BAP = 90°$, $\overline{BP}$ is a diameter of ω. By Power of a Point, we have

$$5^2 = 1 \cdot DP \Longrightarrow DP = 25 \Longrightarrow AP = DP - 1 = 24.$$

Therefore, $BP = \sqrt{24^2 + 10^2} = 26$, so the radius of ω is $\boxed{13}$.

20. **LT1554** Suppose a is a real number such that the equation $x^3 + ax^2 - 1000 = 0$ has three real solutions in x, one of which equals the sum of the other two. Determine the value of a.

Written by: Linus Tang

Answer: $\boxed{20}$

By Vieta's formulas, the sum of all three roots is $-a$. The root that equals the sum of the other two must then be $-\frac{a}{2}$. Plugging this in, $(-\frac{a}{2})^3 + a(-\frac{a}{2})^2 - 1000 = 0$. This simplifies to $\frac{a^3}{8} - 1000 = 0$, so $a = \boxed{20}$.

21. **LT1543** Find the greatest multiple of 11 whose digits are all distinct.

Written by: Linus Tang

Answer: $\boxed{9876524130}$

Since all digits are distinct, the best we can hope for is 10 digits. We want the largest digits in the largest place values, so we start constructing the number $987\ldots$.

By the divisibility by 11 rule, we want the digit sum of $0 + 1 + \cdots + 9 = 45$ to be split into alternating sums equaling 28 and 17.

Note that no number starting with $987654\ldots$ (with the digits $0, 1, 2, 3$ left to place) has this property, since the alternating digit sums are both greater than 17 already.

So, we backtrack a digit, writing 98765 with the digits $0, 1, 2, 3, 4$ left to place. The best way to achieve a sum of 18 in the even positions is to complete the number $\boxed{9876524130}$.

22. **LT1562** Suppose that a, b, and c are positive integers such that $\gcd(a, b, c) = 2024$ and $\operatorname{lcm}(a, b, c) = 2024000$. Let M be the greatest possible value of $\gcd(a, b) \cdot \gcd(b, c) \cdot$

$\gcd(c, a)$. How many positive divisors does M have?

Written by: Linus Tang

Answer: $\boxed{832}$

First, write the prime factorizations $2024 = 2^3 \cdot 11 \cdot 23$ and $2024000 = 2^6 \cdot 5^3 \cdot 11 \cdot 23$.

We first compute the maximum power of 2 that can divide $\gcd(a, b)\gcd(b, c)\gcd(c, a)$. Of these three numbers, at most one can be divisible by 2^k for $k > 3$ (otherwise, 2^k would divide $\gcd(a, b, c)$). None of them can divide 2^k for $k > 6$, otherwise 2^k would divide $\operatorname{lcm}(a, b, c)$. Therefore, the maximum power of 2 that can divide $\gcd(a, b)\gcd(b, c)\gcd(c, a)$ is $2^3 \cdot 2^3 \cdot 2^6 = 2^{12}$.

By similar processes, the maximum powers of 5, 11, and 23 that can divide $\gcd(a, b)\gcd(b, c)\gcd(c, a)$ are 5^3, 11^3, and 23^3.

These powers can be simultaneously achieved, for example, when $a = 2024$ and $b = c = 2024000$.

Therefore, $M = 2^12 \cdot 5^3 \cdot 11^3 \cdot 23^3$, which has $13 \cdot 4 \cdot 4 \cdot 4 = \boxed{832}$ positive divisors.

23. **BF1568** Let $ABCDEF$ be a regular hexagon with side length 1, and let M and N denote the midpoints of $\overline{BC}$ and $\overline{CD}$, respectively. Define P to be the intersection of $\overline{AM}$ and $\overline{BN}$. Find the area of $\triangle BPM$.

Written by: Benjamin Fu

Answer: $\boxed{\dfrac{\sqrt{3}}{56}}$

Draw a line through N parallel to BC intersecting AB and AM at X and Y, respectively. Since N is the midpoint of CD, X and Y must be the midpoints of AB and AM. Therefore, $XY = \frac{1}{2}BM = \frac{1}{4}$. Additionally $XN = \frac{1}{2}(AD + BC) = \frac{3}{2}$, so $YN = XN - XY = \frac{5}{4}$.

Next, because we have $\triangle BPM \sim \triangle NPY$, $BM = \frac{1}{2}$, and $YN = \frac{5}{4}$, the ratio between the distances from P to BM and YN is $2 : 5$. Therefore, the distance from P to BM is $\frac{2}{2+5} = \frac{2}{7}$ the distance from YN to BN. Once again, as N is the midpoint of CD, the distance from BM to YN is a quarter of the distance from BM to EF, which is $\sqrt{3}$, so the distance from P to BM is $\frac{2}{7} \cdot \frac{1}{4} \cdot \sqrt{3} = \frac{\sqrt{3}}{14}$. Thus, the area of $\triangle BPM$ is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{14} = \boxed{\frac{\sqrt{3}}{56}}$.

24. **JES1075** You have just received a plant with 3 special healthy flowers. Each flower independently has a $\frac{1}{3}$ chance to wilt during any given day, after which it will stay permanently wilted. Given that at least one flower is healthy at the end of Day 1, the probability that all flowers have wilted by the end of Day 2 can be written as $\frac{m}{n}$ for relatively prime positive integers m and n. Find $m + n$.

Written by: Jerry Sun

Answer: $\boxed{400}$

We want to find $\frac{P(\text{A and B})}{P(\text{B})}$, where A represents the event in which all flowers have wilted by the end of Day 2, and B represents the event in which at least one flower is healthy at the end of Day 1.

Since each flower is independent of the others, $P(\text{A})$ is equal to the probability that one flower wilts by the end of Day 2, raised to the power of 3. To find this probability, consider two cases: the flower wilts on Day 1 or Day 2, with probabilities $\frac{1}{3}$ and $\left(1-\frac{1}{3}\right)\cdot\frac{1}{3}=\frac{2}{9}$. Adding the probabilities of both cases and cubing yields $\left(\frac{1}{3}+\frac{2}{9}\right)^3=\left(\frac{5}{9}\right)^3=\frac{125}{729}$. To find $P(\text{A and B})$, subtract the probability that all flowers wilt on Day 1, since all such scenarios are considered in $P(\text{A})$: $\frac{125}{729}-\left(\frac{1}{3}\right)^3=\frac{98}{729}$.

Complementary counting gives us $P(\text{B}) = 1-\left(\frac{1}{3}\right)^3=\frac{26}{27}$. Finally, $\frac{\frac{98}{729}}{\frac{26}{27}}=\frac{49}{351}$ and $49+351=\boxed{400}$.

25. **LT1622** Ten distinct cells are chosen randomly from a 100×100 grid. Let p be the probability that there is a pair of chosen cells in the same row or the same column. Estimate the integer nearest $1000p$.

Submit a positive integer N. If the correct answer is A, you will receive $\max(25(2-\max(\frac{A}{N},\frac{N}{A})),0)$ points.

Written by: Linus Tang

Answer: $\boxed{396}$

Will add solution and remarks later

26. **LT1620** Gilbert thinks of a number n, and writes down the equation

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{n}.$$

This equation is satisfied by at least 200 ordered pairs of positive integers (a,b). Estimate the smallest possible value of n.

Submit a positive integer N. If the correct answer is A, you will receive $\max(25-\sqrt{|A-N|},0)$ points.

Written by: Linus Tang

Answer: $\boxed{1260}$

Clearing denominators, the equation is $an+bn=ab$. Using Simon's Favorite Factoring Trick, we can rewrite the equation as $ab-an-bn+n^2=n^2$, which factors as

$$(a-n)(b-n)=n^2.$$

Now, we have that the number of positive integer solutions in (a, b) is the number of positive divisors of n^2. We want to find small n for which the number of positive divisors of n^2 is at least 200. Guessing and checking values of n by their prime factorization, it turns out that $n = 2^2 \cdot 3^2 \cdot 5 \cdot 7 = 1260$ gives $(4+1)(4+1)(2+1)(2+1) = 225$ factors, and is the smallest n that allows us to achieve at least 200.

27. **LT1621** Four points are chosen independently and uniformly at random from the interior of a unit square. Let p be the probability that these points are the vertices of a convex quadrilateral. Estimate the integer nearest $1000p$.

Submit a positive integer N. If the correct answer is A, you will receive $\max(25 - \frac{|A-N|}{6}, 0)$ points.

Written by: Linus Tang

Answer: $\boxed{694}$

Found by computer simulation.

An approach to get an estimate by hand is to first estimate the average area, A, of a triangle determined by three random points in unit square, then use $p = 1 - 4A$.