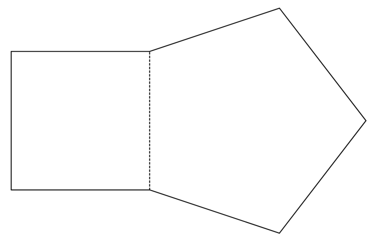
1. **LT1584** A figure is made of a square and a regular pentagon, which share an side of length 2, as shown in the figure below. What is the perimeter of the figure?



Written by: Linus Tang

Answer: $\boxed{14}$

The figure has 7 sides, each of which has length 2, so the perimeter is $7 \times 2 = \boxed{14}$.

2. **LT1623** Amy, Beth, and Carlos want to stand in a line to take a group photo. However, Beth and Carlos refuse to stand directly next to each other. How many ways can Amy, Beth, and Carlos be ordered from left to right?

Written by: Linus Tang

Answer: $\boxed{2}$

Since Beth and Carlos cannot stand directly next to each other, they must be on the two outside positions, and Amy must be in the middle. Beth and Carlos can take the left and right positions in either order, so the answer is $\boxed{2}$.

3. **LT1611** A birthday cake costs \$10.00, plus an additional \$0.50 for every decoration on it. Mr. Li orders two birthday cakes, the first of which has three decorations on it. If the subtotal was \$24.00, how many decorations were on the second cake?

Written by: Linus Tang

Answer: $\boxed{5}$

The first cake cost $\$10.00 + 3 \times \$0.50 = \$11.50$, so the second cake must have cost the remaining \$12.50. The cake itself cost \$10.00, so the decorations added \$2.50 to the cost. This is the cost of $\boxed{5}$ decorations.

4. **LT1585** A square and a circle are drawn on a piece of paper. What is the maximum number of intersection points between the two shapes?

Written by: Linus Tang

Answer: $\boxed{8}$

Each edge of the square intersects the boundary of the circle at most twice. There are four edges, so the number of intersections is at most $4 \times 2 = \boxed{8}$.

(Insert diagram with construction of 8.)

5. **LT1587** The number $2024^2 = 4096576$ has 63 positive divisors. How many of these divisors are greater than 2024?

Written by: Linus Tang

Answer: $\boxed{31}$

The divisors form pairs that multiply to 2024^2.

2024 pairs with itself, and the other 62 divisors form $62/2 = \boxed{31}$ pairs, each of which contains one number greater than 2024 and one number less than 2024.

6. **LT1589** There are initially 1000 bacteria in a petri dish. Every 20 minutes, each bacterium splits into two bacteria. How many bacteria are in the petri dish after 60 minutes?

Written by: Linus Tang

Answer: $\boxed{8000}$

The bacteria split $60/20 = 3$ times. After the first split, there are $1000 \times 2 = 2000$ bacteria. After the second, there are $2000 \times 2 = 4000$. After the third, there are $4000 \times 2 = \boxed{8000}$.

7. **LT1588** A palindrome is a sequence of letters that are in the same order when read from left to right or right to left. For example, *abcba* is a palindrome. How many ways can the seven letters in *pompoms* be rearranged to form a palindrome?

Written by: Linus Tang

Answer: $\boxed{6}$

Trivially $\boxed{6}$.

8. **LT1533** There are 11 balls in a bag, labeled with distinct integers from 1 to 11. Every minute, Ann takes two of the balls from the bag at random, throws away the one with the smaller label, and puts the other back into the bag. After nine minutes, there are two balls left in the bag. What is the probability that one of these balls is the one with the label 10?

Written by: Linus Tang

Answer: $\boxed{\frac{2}{5}}$

This is the same as the probability that the pair $\{10, 11\}$ is never selected.

The probability that $\{10, 11\}$ is not selected during the first draw is $\frac{54}{55}$, since it is one of 55 possible pairs.

If it wasn't selected during the first draw, then the probability that it wasn't selected during the second draw is $\frac{44}{45}$, since there are now 10 balls and therefore 45 possible pairs.

The successive probabilities of not drawing $\{10, 11\}$ are $\frac{35}{36}, \frac{27}{28}, \ldots, \frac{2}{3}$.

Multiplying, the overall probability that $\{10, 11\}$ is not drawn during the first nine minutes is $\frac{54}{55} \cdot \frac{44}{45} \cdot \frac{35}{36} \cdot \frac{27}{28} \cdot \frac{20}{21} \cdot \frac{14}{15} \cdot \frac{9}{10} \cdot \frac{5}{6} \cdot \frac{2}{3}$. Many factors cancel (and in fact, the product telescopes in general), leaving the answer $\boxed{\frac{2}{5}}$.

9. **LT1561** Let p be a prime number. The sum of the positive divisors of $2p$ is 42. What is p?

   Written by: Linus Tang

   Answer: $\boxed{13}$

   Since the positive divisors of $2p$ are $1, 2, p$, and $2p$, we have that $1 + 2 + p + 2p = 42$, or $3p + 3 = 42$. Solving, $p = \boxed{13}$.

10. **LT1527** Tom's favorite number has four digits. The sum of the first three digits is 16, and the sum of the last three digits is 7. What is the first digit?

    Written by: Linus Tang

    Answer: $\boxed{9}$

    Let A, B, C, and D be the four digits in order. Then $A+B+C = 16$ and $B+C+D = 7$. Subtracting, $A - D = 9$. Since A and D are digits, the only way this can happen is if $A = \boxed{9}$ and $D = 0$.

11. **BF1602** Let $ABCD$ be a rectangle with side lengths $AB = 10$ and $BC = 1$. A circle ω passes through A and B and is tangent to $\overline{CD}$. Find the radius of ω.

    Written by: Benjamin Fu

    Answer: $\boxed{13}$

    Let $P \neq A$ denote the second intersection of AD with ω. Note that since $\angle BAP = 90°$, $\overline{BP}$ is a diameter of ω. By Power of a Point, we have

    $$5^2 = 1 \cdot DP \Longrightarrow DP = 25 \Longrightarrow AP = DP - 1 = 24.$$

    Therefore, $BP = \sqrt{24^2 + 10^2} = 26$, so the radius of ω is $\boxed{13}$.

12. **BF1598** Ethan puts five slips of paper into a basket, labelled 1, 2, 3, 4, and 5. He then randomly draws out three slips of paper one by one, without replacement. Determine the probability that the last number Ethan drew was the largest of the three.

Written by: Benjamin Fu

Answer: $\boxed{\frac{1}{3}}$

By symmetry, it is equally likely for each of the three numbers Ethan drew to be the largest, so the answer is simply $\boxed{\frac{1}{3}}$.

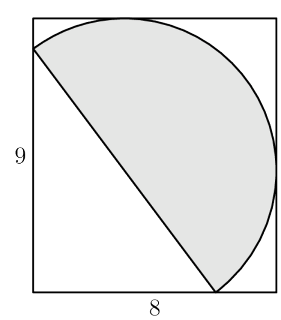
13. **LT1554** Suppose a is a real number such that the equation $x^3 + ax^2 - 1000 = 0$ has three real solutions in x, one of which equals the sum of the other two. Determine the value of a.

Written by: Linus Tang

Answer: $\boxed{20}$

By Vieta's formulas, the sum of all three roots is $-a$. The root that equals the sum of the other two must then be $-\frac{a}{2}$. Plugging this in, $(-\frac{a}{2})^3 + a(-\frac{a}{2})^2 - 1000 = 0$. This simplifies to $\frac{a^3}{8} - 1000 = 0$, so $a = \boxed{20}$.

14. **LT1538** A semicircle is inscribed within an 8×9 rectangle, such that the two endpoints of its diameter lie on two sides of the rectangle, and its arc is tangent to the other two sides. Find the length of the diameter of the semicircle.



Written by: Linus Tang

Answer: $\boxed{10}$

Let $a < b$ be real numbers such that $2a$ and $2b$ are the lengths of the legs of the right triangle between the diameter of the semicircle and the sides of the rectangle, and let r denote the radius of the semicircle. Using the tangency condition, we obtain the two following equations:

$$a + r = 8 \Longrightarrow a = 8 - r$$
$$b + r = 9 \Longrightarrow b = 9 - r.$$

Additionally, the Pythagorean Theorem gives us

$$(2a)^2 + (2b)^2 = (2r)^2 \Longrightarrow (8-r)^2 + (9-r)^2 = r^2.$$

Solving the quadratic yields $r = 5$ and $r = 29$, with the latter clearly being impossible. Therefore, our answer is $2r = \boxed{10}$.

15. **LT1543** Find the greatest multiple of 11 whose digits are all distinct.

Written by: Linus Tang

Answer: $\boxed{9876524130}$

Since all digits are distinct, the best we can hope for is 10 digits. We want the largest digits in the largest place values, so we start constructing the number $987\ldots$.

By the divisibility by 11 rule, we want the digit sum of $0 + 1 + \cdots + 9 = 45$ to be split into alternating sums equaling 28 and 17.

Note that no number starting with $987654\ldots$ (with the digits $0, 1, 2, 3$ left to place) has this property, since the alternating digit sums are both greater than 17 already.

So, we backtrack a digit, writing 98765 with the digits $0, 1, 2, 3, 4$ left to place. The best way to achieve a sum of 18 in the even positions is to complete the number $\boxed{9876524130}$.

16. **LT1562** Suppose that a, b, and c are positive integers such that $\gcd(a, b, c) = 2024$ and $\operatorname{lcm}(a, b, c) = 2024000$. Let M be the greatest possible value of $\gcd(a, b) \cdot \gcd(b, c) \cdot \gcd(c, a)$. How many positive divisors does M have?

Written by: Linus Tang

Answer: $\boxed{832}$

First, write the prime factorizations $2024 = 2^3 \cdot 11 \cdot 23$ and $2024000 = 2^6 \cdot 5^3 \cdot 11 \cdot 23$.

We first compute the maximum power of 2 that can divide $\gcd(a, b)\gcd(b, c)\gcd(c, a)$. Of these three numbers, at most one can be divisible by 2^k for $k > 3$ (otherwise, 2^k would divide $\gcd(a, b, c)$). None of them can divide 2^k for $k > 6$, otherwise 2^k would divide $\operatorname{lcm}(a, b, c)$. Therefore, the maximum power of 2 that can divide $\gcd(a, b)\gcd(b, c)\gcd(c, a)$ is $2^3 \cdot 2^3 \cdot 2^6 = 2^{12}$.

By similar processes, the maximum powers of 5, 11, and 23 that can divide $\gcd(a, b)\gcd(b, c)\gcd(c, a)$ are 5^3, 11^3, and 23^3.

These powers can be simultaneously achieved, for example, when $a = 2024$ and $b = c = 2024000$.

Therefore, $M = 2^12 \cdot 5^3 \cdot 11^3 \cdot 23^3$, which has $13 \cdot 4 \cdot 4 \cdot 4 = \boxed{832}$ positive divisors.

17. **BF1563** Find the number of integers $1 \leq n \leq 2024$ for which the remainder when n^3 is divided by 2025 is odd.

Written by: Benjamin Fu

Answer: $\boxed{990}$

Notice that if $2025 \nmid n^3$, the remainders when n^3 and $(2025-n)^3$ are divided by 2025 have opposite parity since

$$(2025-n)^3 \equiv -n^3 \equiv 2025 - n^3 \pmod{2025}.$$

Therefore, there is a bijection between values of n that leave an odd remainder and those that leave a nonzero even remainder, so our desired answer is simply half the number of values n satisfying $2025 \nmid n^3$. Since $2025 = 3^4 \cdot 5^2$, we have $2025 \mid n^3$ if and only if $3^2 \cdot 5^1 \mid n$, which is satisfied by $\lfloor \frac{2024}{3^2 \cdot 5^1} \rfloor = 44$ values of n. Thus, there are $2024 - 44 = 1980$ values of n satisfying $2025 \nmid n^3$, so our answer is $\frac{1980}{2} = \boxed{990}$.

18. **BF1564** Positive real numbers a, b, and c satisfy the following equations:

$$ab + \frac{1}{c} = 1$$

$$bc + \frac{1}{a} = 2$$

$$ca + \frac{1}{b} = 4$$

Find the least possible value of $a + b + c$.

Written by: Benjamin Fu

Answer: $\boxed{\dfrac{7\sqrt{5}-7}{4}}$

We begin by rewriting the given equations as follows:

$$\begin{aligned} abc + 1 &= c \\ abc + 1 &= 2a \\ abc + 1 &= 4b \end{aligned}$$

Let $n = abc + 1$. Then multiplying the equations above together gives us a cubic polynomial in terms of n:

$$n^3 = 8n - 8 \Longrightarrow (n-2)(n^2 + 2n - 4) = 0.$$

Therefore, $n \in \left\{2, -1+\sqrt{5}, -1-\sqrt{5}\right\}$. We wish to minimize $a + b + c = \frac{7}{4}(abc+1)$ while ensuring each of a, b, and c are positive, which is achieved with $abc + 1 = -1 + \sqrt{5}$. Thus, our answer is $\boxed{\frac{7\sqrt{5}-7}{4}}$.

19. **LT1557** Harry randomly selects six distinct integers between 0 and 9, inclusive. What is the probability that the product of three of these integers equals the product of the

other three?

Written by: Linus Tang

Answer: $\boxed{\dfrac{1}{105}}$

We first count the number of sets of six integers that have the desired property.

Note that 0 cannot be among the six integers, otherwise one product must be zero and the other product must be nonzero. Also, 5 cannot be among the six integers (since no other digits are divisible by 5), otherwise only one product will be divisible by 5. Similarly, 7 cannot be among the six integers.

Thus, all but one of the remaining choices $\{1, 2, 3, 4, 6, 8, 9\}$ must be chosen. Note that for any six integers, if they can be partitioned into two groups with the same product n, then the product of the six integers is a perfect square (n^2). We can check that the unchosen number must be 2 or 8, leaving a product of $1 \cdot 3 \cdot 4 \cdot 6 \cdot 8 \cdot 9 = 72^2$ or $1 \cdot 2 \cdot 3 \cdot 4 \cdot 6 \cdot 9 = 36^2$.

For these two cases, the partitions $1 \cdot 8 \cdot 9 = 3 \cdot 4 \cdot 6$ and $1 \cdot 4 \cdot 9 = 2 \cdot 3 \cdot 6$ work.

Going back to the original problem, there are $\binom{10}{6} = 210$ ways to select the six integers and 2 of them work, so the desired probability is $\frac{2}{210} = \boxed{\dfrac{1}{105}}$.

20. **BF1569** Find the least real number N such that there exist no values of x greater than or equal to N that satisfy

$$\left\lfloor x^2 + \left\lfloor x^2 + \left\lfloor x^2 + \left\lfloor x^2 \right\rfloor \right\rfloor \right\rfloor \right\rfloor = 100,$$

where $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y.

Written by: Benjamin Fu

Answer: $\boxed{\sqrt{26}}$

Observe that since $\lfloor x^2 \rfloor$ is always an integer and $\lfloor x^2 + n \rfloor = \lfloor x^2 \rfloor + n$ for all integers n, we have

$$\left\lfloor x^2 + \left\lfloor x^2 \right\rfloor \right\rfloor = \left\lfloor x^2 \right\rfloor + \left\lfloor x^2 \right\rfloor = 2 \left\lfloor x^2 \right\rfloor.$$

Similarly, we have

$$\left\lfloor x^2 + \left\lfloor x^2 + 2 \left\lfloor x^2 \right\rfloor \right\rfloor \right\rfloor = \left\lfloor x^2 + 3 \left\lfloor x^2 \right\rfloor \right\rfloor = 4 \left\lfloor x^2 \right\rfloor.$$

Thus, we have

$$4 \left\lfloor x^2 \right\rfloor = 100 \Longrightarrow \left\lfloor x^2 \right\rfloor = 25.$$

It is easy to see that $N = \sqrt{26}$.

21. **BF1551** Let $ABCD$ be a rectangle with $AB = 120$ and $BC = 170$, and let $EFGH$ be a unit square within $ABCD$ such that $\overline{AB} \parallel \overline{EF}$ and E is the closest vertex to A.

Given that $\angle ABF = \angle BCG$ and $\angle CDH = \angle DAE$, find the least possible length of $\overline{AE}$.

Written by: Benjamin Fu

Answer: $\boxed{\dfrac{91\sqrt{2}}{2}}$

First, we can get rid of $EFGH$ entirely by collapsing the space between lines EF and GH as well as the space between lines FG and HE. This shortens each side of the rectangle by 1 unit, so we now have $AB = 119$ and $BC = 169$. Additionally, all of E, F, G, and H are consolidated into one single point, which we will name P. The given condition on the angles now become $\angle ABP = \angle BCP$ and $\angle CDP = \angle DAP$.

Consider triangle $\triangle BCP$. Since the sum of the angles in a triangle are equal to $180°$, we have

$$\angle BCP + \angle CPB + \angle PBC = 180°.$$

Furthermore, we also know $90° - \angle PBC = \angle ABP = \angle BCP$, and substituting this into the equation above gives us $\angle CPB = 90°$. Similar reasoning yields $\angle APD = 90°$. By the Inscribed Angle Theorem, P must lie on both a circle with diameter AD and a circle with diameter BC, so P is equidistant from sides AD and BC, with said distance being $\frac{AB}{2} = \frac{119}{2}$.

Define $a = AP$ and $b = DP$. We wish to minimize $AE = AP$, so we let $a < b$. The distance from P to side AD is $\frac{119}{2}$, so the area of $\triangle APD$ is

$$\frac{1}{2} \cdot 169 \cdot \frac{119}{2} = \frac{1}{2} \cdot ab \Longrightarrow 2ab = 169 \cdot 119.$$

Additionally, as $\angle ADP = 90°$, we also have $a^2 + b^2 = 169^2$. Hence, we obtain the two following equations:

$$a + b = \sqrt{a^2 + 2ab + b^2} = \sqrt{169^2 + 169 \cdot 119} = 156\sqrt{2}$$
$$-a + b = \sqrt{a^2 - 2ab + b^2} = \sqrt{169^2 - 169 \cdot 119} = 65\sqrt{2}.$$

Subtracting the second equation from the first, then dividing by 2 gives us our final answer of $a = \boxed{\frac{91\sqrt{2}}{2}}$.

22. **BF1601** There exists a unique positive integer x for which $N = x^3 + 44x^2 + x$ is a perfect square. Find $\sqrt{N}$.

Written by: Benjamin Fu

Answer: $\boxed{124}$

We can factor $x^3 + 44x^2 + x$ as $x(x^2 + 44x + 1)$. Since x and $x^2 + 44x + 1$ are relatively prime by the Euclidean algorithm, the only way for N to be a perfect square is if both factors are a perfect square. For $x^2 + 44x + 1 = (x + 22)^2 - 483$ to be a perfect square, $(x + 22)^2$ must be 483 more than a perfect square. Since consecutive perfect squares differ by odd numbers, we can write 483 as a sum of consecutive odd numbers in the following ways:

$$483 = 242^2 - 241^2 \implies x = 220$$
$$159 + 161 + 163 = 82^2 - 79^2 \implies x = 60$$
$$63 + 65 + \cdots + 75 = 38^2 - 31^2 \implies x = 16$$
$$3 + 5 + \cdots + 43 = 22^2 - 1^2 \implies x = 0$$

Of these, only $x = 16$ yields a positive perfect square with $(16+22)^2 - 483 = 31^2$, so $\sqrt{N} = \sqrt{16 \cdot 31^2} = \boxed{124}$.

23. **LT1560** A birthday cake is in the shape of a triangle with side lengths 27, 28, and 29. A straight line slices the cake into two pieces with equal perimeter. The ratio of the area of the larger piece to the area of the smaller piece is r. What is the maximum possible value of r?

Written by: Linus Tang

Answer: $\boxed{\dfrac{7}{5}}$

Let the original triangle be ABC with $BC = 27$, $CA = 28$, and $AC = 29$. After the cut is made, at least one of the pieces is a triangle. Suppose that the triangle piece has side lengths of x and y which lie along original sides of ABC with lengths s and t (with s, t chosen from $27, 28, 29$), respectively.

Then $x + y$ is half the perimeter of ABC, or 42. The area of the triangular piece is $\frac{xy}{st}$ times the area of ABC, and it is maximized when $x = y = 21$ and $s, t = 27, 28$. Then $\frac{xy}{st} = \frac{7}{12}$, achieving $r = \boxed{\dfrac{7}{5}}$.

We now check that we can't do better by making the triangular piece the smaller piece. Indeed, the smallest the triangular piece can be occurs when $s = 28, t = 29, x = 13, y = 29$, achieving an area of $\frac{13}{28}$ that of the original triangle. This gives $r = \frac{15}{13}$, which is smaller.

24. **BF1600** Let A, B, C, and D be points on a circle ω in that order such that $AB = 7$, $BC = \sqrt{14}$, $CD = \sqrt{34}$, and $DA = \sqrt{69}$. Find the area of ω.

Written by: Benjamin Fu

Answer: $\boxed{\dfrac{83\pi}{4}}$

Notice that swapping the order of the side lengths doesn't change the radius of ω, so we can instead assume that $BC = \sqrt{34}$ and $CD = \sqrt{14}$. Then we have

$$AB^2 + BC^2 = CD^2 + DA^2 = 83,$$

so $\overline{AC}$ must be the diameter (this can be proven rigorously by the Law of Cosines, but intuition should be sufficient here) with a length of $\sqrt{83}$. Therefore, the area of ω is $\boxed{\frac{83\pi}{4}}$.

25. **LT1622** Ten distinct cells are chosen randomly from a 100×100 grid. Let p be the probability that there is a pair of chosen cells in the same row or the same column. Estimate the integer nearest $1000p$.

    Submit a positive integer N. If the correct answer is A, you will receive $\max(25(2 - \max(\frac{A}{N}, \frac{N}{A})), 0)$ points.

    Written by: Linus Tang

    Answer: $\boxed{396}$

    Will add solution and remarks later

26. **LT1620** Gilbert thinks of a number n, and writes down the equation

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{n}.$$

    This equation is satisfied by at least 200 ordered pairs of positive integers (a, b). Estimate the smallest possible value of n.

    Submit a positive integer N. If the correct answer is A, you will receive $\max(25 - \sqrt{|A - N|}, 0)$ points.

    Written by: Linus Tang

    Answer: $\boxed{1260}$

    Clearing denominators, the equation is $an + bn = ab$. Using Simon's Favorite Factoring Trick, we can rewrite the equation as $ab - an - bn + n^2 = n^2$, which factors as

$$(a - n)(b - n) = n^2.$$

    Now, we have that the number of positive integer solutions in (a, b) is the number of positive divisors of n^2. We want to find small n for which the number of positive divisors of n^2 is at least 200. Guessing and checking values of n by their prime factorization, it turns out that $n = 2^2 \cdot 3^2 \cdot 5 \cdot 7 = 1260$ gives $(4 + 1)(4 + 1)(2 + 1)(2 + 1) = 225$ factors, and is the smallest n that allows us to achieve at least 200.

27. **LT1621** Four points are chosen independently and uniformly at random from the interior of a unit square. Let p be the probability that these points are the vertices of a convex quadrilateral. Estimate the integer nearest $1000p$.

    Submit a positive integer N. If the correct answer is A, you will receive $\max(25 - \frac{|A-N|}{6}, 0)$ points.

    Written by: Linus Tang

Answer: $\boxed{694}$

Found by computer simulation.

An approach to get an estimate by hand is to first estimate the average area, A, of a triangle determined by three random points in unit square, then use $p = 1 - 4A$.

