1. **CS1108** Anika has a bag of green and blue gumballs. She eats 1 green gumball and notices there are now 4 times as many blue gumballs as green gumballs. She eats another 14 blue gumballs and notices there are now 2 times as many green gumballs as blue gumballs. How many gumballs were originally in the bag?

   Written by: *Katie Sue*

   Answer: $\boxed{21}$

   Let g equal the number of green gumballs and b equal the number of blue gumballs. From the first statement, we get:

$$4(g-1) = b.$$

   From the second statement, we get:

$$2(b-14) = g-1.$$

   We substitute $b = 4(g-1) = 4g-4$ into the second statement to get

   $2(4g-4-14) = g-1$

   $8g-8-28 = g-1$

   $7g = 35$

   $g = 5$

   Substituting this in the first equation, we get

   $b = 4(5-1)$

   $b = 16$

   Thus, there were originally $16+5 = \boxed{21}$ gumballs.

2. **MX1440** Let x and y be positive integers. If x is 150% of y, and $6y$ is N% of x, what is N?

   Written by: *Max Xie*

   Answer: $\boxed{400}$

   x=1.5y. 6y=4x. 4=400%

3. **LT899** Harry wants to write several positive numbers in a row on a chalkboard. The numbers must follow certain rules:

   - The leftmost number must be 1.
   - The rightmost number must be 24.
   - Every number other than the rightmost must divide the number to its right.

- No two consecutive numbers in the row are the same.

What is the maximum possible number of numbers in the row?

Written by: Linus Tang

Answer: $\boxed{5}$

Because the sum of the exponents in the prime factorization of a number strictly increases from 0 to 4 from left to right, there are at most $\boxed{5}$ numbers. One possible row of numbers is $1, 2, 4, 8, 24$.

4. **CS1016** Worker A can paint a wall in 4 hours, while workers A and B together only take 3 hours to paint the same wall. Assuming each worker paints the wall at a constant rate, how many hours would it take for worker B to paint the wall alone?

Written by: Claire Shen

Answer: $\boxed{12}$

Let A and B denote the number of hours it takes for workers A and B to paint the wall alone respectively. Workers A and B working together will paint $(\frac{1}{4} + \frac{1}{B})$ of the wall every hour. Since we know takes 3 hours for them to paint the wall together,

$$3\left(\frac{1}{4} + \frac{1}{B}\right) = 1 \implies \frac{1}{4} + \frac{1}{B} = \frac{1}{3} \implies \frac{1}{B} = \frac{1}{12} \implies B = \boxed{12}.$$

5. **DG400** Simplify $17^2 + (13 + 23) \cdot 17 + 13 \cdot 23$.

Written by: Daniel Ge

Answer: $\boxed{1200}$

$17^2 + (13 + 23) \cdot 17 + 13 \cdot 23$

$= 17 \cdot 17 + 13 \cdot 17 + 23 \cdot 17 + 13 \cdot 23$

$= 17 \cdot (17 + 13) + 23 \cdot (17 + 13)$

$= (17 + 13)(17 + 23)$

$= 30 \cdot 40$

$= \boxed{1200}$

6. **CS1243** Joe has a cone-shaped pitcher full of pancake batter with a base of radius 6 inches and a height of 16 inches. Each pancake Joe makes is the exact same size: a cylinder with a radius of 4 inches that is $\frac{1}{2}$ inch thick. Assume that Joe's pancakes don't rise. After making breakfast this morning, Joe notices that the pitcher is now $\frac{1}{3}$ full. How many pancakes did Joe make this morning?

Written by: Katie Sue

Answer: $\boxed{16.}$

The volume of the pitcher is:

$$\frac{6^2\pi \cdot 16}{3} = 192\pi$$

The volume of each pancake is:

$$\frac{4^2\pi}{2} = 8\pi$$

Since $\frac{192}{8}$ is 24, Joe can make 24 pancakes if he uses all the batter. However, Joe only uses $\frac{2}{3}$ of the batter so he made $\frac{2}{3} \cdot 24 = \boxed{16}$ pancakes.

7. **HWU1070** How many ways are there to form a 4-letter word from the letters $\{M, U, S, T, A, N, G\}$ with at least one vowel? Note that the vowels are A and U, words cannot contain duplicate letters, and any combination of distinct letters constitutes a word.

   Written by: Han Wu

   Answer: $\boxed{720}$

   We will use complementary counting. There are $7 \cdot 6 \cdot 5 \cdot 4$ possible words when no conditions exist. $MUSTANG$ contains 2 vowels and 5 consonants, so there are $5 \cdot 4 \cdot 3 \cdot 2$ words without a vowel. Thus, there are $7 \cdot 6 \cdot 5 \cdot 4 - 5 \cdot 4 \cdot 3 \cdot 2 = \boxed{720}$ words with at least 1 vowel.

8. **LT745** A palindrome is a positive integer, such as 30303 or 5885, that is the same when read forward or backward. What is the average of all of the 3-digit palindromes?

   Written by: Linus Tang

   Answer: $\boxed{550}$

   The 3-digit palindromes are numbers of the form $\overline{aba} = 101a + 10b$, where a and b are digits and a is nonzero. The average value of a is 5 and the average value of b is 4.5 over all nine choices for a and 10 choices for b, so the average value of $101a + 10b$ is $101 \cdot 5 + 10 \cdot 4.5 = \boxed{550}$.

9. **JES1186** A field of carrots resembles a 4 by 6 rectangular grid of identical square cells, with one carrot planted in each cell. If two rabbits visit the field and randomly select a carrot to eat, one after the other, what is the probability that the two cells no longer containing carrots share a side?

   Written by: Jerry Sun

   Answer: $\boxed{\dfrac{19}{138}}$

The total number of possible ways to choose two cells is $\binom{24}{2} = \frac{24 \cdot 23}{2} = 276$. If the two chosen cells share a side, they will form either a horizontal or vertical 2 by 1. The number of possible horizontal configurations correspond to the number of choices for the left cell, which excludes the rightmost column: $4 \cdot 5 = 20$. Similarly, the number of possible vertical configurations is $3 \cdot 6 = 18$. Thus, the answer is $\frac{20+18}{276} = \boxed{\frac{19}{138}}$.

10. **IZ1069** The fraction $\frac{25}{28}$ can be written in the form of $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$, where a, b, and c are distinct positive integers. What is $a + b + c$?

    Written by: Iris Zheng

    Answer: $\boxed{13}$

    First, we look at the denominator, 28. The prime factorization of 28 is $2^2 \times 7$, meaning that at least one of a, b, and c has to be a multiple of 4, and at least one must be a multiple of 7. Since $\frac{25}{28}$ is relatively greater in value, we can start by checking fractions greater in value such as $\frac{1}{2}$.

    By subtracting $\frac{1}{2}$ from $\frac{25}{28}$, we are left with $\frac{11}{28}$. Now we can try subtracting $\frac{1}{7}$ because we know that we need one of the values to be a multiple of 7, and we are left with $\frac{7}{28}$. $\frac{7}{28}$ simplified is $\frac{1}{4}$, which is perfectly a unit fraction. Now that we know that the values of $\frac{1}{a}$, $\frac{1}{b}$, and $\frac{1}{c}$ are $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{7}$, meaning a, b, and c correspond to 2, 4, and 7. The sum of these values is $2 + 4 + 7 = \boxed{13}$.

11. **MLI847** For integers $a \leq b$, define $S(a, b)$ to be the sum of the integers from a to b, inclusive. For example, $S(2, 4) = 2 + 3 + 4 = 9$ and $S(1, 1) = 1$.

    Consider pairs i, j of positive integers between 1 and 5 inclusive, with $i \leq j$. Find the sum of $S(i, j)$ over all 15 such pairs.

    Written by: Michael Liu

    Answer: $\boxed{105}$

    To get the answer, we wish to find the number of times each integer k from 1 to 5 is included in the sum. There are k ways to choose a value of i and $6 - k$ ways to choose a value of j for which k in included in $S(i, j)$, so summing $k \cdot k \cdot (6 - k)$ over $1 \leq k \leq 5$ gives us

    $$1 \cdot 1 \cdot 5 + 2 \cdot 2 \cdot 4 + 3 \cdot 3 \cdot 3 + 4 \cdot 4 \cdot 2 + 5 \cdot 5 \cdot 1 = \boxed{105}.$$

12. **SV1222** The diagonals of rectangle $ABCD$ intersect at E and $BE = AB = 6$. Points F and G lie on AC and BD, respectively, such that BF is perpendicular to AC and CG is perpendicular to BD. If BF and CG intersect at H, what is BH?

    Written by: Sriram Venkatesh

Answer: $\boxed{6\sqrt{3}.}$

Since $AB = BE = 6$, and $AE = BE$, triangle ABE is equilateral. This means that triangle ABC is a $30-60-90$ triangle, with $BC = 6\sqrt{3}$. Since $\overline{BF}$ is a perpendicular bisector of triangle ABE, $EF = 3$. By symmetry, $\angle FEH = \angle GEH = 60°$. Therefore, $EH = 6$ and $FH = 3\sqrt{3}$. Since $\overline{EH} = \overline{BE}$, triangle BEH is isosceles. This means that $\overline{EF}$ is a perpendicular bisector of triangle BEH, making $BF = FH = 3\sqrt{3}$. Our answer is therefore $\boxed{6\sqrt{3}}$.

13. **BZ1292** The number $717A7B7C$, where A, B, and C are distinct integer digits, is divisible by 12. If x is the maximum possible value of $A+B+C$ and y is the minimum possible value of $A+B+C$, find $x-y$.

Written by: Brian Zhou

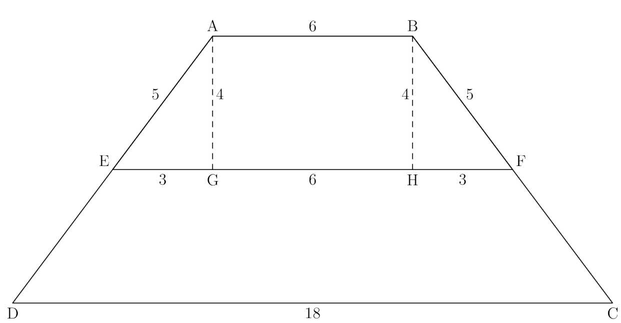
Answer: $\boxed{15}$

Because $12 = 4 \cdot 3$, our number is divisible by both 3 and 4. Divisibility by 3 is achieved if and only if the sum of digits is a multiple of 3, and since $7+1+7+7+7 = 29 \equiv -1 \pmod 3$, $A+B+C \equiv 1 \pmod 3$. Divisibility by 4 is achieved if and only if $7C$ is a multiple of 4, so C is either 2 or 6. Letting $C = 6$, the maximum sum we can achieve satisfying all the constraints is $9+7+6 = 22$. Letting $C = 2$, the minimum sum we can achieve is $0+5+2 = 7$. Note that we cannot get a sum of 4 because A, B, and C must all be distinct. Thus, our answer is $22 - 7 = \boxed{15}$.

14. **CS1006** In trapezoid $ABCD$ with $AB \parallel CD$, E and F are the midpoints of AD and BC, respectively. Given that $CD = 18$, $EF = 12$, and $AE = BF = 5$, what is the area of trapezoid $ABFE$?

Written by: Claire Shen

Answer: $\boxed{36}$



The median of a trapezoid is the average of the two bases. With this,

$\frac{18+AB}{2} = 12$
$18 + AB = 24$

$AB = 6$

Since the altitude of the trapezoid is perpendicular to the base, altitudes from A to EF and from B to EF form right triangle AGE, right triangle BHF, and rectangle $ABGH$. AB and GH are opposite sides of a rectangle, so they are equal in length, meaning GH is also 6. Right triangles AGE and GHF are congruent by HL congruence for right triangles (AE and BF are both 5, and AG and BH are both the height of the trapezoid), meaning EG and HF are also equal in length.

$6 + 2 \cdot EG = 12$

$2 \cdot EG = 6$

$EG = 3$

Using the Pythagorean Theorem,

$AG = \sqrt{5^2 - 3^2}$

$AG = \sqrt{25 - 9}$

$AG = \sqrt{16}$

$AG = 4$

The area of a trapezoid is the average of the two bases multiplied by the height, meaning the final answer is $\frac{12+6}{2} \cdot 4 = \boxed{36}$.

15. **GT1203** A race of aliens live on a flat world, with a source of light of negligible size (called the NUS) 100 km above the ground. One day, the NUS is too bright, and the aliens hope to build a bit of shade. They have a 150 m $\times$ 200 m board, and wish to cover an area of 75 km^2 on the ground. Furthermore, while they have the powers of levitation, the aliens can only levitate the board parallel to the ground, as it becomes unstable otherwise. At how many kilometers above the ground do the aliens need to place the board in order to shade the desired area?

    Written by: Grace Tan

    Answer: $\boxed{98 \text{ km}}$

    The desired area of shade is $\frac{75 \cdot 1000^2}{150 \cdot 200} = 2500$ times the area of the board, so the ratio of the distance between the NUS and the board to the distance between the NUS and the ground is $\sqrt{2500} = 50$. Therefore, the distance between the NUS and the board is $\frac{100}{50} = 2$ km, so the distance between the board and the ground is $100 - 2 = \boxed{98 \text{ km}}$.

16. **LT636** For any positive integer n, define $f(n)$ as the number of primes less than n. For example, $f(7) = 3$ and $f(8) = 4$. Given that there are 25 primes less than 100 and that the sum of these primes is 1060, what is the value of $f(1)+f(2)+f(3)+\cdots+f(100)$?

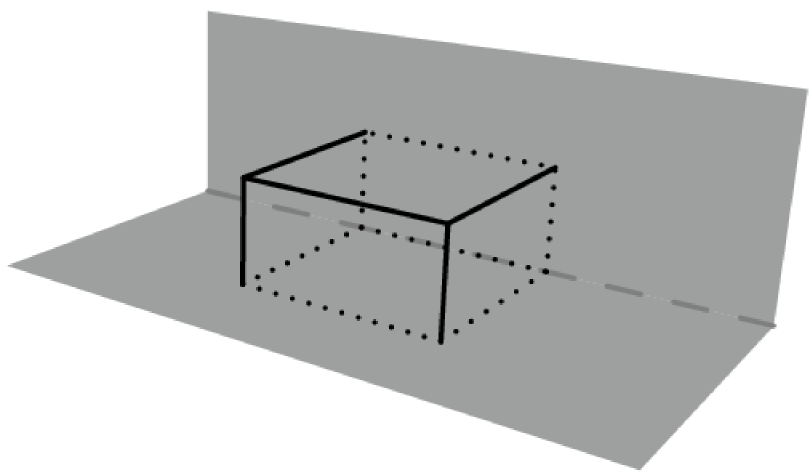
    Written by: Linus Tang

Answer: $\boxed{1440}$

Let p_n denote the nth prime. Note that each of the 25 primes less than 100 contributes $100 - p$ to the total sum. Thus, the answer is

$$\sum_{k=1}^{25} 100 - p_k = 2500 - 1060 = \boxed{1440}.$$

17. **DS1151** Bob is making the outline of a wire cage in the shape of a rectangular prism. As shown in the image below, one face of the cage will lie on the ground and one face will lie against the wall. Bob has 9 meters of wiring, which he will use to make the solid black lines below. What is the maximum volume of the cage, in cubic meters, that Bob can achieve?



Written by: *Dev Saxena*

Answer: $\boxed{\frac{27}{4}.}$

Solution 1: Let the edge lengths of the box be a, b, and c, with a and b appearing twice and c appearing only once. We can see that $2a + 2b + c = 9$, and we wish to maximize abc. We can apply AM-GM with the values $2a$, $2b$, and c:

$$\frac{2a + 2b + c}{3} \geq \sqrt[3]{2a \cdot 2b \cdot c} = \sqrt[3]{4abc}$$

$$27 \geq 4abc$$

$$abc \leq \frac{27}{4}$$

Equality holds when $2a = 2b = c$, which we can solve to find $a = b = \frac{3}{2}$ and $c = 3$. The volume is indeed $\boxed{\frac{27}{4}}$.

Solution 2 (No AM-GM): By the same logic as above, we can see that $2a + 2b + c = 9$ and we must maximize abc. Now, construct a rectangular prism with dimesions $2a$, $2b$, and c. The volume of this prism will be $4abc$, meaning if we can maximize the volume of this prism, we will automatically find the maximum volume of our cage. Since the largest prism that can be constructed from a given perimeter will be a cube, we set $2a = 2b = c$ and solve. This yields an answer of $\boxed{\frac{27}{4}}$.

18. **RRP1007** Owen picks letters randomly from the alphabet with replacement, each with equal probability, until he picks two vowels (5 of the 26 letters in the alphabet are vowels). He arranges the letters he picks in the order he picked them to create a word. Find the probability that his word is a palindrome with an even amount of letters.

    Written by: Ryan Pascual

    Answer: $\boxed{\frac{1}{131}}$

    First, note that the first and last letters must be the same vowel. Then, there can be any amount of consonants in between that form a palindrome. With $2n$ consonants between the vowels, the probability is $\frac{5}{26^2} \cdot \frac{21^n}{26^{2n}}$. Therefore the probability is

$$\sum_{k=0}^{\infty} \left(\frac{5}{26^2} \cdot \frac{21^k}{26^{2k}} \right) = \frac{\frac{5}{26^2}}{1 - \frac{21}{26^2}} = \boxed{\frac{1}{131}}$$

    by geometric series.

19. **SV1183** Equilateral triangle ABC with side length 20 is constructed, and a circle is inscribed in it, tangent to lines AB, BC, and AC. A circular coin with radius 1 is then tossed randomly onto the triangle. Given that the coin lands completely within the triangle, what is the probability that at least part of the coin lands outside the circle?

    Written by: Sriram Venkatesh

    Answer: $\boxed{\frac{3\sqrt{3} - \pi}{3\sqrt{3}}.}$

    We will calculate the complementary probability, which is the probability that the coin lands entirely within the circle. As it is given that the coin must land entirely within the triangle, the probability we are looking for is simply the area of an equilateral triangle's incircle divided by the area of the equilateral triangle. WLOG let the incircle have a radius of 1 and area π. Then, by constructing a 30-60-90 triangle, we find the triangle has a side length of $2\sqrt{3}$, so its area is $s^2\frac{\sqrt{3}}{4} = 3\sqrt{3}$. Thus, the complementary probability is $\frac{\pi}{3\sqrt{3}}$, and the answer is $\boxed{\frac{3\sqrt{3} - \pi}{3\sqrt{3}}}$.

20. **RV1198** Define the function $f(x) = (4x - 6)^{10}$. If $f(x)$ can be represented as $a_{10}x^{10} + a_9x^9 + \cdots + a_1x + a_0$, let $S = a_9 + a_7 + a_5 + a_3 + a_1$. What is the remainder when $|S|$ is divided by 1000?

Written by: Rishabh Venkataramani

Answer: $\boxed{488}$

Notice that letting $x = 1$ and $x = -1$ gives us the following equations:

$$f(1) = a_{10} + a_9 + a_8 + a_7 + a_6 + a_5 + a_4 + a_3 + a_2 + a_1 + a_0 = (4-6)^{10} = (-2)^{10}$$
$$f(-1) = a_{10} - a_9 + a_8 - a_7 + a_6 - a_5 + a_5 - a_3 + a_2 - a_1 + a_0 = (-4-6)^{10} = (-10)^{10}$$

Subtracting the second equation from the first, then dividing by 2, gives us the value of S:

$$S = \frac{(-2)^{10} - (-10)^{10}}{2} = \frac{1024 - 10000000000}{2} = -4999999488.$$

The answer is then $\boxed{488}$.