Mustang Math Tournament 2024

Relay Rodeo Stallion Round

Basic Format

- This round contains 21 problems to be solved in 45 minutes.
- The problems are divided into four suits ($\diamondsuit$, $\heartsuit$, $\clubsuit$, $\spadesuit$) with five problems each, plus one Joker problem.
- Some of the problems refer to the answer to earlier problems in the same suit. For example, problem [$\diamondsuit$2] may begin with, "Let N be the answer to [$\diamondsuit$1]." So, [$\diamondsuit$2] can only be attempted after [$\diamondsuit$1] has been answered. In a similar manner, the Joker problem refers to the answer to the last problem in every suit.
- The first four problems in each suit are worth 2 points each, and the last problem in each suit is worth 3 points. The Joker problem is worth 6 points.
- **Do not** write below the provided answer blank inside each space on the answer sheet (the space is for grading purposes).
- *Be sure to **dot** your i's and **dash** your t's (i.e. carefully check your work) to submit each set with no re**morse**.*

You may use this sheet as scratch paper.

NO ANSWERS ON THIS SHEET WILL BE GRADED.

Diamonds $\diamondsuit$

[$\diamondsuit$1]. Charlie the Colt is standing in line to get ice cream from the ice cream truck. There are 4 colts in front of Charlie and 5 colts behind Charlie. How many colts are in the line, in total?

[$\diamondsuit$2]. Let N be the answer to [$\diamondsuit$1]. The Martian alphabet contains N letters. Half of them are vowels, and the other half of them are consonants. A Martian syllable is any string of two letters that has exactly one vowel. How many different Martian syllables are there?

[$\diamondsuit$3]. Let N be the answer to [$\diamondsuit$1]. A rectangle has integer side lengths and a perimeter of $N-4$. What is the area of the rectangle?

[$\diamondsuit$4]. Let N be the answer to [$\diamondsuit$1]. Sally evaluates the two expressions 2^N+N^3 and 3^N+N^2, and writes both results on the chalkboard. What is the smaller of the two numbers written?

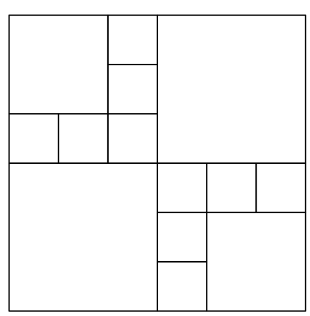
[$\diamondsuit$5]. Let N be the answer to [$\diamondsuit$1]. Suppose m is a positive integer that has exactly N positive divisors, one of which is 20. What is m?

Hearts $\heartsuit$

[$\heartsuit$1]. There are 7 balls placed in a bag, each with a different color. When 3 balls are drawn from the bag with replacement, the probability that all three of them have the same color can be expressed as $\frac{1}{a}$. What is a?

[$\heartsuit$2]. Let N be the answer to [$\heartsuit$1]. N is a perfect square that has the property that the digits of N are positive perfect squares. What is the smallest perfect square greater than N that has the same property?

[$\heartsuit$3]. Let N be the answer to [$\heartsuit$2]. A large square with area N is cut into many smaller squares, as shown below. What is the perimeter of the smallest square?



[$\heartsuit$4]. Let N be the answer to [$\heartsuit$3]. A fair coin is flipped N times. What is the probability that tails is flipped exactly once?

[$\heartsuit$5]. Let N be the answer to [$\heartsuit$4]. A positive integer a has exactly d positive divisors, and exactly d^2N+1 of these divisors are composite numbers. What is the value of d?

Clubs ♣

[♣1]. How many times do the graphs of $y = \lfloor x \rfloor \lceil x \rceil$ and $y = 3x$ intersect? $\lfloor x \rfloor$ is defined as the largest integer less than or equal to x, and $\lceil x \rceil$ is defined as the smallest integer greater than or equal to x.

[♣2]. How many zeroes are at the end of 20! when it is written in base 24? As usual, 20! denotes the product of the integers from 1 to 20, or $1 \times 2 \times 3 \times \cdots \times 20$.

[♣3]. Let N be the answer to [♣1]. There are N breeds of horses in a barn, and there are 8 indistinguishable horses of each breed. Each horse in the barn will be given to one of 6 people, and each person needs at least 1 horse of each breed. Let n be the number of ways to assign the horses to people. What is the sum of the digits of n?

[♣4]. Let N be the answer to [♣2]. A regular tetrahedron is inscribed in a sphere of radius N. The volume of this tetrahedron can be written as $m^2\sqrt{n}$, where m and n are positive integers, and where n is not divisible by any perfect square greater than 1. What is $m + n$?

[♣5]. Let x and y be the answers to [♣3] and [♣4], respectively. Four points A, B, C, and D lie on a circle, in that order. Let diagonals $\overline{AC}$ and $\overline{BD}$ intersect at M. We are given that $MB < MD$ and $MA < MC$. Furthermore, $BD = x$ and $AC = y$, and the lengths MA and MB are both integers. What is the maximum possible area of $\triangle AMD$?

Spades ♠

[♠1]. Let N be the smallest positive integer that has the same number of positive divisors as 675. Find the number of distinct prime factors of N.

[♠2]. A fair 10 sided die is rolled 8 times, and the numbers rolled are recorded in the order that they are rolled in. The probability that the first number is greater than the sum of the other 7 can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. What is the sum of the digits of b?

[♠3]. 8 circles of radius 1 are drawn, each centered at a distinct vertex of a regular octagon with side length 2. The radius of the circle that is externally tangent to each of these 8 circles can be expressed in the form $\sqrt{a + b\sqrt{c}} + d$, where $a, b, c,$ and d are integers, and c is not divisible by any perfect square greater than 1. Determine the value of $a - b + c + d$.

[♠4]. Let $\lfloor y \rfloor$ denote the greatest integer less than or equal to y, and $\{y\}$ denote $y - \lfloor y \rfloor$. For example, $\lfloor 14.3 \rfloor = 14$ and $\{14.3\} = 0.3$. Find the value of $\lfloor x \rfloor$, given that x satisfies

$$\lfloor x \rfloor + \sqrt{\{x\} + \sqrt{\lfloor x \rfloor + \sqrt{\{x\} + \cdots}}} = 5.76$$

[♠5]. Let $a_1, a_2, a_3,$ and a_4 be the answers to [♠1], [♠2], [♠3], and [♠4], respectively. The polynomial $p(x) = a_3x^4 + x^2 + x + a_2$ has complex roots $r_1 \ldots r_4$. Find the integer closest to the value of

$$\sum_{n=1}^{4} \frac{60}{\sqrt{a}_4 - a_1 r_n}.$$

JOKER

[J1]. Let A, B, C, and D be the answers to [$\diamondsuit$5], [$\heartsuit$5], [$\clubsuit$5], and [$\spadesuit$5], respectively. A quadrilateral has vertices at $(A, 0)$, $(0, B)$, $(-C, 0)$, and $(0, -D)$ on the coordinate plane. There is a circle ω on the plane and a positive real number r such that the distance from each vertex to ω is r. If the smallest possible value of r can be written as $\sqrt{X} - \sqrt{Y}$, where X and Y are rational numbers, find $X - Y$.

Acceptable Answers

The following rules provide guidelines for acceptable answers in this round. Please note that any specifications provided in a problem will take precedence over these rules. The decisions of MMT coordinators are final.

- Common fractions are defined as a fraction in the form $\pm\frac{a}{b}$ where a and b are natural numbers and $\gcd(a, b) = 1$.
- Ratios and fractional answers should be expressed as common fractions unless otherwise specified.
- Radicals should be simplified. A simplified radical must satisfy:
  - No square factors, fractions, or nested radicals inside a radical
  - No radicals inside the denominator of a fraction
- Answers must be expressed to the exact accuracy called for in the problem (e.g. 25.0 will not be accepted for 25 and 25 will not be accepted for 25.0).
- Do not make approximations for numbers (e.g. 3.14 or $\frac{22}{7}$ for π) unless otherwise specified.
- Units **do not** need to be included but must be correct if included.