1. **CS1108** Anika has a bag of green and blue gumballs. She eats 1 green gumball and notices there are now 4 times as many blue gumballs as green gumballs. She eats another 14 blue gumballs and notices there are now 2 times as many green gumballs as blue gumballs. How many gumballs were originally in the bag?

   Written by: Katie Sue

   Answer: $\boxed{21}$

   Let g equal the number of green gumballs and b equal the number of blue gumballs. From the first statement, we get:

   $$4(g-1)=b.$$

   From the second statement, we get:

   $$2(b-14)=g-1.$$

   We substitute $b=4(g-1)=4g-4$ into the second statement to get

   $2(4g-4-14)=g-1$

   $8g-8-28=g-1$

   $7g=35$

   $g=5$

   Substituting this in the first equation, we get

   $b=4(5-1)$

   $b=16$

   Thus, there were originally $16+5=\boxed{21}$ gumballs.

2. **DG400** Simplify $17^2+(13+23)\cdot 17+13\cdot 23$.

   Written by: Daniel Ge

   Answer: $\boxed{1200}$

   $17^2+(13+23)\cdot 17+13\cdot 23$

   $=17\cdot 17+13\cdot 17+23\cdot 17+13\cdot 23$

   $=17\cdot(17+13)+23\cdot(17+13)$

   $=(17+13)(17+23)$

   $=30\cdot 40$

   $=\boxed{1200}$

3. **MX1440** Let x and y be positive integers. If x is 150% of y, and $6y$ is $N\%$ of x, what is N?

Written by: Max Xie

Answer: $\boxed{400}$

x=1.5y. 6y=4x. 4=400%

4. **LT899** Harry wants to write several positive numbers in a row on a chalkboard. The numbers must follow certain rules:

- The leftmost number must be 1.
- The rightmost number must be 24.
- Every number other than the rightmost must divide the number to its right.
- No two consecutive numbers in the row are the same.

What is the maximum possible number of numbers in the row?

Written by: Linus Tang

Answer: $\boxed{5}$

Because the sum of the exponents in the prime factorization of a number strictly increases from 0 to 4 from left to right, there are at most $\boxed{5}$ numbers. One possible row of numbers is $1, 2, 4, 8, 24$.

5. **CS1243** Joe has a cone-shaped pitcher full of pancake batter with a base of radius 6 inches and a height of 16 inches. Each pancake Joe makes is the exact same size: a cylinder with a radius of 4 inches that is $\frac{1}{2}$ inch thick. Assume that Joe's pancakes don't rise. After making breakfast this morning, Joe notices that the pitcher is now $\frac{1}{3}$ full. How many pancakes did Joe make this morning?

Written by: Katie Sue

Answer: $\boxed{16.}$

The volume of the pitcher is:

$$\frac{6^2\pi \cdot 16}{3} = 192\pi$$

The volume of each pancake is:

$$\frac{4^2\pi}{2} = 8\pi$$

Since $\frac{192}{8}$ is 24, Joe can make 24 pancakes if he uses all the batter. However, Joe only uses $\frac{2}{3}$ of the batter so he made $\frac{2}{3} \cdot 24 = \boxed{16}$ pancakes.

6. **HWU1070** How many ways are there to form a 4-letter word from the letters $\{M, U, S, T, A, N, G\}$ with at least one vowel? Note that the vowels are A and U, words cannot contain duplicate letters, and any combination of distinct letters constitutes a word.

   Written by: Han Wu

   Answer: $\boxed{720}$

   We will use complementary counting. There are $7 \cdot 6 \cdot 5 \cdot 4$ possible words when no conditions exist. $MUSTANG$ contains 2 vowels and 5 consonants, so there are $5 \cdot 4 \cdot 3 \cdot 2$ words without a vowel. Thus, there are $7 \cdot 6 \cdot 5 \cdot 4 - 5 \cdot 4 \cdot 3 \cdot 2 = \boxed{720}$ words with at least 1 vowel.

7. **SS371** The Mustangs and the Colts are two of the 20 soccer teams in the Horsey Soccer League. The league is randomly split into two divisions with 10 teams in each. Find the probability that the Mustangs and the Colts are in different divisions.

   Written by: Saahil Shah

   Answer: $\boxed{\dfrac{10}{19}}$

   20 teams can be split amongst 2 divisions in a total of $\frac{\binom{20}{10}}{2} = 92378$ ways. For the probability that they end up on different divisions, 9 teams out of the remaining 18 is needed to fill up a division, the other will be filled up automatically with the other teams. The total number of ways this can happen is $\binom{18}{9}$ or 48620. $\frac{48620}{92378} = \boxed{\dfrac{10}{19}}$.

   Solution 2: Let the Mustangs be in Division 1. There are 9 other teams that can be in Division 1 with them, and 10 other teams in Division 2. The chance that the Colts get put in Division 2 is just equal to the chance that the Colts get picked in a group of 10 from a pool of 19. The chance of this happening is the same as every other team, so the probability is $\boxed{\dfrac{10}{19}}$.

8. **MLI847** For integers $a \leq b$, define $S(a, b)$ to be the sum of the integers from a to b, inclusive. For example, $S(2, 4) = 2 + 3 + 4 = 9$ and $S(1, 1) = 1$.

   Consider pairs i, j of positive integers between 1 and 5 inclusive, with $i \leq j$. Find the sum of $S(i, j)$ over all 15 such pairs.

   Written by: Michael Liu

   Answer: $\boxed{105}$

   To get the answer, we wish to find the number of times each integer k from 1 to 5 is included in the sum. There are k ways to choose a value of i and $6 - k$ ways to choose a value of j for which k in included in $S(i, j)$, so summing $k \cdot k \cdot (6 - k)$ over $1 \leq k \leq 5$

gives us

$$1 \cdot 1 \cdot 5 + 2 \cdot 2 \cdot 4 + 3 \cdot 3 \cdot 3 + 4 \cdot 4 \cdot 2 + 5 \cdot 5 \cdot 1 = \boxed{105}.$$

9. **BZ1292** The number $717A7B7C$, where A, B, and C are distinct integer digits, is divisible by 12. If x is the maximum possible value of $A+B+C$ and y is the minimum possible value of $A+B+C$, find $x-y$.

   Written by: Brian Zhou

   Answer: $\boxed{15}$

   Because $12 = 4 \cdot 3$, our number is divisible by both 3 and 4. Divisibility by 3 is achieved if and only if the sum of digits is a multiple of 3, and since $7+1+7+7+7 = 29 \equiv -1 \pmod 3$, $A+B+C \equiv 1 \pmod 3$. Divisibility by 4 is achieved if and only if $7C$ is a multiple of 4, so C is either 2 or 6. Letting $C = 6$, the maximum sum we can achieve satisfying all the constraints is $9+7+6 = 22$. Letting $C = 2$, the minimum sum we can achieve is $0+5+2 = 7$. Note that we cannot get a sum of 4 because A, B, and C must all be distinct. Thus, our answer is $22 - 7 = \boxed{15}$.

10. **GT1203** A race of aliens live on a flat world, with a source of light of negligible size (called the NUS) 100 km above the ground. One day, the NUS is too bright, and the aliens hope to build a bit of shade. They have a 150 m $\times$ 200 m board, and wish to cover an area of 75 km^2 on the ground. Furthermore, while they have the powers of levitation, the aliens can only levitate the board parallel to the ground, as it becomes unstable otherwise. At how many kilometers above the ground do the aliens need to place the board in order to shade the desired area?

    Written by: Grace Tan

    Answer: $\boxed{98 \text{ km}}$

    The desired area of shade is $\frac{75 \cdot 1000^2}{150 \cdot 200} = 2500$ times the area of the board, so the ratio of the distance between the NUS and the board to the distance between the NUS and the ground is $\sqrt{2500} = 50$. Therefore, the distance between the NUS and the board is $\frac{100}{50} = 2$ km, so the distance between the board and the ground is $100 - 2 = \boxed{98 \text{ km}}$.

11. **IZ1069** The fraction $\frac{25}{28}$ can be written in the form of $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$, where a, b, and c are distinct positive integers. What is $a+b+c$?

    Written by: Iris Zheng

    Answer: $\boxed{13}$

    First, we look at the denominator, 28. The prime factorization of 28 is $2^2 \times 7$, meaning that at least one of a, b, and c has to be a multiple of 4, and at least one must be a multiple of 7. Since $\frac{25}{28}$ is relatively greater in value, we can start by checking fractions greater in value such as $\frac{1}{2}$.

    By subtracting $\frac{1}{2}$ from $\frac{25}{28}$, we are left with $\frac{11}{28}$. Now we can try subtracting $\frac{1}{7}$ because we know that we need one of the values to be a multiple of 7, and we are left with $\frac{7}{28}$. $\frac{7}{28}$ simplified is $\frac{1}{4}$, which is perfectly a unit fraction. Now that we know that the values of $\frac{1}{a}$, $\frac{1}{b}$, and $\frac{1}{c}$ are $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{7}$, meaning a, b, and c correspond to 2, 4, and 7. The sum of these values is $2+4+7=\boxed{13}$.

12. **BZ1001** Find the value of x that satisfies the equation

$$\sqrt{2+\sqrt{x+\sqrt{2+\sqrt{x+...}}}}=\frac{15}{2+\frac{15}{2+\frac{15}{2+...}}}.$$

Written by: Brian Zhou

Answer: $\boxed{46}$

We can first simplify the right hand side. Letting the entire infinite fraction equal a variable y, we get

$$\begin{aligned} y &= \frac{15}{2+y}, \\ y^2+2y-15 &= 0, \\ (y+5)(y-3) &= 0. \end{aligned}$$

Clearly, the infinite fraction is positive, so $y=3$. Now, we can solve the original equation, getting

$$\begin{aligned} \sqrt{2+\sqrt{x+\sqrt{2+\sqrt{x+...}}}} &= 3, \\ \sqrt{2+\sqrt{x+3}} &= 3, \\ \sqrt{x+3} &= 7. \end{aligned}$$

Thus, $x=\boxed{46}$.

13. **SV1101** Four consecutive positive integers have a product of 93024. What is their sum?

Written by: Sriram Venkatesh

Answer: $\boxed{70}$

If all of the numbers were greater than 20, their product would be greater than $20^4=160000$, which is impossible. If all of the numbers were less than 15, their product would be less than $15^4=50625$, which is also impossible. Furthermore, none of the numbers can be 15 or 20, as the product is not a multiple of 5. Thus, the numbers have to be $16, 17, 18$, and 19, making their sum $\boxed{70}$.

14. **RRP1007** Owen picks letters randomly from the alphabet with replacement, each with equal probability, until he picks two vowels (5 of the 26 letters in the alphabet are vowels). He arranges the letters he picks in the order he picked them to create a word. Find the probability that his word is a palindrome with an even amount of letters.

Written by: Ryan Pascual

Answer: $\boxed{\dfrac{1}{131}}$

First, note that the first and last letters must be the same vowel. Then, there can be any amount of consonants in between that form a palindrome. With $2n$ consonants between the vowels, the probability is $\frac{5}{26^2} \cdot \frac{21^n}{26^{2n}}$. Therefore the probability is

$$\sum_{k=0}^{\infty} \left(\frac{5}{26^2} \cdot \frac{21^k}{26^{2k}} \right) = \frac{\frac{5}{26^2}}{1 - \frac{21}{26^2}} = \boxed{\frac{1}{131}}$$

by geometric series.

15. **RRP1028** Let $n!$ denote the product of the integers between 1 and n, inclusive. A positive integer divisor d of 41! is chosen at random. What is the probability that d is divisible by 8, but is not divisible by 16?

Written by: Ryan Pascual

Answer: $\boxed{\dfrac{1}{39}.}$

The number of powers of 2 in the prime factorization of 41 is $\sum_{k=1}^{\infty} \lfloor \frac{41}{2^k} \rfloor = 38$. Then let $v_2(k)$ be the highest power of 2 dividing k; that is if $v_2(k) = j$ then $2^j | k$ but $2^{j+1} \not| k$. Then $v_2(n)$ is randomly chosen from the set $\{0, 1, 2, \cdots, 38\}$. The probability that $v_2(d) = 3$ is therefore $\boxed{\dfrac{1}{39}}$.

16. **BZ1314** For any real number z, let $\lfloor z \rfloor$ denote the greatest integer that is less than or equal to z. Find the area of the region formed by all points (x, y) on the coordinate plane satisfying $x \geq 0$, $y \geq 0$, and $\left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{y}{3} \right\rfloor = 9$.

Written by: Brian Zhou

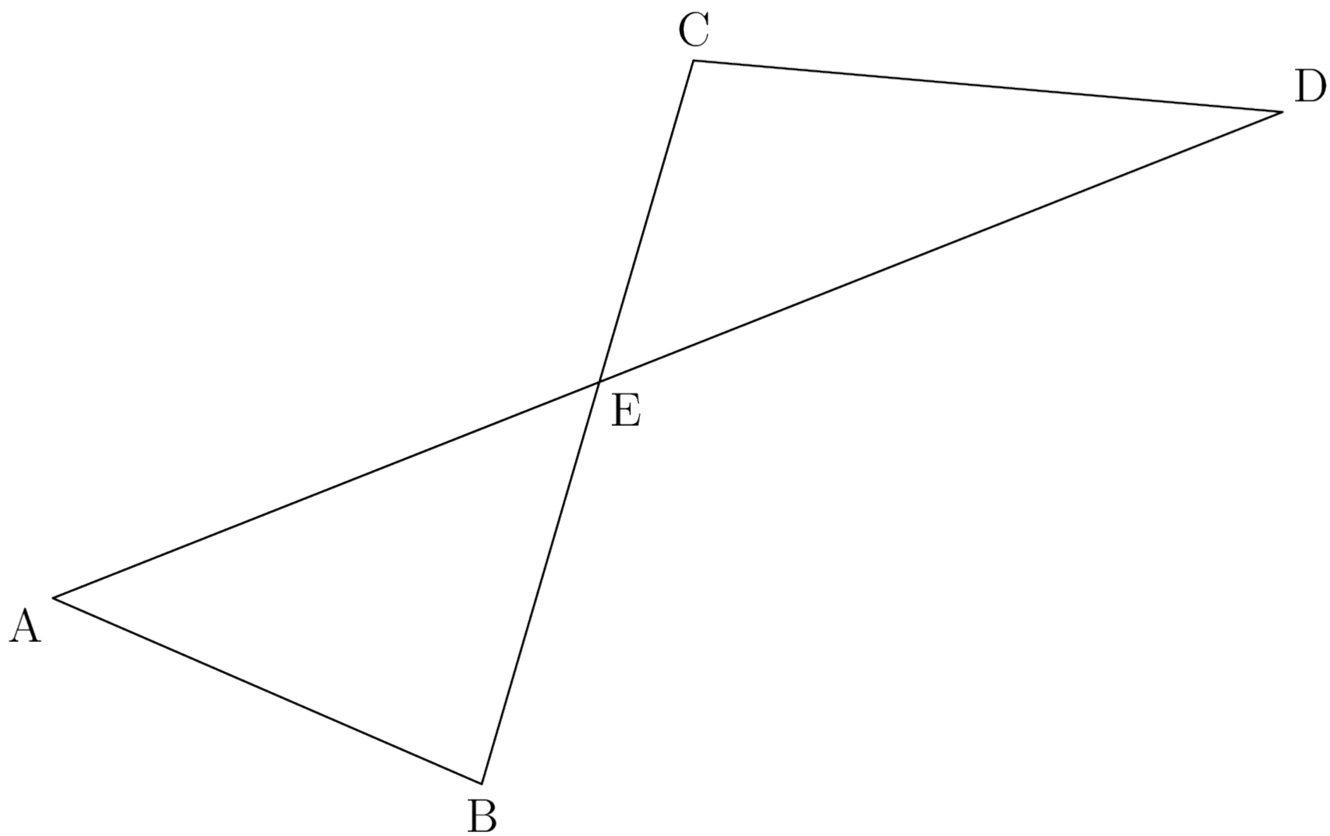
Answer: $\boxed{90}$

Letting $m = \left\lfloor \frac{x}{3} \right\rfloor$ for some nonnegative integer m, we have

$$m \leq \frac{x}{3} < m + 1 \Rightarrow 3m \leq x < 3m + 3.$$

We get the same letting $n = \left\lfloor \frac{y}{3} \right\rfloor$. Thus, each ordered pair (m, n) satisfying $m + n = 9$ corresponds to a square in the coordinate plane with opposite vertices $(3m, 3n)$ and $(3m + 3, 3n + 3)$. Since the side length is 3, each square has an area of 9. There are 10 possible ordered pairs (m, n): $(0, 9), (1, 8), (2, 7) \ldots (9, 0)$, so the total area is $9 \cdot 10 = \boxed{90}$.

17. **DS1034** In the diagram below, E lies on both $\overline{AD}$ and $\overline{BC}$. Given that $\angle DAC = \angle ADB$ and $[ABE] = 2[CDE] - 23$, find $[CDE]$. (Note that $[ABE]$ and $[CDE]$ refer to the areas of triangles $\triangle ABE$ and $\triangle CDE$, respectively.)



Written by: Dev Saxena

Answer: $\boxed{23}$

Since alternate interior angles $\angle DAC$ and $\angle ADB$ are equal, we see that $\overline{AC} \| \overline{BD}$. Draw in $\overline{AC}$, forming two new triangles $\triangle ABC$ and $\triangle CDA$. Now, because the distance from B to $\overline{AC}$ is the same as the distance from D to $\overline{AC}$, and $AC = CA$, we can set the areas of the two new triangles equal:

$$[ABC] = [CDA]$$

Since $[ABC] = [AEC] + [ABE]$ and $[CDA] = [CEA] + [CDE]$

$$[AEC] + [ABE] = [CEA] + [CDE]$$

$$[ABE] = [CDE]$$

Plugging this back into the equation they give us yields $[CDE] = \boxed{23}$.

18. **RV1198** Define the function $f(x) = (4x - 6)^{10}$. If $f(x)$ can be represented as $a_{10}x^{10} + a_9x^9 + \cdots + a_1x + a_0$, let $S = a_9 + a_7 + a_5 + a_3 + a_1$. What is the remainder

when $|S|$ is divided by 1000?

Written by: Rishabh Venkataramani

Answer: $\boxed{488}$

Notice that letting $x = 1$ and $x = -1$ gives us the following equations:

$$f(1) = a_{10} + a_9 + a_8 + a_7 + a_6 + a_5 + a_4 + a_3 + a_2 + a_1 + a_0 = (4 - 6)^{10} = (-2)^{10}$$
$$f(-1) = a_{10} - a_9 + a_8 - a_7 + a_6 - a_5 + a_5 - a_3 + a_2 - a_1 + a_0 = (-4 - 6)^{10} = (-10)^{10}$$

Subtracting the second equation from the first, then dividing by 2, gives us the value of S:

$$S = \frac{(-2)^{10} - (-10)^{10}}{2} = \frac{1024 - 10000000000}{2} = -4999999488.$$

The answer is then $\boxed{488}$.

19. **LT749** A sequence of integers is defined recursively by $a_0 = 0$ and $a_{n+1} = 17a_n + 1$ for all $n \geq 0$. Find the remainder when a_{200} is divided by 2023.

Written by: Linus Tang

Answer: $\boxed{18}$

Notice that for all integers $k \geq 2$, we have $a_k \equiv 18 \pmod{289}$. Furthermore, $a_k \equiv a_{k+6} \pmod 7$ for all integers $k \geq 0$, so $a_{200} \equiv a_2 \equiv 4 \pmod 7$. By the Chinese Remainder Theorem, these two congruences give us a unique residue $\pmod{2023}$, and we find the answer is $\boxed{18}$.

20. **JES1118** Two circles $\odot O_1$ and $\odot O_2$ are drawn such that each circle passes through the center of the other circle. $\odot O_1$ and $\odot O_2$ intersect at points A and B. Points P and Q are chosen independently and uniformly at random on $\odot O_1$ and $\odot O_2$, respectively. What is the probability that line segment $\overline{PQ}$ intersects line segment $\overline{AB}$?

Written by: Jerry Sun

Answer: $\boxed{\frac{4}{9}}$

Let O_1 and O_2 denote the centers of $\odot O_1$ and $\odot O_2$, respectively. We will consider two cases for the location of P: major arc $\overset{\frown}{AB}$ and minor arc $\overset{\frown}{AB}$ of $\odot O_1$.

Case 1: P is on major arc $\overset{\frown}{AB}$, with probability $\frac{2}{3}$. Without loss of generality, assume that $m\overset{\frown}{AP} < m\overset{\frown}{PB}$. Extend $\overline{PA}$ to intersect $\odot O_2$ again at X. Any choice of Q on $\overset{\frown}{BAX}$ (neglecting single points A, B, and X) will cause $\overline{PQ}$ to not intersect $\overline{AB}$, so we need $m\overset{\frown}{BX} = 120° + m\overset{\frown}{AX}$. $\angle APB$ and $\angle AXB$ both inscribe $120°$ arcs, so $m\angle APB = m\angle AXB = 60°$ and triangle $\triangle PXB$ is equilateral. Therefore, $\triangle PO_1B \cong \triangle XO_2B$ by SSS congruence. Let $m\angle PO_1O_2 = \theta$; then $m\angle PO_1B = 300° - \theta$, $m\angle O_1PB =$

$\frac{180^\circ - m\angle PO_1B}{2} = \frac{\theta}{2} - 60^\circ$, $m\angle AXO_2 = 60^\circ + m\angle O_1PB = \frac{\theta}{2}$, and $m\widehat{AX} = m\angle AO_2X = 180^\circ - 2 \cdot m\angle AXO_2 = 180^\circ - \theta$. The probability that Q is chosen on $\widehat{BAX}$ is then $\frac{m\widehat{BX}}{360^\circ} = \frac{300^\circ - \theta}{360^\circ}$, which is linear in θ, so we find its average value by substituting in the average value of θ, which ranges from 60° to 180°: $\frac{300^\circ - 120^\circ}{360^\circ} = \frac{1}{2}$. Therefore, the average probability that Q is not on $\widehat{BAX}$ is $1 - \frac{1}{2} = \frac{1}{2}$.

Case 2: P is on minor arc $\widehat{AB}$, with probability $\frac{1}{3}$. For $\overline{PQ}$ to intersect $\overline{AB}$, Q must be on minor arc $\widehat{AB}$ of $\odot O_2$, with probability $\frac{120^\circ}{360^\circ} = \frac{1}{3}$.

The final answer is $\frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{3} = \boxed{\frac{4}{9}}$.