# Optimization Techniques (MAT-2003)

Lecture-24

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### Degeneracy in the Transportation Problem

# Example:

Find the optimum solution to the following transportation problem in which the cells contain the transportation cost in rupees.

	$W_{I}$	$W_2$	$W_3$	$W_4$	$W_5$	Available
$F_{l}$	7	6	4	5	9	40
$F_2$	8	5	6	7	8	30
$F_{_{\mathfrak{J}}}$	6	8	9	6	5	20
$F_{_{4}}$	5	7	7	8	6	10
Required	30	30	15	20	5	100 (Total)

The initial basic feasible solution using VAM (Vogel's approximation) method is given by:

7		6		4		5		9		40
	(5)				(15)		(20)			40
8		5		6		7		8		
			(20)							30
			(30)							
6		8		9		6		5		
										20
	(15)								(5)	
5		7		7		8		6		
										10
	(10)									
	30		30		15		15	5	5	

Here, the number of allocations are 7.

But the value of m + n - 1 = 4 + 5 - 1 = 8.

Therefore, the number of allocations = 7 < m + n - 1.

The transportation problem is a degeneracy case.

To overcome the degeneracy, we need to add an extra allocation to the problem.

Therefore, here we are adding a positive infinitesimal allocation  $(\epsilon > 0)$  to satisfy the number of allocations equals to m+n-1 at the minimum cost of the un-allocated cell in such a way that the allocations must be independent.

In the given cost matrix the minimum cost among the un-allocated cells is 6 which is at four different locations (1,2), (2,3), (3,4), and (4,5) respectively.

Out of which if we allocate at (3,4) and (4,5) locations, the allocations becomes dependent (forms a closed loop). But, if we allocate at (1,2),(2,3) locations, the allocations becomes independent (doesn't form a closed loop).

Therefore, we have two possibilities where we allocate a new infinitesimal allocation at (1,2) and (2,3) locations. Now, we are allocating  $\epsilon$  at the location (2,3).

7		6		4		5		9	
	(5)				(15)		(20)		
8		5		6		7		8	
			(30)		$(\epsilon)$				
6		8		9		6		5	
	(15)								(5)
5		7		7		8		6	
	(10)								

The number of allocations = 8 = m + n - 1 (= 4 + 5 - 1) and are also at independent positions. Therefore, we can now check whether the allocations are optimal or not.

For the optimal solution, we will apply MODI method or U-V method.

Now, we need to find  $u_i$  and  $v_j$  values corresponding to the allocated cells such that  $u_i + v_j = c_{ij}$ .

Further we need to calculate the netevaluations:

7	6	4	5	9	$\cap$
(5)		(15)	(20)		U
8	5	6	7	8	2
	(30)	$(\epsilon)$			
6	8	9	6	5	<b>—</b> 1
(15)				(5)	
5	7	7	8	6	<b>-</b> 2
(10)					— <b>Z</b>

 $u_i$ 

 $v_i$  7 3 4 5 6

Now, the net evaluations for the non-allocated cells are calculated using the formula

$$c_{ij} - z_{ij} = (u_i + v_j - c_{ij})$$

cell 
$$(1,2) \rightarrow c_{12} - z_{12} = c_{12} - (u_1 + v_2) = 6 - (0+3) = 3 \ge 0$$
  
 $(1,5) \rightarrow c_{15} - z_{15} = c_{15} - (u_1 + v_5) = 9 - (0+6) = 3 \ge 0$   
 $(2,1) \rightarrow c_{21} - z_{21} = c_{21} - (u_2 + v_1) = 8 - (7+2) = -1 \ge 0$   
 $(2,4) \rightarrow c_{24} - z_{24} = c_{24} - (u_2 + v_4) = 7 - (5+2) = 0 \ge 0$   
 $(2,5) \rightarrow c_{25} - z_{25} = c_{25} - (u_2 + v_5) = 8 - (6+2) = 0 \ge 0$   
 $(3,2) \rightarrow c_{32} - z_{32} = c_{32} - (u_3 + v_2) = 8 - (3 + (-1)) = 6 \ge 0$   
 $(3,3) \rightarrow c_{33} - z_{33} = c_{33} - (u_3 + v_3) = 9 - (4 + (-1)) = 6 \ge 0$   
 $(3,4) \rightarrow c_{34} - z_{34} = c_{34} - (u_3 + v_4) = 6 - (5 + (-1)) = 2 \ge 0$ 

$$(4,2) \rightarrow c_{42} - z_{42} = c_{42} - (u_4 + v_2) = 7 - (3 + (-2)) = 6 \ge 0$$
  
 $(4,3) \rightarrow c_{43} - z_{43} = c_{43} - (u_4 + v_3) = 7 - (4 + (-2)) = 5 \le 0$ 

$$(4,4) \rightarrow c_{44} - z_{44} = c_{44} - (u_4 + v_4) = 8 - (5 + (-2)) = 5 \le 0$$

$$(4,5) \rightarrow c_{45} - z_{45} = c_{45} - (u_4 + v_5) = 6 - (6 + (-2)) = 2 \le 0$$

The net-evaluation for the cell (2,1) is < 0. Which fails to satisfy optimality test. Therefore, we need to improve the solution.

To do that, we need to pick the cell (2,1) which is not satisfying the optimality test (or among all such cells we need to take the maximum positive net-evaluation) there we have to make a new allocation (becomes a basic variable).

To make the new allocation we need to draw a loop starting from the cell (2,1) with the help of horizontal and vertical jumps through the allocated cells.

7 0	6	4 28	5	9
(5)	-7	(15)	(20)	
8 ^	5	6 <b>Y</b>	7	8
×9•	(30)	( <i>\epsilon</i> )		
6	8	9	6	5
(15)				(5)
5	7	7	8	6
(10)				

$$\theta = \min\{5, \epsilon\} = \epsilon$$

We need to add  $\epsilon$  at  $+\theta$  and subtract at  $-\theta$  locations.

7	6	4	5	9
(5)		(15)	(20)	
8	5	6	7	8
$(\epsilon)$	(30)			
6	8	9	6	5
(15)				(5)
5	7	7	8	6
(10)				

Now, again we need to check whether the allocation is optimal or not.

					$u_i$
7	6	4	5	9	
(5)		(15)	(20)		0
8	5	6	7	8	4
$(\epsilon)$	(30)				1
6	8	9	6	5	4
(15)				(5)	<b>-</b> 1
5	7	7	8	6	
(10)					<b>-</b> 2
_					
7	4	4	5	6	

Now, we need to calculate the netevaluations for the un-allocated cells. The net evaluations for the non-allocated cells are calculated using the formula

$$c_{ij} - z_{ij} = c_{ij} - (u_i + v_j)$$

cell 
$$(1,2) \rightarrow c_{12} - z_{12} = c_{12} - (u_1 + v_2) = 6 - (0 + 4) = 2 \ge 0$$
  
 $(1,5) \rightarrow c_{15} - z_{15} = c_{15} - (u_1 + v_5) = 9 - (0 + 6) = 3 \ge 0$   
 $(2,3) \rightarrow c_{23} - z_{23} = c_{23} - (u_2 + v_3) = 6 - (4 + 1) = 1 \ge 0$   
 $(2,4) \rightarrow c_{24} - z_{24} = c_{24} - (u_2 + v_4) = 7 - (5 + 1) = 1 \ge 0$   
 $(2,5) \rightarrow c_{25} - z_{25} = c_{25} - (u_2 + v_5) = 8 - (6 + 1) = 1 \ge 0$   
 $(3,2) \rightarrow c_{32} - z_{32} = c_{32} - (u_3 + v_2) = 8 - (4 + (-1)) = 5 \ge 0$   
 $(3,3) \rightarrow c_{33} - z_{33} = c_{33} - (u_3 + v_3) = 9 - (4 + (-1)) = 6 \ge 0$   
 $(3,4) \rightarrow c_{34} - z_{34} = c_{34} - (u_3 + v_4) = 6 - (5 + (-1)) = 2 \ge 0$ 

$$(4,2) \to c_{42} - z_{42} = c_{42} - (u_4 + v_2) = 7 - (4 + (-2)) = 5 \ge 0$$

$$(4,3) \to c_{43} - z_{43} = c_{43} - (u_4 + v_3) = 7 - (4 + (-2)) = 5 \ge 0$$

$$(4,4) \to c_{44} - z_{44} = c_{44} - (u_4 + v_4) = 8 - (5 + (-2)) = 5 \ge 0$$

$$(4,5) \to c_{45} - z_{45} = c_{45} - (u_4 + v_5) = 6 - (6 + (-2)) = 2 \ge 0$$

All the net evaluations are  $\geq 0$ . Therefore, we obtain the optimal allocation.

The optimal (minimum) cost of the transportation problem is given by

$$= 7 \times 5 + 4 \times 15 + 5 \times 20 + 8 \times \epsilon + 5 \times 30 + 6 \times 15 + 5 \times 5 + 5 \times 10 = 510 + 8\epsilon$$

As we know  $\epsilon \to 0$ 

Optimal cost = 510.

# Example:

Given  $x_{13} = 50$  units,  $x_{14} = 20$  units,  $x_{21} = 55$  units,  $x_{31} = 30$  units,  $x_{32} = 35$  units,  $x_{34} = 25$  units. Is it an optimal solution to the transportation problem:

				Av	ailable units
The second secon	6	1	9	3 ]	70
	11	5	2	8	55
E. C. P. T. B. Bright	10	12	4	7	90
Required units:	85	35	50	45	

If not, modify it to obtain a better feasible solution.

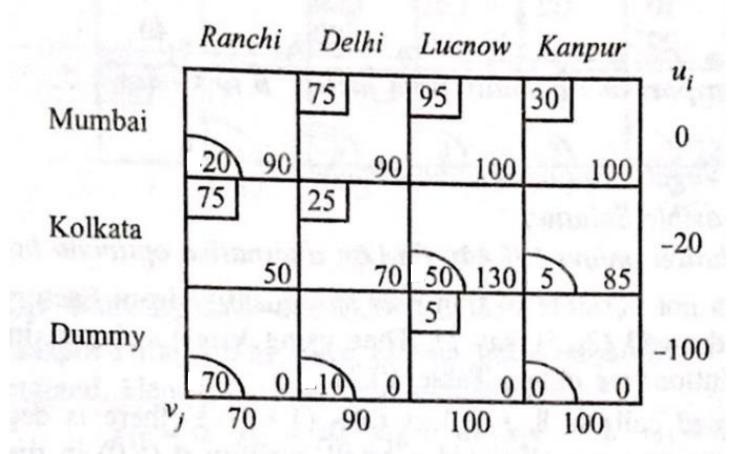
6	1	9		3		70
			(50)		(20)	70
11	5	2		8		55
(55)						33
10	12	4		7		90
(30)	(35)				(25)	30
85	35		50		45	

Do it yourself.

#### Alternative Optimal solution:

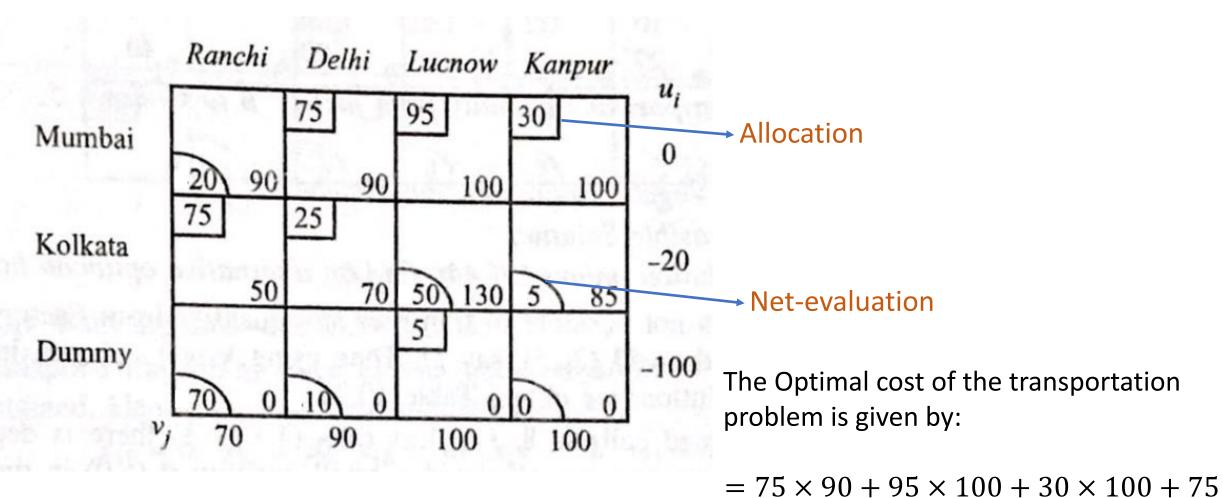
For the optimal solution of transportation problem if at least one net-evaluation of the unallocated cell is 0, then the transportation problem will have the alternative optimal solution.

Which means, the allocations may change however the optimal cost remains same.

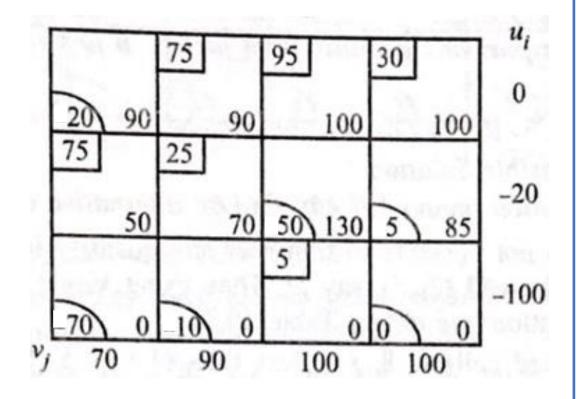


## Find the Alternate Optimal Solution of the following transportation problem:

The final allocations and the net-evaluations are given along with the following cost matrix.



 $\times 50 + 25 \times 70 + 5 \times 0 = 24750$ 



The  $u_i$  and  $v_j$  values corresponding to the basic variables are given above. Which were calculated using the relation

$$u_i + v_j = c_{ij}.$$

The net-evaluations are also given in the cost matrix which are in the cost matrix at the bottom left corner of the cell in the cost matrix.

Among all the net-evaluations the net evaluation in cell (3,4) is 0, which indicate an alternative optimal solution to the transportation problem.

To find the alternative optimal solution of the transportation problem, we need to make a new allocation at the cell having the zero (i.e., cell (3,4)) net-evaluation.

To do that, we need to draw a closed loop starting from the cell (3,4) with the help of horizontal and vertical jumps through the allocated cells.

90		90		100		100	Ģ
			75	Í	95	- 1	30
50		70		130		85	
						i	
	75		25				
0		0		0		0	
				9			9
					5		8

$$\theta = \min\{30,5\} = 5$$

Which we need add at  $+\theta$  and subtract at  $-\theta$  positions.

90		90		100		100	
			75		100		25
50		70		130		85	
	75		25				
0		0		0		0	
							5

Again this solution is going to be a optimal solution. And the optimal cost is given by:

$$= 75 \times 90 + 95 \times 100 + 30 \times 100 + 75 \times 50 + 25 \times 70 + 5 \times 0 = 24750$$

It is clear from the two solutions that the allocation positions are different but the optimal cost remains same.