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Optimization Techniques (MAT-2003)

Lecture-27

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The Travelling Salesman Problem:

The travelling salesman problem has the following Assumptions:

The travelling salesman has to visit n cities and return to the starting point. He has to start from any one of the city and visit each city only once and finally comes back to the initial city from where he started the tour.

Examples: 1) Postal deliveries 2) Inspection 3) School bus routing etc.

Suppose he starts from the k^{th} city and the last city he visited is m.

Mathematical formulation of the travelling salesman problem:

If c_{ij} is the cost going from the city i to the city j and $x_{ij} = 1$, if the salesman directly goes from city i to the city j and zero otherwise. Then the problem is to find x_{ij} which minimize

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij},$$

Subject to
$$\sum_{j=1}^{n} x_{ij} = 1,$$

$$\sum_{i=1}^{n} x_{ij} = 1,$$

and
$$x_{ij} = 0 \text{ or } 1; \; i = 1, 2, \ldots, n; \; j = 1, 2, \ldots, n,$$

with the additional (two) constraints that no city is to be visited twice before tour of all the cities is completed and that going from city i to the city i is not permitted, which mean $c_{ii} = \infty$.

Solution Procedure:

- First, solve the problem as an assignment using Hungarian method.
- If the solutions thus found are cyclic in nature, then that is final solution.
- If it is not cyclic, then select the lowest entry in the table (other than zero)
- Delete the row and column of this lowest entry and again do the zero assignment in the remaining matrix.
- Check whether cyclic assignment is available .
- If not, include (Assign) the next lowest entry in the table and procedure is repeated until a cyclic assignment is obtained.

Example:

A machine operator processes five types of items on his machine each week, and must choose a sequence for them. The set-up cost per change depends on the item presently on the machine and the set-up to be made according to the following table :

From item	To item				
	A	B	C	D	E
A	∞	4	7	3	4
B	4	∞	6	3	4
C	7	6	∞	7	5
D	3	3	7	∞	7
E	4	4	5	7	∞

If he processes each type of item once and only once each week, how should he sequence the items on his machine in order to minimize the total set-up cost?

$$
\begin{matrix}
\infty & 4 & 7 & 3 & 4 \\
4 & \infty & 6 & 3 & 4 \\
7 & 6 & \infty & 7 & 5 \\
3 & 3 & 7 & \infty & 7 \\
4 & 4 & 5 & 7 & \infty
\end{matrix}
$$

Step 1: pick the row minima from each row and subtract from the every element of the respective rows.

$$
\begin{matrix}
\infty & 1 & 4 & 0 & 1 \\
1 & \infty & 3 & 0 & 1 \\
2 & 1 & \infty & 2 & 0 \\
0 & 0 & 4 & \infty & 4 \\
0 & 0 & 1 & 3 & \infty
\end{matrix}
$$

Step 2: pick the column minima from each column and subtract from the every element of the respective columns.

$$\begin{bmatrix} \infty & 1 & 3 & 0 & 1 \\ 1 & \infty & 2 & 0 & 1 \\ 2 & 1 & \infty & 2 & 0 \\ 0 & 0 & 3 & \infty & 4 \\ 0 & 0 & 0 & 3 & \infty \end{bmatrix}$$

Step 3: We need to make assignments to rows and columns having single zero and cross out the zeros in the corresponding assigned row or column respectively.

$$\begin{bmatrix} \infty & 1 & 3 & \boxed{0} & 1 \\ 1 & \infty & 2 & \cancel{0} & 1 \\ 2 & 1 & \infty & 2 & \boxed{0} \\ \boxed{0} & \cancel{0} & 3 & \infty & 4 \\ \cancel{0} & \cancel{0} & \boxed{0} & 3 & \infty \end{bmatrix}$$

Now the total number of assignments are 4 which are less than the order of the matrix $(= 5)$. Therefore, we need to increase the number of assignments.

Step 5: We need to increase the number of zeros to increase the number of assignments in the matrix. To do that, we need to mark the un-assigned rows. In the marked rows, if there are any crossed zeros their respective columns also need to mark. Further if we have any assigned zeros in the marked columns, we need to mark those respective rows and so on.

$$\begin{array}{|ccccc|c}
\cline{1-5}
\infty & 1 & 3 & \boxed{0} & 1 & \checkmark \\
1 & \infty & 2 & \cancel{0} & 1 & \checkmark \\
2 & 1 & \infty & 2 & \boxed{0} & \\
\boxed{0} & \cancel{0} & 3 & \infty & 4 & \\
\cancel{0} & \cancel{0} & \boxed{0} & 3 & \infty & \\
\cline{1-5}
\multicolumn{1}{c}{} & & & \checkmark & \multicolumn{1}{c}{} &
\end{array}$$

Step 6: Now we need to draw the lines over the un-marked rows and marked columns (which are the minimum number of lines that we can draw to cover all the zeros of the matrix).

$$
\begin{array}{c c}
\begin{bmatrix}
\infty & 1 & 3 & \boxed{0} & 1 \\
1 & \infty & 2 & \cancel{0} & 1 \\
2 & 1 & \infty & 2 & \boxed{0} \\
\boxed{0} & \cancel{0} & 3 & \infty & 4 \\
\cancel{0} & \cancel{0} & \boxed{0} & 3 & \infty
\end{bmatrix} &
\begin{array}{c} \checkmark \\ \checkmark \\ \\ \\ \\ \end{array} \\
\begin{array}{ccccc} & & & \checkmark & \end{array} &
\end{array}
$$

Step 7: Now from the un-covered elements by the lines, we need to pick the minimum one and subtract it from all un-covered elements and add the same minimum element to the intersections of the horizontal and vertical lines.

	A	B	C	D	E
A	∞	0	2	0	0
B	0	∞	1	0	0
C	2	1	∞	3	0
D	0	0	3	∞	4
E	0	0	0	4	∞

Step 8: Now we need to make assignments to the rows and (or) or columns having single zeros.

	A	B	C	D	E
A	∞	0	2	0	~~0~~
B	0	∞	1	0	~~0~~
C	2	1	∞	3	$\boxed{0}$
D	0	0	3	∞	4
E	~~0~~	~~0~~	$\boxed{0}$	4	∞

In the rest of the rows and columns, we have multiple zeros. Therefore, we need to make an assignment at any row or column arbitrarily.

	A	B	C	D	E
A	∞	$\boxed{0}$	2	$\cancel{0}$	$\cancel{0}$
B	$\cancel{0}$	∞	1	$\boxed{0}$	$\cancel{0}$
C	2	1	∞	3	$\boxed{0}$
D	$\boxed{0}$	$\cancel{0}$	3	∞	4
E	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	4	∞

Here, the number of assignments are 5 which are equal to the order of the matrix.

Therefore this is the solution of the assignment problem.

The optimal assignments are

$A \to B, B \to D, C \to E, D \to A, E \to C.$

However, This is not the solution of the travelling salesman problem. Because,

$A \to B \to D \to A$, i.e., without covering the items C and E returned to the initial item (A).

Therefore, we need to improve the solution to satisfy the requirement of the travelling salesman problem.

To improve the solution we need to make a new assignment to the next minimum cost (i.e., 1 in this case) of the matrix, and at the same time, we need to cross out all the zeros (including assigned) of the respective and row and column.

We have 1 at two different locations in the matrix where we can make the assignment. Therefore, we have to check both the possibilities to obtain the optimal solution.

Let us make a new assignment at location (3,2) and cross-out all the zeros in second row and third column (including assigned zeros).

	A	B	C	D	E
A	∞	$\cancel{0}$	2	0	0
B	$\cancel{0}$	∞	1	0	0
C	2	$\boxed{1}$	∞	3	$\cancel{0}$
D	$\boxed{0}$	$\cancel{0}$	3	∞	4
E	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	4	∞

Now, we left with multiple zeros in the remaining rows and columns. Therefore, we need to start assignment process by making an arbitrary assignment in any row or column.

Let us make an assignment at cell $(2,4)$ and cross-out the zeros at $(2,5)$ and $(1,4)$. Finally, we left with a single zero at $(1,5)$ where we can make our final assignment.

	A	B	C	D	E
A	∞	$\cancel{0}$	2	$\cancel{0}$	$\boxed{0}$
B	$\cancel{0}$	∞	1	$\boxed{0}$	$\cancel{0}$
C	2	$\boxed{1}$	∞	3	$\cancel{0}$
D	$\boxed{0}$	$\cancel{0}$	3	∞	4
E	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	4	∞

Now, the total number of assignments are again 5 equal to the order of the matrix. Further, check the assignments are in cyclic or not.

$$A \to E \to C \to \mathrm{B} \to \mathrm{D} \to \mathrm{A}$$

The Assignments are in cyclic. Therefore, the this is solution can acceptable for the travelling salesman problem.

The Optimal (minimum) cost is given by

$$= 4 + 3 + 6 + 3 + 5 = 21.$$

Note: The optimal value of the travelling sales problem is greater then or equal to the optimal value of the simple Assignment problem.

Example:

Solve the following travelling salesman problem so as to minimize the cost per cycle.

From	To				
	A	*B*	*C*	*D*	*E*
A	—	3	6	2	3
B	3	—	5	2	3
C	6	5	—	6	4
D	2	2	6	—	6
E	3	3	4	6	—