# Optimization Techniques (MAT-2003)

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Convex linear combination: A linear combination of x and y i.e., ax + by is said to be convex linear combination if  $0 \le a, b \le 1$  and a + b = 1.

Dominance property: In some games, it is possible reduce the size of the payoff matrix by eliminating redundant rows(or columns). If a game has such redundant rows(or columns), those rows or columns are dominated by some other rows(or columns), respectively. Such property is known as dominance property.

## Dominance property by rows:

- ❖ In the payoff matrix of player A, if all the entries in a row (say X) are less than or equal to the corresponding entries of another row(say Y)(i.e.,  $X \le Y$ ), then row X is dominated by row Y. Under such situation, row X of the payoff matrix can be deleted.
- $\clubsuit$  if some convex linear combination of some rows(X and Y) dominate Z row, then the Z row will be deleted. Here, elements of Z row dominated by convex linear combination of X and Y.

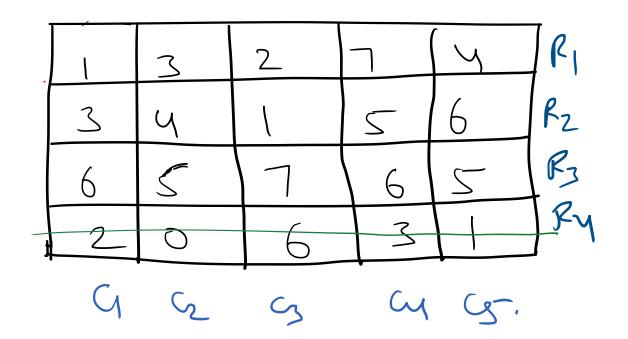
### Dominance property for columns:

- ❖ In the payoff matrix of player A, if all the entries in a column (say X) are greater than or equal to the corresponding entries of another column (say Y) (i.e.,  $X \ge Y$ ), then column X is dominated by column Y. Under such situation, dominated column X of the payoff matrix can be deleted.
- $\clubsuit$  if some convex linear combination of some column (X and Y) dominate Z column, then the Z column will be deleted. Here, elements of Z column dominated by convex linear combination of X and Y.

## Example:

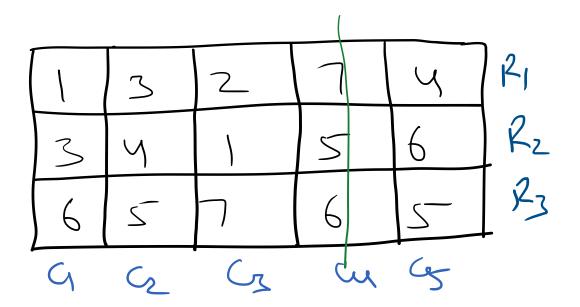
Solve the following payoff matrix using dominance property

			Player B				
		1	2	3	4	5	
	I	I	3	2	7	4	
Player A	II	3	4	1	5	6	
	III	6	5	7	6	5	
	IV	2	0	6	3	1	



Here,  $R_4 \leq R_3$ , therefore by row dominance property we can delete  $R_4$ .

The resultant payoff matrix becomes:

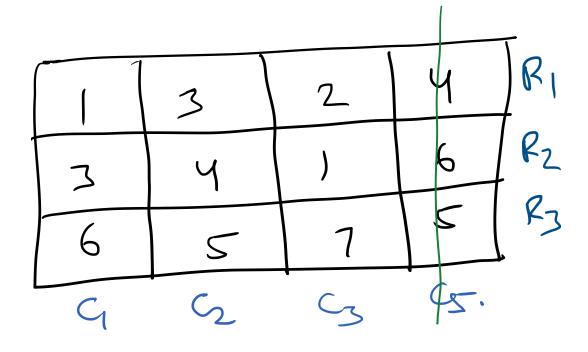


In the above payoff matrix, no two rows are dominated by one another.

Let us look for column dominance property.

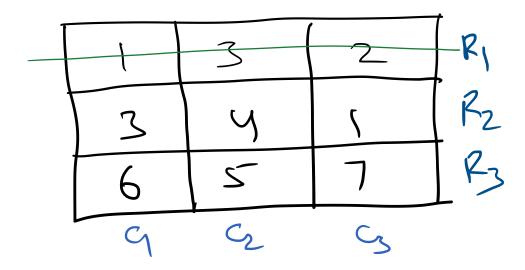
Here we have  $C_4 \ge C_1$ , therefore we can delete  $C_4$ 

The resultant payoff becomes:



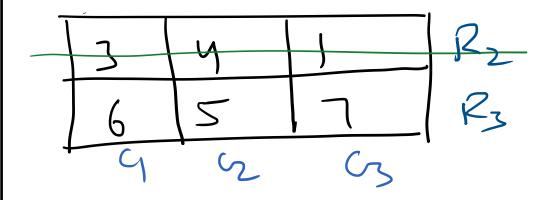
In the above payoff,  $C_5 \ge C_2$ . Therefore, we can delete  $C_5$ .

The payoff matrix becomes:



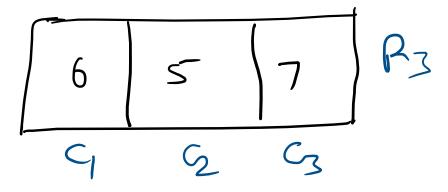
In the above payoff matrix,  $R_1 \leq R_3$ . Therefore, delete  $R_1$  by row dominance property.

The payoff becomes:

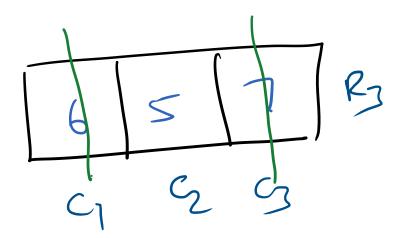


In the above payoff matrix,  $R_2 \le R_3$ . Therefore, delete  $R_3$  by row dominance property.

The payoff becomes:



Now, we have only one row. Therefore we can delete maximum column elements using column dominance property.



Here, we left with only one value which is the saddle point of the game and its value is the value of the game

The game is a pure strategy game.

The optimal strategies of the players A and B are *III* and 2 respectively.

Player A is winning the game.

## Example:

Solve the following payoff matrix using dominance property

		Player B				
		$B_1$	$B_2$	$B_3$	$B_4$	$B_5$
Player $A$	$A_1$	4	6	5	10	6
	$A_2$	7	8	5	9	10
	$A_3$	8	9	11	10	9
	$A_4$	6	4	10	6	4

Δ	6	5	10	6	$-R_1$
•			10		- <sub>1</sub> 1
7	8	5	9	10	$R_2$
8	9	11	10	9	$R_3$
6	4	10	6	4	$R_4$
$C_1$	$C_2$	$C_3$	$C_4$	$C_5$	•

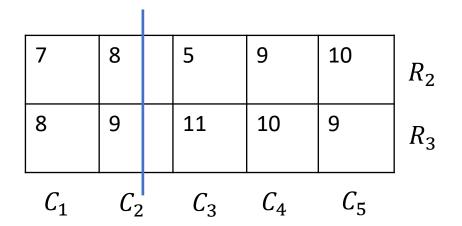
In the above payoff matrix  $R_1 \leq R_3$ , therefore, delete  $R_1$  using row dominance property.

The payoff matrix becomes.

7	8	5	9	10	$R_2$
8	9	11	10	9	$R_3$
6	4	10	6	4	- R <sub>4</sub>
$C_1$	$C_2$	$C_3$	$C_4$	$C_5$	I

In the above payoff matrix  $R_4 \le R_3$ , therefore, delete  $R_4$  using row dominance property.

The payoff matrix becomes.



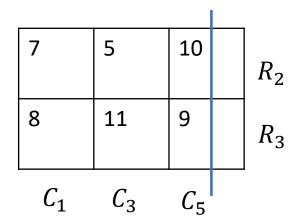
In the above payoff matrix we have  $C_2 \ge C_1$ . Therefore, by column dominance property delete  $C_2$ .

The payoff matrix becomes:

7	5	9	10	$R_2$
8	11	10	9	$R_3$
$C_1$	$C_3$	$C_4$	$C_5$	_

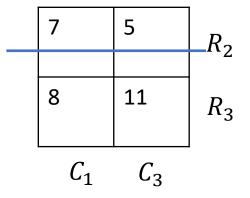
In the above payoff matrix we have  $C_4 \ge C_1$ . Therefore, by column dominance property delete  $C_4$ .

The payoff matrix becomes:



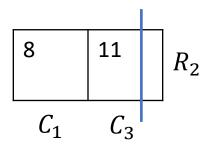
In the above payoff matrix we have  $C_5 \ge C_1$ . Therefore, by column dominance property delete  $C_5$ .

The payoff matrix becomes:



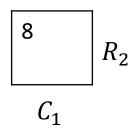
In the above payoff matrix we have  $R_2 \le R_3$ . Therefore, by row dominance property delete  $R_2$ .

The payoff matrix becomes:



In the above payoff matrix we have  $C_3 \ge C_1$ . Therefore, by column dominance property delete  $C_3$ .

The payoff matrix becomes:



Therefore, the game has a pure strategy and the players A and B are coming up with the strategies  $A_2$  and  $B_2$  respectively.

4	6	5	10	6	Saddle point
7	8 _	5	9	10	
8	9	11	10	9	
6	4	10	6	4	

The value of the game V=8

Player *A* is winning the game.

### Mixed Strategy using Dominance property:

## Example:

Solve the following payoff matrix using dominance property

		Player B			
		I	II	III	IV
	I	3	2	4	0
Player A	II	3	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

	3	2	4	0	$R_1$
	3	4	2	4	$R_2$
	4	2	4	0	$R_3$
	0	4	0	8	$R_4$
,	$C_1$	$C_2$	$C_3$	$C_4$	

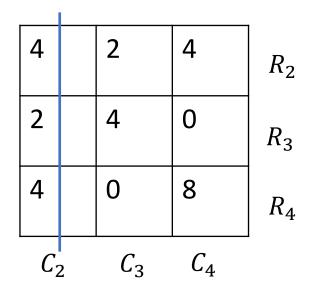
In the above payoff matrix  $R_1 \leq R_3$ , therefore, delete  $R_1$  using row dominance property.

The payoff matrix becomes.

		•	ı	-	Ī
3		4	2	4	$R_2$
4		2	4	0	$R_3$
					113
0		4	0	8	D
					$R_4$
	$C_1$	$C_2$	$C_3$	$C_4$	!
	<b>σ</b> Ι	<u> </u>	03	<b>5</b> 4	

In the above payoff matrix  $C_1 \ge C_3$ , therefore, delete  $C_1$  using column dominance property.

The payoff matrix becomes.



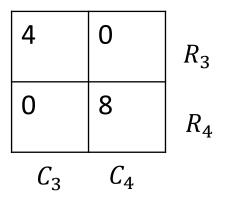
In the above payoff matrix  $C_2 \geq \frac{1}{2}C_3 + \frac{1}{2}C_4$ . Therefore, delete  $C_2$  using column dominance property.

The payoff matrix becomes.

2	4	
4	0	
0	8	

In the above payoff matrix  $R_1 \le \frac{1}{2}R_2 + \frac{1}{2}R_3$ . Therefore, delete  $R_1$  using row dominance property.

The payoff matrix becomes.



Here, the game has no strategy and it is a  $2 \times 2$  pay-off matrix.

Therefore, we know formulae for finding the probabilities of the strategies and value of the game.

Let  $p_3$  and  $p_4$  are the probabilities of the player A to chose the strategies IIIand IV.

Let  $q_3$  and  $q_4$  are the probabilities of the player B to chose the strategies III and IV.

$$p_{3} = \begin{bmatrix} q_{3} & q_{4} \\ 4 & 0 \\ a & b \\ p_{4} & 0 \\ c & d \end{bmatrix} |0 - 8| = 8$$

$$p_{4} = \begin{bmatrix} 0 & 8 & |4 - 0| = 4 \\ c & d \end{bmatrix}$$

$$|0 - 8| = 8 \quad |0 - 4| = 4$$

$$p_3 = \frac{|c-d|}{|a-b|+|c-d|} = \frac{8}{8+4} = \frac{2}{3}$$

$$p_4 = \frac{|a-b|}{|a-b|+|c-d|} = \frac{4}{8+4} = \frac{1}{3}$$

$$q_3 = \frac{|b-d|}{|a-c|+|b-d|} = \frac{8}{8+4} = \frac{2}{3}$$

$$q_4 = \frac{|a-c|}{|a-c|+|b-d|} = \frac{4}{8+4} = \frac{1}{3}$$

And the value of the game is

$$V = \frac{a|c - d| + c|a - b|}{|a - b| + |c - d|}$$

$$= \frac{4|0 - 8| + 0|4 - 0|}{|0 - 8| + |4 - 0|}$$

$$= \frac{32}{12} = \frac{8}{3}$$

Here, the players A and B are using mixed strategies.

Player A is using III and IV strategies with the probabilities  $\frac{2}{3}$  and  $\frac{1}{3}$ .

Player *B* is using *III* and *IV* strategies with the probabilities  $\frac{2}{3}$  and  $\frac{1}{3}$ .

The value of the game,  $V = \frac{8}{3}$ 

Player A is winning the game.