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Optimization Techniques (MAT-2003)

Lecture-20

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Definitions:

1. **Feasible Solution:** A feasible solution to a transportation problem is a set of non-negative allocations, x_{ij} if that satisfies the rim (row and column) restrictions.

2. **Basic Feasible Solution:** A feasible solution to a transportation problem is said to be a basic feasible solution if it contains no more than $m+n-1$ non-negative allocations, where m is the number of rows and n is the number of columns of the transportation problem.

3. **Optimal Solution:** A feasible solution (not necessarily basic) that minimizes (maximizes) the transportation cost (profit) is called an optimal solution.

4. Non-Degenerate Basic Feasible Solution: A basic feasible solution to a $(m \times n)$ transportation problem is said to be non-degenerate if,

(a) the total number of non-negative allocations is exactly $m + n - 1$ (*i.e.*, number of independent constraint equations), and

(b) these $m + n - 1$ allocations are in independent positions.

5. Degenerate Basic Feasible Solution: A basic feasible solution in which the total number of non-negative allocations is less than $m + n - 1$ is called degenerate basic feasible solution.

Note: LPP generated by a transportation model with m sources and n destinations will have $m + n - 1$ basic variables and remaining $(m - 1)(n - 1)$ non-basic variables. Thus, a balanced transportation with m sources and n destinations will have at most

$$\binom{mn}{m+n-1} = \frac{(mn)!}{(m+n-1)!\,(mn-m-n+1)!} \text{ basic feasible solutions.}$$

Example:

A dairy firm has three plants located in a state. Daily milk production at each plant is as follows:

Plant 1 ... 6 million litres,
plant 2 ... 1 million litres, and
plant 3 ... 10 million litres.

Each day the firm must fulfil the needs of its four distribution centres. Milk requirement at each centre is as follows:

Distribution centre 1 ... 7 million litres,
distribution centre 2 ... 5 million litres,
distribution centre 3 ... 3 million litres, and
distribution centre 4 ... 2 million litres.

Cost of shipping one million litres of milk from each plant to each distribution centre is given in the following table in hundreds of rupees:

		Distribution centres			
		1	2	3	4
	1	2	3	11	7
Plants	2	1	0	6	1
	3	5	8	15	9

					Supply
	2	3	11	7	6
	1	0	6	1	1
	5	8	15	9	10
Demand	7	5	3	2	

Cost matrix

Solution of a transportation Model:

The solution involves making a transportation table (in the form of a matrix), finding a feasible solution, performing optimality test and iterating towards optimal solution if required.

Finding the Feasible Solution of a transportation Problem:

There are several methods available for finding the initial feasible solution. Which are as follows,

1) North-West corner method
2) Least cost method
3) Row minima Method
4) Column minima method
5) Vogel's approximation method

North-west corner method

Procedure for North-West Corner Method:

Step 1: After the formation of the transportation matrix, start in the north-west (upper left) corner of the requirements table and compare the supply of plant 1 (call it S_1) with the requirement of distribution centre 1 (call it D_1).

(a) If $D_1 < S_1$ *i.e.*, if the amount required at D_1 is less than the number of units available at S_1, set x_{11} equal to D_1, find the balance supply and demand and proceed to cell (1, 2) (*i.e.*, proceed horizontally)

(b) If $D_1 = S_1$, set x_{11} equal to $D_1(S_1)$, compute the balance supply and demand and proceed to cell (2, 2) (*i.e.*, proceed diagonally).

(c) If $D_1 > S_1$, set x_{11} equal to S_1, compute the balance supply and demand and proceed to cell (2, 1) (*i.e.*, proceed vertically).

Step 2: Continue in this manner, step by step, away from the north-west comer until, finally, a value is reached in the south-east corner.

North-west corner method

Example:

Distribution centres

		1	2	3	4	*Supply*
	1	2	3	11	7	6
Plants	2	1	0	6	1	1
	3	5	8	15	9	10
Requirement		7	5	3	2	

2	3	11	7	6
1	0	6	1	1
5	8	15	9	10
7	5	3	2	17 / 17

total supply $= 6+1+10$

$= 17.$

total demand

$= 7+5+3+2$

$= 17.$

total supply = total Demand

$\Rightarrow$ The given transportation problem is balanced.

Applying north-west corner method.

2 $\boxed{6}$	3	11	7	~~6~~ 0
1	0	6	1	1
5	8	15	9	10
~~7~~ 1	5	3	2	

Start from the north-west corner cell i.e., (1,1) cell. There we have to make our first allocation (minimum of 6, 7 and subtract the minimum value from the other). The respective supply is full-filled completely. Therefore, we need to cross-out the respective row (which implies we should not make any further allocations in that row).

After removing the first row the cost matrix becomes:

1 $\boxed{1}$	0	6	1	~~1~~ 0
5	8	15	9	10
~~1~~ 0	5	3	2	

Next north-west corner cell is second row, first column (2,1) of the original matrix. Here, we have make allocation (minimum of 1, 1).

Now, the supply and demand corresponding to the cell (2,1) is same (1) only. Therefore, cross-out both the corresponding row and column. After removing corresponding row and column, the matrix becomes:

8 $\boxed{5}$	15	9	~~10~~ 5
~~5~~ 0	3	2	

Next, north-west corner cell is (3,2) of the original matrix. There we have to make the next allocation (minimum of 5, 10 and subtract the minimum from the other).

After removing the corresponding row the matrix becomes:

15 $\boxed{3}$	9 $\boxed{2}$	~~5~~ ~~2~~ 0
~~3~~ 0	~~2~~ 0	

Next, north-west corner cell is (3,3) of the original matrix. There we have to make the next allocation (minimum of 5, 3 and subtract this minimum value from the other) and cross-out the respective column. Therefore, we remain with only one cell there we have to make the allocation.

Total cost = $6 \times 2 + 1 \times 1 + 8 \times 5 + 15 \times 3 + 9 \times 2 = 116.$

Least-cost method or Matrix minima Method

Procedure for Least cost Method:

Step1: Find the smallest cost in the transportation table, let it be c_{ij}. Allocate $x_{ij} =$ minimum $(\mathrm{a_i}, \mathrm{b_j})$ in the cell (i, j).

Step 2: If $x_{ij} = a_i$ cross off the i^{th} row of the transportation table and decrease b_j by a_i. Go to Step 3. **(or)**

If $x_{ij} = b_j$ cross off the j^{th} column of the transportation table and decrease b_j by a_i. Go to Step 3. **(or)**

If $x_{ij} = a_i = b_j$ cross off either the i^{th} row or j^{th} column but not both.

Step 3: Repeat Steps 1 and Step 2 for the resulting reduced transportation table until all rim requirements are satisfied. Whenever the minimum cost is not unique, make an arbitrary choice among the minima.

Least-cost method or Matrix minima Method

Example:

		Distribution centres				
		1	2	3	4	Supply
Plants	1	2	3	11	7	6
	2	1	0	6	1	1
	3	5	8	15	9	10
Requirement		7	5	3	2	

2 $\boxed{6}$	3	11	7	~~6~~ 0
1	0 $\boxed{1}$	6	1	~~1~~ 0
5 $\boxed{1}$	8 $\boxed{4}$	15 $\boxed{3}$	9 $\boxed{2}$	~~10~~ ~~9~~ ~~5~~ ~~2~~ 0
~~7~~ ~~1~~ 0	~~5~~ ~~4~~ 0	~~3~~ 0	~~2~~ 0	

We have make the allocation at the least cost of the matrix. Least cost we have we have 0 which is at cell (2,2). Cross-out the corresponding row.

Next minimum is in the matrix is 2 which is at (1,1) cell of the original matrix. There we have make the allocation and cross-out the corresponding row.

Next minimum in the matrix is 5 which is at cell (3,1) of the original matrix. There we have to make the allocation and cross-out the respective column.

Next minimum is 8 which is at cell (3,2), there we have to make the next allocation and cross-out the corresponding column.

Next minimum is 9 which is at cell (3,3). There we have to make the next allocation and cross-out the respective column. Finally, we left with one cell which is (3,4) and cross-out the column.

Now the total cost is given by:

$$= 6 \times 2 + 0 \times 1 + 5 \times 1 + 8 \times 4 + 15 \times 3 + 9 \times 2 = 112.$$

Row Minima method

Procedure for Least cost Method:

This method consists in allocating as much as possible in the lowest cost cell of the first row so that either the capacity of the first plant is exhausted or the requirement at jth distribution centre is satisfied or both. In case of tie among the cost, select arbitrarily. Three cases arise:

(*i*) if the capacity of the first plant is completely exhausted, cross off the first row and proceed to the second row.

(*ii*) if the requirement at jth distribution centre is satisfied, cross off the jth column and reconsider the first row with the remaining capacity.

(*iii*) if the capacity of the first plant as well as the requirement at jth distribution centre are completely satisfied, cross off the row as well as the jth column and move down to the second row.

Continue the process for the resulting reduced transportation table until all the *rim conditions* (supply and requirement conditions) are satisfied.

Row Minima method

Example:

		Distribution centres				
		1	2	3	4	*Supply*
	1	2	3	11	7	6
Plants	2	1	0	6	1	1
	3	5	8	15	9	10
Requirement		7	5	3	2	

2 $\boxed{6}$	3	11	7	~~6~~ 0
1	0 $\boxed{1}$	6	1	~~1~~ 0
5 $\boxed{1}$	8 $\boxed{4}$	15 $\boxed{3}$	9 $\boxed{2}$	~~10~~ ~~9~~ ~~5~~ ~~3~~ 0
~~7~~ ~~1~~ 0	~~5~~ ~~4~~ 0	~~3~~ 0	~~2~~ 0	

In the first row, the minimum cost is 2 which is at cell (1,1). There we have make the first allocation and cross-out the respective row.

In the second row, the minimum cost is 0 which is at cell (2,2). There we have make the second allocation and cross-out the respective row.

In the third row, the minimum cost is 5 which is at (3,1) location. There we have to make the third allocation and cross-out the respective column. In the same row next minimum is 8 which is at (3,2) location. There we have to make the fourth allocation and cross-out the respective column. Similarly in the same row, next minimum is 9 which is at (3,4) location, where we have to make fifth allocation and cross-out the corresponding column. Finally, we left with one cell there we have to make the final(sixth) allocation.

Total cost is given by:

$$= 6 \times 2 + 0 \times 1 + 5 \times 1 + 8 \times 4 + 15 \times 3 + 9 \times 2 = 112.$$

Column Minima method

Procedure for Column Minima Method:

This method consists in allocating as much as possible in the lowest cost cell of the first column so that either the demand of the first distribution centre is satisfied or the capacity of the i^{th} plant is exhausted or both. In case of tie among the lowest cost cells in the column, select arbitrarily. Three cases arise:

(i) if the requirement of the first distribution centre is satisfied, cross off the first column and move right to the second column.

(ii) if the capacity of i^{th} plant is satisfied, cross off i^{th} row and reconsider the first column with the remaining requirement.

(iii) if the requirement of the first distribution centre as well as the capacity of the i^{th} plant are completely satisfied, cross off the column as well as the i^{th} row and move right to the second column.

Continue the process for the resulting reduced transportation table until all the rim (row and column) conditions are satisfied.

Column Minima method

Example:

		Distribution centres				
		1	2	3	4	*Supply*
	1	2	3	11	7	6
Plants	2	1	0	6	1	1
	3	5	8	15	9	10
Requirement		7	5	3	2	

2 $\boxed{6}$	3	11	7	~~6~~ 0
1 $\boxed{1}$	0	6	1	~~1~~ 0
5	8 $\boxed{5}$	15 $\boxed{3}$	9 $\boxed{2}$	~~10~~ ~~5~~ ~~2~~ 0
~~7~~ ~~6~~ 0	~~5~~ 0	~~3~~ 0	~~2~~ 0	

In the first column, the minimum cost is 1 which is at (2,1) position where we have to make the first allocation and cross-out the respective row.

Since the first column not crossed-out completely, in the same column, the next minimum cost is 2 which is at (1,1) position. There we have to make our second allocation. Here both the row and column having the same supply and demand. Therefore, cross-out respective row and columns.

Now in the second column, the minimum cost is 8 which is at (3,2) position, there we have make our third allocation and cross-out the respective column.

Now in the third column, the minimum cost is 15 which is at position (3,3). There we have to make our fourth allocation and cross-out the respective column. Finally we left with one cell which is in fourth column there we have to make out fifth allocation and cross-out the respective column.

The total cost is given by:

$$= 2 \times 6 + 1 \times 1 + 8 \times 5 + 15 \times 3 + 9 \times 2 = 116.$$