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Optimization Techniques (MAT-2003)

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Convex linear combination: A linear combination of x and y i.e., $ax + by$ is said to be convex linear combination if $0 \leq a, b \leq 1$ and $a + b = 1$.

Dominance property: In some games, it is possible reduce the size of the payoff matrix by eliminating redundant rows(or columns). If a game has such redundant rows(or columns), those rows or columns are dominated by some other rows(or columns), respectively. Such property is known as dominance property.

Dominance property by rows:

- In the payoff matrix of player A, if all the entries in a row (say X) are less than or equal to the corresponding entries of another row(say Y)(i.e., $X \leq Y$), then row X is dominated by row Y. Under such situation, row X of the payoff matrix can be deleted.

- if some convex linear combination of some rows(X and Y) dominate Z row, then the Z row will be deleted. Here, elements of Z row dominated by convex linear combination of X and Y.

Dominance property for columns:

- In the payoff matrix of player A, if all the entries in a column (say X) are greater than or equal to the corresponding entries of another column (say Y) (i.e., $X \geq Y$), then column X is dominated by column Y. Under such situation, dominated column X of the payoff matrix can be deleted.

- if some convex linear combination of some column (X and Y) dominate Z column, then the Z column will be deleted. Here, elements of Z column dominated by convex linear combination of X and Y.

Example:

Solve the following payoff matrix using dominance property

		Player B				
		1	2	3	4	5
Player A	I	1	3	2	7	4
	II	3	4	1	5	6
	III	6	5	7	6	5
	IV	2	0	6	3	1

	C_1	C_2	C_3	C_4	C_5
R_1	1	3	2	7	4
R_2	3	4	1	5	6
R_3	6	5	7	6	5
R_4	~~2~~	~~0~~	~~6~~	~~3~~	~~1~~

Here, $R_4 \leq R_3$, therefore by row dominance property we can delete R_4.

The resultant payoff matrix becomes:

	C_1	C_2	C_3	C_4	C_5
R_1	1	3	2	7	4
R_2	3	4	1	5	6
R_3	6	5	7	6	5

In the above payoff matrix, no two rows are dominated by one another.

Let us look for column dominance property.

Here we have $C_4 \geq C_1$, therefore we can delete C_4

The resultant payoff becomes:

	C_1	C_2	C_3	C_5
R_1	1	3	2	4
R_2	3	4	1	6
R_3	6	5	7	5

In the above payoff, $C_5 \geq C_2$. Therefore, we can delete C_5.

The payoff matrix becomes:

$$\begin{array}{cccc} \cancel{1} & \cancel{3} & \cancel{2} & R_1 \\ 3 & 4 & 1 & R_2 \\ 6 & 5 & 7 & R_3 \\ C_1 & C_2 & C_3 & \end{array}$$

In the above payoff matrix, $R_1 \leq R_3$. Therefore, delete R_1 by row dominance property.

The payoff becomes:

$$\begin{array}{cccc} \cancel{3} & \cancel{4} & \cancel{1} & R_2 \\ 6 & 5 & 7 & R_3 \\ C_1 & C_2 & C_3 & \end{array}$$

In the above payoff matrix, $R_2 \leq R_3$. Therefore, delete R_3 by row dominance property.

The payoff becomes:

$$\begin{array}{cccc} 6 & 5 & 7 & R_3 \\ C_1 & C_2 & C_3 & \end{array}$$

Now, we have only one row. Therefore we can delete maximum column elements using column dominance property.

$$\begin{array}{c|ccc|}
 & C_1 & C_2 & C_3 \\
R_3 & \cancel{6} & 5 & \cancel{7}
\end{array}$$

Here, we left with only one value which is the saddle point of the game and its value is the value of the game

The game is a pure strategy game.

The optimal strategies of the players A and B are III and 2 respectively.

Player A is winning the game.

Example:

Solve the following payoff matrix using dominance property

		Player B				
		B_1	B_2	B_3	B_4	B_5
Player A	A_1	4	6	5	10	6
	A_2	7	8	5	9	10
	A_3	8	9	11	10	9
	A_4	6	4	10	6	4

	C_1	C_2	C_3	C_4	C_5
R_1	~~4~~	~~6~~	~~5~~	~~10~~	~~6~~
R_2	7	8	5	9	10
R_3	8	9	11	10	9
R_4	6	4	10	6	4

In the above payoff matrix $R_1 \leq R_3$, therefore, delete R_1 using row dominance property.

The payoff matrix becomes.

	C_1	C_2	C_3	C_4	C_5
R_2	7	8	5	9	10
R_3	8	9	11	10	9
R_4	~~6~~	~~4~~	~~10~~	~~6~~	~~4~~

In the above payoff matrix $R_4 \leq R_3$, therefore, delete R_4 using row dominance property.

The payoff matrix becomes.

7	8	5	9	10	R_2
8	9	11	10	9	R_3
C_1	C_2	C_3	C_4	C_5	

In the above payoff matrix we have $C_2 \geq C_1$. Therefore, by column dominance property delete C_2.

The payoff matrix becomes:

7	5	9	10	R_2
8	11	10	9	R_3
C_1	C_3	C_4	C_5	

In the above payoff matrix we have $C_4 \geq C_1$. Therefore, by column dominance property delete C_4.

The payoff matrix becomes:

	C_1	C_3	C_5
R_2	7	5	10
R_3	8	11	9

In the above payoff matrix we have $C_5 \geq C_1$. Therefore, by column dominance property delete C_5.

The payoff matrix becomes:

	C_1	C_3
R_2	7	5
R_3	8	11

In the above payoff matrix we have $R_2 \leq R_3$.Therefore, by row dominance property delete R_2.

The payoff matrix becomes:

	C_1	C_3
R_2	8	11

In the above payoff matrix we have $C_3 \geq C_1$. Therefore, by column dominance property delete C_3.

The payoff matrix becomes:

	C_1
R_2	8

Therefore, the game has a pure strategy and the players A and B are coming up with the strategies A_2 and B_2 respectively.

4	6	5	10	6
7	8	5	9	10
8	9	11	10	9
6	4	10	6	4

Saddle point

The value of the game $V = 8$

Player A is winning the game.

Mixed Strategy using Dominance property:

Example:

Solve the following payoff matrix using dominance property

		Player B			
		I	II	III	IV
Player A	I	3	2	4	0
	II	3	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

	C_1	C_2	C_3	C_4
R_1	~~3~~	~~2~~	~~4~~	~~0~~
R_2	3	4	2	4
R_3	4	2	4	0
R_4	0	4	0	8

In the above payoff matrix $R_1 \leq R_3$, therefore, delete R_1 using row dominance property.

The payoff matrix becomes.

	C_1	C_2	C_3	C_4
R_2	~~3~~	4	2	4
R_3	~~4~~	2	4	0
R_4	~~0~~	4	0	8

In the above payoff matrix $C_1 \geq C_3$, therefore, delete C_1 using column dominance property.

The payoff matrix becomes.

	C_2	C_3	C_4
R_2	4	2	4
R_3	2	4	0
R_4	4	0	8

In the above payoff matrix $C_2 \geq \frac{1}{2}C_3 + \frac{1}{2}C_4$. Therefore, delete C_2 using column dominance property.

The payoff matrix becomes.

2	4
4	0
0	8

In the above payoff matrix $R_1 \leq \frac{1}{2}R_2 + \frac{1}{2}R_3$. Therefore, delete R_1 using row dominance property.

The payoff matrix becomes.

$$\begin{array}{cc|c} 4 & 0 & R_3 \\ 0 & 8 & R_4 \\ \hline C_3 & C_4 & \end{array}$$

Here, the game has no strategy and it is a 2×2 pay-off matrix.

Therefore, we know formulae for finding the probabilities of the strategies and value of the game.

Let p_3 and p_4 are the probabilities of the player A to chose the strategies III and IV.

Let q_3 and q_4 are the probabilities of the player B to chose the strategies III and IV.

$$\begin{array}{c|cc|c} & q_3 & q_4 & \\ \hline p_3 & 4\ (a) & 0\ (b) & |0-8| = 8 \\ p_4 & 0\ (c) & 8\ (d) & |4-0| = 4 \\ \hline & |0-8| = 8 & |0-4| = 4 & \end{array}$$

$$p_3 = \frac{|c-d|}{|a-b|+|c-d|} = \frac{8}{8+4} = \frac{2}{3}$$

$$p_4 = \frac{|a-b|}{|a-b|+|c-d|} = \frac{4}{8+4} = \frac{1}{3}$$

$$q_3 = \frac{|b-d|}{|a-c|+|b-d|} = \frac{8}{8+4} = \frac{2}{3}$$

$$q_4 = \frac{|a-c|}{|a-c|+|b-d|} = \frac{4}{8+4} = \frac{1}{3}$$

And the value of the game is

$$V = \frac{a|c-d| + c|a-b|}{|a-b| + |c-d|}$$

$$= \frac{4|0-8| + 0|4-0|}{|0-8| + |4-0|}$$

$$= \frac{32}{12} = \frac{8}{3}$$

Here, the players A and B are using mixed strategies.

Player A is using III and IV strategies with the probabilities $\frac{2}{3}$ and $\frac{1}{3}$.

Player B is using III and IV strategies with the probabilities $\frac{2}{3}$ and $\frac{1}{3}$.

The value of the game, $V = \frac{8}{3}$

Player A is winning the game.