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Optimization Techniques (MAT-2003)

Lecture-24

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Degeneracy in the Transportation Problem

Example:

Find the optimum solution to the following transportation problem in which the cells contain the transportation cost in rupees.

	W_1	W_2	W_3	W_4	W_5	Available
F_1	7	6	4	5	9	40
F_2	8	5	6	7	8	30
F_3	6	8	9	6	5	20
F_4	5	7	7	8	6	10
Required	30	30	15	20	5	100 (Total)

The initial basic feasible solution using VAM (Vogel's approximation) method is given by:

7 (5)	6	4 (15)	5 (20)	9	40
8	5 (30)	6	7	8	30
6 (15)	8	9	6	5 (5)	20
5 (10)	7	7	8	6	10
30	30	15	15	5	

Here, the number of allocations are 7.

But the value of $m + n - 1 = 4 + 5 - 1 = 8$.

Therefore, the number of allocations $= 7 < m + n - 1$.

The transportation problem is a degeneracy case.

To overcome the degeneracy, we need to add an extra allocation to the problem.

Therefore, here we are adding a positive infinitesimal allocation $(\epsilon > 0)$ to satisfy the number of allocations equals to $m + n - 1$ at the minimum cost of the un-allocated cell in such a way that the allocations must be independent.

In the given cost matrix the minimum cost among the un-allocated cells is 6 which is at four different locations $(1,2)$, $(2,3)$, $(3,4)$, and (4,5) respectively.

Out of which if we allocate at $(3,4)$ and $(4,5)$ locations, the allocations becomes dependent (forms a closed loop). But, if we allocate at $(1,2)$, $(2,3)$ locations, the allocations becomes independent (doesn't form a closed loop).

Therefore, we have two possibilities where we allocate a new infinitesimal allocation at $(1,2)$ and $(2,3)$ locations. Now, we are allocating ϵ at the location $(2,3)$.

7 (5)	6	4 (15)	5 (20)	9
8	5 (30)	6 (ϵ)	7	8
6 (15)	8	9	6	5 (5)
5 (10)	7	7	8	6

The number of allocations $= 8 = m + n - 1 (= 4 + 5 - 1)$ and are also at independent positions. Therefore, we can now check whether the allocations are optimal or not.

For the optimal solution, we will apply MODI method or U-V method.

Now, we need to find u_i and v_j values corresponding to the allocated cells such that $u_i + v_j = c_{ij}$.

Further we need to calculate the net-evaluations:

					u_i
7 (5)	6	4 (15)	5 (20)	9	0
8	5 (30)	6 (ϵ)	7	8	2
6 (15)	8	9	6	5 (5)	-1
5 (10)	7	7	8	6	-2
v_j: 7	3	4	5	6	

Now, the net evaluations for the non-allocated cells are calculated using the formula

$$c_{ij} - z_{ij} = (u_i + v_j - c_{ij})$$

cell $(1,2) \rightarrow c_{12} - z_{12} = c_{12} - (u_1 + v_2) = 6 - (0 + 3) = 3 \geq 0$

$(1,5) \rightarrow c_{15} - z_{15} = \mathrm{c}_{15} - (u_1 + \mathrm{v}_5) = 9 - (0 + 6) = 3 \geq 0$

$(2,1) \rightarrow c_{21} - z_{21} = c_{21} - (u_2 + v_1) = 8 - (7 + 2) = -1 \ngeq 0$

$(2,4) \rightarrow c_{24} - z_{24} = c_{24} - (u_2 + v_4) = 7 - (5 + 2) = 0 \geq 0$

$(2,5) \rightarrow c_{25} - z_{25} = c_{25} - (u_2 + v_5) = 8 - (6 + 2) = 0 \geq 0$

$(3,2) \rightarrow c_{32} - z_{32} = c_{32} - (u_3 + v_2) = 8 - \big(3 + (-1)\big) = 6 \geq 0$

$(3,3) \rightarrow c_{33} - z_{33} = c_{33} - (u_3 + v_3) = 9 - \big(4 + (-1)\big) = 6 \geq 0$

$(3,4) \rightarrow c_{34} - z_{34} = c_{34} - (u_3 + v_4) = 6 - \big(5 + (-1)\big) = 2 \geq 0$

$$(4,2) \rightarrow c_{42} - z_{42} = c_{42} - (u_4 + v_2) = 7 - \big(3 + (-2)\big) = 6 \geq 0$$

$$(4,3) \rightarrow c_{43} - z_{43} = c_{43} - (u_4 + v_3) = 7 - (4 + (-2)) = 5 \leq 0$$

$$(4,4) \rightarrow c_{44} - z_{44} = c_{44} - (u_4 + v_4) = 8 - (5 + (-2)) = 5 \leq 0$$

$$(4,5) \rightarrow c_{45} - z_{45} = c_{45} - (u_4 + v_5) = 6 - (6 + (-2)) = 2 \leq 0$$

The net-evaluation for the cell $(2,1)$ is < 0. Which fails to satisfy optimality test. Therefore, we need to improve the solution.

To do that, we need to pick the cell $(2,1)$ which is not satisfying the optimality test (or among all such cells we need to take the maximum positive net-evaluation) there we have to make a new allocation (becomes a basic variable).

To make the new allocation we need to draw a loop starting from the cell $(2,1)$ with the help of horizontal and vertical jumps through the allocated cells.

7 $-\theta$ (5)	6	4 $+\theta$ (15)	5 (20)	9
8 $+\theta$ (θ)	5 (30)	6 $-\theta$ (ϵ)	7	8
6 (15)	8	9	6	5 (5)
5 (10)	7	7	8	6

$$\theta = \min\{5, \epsilon\} = \epsilon$$

We need to add ϵ at $+\theta$ and subtract at $-\theta$ locations.

7 (5)	6	4 (15)	5 (20)	9
8 (ϵ)	5 (30)	6	7	8
6 (15)	8	9	6	5 (5)
5 (10)	7	7	8	6

Now, again we need to check whether the allocation is optimal or not.

					u_i
7 (5)	6	4 (15)	5 (20)	9	0
8 (ϵ)	5 (30)	6	7	8	1
6 (15)	8	9	6	5 (5)	-1
5 (10)	7	7	8	6	-2
v_j: 7	4	4	5	6	

Now, we need to calculate the net-evaluations for the un-allocated cells.

The net evaluations for the non-allocated cells are calculated using the formula

$$c_{ij} - z_{ij} = c_{ij} - (u_i + v_j)$$

cell $(1,2) \rightarrow c_{12} - z_{12} = c_{12} - (u_1 + v_2) = 6 - (0 + 4) = 2 \geq 0$

$(1,5) \rightarrow c_{15} - z_{15} = \mathrm{c}_{15} - (u_1 + \mathrm{v}_5) = 9 - (0 + 6) = 3 \geq 0$

$(2,3) \rightarrow c_{23} - z_{23} = c_{23} - (u_2 + v_3) = 6 - (4 + 1) = 1 \geq 0$

$(2,4) \rightarrow c_{24} - z_{24} = c_{24} - (u_2 + v_4) = 7 - (5 + 1) = 1 \geq 0$

$(2,5) \rightarrow c_{25} - z_{25} = c_{25} - (u_2 + v_5) = 8 - (6 + 1) = 1 \geq 0$

$(3,2) \rightarrow c_{32} - z_{32} = c_{32} - (u_3 + v_2) = 8 - \big(4 + (-1)\big) = 5 \geq 0$

$(3,3) \rightarrow c_{33} - z_{33} = c_{33} - (u_3 + v_3) = 9 - \big(4 + (-1)\big) = 6 \geq 0$

$(3,4) \rightarrow c_{34} - z_{34} = c_{34} - (u_3 + v_4) = 6 - \big(5 + (-1)\big) = 2 \geq 0$

$$(4,2) \rightarrow c_{42} - z_{42} = c_{42} - (u_4 + v_2) = 7 - \big(4 + (-2)\big) = 5 \geq 0$$

$$(4,3) \rightarrow c_{43} - z_{43} = c_{43} - (u_4 + v_3) = 7 - \big(4 + (-2)\big) = 5 \geq 0$$

$$(4,4) \rightarrow c_{44} - z_{44} = c_{44} - (u_4 + v_4) = 8 - \big(5 + (-2)\big) = 5 \geq 0$$

$$(4,5) \rightarrow c_{45} - z_{45} = c_{45} - (u_4 + v_5) = 6 - \big(6 + (-2)\big) = 2 \geq 0$$

All the net evaluations are ≥ 0. Therefore, we obtain the optimal allocation.

The optimal (minimum) cost of the transportation problem is given by

$$= 7 \times 5 + 4 \times 15 + 5 \times 20 + 8 \times \epsilon + 5 \times 30 + 6 \times 15 + 5 \times 5 + 5 \times 10 = 510 + 8\epsilon$$

As we know $\epsilon \rightarrow 0$

Optimal cost $= 510$.

Example:

Given $x_{13} = 50$ units, $x_{14} = 20$ units, $x_{21} = 55$ units, $x_{31} =30$ units, $x_{32} = 35$ units, $x_{34} = 25$ units. Is it an optimal solution to the transportation problem:

$$\begin{array}{lcc}
 & & \textit{Available units} \\
 & \begin{bmatrix} 6 & 1 & 9 & 3 \\ 11 & 5 & 2 & 8 \\ 10 & 12 & 4 & 7 \end{bmatrix} & \begin{array}{c} 70 \\ 55 \\ 90 \end{array} \\
\textit{Required units}: & \begin{array}{cccc} 85 & 35 & 50 & 45 \end{array} &
\end{array}$$

If not, modify it to obtain a better feasible solution.

6	1	9 (50)	3 (20)	70
11 (55)	5	2	8	55
10 (30)	12 (35)	4	7 (25)	90
85	35	50	45	

Do it yourself.

Alternative Optimal solution:

For the optimal solution of transportation problem if at least one net-evaluation of the un-allocated cell is 0, then the transportation problem will have the alternative optimal solution.

Which means, the allocations may change however the optimal cost remains same.

	Ranchi	Delhi	Lucnow	Kanpur	u_i
Mumbai	20 90	$\boxed{75}$ 90	$\boxed{95}$ 100	$\boxed{30}$ 100	0
Kolkata	$\boxed{75}$ 50	$\boxed{25}$ 70	50 130	5 85	-20
Dummy	70 0	10 0	$\boxed{5}$ 0	0 0	-100
v_j	70	90	100	100	

Find the Alternate Optimal Solution of the following transportation problem:

The final allocations and the net-evaluations are given along with the following cost matrix.

	Ranchi	Delhi	Lucnow	Kanpur	u_i
Mumbai	20 90	75 90	95 100	30 100	0
Kolkata	75 50	25 70	50 130	5 85	-20
Dummy	70 0	10 0	5 0	0 0	-100
v_j	70	90	100	100	

Allocation

Net-evaluation

The Optimal cost of the transportation problem is given by:

$$= 75 \times 90 + 95 \times 100 + 30 \times 100 + 75 \times 50 + 25 \times 70 + 5 \times 0 = 24750$$

				u_i
20 90	$\boxed{75}$ 90	$\boxed{95}$ 100	$\boxed{30}$ 100	0
$\boxed{75}$ 50	$\boxed{25}$ 70	50 130	5 85	–20
70 0	10 0	$\boxed{5}$ 0	0 0	–100
v_j 70	90	100	100	

The u_i and v_j values corresponding to the basic variables are given above. Which were calculated using the relation

$$u_i + v_j = c_{ij}.$$

The net-evaluations are also given in the cost matrix which are in the cost matrix at the bottom left corner of the cell in the cost matrix.

Among all the net-evaluations the net evaluation in cell $(3,4)$ is 0, which indicate an alternative optimal solution to the transportation problem.

To find the alternative optimal solution of the transportation problem, we need to make a new allocation at the cell having the zero (i.e., cell $(3,4)$) net-evaluation.

To do that, we need to draw a closed loop starting from the cell $(3,4)$ with the help of horizontal and vertical jumps through the allocated cells.

90	90 $\boxed{75}$	100 $+\theta$ $\boxed{95}$	100 $-\theta$ $\boxed{30}$
50 $\boxed{75}$	70 $\boxed{25}$	130	85
0	0	0 $-\theta$ $\boxed{5}$	0 $+\theta$ $\boxed{\theta}$

$$\theta = \min\{30,5\} = 5$$

Which we need add at $+\theta$ and subtract at $-\theta$ positions.

90	90 $\boxed{75}$	100 $\boxed{100}$	100 $\boxed{25}$
50 $\boxed{75}$	70 $\boxed{25}$	130	85
0	0	0	0 $\boxed{5}$

Again this solution is going to be a optimal solution. And the optimal cost is given by:

$$= 75 \times 90 + 95 \times 100 + 30 \times 100 + 75 \times 50 + 25 \times 70 + 5 \times 0 = 24750$$

It is clear from the two solutions that the allocation positions are different but the optimal cost remains same.