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Optimization Techniques (MAT-2003)

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Assignment Problem

The Assignment problem is a special case of the Transportation problem in which the objective is to assign a number of resources to the equal number of activities at a minimum cost (or Maximum profit).

Mathematical Formulation:

$$\text{minimize } Z = \sum_{j=1}^{n} \sum_{i=1}^{n} c_{ij} x_{ij} \left\{ = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \right\},$$

subject to constraints $\sum_{j=1}^{n} x_{ij} = 1, i = 1, 2, 3, \ldots, n,$ (one job is assigned to the ith facility)

$\sum_{i=1}^{n} x_{ij} = 1, j = 1, 2, 3, \ldots, n,$ (one facility is assigned the jth job)

and $x_{ij} = 0$ or 1 (or $x_{ij} = x_{ij}^2$).

Note:

- In an assignment problem, if we add or subtract a constant to every element of any row (or column) of the cost matrix $[c_{ij}]$, then an assignment that minimizes the total cost on one matrix also minimizes the total cost on the other matrix.

- $If\ c_{ij} \geq 0,$ such that $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij} = 0,$ then the feasible solution x_{ij} provides an optimum assignment.

Hungarian Assignment Method:

Example:

A departmental head has four subordinates, and four tasks to be performed. The subordinates differ in efficiency, and the tasks differ in their intrinsic difficulty. His estimate of the time each man would take to perform each task, is given in the matrix below:

Tasks	*Men*			
	E	*F*	*G*	*H*
A	18	26	17	11
B	13	28	14	26
C	38	19	18	15
D	19	26	24	10

How would the tasks be allocated, one to a man, so as to minimize the total man-hours?

Writing the cost matrix of the above assignment problem:

	E	F	G	H
A	18	26	17	11
B	13	28	14	26
C	38	19	18	15
D	19	26	24	10

Step: 1

In the cost matrix, pick the minimum element in the each row and subtract it from each element of the corresponding row.

The required cost matrix becomes:

	E	F	G	H
A	7	15	6	0
B	0	15	1	13
C	23	4	3	0
D	9	16	14	0

Step: 2

Similarly, subtract each element of all columns from their respective column minimum element.

	E	F	G	H
A	7	11	5	0
B	0	11	0	13
C	23	0	2	0
D	9	12	13	0

Examine each row successively until a row with single zero is found and enrectangle that zero and cross off all other zeros in the rectangle element's column.

Now, examine the each column having single zero and enrectangle that zero and cross off all the other zeros in the rectangle elements row.

	E	F	G	H
A	7	11	5	$\boxed{0}$
B	$\boxed{0}$	11	$\cancel{0}$	13
C	23	$\boxed{0}$	2	$\cancel{0}$
D	9	12	13	$\cancel{0}$

Here, the enrectangle zeros (Assigned zeros) are 3 in number which are less than the order of the cost matrix(4).
Therefore, we didn't reach the optimal solution. Therefore we need to proceed further.

To make the more number of zeros, we need to follow the following procedure.

Mark the row (here 4th row) which do not have assigned zero.
Now, mark the column(s) which has crossed zero in the marked 4th row.
Further mark the row (1st row) which has assigned zero in the marked column (4th column).

Now in the marked first row, there are no crossed zeros. Therefore, the process ends here only.

There will be no more further marked rows or columns.

	E	F	G	H	
A	7	11	5	$\boxed{0}$	✓
B	$\boxed{0}$	11	$\cancel{0}$	13	
C	23	$\boxed{0}$	2	$\cancel{0}$	
D	9	12	13	$\cancel{0}$	✓
				✓	

Draw the lines to cover the un marked rows and marked columns.

Now to get more zeros, we have to do the following procedure:

From the un covered elements pick the smallest element (here it is 5) and subtract it from all un covered elements.
Also, add the same to all the elements (at (2,4) and at (3,4)) which are at the intersection of the horizontal and vertical lines. The resultant matrix is in the next slide.

	E	F	G	H
A	7	11	5	$\boxed{0}$
B	$\boxed{0}$	11	~~0~~	13
C	23	$\boxed{0}$	2	~~0~~
D	9	12	13	~~0~~

Now do the assignments again to the cost matrix as follows:

Examine each row successively until a row with single zero is found and Enrectangle that zero and cross off all other zeros in the rectangle element's column. If you have more than one zero in any row we can assign to any one of them arbitrarily.

Similarly do the same procedure for all the columns as well.

$$
\begin{array}{c|cccc|}
 & E & F & G & H \\
\hline
A & 2 & 6 & \boxed{0} & \cancel{0} \\
B & \boxed{0} & 11 & \cancel{0} & 18 \\
C & 23 & \boxed{0} & 2 & 5 \\
D & 4 & 7 & 8 & \boxed{0} \\
\hline
\end{array}
$$

Therefore, the number of assignments are 4 (= order of matrix) which are A to G, B to E, C to F and D to H; and the minimum number of man hours are given by adding the corresponding assignment elements from the initial cost matrix.

$$\text{Min } Z = 17 + 13 + 19 + 10 = 59.$$

Procedure for solving Assignment problem using Hungarian Method:

The method of solving an assignment problem (minimization case) consists of the following steps:

Step 1: Determine the cost table form the given problem.

(i) If the number sources is equal to the number of destinations, go to step 3.
(ii) If the number of sources is not equal to the number of destinations, go to step 2.

Step 2: Add a dummy source or destination, so that the cost table becomes the square matrix. The cost entries of dummy source/destinations are always zero.

Step 3: Locate the smallest element in each row of the given cost matrix and then subtract from each element of that row.

Step 4: In the reduced matrix obtained in step 3, locate smallest element of each column and then subtract the same from the each element of the column. Each column and row now at least one zero.

Step 5: In the modified matrix obtained in step 4, search for an optimal assignment as follows:

(a) Examine the rows successively until a row with a single zero is found. Enrectangle this zero (□) and cross off (X) all other zeros its column. Continue in this manner until all the rows have been taken case of.

(b) Repeat the procedure for each column of the reduced matrix.

(c) If a row and/or column has two or more zeros then skip that particular row/column and you can continue the procedure for other rows and columns and ultimately you will end up with single zero in that particular row or column.

(d) Repeat (a) through (c) above successively until the chain of assigning (□) or cross (X) ends.

Step 6: If the number of assignments (□) is equal to n (the order of the cost matrix), an and optimum) solution is reached.

If the number of assignments is less than n (the order of the matrix), go to next step.

Step 7: Draw the minimum number of horizontal and/or vertical lines to cover all the zeros of the reduced matrix. This can be conveniently done by using a simple procedure.

(a) Mark (✓) the rows that do not have any assigned zero.
(b) Mark (✓) the columns that have zeros in the marked rows.
(c) Mark (✓) the rows that have assigned zeros in the marked columns.
(d) Repeat (b) and (c) above until the chain of marking is completed.
(e) Draw lines through all the unmarked rows and marked columns. Thus gives us the desired minimum number of lines.

Step 8: Develop the new revised cost matrix as follows:
(a) Find the smallest element of the reduced matrix not covered by any of the lines.
(b) Subtract this element from all the uncovered elements and add the same to all the elements lying at the intersection of any two lines.

Step 9: Go to step 6 and repeat the procedure until an optimum solution is attained.

Example:

A machine tool company decides to make four subassemblies through four contractors. Each contractor is to receive only one subassembly. The cost of each subassembly is determined by the bids submitted by each contractor and is shown in table 4. 7 in hundreds of rupees.

Subassemblies \ Contractors	1	2	3	4
1	15	13	14	17
2	11	12	15	13
3	13	12	10	11
4	15	17	14	16

(i) Formulate the mathematical model (LPP) for the problem.

(ii) Show that the assignment model is a special case of the transportation model.

(iii) Assign the different subassemblies to contractors so as to minimize the total cost

15	13	14	17
11	12	15	13
13	12	10	11
15	17	14	16.

Ans: Assignments are

1 to 2
2 to 1
3 to 4
4 to 3

Min $Z = 49(in\ hundreds\ of\ Rs.) = Rs. 4900.$

Verify yourself.