Gradient-based Methods:

We have demonstrated that all the methods described in earlier sections work with direct function values and not with derivative information.

The algorithms discussed in this section require derivative information.

It is recommended to use these algorithms in problems where the derivative information is available or can be calculated easily.

Newton Method Algorithm:

Step 1: Choose initial guess $x^{(1)}$, a small number ϵ and set also set $k = 1$. Compute $f'\left(x^{(k)}\right)$.

Step 2: Compute $f''\left(x^{(k)}\right)$

Step 3: Calculate $x^{(k+1)} = x^{(k)} - \frac{f'\left(x^{(k)}\right)}{f''\left(x^{(k)}\right)}$. Next, Compute $f'\left(x^{(k+1)}\right)$.

Step 4: If $\left|f'\left(x^{(k+1)}\right)\right| < \epsilon$, Terminate. Else set $k = k + 1$ and go to step 2.

Example:

Minimize the function $f(x) = x^2 + \frac{54}{x}$, using Newton method with initial approximation $x^{(1)} = 1$ and $\epsilon = 10^{-3}$.

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Sol^n: $f(x) = x^2 + \frac{54}{x}$

$\boxed{k=1}$

it 1

$x^{(1)} = 1$

$$x^{(k+1)} = x^{(k)} - \frac{f'(x^{(k)})}{f''(x^{(k)})}$$

$$x^{(2)} = x^{(1)} - \frac{f'(x^{(1)})}{f''(x^{(1)})}$$

$$f'(x) = 2x - \frac{54}{x^2}.$$

$$f'(x^{(1)}) = 2(1) - 54 = -52$$

$$f''(x) = 2 + \frac{54 \times 2}{x^3}.$$

$$f''(x^{(1)}) = 2 + 108 = 110$$

$$x^{(2)} = x^{(1)} - \frac{f'(x^{(1)})}{f''(x^{(1)})} = 1 + \frac{52}{110}$$

$$= 1.4727$$

$$f'(x^{(2)}) = 2(1.4727) - 54/1.4727 = -21.944$$

$$\left| f^{(1)}\left(x^{(2)}\right) \right| = 21.944 \nless 10^{-3}$$

$k = 1+1 = 2$

It 2:

$$x^{(3)} = x^{(2)} - \frac{f'\left(x^{(2)}\right)}{f''\left(x^{(2)}\right)}$$

$$f'\left(x^{(2)}\right) = -21.944$$

$$f''\left(x^{(2)}\right) = 2 + \frac{108}{(1.4727)^3} = 35.796$$

$$x^{(3)} = 1.4727 + \frac{21.944}{35.796} = 2.0857.$$

$$\left| f'\left(x^{(3)}\right) \right| < \epsilon.$$

$$\left| 2(2.0857) - \frac{54}{(2.0857)} \right| \nless 10^{-3}$$

$$\left| -8.239 \right| \nless 10^{-3}$$

$k = 2+1 = 3$

It 3

$$x^{(4)} = x^{(3)} - \frac{f'\left(x^{(3)}\right)}{f''\left(x^{(3)}\right)}$$

$$f''\left(x^{3}\right) = 2 + \frac{108}{(2.0857)^3} = 13.898$$

$$x^{(4)} = 2.0857 + \frac{8.239}{13.898} = 2.699$$

$$f'(x^{(4)}) = 2(2.699) - \frac{54}{(2.699)^2}$$

$$= -2.074$$

$$|f'(x^{4})| \nless 10^{-3}.$$

$k = \underline{3+1}$

$\underline{\text{It } 4}$: $$x^{(5)} = x^{(4)} - \frac{f'(x^{(4)})}{f''(x^{(4)})}$$

$$f'(x^{(4)}) = -2.074$$

$$f''(x^{(4)}) = 2 + \frac{108}{(2.699)^3}$$

$$= 7.493.$$

$$x^{(5)} = 2.699 + \frac{2.074}{7.493}$$

$$= 2.9758$$

$$|f'(x^{5})| = \left|2(2.9758) - \frac{54}{(2.9758)^2}\right|$$

$$= |-0.1455| \nless 10^{-3}$$

$k = 4+1 = \underline{5}$

$\underline{\text{It } 5}$

$$x^{(6)} = x^{(5)} - \frac{f'(x^{(5)})}{f''(x^{(5)})}$$

$$f''(x^{(5)}) = 2 + \frac{108}{(2.9758)^3} = 6.098$$

$$x^{(6)} = 2.9758 + \frac{0.1455}{6.098}$$

$$= 2.9998$$

$$\left|f'(x^{(6)})\right| = \left|2(2.9998) - \frac{54}{(2.9998)^2}\right|$$

$$= \left|-0.0012\right| \not< 10^{-3}$$

$k = 5+1 = 6$

$It \leftarrow 6$

$$x^{(7)} = x^{(6)} - \frac{f'(x^{(6)})}{f''(x^{(6)})}$$

$$f''(x^{(6)}) = 2 + \frac{108}{(2.9998)^3} = 6.0008$$

$$x^{(7)} = 2.9998 + \frac{0.0012}{6.0008}$$

$$= 3.0001$$

$$f'(x^{(7)}) = \left|2(3.0001) - \frac{54}{(3.0001)^2}\right|$$

$$\left|-0.0006\right| < 10^{-3} = 0.001$$

we can terminate the procedure

Therefore, the minimum for the given function obtained approximately at $x = 3.0001$.
And the minimum value is $f(3.0001) \approx 27$