Hey Dr., Record the session

Optimization Techniques (MAT-2003)

Dr. Yada Nandukumar
Department of Mathematics
School of Advanced Sciences (SAS)
VIT-AP University

Non classical Optimization Method:

1) Region Elimination Methods
   (i) Fibonacci Search Method
   (ii) Golden Search Method

2) Gradient Based Methods
   (i) Newton Method

Unimodal function: A function $f(x)$ is said to be unimodal function if for some value m it is monotonically increasing for $x > m$ and monotonically decreasing for $x < m$. For function $f(x)$, maximum value is $f(m)$ and there is no other local maximum.

Region of Elimination Methods:

The fundamental rule for the region of elimination method as follows:

Let us consider two points x_1 and x_2 which lie in the interval (a, b) and satisfy $x_1 < x_2$. For unimodal functions for minimization, we can conclude the following:

- If $f(x_1) > f(x_2)$ then the minimum does not lie in (a, x_1).
- If $f(x_1) < f(x_2)$ then the minimum does not lie in (x_2, b).
- If $f(x_1) = f(x_2)$ then minimum does not lie in (a, x_1) and (x_2, b).

Fibonacci Search Method:

- This method is an elimination Technique.
- The function should be unimodal function.
- In this method, the search interval is reduced according to Fibonacci numbers. The property of the Fibonacci numbers is that, given two consecutive numbers f_{n-2} and F_{n-1}, the third number is calculated as follows:

$$F_n = F_{n-1} + F_{n-2}, \quad \ldots\ldots\ldots\ldots\ldots(1)$$
$$\text{where } n = 2, 3, 4, \ldots$$

The first few Fibonacci numbers are $F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_6 = 13, \ldots$

- The property of the Fibonacci numbers can be used to create a search algorithm that requires only one function evaluation at each iteration.

- The principle of Fibonacci search is that out of two points required for the use of region-elimination rule, one is always the previous point and the other point is new.

- Thus, only one function evaluation is required at each iteration.

- When the region-elimination rule eliminates a portion of the search space depending on the function values at these two points, the remaining search is L_k. By defining $L_k^* = (F_{n-k+1}/F_{n+1})L$ and $L_k = (F_{n-k+2}\,/F_{n+1})L$, it can be shown that $L_k - L_k^* = L_{k+1}$, which means that one of the two points used in iteration k remains as one point in iteration $(k+1)$. If the region (a, x_2) is eliminated in the k^{th} iteration, the point x_1 is at a distance $(L_k - L_k^*)$ or L_{k+1}^* from the point x_2 in the $(k+1)$ iteration. Since, the first two Fibonacci numbers are the same, the algorithm usually starts with $k = 2$.

Fibonacci Search Algorithm:

(i) Choose a lower bound a and an upper bound b. Set $L = b - a$. Assume the desired number of function evaluations to be n. Set $k = 2$.

(ii) Compute $L_k^* = (F_{n-k+1}/F_{n+1})L$. Set $x_1 = a + L_k^*$ and $x_2 = b - L_k^*$.

(iii) Compute one of $f(x_1)$ or $f(x_2)$, which was not evaluated earlier. Use the fundamental region-elimination rule to eliminate a region. Set new a and b.

(iv) Is $k = n$? If not, set $k = k + 1$ and go to step (ii), else terminate the procedure.

Note: In this algorithm, the interval reduces to $(2/F_{n+1})L$ after n function evaluations. Thus, for a desired accuracy ϵ, the number of required function evaluations n can be calculated using the following equation:

$$\frac{2}{F_{n+1}}(b-a)=\epsilon.$$

Example:

Find the minimum of $f(x) = x^2 + \frac{54}{x}$ using Fibonacci search method on $[0, 5]$ and $n = 3$.

Solution: Given function $f(x) = x^2 + \frac{54}{x}$

Step 1: Let $a = 0, b = 5$. Thus, the initial interval is $L = b - a = 5$.
Let us choose the number of function evaluations to be three $(n = 3)$. In practice the large number of n is usually chosen to get the more accurate value. Also,. We set $k = 2$.

Step 2: We compute L_2^* as follows:

$$L_2^* = \left(\frac{F_{3-2+1}}{F_{3+1}}\right) L = \left(\frac{F_2}{F_4}\right).5 = \frac{2}{5}.5 = 2.$$

Calculate $x_1 = 0 + 2 = 2$ and $x_2 = 5 - 2 = 3$.

Step 3: Let us compute $f(x_1) = 31$ and $f(x_2) = 27$
We have $f(x_1) > f(x_2)$, we eliminate the region $(0, x_1)$ or $(0, 2)$. Now, we set $a = 2, b = 5$.

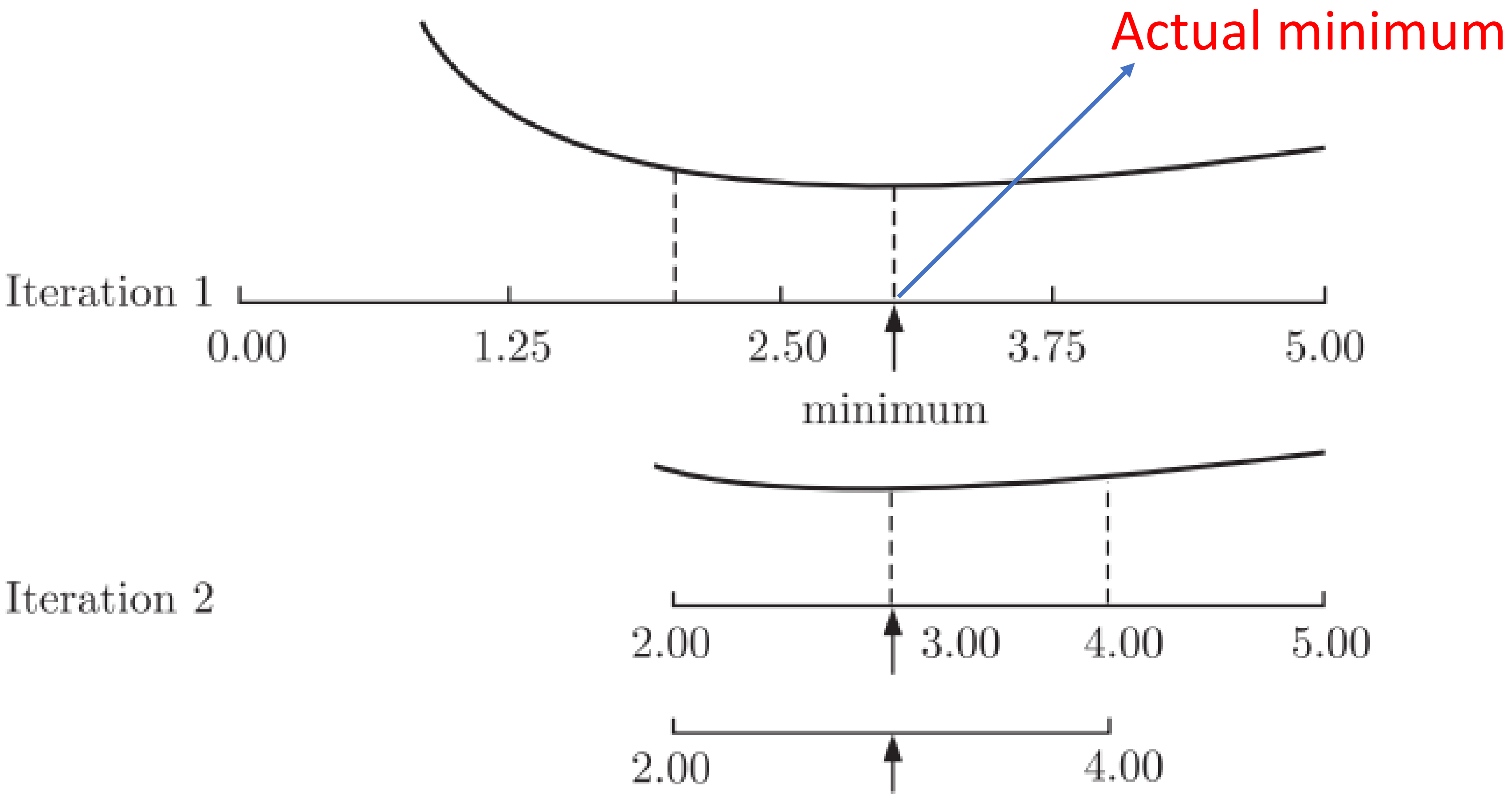
Step 4: Since $k = 2 \neq 3$, we increase $k = k + 1$ and go to step 2. This completes one iteration.

Iteration 2: Now the new interval is $(2, 5)$, i.e., $a = 2$, $b = 5$

Step 2: Compute, $L_3^* = \left(\frac{F_1}{F_4}\right) L = \frac{1}{5}.5 = 1$, $x_1 = 2 + 1 = 3$, and $x_2 = 5 - 1 = 4$.

Step 3: The function evaluation at $x_1 = 3$ is evaluated in the last iteration. Thus, we need to compute the function at $x_2 = 4$ implies $f(x_2) = 29.5$. We have $f(x_1) < f(x_2)$. Therefore, eliminate the region $(4, 5)$.

Step 4: At this iteration $k = n = 3$ and we terminate the algorithm. Therefore, the final interval is $(2,4)$.


Actual minimum
Iteration 1
0.00
1.25
2.50
3.75
5.00
minimum
Iteration 2
2.00
3.00
4.00
5.00
2.00
4.00

Example:

Find the minimum of the function $f(x) = 10 + x^3 - 2x - 5e^x$ using Fibonacci search method on $(-5, 5)$ and $n = 3$.

Golden Search Method

Algorithm:

Step 1: Choose a lower bound a and an upper bound b. Also choose a small number ϵ. Normalize the variable by using the equation $w = (x-a)/(b-a)$. Thus, $a_w = 0, b_w = 1$, and $L_w = 1$. Set $k = 1$.

Step 2: Set $w_1 = a_w + (0.618)L_w$ and $w_2 = b_w - (0.618)L_w$. Compute $f(w_1)$ or $f(w_2)$, depending on whichever of the two was not evaluated earlier. Use the fundamental region-elimination rule to eliminate a region. Set new a_w and b_w.

Step 3: Is $|L_w| < \epsilon$ small? If not, set $k = k + 1$, go to step 2; Else terminate.

Note:

1) Using golden search method, after n function evaluations the interval reduces to $(0.618)^{n-1}$.

2) The number of function evaluations n required to achieve a desired accuracy ϵ is given by $(0.618)^{n-1}(b-a)=\epsilon$.

Relation between Fibonacci method and Golden search method:

$$F_n = F_{n-1} + F_{n-2}$$

$$\frac{F_n}{F_{n-1}} = 1 + \frac{F_{n-2}}{F_{n-1}}$$

If n is large i.e., as $n \to \infty$ then

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = 1 + \lim_{n\to\infty} \frac{F_{n-2}}{F_{n-1}}$$

$$r = 1 + \frac{1}{r}$$

$$r^2 = r + 1$$

$$r^2 - r - 1 = 0$$

$$\text{By solving } r = \frac{1 \pm \sqrt{5}}{2}$$

$$r = 1.618, -0.618.$$

Here, $r = 1.618$ is the golden ratio and $r = -0.618$ is the conjugate golden ratio.

Note: As $n \to \infty$, $\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \frac{1}{\lim_{n\to\infty} \frac{F_{n-2}}{F_{n-1}}}$

Example:

Find the minimum of $f(x) = x^2 + \frac{54}{x}$ using Golden search method on $[0, 5]$ and $\boldsymbol{\epsilon} = \mathbf{0.01}$.

Solution:

Step 1: Given $a = 0, b = 5$. The transformation of the variable x is given by $w = \frac{x-a}{b-a}$

$w = \frac{x}{5}$. Thus, $a_w = 0$, $b_w = 1$, and $L_w = 1$. Since, given function transformed to

$g(w) = 25w^2 + \frac{54}{5w}$, we set $k = 1$.

Iteration 1:

Step 2: We set $w_1 = 0 + (0.618)1 = 0.618$ and $w_2 = 1 - (0.618)1 = 0.382$.

The function values are $g(w_1) = 27.02$ and $g(w_2) = 31.92$.

Here, $g(w_1) < g(w_2)$. Therefore, the minimum cannot lie in any point smaller than 0.382.

Thus, we eliminate the region (a, w_2) or $(0, 0.382)$. Thus, $a_w = 0.382$ and $b_w = 1$. At this stage, $L_w = 1 - 0.382 = 0.618$.

Step 3: Here, $|L_w|$ is not less than ϵ. Therefore, go to next iteration by setting $k = 2$.

Iteration 2:

Step 2: Let us calculate $w_1 = 0.382 + (0.618)0.618 = 0.764.$
$$w_2 = 1 - (0.618)0.618 = 0.618.$$

The function value at $w_2 = 0.618$ is calculated in the previous iteration. Therefore, we need to compute the function value at w_1:

$g(w_1) = 28.73$. And, $g(w_1) > g(w_2)$ we eliminate the interval $(0.763, 1)$ using fundamental region of elimination rule.

The new bounds are $a_w = 0.382$ and $b_w = 0.764$; and $L_w = 0.764 - 0.382 = 0.382.$

Step 3: Here, $|\ L_w\ |$=0.382$\nless 0.01$. Therefore, go to next iteration by setting $k = 3$.

Iteration 3:

Step 2: Here, we observe that $w_1 = 0.382 + (0.618)0.382 = 0.618.$

$$w_2 = 0.764 - (0.618)0.382 = 0.528$$

From above w_1 and w_2, the function value at $w_1 = 0.618$ is evaluated earlier. Thus, we need to compute the function value only at $w_2 = 0.528$ is $g(0.528) = 27.43$. Also, we have $g(w_1) < g(w_2)$ and we eliminate the interval $(0.382, 0.528)$ by fundamental region of elimination. The new interval is $(0.528, 0.764)$ and $L_w = 0.236$.

Step 3: Here, $|L_w| = 0.236 \nless 0.01$. Therefore, go to next iteration by setting k=4.

Iteration 4:

Step 2:

$a_w = 0.528, b_w = 0.764$

$w_1 = 0.528 + (0.618)0.236 = 0.6738$

$w_2 = 0.764 - (0.618)0.236 = 0.6182$

$g(w_1) = 27.3787 \; and \; g(w_2) = 27.0244$

From the above it is clear that $g(w_1) > g(w_2)$ and we eliminate $(0.6738, 0.764)$ by region elimination method. The new interval is $(0.528, 0.6738)$ and $L_w = 0.1458$.

Step 3: $|L_w| = 0.1458 \nless 0.01$. Therefore, go to next iteration by setting k=5.

You can go further iterations until $|L_w| < \epsilon (= 0.01)$.

Note:
The termination condition for the Golden Search method can be depend on the number of functional evaluations or threshold value for the length of the interval (ϵ).

Example:

Find the maximum of $f(x) = \frac{x^4}{4} - \frac{5x^3}{3} - 6x^2 + 19x - 7$, Using Golden section search Method with the interval $[-4, 0]$ and $\epsilon = 0.1$.

Solution:

$$w = \frac{x-(-4)}{4} = \frac{x+4}{4}$$

$$x = 4w - 4$$

$$g(w) = \frac{(4w-4)^4}{4} - \frac{5(4w-4)^3}{3} - 6(4w-4)^2 + 19(4w-4) - 7$$

Do the rest ...

Practice Problems:

Minimize the following functions using Region elimination methods (Fibonacci search method, Golden section search method).

1) $\frac{x^2}{16} - \frac{27x}{4}$ in range $(0, 10)$

2) $x^3 + x^2 - x - 2$ in the interval $(-2, 2)$

3) $-\frac{1.5}{x} + 6(10^{-6})/x^9$ in $(-4, 4)$

4) $f(x) = \exp(x) - x^3$ in $(0, 5)$

5) $f(x) = x^5 - 5x^3 - 20x + 5$ in $(0, 5)$

6. Find the minimum of the following function using golden section search method in terms of the obtained interval after 10 function evaluations in the interval $(-10, 5)$

$$f(x) = x^2 - 10\exp(0.1x)\,.$$

https://www.investopedia.com/articles/technical/04/033104.asp#:~:text=The%20golden%20ratio%20is%20an,introduced%20the%20concept%20to%20Europe.