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Optimization Techniques (MAT-2003)

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Game Theory

Many practical problems require decision-making in a competitive situation where there are two or more opposing parties with conflicting interests and where the action of one depends upon the one taken by the opponent.

Example:

1) Candidates participating in the election.
2) Advertising and marketing campaigns by competing business firms.

Two-person zero-sum games:

When there are two competitors playing a game, it is called a two-person game. In case the number of competitors exceed two, say n, then the game is termed as a ‘n-person game’.

Games having the ‘zero-sum’ character that the algebraic sum of gains and losses of all the players is zero are called **zero-sum games**. i.e., The play does not add a single paisa to the total wealth of all the players. Zero sum games with two players are called two-person zero-sum games. In this case the loss (gain) of one player is exactly equal to the gain (loss) of the other. If the sum of gains or losses is not equal to zero, then the game is of **non-zero-sum game**.

Some Basic Definitions:

1) Player: The competitor(s) in the game is(are) known as player(s). A player may be an individual or group of individuals, or an organization.

2) Strategy: A Strategy for a player is defined as a set of rules or alternative courses of action available to him in advance, by which player decides the course of action that he should adopt.

Let m be the number of strategies of player A, n be the number of strategies of player B, p_i be the probability of selection of the Strategy/Alternative i of player $A, i = 1, 2, \ldots, m$, q_j be the probability of selection of the Strategy/Alternative j of player $B, j = 1, 2, \ldots, n$,

The sum of the probabilities of selection of various alternatives/strategies of each players is equal to 1 that is,

$$\sum_{i=1}^{m} p_i = 1, \qquad \sum_{j=1}^{n} q_j = 1$$

A strategy may be of two types:

a) Pure strategy: If a player selects a particular strategy with a probability of 1, then that strategy is known as a **pure strategy**. This means that the player is **selecting** that **particular strategy alone** ignoring his remaining strategies.

If player A follows a pure strategy , then only one of the *pi* values will be equal to 1 and the remaining *pi* values will be equal to 0. A sample set of probabilities of selection of the alternatives for player A is shown below

$$p_1 = 0, p_2 = 1, p_3 = 0.$$

The sum of these probabilities is equal to 1. That is

$$p_1 + p_2 + p_3 = 1$$

b) Mixed strategy: If a player follows more than one strategy, then the player is said to be follow a mixed strategy. But the **probability** of selection of the individual strategies will be **less than one** and **sum** will be equal to **one**.

If Player B follows a mixed strategy, then a sample set of probabilities of selection of mixed strategy is shown below

$$p_1 = 0.65, p_2 = 0, p_3 = 0.35$$

It is clear that the sum of the probabilities is equal to 1. that is

$$p_1 + p_2 + p_3 = 0.65 + 0 + 0.35 = 1.$$

3) Optimum strategy: A course of action or play which puts the player in the most preferred position, irrespective of the strategy of his competitors, is called an optimum strategy.

4) Pay-off Matrix $[a_{ij}]$:

Each combination of the strategies/alternatives of Player A and B is associated with an outcome

- Pay-off is the outcome of playing the game.
- If a_{ij} is positive, then it represents a gain to player A, and loss to player B.
- If a_{ij} is negative, then it represents a loss to player A, and a gain to player B.
- A_i represents the player at strategy i and B_j represents the player at strategy j.

- Pay-off matrix is a table showing the amount received by the player named at the left-hand side after all possible plays of the game. The payment is made by the player named at the top of the table.

		Player B			
		B_1	B_2	$\cdots$	B_n
Player A	A_1	a_{11}	a_{12}	$\cdots$	a_{1n}
	A_2	a_{21}	a_{22}	$\cdots$	a_{2n}
	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	A_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}

5) Value of the game: It is the expected payoff of play when all the players of the game follow their optimum strategies. The game is called fair if the value of the game is zero and unfair if it is non-zero.

The Maximin and Minimax Principle:

Let us see Maximin-Minimax principle for the selection of optimal strategies by the two players:

Maximin Principle: For the player *A*, minimum value in each row represents the least gain (payoff) to him if he chooses his particular strategy. These are written in the matrix by row minima. He will then select the strategy that maximizes his minimum gains. This choice of payer *A* is called the maximin principle, and the corresponding gain is called the maximum value of the game.

Minimax Principle: Player *B* likes to minimize his losses. The maximum value in each column represents the maximum loss to him if he choses his particular strategy. These are written in the matrix by column maxima. He will then select the strategy that minimizes his maximum losses. This choice of player *B* is called the minimax principle, and the corresponding loss is the minimax value of the game.

Saddle Point: If the maximin value is equal to the minimax value, then the game is said to have a saddle point. The intersecting cell corresponding to these values is known as the saddle point. If the game has a saddle point, then each player has a **pure strategy.**

Value of the Game: In a game, If the game has a saddle point, then the value of the cell at the saddle point is called the **Value of the game**; otherwise, the value of the game is computed based on expected value calculations will be explained later.

Fair and Determinable Game:

We shall denote the maximin value by $\underline{\gamma}$, the minimax of the game by $\overline{\gamma}$ and the value of the game by γ.

A game is said to be **fair** if maximin value = minimax value= 0, i.e., if $\underline{\gamma}=\overline{\gamma}=0$

A game is said to be strictly **determinable** if, maximin value = minimax value$\neq$ 0 and $\underline{\gamma}=\overline{\gamma}=\gamma$

Result: Let (a_{ij}) be the $m \times n$ payoff matrix for a two-person zero sum game. If $\underline{v}$ denotes the maximin value and $\overline{v}$ the minimax value of the game, then $\overline{v} \geq \underline{v}$. That is,

$$\min_{1\leq j\leq n}\left[\max_{1\leq i\leq m}\{a_{ij}\}\right] \geq \max_{1\leq i\leq m}\left[\min_{1\leq j\leq n}\{a_{ij}\}\right].$$

Example:

Find the optimum strategies of the players in the following games:

	B_1	B_2	B_3
A_1	25	20	35
A_2	50	45	55
A_3	58	40	42

Example:

Find the optimum strategies of the players in the following games:

	B_1	B_2	B_3
A_1	25	20	35
A_2	50	45	55
A_3	58	40	42

$\underline{B}$

$$\begin{array}{cc|ccc|c} & & B_1 & B_2 & B_3 & \text{Min} \\ \hline & A_1 & 25 & 20 & 35 & 20 \\ \underline{A} & A_2 & 50 & \boxed{45} & 55 & 45 \leftarrow \\ & A_3 & 58 & 40 & 42 & 40 \\ \hline & \text{Max.} & 58 & 45 & 55 & \\ & & & \uparrow & & \end{array}$$

maximin = 45

minimax = 45.

Maximin = minimax = 45.

= Value of the game

Therefore, the player A coming up with strategy A_2 and player B is coming-up with strategy B_2 which are their optimal strategies.

And the value of the game is $= 45 \neq 0$.

The game is deterministic but unfair. i.e., it’s a strictly deterministic game.

Example:

Determine which of the following two-person zero sum games are strictly determinable and fair. Given the optimum strategy for each player in the case of strictly determinable games.

1.

		Player B		
		B_1	B_2	B_3
Player A	A_1	1	3	1
	A_2	0	-4	-3
	A_3	1	5	-1

2.

		Player B	
		B_1	B_2
Player A	A_1	-5	2
	A_2	-7	-4

Example:

Determine which of the following two-person zero sum games are strictly determinable and fair. Given the optimum strategy for each player in the case of strictly determinable games.

1.

		Player B			
		B_1	B_2	B_3	min
Player A	A_1	$\boxed{1}$	3	$\boxed{1}$	1
	A_2	0	-4	-3	-4
	A_3	1	5	-1	-1
	max	1	5	1	

Maximin=1=minimax.

The game has two saddle points.

It's a pure strategy game.

Maximin=Minimax=value of the game=$1(\neq 0)$.

$\therefore$ the game deterministic but unfair.

The optimal strategies of the players are

$$A \rightarrow A_1$$
$$B \rightarrow B_1 \ or \ B_3$$

2.

		Player B		
		B_1	B_2	min
Player A	A_1	$\boxed{-5}$	2	-5
	A_2	-7	-4	-7
	max	-5	2	

Maximin=-5=minimax.

The game has a saddle point.

It's a pure strategy game.

Maximin=Minimax=value of the game=$-5(\neq 0)$.

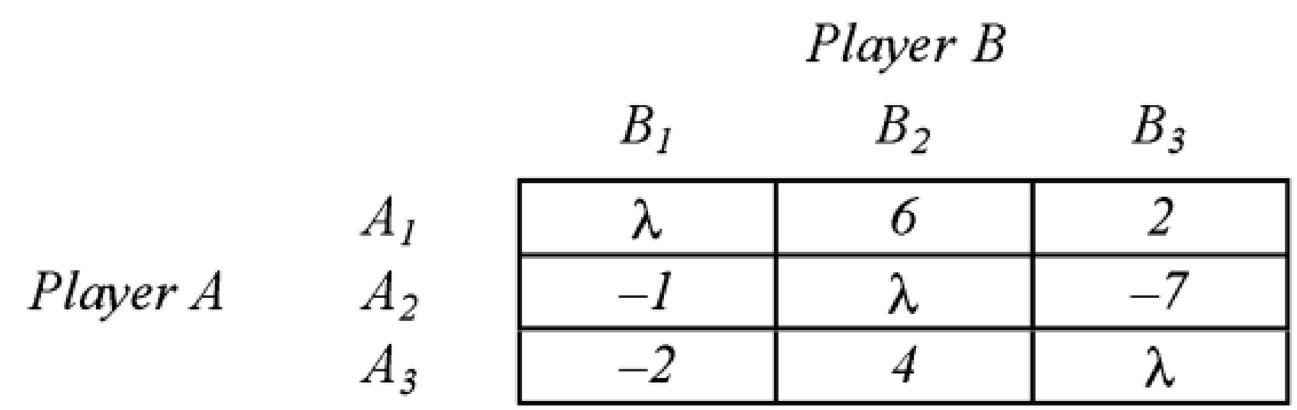
$\therefore$ the game deterministic but unfair.

The optimal strategies of the players are

$$A \rightarrow A_1$$
$$B \rightarrow B_1$$

Example:

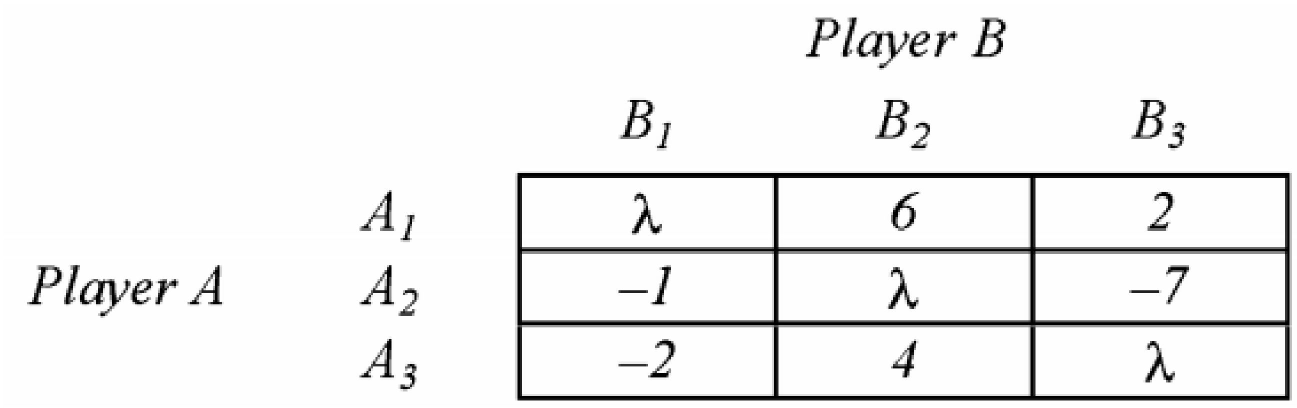
Find the value of λ, the game with the following payoff matrix is strictly determinable.



		Player B		
		B_1	B_2	B_3
Player A	A_1	λ	6	2
	A_2	–1	λ	–7
	A_3	–2	4	λ

Example:

Find the value of λ, the game with the following payoff matrix is strictly determinable.



Player B

		B_1	B_2	B_3	Min
Player A	A_1	λ	6	2	2
	A_2	-1	λ	-7	-7
	A_3	-2	4	λ	-2.
	Max	-1	6	2	

Maximin = 2, minimax = −1.

The value of λ such that the game becomes strictly deterministic is $-1 \leq \lambda \leq 2$