

# Fluid and Diffusion Limits for Bike Sharing Systems

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## Abstract

Bike sharing systems have rapidly developed around the world, and they are served as a promising strategy to improve urban traffic congestion and to decrease polluting gas emissions. So far performance analysis of bike sharing systems always exists many difficulties and challenges under some more general factors. In this paper, a more general large-scale bike sharing system is discussed by means of heavy traffic approximation of multiclass closed queueing networks with non-exponential factors. Based on this, the fluid scaled equations and the diffusion scaled equations are established by means of the numbers of bikes both at the stations and on the roads, respectively. Furthermore, the scaling processes for the numbers of bikes both at the stations and on the roads are proved to converge in distribution to a semimartingale reflecting Brownian motion (SRBM) in a  $N^2$ -dimensional box, and also the fluid and diffusion limit theorems are obtained. Furthermore, performance analysis of the bike sharing system is provided. Thus the results and methodology of this paper provide new highlight in the study of more general large-scale bike sharing systems.

**Keywords:** Bike sharing systems, fluid limit, diffusion limit, semimartingale reflecting Brownian motion.

## 1 Introduction

Bike sharing systems have become an important way of urban transportation due to its accessibility and affordability, and they are widely deployed in more than 600 major cities

around the world. Bike sharing systems are regarded as promising solutions to reduce congestion of traffic and parking, automobile exhaust pollution, transportation noise, and so on. For some survey and development of bike sharing systems, readers may refer to, DeMaio [10], Shaheen et al. [33], Shu et al. [35], Labadi et al. [23], and Meddin and DeMaio [28].

Two major operational issues of bike sharing systems are to care for (i) the non-empty: sufficient bikes parked at each station in order to be able to rent a bike at any time; and (ii) the non-full: suitable bike parking capacity designed for each station in order to be able to return a bike in real time. Thus the empty or full stations are called problematic stations. Up till now, efficient measures are developed in the study of problematic stations, including time-nonhomogeneous demand forecasting, average bike inventory level, timely bike repositioning, and probability analysis of problematic stations.

So far queueing models and Markov processes have been applied to characterizing important steady-state performance of the bike sharing systems. Important prior works on the bike sharing models include the  $M/M/1/C$  queue by Leurent [22] and Schuijbroek et al. [34]; the time-inhomogeneous  $M(t)/M(t)/1/C$  model by Raviv et al. [31] and Raviv and Kolka [30]; the queueing networks by Kochel et al. [20], Savin et al. [32], Adelman [1], George and Xia [14, 15] and Li et al. [26]; the fluid models combining with Markov decision processes by Waserhole and Jost [36, 37]; the mean-field theory by Fricker et al. [11], Fricker and Gast [12] and Fricker and Tibi [13]; the time-inhomogeneous  $M(t)/M(t)/1/K$  and  $MAP(t)/MAP(t)/1/K + 2L + 1$  queues combining with mean-field theory by Li et al. [24] and Li and Fan [25].

An important and realistic feature of bike sharing systems is the time-varying arrivals of bike users and their random travel times. In general, analysis of bike sharing systems with non-Poisson user arrivals and general travel times are always very difficult and challenging because more complicated multiclass closed queueing networks are established to deal with bike sharing systems. See Li et al. [26] for more interpretations. For this, fluid and diffusion approximations may be an effective and better method in the study of more general bike sharing systems. This motivates us in this paper to develop fluid and diffusion limits for more general large-scale bike sharing systems.

Fluid and diffusion approximations are usually applied to analysis of more general large-scale complicated queueing networks, which possibly originate in some practical systems including communication networks, manufacturing systems, transportation networks

and so forth. See excellent monographs by, for example, Harrison [16], Chen and Yao [4], Whitt [38]. For the bike sharing system, further useful information is introduced as follows. **(a)** For heavy traffic approximation of closed queueing networks, readers may refer to, such as, Harrison et al. [19] for a closed queueing network with homogeneous customer population and infinite buffer. Chen and Mandelbaum [3] for a closed Jackson network, Harrison and Williams [18] for a multiclass closed network with two single-server stations and a fixed customer population. Kumar [21] for a two-server closed networks in heavy traffic. **(b)** For heavy traffic approximation of queueing networks with finite buffers, important examples include, Dai and Dai [6] obtained the SRBM of queue-length process relying on a uniform oscillation result for solutions to a family of Skorohod problems. Dai [8] modeled the queueing networks with finite buffers under a communication blocking scheme, showed that the properly normalized queue length process converges weakly to a reflected Brownian motion in a rectangular box, and presented a general implementation via finite element method to compute the stationary distribution of SRBM. Furthermore, Dai [9] analyzed a multiclass queueing networks with finite buffers and a feedforward routing structure under a blocking scheme, and showed a pseudo-heavy-traffic limit theorem which stated that the limit process of queue length is a reflecting Brownian motion. **(c)** There are some available results on heavy traffic approximation of multiclass queueing networks, readers may refer to, for instance, Harrison and Nguyen [17], Dai [5], Bramson [2], Meyn [29] and Majewski [27].

**Contributions of this paper:** The main contributions of this paper are threefold. The first contribution is to propose a more general large-scale bike sharing system having renewal arrival processes of bike users and general travel times, and to establish a multiclass closed queueing network from the practical factors of the bike-sharing system where bikes are abstracted as virtual customers, while both stations and roads are regarded as virtual nodes or servers. Note that the virtual customers (i.e. bikes) at stations are of single class; while the virtual customers (i.e. bikes) on roads are of two different classes due to two classes of different bike travel or return times. The second contribution is to set up the queue-length processes of the multiclass closed queueing network through observing both some bikes parked at stations and the other bikes ridden on roads. Such analysis gives the fluid scaled equations and the diffusion scaled equations by means of the numbers of bikes both at the stations and on the roads. The third contribution is to prove that the scaling processes, corresponding to the numbers of bikes both at the stations (having one class of

virtual customers) and on the roads (having two classes of virtual customers), converge in distribution to a semimartingale reflecting Brownian motion, and the fluid and diffusion limit theorems are obtained in some simple versions. Based on this, performance analysis of the bike sharing system is also given. Therefore, the results and methodology given in this paper provide new highlight on the study of more general large-scale bike sharing systems.

**Organization of this paper:** The structure of this paper is organized as follows. In Section 2, we describe a more general large-scale bike sharing system with  $N$  different stations and with  $N(N-1)$  different roads, while this system has renewal arrival processes of bike users and general travel times on the roads. In Section 3, we establish a multiclass closed queueing network from practical factors of the bike-sharing system where bikes are abstracted as virtual customers, while both stations and roads are regarded as virtual nodes or servers. In Section 4, we set up the queue-length processes of the multiclass closed queueing network by means of the numbers of bikes both at the stations and on the roads, and establish the fluid scaled equations and the diffusion scaled equations. In Sections 5 and 6, we prove that the scaling processes of the bike sharing system converge in distribution to a semimartingale reflecting Brownian motion under heavy traffic conditions, and obtain the fluid limit theorem and the diffusion limit theorem, respectively. In Sections 7, we give performance analysis of the bike sharing system by means of the fluid and diffusion limits. Finally, some concluding remarks are described in Section 8.

**Useful notation:** We now introduce the notation used in the paper. For positive integer  $n$ , the  $n$ -dimensional Euclidean space is denoted by  $\mathcal{R}^n$  and the  $n$ -dimensional positive orthant is denoted by  $\mathcal{R}_+^n = \{x \in \mathcal{R}^n : x_i \geq 0\}$ . We define  $D_{\mathcal{R}^n}[0, T]$  as the path space of all functions  $f : [0, T] \rightarrow \mathcal{R}^n$  which are right continuous and have left limits. Define  $\delta_{j,k} = 1$  if  $j = k$ , else,  $\delta_{j,k} = 0$ . For a set  $\mathcal{K}$ , let  $|\mathcal{K}|$  denote its cardinality. u.o.c. means that the convergence is uniformly on compact set. A triple  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$  is called a filtered space if  $\Omega$  is a set,  $\mathcal{F}$  is a  $\sigma$ -field of subsets of  $\Omega$ , and  $\{\mathcal{F}_t, t \geq 0\}$  is an increasing family of sub- $\sigma$ -fields of  $\mathcal{F}$ , i.e., a filtration. If, in addition,  $P$  is a probability measure on  $(\Omega, \mathcal{F})$ , then  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$  is called a filtered probability space. Let  $P_x$  denote the unique family of probability measures on  $(\Omega, \mathcal{F})$ , and  $E_x$  be the expectation operator under  $P_x$ .

## 2 Model Description

In this section, we describe a more general large-scale bike sharing system with  $N$  different stations and with  $N(N - 1)$  different roads, which has renewal arrival processes of bike users and general travel times.

In the large-scale bike sharing system, a customer arrives at a nonempty station, rents a bike, and uses it for a while, then he returns the bike to a destination station and immediately leaves this system. If a customer arrives at a empty station, then he immediately leaves this system.

Now, we describe the bike sharing system including operations mechanism, system parameters and mathematical notation as follows:

**(1) Stations and roads:** We assume that the bike sharing system contains  $N$  different stations and at most  $N(N - 1)$  different roads, where a pair of directed roads may be designed from any station to another station. Also, we assume that at the initial time  $t = 0$ , each station has  $C_i$  bikes and  $K_i$  parking positions, where  $1 \leq C_i \leq K_i < \infty$  for  $i = 1, \dots, N$  and  $\sum_{i=1}^N C_i > K_j$  for  $j = 1, \dots, N$ . Note that these conditions make that some bikes can result in at least a full station.

**(2) Arrival processes:** The arrivals of outside bike users (or customers) at each station is a general renewal process. For station  $j$ , let  $u_j = \{u_j(n), n \geq 1\}$  be an i.i.d. random sequence of exogenous interarrival times, where  $u_j(n) \geq 0$  is the interarrival time between the  $(n - 1)$ st customer and the  $n$ th customer. We assume that  $u_j(n)$  has the mean  $1/\lambda_j$  and the coefficient of variation  $c_{a,j}$ .

**(3) The bike return times:**

**(3.1) The first return:** Once an outside customer successfully rents a bike from station  $i$ , then he rides on a road directed to station  $j$  with probability  $p_{i \rightarrow j}$  for  $\sum_{j \neq i}^N p_{i \rightarrow j} = 1$ , and his riding-bike time  $v_{i \rightarrow j}^{(1)}$  on the road  $i \rightarrow j$  is a general distribution with the mean  $1/\mu_{i \rightarrow j}^{(1)}$  and the coefficient of variation  $c_{s,i \rightarrow j}^{(1)}$ . If there is at least one available parking position at station  $j$ , then the customer directly returns his bike to station  $j$ , and immediately leaves this system. Let  $r^i = \{r_j^i(n), n \geq 1\}$  be a sequence of routing selections for  $i, j = 1, \dots, N$  with  $i \neq j$ , where  $r_j^i(n) = 1$  means that the  $n$ th customer rents a bike from station  $i$  and rides on a road directed to station  $j$  (i.e., the customer rides on road  $i \rightarrow j$ ), hence  $\Pr\{r_j^i(n) = 1\} = p_{i \rightarrow j}$ .

**(3.2) The second return:** From (3.1), if no parking position is available at station

$j$ , then the customer has to ride the bike to another station  $l_1$  with probability  $\alpha_{j \rightarrow l_1}$  for  $\sum_{l_1 \neq j}^N \alpha_{j \rightarrow l_1} = 1$ , and his riding-bike time  $v_{j \rightarrow l_1}^{(2)}$  on road  $j \rightarrow l_1$  is also a general distribution with the mean  $1/\mu_{j \rightarrow l_1}^{(2)}$  and the coefficient of variation  $c_{s,j \rightarrow l_1}^{(2)}$ . If there is at least one available parking position at station  $l_1$ , then the customer directly returns his bike and immediately leaves this bike sharing system.

**(3.3) The  $(k + 1)$ st return for  $k \geq 2$ :** From (3.2) and more, we assume that this bike has not been returned at any station yet through  $k$  consecutive returns. In this case, the customer has to try his  $(k + 1)$ st lucky return, he will ride bike from the  $l_{k-1}$ th full station to the  $l_k$ th station with probability  $\alpha_{l_{k-1} \rightarrow l_k}$  for  $\sum_{l_k \neq l_{k-1}}^N \alpha_{l_{k-1} \rightarrow l_k} = 1$ , and his riding-bike time  $v_{l_{k-1} \rightarrow l_k}^{(2)}$  on road  $l_{k-1} \rightarrow l_k$  is also a general distribution with the mean  $1/\mu_{l_{k-1} \rightarrow l_k}^{(2)}$  and the coefficient of variation  $c_{s,l_{k-1} \rightarrow l_k}^{(2)}$ . If there is at least one available parking station, then the customer directly returns his bike and immediately leaves this bike sharing system; otherwise he has to continuously try another station again. In the next section, those bikes ridden under their first return are called the first class of virtual customers; while those bikes ridden under the  $k$  ( $k \geq 2$ ) returns are called the second class of virtual customers. Let  $\bar{r}^j = \{\bar{r}_i^j(n), n \geq 1\}$  be a sequence of routing selections for  $i, j = 1, \dots, N$  with  $i \neq j$ , where  $\bar{r}_i^j(n) = 1$  means that the  $n$ th customer who can not return the bike to the full station  $j$  will deflect into road  $j \rightarrow i$ , thus  $\Pr\{\bar{r}_i^j(n) = 1\} = \alpha_{j \rightarrow i}$ . Similarly, let  $r^{j \rightarrow i, (d)} = \{r^{j \rightarrow i, (d)}(n), n \geq 1\}$  be a sequence of routing selections for  $i, j = 1, \dots, N$  with  $i \neq j$ ,  $d = 1, 2$ , where  $r^{j \rightarrow i, (d)}(n) = 1$  means the  $n$ th customer of class  $d$  who completes his short trip on road  $j \rightarrow i$  will return the bike to station  $i$ , hence  $\Pr\{r^{j \rightarrow i, (d)}(n) = 1\} = p_{j \rightarrow i, i} = 1$ .

**(4) Two classes of riding-bike times:** In (3), there are two classes of riding-bike times, who have two general distributions, that is, there are two classes of virtual customers riding on each road. Let  $v_{j \rightarrow i}^{(d)} = \{v_{j \rightarrow i}^{(d)}(n), n \geq 1\}$  be a random sequence of riding-bike times of class  $d$  for  $i, j = 1, \dots, N$  with  $i \neq j$ ,  $d = 1, 2$ , where  $v_{j \rightarrow i}^{(d)}(n)$  is the riding-bike time for the  $n$ th customer of class  $d$  riding on the road  $j \rightarrow i$ . We assume that  $v_{j \rightarrow i}^{(d)}$  has the mean  $1/\mu_{j \rightarrow i}^{(d)}$  and the coefficient of variation  $c_{s,j \rightarrow i}^{(d)}$ . To care for the expected riding-bike times, we set that  $\mu_{j \rightarrow i}^{(d)} = 1/m_{j \rightarrow i}$  for  $d = 1$  and  $\mu_{j \rightarrow i}^{(d)} = 1/\xi_{j \rightarrow i}$  for  $d = 2$ .

**(5) The departure disciplines:** The customer departure has two different cases: (a) an outside customer directly leaves the bike sharing system if he arrives at an empty station; (b) if one customer rents and uses a bike, and he finally returns the bike to a station, then the customer completes his trip and immediately leaves the bike sharing

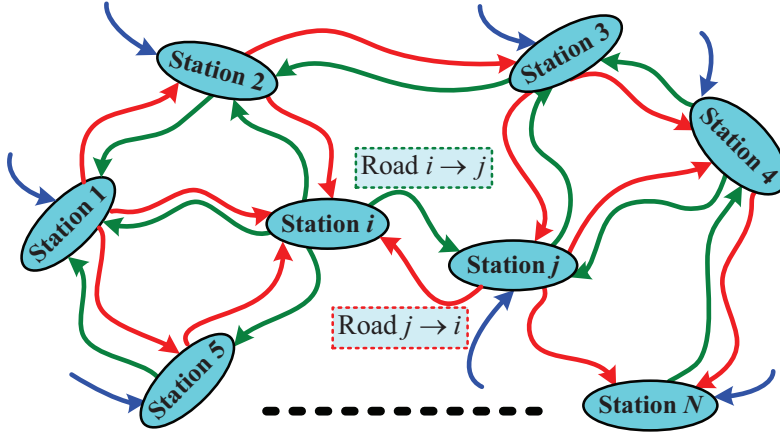


Figure 1: The physical structure of the bike sharing system.

system.

For such a bike sharing system, Figure 1 outlines its physical structure and associated operations.

### 3 The Closed Queueing Network

In this section, we establish a multiclass closed queueing network from the bike-sharing system where bikes are abstracted as virtual customers, and both stations and roads are regarded as virtual nodes or servers. Specifically, the stations contains only one class of virtual customers; while the roads can contains two classes of virtual customers.

In the bike sharing system, there are  $N$  stations and  $N(N-1)$  roads, and each bike can not leave this system, hence, the total number of bikes in this system is fixed as  $\sum_{i=1}^N C_i$ . Base on this, such a system can be regarded as a closed queueing network with multiclass customers due to two types of different travel or return times.

Let  $S_i$  and  $R_{i \rightarrow j}$  denote station  $i$  and road  $i \rightarrow j$ , respectively. Let SN denote the set of nodes abstracted by the stations, and RN the set of nodes abstracted by the roads. Clearly  $\text{SN} = \{S_i, i = 1, \dots, N\}$  and  $\text{RN} = \{R_{i \rightarrow j} : i, j = 1, \dots, N \text{ with } i \neq j\}$ . Let  $n_j$  and  $n_{i \rightarrow j}^{(d)}$  denote the numbers of bikes parking in the  $j$ th station node and of bikes of class  $d$  riding on the road  $i \rightarrow j$  node, respectively.

**(1) Virtual nodes:** Although the stations and the roads have different physical attributes, they are all regarded as abstract nodes in the closed queueing network.

**(2) Virtual customers:** The virtual customers are abstracted by the bikes, which are either parked in the stations or ridden on the roads. It is seen that only one class of virtual customers are packed in the station nodes; while two classes of different virtual customers are ridden on the road nodes due to their different return times.

**(3) The routing matrix  $P$ :** To express the routing matrix, we first define a mapping  $\sigma(\cdot)$  as follow,

$$\begin{cases} \sigma(S_i) = i & \text{for } i = 1, \dots, N, \\ \sigma(R_{i \rightarrow j}) = i\langle j \rangle & \text{for } i, j = 1, \dots, N, \text{ with } i \neq j. \end{cases}$$

It is necessary to understand the mapping  $\sigma(\cdot)$ . For example,  $N = 2$ ,  $\sigma(S_1) = 1$ ,  $\sigma(S_2) = 2$ ,  $\sigma(R_{1 \rightarrow 2}) = 1\langle 2 \rangle$ ,  $\sigma(R_{2 \rightarrow 1}) = 2\langle 1 \rangle$ , thus the routing matrix is written as

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 1\langle 2 \rangle & 2\langle 1 \rangle \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 1\langle 2 \rangle \\ 2\langle 1 \rangle \end{matrix} & \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{matrix}.$$

In this case, the component  $p_{\tilde{i}, \tilde{j}}$  of the routing matrix  $P$  denotes the probability that a customer leaves node  $\tilde{i}$  to node  $\tilde{j}$ , where

$$p_{\tilde{i}, \tilde{j}} = \begin{cases} 1 & \text{if } \tilde{i} = \sigma(R_{i \rightarrow j}), \tilde{j} = \sigma(S_j), \\ p_{i \rightarrow j} & \text{if } \tilde{i} = \sigma(S_i), \tilde{j} = \sigma(R_{i \rightarrow j}), \\ \alpha_{j \rightarrow k} & \text{if } \tilde{i} = \sigma(R_{i \rightarrow j}), \tilde{j} = \sigma(R_{j \rightarrow k}), \\ 0 & \text{otherwise.} \end{cases}$$

**(4) The service processes in the station nodes:** For  $j \in \text{SN}$ , the service process  $S_j = \{S_j(t), t \geq 0\}$  of station node  $j$ , associated with the interarrival time sequence  $u_j = \{u_j(n), n \geq 1\}$  of the outside customers who arrive at station  $j$ , is given by

$$S_j(t) = \sup\{n : U_j(n) \leq t\},$$

where  $U_j(n) = \sum_{l=1}^n u_j(l)$ ,  $n \geq 1$  and  $U_j(0) = 0$ . Let  $b_j = \lambda_j 1_{\{1 \leq n_j \leq K_j\}}$ .

**(5) The service processes in the road nodes:** For  $i, j = 1, \dots, N$  with  $i \neq j$  and  $d = 1, 2$ , the service process  $S_{j \rightarrow i}^{(d)} = \{S_{j \rightarrow i}^{(d)}(t), t \geq 0\}$  of road node  $j \rightarrow i$ , associated with the riding-bike time sequence  $v_{j \rightarrow i}^{(d)} = \{v_{j \rightarrow i}^{(d)}(n), n \geq 1\}$  of the customers of class  $d$  ridden on road  $j \rightarrow i$ , is given by

$$S_{j \rightarrow i}^{(d)}(t) = \sup\{n : V_{j \rightarrow i}^{(d)}(n) \leq t\},$$



where  $V_{j \rightarrow i}^{(d)}(n) = \sum_{l=1}^n v_{j \rightarrow i}^{(d)}(l)$ ,  $n \geq 1$  and  $V_{j \rightarrow i}^{(d)}(0) = 0$ . We write

$$b_{j \rightarrow i}^{(d)} = n_{j \rightarrow i}^{(d)} \mu_{j \rightarrow i}^{(d)} = \begin{cases} n_{j \rightarrow i}^{(1)} \frac{1}{m_{j \rightarrow i}} & d = 1, \\ n_{j \rightarrow i}^{(2)} \frac{1}{\xi_{j \rightarrow i}} & d = 2. \end{cases}$$

**(6) The routing processes in the station nodes:**

*Case one:* For  $j \in \text{SN}$ , the routing process  $R^j = \{R_i^j, i \neq j, i = 1, \dots, N\}$  and  $R_i^j = \{R_i^j(n), n \geq 1\}$ , associated with the routing selecting sequence  $r^i = \{r_i^j(n), n \geq 1\}$  of station  $j$ , is given by

$$R^j(n) = \sum_{l=1}^n r^j(l) \text{ or } R_i^j(n) = \sum_{l=1}^n r_i^j(l), n \geq 1,$$

and the  $i$ th component of  $R^j(n)$  is  $R_i^j(n)$  associated with probability  $p_{j \rightarrow i}$ .

*Case two:* For  $j \in \text{SN}$ , the routing process  $\bar{R}^j = \{\bar{R}_i^j, i \neq j, i = 1, \dots, N\}$  and  $\bar{R}_i^j = \{\bar{R}_i^j(n), n \geq 1\}$ , associated with the routing deflecting sequence  $\bar{r}^j = \{\bar{r}_i^j(n), n \geq 1\}$  of station  $j$ , is given by

$$\bar{R}^j(n) = \sum_{l=1}^n \bar{r}^j(l) \text{ or } \bar{R}_i^j(n) = \sum_{l=1}^n \bar{r}_i^j(l), n \geq 1,$$

and the  $i$ th component of  $\bar{R}^j(n)$  is  $\bar{R}_i^j(n)$  associated with probability  $\alpha_{j \rightarrow i}$ .

**(7) The routing processes in the road nodes:** For  $i, j = 1, \dots, N$  with  $i \neq j$  and  $d = 1, 2$ , the routing process  $R^{j \rightarrow i, (d)} = \{R^{j \rightarrow i, (d)}(n), n \geq 1\}$ , associated with the routing transferring sequence  $r^{j \rightarrow i, (d)} = \{r^{j \rightarrow i, (d)}(n), n \geq 1\}$  of road  $j \rightarrow i$ , is given by

$$R^{j \rightarrow i, (d)}(n) = \sum_{l=1}^n r^{j \rightarrow i, (d)}(l), n \geq 1,$$

and the  $R^{j \rightarrow i, (d)}(n)$  is associated with probability  $p_{j \rightarrow i, i} = 1$ .

**(8) Service disciplines:** The first come first served (FCFS) discipline is assumed for all station nodes. A new processor sharing (PS) is used for all the road nodes, where each customer of either class one or class two is served by a general service time distribution, as described in (4) and (5).

## 4 The Joint Queueing Process

In this section, we set up the queue-length processes of the multiclass closed queueing network by means of the numbers of bikes both at the stations and on the roads, and establish the fluid scaled equations and the diffusion scaled equations.

(1)  $Q(t) = \{(Q_j(t), Q_{j \rightarrow i}^{(d)}(t)), i \neq j, i, j = 1, \dots, N; d = 1, 2; t \geq 0\}$ , where  $Q_j(t)$  and  $Q_{j \rightarrow i}^{(d)}(t)$  are the number of virtual customers at station node  $j$  and the numbers of virtual customers of class  $d$  at the road  $j \rightarrow i$  at time  $t$ , respectively. Specifically,  $Q_j(0)$  and  $Q_{j \rightarrow i}^{(d)}(0)$  are the number of virtual customers at station node  $j$  and the number of virtual customers of class  $d$  on the road node  $j \rightarrow i$  at time  $t = 0$ , respectively.

(2)  $Y^K(t) = \{(Y_j^K(t)), j = 1, \dots, N; t \geq 0\}$ , where  $Y_j^K(t)$  is the cumulative number of virtual customers deflecting from station node  $j$  whose parking positions are full in the time interval  $[0, t]$ .

(3)  $Y^0(t) = \{(Y_j^0(t), Y_{j \rightarrow i}^{0,(d)}(t)), i \neq j, i, j = 1, \dots, N; d = 1, 2; t \geq 0\}$ , where  $Y_j^0(t)$  and  $Y_{j \rightarrow i}^{0,(d)}(t)$  are the cumulative amount of time that station node  $j$  and the road node  $j \rightarrow i$  are idle (no available bike, i.e., empty) in the time interval  $[0, t]$ , respectively.

$$Y_j^0(t) = \int_0^t 1\{Q_j(s) = 0\}ds = t - B_j(t),$$

$$Y_{j \rightarrow i}^{0,(d)}(t) = \int_0^t 1\{Q_{j \rightarrow i}^{(d)}(s) = 0\}ds = t - B_{j \rightarrow i}^{(d)}(t).$$

(4)  $B(t) = \{(B_j(t), B_{j \rightarrow i}^{(d)}(t)), i \neq j, i, j = 1, \dots, N; d = 1, 2; t \geq 0\}$ , where  $B_j(t)$  and  $B_{j \rightarrow i}^{(d)}(t)$  are the cumulative amount of time that the station node  $j$  and the road node  $j \rightarrow i$  are busy (available bike, non-empty) in the time interval  $[0, t]$ , respectively.

$$B_j(t) = \int_0^t 1\{0 < Q_j(s) \leq K_j\}ds,$$

$$B_{j \rightarrow i}^{(d)}(t) = \int_0^t 1\{Q_{j \rightarrow i}^{(d)}(s) > 0\}ds.$$

(5)  $B^F(t) = \{(B_j^F(t)), j = 1, \dots, N; t \geq 0\}$ , where  $B_j^F(t)$  is the cumulative amount of time that station node  $j$  is full (no available parking position) in the time interval  $[0, t]$ ,

$$B_j^F(t) = \int_0^t 1\{Q_j(s) = K_j\}ds.$$

(6)  $S_j(B_j(t))$  denotes the number of virtual customers that have completed service at station node  $j$  during the time interval  $[0, t]$ ;  $S_{j \rightarrow i}^{(d)}(B_{j \rightarrow i}^{(d)}(t))$  denotes the number of virtual customers of class  $d$  that have completed service at road node  $j \rightarrow i$  during the time interval  $[0, t]$ .

(7)  $R_i^j(S_j(B_j(t)))$  denotes the number of virtual customers that enter station node  $i$  (i.e., riding on road  $j \rightarrow i$ ) from station node  $j$  during the time interval  $[0, t]$ ;  $\bar{R}_i^j(Y_j^K(t))$  denotes the number of virtual customers that enter station node  $i$  from station  $j$  whose

parking positions are full during the time interval  $[0, t]$ ; and  $R^{j \rightarrow i, (d)}(S_{j \rightarrow i}^{(d)}(B_{j \rightarrow i}^{(d)}(t)))$  denotes the number of virtual customers of class  $d$  that enter station node  $i$  from road node  $j \rightarrow i$  during the time interval  $[0, t]$ .

Now, we have the following flow balance relations for the station nodes and the road nodes. For station node  $j = 1, \dots, N$ ,

$$Q_j(t) = Q_j(0) + \sum_{d=1}^2 \sum_{i \neq j}^N \left[ R^{i \rightarrow j, (d)}(S_{i \rightarrow j}^{(d)}(B_{i \rightarrow j}^{(d)}(t))) - R^{i \rightarrow j, (d)}(S_{i \rightarrow j}^{(d)}(B_j^F(t))) \right] - S_j(B_j(t)). \quad (1)$$

Note that  $Y_j^K(t) = \sum_{d=1}^2 \sum_{i \neq j}^N R^{i \rightarrow j, (d)}(S_{i \rightarrow j}^{(d)}(B_j^F(t)))$ , we have

$$Q_j(t) = Q_j(0) + \sum_{d=1}^2 \sum_{i \neq j}^N R^{i \rightarrow j, (d)}(S_{i \rightarrow j}^{(d)}(B_{i \rightarrow j}^{(d)}(t))) - S_j(B_j(t)) - Y_j^K(t). \quad (2)$$

For road node  $j \rightarrow i$  for  $i, j = 1, \dots, N$  with  $i \neq j$  and  $d = 1, 2$ , we have

$$Q_{j \rightarrow i}^{(1)}(t) = Q_{j \rightarrow i}^{(1)}(0) + R_i^j(S_j(B_j(t))) - S_{j \rightarrow i}^{(1)}(B_{j \rightarrow i}^{(1)}(t)), \quad (3)$$

$$Q_{j \rightarrow i}^{(2)}(t) = Q_{j \rightarrow i}^{(2)}(0) + \bar{R}_i^j(Y_j^K(t)) - S_{j \rightarrow i}^{(2)}(B_{j \rightarrow i}^{(2)}(t)). \quad (4)$$

Because the total number of bikes in this bike sharing system is fixed as  $\sum_{i=1}^N C_i$ , we get that for  $t \geq 0$

$$\sum_{i=1}^N Q_i(t) + \sum_{d=1}^2 \sum_{i \neq j}^N Q_{i \rightarrow j}^{(d)}(t) = \sum_{i=1}^N C_i. \quad (5)$$

We now elaborate to apply a centering operation to the queue-length representations of the station nodes and of the road nodes, and rewrite (2), (3) and (4) as follows:

$$Q(t) = X(t) + R^0 Y^0(t) + R^K Y^K(t), \quad (6)$$

where  $X(t) = (X_1(t), X_2(t), \dots, X_N(t))$ , and  $X_j(t)$  is given by

$$\begin{aligned} X_j(t) = & Q_j(0) + \sum_{d=1}^2 \sum_{i \neq j}^N \left[ R^{i \rightarrow j, (d)}(S_{i \rightarrow j}^{(d)}(B_{i \rightarrow j}^{(d)}(t))) - S_{i \rightarrow j}^{(d)}(B_{i \rightarrow j}^{(d)}(t)) \right] \\ & + \sum_{d=1}^2 \sum_{i \neq j}^N \left[ S_{i \rightarrow j}^{(d)}(B_{i \rightarrow j}^{(d)}(t)) - b_{i \rightarrow j}^{(d)} B_{i \rightarrow j}^{(d)}(t) \right] - [S_j(B_j(t)) - b_j B_j(t)] \\ & - Y_j^K(t) + \theta_j t, \end{aligned} \quad (7)$$

note that  $R^{i \rightarrow j, (d)}(S_{i \rightarrow j}^{(d)}(B_{i \rightarrow j}^{(d)}(t))) = S_{i \rightarrow j}^{(d)}(B_{i \rightarrow j}^{(d)}(t))$ ,  $X_j(t)$  is simplified as

$$X_j(t) = Q_j(0) + \sum_{d=1}^2 \sum_{i \neq j}^N \left[ S_{i \rightarrow j}^{(d)}(B_{i \rightarrow j}^{(d)}(t)) - b_{i \rightarrow j}^{(d)} B_{i \rightarrow j}^{(d)}(t) \right] - [S_j(B_j(t)) - b_j B_j(t)] - Y_j^K(t) + \theta_j t, \quad (8)$$

$$\theta_j = \sum_{d=1}^2 \sum_{i \neq j}^N b_{i \rightarrow j}^{(d)} - b_j, \quad (9)$$

$$(R^0 Y^0(t))_{\tilde{i}, \tilde{j}} = \begin{cases} b_j Y_j^0(t), & \text{if } \tilde{i} = \sigma(S_i), \text{ and } \tilde{j} = \tilde{i}, \\ -\sum_{d=1}^2 b_{i \rightarrow j}^{(d)} Y_{i \rightarrow j}^{0, (d)}(t), & \text{if } \tilde{i} = \sigma(S_i), \text{ and } \tilde{j} = \sigma(R_{i \rightarrow j}), \\ 0, & \text{otherwise,} \end{cases} \quad (10)$$

$$(R^K Y^K(t))_{\tilde{i}, \tilde{j}} = \begin{cases} -Y_j^K(t), & \text{if } \tilde{i} = \sigma(S_i), \text{ and } \tilde{i} = \tilde{j}, \\ 0, & \text{otherwise.} \end{cases} \quad (11)$$

For road node  $j \rightarrow i$  ( $i, j = 1, \dots, N$  with  $i \neq j$  and  $d = 1, 2$ ),  $X_{j \rightarrow i}^{(d)}(t)$  is given by,

$$X_{j \rightarrow i}^{(1)}(t) = Q_{j \rightarrow i}^{(1)}(0) + \left[ R_i^j(S_j(B_j(t))) - p_{j \rightarrow i} S_j(B_j(t)) \right] + [p_{j \rightarrow i}(S_j(B_j(t)) - b_j B_j(t))] - \left[ S_{j \rightarrow i}^{(1)}(B_{j \rightarrow i}^{(1)}(t)) - b_{j \rightarrow i}^{(1)} B_{j \rightarrow i}^{(1)}(t) \right] + \theta_{j \rightarrow i}^{(1)} t, \quad (12)$$

$$\theta_{j \rightarrow i}^{(1)} = p_{j \rightarrow i} b_j - b_{j \rightarrow i}^{(1)}, \quad (13)$$

$$(R^0 Y^0(t))_{\tilde{i}, \tilde{j}} = \begin{cases} b_{j \rightarrow i}^{(1)} Y_{j \rightarrow i}^{0, (1)}(t), & \text{if } \tilde{i} = \sigma(R_{j \rightarrow i}) \text{ and } \tilde{j} = \tilde{i}, \\ -p_{j \rightarrow i} b_j Y_j^0(t), & \text{if } \tilde{i} = \sigma(R_{j \rightarrow i}) \text{ and } \tilde{j} = \sigma(S_j), \\ 0, & \text{otherwise,} \end{cases} \quad (14)$$

$$(R^K Y^K(t))_{\tilde{i}, \tilde{j}} = 0. \quad (15)$$

$$X_{j \rightarrow i}^{(2)}(t) = Q_{j \rightarrow i}^{(2)}(0) + \left[ \bar{R}_i^j(Y_j^K(t)) - \alpha_{j \rightarrow i} Y_j^K(t) \right] - \left[ S_{j \rightarrow i}^{(2)}(B_{j \rightarrow i}^{(2)}(t)) - b_{j \rightarrow i}^{(2)} B_{j \rightarrow i}^{(2)}(t) \right] + \theta_{j \rightarrow i}^{(2)} t, \quad (16)$$

$$\theta_{j \rightarrow i}^{(2)} = -b_{j \rightarrow i}^{(2)}, \quad (17)$$

$$(R^0 Y^0(t))_{\tilde{i}, \tilde{j}} = \begin{cases} b_{j \rightarrow i}^{(2)} Y_{j \rightarrow i}^{0, (2)}(t), & \text{if } \tilde{i} = \sigma(R_{j \rightarrow i}) \text{ and } \tilde{j} = \tilde{i}, \\ 0, & \text{otherwise,} \end{cases} \quad (18)$$

$$(R^K Y^K(t))_{\tilde{i}, \tilde{j}} = \begin{cases} \alpha_{j \rightarrow i} Y_j^K(t), & \text{if } \tilde{i} = \sigma(R_{j \rightarrow i}) \text{ and } \tilde{j} = \sigma(S_j), \\ 0, & \text{otherwise,} \end{cases} \quad (19)$$

For  $i, j = 1, \dots, N$  with  $i \neq j$ , and  $d = 1, 2$ ,  $Q_j(t)$ ,  $Q_{j \rightarrow i}^{(d)}(t)$ ,  $Y_j^0(t)$ ,  $Y_j^K(t)$ ,  $Y_{j \rightarrow i}^{0, (d)}(t)$  have some important properties as follows:

$$0 \leq Q_j(t) \leq K_j; \quad 0 \leq Q_{j \rightarrow i}^{(d)}(t) \leq \sum_{i=1}^N C_i; \quad t \geq 0, \quad (20)$$

$$Y_j^0(0) = 0, Y_j^0(t) \text{ is continuous and nondecreasing,} \quad (21)$$

$$Y_j^K(0) = 0, Y_j^K(t) \text{ is continuous and nondecreasing,} \quad (22)$$

$$Y_{j \rightarrow i}^{0, (d)}(0) = 0, Y_{j \rightarrow i}^{0, (d)}(t) \text{ is continuous and nondecreasing,} \quad (23)$$

$$Y_j^0(t) \text{ increases at times } t \text{ only when } Q_j(t) = 0, \quad (24)$$

$$Y_j^K(t) \text{ increases at times } t \text{ only when } Q_j(t) = K_j, \quad (25)$$

$$Y_{j \rightarrow i}^{0, (d)}(t) \text{ increases at times } t \text{ only when } Q_{j \rightarrow i}^{(d)}(t) = 0. \quad (26)$$

In the remainder of this section, we provide a lemma to prove that the matrix  $R = (R^0, R^K)$  is an  $\mathcal{S}$  - matrix, which plays a key role in discussing existence and uniqueness of the SRBM through the box polyhedron for the closed queueing network. Note that  $R^0$  and  $R^K$  are defined in (14) and (15) for  $d = 1$ , and in (18) and (19) for  $d = 2$ . Also, the  $i$ th column of  $R$  is denoted as the vector  $v_i$ . To analyze the matrix  $R$ , readers may refer to Theorem 1.3 in Dai and Williams [7] for more details.

The following definition comes from Dai and Williams [7], here we restate it for convenience of readers.

**Definition 1** *A square matrix  $A$  is called an  $\mathcal{S}$  - matrix if there is a vector  $x \geq 0$  such that  $Ax > 0$ . The matrix  $A$  is completely -  $\mathcal{S}$  if and only if each principal submatrix of  $A$  is an  $\mathcal{S}$  - matrix.*

Notice that the capacity of station nodes is finite and the total number of bikes in this bike sharing system is a fixed constant. Without loss of generality, we assume that the capacity of each road node is also finite, and the maximal capacity of each road is  $\sum_{i=1}^N C_i$  due to the fact that the total number of bikes in this bike sharing system is  $\sum_{i=1}^N C_i$ .

Therefore, the state space  $S$  of this close queueing network is a  $N^2$ -dimensional box space with  $2N^2$  boundary faces  $F_i$ , given by

$$S \equiv \{x = (x_1, \dots, x_{N^2})' \in \mathcal{R}_+^{N^2} : 0 \leq x_i \leq \sum_{i=1}^N C_i\}. \quad (27)$$

We write

$$F_i \equiv \{x \in S : x_i = 0\}, F_{i+N^2} \equiv \{x \in S : x_i = K_i\} \text{ for } i \in \text{SN}, \quad (28)$$

$$F_j \equiv \{x \in S : x_j = 0\}, F_{j+N^2} \equiv \{x \in S : x_j = \sum_{i=1}^N C_i\} \text{ for } j \in \text{RN}. \quad (29)$$

Let  $J \equiv \{1, 2, \dots, 2N^2\}$  be the index set of the faces, and for each  $\emptyset \neq \mathcal{K} \subset J$ , define  $F_{\mathcal{K}} = \cap_{i \in \mathcal{K}} F_i$ . We indicate that the set  $\mathcal{K} \subset J$  is maximal if  $\mathcal{K} \neq \emptyset$ ,  $F_{\mathcal{K}} \neq \emptyset$ , and  $F_{\mathcal{K}} \neq F_{\tilde{\mathcal{K}}}$  for any  $\mathcal{K} \subset \tilde{\mathcal{K}}$  such that  $\mathcal{K} \neq \tilde{\mathcal{K}}$ . Thus, we can obtain that the maximal set  $\mathcal{K}$  is precisely the set of indexes of  $N^2$  distinct faces meeting at any vertex of  $S$ . Let  $\mathcal{N}$  be a  $2N^2 \times N^2$  matrix whose  $i$ th row is given by the unit normal of face  $F_i$ , which directs to the interior of  $S$ . We obtain,

$$\mathcal{N} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 1 \\ - & - & - & - \\ -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & -1 \end{bmatrix}.$$

The state space  $S$  has  $2^{N^2}$  vertexes due to its box space and each vertex given by  $(\cap_{i \in \alpha} F_i) \cap (\cap_{i \in \beta} F_{i+N^2})$  for a unique index set  $\alpha \subset \{1, \dots, N^2\}$  with  $\beta = \{1, \dots, N^2\} \setminus \alpha$ . Before we provide a lemma to prove the  $(NR)_{\mathcal{K}}$  (exactly  $|\mathcal{K}|$  distinct faces contain  $F_{\mathcal{K}}$ ) is a special  $\mathcal{S}$ -matrix, we give a geometric interpretation for a  $|\mathcal{K}| \times |\mathcal{K}|$   $\mathcal{S}$ -matrix  $(NR)_{\mathcal{K}}$ . At the each vertex of the box, we should make sure that there is a positive linear combination  $x_i v_i + x_j v_{j+N^2}$ ,  $x_i > 0$  for  $i \in \alpha$  and  $x_j > 0$  for  $j \in \beta$  such that  $x_i v_i + x_j v_{j+N^2}$  directs to the interior of the state space  $S$ .

Now, we provide a lemma to indicate the matrix  $(NR)_{\mathcal{K}}$  is an  $\mathcal{S}$ -matrix.

**Lemma 1** *The matrix  $(NR)_{\mathcal{K}}$  is an  $\mathcal{S}$ -matrix for each maximal  $\mathcal{K} \subset J$ .*

**Proof:** It is easy to check that

$$\mathcal{N}R = \begin{pmatrix} R^0 & R^K \\ -R^0 & -R^K \end{pmatrix}.$$

Because the state space of the closed queueing network is a  $N^2$ -dimensional box space, it has  $2N^2$  faces. Now, let us make a classify of those vertexes in this box space as follows:

**Type-1:** the vertexes are given by  $(\cap_{i \in A_S} F_i) \cap (\cap_{j \in A_R} F_j)$ ;

**Type-2:** the vertexes are given by  $(\cap_{i \in A_S} F_i) \cap (\cap_{k \in B_R} F_k)$ ;

**Type-3:** the vertexes are given by  $(\cap_{l \in B_S} F_l) \cap (\cap_{j \in A_R} F_j)$ ;

**Type-4:** the vertexes are given by  $(\cap_{l \in B_S} F_l) \cap (\cap_{k \in B_R} F_k)$ ;

**Type-5:** the vertexes are given by  $(\cap_{i \in A_S} F_i) \cap (\cap_{j \in A_R} F_j) \cap (\cap_{k \in B_R \setminus A_R} F_k)$ ;

**Type-6:** the vertexes are given by  $(\cap_{l \in B_S} F_l) \cap (\cap_{j \in A_R} F_j) \cap (\cap_{k \in B_R \setminus A_R} F_k)$ ;

**Type-7:** the vertexes are given by  $(\cap_{j \in A_R} F_j) \cap (\cap_{i \in A_S} F_i) \cap (\cap_{l \in B_S \setminus A_S} F_l)$ ;

**Type-8:** the vertexes are given by  $(\cap_{k \in B_R} F_k) \cap (\cap_{i \in A_S} F_i) \cap (\cap_{l \in B_S \setminus A_S} F_l)$ ;

**Type-9:** the vertexes are given by  $(\cap_{i \in A_S} F_i) \cap (\cap_{l \in B_S \setminus A_S} F_l) \cap (\cap_{j \in A_R} F_j) \cap (\cap_{k \in B_R \setminus A_R} F_k)$ ;

where  $A_S$  and  $A_R$  denote the set of index of face  $F_i = \{x_i = 0\}$  for  $i \in \text{SN}$  and  $F_j = \{x_j = 0\}$  for  $j \in \text{RN}$ , respectively;  $B_S$  and  $B_R$  denote the set of index of face  $F_l = \{x_l = K_l\}$  for  $l \in \text{SN}$  and  $F_k = \{x_k = \sum_{i=1}^N C_i + 1\}$  for  $k \in \text{RN}$ , respectively. According to the model description in Section 2, it is seen that the following two cases can not be established:

**Case 1:** All the station nodes are saturated when  $1 \leq C_i < K_i < \infty$ , namely, the reflection direction vector  $v_i$  on face  $F_i$  ( $i \in B_S$ ) can not simultaneously exist in the box state space  $S$  due to  $\sum_{i=1}^N K_i > \sum_{i=1}^N C_i$ . Therefore, at the vertexes of type-3, there must be a positive linear combination  $x_i v_i + x_j v_j > 0$  to direct to the interior of state space  $S$ , where  $x_i \geq 0$  for  $i \in A_R$  and  $x_j \geq 0$  for  $j \in B_S$ .

**Case 2:** Any road node is full, namely, the faces  $F_i$  ( $i \in B_R$ ) does not have the reflection direction vector  $v_i$  in the box state space  $S$ . In other word, the reflection direction vector  $v_i$  on face  $F_i$  ( $i \in B_R$ ) is zero vector. Therefore, at the vertexes of type-2, type-4, type-5, type-6, type-8 and type-9, there must be a positive linear combination who directs to the interior of state space  $S$ .

Now, we should only prove that at these vertexes of type-1, type-7 and type-3, where  $C_i = K_i$ , there also is a positive linear combination who directs to the interior of the state space  $S$ .

At the vertexes of type-1, we only should prove that the matrix  $R^0$  in the matrix  $\mathcal{N}R$

is an  $\mathcal{S}$ -matrix for  $d = 1, 2$ . It is clear that the matrix  $R^0$  is an  $\mathcal{S}$ -matrix due to the fact that all the diagonal elements of  $R^0$  are positive.

At the vertexes of type-7 and of type-3, for  $C_i = K_i$  and  $d = 1, 2$ , we can rewrite the  $(\mathcal{N}R)_\mathcal{K}$  as the following form:

$$M = (\mathcal{N}R)_\mathcal{K} = \begin{pmatrix} M_1 & M_2 \\ M_1 & M_4 \end{pmatrix} = \begin{pmatrix} M_1 & 0 \\ 0 & M_4 \end{pmatrix} + \begin{pmatrix} 0 & M_2 \\ M_3 & 0 \end{pmatrix}.$$

where  $M_1$  is a submatrix of  $R^0$ , which contains  $i$ th row (column) and  $i$ th column (row) of  $R^0$  simultaneously with  $i \in \alpha \subset \{1, \dots, N^2\}$ . Because the  $R^0$  is a complete  $\mathcal{S}$ -matrix,  $M_1$  is an  $\mathcal{S}$ -matrix.  $M_4$  is also a submatrix of  $-R^K$ , which also contains  $i + N^2$ th row (column) and  $i + N^2$ th column (row) of  $-R^K$  simultaneously with  $i \in \beta = \{1, \dots, N^2\} \setminus \alpha$ . At the same time,  $M_4$  is a diagonal matrix whose diagonal element is unit one, hence  $M_4$  is also an  $\mathcal{S}$ -matrix.  $M_2$  is a submatrix of  $R^K$  and  $M_3$  is a submatrix of  $-R^0$ . Because  $M_2$  and  $M_3$  do not contain any diagonal elements of  $R^K$  and  $-R^0$ ,  $M_2$  and  $M_3$  are both nonnegative matrices. Therefore, there must be a positive linear combination who direct to the interior of the state space  $S$  at the vertexes of type-7 and type-3, for  $C_i = K_i$  and  $d = 1, 2$ . This completes the proof. ■

## 5 Fluid Limits

In this section, we provide a fluid limit theorem for the queueing processes of the closed queueing network corresponding to the bike sharing system.

It follows from the functional strong law of large numbers (FSLN) that as  $t \rightarrow \infty$

$$\left(\frac{1}{t}S_j(t), \frac{1}{t}S_{j \rightarrow i}^{(d)}(t)\right) \rightarrow (b_j, b_{j \rightarrow i}^{(d)}), \quad d = 1, 2, \quad (30)$$

and as  $n \rightarrow \infty$

$$\left(\frac{1}{n}R_i^j(n), \frac{1}{n}\bar{R}_i^j(n), \frac{1}{n}R^{i \rightarrow j, (d)}(n)\right) \rightarrow (p_{j \rightarrow i}, \alpha_{j \rightarrow i}, 1), \quad d = 1, 2. \quad (31)$$

We consider a sequence of closed queueing networks, indexed by  $n = 1, 2, \dots$ , as described in Section 3. Let  $(\Omega^n, \mathcal{F}^n, P^n)$  be the probability space on which the  $n$ th closed queueing network is defined for the bike sharing system. All the processes and parameters associated with the  $n$ th network are appended with a superscript  $n$ .



For the  $n$ th network, the renewal service processes of the station nodes and of the road nodes are expressed by  $S_j^n = \{S_j^n(t), t \geq 0\}$  and  $S_{j \rightarrow i}^{(d),n} = \{S_{j \rightarrow i}^{(d),n}(t), t \geq 0\}$ , respectively. Let  $b_j^n$  and  $b_{j \rightarrow i}^{(d),n}$  be the long run average service rates of  $S_j^n(t)$  and  $S_{j \rightarrow i}^{(d),n}(t)$ , respectively. The vectors of the  $N$  station capacities and of their initial bike numbers are denoted as  $K^n = (K_1^n, \dots, K_N^n)'$  and  $C^n = (C_1^n, \dots, C_N^n)'$ , respectively, where  $1 \leq C_i^n \leq K_N^n < \infty$ . For simplicity of description, we write  $R^{j,n}$  as  $R^j$ ,  $\bar{R}^{j,n}$  as  $\bar{R}^j$  and  $R^{j \rightarrow i, (d), n}$  as  $R^{j \rightarrow i, (d)}$  for all  $n \geq 1$ , i.e., the routing processes of the station nodes and of the road nodes are compressed the number  $n$ . We append a superscript  $n$  to the performance indexes such as  $Y_j^{0,n}(t)$ ,  $Y_{j \rightarrow i}^{0, (d), n}(t)$ ,  $B_j^n(t)$  and  $B_{j \rightarrow i}^n(t)$ , and the interesting processes  $Q^n = ((Q_j^n(t), Q_{j \rightarrow i}^{(d),n}(t)))'$  and  $Y_j^{K,n}(t)$ .

**The heavy traffic conditions:** We assume that as  $n \rightarrow \infty$

$$(b_j^n, b_{j \rightarrow i}^{(d),n}, \sqrt{n}\theta_j^n, \sqrt{n}\theta_{j \rightarrow i}^{(d),n}, \frac{1}{\sqrt{n}}C_i^n, \frac{1}{\sqrt{n}}K_i^n) \rightarrow (b_j, b_{j \rightarrow i}^{(d)}, \theta_j, \theta_{j \rightarrow i}^{(d)}, C_i, K_i), \quad (32)$$

where  $\theta_j^n = \sum_{d=1}^2 \sum_{j \neq i}^N b_{j \rightarrow i}^{(d),n} - b_j^n$ ;  $\theta_{j \rightarrow i}^{(1),n} = p_{j \rightarrow i} b_j^n - b_{j \rightarrow i}^{(1),n}$  and  $\theta_{j \rightarrow i}^{(2),n} = -b_{j \rightarrow i}^{(2),n}$ . At the same time, we assume that for  $i, j = 1, \dots, N$  with  $i \neq j$ ,  $d = 1, 2$ , all these limits are finite.

For the initial queue lengths  $Q_j^n(0)$  and  $Q_{j \rightarrow i}^{(d),n}(0)$ , we assume that as  $n \rightarrow \infty$

$$\bar{Q}_j^n(0) \equiv \frac{1}{n}Q_j^n(0) \rightarrow 0 \text{ and } \bar{Q}_{j \rightarrow i}^{(d),n}(0) \equiv \frac{1}{n}Q_{j \rightarrow i}^{(d),n}(0) \rightarrow 0. \quad (33)$$

It follows from the functional strong law of large numbers that for  $d = 1, 2$ , as  $n \rightarrow \infty$

$$\begin{aligned} &(\bar{S}_j^n(t), \bar{S}_{j \rightarrow i}^{(d),n}(t), \bar{R}_i^{j,n}(t), \bar{R}_i^{j,n}(t), \bar{R}^{j \rightarrow i, (d), n}(t)) \\ &\rightarrow (b_j t, b_{j \rightarrow i}^{(d)} t, p_{j \rightarrow i} t, \alpha_{j \rightarrow i} t, t), \text{ u.o.c.,} \end{aligned} \quad (34)$$

where

$$\begin{aligned} \bar{S}_j^n(t) &= \frac{1}{n}S_j^n(nt), \bar{S}_{j \rightarrow i}^{(d),n}(t) = \frac{1}{n}S_{j \rightarrow i}^{(d),n}(nt), \bar{R}_i^{j,n}(t) = \frac{1}{n}R_i^j(\lfloor nt \rfloor), \\ \bar{R}_i^{j,n}(t) &= \frac{1}{n}\bar{R}_i^j(\lfloor nt \rfloor), \bar{R}^{j \rightarrow i, (d), n}(t) = \frac{1}{n}R^{j \rightarrow i, (d)}(\lfloor nt \rfloor), \end{aligned}$$

and  $\lfloor x \rfloor$  is the maximal integer part of the real number  $x$ .

We give a notation: for any process  $W^n = \{W^n(t), t \geq 0\}$ , we define its centered processes  $\hat{W}^n = \{\hat{W}^n(t), t \geq 0\}$  by

$$\hat{W}^n(nt) = W^n(nt) - w^n nt,$$

where  $w^n$  is the mean of the process  $W^n$ .

For the station nodes and road nodes, we write some centered processes as

$$\hat{S}_j^n(nt) = S_j^n(nt) - b_j^n nt, \quad \hat{S}_{j \rightarrow i}^{(d),n}(t) = S_{j \rightarrow i}^{(d),n}(nt) - b_{j \rightarrow i}^{(d),n} nt, \quad (35)$$

$$\hat{R}_i^{j,n}(t) = R_i^{j,n}(\lfloor nt \rfloor) - p_{j \rightarrow i} \lfloor nt \rfloor, \quad \hat{\bar{R}}_i^{j,n}(t) = \bar{R}_i^{j,n}(\lfloor nt \rfloor) - \alpha_{j \rightarrow i} \lfloor nt \rfloor. \quad (36)$$

For convenience of readers, we restate a lemma for the oscillation result of a sequence of  $(S^n, R^n)$ -regulation problems in convex polyhedrons, which is a summary restatement of Lemma 4.3 of Dai and Williams [7] and the Theorem 3.1 of Dai [8], whose proof is omitted here and can easily be referred to Dai and Williams [7] and Dai [8] for more details.

This lemma prevails due to the fact that the state space of the box polyhedron of this bike sharing system belongs to a simple convex polyhedrons as analyzed in the last of Section 4. For a function  $f$  defined from  $[t_1, t_2] \subset [0, \infty]$  into  $\mathcal{R}^k$  for some  $k \geq 1$ , let

$$Osc(f, [t_1, t_2]) = \sup_{t_1 \leq s \leq t \leq t_2} |f(t) - f(s)|.$$

**Lemma 2** *For any  $T > 0$ , given a sequence of  $\{x^n\}_{n=1}^\infty \in D_{\mathcal{R}^{N^2}}[0, T]$  with the initial values  $x^n(0) \in S^n$ . Let  $(z^n, y^n)$  be an  $(S^n, R^n)$ -regulation of  $x^n$  over  $[0, T]$ , where  $(z^n, y^n) \in D_{\mathcal{R}^{N^2}}[0, T] \times D_{\mathcal{R}^{2N^2}}[0, T]$ . Assuming that all  $S^n$  have the same shape, i.e., the only difference is the corresponding boundary size  $K_i^n$ . Assuming that  $\{K_i^n\}$  belongs to some bounded set, and the jump sizes of  $y^n$  are bounded by  $\Gamma^n$  for each  $n$ . Then if  $(\mathcal{N}R)_K$  is an  $\mathcal{S}$  - matrix and  $R^n \rightarrow R$  as  $n \rightarrow \infty$ , we have*

$$Osc(z^n, [t_1, t_2]) \leq C \max\{Osc(x^n, [t_1, t_2]), \Gamma^n\},$$

$$Osc(y^n, [t_1, t_2]) \leq C \max\{Osc(x^n, [t_1, t_2]), \Gamma^n\},$$

where  $C$  depends only on  $(\mathcal{N}, R, |\mathcal{K}|)$  for all  $\mathcal{K} \subset \Xi$ , where  $\Xi$  denotes the collection of subsets of  $J \equiv \{1, 2, \dots, 2N^2\}$  consisting of all maximal sets in  $J$  together with the empty set.

**Theorem 1** (Fluid Limit Theorem) *Under Assumptions (32) to (34), as  $n \rightarrow \infty$ , we have*

$$\left( \bar{B}_j^n(t), \bar{B}_{j \rightarrow i}^{(d),n}(t), \bar{Y}_j^{0,n}(t), \bar{Y}_{j \rightarrow i}^{0,(d),n}(t) \right) \rightarrow \left( \bar{\tau}_j(t), \bar{\tau}_{j \rightarrow i}^{(d)}(t), \bar{Y}_j^0(t), \bar{Y}_{j \rightarrow i}^{0,(d)}(t) \right) \quad u.o.c.,$$

where  $\bar{\tau}_j(t) \equiv et$ ,  $\bar{\tau}_{j \rightarrow i}^{(d)}(t) \equiv et$ ,  $\bar{Y}_j(t) \equiv 0$  and  $\bar{Y}_{j \rightarrow i}^{(d)}(t) \equiv 0$ ;  $\bar{Y}_j^{0,n}(t) = \frac{1}{n} Y_j^{0,n}(nt)$ ,  $\bar{Y}_{j \rightarrow i}^{0,(d),n}(t) = \frac{1}{n} Y_{j \rightarrow i}^{0,(d),n}(nt)$ ,  $\bar{B}_j^n(t) = \frac{1}{n} B_j^n(nt)$  and  $\bar{B}_{j \rightarrow i}^{(d),n}(t) = \frac{1}{n} B_{j \rightarrow i}^{(d),n}(nt)$  for  $i, j = 1, \dots, N$  with  $i \neq j$ ,  $d = 1, 2$ .

**Proof:** Recall the queue length process :  $Q(t) = X(t) + R^0 Y^0(t) + R^K Y^K(t)$ , where  $X(t)$  is given by (8), (12) and (16) in Section 4. It follows from (2) to (4) that the scaling queueing processes for the station nodes and the road nodes are given by

$$\bar{Q}^n(t) = \bar{Q}^n(0) + \bar{X}^n(t) + R^{0,n} \bar{Y}^{0,n}(t) + R^{K,n} \bar{Y}^{K,n}(t),$$

where  $\bar{Q}^n(t) = \frac{1}{n} Q^n(nt)$ ,  $\bar{Q}^n(t) = \{(\bar{Q}_j^n(t), \bar{Q}_{j \rightarrow i}^{(d),n}(t)), i \neq j, i, j = 1, \dots, N; d = 1, 2; t \geq 0\}$ ;  $\bar{X}^n(t) = \frac{1}{n} X^n(nt)$ ,  $\bar{X}^n(t) = \{(\bar{X}_j^n(t), \bar{X}_{j \rightarrow i}^{(d),n}(t)), i \neq j, i, j = 1, \dots, N; d = 1, 2; t \geq 0\}$ ;  $\bar{Y}^{0,n}(t) = \frac{1}{n} Y^{0,n}(nt)$ ,  $\bar{Y}^{0,n}(t) = \{(\bar{Y}_j^{0,n}(t), \bar{Y}_{j \rightarrow i}^{0,(d),n}(t)), i \neq j, i, j = 1, \dots, N; d = 1, 2; t \geq 0\}$ ;  $\bar{Y}^{K,n}(t) = \frac{1}{n} Y^{K,n}(nt)$ ,  $\bar{Y}^{K,n}(t) = \{(\bar{Y}_j^{K,n}(t)), j = 1, \dots, N\}$ . For each  $n$ ,  $\bar{Q}^n(t)$ ,  $\bar{Y}^n(t)$  and  $\bar{Y}^{K,n}(t)$  satisfy the properties (20) to (26) with the state space  $S^n$ , given by

$$S^n \equiv \left\{ x = (x_1, \dots, x_{N^2})' \in R_+^{N^2} : x_i \leq \bar{K}_i^n = \frac{K_i^n}{n} \text{ for } i \in \text{SN}; \right. \\ \left. \text{and } x_i \leq \frac{\sum_{i=1}^N C_i^n}{n} + 1 \text{ for } i \in \text{RN} \right\}.$$

For station node  $j = 1, \dots, N$ , by using (2), (8), (35) and (36), we have

$$\bar{X}_j^n(t) \equiv \frac{1}{n} Q_j^n(0) + \frac{1}{n} \sum_{d=1}^2 \sum_{i \neq j}^N \hat{S}_{i \rightarrow j}^{(d),n}(n \bar{B}_{i \rightarrow j}^{(d),n}(t)) - \frac{1}{n} \hat{S}_j^n(n \bar{B}_j^n(t)) + \frac{1}{n} \theta_j^n nt. \quad (37)$$

For road node  $j \rightarrow i$  ( $i, j = 1, \dots, N$  with  $i \neq j$ ), by using (12), (16), (35) and (36), we have,

$$\bar{X}_{j \rightarrow i}^{(1),n}(t) \equiv \frac{1}{n} Q_{j \rightarrow i}^{(1),n}(0) + \frac{1}{n} \hat{R}_i^{j,n}(n \bar{S}_j^n(\bar{B}_j^n(t))) \\ + \frac{1}{n} p_{j \rightarrow i} \hat{S}_j^n(n \bar{B}_j^n(t)) - \frac{1}{n} \hat{S}_{j \rightarrow i}^{(1),n}(n \bar{B}_{j \rightarrow i}^{(1),n}(t)) + \frac{1}{n} \theta_{j \rightarrow i}^{(1),n} nt, \quad (38)$$

$$\bar{X}_{j \rightarrow i}^{(2),n}(t) \equiv \frac{1}{n} Q_{j \rightarrow i}^{(2),n}(0) + \frac{1}{n} \hat{R}_i^{j,n}(n \bar{Y}_j^{K,n}(t)) - \frac{1}{n} \hat{S}_{j \rightarrow i}^{(2),n}(n \bar{B}_{j \rightarrow i}^{(2),n}(t)) + \frac{1}{n} \theta_{j \rightarrow i}^{(2),n} nt. \quad (39)$$

Note that  $\bar{B}_{j \rightarrow i}^{(1),n}(t) \leq t$ ,  $\bar{B}_j^n(t) \leq t$ ,  $\bar{Y}_j^{K,n}(t) \leq \sum_{i=1}^N C_i^n - K_j^n$ , by using (32) to (34) and the Skorohod Representation Theorem, as  $n \rightarrow \infty$ , we have

$$\bar{X}^n(t) = (\bar{X}_j^n(t), \bar{X}_{j \rightarrow i}^{(d),n}(t)) \rightarrow 0, \quad \text{u.o.c.}$$

Since the state space  $S^n$  of this bike sharing system are the boxes of the same shape in the  $N^2$ -dimensional space,  $(\mathcal{NR})_\mathcal{K}$  is an  $S$ -matrix and  $R^n \rightarrow R$  as  $n \rightarrow \infty$ . Then by Lemma 2 we have

$$Osc(\bar{Y}^{0,n}, [s, t] \subseteq [0, T]) \leq C Osc(\bar{X}^n, [s, t] \subseteq [0, T]),$$

for any  $T \geq 0$ , where  $C$  depends only on  $R$  and  $\mathcal{N}$  for  $n$  large enough.

$$\begin{aligned}
0 &\leq \lim_{n \rightarrow \infty} \inf \text{Osc}(\bar{Y}^{0,n}, [s, t] \subseteq [0, T]) \\
&\leq \lim_{n \rightarrow \infty} \sup \text{Osc}(\bar{Y}^{0,n}, [s, t] \subseteq [0, T]) \\
&\leq C \lim_{n \rightarrow \infty} \text{Osc}(\bar{X}^n, [s, t] \subseteq [0, T]) \\
&= 0, \text{ a.s.}
\end{aligned}$$

where  $\bar{Y}^n(t) = (\bar{Y}_j^{0,n}(t)', \bar{Y}_{j \rightarrow i}^{(d),0,n}(t)')'$ . Notice that  $Y^n(0) = 0$  for all  $n$ , we have

$$\lim_{n \rightarrow \infty} \bar{Y}^n(t) = 0, \text{ u.o.c.} \quad (40)$$

Since  $\bar{B}_j^n(t) = t - \bar{Y}_j^{0,n}(t)$  and  $\bar{B}_{j \rightarrow i}^{(d),n}(t) = t - \bar{Y}_{j \rightarrow i}^{(d),0,n}(t)$ , we obtain the convergence of  $\bar{B}_j^n(t)$  and  $\bar{B}_{j \rightarrow i}^{(d),n}(t)$  for  $i, j = 1, \dots, N$  with  $i \neq j$ ,  $d = 1, 2$ . This completes the proof. ■

## 6 Diffusion limits

In this section, we set up the diffusion scaled processes of the queueing processes, and give their weak convergence results for the multiclass closed queueing network corresponding to the bike sharing system.

We introduce the diffusion scaling process for the process  $\hat{W}^n = \{\hat{W}^n(nt), t \geq 0\}$ , given by

$$\tilde{W}^n(t) \equiv \frac{1}{\sqrt{n}} \hat{W}^n(nt) = \frac{1}{\sqrt{n}} (W^n(nt) - w^n nt).$$

For the station nodes and the road nodes, we write

$$\tilde{S}_j^n(t) = \sqrt{n} \left( \frac{S_j^n(nt)}{n} - b_j^n t \right), \quad \tilde{S}_{j \rightarrow i}^{(d),n}(t) = \sqrt{n} \left( \frac{S_{j \rightarrow i}^{(d),n}(nt)}{n} - b_{j \rightarrow i}^{(d),n} t \right), \quad (41)$$

$$\tilde{R}_i^{j,n}(t) = \sqrt{n} \left( \frac{R_i^{j,n}(nt)}{n} - p_{j \rightarrow i} t \right), \quad \tilde{\bar{R}}_i^{j,n}(t) = \sqrt{n} \left( \frac{\bar{R}_i^{j,n}(nt)}{n} - \alpha_{j \rightarrow i} t \right), \quad (42)$$

$$\tilde{R}^{j \rightarrow i, (d), n}(t) = \sqrt{n} \left( \frac{R^{j \rightarrow i, (d), n}(nt)}{n} - t \right). \quad (43)$$

For the initial queueing processes  $Q_j^n(0)$  and  $Q_{j \rightarrow i}^{n, (d)}(0)$  for  $i, j = 1, \dots, N$  with  $i \neq j$ ,  $d = 1, 2$ , we assume that as  $n \rightarrow \infty$

$$\tilde{Q}_j^n(0) \equiv \frac{1}{\sqrt{n}} Q_j^n(0) \Rightarrow \tilde{Q}(0), \quad (44)$$

$$\tilde{Q}_{j \rightarrow i}^{(d),n}(0) \equiv \frac{1}{\sqrt{n}} Q_{j \rightarrow i}^{(d),n}(0) \Rightarrow \tilde{Q}_{j \rightarrow i}^{(d)}(0). \quad (45)$$

It follows from the Skorohod Representation Theorem and the Donsker's Theorem that

$$\begin{aligned} (\tilde{S}_j^n(t), \tilde{S}_{j \rightarrow i}^{(d),n}(t), \tilde{R}_i^{j,n}(t), \tilde{\tilde{R}}_i^{j,n}(t), \tilde{R}^{j \rightarrow i,(d),n}(t)) \\ \Rightarrow (\tilde{S}_j(t), \tilde{S}_{j \rightarrow i}^{(d)}(t), \tilde{R}_i^j(t), \tilde{\tilde{R}}_i^j(t), \tilde{R}^{j \rightarrow i,(d)}(t)), \end{aligned} \quad (46)$$

where  $\Rightarrow$  denotes weak convergence, and  $\tilde{S}_j(t)$ ,  $\tilde{S}_{j \rightarrow i}^{(d)}(t)$ ,  $\tilde{R}_i^j(t)$ ,  $\tilde{\tilde{R}}_i^j(t)$  and  $\tilde{R}^{j \rightarrow i,(d)}(t)$  are all the Brownian motions with drift zero and covariance matrices  $\Gamma^S$ ,  $\Gamma^{R,S,l}$ ,  $\Gamma^{\bar{R},S,l}$  and  $\Gamma^{R,S,j \rightarrow i}$ , which are given by

(1) The covariance matrix of  $\tilde{S}(t) = (\tilde{S}_j(t), \tilde{S}_{j \rightarrow i}^{(d)}(t))$  for  $i, j = 1, \dots, N$  with  $i \neq j$ ,  $d = 1, 2$ , is given by

$$\Gamma^S = \begin{pmatrix} (\Gamma^{S,S})_{N \times N} & 0 \\ 0 & (\Gamma^{S,R,(d)})_{(N^2-N) \times (N^2-N)} \end{pmatrix}_{N^2 \times N^2},$$

where

$$\begin{aligned} (\Gamma^{S,S})_{\tilde{i},\tilde{j}} &= \begin{cases} b_i c_{a,i}^2 \delta_{\tilde{i},\tilde{j}}, & \sigma(S_i) = \tilde{i}, \\ 0, & \text{otherwise,} \end{cases} \\ (\Gamma^{S,R,(d)})_{\tilde{i},\tilde{j}} &= \begin{cases} b_{i \rightarrow j}^{(d)} (c_{s,i \rightarrow j}^{(d)})^2 \delta_{\tilde{i},\tilde{j}}, & \sigma(R_{i \rightarrow j}) = \tilde{i}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

(2) The covariance matrix of  $\tilde{R}(t) = (\tilde{R}^l(t))$  for  $l = 1, \dots, N$ , is given by

$$\Gamma^{R,S,l} = \begin{pmatrix} 0 & 0 \\ 0 & (\Gamma^{R,S,l})_{(N-1) \times (N-1)} \end{pmatrix}_{N^2 \times N^2},$$

where

$$(\Gamma^{R,S,l})_{\tilde{i},\tilde{j}} = \begin{cases} p_{l \rightarrow k_1} (\delta_{\tilde{i},\tilde{j}} - p_{l \rightarrow k_2}), & \sigma(R_{l \rightarrow k_1}) = \tilde{i}, \quad \sigma(R_{l \rightarrow k_2}) = \tilde{j}, \\ 0, & \text{otherwise.} \end{cases}$$

(3) The covariance matrix of  $\tilde{\tilde{R}}(t) = (\tilde{\tilde{R}}^l(t))$  for  $l = 1, \dots, N$ , is given by

$$\Gamma^{\bar{R},S,l} = \begin{pmatrix} 0 & 0 \\ 0 & (\Gamma^{\bar{R},S,l})_{(N-1) \times (N-1)} \end{pmatrix}_{N^2 \times N^2},$$

where

$$(\Gamma^{\bar{R},S,l})_{\tilde{i},\tilde{j}} = \begin{cases} \alpha_{l \rightarrow k_1} (\delta_{\tilde{i},\tilde{j}} - \alpha_{l \rightarrow k_2}), & \sigma(R_{l \rightarrow k_1}) = \tilde{i}, \quad \sigma(R_{l \rightarrow k_2}) = \tilde{j}, \\ 0, & \text{otherwise.} \end{cases}$$

(4) The covariance matrix of  $\tilde{R}(t) = (\tilde{R}^{j \rightarrow i, (d)}(t))$  for  $i, j = 1, \dots, N$  with  $i \neq j$ ,  $d = 1, 2$ , is given by

$$\Gamma^{R, S, j \rightarrow i} = \begin{pmatrix} (\Gamma^{R, R, j \rightarrow i})_{N \times N} & 0 \\ 0 & 0 \end{pmatrix}_{N^2 \times N^2},$$

where

$$(\Gamma^{R, R, j \rightarrow i})_{\tilde{l}, \tilde{k}} = \begin{cases} p_{j \rightarrow i, l}(\delta_{\tilde{l}, \tilde{k}} - p_{j \rightarrow i, k}) = 0, & \sigma(S_l) = \tilde{l}, \sigma(S_k) = \tilde{k}, \\ 0, & \text{otherwise.}, \end{cases}$$

Now, we prove adaptedness properties of the diffusion scaling processes  $(\tilde{Q}^n(t), \tilde{X}^n(t), \tilde{Y}^n(t))$ , where  $\tilde{Q}^n(t) = \frac{1}{\sqrt{n}}Q^n(nt)$ ,  $\tilde{Q}^n(t) = (\tilde{Q}_j^n(t), \tilde{Q}_{j \rightarrow i}^{(d), n}(t))$ ;  $\tilde{X}^n(t) = \frac{1}{\sqrt{n}}X^n(nt)$ ,  $\tilde{X}^n(t) = (\tilde{X}_j^n(t), \tilde{X}_{j \rightarrow i}^{(d), n}(t))$ ;  $\tilde{Y}^{0, n}(t) = \frac{1}{\sqrt{n}}Y^{0, n}(nt)$ ,  $\tilde{Y}^{0, n}(t) = (\tilde{Y}_j^{0, n}(t), \tilde{Y}_{j \rightarrow i}^{0, (d), n}(t))$ ,  $\tilde{Y}_j^{K, n}(t) = \frac{1}{\sqrt{n}}Y_j^{K, n}(nt)$ .

Define

$$\varsigma_t^n = \sigma\{\tilde{Q}^n(0), \tilde{S}^n(s), \tilde{Y}^{0, n}(s), \tilde{Y}^{K, n}(t), s \leq t\}, \quad (47)$$

where  $\tilde{Q}^n(0), \tilde{S}^{(d), n}(s), \tilde{R}^{(d), n}(s)$  and  $\tilde{R}^n(s)$  are defined in (41) to (45). Define  $T_k^n = (T_k^{j, n}, T_k^{j \rightarrow i, (d), n})$ , where  $T_k^{j, n}$  and  $T_k^{j \rightarrow i, (d), n}$  denote the partial sum of the service time sequence at station node  $j$  and road node  $j \rightarrow i$ , respectively, for the  $n$ th network, that is,

$$T_k^{j, n} = \sum_{l=1}^k u_j^n(l), \quad T_k^{j \rightarrow i, (d), n} = \sum_{l=1}^k v_{j \rightarrow i}^{(d), n}(l),$$

with the initial condition  $T_0^n \equiv 0$ . Notice that  $T_k^n = (T_k^{j, n}, T_k^{j \rightarrow i, (d), n})$  is a  $\varsigma_t^n$  - *stopping time*, and,  $0 = T_0^n < T_1^n < T_2^n < \dots < T_k^n \rightarrow \infty$  a.s. as  $k \rightarrow \infty$  for each  $n$  and  $i, j = 1, \dots, N$  with  $i \neq j$ ,  $d = 1, 2$ . Let  $\varsigma_{T_k^n}^{(n)-}$  denote the strict past at time  $T_k^n$ . Then

$$\varsigma_{T_k^n}^{(n)-} = \sigma(A_t \cap \{t < T_k^n\}, A_t \in \varsigma_t^n, t \geq 0).$$

Because  $T_k^n$  is a  $\varsigma_t^n$ -stopping time,  $u_j^n(k+1)$  and  $v_{j \rightarrow i}^{(d), n}(k+1)$  are independent of the history of the network before the time at which the  $k$ th customer is served at station node  $j$  and road node  $j \rightarrow i$ . Therefore,  $T_k^n$  is  $\varsigma_{T_k^n}^{(n)-}$ -measurable,  $u_j^n(k+1)$  is independent of  $\varsigma_{T_k^n}^{(j, n)-}$ , and  $v_{j \rightarrow i}^{(d), n}(k+1)$  is independent of  $\varsigma_{T_k^n}^{(j \rightarrow i, (d), n)-}$ .

**Theorem 2** *Under Assumption (32), we have that*

$$(\tilde{Q}^n(t), \tilde{X}^n(t), \tilde{Y}^{0, n}(t), \tilde{Y}^{K, n}(t)) \Rightarrow (\tilde{Q}(t), \tilde{X}(t), \tilde{Y}^0(t), \tilde{Y}^K(t)), \text{ as } n \rightarrow \infty,$$

or, in component form,

$$\begin{aligned} & \left( \tilde{Q}_j^n(t), \tilde{Q}_{j \rightarrow i}^{(d),n}(t), \tilde{X}_j^n(t), \tilde{X}_{j \rightarrow i}^{(d),n}(t), \tilde{Y}_j^{0,n}(t), \tilde{Y}_{j \rightarrow i}^{0,(d),n}(t), \tilde{Y}_j^{K,n}(t) \right) \\ & \Rightarrow \left( \tilde{Q}_j(t), \tilde{Q}_{j \rightarrow i}^{(d)}(t), \tilde{X}_j(t), \tilde{X}_{j \rightarrow i}^{(d)}(t), \tilde{Y}_j^0(t), \tilde{Y}_{j \rightarrow i}^{0,(d)}(t), \tilde{Y}_j^K(t) \right), \quad \text{as } n \rightarrow \infty, \end{aligned}$$

where  $\tilde{X}(t)$  is a Brownian motion with covariance matrix  $\Gamma$ . Moreover,  $\tilde{X}(t) - \theta t$  is a martingale with respect to the filtration  $\mathcal{F}_t = \sigma(\tilde{Q}(s), \tilde{Y}^0(s), \tilde{Y}^K(s), s \leq t)$ .

**Proof.** First, we define

$$\tau_+^n(t) = \min\{T_k^n : T_k^n > t\} \text{ and } \tau_-^n(t) = \max\{T_k^n : T_k^n \leq t\}. \quad (48)$$

For the station node  $j \in \text{SN}$ , when  $\tau_+^{j,n}(nt)$  approximates  $nt$  from its right side, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} E \left[ \left| \frac{1}{\sqrt{n}} (S_j^n(\tau_+^{j,n}(nt)) - b_j^n \tau_+^{j,n}(nt)) - \tilde{S}_j^n(t) \right| \right] \\ &= \lim_{n \rightarrow \infty} E \left[ \left| \frac{1}{\sqrt{n}} (1 - b_j^n(\tau_+^{j,n}(nt)) - nt) \right| \right] \\ &\leq \frac{1}{\sqrt{n}} \lim_{n \rightarrow \infty} b_j^n E [\tau_+^{j,n}(nt) - \tau_-^{j,n}(nt)] \\ &= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} b_j^n E [u_j^n(1)] = 0. \end{aligned} \quad (49)$$

Similarly, when  $\tau_-^{j,n}(nt)$  approximates  $nt$  from its left side, we have

$$\lim_{n \rightarrow \infty} E \left[ \left| \frac{1}{\sqrt{n}} (S_j^n(\tau_-^{j,n}(nt)) - b_j^n \tau_-^{j,n}(nt)) - \tilde{S}_j^n(t) \right| \right] = 0. \quad (50)$$

Moreover, we obtain

$$E[\tilde{S}_j^n(T_{k+1}^{j,n}) - \tilde{S}_j^n(T_k^{j,n}) | \varsigma_{T_k^{j,n}}^n] = \frac{1}{\sqrt{n}} \{1 - b_j^n E[u_j^n(k+1) | \varsigma_{T_k^{j,n}}^n]\} = 0, \quad (51)$$

where the filtration  $\{\varsigma_t^n\}$  is defined in (47). Notice that for any  $\{\varsigma_t^n\}$ -stopping time  $T$  and any random variable  $X$  with  $E[|X|] < \infty$ ,

$$E[E[X | \varsigma_t^n] | \varsigma_t^n] I_{\{T > t\}} = E[X | \varsigma_t^n] I_{\{T > t\}} = E[X I_{\{T > t\}} | \varsigma_t^n]. \quad (52)$$

Also, for each  $j \in \text{SN}$  and all  $s, t \geq 0$ ,

$$\begin{aligned} & E[\tilde{S}_j^n(t+s) - \tilde{S}_j^n(t) | \varsigma_t^n] \\ &= E \left[ \tilde{S}_j^n(t+s) - \frac{1}{\sqrt{n}} \left( S_j^n(\tau_-^{j,n}(n(t+s))) - b_j^n \tau_-^{j,n}(n(t+s)) \right) \middle| \varsigma_t^n \right] \\ &+ E \left[ \frac{1}{\sqrt{n}} \left( S_j^n(\tau_+^{j,n}(nt)) - b_j^n \tau_+^{j,n}(nt) \right) - \tilde{S}_j^n(t) \middle| \varsigma_t^n \right] \\ &- \sum_k E \left[ E[\tilde{S}_j^n(T_{k+1}^{j,n}) - \tilde{S}_j^n(T_k^{j,n}) | \varsigma_{T_k^{j,n}}^n] I_{\{nt < T_k^{j,n} \leq n(t+s)\}} \middle| \varsigma_t^n \right]. \end{aligned}$$

Hence, it follows from (49) to (52) that

$$\lim_{n \rightarrow \infty} E \left[ \left| E \left[ \tilde{S}_j^n(t+s) - \tilde{S}_j^n(t) \mid \varsigma_t^n \right] \right| \right] = 0. \quad (53)$$

For road node  $j \rightarrow i$  ( $i, j = 1, \dots, N$  with  $i \neq j$ ). When we approximate  $nt$  from both sides, a similar analysis to the proof of (53) for station node  $j$ . For all  $s, t \geq 0$ , we have

$$\lim_{n \rightarrow \infty} E \left[ \left| E \left[ \tilde{S}_{j \rightarrow i}^{(d),n}(t+s) - \tilde{S}_{j \rightarrow i}^{(d),n}(t) \mid \varsigma_t^n \right] \right| \right] = 0. \quad (54)$$

Next, we can set up the scaling queueing processes by mean of (2) to (4) for the station nodes and of the road nodes through the scaling processes (41) to (45), given by:

$$\tilde{Q}^n(t) = \tilde{Q}^n(0) + \tilde{X}^n(t) + R^{0,n} \tilde{Y}^{0,n}(t) + R^{K,n} \tilde{Y}^{K,n}(t), \quad (55)$$

and for each  $n$ ,  $(\tilde{Q}^n(t), \tilde{Y}^{0,n}(t), \tilde{Y}^{K,n}(t))$  has the properties (20) to (26) with the state space  $S^n$  as follow:

$$S^n \equiv \left\{ x = (x_1, \dots, x_{N^2})' \in R_+^{N^2} : x_i \leq \tilde{K}_i^n = \frac{K_i^n}{\sqrt{n}} \text{ for } i \in \text{SN}, \right. \\ \left. \text{and } x_i \leq \frac{\sum_{i=1}^N C_i^n}{\sqrt{n}} + 1 \text{ for } i \in \text{RN} \right\}.$$

For station node  $j = 1, \dots, N$ , by using (3), (12), (41) to (45) and  $\tilde{X}_j^n(t) = \frac{1}{\sqrt{n}} X_j^n(nt) = \sqrt{n} \bar{X}_j^n(t)$ , we have

$$\tilde{X}_j^n(t) = \tilde{Q}_j^n(0) + \frac{1}{\sqrt{n}} \sum_{d=1}^2 \sum_{i \neq j}^N \hat{S}_{i \rightarrow j}^{(d),n}(n \bar{B}_{i \rightarrow j}^{(d),n}(t)) - \frac{1}{\sqrt{n}} \hat{S}_j^n(n \bar{B}_j^n(t)) + \sqrt{n} \theta_j^n t. \quad (56)$$

For road node  $j \rightarrow i$  ( $i, j = 1, \dots, N$  with  $i \neq j$ ), by using (12), (16), (41) to (45) and  $\tilde{X}_{j \rightarrow i}^{(d),n}(t) = \frac{1}{\sqrt{n}} X_{j \rightarrow i}^{(d),n}(nt) = \sqrt{n} \bar{X}_{j \rightarrow i}^{(d),n}(t)$ , we have,

$$\tilde{X}_{j \rightarrow i}^{(1),n}(t) = \tilde{Q}_{j \rightarrow i}^{(1),n}(0) + \frac{1}{\sqrt{n}} \hat{R}_i^{j,n}(n \bar{S}_j^n(\bar{B}_j(t))) \\ + \frac{1}{\sqrt{n}} p_{j \rightarrow i} \hat{S}_j^n(n \bar{B}_j^n(t)) - \frac{1}{\sqrt{n}} \hat{S}_{j \rightarrow i}^{(1),n}(n \bar{B}_{j \rightarrow i}^{(1),n}(t)) + \frac{1}{\sqrt{n}} \theta_{j \rightarrow i}^{(1),n} nt \quad (57)$$

and

$$\tilde{X}_{j \rightarrow i}^{(2),n}(t) = \tilde{Q}_{j \rightarrow i}^{(2),n}(0) + \frac{1}{\sqrt{n}} \hat{R}_i^{j,n}(n \bar{Y}_j^{K,n}(t)) - \frac{1}{\sqrt{n}} \hat{S}_{j \rightarrow i}^{(2),n}(n \bar{B}_{j \rightarrow i}^{(2),n}(t)) + \frac{1}{\sqrt{n}} \theta_{j \rightarrow i}^{(2),n} nt. \quad (58)$$

From Assumption (32), using the Continuous Mapping Theorem and Theorem 1 (Fluid Limit), we obtain that for station node  $j$ ,

$$\tilde{X}_j^n(t) \Rightarrow \tilde{X}_j(t) = \tilde{Q}_j(0) + \sum_{d=1}^2 \sum_{i \neq j}^N \tilde{S}_{i \rightarrow j}^{(d)}(t) - \tilde{S}_j(t) + \theta_j t, \quad (59)$$



where  $\tilde{X}_j(t)$  is an Brownian motion with the initial queue length  $\tilde{Q}_j(0)$  and the drift  $\theta_j$ . For road station  $j \rightarrow i$ ,

$$\tilde{X}_{j \rightarrow i}^{(1),n}(t) \Rightarrow \tilde{X}_{j \rightarrow i}^{(1)}(t) = \tilde{Q}_{j \rightarrow i}^{(1)}(0) + \tilde{R}_i^j(b_j t) + p_{j \rightarrow i} \tilde{S}_j(t) - \tilde{S}_{j \rightarrow i}^{(1)}(t) + \theta_{j \rightarrow i}^{(1)} t, \quad (60)$$

where  $\tilde{X}_{j \rightarrow i}^{(1)}(t)$  is an Brownian motion with the initial queue length  $\tilde{Q}_{j \rightarrow i}^{(1)}(0)$  and the drift  $\theta_{j \rightarrow i}^{(1)}$ . Similarly we have

$$\tilde{X}_{j \rightarrow i}^{(2),n}(t) \Rightarrow \tilde{X}_{j \rightarrow i}^{(2)}(t) = \tilde{Q}_{j \rightarrow i}^{(2)}(0) + \tilde{R}_i^{j,n}(\tilde{Y}_j^{K,n}(t)) - \tilde{S}_{j \rightarrow i}^{(2)}(t) + \theta_{j \rightarrow i}^{(2)} t, \quad (61)$$

where  $\tilde{X}_{j \rightarrow i}^{(2)}(t)$  is an Brownian motion with the initial queue length  $\tilde{Q}_{j \rightarrow i}^{(2)}(0)$  and the drift  $\theta_{j \rightarrow i}^{(2)}$ . The covariance matrix  $\Gamma = (\Gamma_{\tilde{k}, \tilde{l}})_{N^2 \times N^2}$  of  $\tilde{X}(t) = (\tilde{X}_j(t), \tilde{X}_{j \rightarrow i}^{(d)}(t))$  is given by

$$\Gamma_{\tilde{k}, \tilde{l}} = \begin{cases} \sum_{d=1}^2 \sum_{i \neq k}^N b_{i \rightarrow k}^{(d)} (c_{s, i \rightarrow k}^{(d)})^2 \delta_{\tilde{k}, \tilde{l}} & \text{if } \sigma(S_k) = \tilde{k}, \sigma(S_l) = \tilde{l}; \\ + b_l c_{a, l}^2 \delta_{\tilde{k}, \tilde{l}}, & \\ p_{k \rightarrow l} b_k c_{a, k}^2, & \text{if } \sigma(S_k) = \tilde{k}, \sigma(R_{k \rightarrow l}) = \tilde{l}, d = 1; \\ b_i p_{i \rightarrow k} (\delta_{\tilde{k}, \tilde{l}} - p_{i \rightarrow l}) & \text{if } \sigma(R_{k \rightarrow l}) = \tilde{k}, \tilde{l} = \tilde{k}, d = 1; \\ + p_{k \rightarrow l} b_k c_{a, k}^2 + b_{k \rightarrow l}^{(d)} (c_{s, k \rightarrow l}^{(d)})^2, & \\ b_k \alpha_{i \rightarrow k} (\delta_{\tilde{k}, \tilde{l}} - \alpha_{i \rightarrow l}) & \text{if } \sigma(R_{k \rightarrow l}) = \tilde{k}, \tilde{l} = \tilde{k}, d = 2; \\ + b_{k \rightarrow l}^{(d)} (c_{s, k \rightarrow l}^{(d)})^2, & \\ 0, & \text{otherwise.} \end{cases} \quad (62)$$

Now, let  $h(t)$  be an arbitrary real, bounded and continuous function. For an arbitrary positive integer  $m$ , let  $t_i \leq t \leq t + s, i \leq m$ . Define

$$\tilde{H}^n(t) = \left( \tilde{Q}^n(t), \tilde{Y}^{0,n}(t), \tilde{Y}^{K,n}(t) \right), \quad \tilde{H}(t) = \left( \tilde{Q}(t), \tilde{Y}^0(t), \tilde{Y}^K(t) \right),$$

$$G^n(t, s) = \left( G_j^n(t, s), G_{j \rightarrow i}^{(d),n}(t, s) \right),$$

$$G_j^n(t, s) = \tilde{X}_j^n(t + s) - \tilde{X}_j^n(t), \quad G_{j \rightarrow i}^{(d),n}(t, s) = \tilde{X}_{j \rightarrow i}^{(d),n}(t + s) - \tilde{X}_{j \rightarrow i}^{(d),n}(t).$$

Notice that

$$\tilde{S}_j^m(t) = \frac{1}{\sqrt{n}} \left( \sup \left\{ k : \sum_{l=1}^k u_j^n(l) \leq b_j^n n t \right\} - b_j^n n t \right),$$

$$\tilde{S}_{j \rightarrow i}^{(d),n}(t) = \frac{1}{\sqrt{n}} \left( \sup \left\{ k : \sum_{l=1}^k v_{j \rightarrow i}^{(d),n}(l) \leq b_{j \rightarrow i}^{(d),n} n t \right\} - b_{j \rightarrow i}^{(d),n} n t \right),$$

by using the Assumption (32), there exist some nonnegative constants  $C_1$  and  $C_2$  such that  $b_j^n \leq C_1$  and  $b_{j \rightarrow i}^{(d),n} \leq C_2$ . From the convergences of (53) and (54), we have

$$\begin{aligned}
& \left| E \left[ h \left( \tilde{H}(t_i), i \leq m \right) \left( \tilde{X}(t+s) - \tilde{X}(t) - \theta s \right) \right] \right| \\
&= \left| \lim_{n \rightarrow \infty} E \left[ h \left( \tilde{H}^n(t_i), i \leq m \right) G^n(t, s) \right] \right| \\
&= \lim_{n \rightarrow \infty} \left| E \left[ h \left( \tilde{H}^n(t_i), i \leq m \right) E[G^n(t, s) | \varsigma_t^n] \right] \right| \\
&\leq M \lim_{n \rightarrow \infty} E[|E[G^n(t, s) | \varsigma_t^n]|] \\
&= 0,
\end{aligned}$$

where  $M$  is some positive constant. The arbitrariness of  $h(t)$ ,  $t_i$ ,  $t$  and  $t+s$  implies that

$$E \left[ \tilde{X}(t+s) - \tilde{X}(t) - \theta s | \mathcal{F}_u, u \leq t \right] = 0.$$

This shows that  $\tilde{X}(t) - \theta t$  is an  $\{\mathcal{F}_t\}$ -martingale. This completes the proof. ■

**Remark 1** *Note that Dai [8] discussed the queueing networks with finite buffers, this paper is related well to fluid and diffusion limits in Dai [8] in order to deal with a two-class closed queueing network.*

Now, we give the diffusion limit for the bike sharing system. In Section 5, we set up a sequence of closed queueing networks corresponding to the bike sharing systems, and prove the limit theorems of the fluid scaled equations of the busy period processes and the idle period processes through the functional strong law of large numbers and the oscillation property of an  $(S^n, R^n)$ -regulation. This is summarized as the Fluid Limit Theorem 1. Furthermore, based on the Fluid Limit Theorem, we prove the weak limit of the diffusion scaled processes of some performance measures and obtain a key martingale. Also see Theorem 2.

The following theorem provides a diffusion limit, and its proof is easy by means of some similar analysis to Theorems 3.2 and 3.3 in Dai [8] or Theorem 3.1 in Dai and Dai [6].

**Theorem 3** (*Diffusion Limit Theorem*) *Under Assumption (32), we have*

$$\left( \frac{1}{\sqrt{n}} Q^n(nt), \frac{1}{\sqrt{n}} Y^{0,n}(nt), \frac{1}{\sqrt{n}} Y^{K,n}(nt) \right) \Rightarrow \left( \tilde{Q}(t), \tilde{Y}^0(t), \tilde{Y}^K(t) \right),$$

where  $\tilde{Q}(t) = (\tilde{Q}_j(t), \tilde{Q}_{j \rightarrow i}^{(d)}(t))$ ,  $\tilde{Y}^0(t) = (\tilde{Y}_j^0(t), \tilde{Y}_{j \rightarrow i}^{0,(d)}(t))$ ;  $\tilde{Q}(t)$  together with  $\tilde{Y}^0(t)$  and  $\tilde{Y}^K(t)$  are an  $(S, \theta, \Gamma, R)$ -semimartingale reflecting Brownian motion with  $\tilde{Q}(t) = \tilde{Q}(0) + \tilde{X}(t) + R^0 \tilde{Y}^0(t) + R^K \tilde{Y}^K(t)$ . The state space  $S$  is given by (27) to (29). For station node  $j$ ,  $\tilde{X}_j(t)$  is given by (59),  $R^0$  and  $R^K$  are given by (10), (11). For road node  $j \rightarrow i$ , when  $d = 1$ ,  $\tilde{X}_{j \rightarrow i}^{(1)}(t)$  is given by (60),  $R^0$  and  $R^K$  are given by (14) and (15); when  $d = 2$ ,  $\tilde{X}_{j \rightarrow i}^{(2),n}(t)$  is given by (61),  $R^0$  and  $R^K$  are given by (18) and (19), and the covariance matrix  $\Gamma = (\Gamma_{\tilde{k}, \tilde{l}})_{N^2 \times N^2}$  of  $\tilde{X}(t) = (\tilde{X}_j(t), \tilde{X}_{j \rightarrow i}^{(d)}(t))$  is given by (62).

## 7 Performance analysis

In this section, we first set up a basic adjoint relationship for the steady-state probabilities of  $N$  station nodes and of  $N(N - 1)$  road nodes in the multiclass closed queueing network. Then we analyze some key performance measures of the bike sharing system.

From Theorem 3, it is seen that the scaling queueing processes, for the numbers of bikes in the stations and on the roads, converge in distribution to a semimartingale reflecting Brownian motion  $\tilde{Q}(t) = (\tilde{Q}_{\tilde{i}}(t), \tilde{Q}_{\tilde{j}}^{(d)}(t))$  for  $\tilde{i} = \sigma(S_i)$  ( $i = 1, \dots, N$ ) and  $\tilde{j} = \sigma(R_{j \rightarrow i})$  ( $i, j = 1, \dots, N$  with  $i \neq j, d = 1, 2$ ), where the state space  $S$ , the drift vector  $\theta = (\theta_{\tilde{i}}, \theta_{\tilde{j}}^{(d)})$  for  $\tilde{i} = \sigma(S_i), \tilde{j} = \sigma(R_{j \rightarrow l})$ , the covariance matrix

$$\Gamma = \begin{pmatrix} (\Gamma_{\tilde{i}, \tilde{k}}) & (\Gamma_{\tilde{i}, \tilde{j}}^{(d)}) \\ (\Gamma_{\tilde{l}, \tilde{k}}^{(d)}) & (\Gamma_{\tilde{l}, \tilde{j}}^{(d)}) \end{pmatrix}_{N^2 \times N^2}$$

for  $\tilde{i} = \sigma(S_i), \tilde{k} = \sigma(S_k), \tilde{j} = \sigma(R_{j \rightarrow h}), \tilde{l} = \sigma(R_{l \rightarrow g})$  and the reflecting matrix  $R = \left( (R_{\tilde{i}}^0), (R_{\tilde{j}}^{K,(d)}) \right)$  for  $\tilde{i} = \sigma(S_i), \tilde{j} = \sigma(R_{j \rightarrow l})$ , as seen in those previous sections. Hence, it is natural to approximate the steady-state distribution of the queue-length process by means of the steady-state distribution of the semimartingale reflecting Brownian motion.

From Lemma 1 and Theorem 1.3 in Dai and Williams [7], it is seen that there exists a unique stationary distribution  $\pi = (\pi_{\tilde{i}}, \pi_{\tilde{j}}^{(d)})$  on  $(S, \mathcal{B}_S)$  for the SRBM  $\tilde{Q}(t) = (\tilde{Q}_{\tilde{i}}(t), \tilde{Q}_{\tilde{j}}^{(d)}(t))$ . Furthermore,  $\pi = (\pi_{\tilde{i}}, \pi_{\tilde{j}}^{(d)})$  is equivalent to the Lebesgue measure on the state space  $S$ , thus for every bounded Borel function  $f$  on  $S$  and for  $t \geq 0$ , we have

$$E_\pi [f(\tilde{Q}(t))] \equiv \int_S (E_x [f(\tilde{Q}(t))]) \pi(dx) = \int_S f(x) \pi(dx).$$

Then for each  $\tilde{i} = 1, \dots, N$  (i.e.,  $\tilde{i} = \sigma(S_i), i = 1, \dots, N$ ) and  $\tilde{j} = 1, \dots, N(N - 1)$  (i.e.,

$\tilde{j} = \sigma(R_{j \rightarrow i}), i, j = 1, \dots, N$  with  $i \neq j$ ), let  $\delta = (\delta_{\tilde{i}}, \delta_{\tilde{j}}^{(d)})$  denote  $(N^2 - 1)$ -dimensional Lebesgue measure (surface measure) vector on face  $(F, \mathcal{B}_F)$ . Thus, there is a finite Borel measure vector  $\beta^F = (\beta_i^F, \beta_j^{F,(d)})$  on face  $F = (F_i, F_{\tilde{j}})$  such that  $\beta^F \approx \delta$  and

$$E_\pi \left\{ \int_0^t 1_A(\tilde{Q}(s)) d\tilde{Y}(s) \right\} = t\beta^F(A), \quad t \geq 0, A \in \mathcal{B}_F,$$

where  $\tilde{Y}(t) = (\tilde{Y}^0(t), \tilde{Y}^K(t))$ . Notice that the SRBM  $\tilde{Q}(t) = (\tilde{Q}_{\tilde{i}}(t), \tilde{Q}_{\tilde{j}}^{(d)}(t))$  is a strong Markov process with continuous sample paths. Furthermore, let  $p(x) = (p_{\tilde{i}}(x_{\tilde{i}}), p_{\tilde{j}}^{(d)}(x_{\tilde{j}}^{(d)}))$ ,  $p^F(x) = (p_i^F(\delta_{\tilde{i}}), p_j^{F,(d)}(\delta_{\tilde{j}}^{(d)}))$ , and define  $d\pi = p dx$ , i.e.,  $d\pi_{\tilde{i}} = p_{\tilde{i}} dx_{\tilde{i}}$  for  $\tilde{i} = \sigma(S_i)$  ( $i = 1, \dots, N$ ) and  $d\pi_{\tilde{j}}^{(d)} = p_{\tilde{j}}^{(d)} dx_{\tilde{j}}^{(d)}$  for  $\tilde{j} = \sigma(R_{j \rightarrow i})$  ( $i, j = 1, \dots, N$  with  $i \neq j, d = 1, 2$ ). Further, we define  $d\beta^F = p^F d\delta$ , i.e.,  $d\beta_{\tilde{i}}^F = p_i^F d\delta_{\tilde{i}}$  for  $\tilde{i} = \sigma(S_i)$  ( $i = 1, \dots, N$ ) and  $d\beta_{\tilde{j}}^{F,(d)} = p_j^{F,(d)} d\delta_{\tilde{j}}^{(d)}$  for  $\tilde{j} = \sigma(R_{j \rightarrow i})$  ( $i, j = 1, \dots, N$  with  $i \neq j, d = 1, 2$ ). Let  $\nabla f(x)$  be the gradient of  $f$ , and  $C_b^2(S)$  the space of twice differentiable functions whose first and second order partial derivative are continuous and bounded on the state space  $S$ . Base on this, it follows from the Ito's formula that the probability measures  $p(x)$  and  $p^F(x)$  have a basic adjoint relationship as follows: for  $\forall f \in C_b^2(S)$ ,

$$\begin{aligned} \int_S (\mathcal{L}f(x)p(x)) dx + \sum_{\tilde{i}=1}^N \int_{F_{\tilde{i}}^\bullet} (\mathcal{D}_{\tilde{i}}f(\delta_{\tilde{i}})p_{\tilde{i}}^F(\delta_{\tilde{i}})) d\delta_{\tilde{i}} + \sum_{d=1}^2 \sum_{\tilde{j}=1}^{N^2-N} \int_{F_{\tilde{j}}^\bullet} (\mathcal{D}_{\tilde{j}}f(\delta_{\tilde{j}}^{(d)})p_{\tilde{j}}^{F,(d)}(\delta_{\tilde{j}}^{(d)})) d\delta_{\tilde{j}}^{(d)} \\ + \sum_{\tilde{i}=1}^N \int_{F_{\tilde{i}}^\bullet} (\mathcal{D}_{\tilde{i}}f(\delta_{\tilde{i}})p_{\tilde{i}}^F(\delta_{\tilde{i}})) d\delta_{\tilde{i}} + \sum_{d=1}^2 \sum_{\tilde{j}=1}^{N^2-N} \int_{F_{\tilde{j}}^\bullet} (\mathcal{D}_{\tilde{j}}f(\delta_{\tilde{j}}^{(d)})p_{\tilde{j}}^{F,(d)}(\delta_{\tilde{j}}^{(d)})) d\delta_{\tilde{j}}^{(d)} = 0, \end{aligned} \quad (63)$$

where

$$\mathcal{L}f = \sum_{\tilde{i}=1}^N \mathcal{L}f(x_{\tilde{i}}) + \sum_{d=1}^2 \sum_{\tilde{j}=1}^{N^2-N} \mathcal{L}f(x_{\tilde{j}}^{(d)}),$$

for  $i, k, j = 1, \dots, N$  with  $i \neq j, d = 1, 2$ , and  $\tilde{k} = \sigma(S_k), \tilde{j} = \sigma(R_{j \rightarrow i}), \tilde{i} = \sigma(S_i) \in \{1, 2, \dots, N\}$ ,

$$\begin{aligned} \mathcal{L}f(x_{\tilde{i}}) &= \frac{1}{2} \sum_{\tilde{k}=1}^N \Gamma_{\tilde{i}, \tilde{k}} \frac{\partial^2 f(x_{\tilde{i}})}{\partial x_{\tilde{i}} \partial x_{\tilde{k}}} + \frac{1}{2} \sum_{d=1}^2 \sum_{\tilde{j}=1}^{N^2-N} \Gamma_{\tilde{i}, \tilde{j}}^{(d)} \frac{\partial^2 f(x_{\tilde{i}})}{\partial x_{\tilde{i}} \partial x_{\tilde{j}}^{(d)}} + \theta_{\tilde{i}} \frac{\partial f(x_{\tilde{i}})}{\partial x_{\tilde{i}}}, \\ \mathcal{D}_{\tilde{i}}f(\delta_{\tilde{i}}) &\equiv v_{\tilde{i}}' \nabla f(\delta_{\tilde{i}}) = \sum_{\tilde{k}=1}^N v_{\tilde{k}, \tilde{i}} \frac{\partial}{\partial \delta_{\tilde{k}}} f(\delta_{\tilde{i}}) + \sum_{d=1}^2 \sum_{\tilde{j}=1}^{N^2-N} v_{\tilde{j}, \tilde{i}} \frac{\partial}{\partial \delta_{\tilde{j}}^{(d)}} f(\delta_{\tilde{i}}), \end{aligned}$$

for  $l, k, i, j, h = 1, \dots, N$  with  $j \neq i, l \neq k$  and  $d = 1, 2$ , and  $\tilde{l} = \sigma(R_{l \rightarrow k}), \tilde{h} = \sigma(S_h), \tilde{j} = \sigma(R_{j \rightarrow i}) \in \{1, 2, \dots, N^2 - N\}$ ,

$$\mathcal{L}f(x_{\tilde{j}}^{(d)}) = \frac{1}{2} \sum_{\tilde{h}=1}^N \Gamma_{\tilde{j},\tilde{h}}^{(d)} \frac{\partial^2 f(x_{\tilde{j}}^{(d)})}{\partial x_{\tilde{j}}^{(d)} \partial x_{\tilde{h}}^{(d)}} + \sum_{\tilde{l}=1}^{N^2-N} \Gamma_{\tilde{j},\tilde{l}}^{(d)} \frac{\partial^2 f(x_{\tilde{j}}^{(d)})}{\partial x_{\tilde{j}}^{(d)} \partial x_{\tilde{l}}^{(d)}} + \theta_{\tilde{j}}^{(d)} \frac{\partial f(x_{\tilde{j}}^{(d)})}{\partial x_{\tilde{j}}^{(d)}},$$

$$\mathcal{D}_{\tilde{j}}f(\delta_{\tilde{j}}^{(d)}) \equiv v_{\tilde{j}}' \nabla f(\delta_{\tilde{j}}^{(d)}) = \sum_{\tilde{l}=1}^{N(N-1)} v_{\tilde{l},\tilde{j}} \frac{\partial}{\partial \delta_{\tilde{l}}^{(d)}} f(\delta_{\tilde{j}}^{(d)}) + \sum_{\tilde{h}=1}^N v_{\tilde{h},\tilde{j}} \frac{\partial}{\partial \delta_{\tilde{h}}^{(d)}} f(\delta_{\tilde{j}}^{(d)}),$$

$F_{\tilde{i}}^{\blacktriangledown}$  and  $F_{\tilde{i}}^{\blacktriangle}$  denote the “bottom face” and the “top face” in this box state space  $S$  corresponding to empty station  $i$  and full station  $i$ , respectively. As a similar expression, it is clear that  $F_{\tilde{j}}^{\blacktriangledown}$  and  $F_{\tilde{j}}^{\blacktriangle}$  are related to road  $j \rightarrow i$ ;  $v_{\tilde{k}}$  is the  $\tilde{k}$ th column of the reflection matrix  $R = \left( \left( R_{\tilde{i}}^0 \right), \left( R_{\tilde{j}}^{K,(d)} \right) \right)$ .

Now, we consider some key performance measures of the bike sharing system in terms of the steady-state probability density function  $p$  on  $(S, \mathcal{B}_S)$  and an nonnegative integrable Borel function  $p^F$  on  $(F, \mathcal{B}_F)$ . Here, it is easy to see that for  $\tilde{i} = 1, \dots, N$  and  $\tilde{j} = 1, \dots, N(N-1)$ , the “bottom face”  $F_{\tilde{i}}^{\blacktriangledown}$  ( $F_{\tilde{j}}^{\blacktriangledown}$ ) and the “top face”  $F_{\tilde{i}}^{\blacktriangle}$  ( $F_{\tilde{j}}^{\blacktriangle}$ ) are precisely parallel in this box state space  $S$ .

- (1) The steady-state probability that station  $i$  is empty is given by

$$\int_S p_{\tilde{i}}^F 1_{\{x_{\tilde{i}} \in F_{\tilde{i}}^{\blacktriangledown}\}} dx_{\tilde{i}}, \quad \text{for } \tilde{i} = \sigma(S_i).$$

- (2) The steady-state probability that station  $i$  is full is given by

$$\int_S p_{\tilde{i}}^F 1_{\{x_{\tilde{i}} \in F_{\tilde{i}}^{\blacktriangle}\}} dx_{\tilde{i}}, \quad \text{for } \tilde{i} = \sigma(S_i).$$

- (3) The steady-state probability that road  $j \rightarrow i$  is empty for bikes of class  $d$  is given by

$$\int_S p_{\tilde{j}}^{F,(d)} 1_{\{x_{\tilde{j}}^{(d)} \in F_{\tilde{j}}^{\blacktriangledown}\}} dx_{\tilde{j}}^{(d)}, \quad \text{for } \tilde{j} = \sigma(R_{j \rightarrow i}), d = 1, 2.$$

- (4) The steady-state probability that road  $j \rightarrow i$  is full for bikes of class  $d$  is given by

$$\int_S p_{\tilde{j}}^{F,(d)} 1_{\{x_{\tilde{j}}^{(d)} \in F_{\tilde{j}}^{\blacktriangle}\}} dx_{\tilde{j}}^{(d)}, \quad \text{for } \tilde{j} = \sigma(R_{j \rightarrow i}), d = 1, 2.$$

- (5) The steady-state means of the number of bikes parked at the station  $i$  and the number of bikes of class  $d$  ridden on road  $j \rightarrow i$  are respectively given by

$$\mathcal{Q}_{\tilde{i}} = \int_S x_{\tilde{i}} p_{\tilde{i}}(x_{\tilde{i}}) dx_{\tilde{i}}, \quad \text{for } \tilde{i} = \sigma(S_i),$$

$$\mathcal{Q}_{\tilde{j}}^{(d)} = \int_S x_{\tilde{j}}^{(d)} p_{\tilde{j}}^{(d)}(x_{\tilde{j}}^{(d)}) dx_{\tilde{j}}^{(d)}, \quad \text{for } \tilde{j} = \sigma(R_{j \rightarrow i}), d = 1, 2.$$

(6) The steady-state mean of the number of bikes of class  $d$  deflecting from the full station  $i$  is given by

$$\mathcal{E}_i^{(d)} = \int_{F_i^\Delta} x_i^{(d)} p_i^{F, (d)} \left( x_i^{(d)} \right) dx_i^{(d)}, \quad \text{for } \tilde{i} = \sigma(S_i), d = 1, 2.$$

## 8 Concluding Remarks

In this paper, we describe a more general large-scale bike sharing system having renewal arrival processes and general travel times, and develop fluid and diffusion approximation of a multiclass closed queuing network which is established from the bike sharing system where bikes are regarded as virtual customers, and stations and roads are viewed as virtual nodes or servers. From the multiclass closed queuing network, we show that the scaling queue-length processes, which are set up by means of the number of bikes both at stations and on roads, converge in distribution to a semimartingale reflecting Brownian motion. Also, we obtain the Fluid Limit Theorem and the Diffusion Limit Theorem. Based on this, we provide performance analysis of the bike sharing system. Therefore, the results of this paper give new highlight in the study of more general large-scale bike sharing systems. The methodology developed here can be applicable to deal with more general bike sharing systems by means of the fluid and diffusion approximation. Along such a line, there are some interesting directions in our future research, for example,

- analyzing bike repositioning policies through several fleets of trucks under information technologies;
- making price regulation of bike sharing systems through Brownian approximation of multiclass closed queuing network;
- developing heavy traffic approximation for time-varying or periodic bike sharing systems; and
- developing heavy traffic approximation for new ride sharing (bike or car) systems with scheduling, matching and control.

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