CS201 Assignment 1
The Concept of Numbers

Mahaarajan J - 220600

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Maximum Marks: $20 \times 5 = 100$

Before we start discussion on numbers, let us examine the axioms of set theory and why they are required. Define U to be the collection of all sets.

- Show that U is not a set as per the Zermalo Fraenkel Axioms.

Ans: As U is a collection of all sets by axiom of pairing {U,U} is a set which means {U} is a set Also U is the only element in the set U hence

$$U \epsilon U$$

which is a violation of the axiom of regularity. Hence according to the Zermalo Frankel Axioms U is not a set

The motivation to define these axioms was a paradox discovered by Bertrand Russell: Suppose we allow U to be a set. Then $U \in U$ by definition. Define:

$$V = \{A \mid A \notin A\}.$$

- Derive a contradiction using the question "is $V \in V$?".

Ans: Case 1 ($V \epsilon V$): As $V \epsilon V$ it must satisfy the property of the definition of V, i.e $V \notin V$(As V is now an element of the set V); hence we have arrived at a contradiction.
Case 2 ($V \notin V$): As $V \notin V$ V satisfies the property of definition of V hence $V \epsilon V$ hence we have arrived at a contradiction

Hence for both cases V can't exist as we have contradictions

This is the reason that circularity in definition of sets was explicitly not permitted by the axioms.

Let us now move to numbers. In the class, we discussed the definition of natural numbers through Peano's Axioms. How does one define numbers in general? One possible way is to define numbers as any set that admits four arithmetic operations: addition, subtraction, multiplication, and division. But to define arithmetic operations, we need numbers! This is resolved by defining both together. Let us develop axioms for this. Consider addition and subtraction first.

Define set of *numbers with addition* $(N, +)$ as:

1. $+ : N \times N \mapsto N$. We will write $+(a, b)$ as $a + b$.
2. $(a + b) + c = a + (b + c)$ for all $a, b, c \in N$.
3. There is an element $0 \in N$ such that $a + 0 = 0 + a = a$ for all $a \in N$.
4. For all $a \in N$, there is an element $b \in N$ such that $a + b = 0$.
5. $a + b = b + a$ for all $a, b \in N$.

With above definition, subtraction can be defined as: $a - b = a + c$ where c is such that $b + c = 0$. Does this capture the addition and subtraction properly? Show that:

- There is a unique number 0 satisfying third axiom.

  Ans: Let us assume there be 2 distinct zeroes 0_1 and 0_2 from the definition of additive identity we have

  $$0_1 + 0_2 = 0_1$$

  by using 0_2 as the additive identity. Similarly we have

  $$0_1 + 0_2 = 0_2$$

  on using 0_1 as additive identity.
  Hence we have $0_1 = 0_2$ which is a contradiction (from the 2 equations). Therefore we have proved that there exists only a unique number 0 satisfying the third axiom.

- For every $a \in N$, there is a unique b satisfying fourth axiom.

Ans: Let us assume there be 2 distinct additive inverses b_1 and b_2 and we know that

$$a + b_1 = 0 = a + b_2$$

by definition of additive inverse. Now let us add b_2 to both the sides of the 1st equation

$$a + b_2 + b_1 = b_2$$

Now from 2nd equation $a + b_2 = 0$ the above equation simplifies to

$$b_1 = b_2$$

which is a contradiction to our initial assumption. Hence we have proved that exists a unique b for every a satisfying the 4th axiom.

- Define $-a$ to be the number such that $a + (-a) = 0$. For every $a, b \in N$, $a - b = -(b - a)$.

Ans: Let us define $k_1 = a - b$ and $k_2 = -(b - a)$ by definition of inverse

$$k_2 + b - a = 0$$

let us add k_1 on both sides

$$k_2 + b - a + a - b = k_1$$

from definition of inverse we have

$$a + (-a) = b + (-b) = 0$$

using these equations and the property of associativity we get

$$k_1 = k_2$$

Hence Proved

Now let us add multiplication and division. Define set of *numbers with multiplication* $(N, *)$ as:

1. $* : N \times N \mapsto N$. We will write $*(a, b)$ as $a * b$.
2. $(a * b) * c = a * (b * c)$ for all $a, b, c \in N$.
3. There is an element $1 \in N$ such that $a * 1 = 1 * a = a$ for all $a \in N$.
4. For all $a \in N$, there is an element $b \in N$ such that $a * b = 1$.

5. $a * b = b * a$ for all $a, b \in N$.

These axioms are identical to first ones except for the name of operation and replacement of 0 by 1. Division operation is defined analogously to subtraction. It is easy to see that the definition of '$-$' and '$/$' is entirely determined by the definition of $+$ and $*$ respectively.

Finally define set of *numbers with addition and multiplication* $(N, +, *)$ as:

1. $(N, +)$ is a set of numbers with addition.
2. $(N \backslash \{0\}, *)$ is a set of numbers with multiplication.
3. For all $a, b, c \in N$, $a * (b + c) = a * b + a * c$.

Why is the number '0' excluded from N in second axiom above? It is to avoid division by zero. Show that:

- If 0 is included in N for the second axiom, then $1 = 0$.

Ans: Let a be such that a*0=1 as 0 satisfies the 2nd axiom. As 0 is the additive identity we have 0-0=0. Using the distributive property

$$a * (0 - 0) = a * 0$$

$$a * 0 - a * 0 = 1$$

$$1 - 1 = 1$$

Hence we have

$$1 = 0$$

Hence proved that if 0 is included in N 1 will be equal to 0 which is clearly False hence 0 must be excluded from N in the 2nd axiom

The addition and multiplication operations can be different for different sets of numbers:

- Give two examples of sets of numbers with different addition and multiplication operations.

1) Let

$$a +_2 b = a + b$$

and

$$a *_2 b = 2 * a * b$$

i)Now as addition is the same all addition axioms will be satisfied and N will be a set of numbers with addition.

ii) For multiplication we have:

1. $a*_2$ b ϵ N as 2*a*b does belong to N.

2. $a *_2 (b *_2 c) = (a *_2 b) *_2 c$ as
$b *_2 c = 2 * b * c$
and $a *_2 (b *_2 c) = 4 * a * (b * c)$
similarly $(a *_2 b) *_2 c = 4 * (a * b) * c$
Now as * is associative both are equal. Hence this axiom is satisfied

3. We have an element 1/2 belonging to N such that $a *_2 (1/2) = a$ from definition of $*_2$

4. For every a belonging to N (except 0) we have b belonging to N such that a$*_2$b=1/2 b will be 1/(4*a) which does belong to N

5. $a *_2 b = b *_2 a$ will be true as 2*a*b=2*b*a by associative property of *
Hence N excluding 0 is a set of numbers wrt multiplication

iii) For a combination of both we have:
$a *_2 (b + c) = a *_2 b + a *_2 c$ which is true as
$a*_2(b+c) = 2*a*(b+c) = 2*a*b+2*a*c$ which is the same as $a*_2b+a*_2c$

Hence N is a set of numbers wrt $+_2$ and $*_2$

2) Let $a +_3 b = (a + b)mod(3)$ and $a *_3 b = (a * b)mod(3)$ where 3 is a natural prime number > 1 Now we will get N to be the set {0,1,2}

i) For addition

1) $a +_3 b$ will belong to N as (a+b)mod(3) will always be an integer from 0 to 2

2)Associativity $(a +_3 (b +_3 c) = (a +_3 b) +_3 c)$: as we can write (a+b)mod(3) as ((a)mod(3)+(b)mod(3))mod(3) from definition of modulus similarly we can write ((a+b)mod(3)+c)mod(3) as (a+b+c)mod(3) from property of modulus In the same way LHS can also be simplified to (a+b+c)mod(3) hence the axiom is satisfied

3) There is an identity element 0 belonging to the set N such that $a+_30$=(a+0)mod(3)=(a)mod(3)=a as a< 3

4) There exists b in the set where b=(3-a)mod(3) for every a belonging to N where $a+_3b$=0 as (3)mod(3)=0

5) as + is commutative $+_3$ is also commutative as (a+b)mod(3)=(b+a)mod(3)

Hence N is a set of numbers wrt $+_3$.

ii) For multiplication

1) $a*_3b$=(a*b)mod(3) will belong to N as mod(3) will always be an integer from 0 to 3-1

2) Associativity $(a *_3 (b *_3 c) = (a *_3 b) *_3 c)$ Now using properties of modulus (a*(b*c)mod(3))mod(3) can be written as (a*b*c)mod(3) hence as * is associative we can say that LHS=RHS hence $*_3$ is also associative

3) There is 1 in the set N such that $1*_3$a=(1*a)mod(3)=(a)mod(3)=a hence 1 is the multiplicative identity

4) There exists inverse for each element except 0 as inverse of 1 is 1 and inverse of 2 is 2 as can be verified using the $*_3$ operation

5) Commutative as $a*_3$b=(a*b)mod(3) and * is commutative hence (a*b)mod(3)=(b*a)mod(3).

Hence N is a set wrt multiplication

iii) Combination of both $a *_3 (b + c) = a *_3 b + a *_3 c$
(a*((b+c)mod(3))mod(3)=(a*(b+c))mod(3)=(a*b+a*c)mod(3) as *,+ is distributive similarly simplify RHS to get LHS=RHS

Hence N is a set of numbers wrt $+_3$ and $*_3$.

Does a set of numbers defined as above contains natural numbers? Show that:

- There is a set of numbers $(N, +, *)$ such that N is finite.

> refer to the 2nd example of the previous question where N was a set of 3 numbers. Thus there is a set of numbers $(N, +, *)$ such that N is finite

Does this mean that we have not been able to capture the notion of numbers properly? Later in the course, we will show that it is not so. A set of numbers *can* be finite, and such numbers are extremely useful!

In order to identify set of numbers that contain $\mathbb{N}$, define *multiplicity* of set $(N, +, *)$ to be the smallest 3 for which $\underbrace{1 + 1 + \cdots + 1}_{3 \text{ times}} = 0$. When there is no such 3, then we set multiplicity of $(N, +, *)$ to 0. Show that:

- Multiplicity of $(N, +, *)$ is either 0 or a prime number.

Let us prove this by contradiction. Multiplicity can't be 1 as then we would have 1=0 which is wrong.
Let us assume the composite number k can be expressed as k=a*b where a and b are natural numbers
let 1+1+1...a times =c and 1+1+1....b times=d now using the 1st equation it reduces to c+c+c...b times
Let us define a lemma that a*(1+1+1...k times)=a+a+a...k times

Proof of this lemma by induction:
For k=1 it is trivially true. Assume it is true for k then prove it for k+1 we have:
a+a+a...k times +a= a*(1+1+1...k times)+a*1
In LHS by applying axiom 3 of Combination of addition and multiplication we have
LHS=a*(1+1+1...k times+1) hence we have proved
a+a+...k+1 times=a*(1+1+1...k+1 times). Therefore by principle of mathematical induction the lemma defined as above is true

Hence c+c+c...b times can be written as c*(1+1+...b times)=c*d.
Now from definition of multiplicity as k is the multiplicity c,d can't be 0 as a,b both are $< k$
so since c is not 0 let uis define e such that e*c=1 from the multiplicative inverse axiom.
Multiplying the equation with e on both sides we have
e*c*d=e*0 as c*d=0 from the question
hence we get d=0 which is a contradiction as b<k

Hence we have proved that k must be either 0 or a prime number.

- Any set of numbers $(N, +, *)$ of multiplicity 0 contains $\mathbb{N}$.

  As multiplicity is 0, 1+1 is not 0 hence there exisits an additive inverse -1 such that 1+(-1)=0. Hence we have the existence of 3 numbers 1,0 and -1. Similarly 1+1 is not equal to either of 1 or 0 or -1

  1+1=1 add -1 on both sides we get 1=0 False
  1+1=0 not possible as multiplicity is 0
  1+1=-1 add 1 on both sides 1+1+1=0 not possile as multiplicity is 0

  Hence let us define 1+1 to be a new number 2. Similarly one can extend the arguments that 1+1+1...k times will be distinct (or else we will raise a contradiction to the multiplicity being 0 as seen before) and can define it to be k for every such k. Therefore we have the set of natural numbers to be contained in any set of numbers of multiplicity 0

  Hence Proved

- For any set of numbers $(N, +, *)$ of multiplicity 0, for any $k \in \mathbb{N} \subseteq N$, for any $a \in N$, $k * a = \underbrace{a + a + \cdots + a}_{k \text{ times}}$.

  From the Lemma defined in the previous question we can write a+a+...k times as a*(1+1+...k times).
  Now as multiplicity is 0 and from the analysis of the previous question we can write 1+1+...k times=k.
  Hence a+a+...k times will be a*k.

  Hence Proved.

As was done in the class with $\mathbb{N}$, is there way to identify a unique set of numbers using equivalence classes? The answer is no, as there can be finite as well as infinite set of numbers. Moreover, there are binary operations defined on numbers and any equivalence between two sets of numbers must equate the operations as well. Define an *isomorphism* h between two sets of numbers $(N_1, +_1, *_1)$ and $(N_2, +_2, *_2)$ as:

1. $h : N_1 \mapsto N_2$ is a bijection,
2. For all $a, b \in N_1$, $h(a +_1 b) = h(a) +_2 h(b)$,
3. For all $a, b \in N_1$, $h(a *_1 b) = h(a) *_2 h(b)$.

Show that:

- The relation defined by isomorphism between two sets of numbers is an equivalence relation on the set of all sets of numbers.

We will prove its an equivalence relation by proving it is reflexive, transitive and symmetric:

i) Reflexive: One can trivially define a bijection that maps every element to itself for any set N.thus proving $N_1 \mathrm{R} N_1$

ii) Transitive: Consider 3 sets of Numbers $(N_1(+_1, *_1), N_2(+_2, *_2), N_3(+_3, *_3))$ and let f :$N_1 \mapsto N_2$ and g : $N_2 \mapsto N_3$ be bijections satisfying the above properties. Let us define h :$N_1 \mapsto N_3$ such that h=f$\circ$g Now let us check if h satisfies the properties:
1) h is a bijection as both f,g are bijections
2)Property of addition from definition of f,g we have
$f(a +_1 b) = f(a) +_2 f(b)$,
$g(c +_2 d) = g(c) +_3 g(d)$
where a,b ϵN_1 and c,d ϵN_2
$h(a +_1 b) = g(f(a +_1 b)) = g(f(a) +_2 f(b)) = g(f(a)) +_3 g(f(b))$
which is same as $h(a) +_3 h(b)$ Hence h satisfies the 2nd property
3) Similarly for multiplication we have
$f(a *_1 b) = f(a) *_2 f(b)$,
$g(c *_2 d) = g(c) *_3 g(d)$
where a,b ϵN_1 and c,d ϵN_2
$h(a *_1 b) = g(f(a *_1 b)) = g(f(a) *_2 f(b)) = g(f(a)) *_3 g(f(b))$
which is same as $h(a) *_3 h(b)$ Hence h satisfies the 3rd property
Therefore it is transitive i.e $N_1 \mathrm{R} N_2$ and $N_2 \mathrm{R} N_3$ implies $N_1 \mathrm{R} N_3$

iii) Symmetric: Consider a bijection $f : N_1 \mapsto N_2$ and its inverse $f^{-1} : N_2 \mapsto N_1$ Given f is an isomorphism we will try to prove that f^{-1} is also an isomorphism thus proving $N_1 \mathrm{R} N_2$ implies $N_2 \mathrm{R} N_1$
1) As f is a bijection f^{-1} is also a bijection trivially
2) Property of addition from definition of f we have
$f(a +_1 b) = f(a) +_2 f(b)$ applying f^{-1} on both the sides we get
$f^{-1}(c +_2 d) = f^{-1}(c) +_1 f^{-1}(d)$ where c,d ϵN_2 c=f(a) and d=f(b)
3)Property of multiplication from definition of f we have
$f(a *_1 b) = f(a) *_2 f(b)$ applying f^{-1} on both the sides we get
$f^{-1}(c *_2 d) = f^{-1}(c) *_1 f^{-1}(d)$ where c,d ϵN_2 c=f(a) and d=f(b)

Hence we have proved that the realtion is reflexive, transitive and symmetric therefore it is an equivalence relation.
Hence Proved.

- If h is an isomorphism from $(N_1, +_1, *_1)$ to $(N_2, +_2, *_2)$ then $h(0_1) = 0_2$ and $h(1_1) = 1_2$.

  Consider 2 sets of Numbers $(N_1(+_1, *_1), N_2(+_2, *_2))$ and let
  h :$N_1 \mapsto N_2$ be an isomorphism now by properties of isomorphisms we have $h(0_1 +_1 0_1) = h(0_1) +_2 h(0_1)$ now we know that $0_1 +_1 0_1 = 0_1$ so we have
  $h(0_1) = h(0_1) +_2 h(0_1)$ let additive inverse of h(0_1) be d such that $h(0_1) + d = 0_2$ adding d on both sides we have
  $0_2 = 0_2 +_2 h(0_1)$ as 0_2 is additive identity we have
  $h(0_1) = 0_2$.

  we have $h(1_1 *_1 1_1) = h(1_1) *_2 h(1_1)$ now we know that $1_1 *_1 1_1 = 1_1$ so we have
  $h(1_1) = h(1_1) *_2 h(1_1)$ let multiplicative inverse of h(1_1) be d (we are considering $h(1_1)! = 0_2$such that $h(1_1) * d = 1_2$ multiplying d on both sides we have
  $1_2 = 1_2 *_2 h(1_1)$ as 1_2 is multiplicative identity we have
  $h(1_1) = 1_2$.

  Hence Proved

- If h is an isomorphism from $(N_1, +_1, *_1)$ to $(N_2, +_2, *_2)$ then $h(a -_1 b) = h(a) -_2 h(b)$ and $h(a/_1 b) = h(a)/_2 h(b)$.

Consider 2 sets of Numbers $(N_1(+_1, *_1), N_2(+_2, *_2))$ and let h $:N_1 \mapsto N_2$ be an isomorphism now from previous solution we have

$$h(0_1) = 0_2$$

$$h(a +_1 (-_1 a)) = 0_2$$

$$h(a) +_2 h(-_1 a) = 0_2$$

Hence $h(a) = -_2 h(-_1 a)$ Now consider the equation $h(a +_1 (-_1 b)) = h(a) +_2 h(-_1 b)$ from property of h now as $-_2 h(b) = h(-_1 b)$ we have
$h(a -_1 b) = h(a) -_2 h(b)$ Hence proved

Similarly for multiplication we have

$$h(1_1) = 1_2$$

$$h(a *_1 (1_1/_1 a)) = 1_2$$

$$h(a) *_2 h(1_1/_1 a) = 1_2$$

Hence $h(a) = 1_2/_2 h(1_1/_1 a)$ Now consider the equation $h(a *_1 (1_1/_1 b)) = h(a) *_2 h(1_1/_1 b)$ from property of h now as $1_2/_2 h(b) = h(1_1/_1 b)$ we have
$h(a/_1 b) = h(a)/_2 h(b)$ Hence proved

Do two sets of numbers of same cardinality always have isomorphism between them? The answer is no. Define a 0-1 polynomial to be $\sum_{i=0}^{3} c_i x^i$ with $c_i = 0, 1$. Define addition of these polynomials as $x^i + x^i = 0$ for every i. Let $F_2(x)$ be defined as rational functions of the kind p(x)/q(x) where both p,q are 0-1 polynomials as defined and q(x) is not 0

- Prove that the set of 0-1 polynomials with addition defined as above and usual multiplication of polynomials is a set of numbers. It is represented as $F_2(x)$.

For the set of numbers $F_2(x)$ the additive identity is obviously 0 itself as 0+P(x)=P(x)+0=P(x) for all P(x) belonging to our set.
Similarly the multiplicative identity is 1 as 1*P(x)=P(x)*1=P(x) for all P(x) belonging to our set.
Let us define addition as
$P_1(x)/Q_1(x) + P_2(x)/Q_2(x) = (P_1(x) * Q_2(x) + P_2(x) * Q_1(x))/(Q_1(x) * Q_2(x))$
Clearly additive inverse will be the polynomial itself as
$P_1(x)/Q_1(x) + P_1(x)/Q_1(x) = (P_1(x) * Q_1(x) + P_1(x) * Q_1(x))/(Q_1(x)*Q_1(x)) = Q_1(x)*(P_1(x)+P_1(x))/(Q_1(x)*Q_1(x))//$
Now from definition of 0-1 polynomials $P_1(x) + P_1(x) = 0$ Hence the additive inverse of a rational polynomial in F_2 is itself.
Other addition axioms (closure, associativity, commutativity) are

Closure:
Consider the addition of any two 0-1 polynomials,a(x) and b(x). Suppose the sum is the polynomial c(x). For any term of power i in c(x), three cases arise-
Case 1: term is not present in both a(x) and b(x).
In this case, c(x) doesn't have the term either. Thus, the coefficient is 0
Case 2: term is present in either a(x) or b(x)
In this case, c(x) does have the term and its coefficient is 1.
Case 3: term is present in both a(x) and b(x)
In this case, keeping in lines with the definition of addition, c(x) doesn't contain the term. Thus, its coefficient is 0. Thus, the sum of any two 0-1 polynomials will be a 0-1 polynomial. Since multiplication is repeated addition, the product of two 0-1 polynomials will be a 0-1 polynomial as well.

Now consider the addition of two F_2(x) polynomials, $\frac{a(x)}{b(x)}$ and $\frac{c(x)}{d(x)}$. The numerator is ((a(x)*d(x)) + (c(x)*b(x))). Since we have already proven that addition and multiplication of two 0-1 polynomials lead to 0-1 polynomials, the numerator is a 0-1 polynomial.
Similarly, the denominator b(x)*d(x) is a 0-1 polynomial.
Thus, the addition of two F_2(x) numbers results in another member of F_2(x)
Hence closure property is satisfied

Associative property:
Consider $\frac{a(x)}{b(x)}$, $\frac{c(x)}{d(x)}$ and $\frac{e(x)}{f(x)}$.
$\frac{a(x)}{b(x)}+(\frac{c(x)}{d(x)} + \frac{e(x)}{f(x)}) = \frac{a(x)*d(x)*f(x)+c(x)*b(x)*f(x)+e(x)*d(x)*b(x)}{f(x)*d(x)*b(x)}$.
$(\frac{a(x)}{b(x)}+ \frac{c(x)}{d(x)}) + \frac{e(x)}{f(x)}$ upon expansion gives the same.
Hence associative property is satisfied

Commutative property:
Consider two numbers $\frac{a(x)}{b(x)}$ and $\frac{c(x)}{d(x)}$
$(\frac{a(x)}{b(x)}) + (\frac{c(x)}{d(x)}) = \frac{a(x)*d(x)+b(x)*c(x)}{d(x)*b(x)}$
$(\frac{c(x)}{d(x)}) + (\frac{a(x)}{b(x)}) = \frac{b(x)*c(x)+a(x)*d(x)}{d(x)*b(x)}$
Using the cases in the first axiom, we can prove that the addition of two 0-1 polynomials is commutative. Similarly, multiplication is commutative as well.
Thus, commutativity is satisfied for F_2(x) polynomials.

Multiplication axioms

Closure:
Consider $\frac{a(x)}{b(x)}$ and $\frac{c(x)}{d(x)}$. $*(\frac{a(x)}{b(x)},\frac{c(x)}{d(x)}) = \frac{a(x)*c(x)}{b(x)*d(x)}$
We have already proven that both the denominator and the numerator will be 0-1 polynomials.
Thus, closure axiom is satisfied.

Associativity:
Consider $\frac{a(x)}{b(x)}$, $\frac{c(x)}{d(x)}$ and $\frac{e(x)}{f(x)}$.
$\frac{a(x)}{b(x)} * (\frac{c(x)}{d(x)} * \frac{e(x)}{f(x)}) = \frac{a(x)}{b(x)} * (\frac{c(x)*e(x)}{d(x)*f(x)}) = \frac{a(x)*c(x)*e(x)}{b(x)*d(x)*f(x)}$
$(\frac{a(x)}{b(x)} * \frac{c(x)}{d(x)}) * \frac{e(x)}{f(x)}$ upon expansion produces the same.
Thus, Associativity is satisfied.

Multiplicative inverse:
for every element in F_2(x) of the form $\frac{a(x)}{b(x)}$ where a(x) and b(x) are 0-1 polynomials, consider $\frac{b(x)}{a(x)}$.
Hence, $*(\frac{a(x)}{b(x)},\frac{b(x)}{a(x)})$=1, where 1 is as defined above.
Hence every element except 0 has a multiplicative inverse.

Commutative property:
$\forall \frac{a(x)}{b(x)},\frac{c(x)}{d(x)} \in F_2(\text{x}), *(\frac{a(x)}{b(x)},\frac{c(x)}{d(x)}) = \frac{a(x)*c(x)}{b(x)*d(x)} = \frac{c(x)*a(x)}{d(x)*b(x)} = *(\frac{c(x)}{d(x)},\frac{a(x)}{b(x)})$
Hence commutative property is satisfied

Distributive property:
$\forall \frac{a(x)}{b(x)},\frac{c(x)}{d(x)},\frac{e(x)}{f(x)} \in F_2(\text{x}),$:
$\frac{a(x)}{b(x)} * (\frac{c(x)}{d(x)} + \frac{e(x)}{f(x)}) = \frac{a(x)}{b(x)} * (\frac{c(x)*f(x)+e(x)*d(x)}{d(x)*f(x)}) = \frac{c(x)*f(x)*a(x)+e(x)*d(x)*a(x)}{d(x)*f(x)*b(x)} = \frac{a(x)*c(x)}{b(x)*d(x)} + \frac{a(x)*e(x)}{b(x)*f(x)} = \frac{a(x)}{b(x)} * \frac{c(x)}{d(x)} + \frac{a(x)}{b(x)} * \frac{e(x)}{f(x)}$
Thus, it satisfies the distributive property

Hence as it satisfies all the axioms of numbers wrt the addition, multiplication and combination of both as defined above. Hence the set $F_2(x)$ it is a set of numbers.

- Show that there is a bijection between rational numbers $\mathbb{Q}$ and $F_2(x)$.

  From theorm we can show that a bijection exists if we can define 2 one-one functions from $Q \mapsto F_2(x)$ and $F_2(x) \mapsto Q$

  1) $F_2(x) \mapsto Q$:
  we know that all elements of $F_2(x)$ are of the form P(x)/Q(x) now let the coefficients of P(x) when powers are arranged in decreasing order be $a_1, a_2, ...$ Let us define a binary number (as a_i= {0,1} for all values of i) $a_1a_2a_3....$=A similarly let the coefficients of Q(x) when powers are arranged in decreasing order be $b_1, b_2, ...$ Let us define a binary number (as b_i= {0,1} for all values of i) $b_1b_2b_3....$=B. Let C=$a_1b_1a_2b_2....$ be a binary number. Now since there exists a bijection between binary numbers and rational numbers we can say that for every C there is a unique rational number. Also for every pair of P,Q we will have a unique binary representation C. Hence this is a one-one function

  2) $Q \mapsto F_2(x)$
  Rational number is of the form a/b where a ϵ Z and bϵ N. Let us convert —a— and —b— to binary and define a sequence $sign(a), a_1, b_1, a_2, b_2, ...$ where sign(a) returns 0 if a is positive and 1 if a is negative. Now we define a 0-1 polynomial with its coefficients in the following order. Thus this polynomial will be a unique $F_2(x)$ polynomial with Q(x)=1 for every unique rational number. Hence we have defined a one-one function

  Therefore there exists a bijection between Q and $F_2(x)$. Hence Proved

- Show that there is no isomorphism between $\mathbb{Q}$ and $F_2(x)$.

  Let there be an isomorphism $h : Q \mapsto F_2(x)$ then:
  h(0)=0 and h(1)=1 as the additive and multiplicative identity are same from previous solutions. Now h(1+1)=h(1)+$_2$h(1) as h is an isomorphism.
  Hence h(2)=1 +$_2$ 1=0 from definition of additive inverse and 0-1 polynomials. Now we already know that h(0)=0
  Hence h is not a bijection and therefore not an isomorphism. Hence we arrive at a contradiction and therefore there exists no isomorphism between Q and $F_2(x)$

As per the definition above, the set of integers $\mathbb{Z}$ is not a set of numbers. This is unsatisfactory. The problem is that division is generally not possible in

$\mathbb{Z}$. To address this, define a set of *numbers without division* $(N, +, *)$ to be a set of numbers in which the fourth axiom for $(N, *)$ is removed. Show that:

- $(\mathbb{Z}, +, *)$ is a set of numbers without division.

  For checking if $(\mathbb{Z}, +, *)$ is a set of numbers without division we will check all the axioms.

  Addition Axioms:
  Since 0 is already a part of (Z, +) and no changes are made to the axioms as well as we have defined addition to be the normal addition, all the axioms of addition are satisfied in the same way as before.

  Multiplication axioms:
  For all members of $\mathbb{Z}$ apart from 0, the remaining 4 multiplication axioms are satisfied as before as multiplication is just the normal multiplication. Only multiplication of 0 can now be explicitly defined.
  Multiplication with 0
  Consider a*(0+0) for some a in Z. Since 0 was already included in the set of numbers while defining addition and multiplication taking place together, the third axiom is satisfied. Thus,a*0 = a*(0+0) = a*0 + a*0 (as 0 is the additive identity)
  $\forall$a$\in\mathbb{Z}$, a*0 = 0.

  Distributive property:
  Trivially true as we have normal addition and multiplication

  Therefore since all axioms except for axiom 4 of multiplication are satisfied we can define $(\mathbb{Z}, +, *)$ as a set of numbers without division.
  Hence Proved.

Such set of numbers can also have unexpected properties. Show that:

- There is a set of numbers without division $(N, +, *)$ such that there are $a, b \in N$, $a \neq 0$, $b \neq 0$, but $a * b = 0$.

  Consider the set of numbers $N(+_1, *_2)$ without division such that N={0,1,2,3,4,5} and $+_1(a, b) = (a + b) mod(6)$ and $*_1(a, b) = (a * b) mod(6)$. Let a=2 and b=3 both of which are not 0 and belong to the set N defined above. but $a *_1 b$ is 0 (6mod(6)=0)
  Hence proved.

- There is a set of numbers without division $(N, +, *)$ such that there is $a \in N$, $a \neq 0$, but $a^3 = a * a * a = 0$.

  Consider the set of numbers $N(+_1, *_2)$ without division such that N=$\{0,1,2,3\}$ and $+_1(a, b) = (a + b) mod(4)$ and $*_1(a, b) = (a * b) mod(4)$. Let a=2 which is not 0 and belongs to the set N as defined above. but $a *_1 a *_1 a$ is 0 (8mod(8)=0)
  Hence proved.

Later in the course, we will see utility of these types of numbers as well.