ON FORMULAE FOR THE NTH PRIME NUMBER

By C. P. Willans

Let p_n denote the nth prime number. ($p_1=2$, $p_2=3$, etc.)
Let $[x]$ denote the greatest integer which is not greater than x.

From Wilson's theorem, $\dfrac{(x-1)!+1}{x}$ is an integer for $x=1$ and for all prime values of x; but is fractional for all composite values of x.

Then

$$F(x)=\left[\cos^2 \pi \frac{(x-1)!+1}{x}\right]=1 \quad \text{for } x=1 \text{ and for } x \text{ prime},$$
$$=0 \quad \text{for } x \text{ composite.} \tag{1}$$

It follows that if $\pi(m)$ denotes the number of primes $\leqslant m$,

$$\pi(m)=-1+\sum_{x=1}^{m} F(x) \tag{2}$$

(For references to other formulae of this nature see Dickson, *History of the Theory of Numbers*, Vol. I, Ch. XVIII.)

Let $\quad A_n(a)=\left[\sqrt[n]{\dfrac{n}{1+a}}\right] \quad$ for $n=1, 2, \ldots$; $a=0, 1, 2, \ldots$

This function has the properties

$$A_n(a)=1 \quad \text{for } a<n; \quad A_n(a)=0 \quad \text{for } a\geqslant n.$$

(For if $a<n$, then $1\leqslant\dfrac{n}{1+a}\leqslant n$, and so $1\leqslant\sqrt[n]{\dfrac{n}{1+a}}\leqslant\sqrt[n]{n}<2$; and if $a\geqslant n$, then $0<\dfrac{n}{1+a}<1$, and so $0<\sqrt[n]{\dfrac{n}{1+a}}<1$.)

We thus obtain the formula

$$p_n=1+\sum_{m=1}^{N} A_n(\pi(m)) \tag{3}$$

where N is any sufficiently large integer ($N=2^n$ will suffice since $p_n\leqslant 2^n$ for all n).

The formula written out in full is

$$p_n=1+\sum_{m=1}^{2^n}\left[\sqrt[n]{n}\left(\sum_{x=1}^{m}\left[\cos^2 \pi \frac{(x-1)!+1}{x}\right]\right)^{-1/n}\right] \tag{4}$$

The following example should indicate how the formula works.

$$\begin{aligned} p_5 &= 1 + A_5(\pi(1)) + A_5(\pi(2)) + \ldots + A_5(\pi(10)) + A_5(\pi(11)) + \ldots + A_5(\pi(32)) \\ &= 1 + A_5(0) + A_5(1) + \ldots + A_5(4) + A_5(5) + \ldots + A_5(11) \\ &= 1 + 1 + 1 + \ldots + 1 + 0 + \ldots + 0 = 11. \end{aligned}$$

We may obtain an alternative formula as follows:

Let $G(x) = xF(x)$, where $F(x)$ is defined as in (1).

Let $B_n(a) = [2^{-|a-n|}]$, for $n = 1, 2, \ldots$; $a = 0, 1, 2, \ldots$.

We note that $B_n(a) = 0$ for all $a \neq n$; $B_n(a) = 1$ for $a = n$.

Then
$$p_n = \sum_{m=1}^{2^n} G(m) B_n(\pi(m)) \tag{5}$$

As before, it is helpful to consider a numerical example.

We shall now find a formula for p_n which, although more complicated, does not involve the use of the somewhat artificial functions $[x]$ and $|x|$.

Consider the function

$$\frac{\{(x-1)!\}^2}{x} \quad \text{for } x = 2, 3, \ldots$$

If x is prime,

$$\begin{aligned} \frac{\{(x-1)!\}^2}{x} &= \frac{\{(x-1)!+1\}\{(x-1)!-1\}}{x} + \frac{1}{x} \\ &= \text{an integer} + \frac{1}{x}, \quad \text{by Wilson's theorem.} \end{aligned}$$

If x is composite, $\dfrac{\{(x-1)!\}^2}{x}$ is an integer.

(For x is the product of two integers a, b less than x; then since a divides $(x-1)!$ and b divides $(x-1)!$, ab must divide $\{(x-1)!\}^2$.)

Then
$$H(x) = \frac{\sin^2 \pi \dfrac{\{(x-1)!\}^2}{x}}{\sin^2 \dfrac{\pi}{x}} \begin{cases} = 1 \text{ for } x \text{ prime,} \\ = 0 \text{ for } x \text{ composite.} \end{cases}$$

It follows that

$$\pi(m) = \sum_{x=2}^{m} H(x) \quad \text{for } m = 2, 3, \ldots$$

Let $C_n(a) = \sin \pi \,.\, 2^{y-1}$, where $y = a^2(a-1)^2(a-2)^2 \ldots (a-\overline{n-1})^2$, and $n = 1, 2, \ldots;\ a = 1, 2, \ldots$

Now $\qquad C_n(a) = 1 \quad \text{for } a < n;\ C_n(a) = 0 \quad \text{for } a \geqslant n.$

Thus
$$p_n = 2 + \sum_{m=2}^{2^n} C_n(\pi(m)) \tag{6}$$

Finally we consider the related problem of expressing a prime q in terms of the prime p immediately preceding it.

Let
$$f(x) = \left[\cos^2 \pi \frac{\{(x-1)!\}^2}{x}\right] = 0 \quad \text{for } x \text{ prime,}$$
$$= 1 \quad \text{for } x \text{ composite.}$$

Then
$$q = 1 + p + f(p+1) + f(p+1) \,.\, f(p+2) + \ldots + f(p+1) \,.\, f(p+2) \ldots N \tag{7}$$

The value $N = 2p$ suffices, since it is known that $p < q < 2p$.

The only previous formula for the nth prime number which I have been able to trace is that given by Isenkrahe (*Math. Annalen*, **53**, 1900, 42), which expresses a prime in terms of all the preceding primes. Formulae (3) to (6) would appear to be the first formulae for p_n in terms of n alone.

While the formulae in this article are unsuitable for application to problems in prime number theory, they at least provide definite answers to the questions (see e.g. Hardy and Wright, *An Introduction to the Theory of Numbers*, § 1.5):

Is there a formula for the nth prime number?

Is there a formula for a prime, given the preceding prime?

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