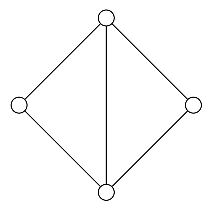
Discussion 2C

CS 70, Summer 2024

1 Degree Sequences

(a) Yes. The following graph has degree sequence $(3,3,2,2)$.



(b) No. For any graph G, the sum of the vertex degrees is even. Therefore the number of vertices with odd degree must be even.The given degree sequence has 3 odd-degree vertices.

(c) No. The total number of vertices is 4. Hence there cannot be a vertex with degree 6.

(d) No. The total number of vertices is 5. Hence, any degree 4 vertex must have an edge with every other vertex. Since there are two degree 4 vertices, there cannot be a vertex with degree 1.

2 Eulerian Tour and Eulerian Walk

(a) By Note 6, a connected graph has an Eulerian walk if and only if evvery vertex has even degree. In this graph, two vertices have odd degree, so there is no Eulerian walk.

(b) There is an Eulerian walk. One of the two vertices with odd degree must be the starting vertex, and the other one must be the ending vertex.

For example, $1 \to 2 \to 3 \to 4 \to 1 \to 3 \to 6 \to 4 \to 5 \to 2 \to 4 \to 7 \to 5 \to 6 \to 7$ is an Eulerian walk on the graph.

(c) If an undirected graph is connected (except for isolated vertices) and has at most two odd-degree vertices, then it has an Eulerian walk.

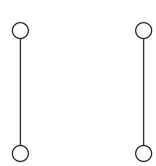
If the graph has no odd-degree vertices, we have seen in Note 6 that the graph has an Eulerian tour; every Eulerian tour is an Eulerian walk, so we have the desired result. By the handshaking lemma, there is no graph with only one odd-degree vertex, so we do not need to consider that case. It remains to consider the case when the graph has exactly two odd-degree vertices.

Take the two odd-degree vertices u and v and add a vertex w with two edges $\{u,w\}$ and $\{w,v\}$. The resulting graph G' has only vertices of even degree, since we increased the degrees of each of of u and v by one and introduced a vertex of degree 2. It is still connected, since we did not remove any edges.

Therefore, there is an Eulerian tour on G'. Delete the component of the tour that uses edges $\{u,w\}$ and $\{w,v\}$. The part of the tour that is left is now an Eulerian walk from u to v on the original graph, since it traverses every edge on the original graph.

3 Build-Up Error?

(a) Consider the following disconnected graph in which every vertex has positive degree.



(b) The argument is assuming that every $(n+1)$-vertex graph with a minimum degree of one can by obtained by adding a vertex from an n-vertex graph with a minimum degree of one.

This is not true; there is no way to build the graph from part **(a)** from a three-vertex graph of positive degree vertices. Therefore the claim has only been proved for graphs which can be built up from smaller graphs this way, rather than for all graphs.

This "build-up" error arises in any graph induction proof which assumes that every graph with size $n+1$ and some property can be "built up" from a graph with size n and the same property. This assumption may be correct for some properties, but it is not generally true, such as in this example.

(c) To avoid these build-up errors, the induction step of an induction proof should always start with an arbitrary example of the $n+1$ case of the problem. This is known as the "shrink down, grow back" approach. In the case of graph induction, this means that the induction step should start with an arbitrary graph of size $n+1$, remove a vertex or edge, apply the induction hypothesis to the smaller graph, and then add back the vertex or edge and argue that the desired property still holds.

Consider another attempt at the same proof using this approach.

Base case. $n = 1$. This is the graph with only a single vertex. In this case, the hypothesis is false, and thus the claim is vacuously true.

Induction case.

Induction hypothesis. Suppose that for some $n \geq 1$, if every vertex in an undirected graph with n vertices has degree at least one, then the graph is connected.

Induction step. Consider any undirected graph with $n+1$ vertices, where every vertex has degree at least one. Remove an arbitrary vertex v to obtain a graph with n vertices. However, removing v means that all the vertices in this new graph do not necessarily have positive degree. Therefore we cannot apply the induction hypothesis, and the proof cannot continue.

4 Odd-Degree Vertices

Let V_{odd} be the set of vertices in G that have odd degree.

(a) By the handshaking lemma,

$$\sum_{v \in V} \deg v = 2m,$$

where $m = |E|$ is the number of edges in G.

Partition V into the odd-degree vertices V_{odd} and the even-degree vertices $V \setminus V_{\text{odd}}$. Therefore

$$\sum_{v \in V} \deg v = \sum_{v \in V_{\text{odd}}} \deg v + \sum_{v \notin V_{\text{odd}}} \deg v = 2m \iff \sum_{v \in V_{\text{odd}}} \deg v = 2m - \sum_{v \notin V_{\text{odd}}} \deg v.$$

Each of the terms on the right-hand side are even: $2m$ is even and the sum of the degrees of the even-degree vertices $v \notin V_{\text{odd}}$ is even.

Therefore the left-hand side is even. For this to happen the sum must have an even number of terms, since each term in the summation on the left-hand side is odd. Therefore, there must be an even number of odd-degree vertices.

(b) By induction on $m = |E|$.

Base case. $m = 0$. If there are no vertices in G, then all vertices have degree zero. Therefore there are zero odd-degree vertices, which is an even number.

Induction case.

Induction hypothesis. Suppose that for some $m \in \mathbb{N}$, every graph with m edges has an even number of odd-degree vertices.

Induction step. Let G be a graph with $m+1$ edges. Remove an arbitrary edge $\{u, v\}$ from G to yield a new graph G' with m edges. By the induction hypothesis, G' has an even number of odd-degree vertices.

Add back the edge $\{u, v\}$ to get back G. Note that in G, both u and v have one more incident edge than it hand in G'. These are the only vertices whose degree has changed. There are four cases.

(1) Both u and v had odd degree in G'. Then G has two fewer odd-degree vertices than G', so G also has an even number of odd-degree vertices.

(2) While u had odd degree in G', v had even degree. Then u has even degree in G and v has odd degree, so the number of odd-degree vertices in G is the same as in G', so G also has an even number of odd-degree vertices.

(3) While v had odd degree in G', u had even degree. This is identical to Case (2) with the roles of u and v flipped.

(4) Both u and v had even degree in G'. Then G has two more odd-degree vertices than G', so G also has an even number of odd-degree vertices.

In all cases, G has an even number of odd-degree vertices, as desired.

By the principle of mathematical induction, every undirected graph G must have an even number of vertices with odd degree.

(c) By induction on $n = |V|$.

Base case. $n = 1$. If G has one vertex, then that vertex has degree zero. Therefore there are zero vertices with odd degree, which is an even number.

Induction case.

Induction hypothesis. Suppose that some $n \in \mathbb{N}^+$, every graph with n vertices has an even number of odd-degree vertices.

Induction step. Let G be a graph with $n + 1$ vertices. Remove an arbitrary vertex v and all its incident edges from G to yield a new graph G' with n vertices.

Add v and its incident edges back to get back G. Let $N_v = \{u : \{u, v\} \in E\}$ be the set of vertices adjacent to v, let V'_{odd} be the vertices in G' with odd degree, and let V_{odd} be the vertices in G with odd degree. By the induction hypothesis, G' has an even number of vertices with odd degree, so $|V'_{\text{odd}}|$ is even.

Any vertex $u \in A = N_v \cap V'_{\text{odd}}$ had odd degree in G', and therefore have even degree in G by the addition of one adjacent vertex. Similarly, any vertex $w \in B = N_v \cap (V \setminus V'_{\text{odd}})$ had even degree in G', and therefore has odd degree in G. We will build V_{odd} from V'_{odd} by removing the vertices which now have even degree G and adding the vertices which now have odd degree in G.

We split into cases based on the parity of $|N_v|$.

(1) $|N_v|$ is even. Then v has even degree, so $v \notin V_{\text{odd}}$ and $V_{\text{odd}} = (V'_{\text{odd}} \setminus A) \cup B$. Since $A \subseteq V'_{\text{odd}}$ and $B \cap (V'_{\text{odd}} \setminus A) = \varnothing$,

$$|V_{\text{odd}}| = |V'_{\text{odd}}| - |A| + |B|.$$

Since $A \cup B = N_v$ and $A \cap B = \varnothing$, $|A| + |B| = |N_v|$ and therefore

$$|V_{\text{odd}}| = |V'_{\text{odd}}| - |A| + (|N_v| - |A|) = |V_{\text{odd}}| + |N_v| - 2|A|.$$

All of the terms on the right-hand side are even, so $|V_{\text{odd}}|$ is also even, as desired.

(2) $|N_v|$ is odd. Then v has odd degree, so $v \in V_{\text{odd}}$ and $V_{\text{odd}} = (V'_{\text{odd}} \setminus A) \cup B \cup \{v\}$. By the same logic as in Case (1) and the fact that $\{v\}$ is disjoint with V'_{odd} and B,

$$|V_{\text{odd}}| = |V'_{\text{odd}}| - |A| + |B| + 1 = |V'_{\text{odd}}| - |A| + (|N_v - |A|) + 1 = |V'_{\text{odd}}| + (|N_v| + 1) - 2|A|.$$

Each of the terms $|V'_{\text{odd}}|$, $|N_v| + 1$, and $2|A|$ on the right-hand side are even, so $|V_{\text{odd}}|$ is also even, as desire.

In either case $|V_{\text{odd}}|$ is even. Therefore G has an even number of odd-degree vertices.

By the principle of mathematical induction, every undirected graph has an even number of vertices with odd degree.

Note that this proof is much more complicated than the proof which uses induction on the number of edges in the graph. Choosing the right variable for induction can help simplify a proof.