$$= -\left[\frac{1}{D-2D'+1} e^{x+y}\right] = -e^{x+y} \cdot \frac{1}{D+1-2(D'+1)+1}(1)$$

$$= -e^{x+y}\left[\frac{1}{D-2D'}(1)\right] = -e^{x+y}\left[\frac{1}{D-2D'}(e^{0x+0y})\right]$$

$$= -xe^{x+y}.$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y+2x) + e^{-x} f_2(y+2x) - xe^{x+y}$$

where f_1 and f_2 are arbitrary functions.

Example 5. *Find the particular integral of* $(D^2 - D')z = xe^{ax+a^2y}$.

Sol.
$$\text{P.I.} = \frac{1}{D^2 - D'}(xe^{ax+a^2y}) = e^{ax+a^2y} \cdot \frac{1}{(D+a)^2 - (D'+a^2)}(x)$$

$$= e^{ax+a^2y} \cdot \frac{1}{D^2+2aD-D'}(x) = e^{ax+a^2y} \cdot \frac{1}{2aD}\left(1+\frac{D}{2a}-\frac{D'}{2aD}\right)^{-1}(x)$$

$$= e^{ax+a^2y} \cdot \frac{1}{2aD}\left\{1-\left(\frac{D}{2a}-\frac{D'}{2aD}\right)+\ldots\right\} x = e^{ax+a^2y} \cdot \frac{1}{2aD}\left(x-\frac{1}{2a}\right)$$

$$= e^{ax+a^2y} \cdot \left(\frac{x^2}{4a}-\frac{x}{4a^2}\right).$$

TEST YOUR KNOWLEDGE

Solve the following partial differential equations:

1. $(3D^2 - 2D'^2 + D - 1)z = 4e^{x+y} \cos(x+y)$
2. $(D^2 + DD' + D + D' - 1)z = e^{-2x}(x^2 + y^2)$
3. $(D + D' - 1)(D + D' - 3)(D + D')z = e^{x+y} \sin(2x+y)$
4. $(D^2 - DD' + D' - 1)z = \cos(x+2y) + e^x$.

Answers

1. $z = \Sigma Ae^{hx+ky} + \frac{4}{3}e^{x+y} \sin(x+y)$, where $3h^2 - 2k^2 + h - 1 = 0$
2. $z = \Sigma Ae^{hx+ky} + \frac{1}{27}e^{-2x}(9x^2 + 9y^2 + 18x + 6y + 14)$, where $h^2 + hk + h + k + 1 = 0$
3. $z = e^x f_1(y-x) + e^{3x} f_2(y-x) + f_3(y-x) + \frac{1}{130}[3\cos(2x+y) - 2\sin(2x+y)]e^{x+y}$
4. $z = e^x f_1(y) + e^{-x} f_2(y+x) + \frac{1}{2}\sin(x+2y) + \frac{1}{2}xe^x$.

1.25 EQUATIONS REDUCIBLE TO PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

An equation in which the coefficient of derivative of any order say k is a multiple of the variables of the degree k then it can be reduced to partial differential equation with constant coefficients in the following way.

Let
$$x = e^{X},\ y = e^{Y} \text{ so that } X = \log x,\ Y = \log y$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{1}{x}\frac{\partial z}{\partial X}$$

or
$$x\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \qquad \therefore \quad x\frac{\partial}{\partial x} = D\left(\equiv \frac{\partial}{\partial X}\right)$$

Now $x\dfrac{\partial}{\partial x}\left(x^{k-1}\dfrac{\partial^{k-1} z}{\partial x^{k-1}}\right) = x^k\dfrac{\partial^k z}{\partial x^k} + (k-1)x^{k-1}\dfrac{\partial^{k-1} z}{\partial x^{k-1}}$

or
$$x^k\frac{\partial^k z}{\partial x^k} = \left(x\frac{\partial}{\partial x} - k + 1\right)x^{k-1}\frac{\partial^{k-1} z}{\partial x^{k-1}}$$

Putting $k = 2, 3, \ldots\ldots$, we get

$$x^2\frac{\partial^2 z}{\partial x^2} = (D-1)x\frac{\partial z}{\partial x} = (D-1)Dz$$

$$x^3\frac{\partial^3 z}{\partial x^3} = (D-2)x^2\frac{\partial^2 z}{\partial x^2} = (D-2)(D-1)Dz \text{ etc.}$$

Similarly,
$$y\frac{\partial z}{\partial y} = D'z,\quad y^2\frac{\partial^2 z}{\partial y^2} = (D'-1)D'z,\quad y^3\frac{\partial^3 z}{\partial y^3} = (D'-2)(D'-1)D'z \text{ etc.}$$

and
$$xy\frac{\partial^2 z}{\partial x \partial y} = DD'z \ \ldots\ldots$$

Substituting in the given equation, it reduces to $\psi(D, D')z = V$ which is an equation with constant coefficients.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the linear partial differential equation*

$$x^2\frac{\partial^2 z}{\partial x^2} - 4xy\frac{\partial^2 z}{\partial x \partial y} + 4y^2\frac{\partial^2 z}{\partial y^2} + 6y\frac{\partial z}{\partial y} = x^3 y^4.$$

Sol. Put $x = e^{X}, y = e^{Y}$ so that $X = \log x$ and $Y = \log y$ and Let $D \equiv \dfrac{\partial}{\partial X}, D' \equiv \dfrac{\partial}{\partial Y}$

and $DD' \equiv \dfrac{\partial^2}{\partial X \partial Y}$ then the given equation reduces to

$$[D(D-1)-4DD'+4D'(D'-1)+6D']z=e^{3X+4Y}$$

$\Rightarrow$ $$[(D^2-4DD'+4D'^2)-(D-2D')]z=e^{3X+4Y}$$

$\Rightarrow$ $$(D-2D')(D-2D'-1)z=e^{3X+4Y}$$

Its $$\text{C.F.}=f_1(Y+2X)+e^X f_2(Y+2X)$$

$$=f_1(\log y+2\log x)+xf_2(\log y+2\log x)$$

$$=f_1(\log yx^2)+x f_2(\log yx^2)=g_1(yx^2)+x\,g_2(yx^2)$$

$$\text{P.I.}=\frac{1}{D-2D'-1}\left[\frac{1}{D-2D'}e^{3X+4Y}\right]$$

$$=\frac{1}{D-2D'-1}\left[\frac{1}{3-8}\int e^u\,du\right] \quad \text{where } 3X+4Y=u$$

$$=\frac{1}{D-2D'-1}\left[-\frac{1}{5}e^{3X+4Y}\right]=-\frac{1}{5}\left[\frac{1}{D-2D'-1}e^{3X+4Y}\right]$$

$$=-\frac{1}{5}\left[\frac{1}{3-8-1}e^{3X+4Y}\right]=\frac{1}{30}e^{3X+4Y}=\frac{1}{30}x^3y^4.$$

Hence the complete solution is

$$z=\text{C.F.}+\text{P.I.}=g_1(yx^2)+x\,g_2(yx^2)+\frac{1}{30}x^3y^4$$

where g_1 and g_2 are arbitrary functions.

Example 2. *Solve:* $x^2r-y^2t+px-qy=\log x$.

Sol. Let $x=e^X$, $y=e^Y$ so that $X=\log x$ and $Y=\log y$ and let $D\equiv\frac{\partial}{\partial X}$ and $D'\equiv\frac{\partial}{\partial Y}$, then the given equation reduces to

$$[D(D-1)-D'(D'-1)+D-D']z=X$$

$\Rightarrow$ $$(D^2-D'^2)z=X \quad \text{...(1)}$$

which is a homogeneous linear p.d.e. with constant coefficients.

$\therefore$ $$\text{C.F.}=\phi_1(Y+X)+\phi_2(Y-X)$$

and $$\text{P.I.}=\frac{1}{D^2-D'^2}(X)=\frac{1}{(1)^2-(0)^2}\iint u\,du\,du \quad \text{where } X=u$$

$$=\int\frac{u^2}{2}du=\frac{u^3}{6}=\frac{X^3}{6}$$

Hence solution to (1) is

$$z=\phi_1(Y+X)+\phi_2(Y-X)+\frac{X^3}{6}$$

$$=\phi_1(\log y+\log x)+\phi_2(\log y-\log x)+\frac{(\log x)^3}{6}.$$

Therefore, the complete solution to the given differential equation is

$$z = \phi_1(\log xy) + \phi_2\left(\log \frac{y}{x}\right) + \frac{1}{6}(\log x)^3$$

$$z = f_1(xy) + f_2\left(\frac{y}{x}\right) + \frac{1}{6}(\log x)^3$$

where f_1 and f_2 are arbitrary functions.

Example 3. *Solve:*

(*i*) $x^2 \dfrac{\partial^2 z}{\partial x^2} - y^2 \dfrac{\partial^2 z}{\partial y^2} = xy$ (*ii*) $(x^2 D^2 + 2xy\, DD' + y^2 D'^2)z = x^m y^n$.

Sol. (*i*) Let $x = e^X$, $y = e^Y$ so that $X = \log x$ and $Y = \log y$ and Let $D \equiv \dfrac{\partial}{\partial X}$, $D' \equiv \dfrac{\partial}{\partial Y}$ and $DD' \equiv \dfrac{\partial^2}{\partial X \partial Y}$ then the given equation reduces to

$$[D(D-1) - D'(D'-1)]z = e^{X+Y}$$

$$\Rightarrow \quad (D^2 - D'^2 - D + D')z = e^{X+Y}$$

$$\Rightarrow \quad (D - D')(D + D' - 1)z = e^{X+Y}$$

$$\text{C.F.} = f_1(Y + X) + e^X f_2(Y - X)$$

$$= f_1(\log y + \log x) + x f_2(\log y - \log x) = g_1(xy) + x g_2\left(\frac{y}{x}\right)$$

$$\text{P.I.} = \frac{1}{(D-D')(D+D'-1)} e^{X+Y} = \frac{1}{D-D'}\left[\frac{1}{1+1-1} e^{X+Y}\right]$$

$$= \frac{1}{D-D'} e^{X+Y} = X.e^{X+Y} = xy \log x$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = g_1(xy) + x g_2\left(\frac{y}{x}\right) + xy \log x$$

where g_1 and g_2 are arbitrary functions.

(*ii*) Let $x = e^X$, $y = e^Y$ so that $X = \log x$, $Y = \log y$ and Let $D \equiv \dfrac{\partial}{\partial X}$, $D' \equiv \dfrac{\partial}{\partial Y}$ and $DD' \equiv \dfrac{\partial^2}{\partial X \partial Y}$

then the given equation reduces to

$$[D(D-1) + 2DD' + D'(D'-1)]z = e^{mX+nY}$$

$$\Rightarrow \quad (D^2 + 2DD' + D'^2 - D - D')z = e^{mX+nY}$$

$$\Rightarrow \quad [(D + D')^2 - (D + D')]z = e^{mX+nY}$$

$$\Rightarrow \quad (D + D')(D + D' - 1)z = e^{mX+nY}$$

$$\therefore \quad \text{C.F.} = f_1(Y - X) + e^X f_2(Y - X)$$

$$= f_1(\log y - \log x) + x f_2(\log y - \log x)$$

$$= f_1\left(\log \frac{y}{x}\right) + x f_2\left(\log \frac{y}{x}\right) = g_1\left(\frac{y}{x}\right) + x g_2\left(\frac{y}{x}\right)$$

$$\text{P.I.} = \frac{1}{(D+D')(D+D'-1)} e^{mX+nY}$$

$$= \frac{1}{(m+n)(m+n-1)} e^{mX+nY} = \frac{x^m y^n}{(m+n)(m+n-1)}$$

Hence complete solution is

$$z = \text{C.F.} + \text{P.I.} = g_1\,(y/x) + x\,g_2\,(y/x) + \frac{x^m y^n}{(m+n)(m+n-1)}$$

where g_1 and g_2 are arbitrary functions.

ASSIGNMENT-I

(2 Marks Questions for Section–A)

1. Show that $z = f(x^2 + y^2)$ is a solution of $y\dfrac{\partial z}{\partial x} - x\dfrac{\partial z}{\partial y} = 0$. [G.B.T.U. (AG) 2012]

2. Find the particular integral of $(D^2 + DD')z = \sin(x + y)$ where $D = \dfrac{\partial}{\partial x}$, $D' = \dfrac{\partial}{\partial y}$.
(G.B.T.U. 2012)

3. Solve:
$(D - 5D' + 1)^2 z = 0$ (U.P.T.U. 2015)

4. Solve:
$(D - 5D' + 4)^3 z = 0$ (U.P.T.U. 2015)

5. Form the partial differential equation by eliminating a and b from $z = (x^2 + a)(y^2 + b)$.
[G.B.T.U. (AG) 2012]

6. Formulate the PDE by eliminating the arbitrary function from $\phi\,(x^2 + y^2, y^2 + z^2) = 0$.

7. Find the particular integral of $(D^2 + DD' - 6D'^2)\,z = \cos(2x + y)$ (A.K.T.U. 2016)

8. Solve: $(D^2 - 2DD' + D'^2)\,z = 0$ (U.P.T.U. 2015)

9. Give two examples of non-linear partial differential equation of the first order.
(M.T.U. 2013)

10. Find the P.I. of $(D^2 - D'^2)z = \cos(x + y)$.

11. Find the P.I. of $(2D^2 - 3DD' + D'^2)z = e^{x+2y}$. [M.T.U. (SUM) 2011, G.B.T.U. 2011]

42. Find the partial differential equation which is satisfied by the relation $z = c_1 xy + c_2$ where c_1 and c_2 are constants. (M.T.U. 2013)
(U.P.T.U. 2014)

13. Solve: $(D^2 + DD')z = 0$

14. Form a partial differential equation from $z = (a + x)^2 + y$.

15. Find the solution of $xp + yq = z$.

16. Solve: $p + q = z$.

17. Solve: $(y - z)\,p + (z - x)\,q = x - y$.

18. Solve: $\dfrac{\partial^3 z}{\partial x^3} = 0$.

19. Find the partial differential equation of all spheres whose centres lie on z-axis and given by equations $x^2 + y^2 + (z - a)^2 = b^2$; a and b being constants. (A.K.T.U. 2017)

20. Write the complementary function of the partial differential equation $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = \dfrac{1}{x^2}$.

[G.B.T.U. (AG) 2011]

21. Write down the auxiliary equations for the differential equation of the type $Pq + Qq = R$.

22. Find the particular integral of $(D^2 - 2DD' + D'^2)z = e^{x+2y}$ [G.B.T.U. (AG) 2011]

23. Solve: $\dfrac{\partial^2 z}{\partial x \partial y} = 0$.

24. What is the general solution of I order equation $Pq + Qq = R$? [G.B.T.U. (AG) 2011]

25. Solve: $p - q = \log(x + y)$ (M.T.U. 2011)

26. Solve: $(D^3 + D^2D' - DD'^2 - D'^3)z = 0$.

27. Find the order of the partial differential equation obtained by eliminating F from

$$z = F(x^2 + y^2).$$

28. Solve: $(D^3 - 3D^2D' + 2DD'^2)z = 0$.

29. Solve: $r + 6s + 9t = 0$.

30. Form a partial differential equation by eliminating the constants a and b from $z = (x + a)(y + b)$. [M.T.U. (SUM) 2011]

31. Solve: $p - 2q = \sin(x + 2y)$.

Answers

2. $-\dfrac{1}{2}\sin(x+y)$

3. $z = e^{-x} f_1(y + 5x) + xe^{-x}f_2(y + 5x)$

4. $z = e^{-4x} f_1(y + 5x) + xe^{-4x}f_2(y + 5x) + x^2e^{-4x} f_3(y + 5x)$

5. $pq = 4xyz$

6. $yzp - xzq = xy$ where $p = \dfrac{\partial z}{\partial x}$ and $q = \dfrac{\partial z}{\partial y}$

7. $\dfrac{x}{5}\sin(2x+y)$

8. $z = f_1(y + x) + x f_2(y + x)$

10. $\dfrac{x}{2}\sin(x+y)$

11. $-\dfrac{x}{2}e^{x+2y}$

12. $y\dfrac{\partial z}{\partial y} = x\dfrac{\partial z}{\partial x}$

13. $z = f_1(y) + f_2(y - x)$

14. $z = \dfrac{1}{4}\left(\dfrac{\partial z}{\partial x}\right)^2 + y$

15. $f\left(\dfrac{x}{y}, \dfrac{y}{z}\right) = 0$

16. $f(x - y, y - \log z) = 0$

17. $f(x + y + z) = x^2 + y^2 + z^2$

18. $z = f_1(y) + xf_2(y) + x^2f_3(y)$

19. $x\dfrac{\partial z}{\partial y} - y\dfrac{\partial z}{\partial x} = 0$

20. $z = f_1(y + x) + f_2(y - x)$

21. $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

22. e^{x+2y}

23. $z = \phi(y) + \psi(x)$

24. $\phi(u, v) = 0$; $u = f_1(x, y, z)$, $v = f_2(x, y, z)$

25. $\phi\,[x + y, z - x\log(x + y)] = 0$

26. $z = f_1(y + x) + f_2(y - x) + xf_3(y - x)$

27. First

28. $z = f_1(y) + f_2(y + x) + f_3(y + 2x)$.

29. $z = f_1(y - 3x) + xf_2(y - 3x)$.

30. $z = pq$

31. $3z = \cos(x + 2y)$