$$= \frac{12}{1^2 - 6(1)(0) + 9(0)^2} \iint u^2 \, du \, du, \quad \text{where } x = u$$

$$= 12 \cdot \frac{u^4}{12} = u^4 = x^4$$

$$P_2 = \frac{1}{D^2 - 6DD' + 9D'^2}(36xy)$$

$$= \frac{36}{(D - 3D')^2}(xy) = \frac{36}{D^2}\left(1 - \frac{3D'}{D}\right)^{-2}(xy)$$

$$= \frac{36}{D^2}\left(1 + \frac{6D'}{D}\right)(xy) \qquad | \text{ Leaving higher power terms}$$

$$= \frac{36}{D^2}\left[xy + \frac{6}{D}(x)\right] = \frac{36}{D^2}(xy + 3x^2)$$

$$= 36\left(\frac{x^3}{6}y + \frac{x^4}{4}\right) = 6x^3y + 9x^4$$

$$\text{P.I.} = P_1 + P_2 = 6x^3y + 10x^4$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y + 3x) + x f_2(y + 3x) + 6x^3y + 10x^4$$

where f_1 and f_2 are arbitrary functions.

TEST YOUR KNOWLEDGE

Solve:

1. $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 12xy$ (U.P.T.U. 2015)
2. $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = x^2 + xy + y^2$
3. $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x} + 3x^2y$
4. $(D^2 - a^2 D'^2)z = x$
5. $\{D^2 + (a + b)DD' + ab D'^2\}z = xy$ [G.B.T.U. (AG) 2011, G.B.T.U. 2013]
6. $(D^2 - 2DD' + D'^2)z = e^{x+2y} + x^3$.
7. $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} - 15\frac{\partial^2 z}{\partial y^2} = 12xy$ (U.P.T.U. 2013)

Answers

1. $z = f_1(y - x) + f_2(y - 2x) + 2x^3y - \frac{3}{2}x^4$
2. $z = f_1(y - x) + x f_2(y - x) + \frac{1}{4}(x^4 - 2x^3y + 2x^2y^2)$
3. $z = f_1(y) + x f_2(y) + f_3(y + 2x) + \frac{1}{4}e^{2x} + \frac{1}{60}(3x^5y + x^6)$
4. $z = f_1(y + ax) + f_2(y - ax) + \frac{x^3}{6}$
5. $z = f_1(y - ax) + f_2(y - bx) + \frac{1}{6}x^3y - \left(\frac{a + b}{24}\right)x^4$
6. $z = f_1(y + x) + x f_2(y + x) + e^{x+2y} + \frac{x^5}{20}$
7. $z = f_1(y - 3x) + f_2(y + 5x) + 2x^3y + x^4$.

1.21 GENERAL METHOD TO FIND THE P.I.

$\phi(x, y)$ is not always of the form given above. The general method is applicable to all cases, where $\phi(x, y)$ is not of the form given above.

Now, $F(D, D')$ can be factorised, in general, into n linear factors.

$$\therefore \quad \text{P.I.} = \frac{1}{F(D, D')}\phi(x, y)$$

$$= \frac{1}{(D - m_1D')(D - m_2D')\ldots(D - m_nD')}\phi(x, y)$$

$$= \frac{1}{D - m_1D'} \cdot \frac{1}{D - m_2D'} \cdots\cdots \frac{1}{D - m_nD'}\phi(x, y)$$

Let us evaluate $\dfrac{1}{D - mD'}\phi(x, y)$

Consider the equation,

$$(D - mD')z = \phi(x, y) \quad \text{or} \quad p - mq = \phi(x, y) \qquad \text{[Lagrange's form]}$$

The subsidiary equations are $\dfrac{dx}{1} = \dfrac{dy}{-m} = \dfrac{dz}{\phi(x, y)}$

From the first two members $dy + mdx = 0$ or $y + mx = c$

From the first and last members, we have

$$dz = \phi(x, y)dx = \phi(x, c - mx)dx$$

$$\therefore \quad z = \int \phi(x, c - mx)dx$$

or

$$\boxed{\frac{1}{D - mD'}\phi(x, y) = \int \phi(x, c - mx)\,dx}$$

where c is replaced by $y + mx$ after integration.

By repeated application of the above rule, the P.I. can be evaluated.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the linear partial differential equation*

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y \cos x.$$

Sol. The given equation is

$$(D^2 + DD' - 6D'^2)z = y \cos x$$

The auxiliary equation is

$$m^2 + m - 6 = 0$$

$$\Rightarrow \quad (m - 2)(m + 3) = 0$$

$$\Rightarrow \quad m = 2, -3.$$

$$\therefore \quad \text{C.F.} = f_1(y + 2x) + f_2(y - 3x)$$

$$\text{P.I.} = \frac{1}{D^2 + DD' - 6D'^2} y \cos x = \frac{1}{(D - 2D')(D + 3D')} y \cos x$$

$$= \frac{1}{D - 2D'} \int (c + 3x) \cos x \, dx, \quad \text{where } y = c + 3x$$

$$= \frac{1}{D - 2D'} \{c \sin x + 3 \int x \cos x \, dx\}$$

$$= \frac{1}{D - 2D'} [c \sin x + 3 \{x \sin x - \int 1 \,.\, \sin x \, dx\}]$$

$$= \frac{1}{D - 2D'} [(c + 3x) \sin x + 3 \cos x]$$

$$= \frac{1}{D - 2D'} (y \sin x + 3 \cos x), \quad \text{where } c = y - 3x$$

$$= \int (b - 2x) \sin x \, dx + 3 \sin x, \quad \text{where } y = b - 2x$$

$$= - b \cos x - 2 \{x (- \cos x) - \int 1 \,.\, (- \cos x) \, dx\} + 3 \sin x$$

$$= - b \cos x + 2x \cos x - 2 \sin x + 3 \sin x$$

$$= - (b - 2x) \cos x + \sin x$$

$$= - y \cos x + \sin x, \quad \text{where } b = y + 2x$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1 (y + 2x) + f_2 (y - 3x) - y \cos x + \sin x$$

where f_1 and f_2 are arbitrary functions.

Example 2. *Solve the linear partial differential equation*

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 2 \frac{\partial^2 z}{\partial y^2} = (y - 1) e^x.$$

Sol. The given equation is

$$(D^2 + DD' - 2D'^2) z = (y - 1) e^x$$

The auxiliary equation is

$$m^2 + m - 2 = 0$$

$$\Rightarrow \quad (m - 1)(m + 2) = 0$$

$$\Rightarrow \quad m = 1, - 2$$

$$\text{C.F.} = f_1 (y + x) + f_2 (y - 2x)$$

$$\text{P.I.} = \frac{1}{D^2 + DD' - 2D'^2} (y - 1) e^x$$

$$= \frac{1}{(D - D')(D + 2D')} (y - 1) e^x$$

$$= \frac{1}{D - D'} \int (c + 2x - 1) e^x \, dx, \quad \text{where } y = c + 2x$$

$$= \frac{1}{D - D'} [(c - 1) e^x + 2 (x - 1) e^x]$$

$$= \frac{1}{D - D'} [(c + 2x) e^x - 3e^x]$$

$$= \frac{1}{D - D'}(ye^x - 3e^x), \qquad \text{where } c = y - 2x$$

$$= \int (b - x)\, e^x\, dx - 3e^x, \qquad \text{where } y = b - x$$

$$= be^x - (x - 1)\, e^x - 3e^x$$

$$= (b - x - 2)\, e^x = (y - 2)\, e^x, \qquad \text{where } b = y + x$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y + x) + f_2(y - 2x) + (y - 2)\, e^x$$

where f_1 and f_2 are arbitrary functions.

Example 3. *Solve the partial differential equation:*

$$r - t = \tan^3 x \tan y - \tan x \tan^3 y.$$

Sol. The given equation is

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \tan x \tan y\,(\sec^2 x - \sec^2 y).$$

Auxiliary equation is $\quad m^2 - 1 = 0 \quad \Rightarrow \quad m = \pm 1.$

$$\text{C.F.} = f_1(y + x) + f_2(y - x)$$

$$\text{P.I.} = \frac{1}{D^2 - D'^2} \tan x \tan y\,(\sec^2 x - \sec^2 y)$$

$$= \frac{1}{D + D'}\left[\frac{1}{D - D'}(\tan x \sec^2 x \tan y - \tan x \tan y \sec^2 y)\right]$$

$$= \frac{1}{D + D'}\left[\int \tan x \sec^2 x \tan(c - x)\, dx - \int \tan x \tan(c - x) \sec^2(c - x)\, dx\right]$$

where $y = c - x$

$$= \frac{1}{D + D'}\left[\tan(c - x)\frac{\tan^2 x}{2} + \int \sec^2(c - x)\frac{\tan^2 x}{2}\, dx\right.$$
$$\left. + \tan x \cdot \frac{\tan^2(c - x)}{2} - \int \sec^2 x\, \frac{\tan^2(c - x)}{2}\, dx\right]$$

$$= \frac{1}{2(D + D')}[\tan(c - x) \tan^2 x + \tan x \tan^2(c - x]$$
$$+ \int \sec^2(c - x)(\sec^2 x - 1)\, dx - \int \sec^2 x\,\{\sec^2(c - x) - 1\}\, dx]$$

$$= \frac{1}{2(D + D')}[\tan(c - x) \tan^2 x + \tan x \tan^2(c - x) + \int \{\sec^2 x - \sec^2(c - x)\}\, dx]$$

$$= \frac{1}{2(D + D')}[\tan(c - x) \tan^2 x + \tan x \tan^2(c - x) + \tan x + \tan(c - x)]$$

$$= \frac{1}{2(D + D')}[\tan x \sec^2(c - x) + \tan(c - x) \sec^2 x]$$

$$= \frac{1}{2(D + D')}(\tan x \sec^2 y + \tan y \sec^2 x) \qquad \text{where } c = y + x$$

$$= \frac{1}{2}\left[\int \tan x \sec^2 (b+x)\, dx + \int \tan (b+x) \sec^2 x\, dx\right], \quad \text{where } y = b + x$$

$$= \frac{1}{2}\left[\tan x \,.\, \tan (b+x) - \int \sec^2 x \,.\, \tan (b+x)\, dx + \int \tan (b+x) \sec^2 x\, dx\right]$$

$$= \frac{1}{2} \tan x \tan (b+x) = \frac{1}{2} \tan x \tan y \quad \text{where } b = y - x.$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1 (y+x) + f_2 (y-x) + \frac{1}{2} \tan x \tan y$$

where f_1 and f_2 are arbitrary functions.

Example 4. *Solve:* $r - s - 2t = (2x^2 + xy - y^2) \sin xy - \cos xy$.

Sol. The given equation is

$$(\text{D}^2 - \text{DD}' - 2\text{D}'^2)z = (2x^2 + xy - y^2) \sin xy - \cos xy$$

Auxiliary equation is $\quad m^2 - m - 2 = 0$

$\Rightarrow \quad (m+1)(m-2) = 0 \quad \Rightarrow \quad m = -1, 2$

$\therefore \quad \text{C.F.} = f_1(y - x) + f_2(y + 2x)$

$$\text{P.I.} = \frac{1}{(\text{D}+\text{D}')(\text{D}-2\text{D}')}[(2x-y)(x+y) \sin xy - \cos xy]$$

$$= \frac{1}{(\text{D}+\text{D}')}\int [(4x-c)(c-x) \sin (cx - 2x^2) - \cos (cx - 2x^2)]\, dx, \text{ where } y = c - 2x$$

$$= \frac{1}{(\text{D}+\text{D}')}\int [(x-c)(c-4x) \sin (cx - 2x^2) - \cos (cx - 2x^2)]\, dx$$

$$= \frac{1}{(\text{D}+\text{D}')}\left[(x-c)\{-\cos (cx - 2x^2)\} + \int \cos (cx - 2x^2)\, dx - \int \cos (cx - 2x^2)\, dx\right]$$

$$= \frac{1}{(\text{D}+\text{D}')}(c-x) \cos \{x(c-2x)\}$$

$$= \frac{1}{\text{D}+\text{D}'}(y+x) \cos xy, \qquad \text{where } c = y + 2x$$

$$= \int (b + 2x) \cos (bx + x^2)\, dx, \qquad \text{where } y = b + x$$

$$= \sin (bx + x^2) = \sin xy, \qquad \text{where } b = y - x$$

$\therefore$ Complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y - x) + f_2(y + 2x) + \sin xy$$

where f_1 and f_2 are arbitrary functions.

Example 5. *Solve:* $\quad (D^2 + 2DD' + D'^2)z = 2 \cos y - x \sin y.$

Sol. Auxiliary equation is

$$m^2 + 2m + 1 = 0$$

$\Rightarrow \quad m = -1, -1$

$\therefore \quad \text{C.F.} = f_1 (y - x) + x f_2 (y - x)$

$$\text{P.I.} = \frac{1}{(D+D')^2}\, 2\cos y - \frac{1}{(D+D')^2}\,(x \sin y) = P_1 - P_2$$

where
$$P_1 = \frac{1}{(D+D')^2}\, 2\cos y = \frac{2}{(0+1)^2} \iint \cos u \, du \, du, \text{ where } y = u$$

$$= 2\,(-\cos y) = -2\cos y$$

$$P_2 = \frac{1}{(D+D')^2}\, x \sin y = \frac{1}{D+D'} \int x \sin (c+x)\, dx, \text{ where } y = c + x$$

$$= \frac{1}{D+D'}\left[x\{-\cos (c+x)\} - \int 1.\{-\cos (c+x)\}\, dx\right]$$

$$= \frac{1}{D+D'}\,[-x\cos (c+x) + \sin (c+x)]$$

$$= \frac{1}{D+D'}\,(-x\cos y + \sin y), \qquad \text{where } c = y - x$$

$$= \int -x\cos (b+x)\, dx + \int \sin (b+x)\, dx, \quad \text{where } y = b + x$$

$$= (-x)\,.\,\sin (b+x) - \int (-1)\,.\,\sin (b+x)\, dx + \int \sin (b+x)\, dx$$

$$= -x\sin (b+x) - 2\cos (b+x)$$

$$= -x \sin y - 2\cos y, \qquad \text{where } b = y - x$$

$$\therefore \quad \text{P.I.} = P_1 - P_2 = -2\cos y + x\sin y + 2\cos y = x \sin y$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y-x) + x\, f_2(y-x) + x \sin y$$

where f_1 and f_2 are arbitrary functions.

TEST YOUR KNOWLEDGE

Solve:

1. $(D^2 - DD' - 2D'^2)z = (y-1)\,e^x$ (U.P.T.U. 2014)
2. $(D - D')(D + 2D')z = (y+1)\,e^x$
3. $(D^3 + 2D^2D' - DD'^2 - 2D'^3)z = (y+2)\,e^x$
4. $r - 4t = \dfrac{4x}{y^2} - \dfrac{y}{x^2}$
5. $(D^3 + D^2D' - DD'^2 - D'^3)z = e^x \cos 2y$
6. $(D^2 + DD' - 6D'^2)\,z = y \sin x$ [M.T.U. (SUM) 2011; M.T.U. 2012]

Answers

1. $z = f_1(y-x) + f_2(y+2x) + ye^x$
2. $z = f_1(y+x) + f_2(y-2x) + ye^x$
3. $z = f_1(y+x) + f_2(y-x) + f_3(y-2x) + ye^x$
4. $z = f_1(y+2x) + f_2(y-2x) + x\log y + y \log x + 3x$
5. $z = f_1(y-x) + x\, f_2(y-x) + f_3(y+x) + \dfrac{e^x}{25}(\cos 2y + 2\sin 2y)$
6. $z = f_1(y+2x) + f_2(y-3x) - y\sin x - \cos x.$

1.22 NON-HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

In the equation $\phi(D, D')z = F(x, y)$ ...(1)

if the polynomial $\phi(D, D')$ in D, D′ is not homogeneous, then (1) is called a non-homogeneous linear partial differential equation.

Its complete solution is $z = \text{C.F.} + \text{P.I.}$

1.23 METHODS FOR FINDING OUT C.F.

(*a*) We resolve $\phi(D, D')$ into linear factors of the form $\mathbf{D - mD' - a}$.

Now consider the equation $(D - mD' - a)z = 0$ ...(2)

or $$p - mq = az$$

Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{az}$$

From the first two members $dy + m\,dx = 0 \quad \therefore \quad y + mx = b$

From the first and last members $\dfrac{dz}{z} = a\,dx \quad \therefore \quad \log z = ax + \log c$ or $z = ce^{ax}$

$\therefore$ The complete solution of (2) is $z = e^{ax} f(y + mx)$

Hence the C.F. of (1), *i.e.*, the complete solution of

$$(D - m_1D' - a_1)(D - m_2D' - a_2)......(D - m_nD' - a_n)z = 0 \text{ is}$$

$$z = e^{a_1x} f_1(y + m_1x) + e^{a_2x} f_2(y + m_2x) + ... + e^{a_nx} f_n(y + m_nx).$$

1.23.1. In the Case of Repeated Factors, *e.g.*, $(D - mD' - a)^3z = 0$

We have $z = e^{ax} f_1(y + mx) + xe^{ax} f_2(y + mx) + x^2e^{ax} f_3(y + mx)$.

1.23.2. If the Equation is of the Form

$$(\alpha D + \beta D' + \gamma)z = 0 \qquad ...(3)$$

$$\Rightarrow \qquad \alpha p + \beta q = -\gamma z$$

It is of Lagrange's form.

Lagrange's subsidiary equations are

$$\frac{dx}{\alpha} = \frac{dy}{\beta} = \frac{dz}{-\gamma z} \qquad ...(4)$$

First two will give $\alpha y - \beta x = c_1$ ...(5)

First and last will give, $\dfrac{dz}{z} = -\dfrac{\gamma}{\alpha}\,dx$

Integration gives, $\log z = -\dfrac{\gamma}{\alpha}x + \log c_2$

$$\Rightarrow \qquad z = c_2\, e^{-\frac{\gamma}{\alpha}x} = \phi(c_1)\, e^{-\frac{\gamma}{\alpha}x}$$

$$\Rightarrow \qquad z = e^{-\frac{\gamma}{\alpha}x}\, \phi(\alpha y - \beta x)$$

where ϕ is an arbitrary function.

Note. The above result is not applicable in the absence of the first term *i.e.*, D or αD and also when $\alpha = 0$.

Remark 1. Corresponding to each non-repeated factor $(aD' + b)$, the part of C.F. is $e^{-(by/a)}\,\phi(ax)$ when $a \neq 0$.

Remark 2. Corresponding to repeated factor $(aD' + b)^r$, the part of C.F. is

$$e^{-\left(\frac{by}{a}\right)}\,[\phi_1(ax) + y\,\phi_2(ax) + y^2\,\phi_3(ax) + \ldots\ldots + y^{r-1}\,\phi_r(ax)].$$

Remark 3. As a particular case of remark 1 with $b = 0$, $a = 1$, corresponding to non-repeated factor D′, the part of C.F. $= \phi_1(x)$.

Remark 4. As a particular case of remark 2 with $b = 0$, $a = 1$, corresponding to repeated factor D'^r, part of C.F. $= \phi_1(x) + y\,\phi_2(x) + y^2\,\phi_3(x) + \ldots\ldots + y^{r-1}\,\phi_r(x)$

(*b*) **When F(D, D′) cannot be factorized into linear factors:**

In such cases, we use a trial method.

Let the equation be $\qquad (D - D'^2)z = 0 \qquad$ ...(1)

Let the trial solution of (1) be $\qquad z = Ae^{hx+ky}$, $\quad$ where A, h and k are constants. $\qquad$ ...(2)

From (2),

$$Dz = \frac{\partial z}{\partial x} = Ahe^{hx+ky}$$

$$D'z = \frac{\partial z}{\partial y} = A\,ke^{hx+ky}$$

$$D'^2z = \frac{\partial^2 z}{\partial y^2} = A\,k^2\,e^{hx+ky}$$

Putting in (1), we get $Ahe^{hx+ky} - A\,k^2\,e^{hx+ky} = 0$

$$\Rightarrow \qquad A(h - k^2)\,e^{hx+ky} = 0$$

or $\qquad h = k^2 \qquad$ ...(3)

Equation (2) gives, $\qquad z = Ae^{k^2x + ky} \qquad$ ...(4)

Since all values of k will satisfy eqn. (1), a more general solution of (1) is given by

$$z = \Sigma\, Ae^{k^2x + ky}$$

where A and k are arbitrary constants and Σ denotes that any number of terms may be taken.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the linear partial differential equation*

$$(D + D' - 1)\,(D + 2D' - 2)z = 0.$$

Sol. The given equation is

$$(D + D' - 1)\,(D + 2D' - 2)z = 0.$$

Its $\quad$ C.F. $= e^x f_1\,(y - x) + e^{2x} f_2\,(y - 2x)$

and $\quad$ P.I. $= 0$

Hence complete solution is

$$z = \text{C.F.} + \text{P.I.} = e^{x} f_1 (y - x) + e^{2x} f_2 (y - 2x)$$

where f_1 and f_2 are arbitrary functions.

Example 2. *Solve: $DD' (D + 2D' + 1)z = 0$.*

Sol. The given equation is

$$DD'(D + 2D' + 1)z = 0$$

Corresponding to the factor D, part of C.F. $= f_1(y)$

Corresponding to the factor D′, part of C.F. $= f_2(x)$

Corresponding to the factor $(D + 2D' + 1)$ part of C.F. $= e^{-x} f_3 (y - 2x)$

Hence combined $\quad \text{C.F.} = f_1(y) + f_2(x) + e^{-x} f_3 (y - 2x)$

$\text{P.I.} = 0$

Hence complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y) + f_2(x) + e^{-x} f_3 (y - 2x)$$

where f_1, f_2 and f_3 are arbitrary functions.

Example 3. *Solve: $r + 2s + t + 2p + 2q + z = 0$.*

Sol. The given equation is

$$[D^2 + 2DD' + D'^2 + 2D + 2D' + 1]z = 0$$

$$\Rightarrow \quad \{(D + D')^2 + 2(D + D') + 1\}z = 0$$

$$\Rightarrow \quad (D + D' + 1)^2 z = 0$$

Its $\quad \text{C.F.} = e^{-x} f_1 (y - x) + x\, e^{-x} f_2 (y - x)$

$\text{P.I.} = 0$

Hence complete solution is

$$z = \text{C.F.} + \text{P.I.} = e^{-x} f_1 (y - x) + x\, e^{-x} f_2 (y - x)$$

where f_1 and f_2 are arbitrary functions.

Example 4. *Solve:*

(*i*) $r - t + p - q = 0$ (U.P.T.U. 2013, 2014) $\qquad$ (*ii*) $(D + 4D' + 5)^2 z = 0$ [A.K.T.U. 2018]

(*iii*) $(D^2 - DD' - 2D)z = 0$ $\qquad$ (*iv*) $(D + 1)(D + D' - 1)z = 0$.

Sol. (*i*) The given equation is

$$(D^2 - D'^2 + D - D')z = 0$$

$$\Rightarrow \quad (D - D')(D + D' + 1)z = 0$$

Its $\quad \text{C.F.} = f_1 (y + x) + e^{-x} f_2 (y - x)$

$\text{P.I.} = 0$

Hence complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1 (y + x) + e^{-x} f_2 (y - x)$$

where f_1 and f_2 are arbitrary functions.

(*ii*) The given equation is

$$(D + 4D' + 5)^2 z = 0$$

Its $\quad \text{C.F.} = e^{-5x} f_1 (y - 4x) + x\, e^{-5x} f_2 (y - 4x)$

$\text{P.I.} = 0$

Hence complete solution is

$$z = \text{C.F.} + \text{P.I.} = e^{-5x} f_1 (y - 4x) + x\, e^{-5x} f_2 (y - 4x)$$

where f_1 and f_2 are arbitrary functions.

(*iii*) The given equation is

$$(D^2 - DD' - 2D)z = 0$$

$$\Rightarrow \quad D(D - D' - 2)z = 0$$

Its $\quad \text{C.F.} = f_1(y) + e^{2x} f_2 (y + x)$

$\text{P.I.} = 0$

Hence complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y) + e^{2x} f_2 (y + x)$$

where f_1 and f_2 are arbitrary functions.

(*iv*) The given equation is

$$(D + 1)(D + D' - 1)z = 0$$

Its $\quad \text{C.F.} = e^{-x} f_1(y) + e^{x} f_2 (y - x)$

$\text{P.I.} = 0$

Hence complete solution is

$$z = \text{C.F.} + \text{P.I.} = e^{-x} f_1(y) + e^{x} f_2 (y - x)$$

where f_1 and f_2 are arbitrary functions.

Example 5. *Solve:* (*i*) $(D^2 - D'^2 + D + 3D' - 2)z = 0$

(*ii*) $(D^2 - DD' - 2D'^2 + 2D + 2D')z = 0.$

Sol. (*i*) The given equation is

$$(D^2 - D'^2 + D + 3D' - 2)z = 0$$

$$\Rightarrow \quad (D^2 - D'^2 + 2D - D + 2D' + D' - 2 + DD' - DD')z = 0$$

$$\Rightarrow \quad [D^2 - DD' + 2D + DD' - D'^2 + 2D' - D + D' - 2]z = 0$$

$$\Rightarrow \quad (D - D' + 2)(D + D' - 1)z = 0$$

Its $\quad \text{C.F.} = e^{-2x} f_1 (y + x) + e^{x} f_2 (y - x)$

$\text{P.I.} = 0$

Hence complete solution is $\quad z = \text{C.F.} + \text{P.I.} = e^{-2x} f_1 (y + x) + e^{x} f_2 (y - x)$

where f_1 and f_2 are arbitrary functions.

(*ii*) The given equation is

$$(D^2 - DD' - 2D'^2 + 2D + 2D')z = 0$$

$$\Rightarrow \quad [(D + D')(D - 2D') + 2(D + D')]z = 0$$

$$\Rightarrow \quad (D + D')(D - 2D' + 2)z = 0$$

Its $\quad \text{C.F.} = f_1 (y - x) + e^{-2x} f_2 (y + 2x)$

$\text{P.I.} = 0$

Hence complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1 (y - x) + e^{-2x} f_2 (y + 2x)$$

where f_1 and f_2 are arbitrary functions.

Example 6. *Solve:* $(D^2 + D'^2 - p^2)z = 0.$

Sol. Here $D^2 + D'^2 - p^2$ cannot be resolved into linear factors in D and D′.

Let $\quad z = Ae^{hx + ky}$

$\therefore \quad D^2z = Ah^2 e^{hx + ky}$

$D'^2z = Ak^2 e^{hx + ky}$

$\therefore \quad (D^2 + D'^2 - p^2)z = A(h^2 + k^2 - p^2)\, e^{hx + ky}$

Then, $\quad (D^2 + D'^2 - p^2)z = 0$

$\Rightarrow \quad A(h^2 + k^2 - p^2)\, e^{hx + ky} = 0$

$\Rightarrow \quad h^2 + k^2 - p^2 = 0 \quad \text{or} \quad h^2 + k^2 = p^2$

$\therefore \quad \text{C.F.} = \Sigma\, Ae^{hx + ky}$, where $h^2 + k^2 - p^2 = 0$

$\text{P.I.} = 0$

$\therefore \quad z = \Sigma\, Ae^{hx + ky}, \quad$ where $h^2 + k^2 - p^2 = 0$

Now, h may be taken as $p \cos \alpha$ and k may be taken as $p \sin \alpha$. Therefore
The complete solution is

$$z = \Sigma\, Ae^{p(x \cos \alpha + y \sin \alpha)}$$

where A is arbitrary constant and Σ denotes that any number of terms may be taken.

TEST YOUR KNOWLEDGE

Solve the following partial differential equations:

1. $(D + D' - 1)(D + 2D' - 3)z = 0$
2. $r - 3s + 2t - p + 2q = 0$
3. $DD'(D - 2D' - 3)z = 0$
4. $t + s + q = 0$
5. $(D^2 - a^2D'^2 + 2abD + 2a^2bD')z = 0$
6. $(D - 2D' - 1)(D - 2D'^2 - 1)z = 0$
7. $(2D^4 - 3D^2D' + D'^2)z = 0$
8. $(D^3 - 3DD' + D' + 4)z = 0$
9. $2s + t - 3q = 0$
10. $(DD' + aD + bD' + ab)z = 0.$

Answers

1. $z = e^x f_1(y - x) + e^{3x} f_2(y - 2x)$
2. $z = f_1(y + 2x) + e^x f_2(y + x)$
3. $z = f_1(y) + f_2(x) + e^{3x} f_3(y + 2x)$
4. $z = f_1(x) + e^{-x} f_2(y - x)$
5. $z = f_1(y - ax) + e^{-2abx} f_2(y + ax)$
6. $z = e^x f_1(y + 2x) + \Sigma\, Ae^{(2k^2 + 1)x + ky}$
7. $z = \Sigma\, Ae^{hx + h^2 y} + \Sigma\, A' e^{h'x + h'^2 y}$
8. $z = \Sigma\, A\, e^{hx + ky}$, where $h^3 - 3hk + k + 4 = 0$
9. $z = f_1(y) + e^{(3x/2)} f_2(2y - x).$
10. $z = e^{-ay} f_1(x) + e^{-bx} f_2(y)$

1.24 P.I. OF NON-HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

Let the given equation be $\quad \phi(D, D')z = F(x, y)$

then $$\text{P.I.} = \frac{1}{\phi(D, D')} F(x, y)$$

The methods of finding out P.I. of these equations quite resemble to those of ordinary linear differential equation with constant coefficients.