1.24.1. Case I. *When* $F(x, y) = e^{ax+by}$ *and* $\phi(a, b) \neq 0$

$$\text{P.I.} = \frac{1}{\phi(D, D')} e^{ax+by}$$

$$= \frac{1}{\phi(a, b)} e^{ax+by} \qquad \text{| Replacing D by } a \text{ and D}' \text{ by } b$$

ILLUSTRATIVE EXAMPLES

Example 1. *Solve:* $s + ap + bq + abz = e^{mx + ny}$.

Sol. The given equation is

$$(DD' + aD + bD' + ab)z = e^{mx + ny}$$

$$\Rightarrow \qquad (D + b)(D' + a)z = e^{mx + ny}$$

Its $\qquad \text{C.F.} = e^{-bx} f_1(y) + e^{-ay} f_2(x)$

$$\text{P.I.} = \frac{1}{(D + b)(D' + a)} e^{mx + ny} = \frac{e^{mx + ny}}{(m + b)(n + a)}$$

Hence complete solution is

$$z = \text{C.F.} + \text{P.I.} = e^{-bx} f_1(y) + e^{-ay} f_2(x) + \frac{e^{mx + ny}}{(m + b)(n + a)}$$

where f_1 and f_2 are arbitrary functions.

Example 2. *Solve:* $(D^3 - 3DD' + D' + 4)z = e^{2x + y}$.

Sol. Here $D^3 - 3DD' + D' + 4$ cannot be resolved into linear factors in D and D′.

Let $\qquad z = Ae^{hx + ky}$

$$\therefore \qquad (D^3 - 3DD' + D' + 4)z = A(h^3 - 3hk + k + 4)\, e^{hx+ky}$$

Then $\qquad (D^3 - 3DD' + D' + 4)z = 0$ iff $h^3 - 3hk + k + 4 = 0$

$\therefore$ The C.F. is $z = \Sigma\, Ae^{hx + ky}$, where $h^3 - 3hk + k + 4 = 0$

$$\text{P.I.} = \frac{1}{D^3 - 3DD' + D' + 4} e^{2x + y} = \frac{e^{2x + y}}{2^3 - 3(2)(1) + 1 + 4} = \frac{1}{7} e^{2x + y}$$

Hence complete solution is $z = \Sigma\, Ae^{hx + ky} + \frac{1}{7} e^{2x + y}$, where $h^3 - 3hk + k + 4 = 0$.

Example 3. *Solve:* $D(D - 2D' - 3)z = e^{x + 2y}$.

Sol. $\qquad \text{C.F.} = f_1(y) + e^{3x} f_2(y + 2x)$

$$\text{P.I.} = \frac{1}{D(D - 2D' - 3)} e^{x+2y} = \frac{e^{x + 2y}}{1\{1 - 2(2) - 3\}} = -\frac{1}{6} e^{x+2y}$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y) + e^{3x} f_2(y + 2x) - \frac{1}{6} e^{x + 2y}$$

where f_1 and f_2 are arbitrary functions.

TEST YOUR KNOWLEDGE

Solve:

1. $(D^3 - 3DD' + D + 1)z = e^{2x+3y}$
2. $(D^2 - 4DD' + D - 1)z = e^{3x-2y}$
3. $(D^2 - D'^2 - 3D + 3D')z = e^{x-2y}$
4. $(D - D' - 1)(D + D' - 2)z = e^{2x-y}$
5. $(D^2 - D'^2 + D - D')z = e^{2x+3y}$
6. $(D^2 - 4DD' + 4D'^2 - D + 2D')z = e^{3x+4y}$. *(U.P.T.U. 2013)*

Answers

1. $z = \Sigma Ae^{hx+ky} - \frac{1}{7}e^{2x+3y}$, where $h^3 - 3hk + h + 1 = 0$
2. $z = \Sigma Ae^{hx+ky} + \frac{1}{35}e^{3x-2y}$, where $h^2 - 4hk + h - 1 = 0$
3. $z = f_1(y+x) + e^{3x}f_2(y-x) - \frac{1}{12}e^{x-2y}$
4. $z = e^x f_1(y+x) + e^{2x} f_2(y-x) - \frac{1}{2}e^{2x-y}$
5. $z = f_1(y+x) + e^{-x}f_2(y-x) - \frac{1}{6}e^{2x+3y}$
6. $z = f_1(y+2x) + e^x f_2(y+2x) + \frac{1}{30}e^{3x+4y}$.

1.24.2. Case II. *When $F(x, y) = \sin(ax + by)$ or $\cos(ax + by)$*

$$\text{P.I.} = \frac{1}{\phi(D, D')}\{\sin(ax+by) \text{ or } \cos(ax+by)\}$$

$$= \frac{1}{\phi(D^2, DD', D'^2)}\{\sin(ax+by) \text{ or } \cos(ax+by)\}$$

$$= \frac{1}{\phi(-a^2, -ab, -b^2)}\{\sin(ax+by) \text{ or } \cos(ax+by)\}$$

where $\phi(-a^2, -ab, -b^2) \neq 0$. It is to be noted that here D^2 is replaced by $-a^2$, DD' is replaced by $-ab$ and D'^2 is replaced by $-b^2$.

If $\phi(D, D') = \phi(D^2, DD', D'^2, D, D')$, then

$$\text{P.I.} = \frac{1}{\phi(-a^2, -ab, -b^2, D, D')}\{\sin(ax+by) \text{ or } \cos(ax+by)\}$$

which can be evaluated further by operating N^r and D^r by the suitable conjugate operator.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve: $(D^2 - DD' + D' - 1)z = \sin(x + 2y)$.*

Sol. The given equation is $(D^2 - DD' + D' - 1)z = \sin(x + 2y)$

$\Rightarrow \quad \{(D+1)(D-1) - D'(D-1)\}z = \sin(x+2y)$

$\Rightarrow \quad (D-1)(D-D'+1)z = \sin(x+2y)$

$\therefore \quad \text{C.F.} = e^x f_1(y) + e^{-x} f_2(y+x)$

$$\text{P.I.} = \frac{1}{D^2 - DD' + D' - 1} \sin(x + 2y) = \frac{1}{-1 + 2 + D' - 1} \sin(x + 2y)$$

$$= \frac{1}{D'} \sin(x + 2y) = -\frac{\cos(x + 2y)}{2}$$

Hence complete solution is

$$z = \text{C.F.} + \text{P.I.} = e^{x} f_1(y) + e^{-x} f_2(y + x) - \frac{\cos(x + 2y)}{2}$$

where f_1 and f_2 are arbitrary functions.

Example 2. *Solve : $(D - D' - 1)(D - D' - 2)z = \sin(2x + 3y)$.* (U.K.T.U. 2011)

Sol. $\text{C.F.} = e^{x} f_1(y + x) + e^{2x} f_2(y + x)$

$$\text{P.I.} = \frac{1}{(D - D' - 1)(D - D' - 2)} \sin(2x + 3y)$$

$$= \frac{1}{D^2 - 2DD' + D'^2 - 3D + 3D' + 2} \sin(2x + 3y)$$

$$= \frac{1}{-4 + 12 - 9 - 3D + 3D' + 2} \sin(2x + 3y)$$

$$= \frac{1}{-3D + 3D' + 1} \sin(2x + 3y)$$

$$= -\left[\frac{(3D - 3D') + 1}{\{(3D - 3D') + 1\}\{3D - 3D' - 1\}} \sin(2x + 3y)\right]$$

$$= -\left[\frac{(3D - 3D') + 1}{9D^2 + 9D'^2 - 18DD' - 1} \sin(2x + 3y)\right]$$

$$= -\left[\frac{3D - 3D' + 1}{-36 - 81 + 108 - 1} \sin(2x + 3y)\right]$$

$$= \frac{1}{10}(3D - 3D' + 1) \sin(2x + 3y)$$

$$= \frac{1}{10}[6\cos(2x + 3y) - 9\cos(2x + 3y) + \sin(2x + 3y)]$$

$$= \frac{1}{10}.[\sin(2x + 3y) - 3\cos(2x + 3y)]$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = e^{x} f_1(y + x) + e^{2x} f_2(y + x) + \frac{1}{10}[\sin(2x + 3y) - 3\cos(2x + 3y)]$$

where f_1 and f_2 are arbitrary functions.

Example 3. *Find the particular integral of $2s + t - 3q = 5 \cos (3x - 2y)$.*

Sol. The given equation is

$$(2DD' + D'^2 - 3D')z = 5 \cos (3x - 2y)$$

$$\text{P.I.} = \frac{1}{2DD' + D'^2 - 3D'} 5 \cos (3x - 2y)$$

$$= \frac{1}{2(6) + (-4) - 3D'} 5 \cos (3x - 2y) = \frac{1}{(8 - 3D')} 5 \cos (3x - 2y)$$

$$= 5 \left[\frac{8 + 3D'}{64 - 9D'^2} \cos (3x - 2y) \right] = 5 \left[\frac{8 + 3D'}{64 - 9(-4)} \cos (3x - 2y) \right]$$

$$= \frac{1}{20} [8 \cos (3x - 2y) + 6 \sin (3x - 2y)]$$

$$= \frac{1}{10} [4 \cos (3x - 2y) + 3 \sin (3x - 2y)].$$

TEST YOUR KNOWLEDGE

Solve the following partial differential equations:

1. $(D^2 - DD' + D' - 1)z = \cos (x + 2y)$
2. $(D^2 - DD' - 2D'^2 + 2D + 2D')z = \sin (2x + y)$ *(A.K.T.U. 2016)*
3. $(D^2 + DD' + D' - 1)z = \sin (x + 2y)$
4. $(D^2 - DD' - 2D)z = \sin (3x + 4y) - e^{2x+y}$
5. $(D - D'^2)z = \cos(x - 3y)$
6. $(D^2 + 2DD' + D'^2 - 2D - 2D')z = \sin (x + 2y)$.

Answers

1. $z = e^x f_1(y) + e^{-x} f_2 (y + x) + \frac{1}{2} \sin (x + 2y)$
2. $z = f_1 (y - x) + e^{-2x} f_2 (2x + y) - \frac{1}{6} \cos (2x + y)$
3. $z = e^{-x} f_1(y) + e^x f_2 (y - x) - \frac{1}{10} [2 \sin (x + 2y) + \cos (x + 2y)]$
4. $z = f_1(y) + e^{2x} f_2 (y + x) + \frac{1}{15} [\sin (3x + 4y) + 2 \cos (3x + 4y)] + \frac{1}{2} e^{2x+y}$
5. $z = \Sigma A e^{k^2 x + ky} + \frac{1}{82} [\sin (x - 3y) + 9 \cos (x - 3y)]$
6. $z = f_1 (y - x) + e^{2x} f_2 (y - x) + \frac{1}{39} [2 \cos (x + 2y) - 3 \sin (x + 2y)]$.

1.24.3. Case III. *When $F(x, y) = x^m y^n$ where m and n being positive integers*

$$\text{P.I.} = \frac{1}{\phi(D, D')} x^m y^n = [\phi (D, D')]^{-1} (x^m y^n)$$

which can be evaluated after expanding $[\phi(D, D')]^{-1}$ in ascending powers of $\frac{D'}{D}$ (when $m > n$) or $\frac{D}{D'}$ (when $m < n$) or D or D′ as the case may be.

If a separate constant is present in $\phi(D, D')$ then it should be given preference in taking the term outside the bracket. It is to be noted that if P.I. is obtained by expanding $\phi(D, D')$ in two or more different ways, then difference in P.I. will be immaterial.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the linear partial differential equation:*

$$(D - D' - 1)(D - D' - 2)z = e^{3x-y} + x.$$

Sol. C.F. $= e^x f_1(y + x) + e^{2x} f_2(y + x)$

$$\text{P.I.} = \frac{1}{(D - D' - 1)(D - D' - 2)} e^{3x-y} + \frac{1}{(D - D' - 1)(D - D' - 2)}(x) = P_1 + P_2$$

where
$$P_1 = \frac{1}{(D - D' - 1)(D - D' - 2)} e^{3x-y} = \frac{1}{(3+1-1)(3+1-2)} e^{3x-y} = \frac{1}{6} e^{3x-y},$$

$$P_2 = \frac{1}{(D - D' - 1)(D - D' - 2)}(x) = \frac{1}{(1 - D + D')(2 - D + D')}(x)$$

$$= \frac{1}{\{1 - (D - D')\}2\left\{1 - \left(\frac{D - D'}{2}\right)\right\}}(x)$$

$$= \frac{1}{2}\left[\{1 - (D - D')\}^{-1}\left\{1 - \left(\frac{D - D'}{2}\right)\right\}^{-1}\right](x)$$

$$= \frac{1}{2}\left[(1 + D - D')\left(1 + \frac{D - D'}{2}\right)\right](x) \qquad | \text{ Leaving higher powers}$$

$$= \frac{1}{2}\left[1 + \frac{D}{2} - \frac{D'}{2} + D + \frac{D^2}{2} - \frac{DD'}{2} - D' - \frac{D'D}{2} + \frac{D'^2}{2}\right](x)$$

$$= \frac{1}{2}\left[x + \frac{1}{2} - 0 + 1 + 0 - 0 - 0 - 0 - 0 + 0\right] = \frac{1}{2}\left(x + \frac{3}{2}\right)$$

$$\therefore \quad \text{P.I.} = \frac{1}{6} e^{3x-y} + \frac{1}{2}\left(x + \frac{3}{2}\right).$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = e^x f_1(y + x) + e^{2x} f_2(y + x) + \frac{1}{6} e^{3x-y} + \frac{1}{2}\left(x + \frac{3}{2}\right)$$

where f_1 and f_2 are arbitrary functions.

Example 2. *Solve the linear partial differential equation*

$$(D + D' - 1)(D + 2D' - 3)z = 4 + 3x + 6y.$$

Sol. C.F. $= e^x f_1(y - x) + e^{3x} f_2(y - 2x)$

$$\text{P.I.} = \frac{1}{(D + D' - 1)(D + 2D' - 3)}(4 + 3x + 6y)$$

$$= \frac{1}{(1 - D - D')(3 - D - 2D')}(4 + 3x + 6y)$$

$$= \frac{1}{\{1 - (D + D')\}\, 3\left\{1 - \left(\frac{D + 2D'}{3}\right)\right\}}(4 + 3x + 6y)$$

$$= \frac{1}{3}\left[\{1 - (D + D')\}^{-1}\left\{1 - \left(\frac{D + 2D'}{3}\right)\right\}^{-1}\right](4 + 3x + 6y)$$

$$= \frac{1}{3}\left[(1 + D + D')\left(1 + \frac{D}{3} + \frac{2D'}{3}\right)\right](4 + 3x + 6y)$$

$$= \frac{1}{3}\left[1 + \frac{4D}{3} + \frac{5D'}{3} + \frac{D^2}{3} + DD' + \frac{2D'^2}{3}\right](4 + 3x + 6y)$$

$$= \frac{1}{3}\left[(4 + 3x + 6y) + \frac{4}{3}(3) + \frac{5}{3}(6) + 0 + 0 + 0\right] = 6 + x + 2y.$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = e^x f_1(y - x) + e^{3x} f_2(y - 2x) + 6 + x + 2y$$

where f_1 and f_2 are arbitrary functions.

Example 3. *Solve the partial differential equation*

$$(D^2 - D'^2 - 3D + 3D')z = xy + e^{x + 2y}.$$ [G.B.T.U. (C.O.) 2011]

Sol. The given equation is

$$(D^2 - D'^2 - 3D + 3D')\, z = 0$$

$$(D - D')(D + D' - 3)z = 0$$

$\therefore$ C.F. $= f_1(y + x) + e^{3x} f_2(y - x)$

P.I. corresponding to $xy = \dfrac{1}{(D - D')(D + D' - 3)}(xy)$

$$= -\frac{1}{3D}\left(1 - \frac{D'}{D}\right)^{-1}\left(1 - \frac{D}{3} - \frac{D'}{3}\right)^{-1}(xy)$$

$$= -\frac{1}{3D}\left(1 + \frac{D'}{D} + \ldots\right)\left(1 + \frac{D}{3} + \frac{D'}{3} + \frac{2DD'}{9} + \ldots\right)(xy)$$

$$= -\frac{1}{3D}\left(1 + \frac{D}{3} + \frac{D'}{3} + \frac{D'}{D} + \frac{D'}{3} + \frac{2DD'}{9} + \ldots\right)(xy)$$

$$= -\frac{1}{3D}\left(xy + \frac{y}{3} + \frac{2}{3}x + \frac{x^2}{2} + \frac{2}{9}\right) \qquad | \text{ Leaving higher powers}$$

$$= -\frac{1}{3}\left[\frac{x^2}{2}y + \frac{xy}{3} + \frac{x^2}{3} + \frac{x^3}{6} + \frac{2}{9}x\right]$$

P.I. corresponding to e^{x+2y}

$$= \frac{1}{(D-D')(D+D'-3)}e^{x+2y} = \frac{1}{(1-2)(D+D'-3)}e^{x+2y}$$

$$= -e^{x+2y} \cdot \frac{1}{D+1+D'+2-3}(1) = -e^{x+2y} \cdot \frac{1}{D+D'}e^{0x+0y}$$

$$= -e^{x+2y} \,.\, x\, e^{0x+0y} = -x\, e^{x+2y}$$

$$\therefore \qquad \text{P.I.} = -\frac{1}{3}\left[\frac{x^2y}{2} + \frac{xy}{3} + \frac{x^2}{3} + \frac{x^3}{6} + \frac{2x}{9}\right] - x\, e^{x+2y}$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y+x) + e^{3x} f_2(y-x) - \frac{1}{3}\left[\frac{x^2y}{2} + \frac{xy}{3} + \frac{x^2}{3} + \frac{x^3}{6} + \frac{2x}{9}\right] - x\, e^{x+2y}$$

where f_1 and f_2 are arbitrary funtions.

Example 4. *Solve*: $s + p - q = z + xy$.

Sol. The given equation is

$$(DD' + D - D' - 1)z = xy$$

$$\Rightarrow \qquad (D-1)(D'+1)\, z = xy$$

$$\therefore \qquad \text{C.F.} = e^x f_1(y) + e^{-y} f_2(x)$$

$$\text{P.I.} = \frac{1}{(D-1)(D'+1)}(xy) = -[(1-D)^{-1}(1+D')^{-1}]\,(xy)$$

$$= -[(1 + D + D^2 + \ldots\ldots)(1 - D' + \ldots\ldots)]\,(xy)$$

$$= -(1 + D - D' - DD')\, xy \qquad | \text{ Leaving higher powers}$$

$$= -(xy + y - x - 1) = x + 1 - xy - y$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = e^x f_1(y) + e^{-y} f_2(x) + x + 1 - xy - y$$

where f_1 and f_2 are arbitrary functions.

Example 5. *Solve:*

(*a*) $r - s + p = 1$ (*b*) $D(D + D' - 1)(D + 3D' - 2)z = x^2 - 4xy + 2y^2$.

Sol. (*a*) The given equation is

$$(D^2 - DD' + D)z = 1$$

$$\Rightarrow \qquad D(D - D' + 1)z = 1$$

$$\therefore \qquad \text{C.F} = f_1(y) + e^{-x} f_2(y + x)$$

$$\text{P.I.} = \frac{1}{D(D - D' + 1)}(1) = \frac{1}{D}[1 + (D - D')]^{-1}(1) = \frac{1}{D}(1) = x$$

$\therefore$ Complete solution is

$$z = \text{C.F} + \text{P.I.} = f_1(y) + e^{-x} f_2(y + x) + x$$

where f_1 and f_2 are arbitrary functions.

(b) $\text{C.F.} = f_1(y) + e^x f_2(y - x) + e^{2x} f_3(y - 3x)$

$$\text{P.I.} = \frac{1}{D(D + D' - 1)(D + 3D' - 2)}(x^2 - 4xy + 2y^2)$$

$$= \frac{1}{2D}\{1 - (D + D')\}^{-1}\left\{1 - \frac{D + 3D'}{2}\right\}^{-1}(x^2 - 4xy + 2y^2)$$

$$= \frac{1}{2D}\{1 + D + D' + (D + D')^2 + \ldots\} . \left\{1 + \frac{D + 3D'}{2} + \left(\frac{D + 3D'}{2}\right)^2 + \ldots\right\}(x^2 - 4xy + 2y^2)$$

$$= \frac{1}{2D}\left[1 + \frac{3D}{2} + \frac{5D'}{2} + \frac{7D^2}{4} + \frac{19D'^2}{4} + \frac{11DD'}{2} + \ldots\right](x^2 - 4xy + 2y^2)$$

$$= \frac{1}{2D}\left[x^2 - 4xy + 2y^2 + 3(x - 2y) + 5(2y - 2x) + \frac{7}{2} + 19 - 22\right]$$

$$= \frac{1}{2D}\left(x^2 - 4xy + 2y^2 - 7x + 4y + \frac{1}{2}\right) = \frac{1}{2}\left(\frac{x^3}{3} - 2x^2y + 2y^2x - \frac{7x^2}{2} + 4yx + \frac{x}{2}\right)$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y) + e^x f_2(y - x) + e^{2x} f_3(y - 3x) + \frac{1}{6}x^3 - x^2y + xy^2 - \frac{7}{4}x^2 + 2xy + \frac{x}{4}$$

where f_1, f_2 and f_3 are arbitrary functions.

TEST YOUR KNOWLEDGE

Solve:

1. $(D - D' - 1)(D - D' - 2)z = e^{2x-y} + x$
2. $(D + D' - 1)(D + 2D' - 3)z = 2x + 3y$
3. $(D^2 - D' - 1)z = x^2y$
4. $(D + D' - 1)^2 z = xy$ [G.B.T.U. (AG) 2011]
5. $(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x + 3y} + \sin(2x + y) + xy.$

Answers

1. $z = e^x f_1(y + x) + e^{2x} f_2(y + x) + \dfrac{e^{2x - y}}{2} + \dfrac{2x + 3}{4}$
2. $z = e^x f_1(y - x) + e^{3x} f_2(y - 2x) + \dfrac{2}{3}x + y + \dfrac{23}{9}$
3. $z = \Sigma\, Ae^{hx + (h^2 - 1)y} + x^2 - x^2y - 2y + 4$

4. $z = e^x f_1(y - x) + xe^x f_2(y - x) + xy + 2y + 2x + 6$

5. $z = f_1(y - x) + e^{-2x} f_2(y + 2x) - \dfrac{1}{10} e^{2x+3y} - \dfrac{1}{6}\cos(2x + y) + \dfrac{x^2 y}{4} - \dfrac{xy}{4} - \dfrac{x^3}{12} + \dfrac{3x^2}{8} - \dfrac{x}{2}$.

1.24.4. Case IV. When $F(x, y) = e^{ax+by} \,.\, V$, where V is a function of x and y

$$\text{P.I.} = \frac{1}{\phi(D, D')} e^{ax+by} \,.\, V = e^{ax+by} \,.\, \frac{1}{\phi(D + a, D' + b)} V$$

which can be evaluated further by using any one of the previous cases discussed.

Remark. V can be either

(*i*) $e^{a_1 x + b_1 y}$ (*ii*) $\sin(ax + by)$ or $\cos(ax + by)$

(*iii*) $x^m y^n$ (*iv*) constant (say 1)

or any other function of x and y in some special cases.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve:* $(D - 3D' - 2)^3 z = 6e^{2x} \sin(3x + y)$.

Sol. Here C.F. $= e^{2x} f_1(y + 3x) + xe^{2x} f_2(y + 3x) + x^2 e^{2x} f_3(y + 3x)$

$$\text{P.I.} = \frac{1}{(D - 3D' - 2)^3} 6e^{2x} \sin(3x + y)$$

$$= 6e^{2x} \frac{1}{(D + 2 - 3D' - 2)^3} \sin(3x + y) = 6e^{2x} \frac{1}{(D - 3D')^3} \sin(3x + y)$$

$$= 6e^{2x} \,.\, \frac{x^3}{6} \sin(3x + y) = x^3 e^{2x} \sin(3x + y)$$

Hence complete solution is

$$z = e^{2x} f_1(y + 3x) + xe^{2x} f_2(y + 3x) + x^2 e^{2x} f_3(y + 3x) + x^3 e^{2x} \sin(3x + y)$$

where f_1, f_2 and f_3 are arbitrary functions.

Example 2. *Solve:* $(D - 3D' - 2)^2 z = 2e^{2x} \tan(y + 3x)$.

Sol. C.F. $= e^{2x} f_1(y + 3x) + xe^{2x} f_2(y + 3x)$

$$\text{P.I.} = \frac{1}{(D - 3D' - 2)^2} 2e^{2x} \tan(y + 3x)$$

$$= 2e^{2x} \,.\, \frac{1}{(D + 2 - 3D' - 2)^2} \tan(y + 3x) = 2e^{2x} \,.\, \left[\frac{1}{(D - 3D')^2} \tan(y + 3x)\right]$$

$$= 2e^{2x} \,.\, \left[x \,.\, \frac{1}{2(D - 3D')} \tan(y + 3x)\right] = 2e^{2x} \,.\, x^2 \,.\, \frac{1}{2} \tan(y + 3x)$$

$$= x^2 e^{2x} \tan(y + 3x)$$

Hence complete solution is

$$z = \text{C.F.} + \text{P.I.} = e^{2x} f_1(y + 3x) + xe^{2x} f_2(y + 3x) + x^2 e^{2x} \tan(y + 3x)$$

where f_1 and f_2 are arbitrary functions.

Example 3. *Solve the linear partial differential equation*

$$(D^2 - DD' + D' - 1)\, z = \cos (x + 2y) + e^y.$$

Sol. The given equation is

$$(D^2 - DD' + D' - 1)z = \cos (x + 2y) + e^y$$

$$\Rightarrow \quad (D - 1)\,(D - D' + 1)z = \cos (x + 2y) + e^y$$

$$\text{C.F.} = e^x f_1 (y) + e^{-x} f_2 (y + x)$$

$$\text{P.I.} = \frac{1}{D^2 - DD' + D' - 1} \cos (x + 2y) + \frac{1}{(D - D' + 1)(D - 1)} e^y = P_1 + P_2$$

where
$$P_1 = \frac{1}{D^2 - DD' + D' - 1} \cos (x + 2y)$$

$$= \frac{1}{-(1)^2 - (-2) + D' - 1} \cos (x + 2y) = \frac{1}{D'} \cos (x + 2y) = \frac{\sin (x + 2y)}{2}$$

and
$$P_2 = \frac{1}{D - D' + 1}\left[\frac{1}{D - 1} e^y\right] = \frac{1}{D - D' + 1}\left[\frac{1}{0 - 1} e^y\right]$$

$$= \frac{1}{D - D' + 1}(-e^y) = -e^y \cdot \frac{1}{(D + 0) - (D' + 1) + 1}(1)$$

$$= -e^y \cdot \frac{1}{D - D'}(1) = -e^y \cdot \frac{1}{D - D'}(e^{0x + 0y}) = -e^y \cdot x \cdot e^{0x + 0y} = -xe^y.$$

$$\therefore \quad \text{P.I.} = \frac{1}{2} \sin (x + 2y) - xe^y.$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = e^x f_1 (y) + e^{-x} f_2 (y + x) + \frac{1}{2} \sin (x + 2y) - xe^y$$

where f_1 and f_2 are arbitrary functions.

Example 4. *Solve:* $r - 4s + 4t + p - 2q = e^{x+y}$.

Sol. The given equation is

$$(D^2 - 4DD' + 4D'^2 + D - 2D')z = e^{x+y}$$

$$\Rightarrow \quad [(D - 2D')^2 + (D - 2D')]z = e^{x+y}$$

$$\Rightarrow \quad (D - 2D')\,(D - 2D' + 1)z = e^{x+y}$$

$$\therefore \quad \text{C.F.} = f_1 (y + 2x) + e^{-x} f_2 (y + 2x)$$

$$\text{P.I.} = \frac{1}{(D - 2D')(D - 2D' + 1)} e^{x+y} = \frac{1}{D - 2D' + 1}\left[\frac{1}{D - 2D'} e^{x + y}\right]$$

$$= \frac{1}{D - 2D' + 1}\left[\frac{1}{1 - 2} \int e^u \, du\right] \qquad \text{where } x + y = u$$

$$= -\left[\frac{1}{D - 2D' + 1} e^{x+y}\right] = -e^{x+y} \cdot \frac{1}{D + 1 - 2(D' + 1) + 1}(1)$$

$$= -e^{x+y}\left[\frac{1}{D - 2D'}(1)\right] = -e^{x+y}\left[\frac{1}{D - 2D'}(e^{0x+0y})\right]$$

$$= -xe^{x+y}.$$

Hence the complete solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y + 2x) + e^{-x} f_2(y + 2x) - xe^{x+y}$$

where f_1 and f_2 are arbitrary functions.

Example 5. *Find the particular integral of* $(D^2 - D')z = xe^{ax+a^2y}$.

Sol. $$\text{P.I.} = \frac{1}{D^2 - D'}(xe^{ax+a^2y}) = e^{ax+a^2y} \cdot \frac{1}{(D+a)^2 - (D' + a^2)}(x)$$

$$= e^{ax+a^2y} \cdot \frac{1}{D^2 + 2aD - D'}(x) = e^{ax+a^2y} \cdot \frac{1}{2aD}\left(1 + \frac{D}{2a} - \frac{D'}{2aD}\right)^{-1}(x)$$

$$= e^{ax+a^2y} \cdot \frac{1}{2aD}\left\{1 - \left(\frac{D}{2a} - \frac{D'}{2aD}\right) + \ldots\right\} x = e^{ax+a^2y} \cdot \frac{1}{2aD}\left(x - \frac{1}{2a}\right)$$

$$= e^{ax+a^2y} \cdot \left(\frac{x^2}{4a} - \frac{x}{4a^2}\right).$$

TEST YOUR KNOWLEDGE

Solve the following partial differential equations:

1. $(3D^2 - 2D'^2 + D - 1)z = 4e^{x+y}\cos(x + y)$
2. $(D^2 + DD' + D + D' - 1)z = e^{-2x}(x^2 + y^2)$
3. $(D + D' - 1)(D + D' - 3)(D + D')z = e^{x+y}\sin(2x + y)$
4. $(D^2 - DD' + D' - 1)z = \cos(x + 2y) + e^x$.

Answers

1. $z = \Sigma\, Ae^{hx+ky} + \frac{4}{3}e^{x+y}\sin(x + y)$, where $3h^2 - 2k^2 + h - 1 = 0$
2. $z = \Sigma\, Ae^{hx+ky} + \frac{1}{27}e^{-2x}(9x^2 + 9y^2 + 18x + 6y + 14)$, where $h^2 + hk + h + k + 1 = 0$
3. $z = e^x f_1(y - x) + e^{3x} f_2(y - x) + f_3(y - x) + \frac{1}{130}[3\cos(2x + y) - 2\sin(2x + y)]\, e^{x+y}$
4. $z = e^x f_1(y) + e^{-x} f_2(y + x) + \frac{1}{2}\sin(x + 2y) + \frac{1}{2}xe^x$.