Since, the string is released from rest hence its initial velocity will be zero

$$\therefore \qquad \frac{\partial y}{\partial t} = 0 \quad \text{at} \quad t = 0 \qquad \ldots(4)$$

Since, the string is displaced from its initial position at time $t = 0$ hence the initial displacement is

$$y(x, 0) = \text{A} \sin \frac{\pi x}{l} \qquad \ldots(5)$$

Conditions (2), (3), (4) and (5) are the boundary conditions.

Let us now proceed to solve equation (1),

Let $\qquad y = \text{XT}. \qquad \ldots(6)$

where X is a function of x only and T is a function of t only.

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}(\text{XT}) = \text{X}\frac{d\text{T}}{dt}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(\text{X}\frac{d\text{T}}{dt}\right) = \text{X}\frac{d^2\text{T}}{dt^2}.$$

Similarly, $\qquad \dfrac{\partial^2 y}{\partial x^2} = \text{T}\dfrac{d^2\text{X}}{dx^2}.$

Substituting the above in equation (1), we get

$$\text{X}\frac{d^2\text{T}}{dt^2} = c^2\text{T}\frac{d^2\text{X}}{dx^2} \quad \Rightarrow \quad \text{XT}'' = c^2\text{TX}''$$

Case I. $\qquad \dfrac{1}{c^2}\dfrac{\text{T}''}{\text{T}} = \dfrac{\text{X}''}{\text{X}} = -p^2 \text{ (say)}$

(*i*) $\qquad \dfrac{1}{c^2}\dfrac{\text{T}''}{\text{T}} = -p^2$

$$\frac{d^2\text{T}}{dt^2} + c^2p^2\text{T} = 0.$$

Auxiliary equation is $\qquad m^2 + c^2p^2 = 0$

$$m^2 = c^2p^2i^2$$

$$m = \pm\, cpi$$

$$\therefore \qquad \text{C.F.} = c_1 \cos cpt + c_2 \sin cpt$$

$$\text{P.I.} = 0$$

$$\therefore \qquad \text{T} = \text{C.F.} + \text{P.I.} = c_1 \cos cpt + c_2 \sin cpt \qquad \ldots(7)$$

(*ii*) $\qquad \dfrac{\text{X}''}{\text{X}} = -p^2 \quad \Rightarrow \quad \dfrac{d^2\text{X}}{dx^2} + p^2\text{X} = 0.$

Auxiliary equation is $\qquad m^2 + p^2 = 0$

$$m = \pm\, pi$$

$$\text{C.F.} = c_3 \cos px + c_4 \sin px$$

$$\text{P.I.} = 0$$

$\therefore$ $$X = c_3 \cos px + c_4 \sin px \qquad ...(8)$$

Hence, $$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px) \qquad ...(9)$$

Case II. $$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = p^2 \text{ (say)}$$

(*i*) $$\frac{1}{c^2}\frac{T''}{T} = p^2 \Rightarrow \frac{d^2T}{dt^2} - p^2c^2T = 0$$

Auxiliary equation is $m^2 - p^2c^2 = 0 \Rightarrow m = \pm pc$

$\therefore$ $$\text{C.F.} = c_5 e^{pct} + c_6 e^{-pct}$$
$$\text{P.I.} = 0$$

$\therefore$ $$T = c_5 e^{pct} + c_6 e^{-pct}.$$

(*ii*) $$\frac{X''}{X} = p^2 \Rightarrow \frac{d^2X}{dx^2} - p^2X = 0$$

Auxiliary equation is

$$m^2 - p^2 = 0 \Rightarrow m = \pm p$$

$\therefore$ $$\text{C.F.} = c_7 e^{px} + c_8 e^{-px}$$
$$\text{P.I.} = 0$$

$\therefore$ $$X = c_7 e^{px} + c_8 e^{-px}$$

Hence, $$y(x, t) = (c_5 e^{pct} + c_6 e^{-pct})(c_7 e^{px} + c_8 e^{-px}) \qquad ...(10)$$

Case III. $$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = 0 \text{ (say)}$$

(*i*) $$\frac{1}{c^2}\frac{T''}{T} = 0 \Rightarrow T'' = 0 \text{ or } \frac{d^2T}{dt^2} = 0$$

Auxiliary equation is

$$m^2 = 0 \Rightarrow m = 0, 0$$

$\therefore$ $$\text{C.F.} = c_9 + c_{10} t$$
$$\text{P.I.} = 0$$

$\therefore$ $$T = c_9 + c_{10} t$$

(*ii*) $$\frac{X''}{X} = 0 \Rightarrow X'' = 0 \text{ or } \frac{d^2X}{dx^2} = 0$$

Auxiliary equation is

$$m^2 = 0 \Rightarrow m = 0, 0$$

$\therefore$ $$\text{C.F.} = c_{11} + c_{12}x$$
$$\text{P.I.} = 0$$

$\therefore$ $$X = c_{11} + c_{12}x$$

Hence, $$y(x, t) = (c_9 + c_{10} t)(c_{11} + c_{12}x) \qquad ...(11)$$

Out of these three above solutions (9), (10) and (11), we have to choose the solution which is consistent with the physical nature of the problem. Since, we are dealing with a problem on vibrations, the solution must contain periodic functions. Hence the solution which contains trigonometric terms must be the required solution.

Hence solution (9) is the general solution of one-dimensional wave equation given by equation (1).

Now, $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px)$

Applying the boundary condition,

$$y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$$

$\Rightarrow$ $c_3 = 0.$

$\therefore$ From (9), $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px$ ...(12)

Again, $y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$

$\Rightarrow$ $\sin pl = 0 = \sin n\pi \; (n \in \mathrm{I})$

$\therefore$ $p = \dfrac{n\pi}{l}.$

Hence from (12), $y(x, t) = \left(c_1 \cos \dfrac{n\pi ct}{l} + c_2 \sin \dfrac{n\pi ct}{l}\right) c_4 \sin \dfrac{n\pi x}{l}$ ...(13)

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left[-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right] c_4 \sin \frac{n\pi x}{l}$$

At $t = 0$,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l}\left[c_2 c_4 \sin \frac{n\pi x}{l}\right]$$

$\Rightarrow$ $c_2 = 0,$

$\therefore$ From (13), $y(x, t) = c_1 c_4 \cos \dfrac{n\pi ct}{l} \sin \dfrac{n\pi x}{l}$ ...(14)

$$y(x, 0) = \mathrm{A} \sin \frac{\pi x}{l} = c_1 c_4 \sin \frac{n\pi x}{l}$$

$\Rightarrow$ $c_1 c_4 = \mathrm{A},\; n = 1.$ | Comparing

Hence from (14), $y(x, t) = \mathrm{A} \cos \dfrac{\pi ct}{l} \sin \dfrac{\pi x}{l}$

which is the required solution.

Example 2. *Show how the wave equation* $c^2 \dfrac{\partial^2 y}{\partial x^2} = \dfrac{\partial^2 y}{\partial t^2}$ *can be solved by the method of separation of variables. If the initial displacement and velocity of a string stretched between* $x = 0$ *and* $x = l$ *are given by* $y = f(x)$ *and* $\dfrac{\partial y}{\partial t} = g(x)$*, determine the constants in the series solution.*

Sol. The wave equation is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(1)

Let $y = \mathrm{XT}$ ...(2)

where X is a function of x only and T is a function of t only.

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}(\mathrm{XT}) = \mathrm{X}\frac{d\mathrm{T}}{dt}$$

$$\frac{\partial^2 y}{\partial t^2} = X \frac{d^2T}{dt^2}$$

Similarly, $$\frac{\partial^2 y}{\partial x^2} = T \frac{d^2X}{dx^2}.$$

Substituting in (1), we get

$$X\frac{d^2T}{dt^2} = c^2 T \frac{d^2X}{dx^2} \quad \Rightarrow \quad XT'' = c^2 TX''$$

$$\Rightarrow \quad \frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} \qquad \text{...(3)}$$

Case I. When $\dfrac{1}{c^2}\dfrac{T''}{T} = \dfrac{X''}{X} = p^2$ (say)

(*i*) $$\frac{1}{c^2}\frac{T''}{T} = p^2 \quad \Rightarrow \quad \frac{d^2T}{dt^2} - p^2c^2T = 0.$$

Auxiliary equation is

$$m^2 - p^2c^2 = 0$$
$$m = \pm pc$$
$$\text{C.F.} = c_1 e^{pct} + c_2 e^{-pct}$$
$$\text{P.I.} = 0$$
$$\therefore \quad T = \text{C.F.} + \text{P.I.} = c_1 e^{pct} + c_2 e^{-pct}$$

(*ii*) $$\frac{X''}{X} = p^2 \quad \Rightarrow \quad \frac{d^2X}{dx^2} - p^2X = 0.$$

Auxiliary equation is

$$m^2 - p^2 = 0$$
$$m = \pm p$$
$$\text{C.F.} = c_3 e^{px} + c_4 e^{-px}$$
$$\text{P.I.} = 0.$$
$$\therefore \quad X = \text{C.F.} + \text{P.I.} = c_3 e^{px} + c_4 e^{-px}.$$

Hence, the solution is

$$y = XT = (c_1 e^{pct} + c_2 e^{-pct})(c_3 e^{px} + c_4 e^{-px}). \qquad \text{...(4)}$$

Case II. When

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = -p^2 \text{ (say)}$$

(*i*) $$\frac{1}{c^2}\frac{T''}{T} = -p^2 \quad \Rightarrow \quad \frac{d^2T}{dt^2} + p^2c^2T = 0.$$

Auxiliary equation is

$$m^2 + p^2c^2 = 0 \quad \Rightarrow \quad m = \pm pci$$
$$\therefore \quad \text{C.F.} = (c_5 \cos pct + c_6 \sin cpt)$$
$$\text{P.I.} = 0.$$
$$\therefore \quad T = \text{C.F.} + \text{P.I.} = c_5 \cos cpt + c_6 \sin cpt$$

(ii) $$\frac{X''}{X} = -p^2 \Rightarrow \frac{d^2X}{dx^2} + p^2X = 0.$$

Auxiliary equation is

$$m^2 + p^2 = 0 \Rightarrow m = \pm pi$$

$\therefore$ $$\text{C.F.} = c_7 \cos px + c_8 \sin px$$

$$\text{P.I.} = 0$$

$\therefore$ $$X = c_7 \cos px + c_8 \sin px.$$

Hence, the solution is

$$y = XT = (c_5 \cos cpt + c_6 \sin cpt)(c_7 \cos px + c_8 \sin px) \quad ...(5)$$

Case III. When, $\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = 0$

(i) $$\frac{1}{c^2}\frac{T''}{T} = 0 \Rightarrow \frac{d^2T}{dt^2} = 0$$

$\Rightarrow$ $$T = c_9 + c_{10}t$$

(ii) $$\frac{X''}{X} = 0 \Rightarrow \frac{d^2X}{dx^2} = 0$$

$\Rightarrow$ $$X = c_{11} + c_{12}x.$$

Hence, the solution is

$$y(x, t) = (c_9 + c_{10}t)(c_{11} + c_{12}x) \quad ...(6)$$

Of the above three solutions given by (4), (5) and (6), we have to choose the solution which is consistent with the physical nature of the problem. Since, we are dealing with a problem on vibrations, y must be a periodic function of x and t therefore the solution must involve trigonometric terms hence solution (5) is the required solution.

Boundary conditions are

$$y(0, t) = 0, \qquad y(l, t) = 0$$

$$y = f(x) \qquad \text{when } t = 0$$

$$\frac{\partial y}{\partial t} = g(x) \qquad \text{when } t = 0$$

From equation (5), $y(0, t) = (c_5 \cos cpt + c_6 \sin cpt)\, c_7$

$$0 = (c_5 \cos cpt + c_6 \sin cpt)\, c_7$$

$\Rightarrow$ $$c_7 = 0.$$

Hence from (5), $$y(x, t) = (c_5 \cos cpt + c_6 \sin cpt)\, c_8 \sin px \quad ...(7)$$

$$y(l, t) = 0 = (c_5 \cos cpt + c_6 \sin cpt)\, c_8 \sin pl$$

$\Rightarrow$ $$\sin pl = 0 = \sin n\pi \;(n \in I) \Rightarrow p = \frac{n\pi}{l}.$$

$\therefore$ From (7), $$y(x, t) = \left(c_5 \cos \frac{n\pi ct}{l} + c_6 \sin \frac{n\pi ct}{l}\right) c_8 \sin \frac{n\pi x}{l} \quad ...(8)$$

$$= \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}$$

where $c_5c_8 = a_n$ and $c_6c_8 = b_n$

The general solution is

$$y(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l} \quad ...(9)$$

$$y(x, 0) = f(x) = \sum_{1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

where $$a_n = \frac{2}{l} \int_0^l f(x) . \sin \frac{n\pi x}{l} \, dx \quad ...(10)$$

From (9), $$\frac{\partial y}{\partial t} = \frac{\pi c}{l} \sum_{1}^{\infty} \left(-n\, a_n \sin \frac{n\pi ct}{l} + n\, b_n \cos \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}$$

At $t = 0$, $$\left(\frac{\partial y}{\partial t} \right)_{t=0} = g(x) = \frac{\pi c}{l} \sum_{1}^{\infty} n b_n \sin \frac{n\pi x}{l}$$

where $$\frac{n\pi c}{l} b_n = \frac{2}{l} \int_0^l g(x) . \sin \frac{n\pi x}{l} \, dx$$

$\Rightarrow$ $$b_n = \frac{2}{n\pi c} \int_0^l g(x) . \sin \frac{n\pi x}{l} \, dx. \quad ...(11)$$

Hence, the required solution is

$$y(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}$$

where $$a_n = \frac{2}{l} \int_0^l f(x) . \sin \frac{n\pi x}{l} \, dx$$

and $$b_n = \frac{2}{n\pi c} \int_0^l g(x) \sin \frac{n\pi x}{l} \, dx.$$

Example 3. *A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3 \frac{\pi x}{l}$. If it is released from rest from this position, find the displacement $y(x, t)$.* [G.B.T.U. (C.O.) 2011]

Sol. The equation of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad ...(1)$$

The solution of eqn. (1) is

$$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px) \quad ...(2)$$

Boundary conditions are | Refer Sol. of Ex. 1

$$y(0, t) = 0 \quad ...(3)$$

$$y(l, t) = 0 \quad ...(4)$$

$$\left(\frac{\partial y}{\partial t} \right)_{t=0} = 0 \quad ...(5)$$

$$y(x, 0) = y_0 \sin^3 \frac{\pi x}{l} \quad ...(6)$$

Applying boundary condition in (2),

$$y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$$

$\Rightarrow$ $$c_3 = 0$$

$\therefore$ From (2), $$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px \quad ...(7)$$

Again, $$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$$

$\Rightarrow$ $$\sin pl = 0 = \sin n\pi \quad (n \in I)$$

$\therefore$ $$p = \frac{n\pi}{l}$$

Hence, from (7),

$$y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l} \quad ...(8)$$

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left[-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right] c_4 \sin \frac{n\pi x}{l}$$

At $t = 0$,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l} c_2 c_4 \sin \frac{n\pi x}{l}$$

$\Rightarrow$ $$c_2 = 0.$$

$\therefore$ From (8),

$$y(x, t) = c_1 c_4 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

Most general solution is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \quad ...(9)$$

$$y(x, 0) = y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$\Rightarrow$ $$y_0 \left(\frac{3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}}{4}\right) = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + \dots$$

Comparing, we get

$$b_1 = \frac{3y_0}{4},\ b_2 = 0,\ b_3 = \frac{-y_0}{4},\ b_4 = b_5 = \dots = 0$$

Hence, from (9),

$$y(x, t) = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l}.$$

Example 4. *A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $F(x) = \mu x(l - x)$, μ is a constant and then released. Find the displacement $y(x, t)$ of any point x of the string at any time $t > 0$.*

Sol. The wave equation is $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{dx^2} \quad ...(1)$$

The solution of equation (1) is

$$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px) \qquad \text{...(2)} \quad \text{(Refer sol. of Ex. 1)}$$

Boundary conditions are

$$y(0, t) = 0 \qquad \text{...(3)}$$

$$y(l, t) = 0 \qquad \text{..(4)}$$

$$\frac{\partial y}{\partial t} = 0 \text{ at } t = 0 \qquad \text{...(5)}$$

and

$$y(x, 0) = \mu x(l - x) \qquad \text{...(6)}$$

From (2), $y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)c_3$

$\Rightarrow$ $c_3 = 0.$

$\therefore$ From (2),

$$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px \qquad \text{...(7)}$$

$$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$$

$\Rightarrow$ $\sin pl = 0 = \sin n\pi \ (n \in \mathrm{I})$

$$p = \frac{n\pi}{l}.$$

From (7),

$$y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l} \qquad \text{...(8)}$$

Now from (7),

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left[-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right] . c_4 \sin \frac{n\pi x}{l}$$

At $t = 0$,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l} c_2 c_4 \sin \frac{n\pi x}{l}$$

$\Rightarrow$ $c_2 = 0.$

$\therefore$ From (8),

$$y(x, t) = c_1 c_4 \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

$\Rightarrow$

$$y(x, t) = b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \quad \text{where } c_1 c_4 = b_n.$$

The most general solution is

$$y(x, t) = \sum_1^{\infty} b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \qquad \text{...(9)}$$

$$y(x, 0) = \mu(lx - x^2) = \sum_1^{\infty} b_n \sin \frac{n\pi x}{l}$$

where

$$b_n = \frac{2}{l} \int_0^l \mu\, (lx - x^2) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2\mu}{l}\left[\left\{(lx - x^2)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_0^l - \int_0^l (l - 2x)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) dx\right]$$

$$= \frac{2\mu}{l}\left[\frac{l}{n\pi} \int_0^l (l - 2x) \cos \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{2\mu}{n\pi}\left[\left\{(l-2x)\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}_0^l - \int_0^l (-2)\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\,dx\right] = \frac{2\mu}{n\pi}\cdot\frac{2l}{n\pi}\int_0^l \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{4\mu l}{n^2\pi^2}\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^l = \frac{4\mu l^2}{n^3\pi^3}(-\cos n\pi + 1) = \frac{4\mu l^2}{n^3\pi^3}[1-(-1)^n].$$

$\therefore$ From (9),

$$y(x, t) = \frac{4\mu l^2}{\pi^3}\sum_1^\infty \frac{[1-(-1)^n]}{n^3}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$$

$$= \frac{8\mu l^2}{\pi^3}\sum_{n=1}^\infty \frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}\cos\frac{(2n-1)\pi ct}{l}.$$

Example 5. *A string is stretched between two fixed points (0, 0) and (l, 0) and released at rest from the initial deflection given by*

and
$$f(x) = \begin{cases} \left(\frac{2k}{l}\right)x, & 0 < x < \frac{l}{2} \\ \left(\frac{2k}{l}\right)(l-x), & \frac{l}{2} < x < l \end{cases}$$

Find the deflection of the string at any time.

Sol. The equation for the vibrations of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

The solution of eqn. (1) is

$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px)$ ...(2) [Refer Sol. of Ex. 1]

Boundary conditions are, $y(0, t) = 0$, $y(l, t) = 0$

$$\frac{\partial y}{\partial t} = 0 \quad \text{at} \quad t = 0$$

$$y(x, 0) = \begin{cases} \frac{2k}{l}x, & 0 < x < \frac{l}{2} \\ \frac{2k}{l}(l-x), & \frac{l}{2} < x < l \end{cases}$$

From (2), $y(0, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$

$0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$

$\Rightarrow$ $c_3 = 0.$

$\therefore$ From (2), $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px$ ...(3)

$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$

$\Rightarrow$ $\sin pl = 0 = \sin n\pi\ ;\ n \in I$

$$p = \frac{n\pi}{l}.$$

$\therefore$ From (3), $$y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l} \quad \text{...(4)}$$

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left[-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l}$$

At $t = 0$, $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l}\left[c_2 c_4 \sin \frac{n\pi x}{l}\right]$$

$\Rightarrow$ $c_2 = 0$.

$\therefore$ From (4), $$y(x, t) = c_1 c_4 \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

$$= b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \qquad (\text{where } c_1 c_4 = b_n) \quad \text{...(5)}$$

The most general solution is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \quad \text{...(6)}$$

$$y(x, 0) = \sum_{1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad [\text{From (6)}]$$

where $$b_n = \frac{2}{l}\int_0^l y(x, 0) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} \frac{2k}{l} x \sin \frac{n\pi x}{l}\, dx + \int_{l/2}^{l} \frac{2k}{l}(l - x) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{4k}{l^2}\left[\int_0^{l/2} x \sin \frac{n\pi x}{l}\, dx + \int_{l/2}^{l} (l - x) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{4k}{l^2}\left[x\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^{l/2} - \int_0^{l/2} 1 . \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) dx\right.$$

$$\left. + \left\{(l - x)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_{l/2}^{l} - \int_{l/2}^{l} (-1) . \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) dx\right]$$

$$= \frac{4k}{l^2}\left[-\frac{l}{n\pi} . \frac{l}{2} \cos \frac{n\pi}{2} + \frac{l}{n\pi}\left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^{l/2} + \frac{l}{2} . \frac{l}{n\pi} \cos \frac{n\pi}{2} - \frac{l}{n\pi}\left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_{l/2}^{l}\right]$$

$$= \frac{4k}{l^2}\left[\frac{l^2}{n^2\pi^2} \sin \frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}\left(\sin n\pi - \sin \frac{n\pi}{2}\right)\right]$$

$$= \frac{4k}{l^2}\left[\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

$\therefore$ From (6), $$y(x, t) = \frac{8k}{\pi^2}\sum_{1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}.$$

Example 6. *A tightly stretched violin string of length l and fixed at both ends is plucked at $x = \frac{l}{3}$ and assumes initially the shape of a triangle of height a. Find the displacement y at any distance x and any time t after the string is released from rest.*

Sol. One-Dimensional wave equation is

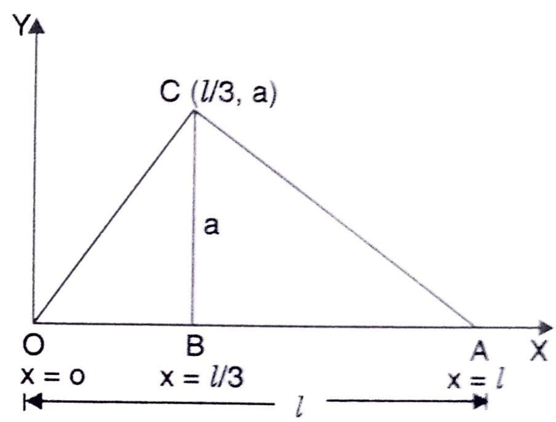
$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \quad ...(1)$$

The solution of eqn. (1) is

$$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px) \quad ...(2) \quad \text{(Refer Sol. of Ex. 1)}$$

Eqn. of line OC is $$y - 0 = \frac{a-0}{\frac{l}{3}-0}(x-0)$$

$$y = \frac{3a}{l}x \quad ...(3)$$



Eqn. of line CA is $$y - a = \frac{0-a}{l - l/3}\left(x - \frac{l}{3}\right)$$

$$y - a = \frac{-a}{\left(\frac{2l}{3}\right)}\left(x - \frac{l}{3}\right) = -\frac{3a}{2l}\left(x - \frac{l}{3}\right)$$

$$y - a = -\frac{3ax}{2l} + \frac{a}{2}$$

$$y = -\frac{3ax}{2l} + \frac{3a}{2} = \frac{3a}{2}\left(1 - \frac{x}{l}\right) \quad ...(4)$$

Hence the boundary conditions are

$$y(0, t) = 0 \quad ...(5)$$

$$y(l, t) = 0 \quad ...(6)$$

$$\frac{\partial y}{\partial t} = 0 \text{ at } t = 0 \quad ...(7)$$

and $$y(x, 0) = \begin{cases} \frac{3ax}{l}, & 0 < x < l/3 \\ \frac{3a}{2}\left(1 - \frac{x}{l}\right), & \frac{l}{3} < x < l \end{cases} \quad ...(8)$$

From (2), $y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$

$\Rightarrow$ $c_3 = 0.$

$\therefore$ From (2), $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px$ ...(9)

$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$

$\Rightarrow \quad \sin pl = 0 = \sin n\pi (n \in I).$

$\Rightarrow \quad p = \dfrac{n\pi}{l}.$

$\therefore$
$$y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l} \quad ...(10)$$

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left[-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right] c_4 \sin \frac{n\pi x}{l}.$$

At $t = 0$,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l}\left[c_2 c_4 \sin \frac{n\pi x}{l}\right]$$

$\Rightarrow \quad c_2 = 0.$

$\therefore$
$$y(x, t) = c_1 c_4 \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} = b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}.$$

The most general solution is

$$y(x, t) = \sum_1^{\infty} b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \quad ...(11)$$

From (11), $\quad y(x, 0) = \displaystyle\sum_1^{\infty} b_n \sin \frac{n\pi x}{l}$, where

$$b_n = \frac{2}{l}\int_0^l y(x, 0) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[\int_0^{l/3} \frac{3ax}{l} \sin \frac{n\pi x}{l}\, dx + \int_{l/3}^{l} \frac{3a}{2}\left(1 - \frac{x}{l}\right) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{2}{l}\left[\frac{3a}{l}\int_0^{l/3} x \sin \frac{n\pi x}{l}\, dx + \frac{3a}{2}\int_{l/3}^{l}\left(1 - \frac{x}{l}\right) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{6a}{l^2}\left[\left(x\left\{\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}\right)_0^{l/3} - \int_0^{l/3} 1.\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) dx\right]$$

$$+ \frac{3a}{l}\left[\left(\left(1 - \frac{x}{l}\right)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right)_{l/3}^{l} - \int_{l/3}^{l}\left(-\frac{1}{l}\right)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) dx\right]$$

$$= \frac{6a}{l^2}\left[-\frac{l}{n\pi}\cdot\frac{l}{3}\cos \frac{n\pi}{3} + \frac{l}{n\pi}\left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^{l/3}\right] + \frac{3a}{l}\left[\frac{l}{n\pi}\cdot\frac{2}{3}\cos \frac{n\pi}{3} - \frac{1}{n\pi}\cdot\left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_{l/3}^{l}\right]$$