$$= \frac{6a}{l^2}\left[-\frac{l^2}{3n\pi}\cos\frac{n\pi}{3}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3}\right]+\frac{3a}{l}\left[\frac{2l}{3n\pi}\cos\frac{n\pi}{3}-\frac{l}{n^2\pi^2}\left(0-\sin\frac{n\pi}{3}\right)\right]$$

$$= \frac{6a}{n\pi}\left[\frac{-1}{3}\cos\frac{n\pi}{3}+\frac{1}{n\pi}\sin\frac{n\pi}{3}\right]+\frac{6a}{n\pi}\left[\frac{1}{3}\cos\frac{n\pi}{3}\right]+\frac{3a}{n^2\pi^2}\sin\frac{n\pi}{3}$$

$$\Rightarrow \qquad b_n = \frac{9a}{n^2\pi^2}\sin\frac{n\pi}{3}$$

$\therefore$ From (11), $$y(x,t) = \frac{9a}{\pi^2}\sum_{1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}.$$

Example 7. *The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid-point of the string always remains at rest.*

Sol. The equation for the vibration of the string is

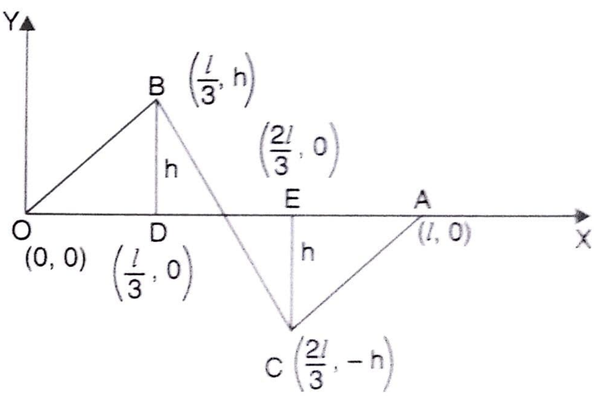
$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad \text{...(1)}$$

The solution of eqn. (1) is

$$y(x,t) = (c_1\cos cpt + c_2\sin cpt)(c_3\cos px + c_4\sin px) \qquad \text{...(2)}$$

Let l be the length of string (Refer Sol. of Ex. 1)

Eqn. of OB is,



$$y - 0 = \frac{h-0}{\frac{l}{3}-0}(x-0)$$

$$\Rightarrow \qquad y = \frac{3h}{l}x \qquad \text{...(3)}$$

Eqn. of BC is,

$$y - h = \frac{-h-h}{\frac{2l}{3}-\frac{l}{3}}\left(x-\frac{l}{3}\right)$$

$$= \frac{-2h}{\left(\frac{l}{3}\right)}\left(x-\frac{l}{3}\right) = -\frac{6h}{l}\left(x-\frac{l}{3}\right)$$

$$y - h = -\frac{6hx}{l} + 2h$$

$$y = 3h - \frac{6hx}{l} = 3h\left(1-\frac{2x}{l}\right) \qquad \text{...(4)}$$

Eqn. of CA is, $$y + h = \frac{0+h}{l-\frac{2l}{3}}\left(x-\frac{2l}{3}\right) = \frac{3h}{l}\left(x-\frac{2l}{3}\right) = \frac{3hx}{l} - 2h$$

$$y = \frac{3hx}{l} - 3h = 3h\left(\frac{x}{l}-1\right) \qquad \text{...(5)}$$

Hence Boundary conditions are

$$y(0, t) = 0, \; y(l, t) = 0$$

$$\frac{\partial y}{\partial t} = 0 \qquad \text{when } t = 0$$

and
$$y(x, 0) = \begin{cases} \dfrac{3h}{l} x, & 0 \le x \le l/3 \\ \dfrac{3h}{l}(l - 2x), & \dfrac{l}{3} \le x \le \dfrac{2l}{3} \\ \dfrac{3h}{l}(x - l), & \dfrac{2l}{3} \le x \le l \end{cases}$$

From (2), $y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$

$\Rightarrow$ $c_3 = 0.$

$\therefore$ From (2),

$$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px \qquad ...(6)$$

$$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$$

$\Rightarrow$ $\sin pl = 0 = \sin n\pi \; (n \in \text{I})$

$\therefore$ $p = \dfrac{n\pi}{l}.$

$\therefore$ From (6),
$$y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l} \qquad ...(7)$$

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left(-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l}.$$

At $t = 0$,
$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l} c_2 c_4 \sin \frac{n\pi x}{l}$$

$\Rightarrow$ $c_2 = 0$

$\therefore$ From (7),
$$y(x, t) = c_1 c_4 \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} = b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}.$$

The most general solution is

$$y(x, t) = \sum_1^\infty b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \qquad ...(8)$$

$$y(x, 0) = \sum_1^\infty b_n \sin \frac{n\pi x}{l}, \text{ where}$$

$$b_n = \frac{2}{l}\int_0^l y(x, 0) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[\int_0^{l/3} \frac{3h}{l} x \sin \frac{n\pi x}{l}\, dx + \int_{l/3}^{2l/3} \frac{3h}{l}(l - 2x) \sin \frac{n\pi x}{l}\, dx + \int_{2l/3}^{l} \frac{3h}{l}(x - l) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{2}{l} \cdot \frac{3h}{l} \int_0^{l/3} x \sin \frac{n\pi x}{l} \, dx + \frac{2}{l} \cdot \frac{3h}{l} \int_{l/3}^{2l/3} (l - 2x) \sin \frac{n\pi x}{l} \, dx + \frac{2}{l} \cdot \frac{3h}{l} \int_{2l/3}^{l} (x - l) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{6h}{l^2} \left[\left\{ x \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) \right\}_0^{l/3} - \int_0^{l/3} 1 . \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) dx \right]$$

$$+ \frac{6h}{l^2} \left[\left\{ (l - 2x) \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) \right\}_{l/3}^{2l/3} - \int_{l/3}^{2l/3} (-2) \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) dx \right]$$

$$+ \frac{6h}{l^2} \left[\left\{ (x - l) \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) \right\}_{2l/3}^{l} - \int_{2l/3}^{l} 1 . \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) dx \right]$$

$$= \frac{6h}{l^2} \left[\frac{-l}{n\pi} \cdot \frac{l}{3} \cos \frac{n\pi}{3} + \frac{l}{n\pi} \left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right)_0^{l/3} \right]$$

$$+ \frac{6h}{l^2} \left[\frac{-l}{3} \cdot \frac{l}{n\pi} \left(-\cos \frac{2n\pi}{3} \right) + \left(\cos \frac{n\pi}{3} \right) \frac{l}{3} \cdot \frac{l}{n\pi} - \frac{2l}{n\pi} \left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right)_{l/3}^{2l/3} \right]$$

$$+ \frac{6h}{l^2} \left[\frac{-l}{3} \cdot \frac{l}{n\pi} \cos \frac{2n\pi}{3} + \frac{l}{n\pi} \left(\frac{\sin n\pi x/l}{n\pi/l} \right)_{2l/3}^{l} \right]$$

$$= \frac{-2h}{n\pi} \cos \frac{n\pi}{3} + \frac{6h}{n^2\pi^2} \sin \frac{n\pi}{3} + \frac{2h}{n\pi} \cos \frac{2n\pi}{3}$$

$$+ \frac{2h}{n\pi} \cos \frac{n\pi}{3} - \frac{12h}{n^2\pi^2} \left(\sin \frac{2n\pi}{3} - \sin \frac{n\pi}{3} \right) - \frac{2h}{n\pi} \cos \frac{2n\pi}{3} + \frac{6h}{n^2\pi^2} \left(0 - \sin \frac{2n\pi}{3} \right)$$

$$= \frac{18h}{n^2\pi^2} \sin \frac{n\pi}{3} - \frac{18h}{n^2\pi^2} \sin \frac{2n\pi}{3} = \frac{18h}{n^2\pi^2} \sin \frac{n\pi}{3} - \frac{18h}{n^2\pi^2} \sin \left(n\pi - \frac{n\pi}{3} \right)$$

$$= \frac{18h}{n^2\pi^2} \sin \frac{n\pi}{3} + \frac{18h}{n^2\pi^2} \sin \frac{n\pi}{3} \cos n\pi$$

$$= \begin{cases} \frac{36h}{n^2\pi^2} \sin \frac{n\pi}{3}, & \text{when } n \text{ is even} \\ 0, & \text{when } n \text{ is odd} \end{cases}$$

$\therefore$ From (8),
$$y(x, t) = \frac{36h}{\pi^2} \sum_{n = 2, 4, \ldots}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{3} \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

$$y(x, t) = \frac{9h}{\pi^2} \sum_{m=1,2,\ldots}^{\infty} \frac{1}{m^2} \sin \frac{2m\pi}{3} \cos \frac{2m\pi ct}{l} \sin \frac{2m\pi x}{l} \quad \ldots(9)$$

(where $n = 2m$)

Putting $x = \frac{l}{2}$ in eqn. (6), we get

$$y\left(\frac{l}{2}, t\right) = \frac{9h}{\pi^2} \sum_{m=1}^{\infty} \sin\left(\frac{2m\pi}{3}\right) \cdot \frac{1}{m^2} \cdot \cos \frac{2m\pi ct}{l} \cdot \sin m\pi = 0.$$

Hence, midpoint of the string is always at rest.

Example 8. *If a string of length l is initially at rest in equilibrium position and each of its points is given the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = b \sin^3 \frac{\pi x}{l}$, find the displacement $y(x, t)$.*

Sol. The equation for the vibrations of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \ldots(1)$$

The solution of equation (1) is

$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px)$ ...(2) [Refer Sol. of Ex. 1]

Boundary conditions are,

$$y(0, t) = 0 \quad \ldots(3)$$

$$y(l, t) = 0 \quad \ldots(4)$$

$$y(x, 0) = 0 \quad \ldots(5)$$

$$\left(\frac{\partial y}{\partial t}\right) = b \sin^3 \frac{\pi x}{l} \text{ at } t = 0 \quad \ldots(6)$$

From eqn. (2), $y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$

$\Rightarrow$ $c_3 = 0.$

$\therefore$ From (2), $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px$ ...(7)

$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$

$\Rightarrow$ $\sin pl = 0 = \sin n\pi \ (n \in \text{I})$

$\therefore$ $p = \frac{n\pi}{l}.$

$\therefore$ From (7), $y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l}$ ...(8)

$y(x, 0) = 0 = c_1 c_4 \sin \frac{n\pi x}{l}$

$\Rightarrow$ $c_1 = 0.$

$\therefore$ From (8), $y(x, t) = c_2 c_4 \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$

$$= b_n \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \quad \text{where } c_2c_4 = b_n$$

The general solution is

$$y(x, t) = \sum_{1}^{\infty} b_n \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \qquad \text{...(9)}$$

$$\frac{\partial y}{\partial t} = \sum_{1}^{\infty} b_n \cdot \frac{n\pi c}{l} \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

At $t = 0$,
$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{1}^{\infty} b_n \cdot \frac{n\pi c}{l} \sin \frac{n\pi x}{l}$$

$$b \sin^3 \frac{\pi x}{l} = \sum_{1}^{\infty} b_n \cdot \frac{n\pi c}{l} \sin \frac{n\pi x}{l}$$

$$\frac{b}{4}\left[3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right] = b_1 \frac{\pi c}{l} \sin \frac{\pi x}{l} + \frac{2b_2\pi c}{l} \sin \frac{2\pi x}{l} + 3b_3 \frac{\pi c}{l} \sin \frac{3\pi x}{l} + \ldots$$

$$\Rightarrow \quad b_1 \frac{\pi c}{l} = \frac{3b}{4} \quad \Rightarrow \quad b_1 = \frac{3bl}{4\pi c}$$

$$b_2 = 0 \quad \text{and} \quad \frac{3b_3\pi c}{l} = -\frac{b}{4} \quad \Rightarrow \quad b_3 = -\frac{bl}{12\pi c}$$

Also, $b_4 = 0 = b_5 = \ldots$ etc.

Hence from (9),
$$y(x, t) = \frac{3bl}{4\pi c} \sin \frac{\pi ct}{l} \sin \frac{\pi x}{l} - \frac{bl}{12\pi c} \sin \frac{3\pi ct}{l} \sin \frac{3\pi x}{l}$$
$$= \frac{bl}{12\pi c}\left[9 \sin \frac{\pi x}{l} \sin \frac{\pi ct}{l} - \sin \frac{3\pi x}{l} \sin \frac{3\pi ct}{l}\right].$$

Example 9. *A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points an initial velocity $\lambda x(l - x)$, find the displacement of the string at any distance x from one end at any time t.* [M.T.U. (SUM) 2011]

Sol. Here the boundary conditions are $y(0, t) = y(l, t) = 0$

$$y(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l} \qquad \text{...(1)} \quad | \text{ Refer Sol. of Ex. 2}$$

Since the string was at rest initially, $y(x, 0) = 0$

$\therefore$ From (1),
$$0 = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \quad \Rightarrow \quad a_n = 0$$

$\therefore$
$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \qquad \text{...(2)}$$

and
$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \frac{n\pi c}{l} b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} = \frac{\pi c}{l} \sum_{n=1}^{\infty} nb_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

But
$$\frac{\partial y}{\partial t} = \lambda x(l - x) \quad \text{when } t = 0$$

$$\therefore \quad \lambda x(l-x) = \frac{\pi c}{l} \sum_{n=1}^{\infty} n b_n \sin \frac{n\pi x}{l}$$

$$\Rightarrow \quad \frac{\pi c}{l} n b_n = \frac{2}{l} \int_0^l \lambda x(l-x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2\lambda}{l}\left[x(l-x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (l-2x)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{4\lambda l^2}{n^3\pi^3}(1-\cos n\pi) = \frac{4\lambda l^2}{n^3\pi^3}[1-(-1)^n]$$

$$= \begin{cases} 0, & \text{when } n \text{ is even} \\ \dfrac{8\lambda l^2}{n^3\pi^3}, & \text{when } n \text{ is odd} \end{cases} \quad i.e., \quad \frac{8\lambda l^2}{\pi^3 (2m-1)^3}, \text{ taking } n = 2m-1$$

$$\Rightarrow \quad b_n = \frac{8\lambda l^3}{c\pi^4 (2m-1)^4}$$

$\therefore$ From (2), the required solution is

$$y(x, t) = \frac{8\lambda l^3}{c\pi^4} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^4} \sin \frac{(2m-1)\pi ct}{l} \sin \frac{(2m-1)\pi x}{l}.$$

Example 10. *Transform the equation* $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ *to its normal form using the transformation* $u = x + ct$, $v = x - ct$ *and hence solve it. Show that the solution may be put in the form* $y = \dfrac{1}{2}[f(x+ct) + f(x-ct)]$. (M.T.U. 2013)

Assume initial conditions $y = f(x)$ *and* $(\partial y/\partial t) = 0$ *at* $t = 0$.

Sol. One-dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

Let us introduce two new independent variables

$$u = x + ct \qquad ...(2)$$

and

$$v = x - ct \qquad ...(3)$$

so that y becomes a function of u and v.

Then,
$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \qquad ...(4) \quad \text{[Using (2) and (3)]}$$

Also,
$$\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \qquad ...(5)$$

$$\therefore \quad \frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right)$$

$$= \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \qquad ...(6)$$

Also, $$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial y}{\partial v} \cdot \frac{\partial v}{\partial t} = c\,\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}(-c) = c\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right) \qquad ...(7)$$

$$\Rightarrow \qquad \frac{\partial}{\partial t} \equiv c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right) \qquad ...(8)$$

$$\therefore \qquad \frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right) = c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right) c\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right)$$

$$= c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right) \qquad ...(9)$$

From (1), (6) and (9), we have

$$c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right) = c^2\left(\frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right)$$

$$\Rightarrow \qquad -4c^2 \frac{\partial^2 y}{\partial u \partial v} = 0$$

$$\Rightarrow \qquad \frac{\partial^2 y}{\partial u \partial v} = 0 \qquad ...(10) \quad (\because \; c^2 \neq 0)$$

Integrating eqn. (10) partially, w.r.t. u, we get

$$\frac{\partial y}{\partial v} = f_1(v).$$

Integrating again w.r.t. v partially, we get

$$y = \int f_1(v)\, \partial v + \psi(u) = \phi(v) + \psi(u)$$

$$\Rightarrow \qquad y(x, t) = \phi(x - ct) + \psi(x + ct) \qquad ...(11)$$

which is *d' Alembert's* solution of wave equation.

Applying initial conditions $y = f(x)$ and $\frac{\partial y}{\partial t} = 0$ at $t = 0$ in (11), we get

$$f(x) = \phi(x) + \psi(x) \text{ and } 0 = -\phi'(x) + \psi'(x)$$

Hence, $$\phi(x) = \psi(x) = \frac{1}{2} f(x)$$

$$\therefore \qquad y = \frac{1}{2}[f(x + ct) + f(x - ct)].$$

Example 11. *A tightly stretched string with fixed end points $x = 0$ and $x = \pi$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points an initial velocity* $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0.03 \sin x - 0.04 \sin 3x$

then find the displacement $y(x, t)$ at any point of string at any time t.

Sol. The equation for the vibrations of a string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

Its solution is

$$y(x, t) = (c_1 \cos pct + c_2 \sin pct)(c_3 \cos px + c_4 \sin px) \qquad ...(2)$$

Boundary conditions are

$$y(0, t) = 0 = y(\pi, t)$$

$$y(x, 0) = 0$$

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0.03 \sin x - 0.04 \sin 3x.$$

From (2), $$y(0, t) = 0 = (c_1 \cos pct + c_2 \sin pct)\, c_3$$

$\Rightarrow$ $$c_3 = 0$$

From (2), $$y(x, t) = (c_1 \cos pct + c_2 \sin pct)\, c_4 \sin px \qquad \text{...(3)}$$

$$y(\pi, t) = 0 = (c_1 \cos pct + c_2 \sin pct)\, c_4 \sin p\pi$$

$\Rightarrow$ $$\sin p\pi = 0 = \sin n\pi \ (n \in \mathrm{I})$$

$\Rightarrow$ $$p = n$$

From (3), $$y(x, t) = (c_1 \cos nct + c_2 \sin nct)\, c_4 \sin nx \qquad \text{...(4)}$$

$$y(x, 0) = 0 = c_1 c_4 \sin nx$$

$\Rightarrow$ $$c_1 = 0.$$

$\therefore$ From (4), $$y(x, t) = c_2 c_4 \sin nct \sin nx = b_n \sin nct \sin nx \qquad \text{...(5)}$$

where $$c_2 c_4 = b_n$$

The most general solution is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin nct \sin nx$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} nc\, b_n \cos nct \sin nx$$

At $t = 0$,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} nc\, b_n \sin nx$$

$$0.03 \sin x - 0.04 \sin 3x = cb_1 \sin x + 2cb_2 \sin 2x + 3cb_3 \sin 3x + \ldots$$

$\Rightarrow$ $$cb_1 = 0.03 \ \Rightarrow\ b_1 = \frac{0.03}{c}$$

$$b_2 = 0$$

and $$3cb_3 = -0.04 \ \Rightarrow\ b_3 = \frac{-0.0133}{c}.$$

$\therefore$ From (6), $$y(x, t) = \frac{0.03}{c} \sin ct \sin x - \frac{0.0133}{c} \sin 3ct \sin 3x$$

$$= \frac{1}{c}\,[0.03 \sin x \sin ct - 0.0133 \sin 3x \sin 3ct].$$