3D Data Processing
Linear Algebra

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Glossary

- **Vector**: 3-vector, 4-vector, column/row vector
- **Matrix**: Rows and columns
- **Matrix computation**: Arithmetic, 교환/결합/분배법칙
- **Identity, transpose, inverse**
- **Square, pseudo-inverse**
- **Upper/lower triangle, symmetric, diagonal**
- **Span, rank, null space, range**
- **Eigen value, eigen vector**
- **Orthogonal, orthonormal**
- **SVD, Cholesky decomposition**

Linear system

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\boldsymbol{Ax} = \boldsymbol{b}$$

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}_{2\times \mathrm{n}}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}_{2\times 1}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}_{\mathrm{n}\times 1}$$

A linear system is a collection of one or more linear equations involving the same variables $(x_1, \ldots, x_n)$.

Linear system

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

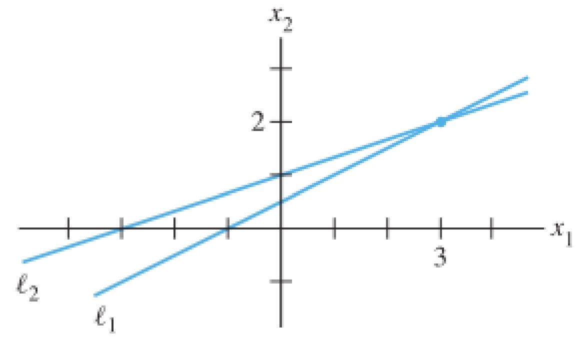


FIGURE 1 Exactly one solution.

(a) $\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$ (b) $\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$

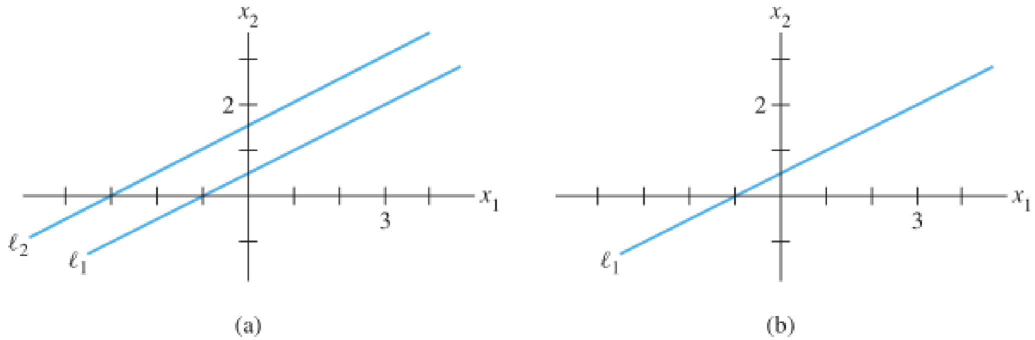


FIGURE 2 (a) No solution. (b) Infinitely many solutions.

Vector

- Column vector

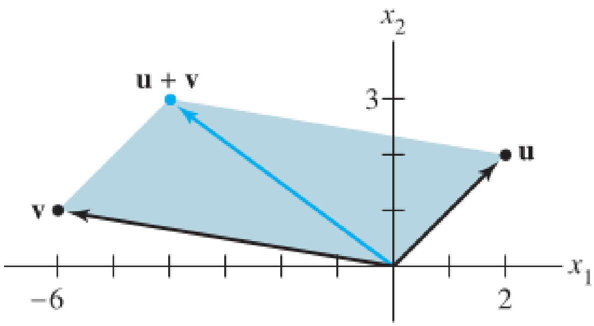
$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- Row vector

$$\boldsymbol{y} = [y_1, y_2, y_3]$$

- $[-3, 1] \neq \begin{bmatrix} -3 \\ 1 \end{bmatrix}$
- $[-3, 1]^T = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$
- 3-vector: $[x_1, x_2, x_3]^T$
- 4-vector: $[y_1, y_2, y_3, y_4]^T$

Vector



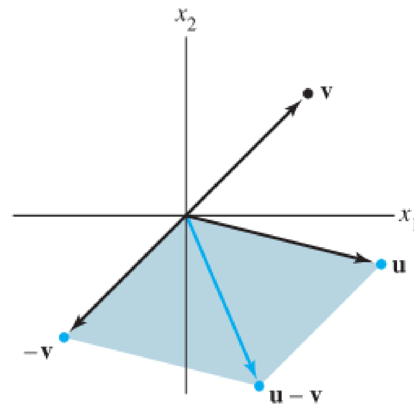


FIGURE 7
Vector subtraction.

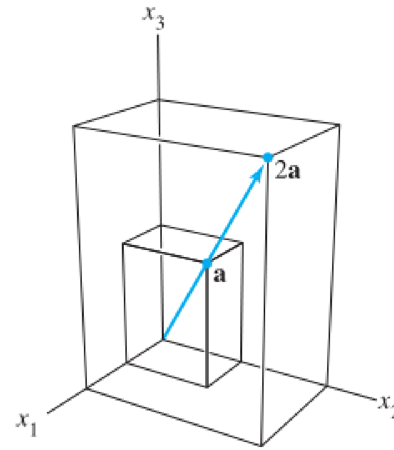
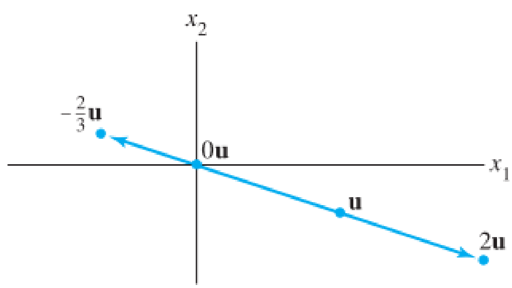
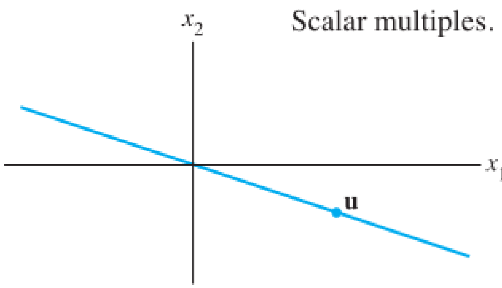


FIGURE 6
Scalar multiples.



Typical multiples of **u**



The set of all multiples of **u**

For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(iv) $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$,
where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(v) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(vi) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(vii) $c(d\mathbf{u})=(cd)\mathbf{u}$

(viii) $1\mathbf{u}=\mathbf{u}$

Linear combinations, Span

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with **weights** $c_1, \ldots, c_p$

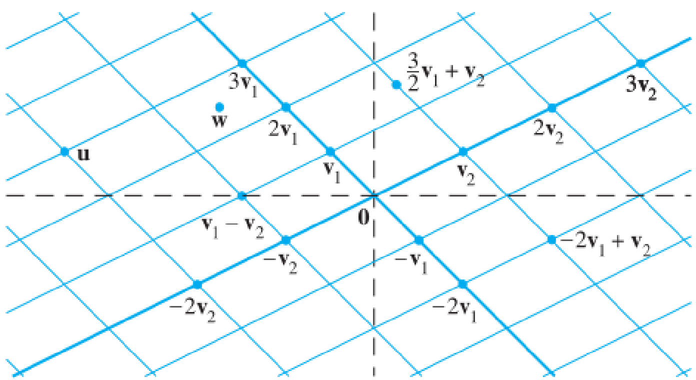


FIGURE 8 Linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$.

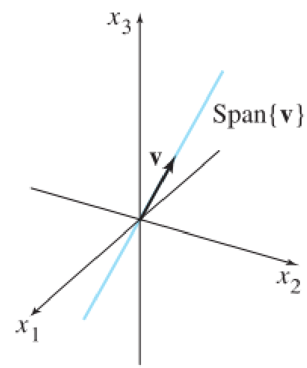


FIGURE 10 Span $\{\mathbf{v}\}$ as a line through the origin.

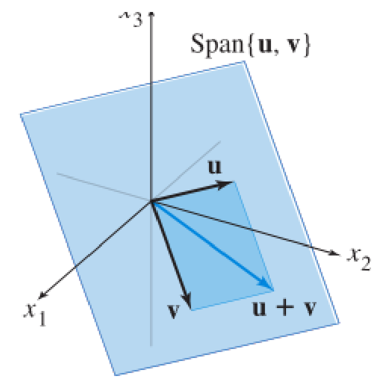


FIGURE 11 Span $\{\mathbf{u}, \mathbf{v}\}$ as a plane through the origin.

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned** (or **generated**) **by** $\mathbf{v}_1, \ldots, \mathbf{v}_p$. That is, Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

with $c_1, \ldots, c_p$ scalars.

Matrix

- m×n matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

- m rows(행) and n columns(열)

- i-th row vector: $[a_{i1} \quad \cdots \quad a_{in}]$, $i = 1, \ldots, m$

- j-th column vector: $\begin{bmatrix} a_{1j} \\ \cdots \\ a_{mj} \end{bmatrix}$, $j = 1, \ldots, n$

Ax=b

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of A and $\mathbf{x}$**, denoted by $A\mathbf{x}$, is **the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights**; that is,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b} \tag{4}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \tag{5}$$

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

- Is $\boldsymbol{b}$ in $Span\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\}$?

Ax=b

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbb{R}^m$.

- Matrix-Vector product Ax

If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:

a. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;

b. $A(c\mathbf{u}) = c(A\mathbf{u})$.

Homogeneous linear systems

- Ax=0 (zero vector)
- Trivial solution: x=0
- Nontrivial solution: a nonzero vector x that satisfies Ax=0

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

Linear independence

An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{2}$$

EXAMPLE 1 Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

a. Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

b. If possible, find a linear dependence relation among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Linear independence

The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution. (3)

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

If a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

Linear transformation

For instance, the equations

$$\underbrace{\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 5 \\ 8 \end{bmatrix}}_{\mathbf{b}} \quad \text{and} \quad \underbrace{\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\mathbf{0}}$$

say that multiplication by A transforms $\mathbf{x}$ into $\mathbf{b}$ and transforms $\mathbf{u}$ into the zero vector. See Figure 1.

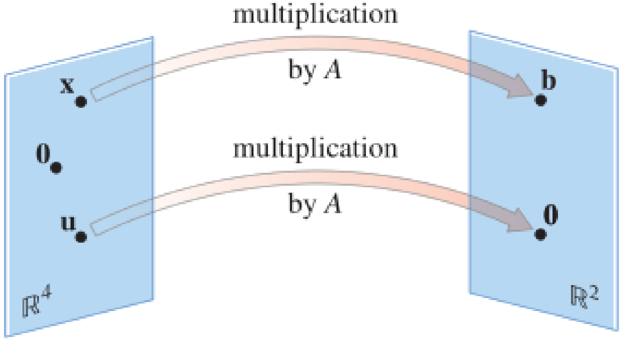


FIGURE 1 Transforming vectors via matrix multiplication.

Linear transformation

A **transformation** (or **function** or **mapping**) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the **domain** of T, and $\mathbb{R}^m$ is called the **codomain** of T. The notation $T : \mathbb{R}^n \to \mathbb{R}^m$ indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$. For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the **image** of $\mathbf{x}$ (under the action of T). The set of all images $T(\mathbf{x})$ is called the **range** of T. See Figure 2.

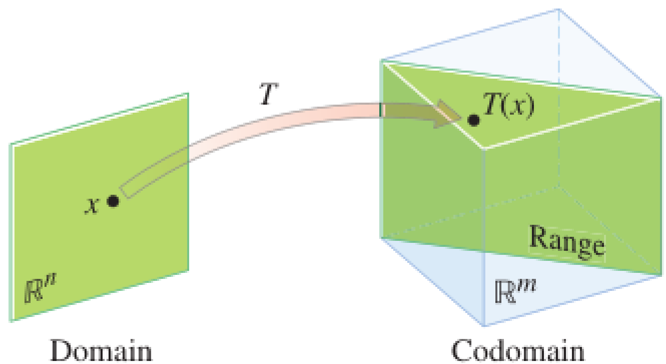


FIGURE 2 Domain, codomain, and range of $T : \mathbb{R}^n \to \mathbb{R}^m$.

Matrix transformation example

sheep

sheared sheep

EXAMPLE 3 Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is called a **shear transformation**. It can be shown that if T acts on each point in the 2×2 square shown in Figure 4, then the set of images forms the shaded parallelogram. The key idea is to show that T maps line segments onto line segments (as shown in Exercise 27) and then to check that the corners of the square map onto the vertices of the parallelogram. For instance, the image of the point $\mathbf{u} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ is $T(\mathbf{u}) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$, and the image of $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$. T deforms the square as if the top of the square were pushed to the right while the base is held fixed. Shear transformations appear in physics, geology, and crystallography. ■

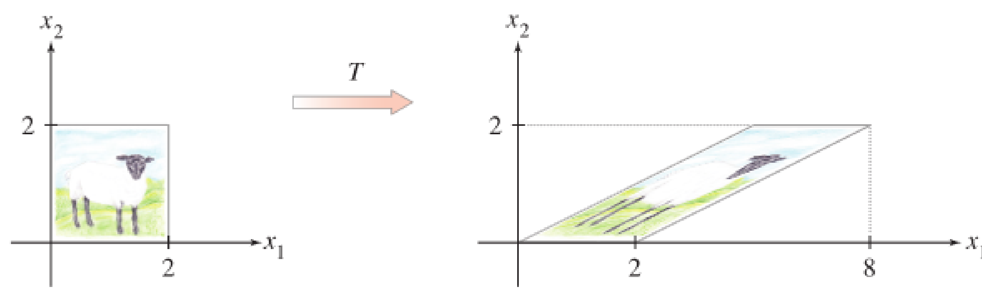


FIGURE 4 A shear transformation.

Linear transformation

A transformation (or mapping) T is **linear** if:

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T;

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T.

- Every matrix transformation is a linear transformation.

If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0} \tag{3}$$

and

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v}) \tag{4}$$

for all vectors $\mathbf{u}, \mathbf{v}$ in the domain of T and all scalars c, d.

$$T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p)$$

Geometric linear transformations of $\mathbb{R}^2$

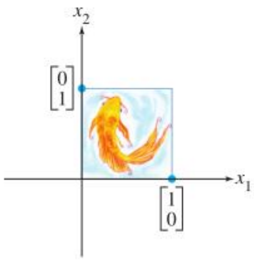


FIGURE 2
The unit square.

TABLE 1 Reflections

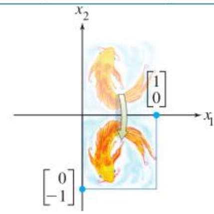
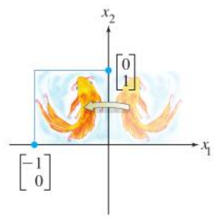
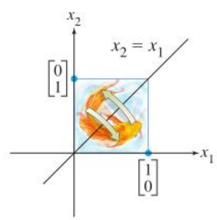
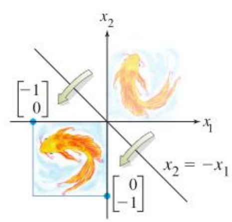
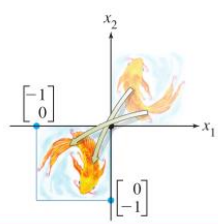
Transformation	Image of the Unit Square	Standard Matrix
Reflection through the x_1-axis		$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection through the x_2-axis		$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection through the line $x_2 = x_1$		$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Reflection through the line $x_2 = -x_1$		$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
Reflection through the origin		$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

TABLE 2 Contractions and Expansions

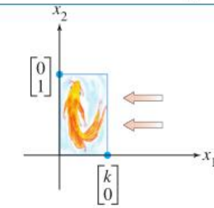
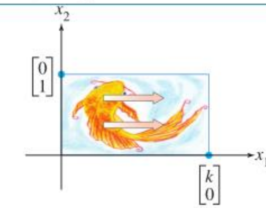
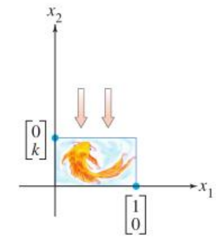
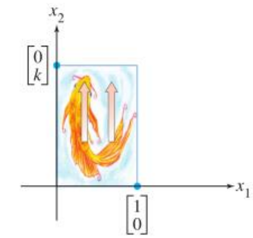
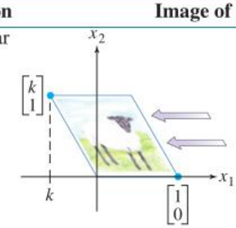
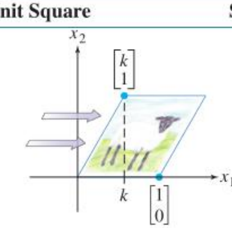
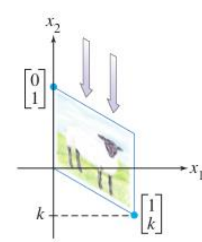
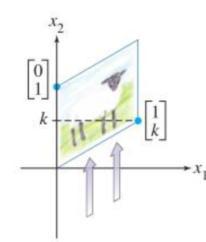
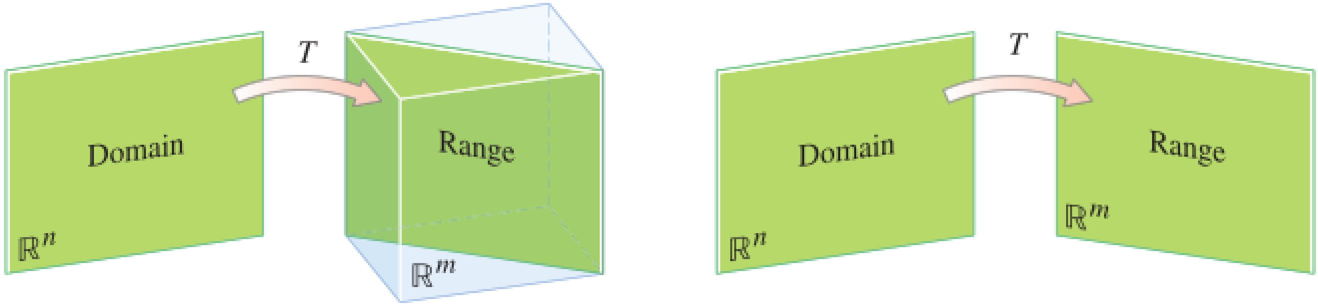
Transformation	Image of the Unit Square		Standard Matrix
Horizontal contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Vertical contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

TABLE 3 Shears

Transformation	Image of the Unit Square		Standard Matrix
Horizontal shear	 $k < 0$	 $k > 0$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Vertical shear	 $k < 0$	 $k > 0$	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Mapping properties: Existence and uniqueness

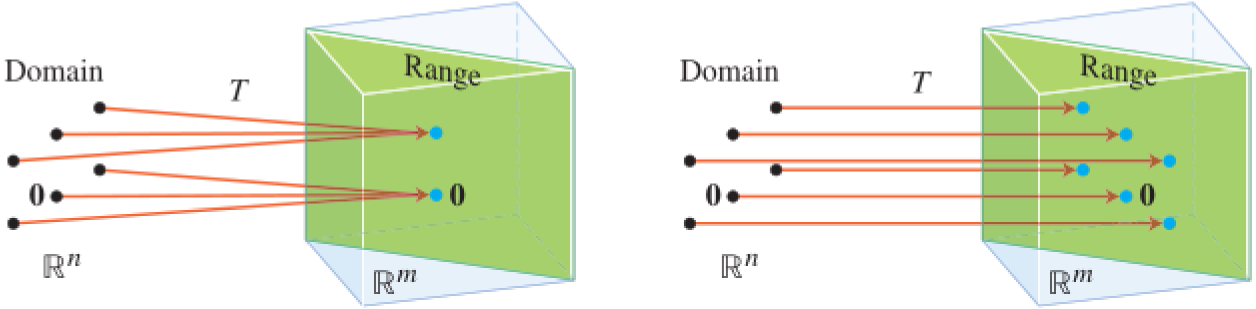
A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at least one* $\mathbf{x}$ in $\mathbb{R}^n$.



T is *not* onto $\mathbb{R}^m$ | T is onto $\mathbb{R}^m$

FIGURE 3 Is the range of T all of $\mathbb{R}^m$?

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at most one* $\mathbf{x}$ in $\mathbb{R}^n$.



T is *not* one-to-one | T is one-to-one

FIGURE 4 Is every $\mathbf{b}$ the image of at most one vector?

Mapping properties

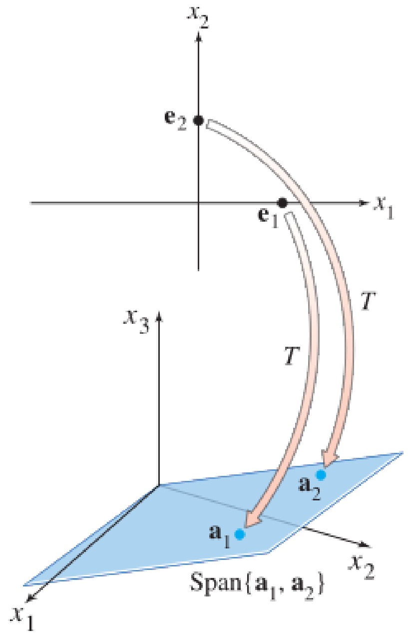
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then:

a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$;

b. T is one-to-one if and only if the columns of A are linearly independent.

One-to-one but not onto example



The transformation T is not onto $\mathbb{R}^3$.

EXAMPLE 5 Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show that T is a one-to-one linear transformation. Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$?

SOLUTION When $\mathbf{x}$ and $T(\mathbf{x})$ are written as column vectors, you can determine the standard matrix of T by inspection, visualizing the row–vector computation of each entry in $A\mathbf{x}$.

$$T(\mathbf{x}) = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4}$$

So T is indeed a linear transformation, with its standard matrix A shown in (4). The columns of A are linearly independent because they are not multiples. By Theorem 12(b), T is one-to-one. To decide if T is onto $\mathbb{R}^3$, examine the span of the columns of A. Since A is 3×2, the columns of A span $\mathbb{R}^3$ if and only if A has 3 pivot positions, by Theorem 4. This is impossible, since A has only 2 columns. So the columns of A do not span $\mathbb{R}^3$, and the associated linear transformation is not onto $\mathbb{R}^3$. ■

Matrix

- m×n matrix

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

- m rows(행) and n columns(열)

- i-th row vector: $[a_{i1} \quad \cdots \quad a_{in}]$, $i = 1, \ldots, m$

- j-th column vector: $\begin{bmatrix} a_{1j} \\ \cdots \\ a_{mj} \end{bmatrix}$, $j = 1, \ldots, n$

Matrix operation

- Sum and scalar multiples: element-wise
- $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$
- $r\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix}$

Let A, B, and C be matrices of the same size, and let r and s be scalars.

a. $A + B = B + A$
b. $(A + B) + C = A + (B + C)$
c. $A + 0 = A$
d. $r(A + B) = rA + rB$
e. $(r + s)A = rA + sA$
f. $r(sA) = (rs)A$

a. 교환법칙 성립

b.f. 결합법칙 성립

d.e. 분배법칙 성립

Matrix multiplication

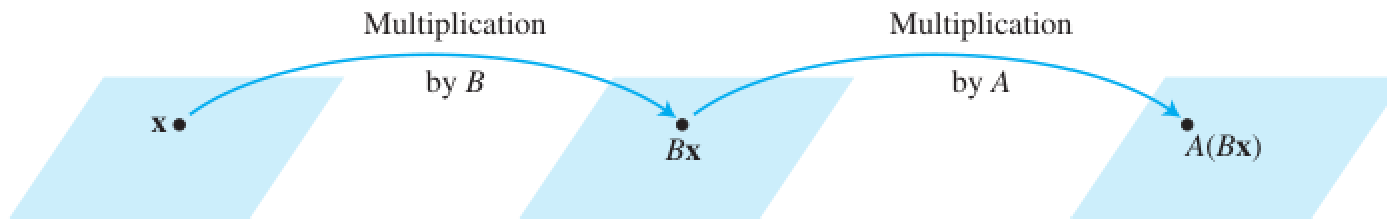


FIGURE 2 Multiplication by B and then A.

$$A(B\mathbf{x}) = (AB)\mathbf{x}$$

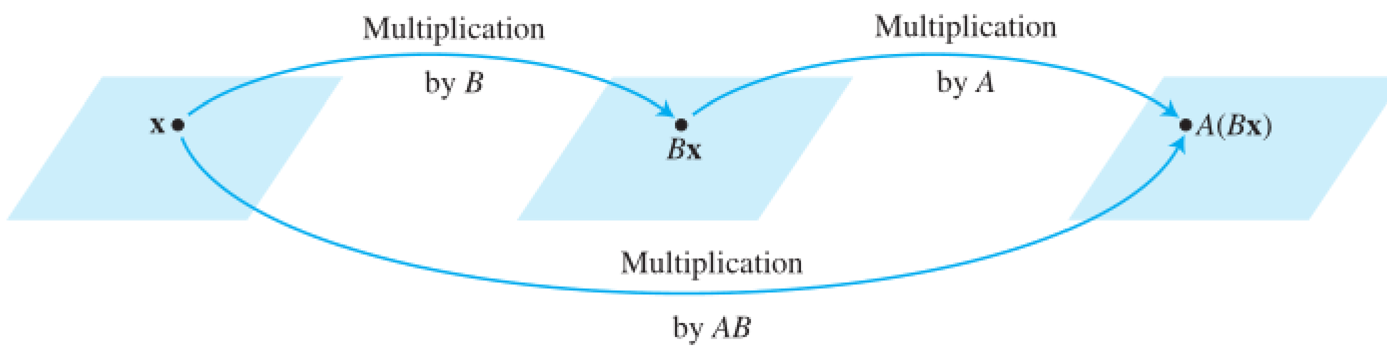


FIGURE 3 Multiplication by AB.

If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$. That is,

$$AB = A\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}$$

Matrix multiplication

$$\overset{A}{\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}} \overset{B}{\begin{bmatrix} * & * \\ * & * \\ * & * \\ * & * \\ * & * \end{bmatrix}} = \overset{AB}{\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}}$$

$$3 \times 5 \qquad 5 \times 2 \qquad 3 \times 2$$

Match

Size of AB

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} \square & \square & 2(6)+3(3) \\ \square & \square & \square \end{bmatrix} = \begin{bmatrix} \square & \square & 21 \\ \square & \square & \square \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} \square & \square & 21 \\ \square & 1(3)+-5(-2) & \square \end{bmatrix} = \begin{bmatrix} \square & \square & 21 \\ \square & 13 & \square \end{bmatrix}$$

Matrix multiplication, Identity matrix

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B + C) = AB + AC$ (left distributive law)

c. $(B + C)A = BA + CA$ (right distributive law)

d. $r(AB) = (rA)B = A(rB)$ for any scalar r

e. $I_m A = A = A I_n$ (identity for matrix multiplication)

WARNINGS:

1. In general, $AB \neq BA$.

2. The cancellation laws do *not* hold for matrix multiplication. That is, if $AB = AC$, then it is *not* true in general that $B = C$. (See Exercise 10.)

3. If a product AB is the zero matrix, you *cannot* conclude in general that either $A = 0$ or $B = 0$. (See Exercise 12.)

Transpose

EXAMPLE 8 Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 5 & -2 & 7 \end{bmatrix}$$

Then

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad B^T = \begin{bmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 & -3 \\ 1 & 5 \\ 1 & -2 \\ 1 & 7 \end{bmatrix}$$ ■

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$

b. $(A + B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$

Inverse, determinant

An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C such that

$$CA = I \quad \text{and} \quad AC = I$$

where $I = I_n$, the $n \times n$ identity matrix. In this case, C is an **inverse** of A. In fact, C is uniquely determined by A, because if B were another inverse of A, then $B = BI = B(AC) = (BA)C = IC = C$. This unique inverse is denoted by A^{-1}, so that

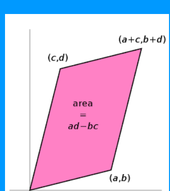
$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

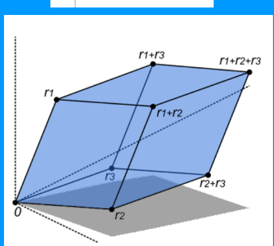
A matrix that is *not* invertible is sometimes called a **singular matrix**, and an invertible matrix is called a **nonsingular matrix**.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.





The simple proof of Theorem 4 is outlined in Exercises 25 and 26. The quantity $ad - bc$ is called the **determinant** of A, and we write

$$\det A = ad - bc$$

Invertible matrix

a. If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

b. If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

c. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,

$$(A^T)^{-1} = (A^{-1})^T$$

The product of $n \times n$ invertible matrices is invertible, and the inverse is the product of their inverses in the reverse order.

Invertible matrix theorem

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

Vector space

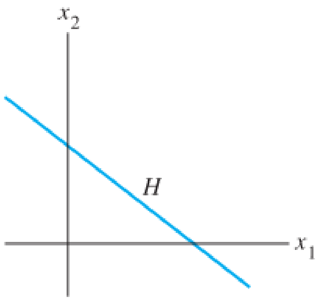


FIGURE 8
A line that is not a vector space.

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below.[1] The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

For each $\mathbf{u}$ in V and scalar c,

$$0\mathbf{u}=\mathbf{0}$$
$$c\mathbf{0}=\mathbf{0}$$
$$-\mathbf{u}=(-1)\mathbf{u}$$

Subspace

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.[2]

b. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.

c. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in a vector space V, then Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a subspace of V.

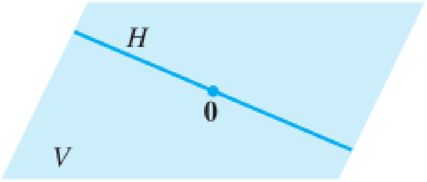


FIGURE 6
A subspace of V.

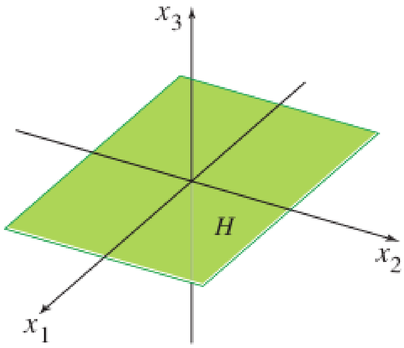


FIGURE 7
The x_1x_2-plane as a subspace of $\mathbb{R}^3$.

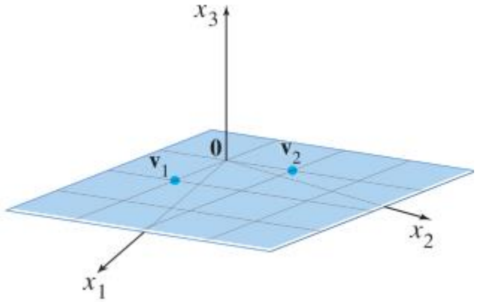


FIGURE 9
An example of a subspace.

Null space

The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$

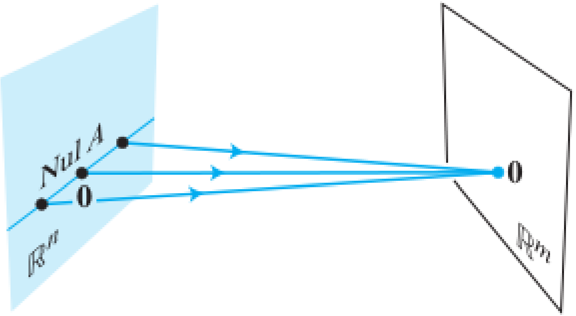


FIGURE 1

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Column space

The **column space** of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A. If $A = [\, \mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n \,]$, then

$$\text{Col } A = \text{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$$

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^m$.

Nul A and Col A

Contrast Between Nul *A* and Col *A* for an *m* x *n* Matrix *A*

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	**1.** Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	**2.** Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\,A \quad \mathbf{0}\,]$ are required.	**3.** It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	**4.** There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \quad \mathbf{v}\,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

Kernel and Range

A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in V and all scalars c.

- The kernel (= null space) of T is the set of all u in V such that T(u)=0 (the zero vector in W).
- The range (=column space) of T is the set of all vectors in W of the form T(x) for some x in V.

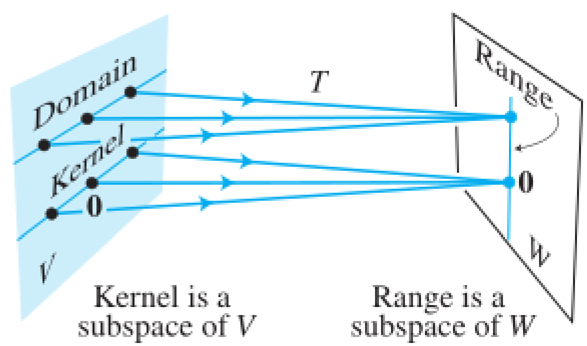


FIGURE 2 Subspaces associated with a linear transformation.

Linearly independent sets: Bases

An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V is said to be **linearly independent** if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \qquad (1)$$

has *only* the trivial solution, $c_1 = 0, \ldots, c_p = 0$.[1]

The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if (1) has a nontrivial solution,

An indexed set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** for H if

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) the subspace spanned by $\mathcal{B}$ coincides with H; that is,

$$H = \text{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$$

Coordinate systems

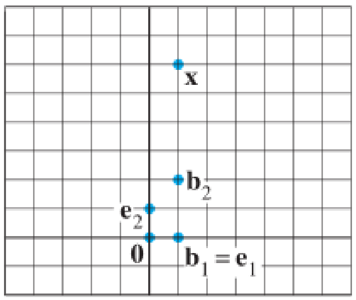


FIGURE 1 Standard graph paper.

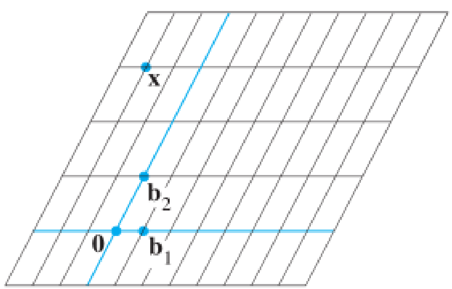


FIGURE 2 $\mathcal{B}$-graph paper.

The Unique Representation Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \dots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n \tag{1}$$

Suppose $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for V and $\mathbf{x}$ is in V. The **coordinates of x relative to the basis $\mathcal{B}$** (or the **$\mathcal{B}$-coordinates of x**) are the weights $c_1, \dots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

If $c_1, \dots, c_n$ are the $\mathcal{B}$-coordinates of $\mathbf{x}$, then the vector in $\mathbb{R}^n$

$$[\,\mathbf{x}\,]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the **coordinate vector of x (relative to $\mathcal{B}$)**, or the **$\mathcal{B}$-coordinate vector of x**. The mapping $\mathbf{x} \mapsto [\,\mathbf{x}\,]_{\mathcal{B}}$ is the **coordinate mapping (determined by $\mathcal{B}$)**.[1]

Coordinate mapping

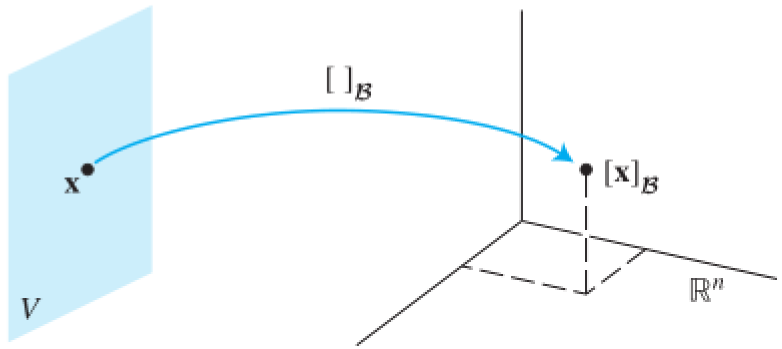


FIGURE 5 The coordinate mapping from V onto $\mathbb{R}^n$.

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

Vector space dimension

If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

If V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written as $\dim V$, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.

The Basis Theorem

Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

Rank

The **rank** of A is the dimension of the column space of A.

The Rank Theorem

The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n$$

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.

n. $\operatorname{Col} A = \mathbb{R}^n$

o. $\dim \operatorname{Col} A = n$

p. $\operatorname{rank} A = n$

q. $\operatorname{Nul} A = \{\mathbf{0}\}$

r. $\dim \operatorname{Nul} A = 0$

Eigenvalues and eigenvectors

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.[1]

If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then A is invertible if and only if:

s. The number 0 is *not* an eigenvalue of A.

t. The determinant of A is *not* zero.

Diagonalization

EXAMPLE 1 If $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$, then $D^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}$

and

$$D^3 = DD^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix} = \begin{bmatrix} 5^3 & 0 \\ 0 & 3^3 \end{bmatrix}$$

In general,

$$D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} \quad \text{for } k \geq 1$$

EXAMPLE 2 Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a formula for A^k, given that $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

In general, for $k \geq 1$,

$$A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 5^k - 3^k & 5^k - 3^k \\ 2 \cdot 3^k - 2 \cdot 5^k & 2 \cdot 3^k - 5^k \end{bmatrix}$$

Diagonalization

The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

Inner product

- Dot product (내적)

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

then the inner product of **u** and **v** is

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Let **u**, **v**, and **w** be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

b. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

d. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

Vector length

The **length** (or **norm**) of $\mathbf{v}$ is the nonnegative scalar $\|\mathbf{v}\|$ defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}, \quad \text{and} \quad \|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}$$

- A vector whose length is 1 is called a unit vector.

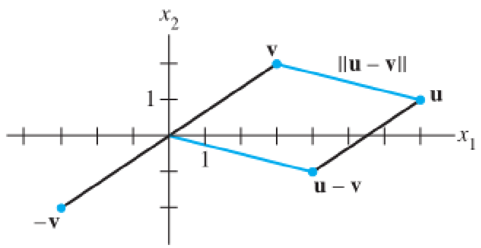


FIGURE 4 The distance between $\mathbf{u}$ and $\mathbf{v}$ is the length of $\mathbf{u} - \mathbf{v}$.

For $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the **distance between u and v**, written as $\text{dist}(\mathbf{u}, \mathbf{v})$, is the length of the vector $\mathbf{u} - \mathbf{v}$. That is,

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

Orthogonal vectors

Two vectors **u** and **v** in $\mathbb{R}^n$ are **orthogonal** (to each other) if $\mathbf{u} \cdot \mathbf{v} = 0$.

The Pythagorean Theorem

Two vectors **u** and **v** are orthogonal if and only if $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

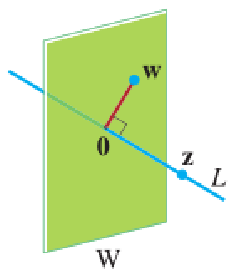


FIGURE 7
A plane and line through **0** as orthogonal complements.

$$L = W^{\perp} \quad \text{and} \quad W = L^{\perp}$$

$W^{\perp}$ (and read as "W perpendicular" or simply "W perp").

1. A vector **x** is in $W^{\perp}$ if and only if **x** is orthogonal to every vector in a set that spans W.
2. $W^{\perp}$ is a subspace of $\mathbb{R}^n$.

Orthogonal sets

A set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is said to be an **orthogonal set** if each pair of distinct vectors from the set is orthogonal, that is, if $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$.

If $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

- Orthogonal projection

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

$$\boxed{\hat{\mathbf{y}} = \text{proj}_L\, \mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u}}$$

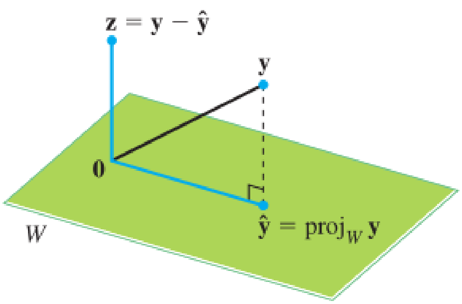


FIGURE 2
Finding α to make $\mathbf{y} - \hat{\mathbf{y}}$ orthogonal to $\mathbf{u}$.

Orthonormal sets

A set $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an **orthonormal set** if it is an orthogonal set of unit vectors. If W is the subspace spanned by such a set, then $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an **orthonormal basis** for W

- An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$.

  Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then

  a. $\|U\mathbf{x}\| = \|\mathbf{x}\|$

  b. $(U\mathbf{x})\cdot(U\mathbf{y}) = \mathbf{x}\cdot\mathbf{y}$

  c. $(U\mathbf{x})\cdot(U\mathbf{y}) = 0$ if and only if $\mathbf{x}\cdot\mathbf{y} = 0$

- An orthogonal matrix is a square invertible matrix U such that $U^{-1} = U^T$. Such a matrix has orthonormal columns. Any square matrix with orthonormal columns is an orthogonal matrix. An orthogonal matrix has orthonormal rows, too.

Least-squares

- When Ax=b has no solution?
- Let's find the best approximation!

If A is $m \times n$ and $\mathbf{b}$ is in $\mathbb{R}^m$, a **least-squares solution** of $A\mathbf{x} = \mathbf{b}$ is an $\hat{\mathbf{x}}$ in $\mathbb{R}^n$ such that

$$\|\mathbf{b} - A\hat{\mathbf{x}}\| \leq \|\mathbf{b} - A\mathbf{x}\|$$

for all $\mathbf{x}$ in $\mathbb{R}^n$.

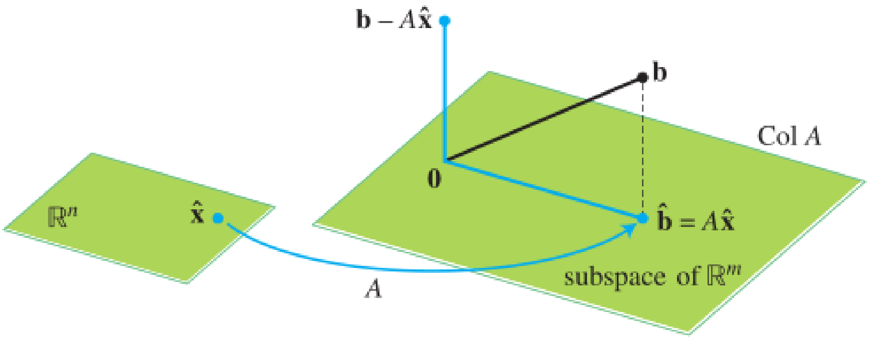


FIGURE 2 The least-squares solution $\hat{\mathbf{x}}$ is in $\mathbb{R}^n$.

Pseudo-inverse

The set of least-squares solutions of $A\mathbf{x} = \mathbf{b}$ coincides with the nonempty set of solutions of the normal equations $A^TA\mathbf{x} = A^T\mathbf{b}$.

Let A be an $m \times n$ matrix. The following statements are logically equivalent:

a. The equation $A\mathbf{x} = \mathbf{b}$ has a unique least-squares solution for each $\mathbf{b}$ in $\mathbb{R}^m$.

b. The columns of A are linearly independent.

c. The matrix A^TA is invertible.

When these statements are true, the least-squares solution $\hat{\mathbf{x}}$ is given by

$$\hat{\mathbf{x}} = \underline{(A^TA)^{-1}}A^T\mathbf{b} \tag{4}$$

When a least-squares solution $\hat{\mathbf{x}}$ is used to produce $A\hat{\mathbf{x}}$ as an approximation to $\mathbf{b}$, the distance from $\mathbf{b}$ to $A\hat{\mathbf{x}}$ is called the **least-squares error** of this approximation.

Rotation Matrix

- A Rotation matrix is an orthonormal matrix with det =+1
  - 2D Rotations $$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$
  - 3D Rotations along the main axes

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \quad R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

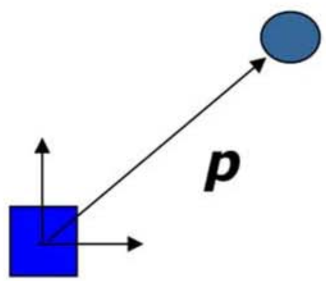
- **IMPORTANT: Rotations are not commutative**

$$R_x(\frac{\pi}{4}) \cdot R_y(\frac{\pi}{4}) = \begin{bmatrix} 0.707 & 0 & -0.707 \\ -0.5 & 0.707 & -0.5 \\ 0.5 & 0.707 & 0.5 \end{bmatrix}, \; R_x(\frac{\pi}{4}) \cdot R_y(\frac{\pi}{4}) \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1.414 \\ 0.586 \\ 3.414 \end{bmatrix}$$

$$R_y(\frac{\pi}{4}) \cdot R_x(\frac{\pi}{4}) = \begin{bmatrix} 0.707 & -0.5 & -0.5 \\ 0 & 0.707 & -0.707 \\ 0.707 & 0.5 & 0.5 \end{bmatrix}, \; R_y(\frac{\pi}{4}) \cdot R_x(\frac{\pi}{4}) \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1.793 \\ 0.707 \\ 3.207 \end{bmatrix}$$

Combining Transformations

- A simple interpretation: chaining of transformations (represented as homogeneous matrices)
  - Matrix ***A*** represents the pose of a **robot** in the space
  - Matrix ***B*** represents the position of a sensor on the robot
  - The **sensor** perceives an **object** at a given location ***p***, in its own frame [the sensor has no clue on where it is in the world]
  - Where is the object in the global frame?



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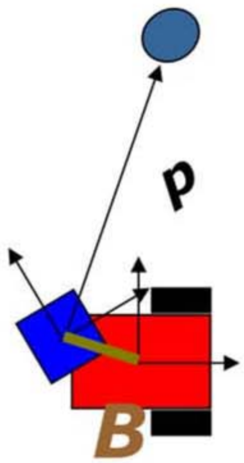

Bp gives the pose of the object wrt the robot

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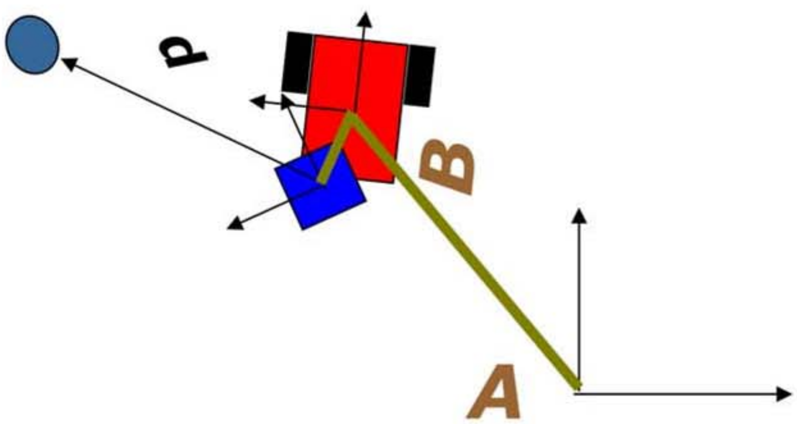


Bp gives the pose of the object wrt the robot

ABp gives the pose of the object wrt the world

Positive Definite Matrix

- The analogous of positive number

- Definition $M > 0$ iff $z^T M z > 0 \forall z \neq 0$

- Example

  - $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = z_1^2 + z_2^2 > 0$

Positive Definite Matrix

- Properties
  - **Invertible**, with positive definite inverse
  - All real **eigenvalues** > 0
  - **Trace** is > 0
  - **Cholesky** decomposition $A = LL^T$

Jacobian Matrix

- It is a **non-square matrix** $n \times m$ in general
- Given a vector-valued function

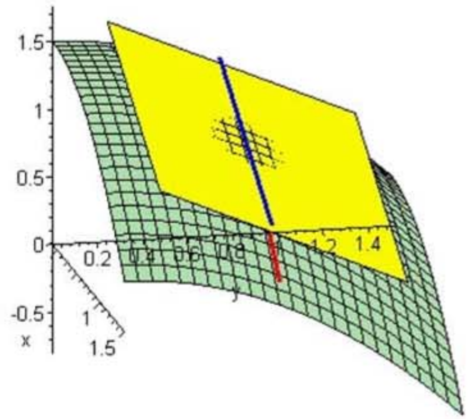
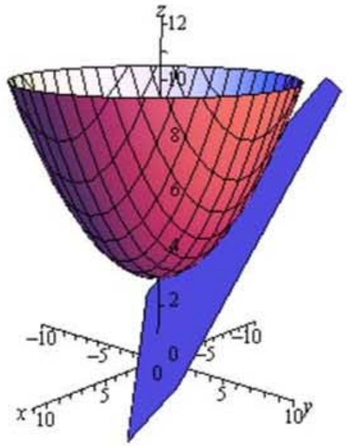
$$f(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{bmatrix}$$

- Then, the **Jacobian matrix** is defined as

$$\mathbf{F_x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Jacobian Matrix

- It is the orientation of the **tangent plane** to the vector-valued function at a given point

- **Generalizes the gradient** of a scalar valued function