3D Data Processing

Homogeneous Coordinate

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Many slides from Probabilistic Robotics

Motivation

- Cameras generate a projected image of the world
- Euclidian geometry is suboptimal to describe the central projection
- In Euclidian geometry, the math can get difficult
- Projective geometry is an alternative algebraic representation of geometric objects and transformations
- Math becomes simpler

Projective Geometry

- A spatial mapping and representation method used to project spatial relationships in N+1 dimensions onto a plane in N dimensions.
  - Possible to represent point and line at infinity, which are impossible to represent with Euclidian Geometry.
  - Preserve geometric relationship when applying Projective transformation (Perspective transformation, 3D→ 2D)

- Projective geometry does not change the geometric relations
- Computations can also be done in Euclidian geometry (but more difficult)

  - Euclidian geometry → Cartesian coordinates
  - Projective geometry → Homogeneous coordinates

Coordinates systems

- The unique Representation Theorem

The Unique Representation Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n \tag{1}$$

- There are various coordinate systems
  - Euclidian, Polar, Cylindrical, Spherical coordinate systems
- Important thing
  - Center is not determined

Homogeneous Coordinates

- Homogeneous Coordinates are a system of coordinates used in projective geometry
- Formulas involving Homogeneous Coordinates are often simpler than in the Cartesian world
- Points at infinity can be represented using finite coordinates
- A single matrix can represent affine transformations and projective transformations

Homogeneous Coordinates

- Definition
  - The representation x of a geometric object is homogeneous if x and λx represent the same object for $\lambda \neq 0$
- Example

$$\mathbf{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} wx \\ wy \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

- From Homogeneous to Euclidian Coordinates

homogeneous

$$\mathbf{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} wx \\ wy \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

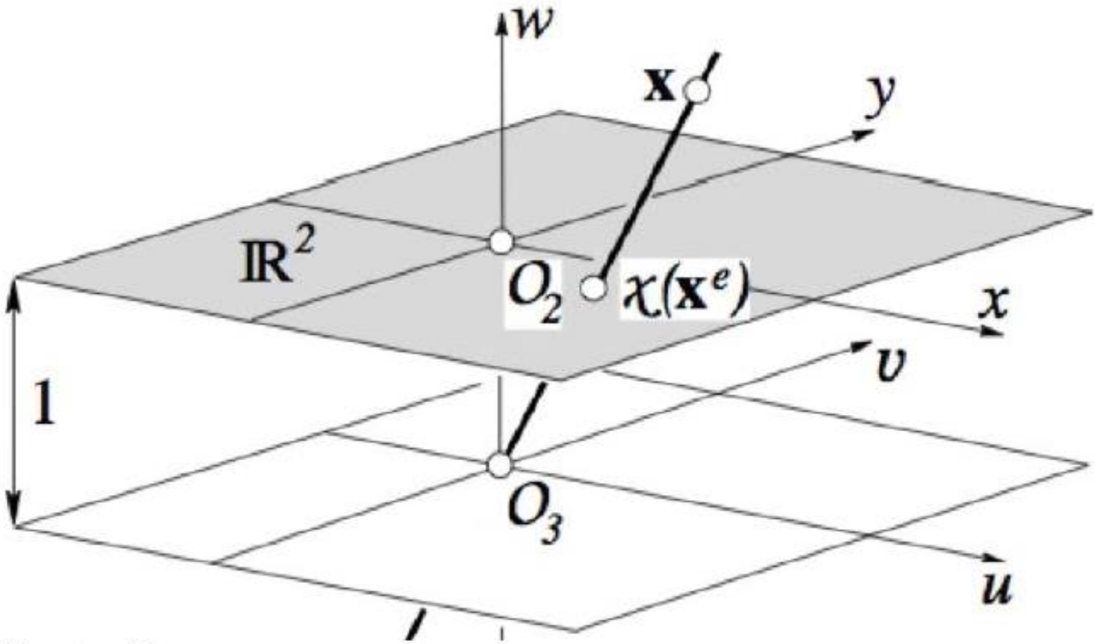
Euclidian

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u/w \\ v/w \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} u/w \\ v/w \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Homogeneous Coordinates

- From Homogeneous to Euclidian Coordinates

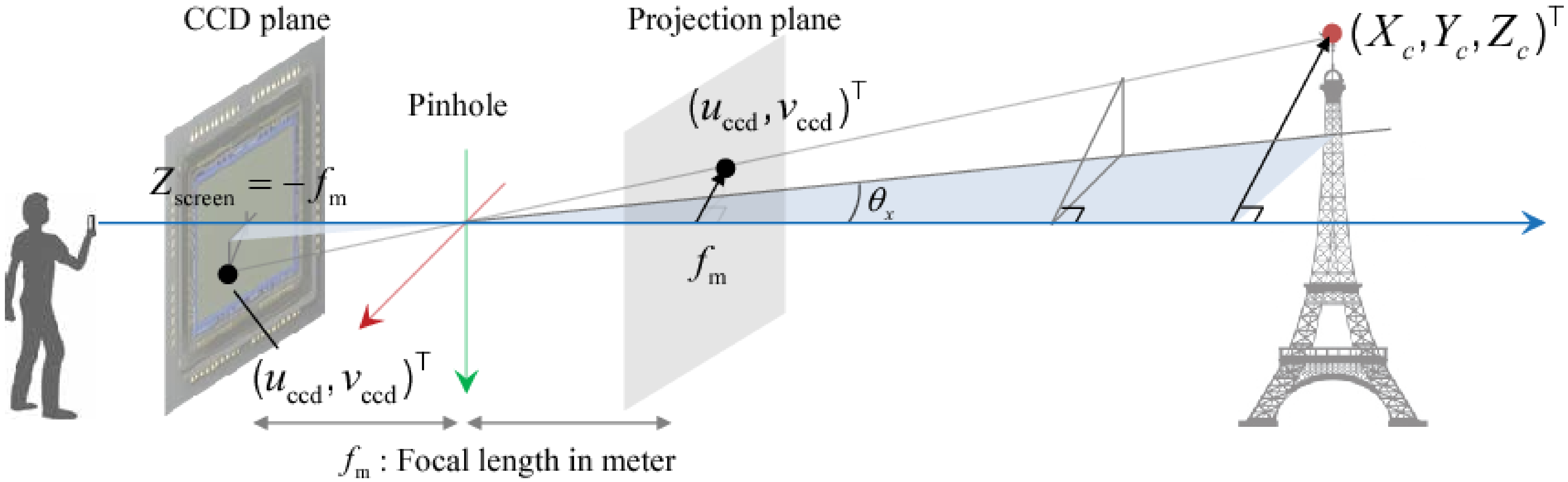


$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u/w \\ v/w \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} u/w \\ v/w \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

[Courtesy by K. Schindler]

Homogeneous Coordinates

- Example) camera projection



Homogeneous Coordinates

- Center of the Coordinate System

$$\mathbf{O}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{O}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Infinitively Distant Objects

- It is possible to explicitly model infinitively distant points with finite coordinates

$$\mathbf{x}_{\infty} = \begin{bmatrix} u \\ v \\ 0 \end{bmatrix}$$

3D points

- Analogous for 3D points

$$\text{homogeneous} \qquad\qquad\qquad \text{Euclidian}$$

$$\mathbf{x} = \begin{bmatrix} u \\ v \\ w \\ t \end{bmatrix} = \begin{bmatrix} u/t \\ v/t \\ w/t \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} u/t \\ v/t \\ w/t \end{bmatrix}$$

Transformations

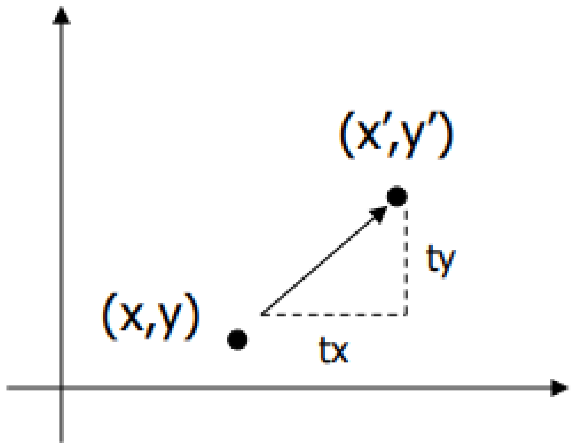
- A projective transformation is a invertible linear mapping

$$\mathbf{x}' = M\mathbf{x}$$

Transformations in 2D

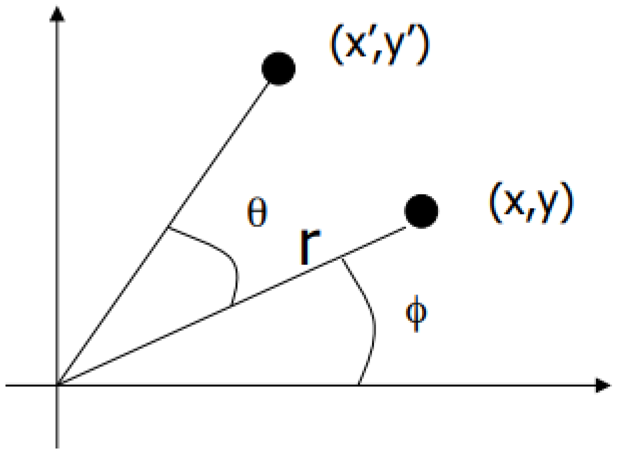
- Translation
  - Re-position a point along a straight line
  - Given a point (x,y), and the translation distance (tx,ty)
  - The new point: (x′, y′)
    - x′ = x + tx
    - y′ = y + ty
  - Using homogeneous Coordinate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



Transformations in 2D

- Rotation
  - Default rotation center: Origin (0,0)
  - Given a point (x,y), and rotate θ deg (C.C.W)
  - The new point: (x′, y′)
    - $x' = x\cos(\theta) + y\sin(\theta)$
    - $y' = -x\sin(\theta) + y\cos(\theta)$



  - Using homogeneous Coordinate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Transformations in 2D

- 2D Scaling
  - Scale: Alter the size of an object by a scaling factor The new point: (Sx, Sy)
    - x′ = x Sx
    - y′ = y Sy
  - Using homogeneous Coordinate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

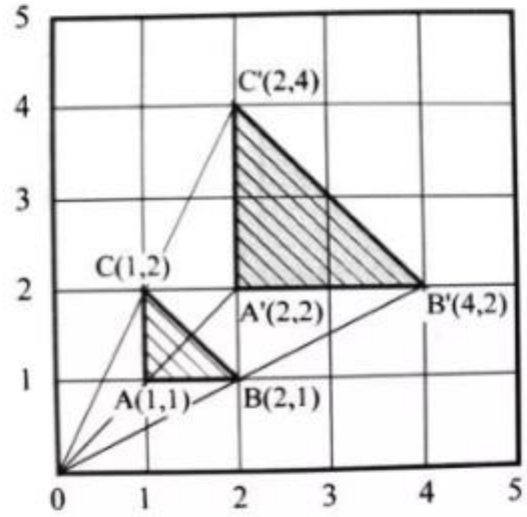


Image Reference:- CAD/CAM AND AUTOMATION By Farazdak Haideri, Nirali Prakashan, Ninth Edition

Transformations in 2D

- Arbitrary Rotation Center
  - Translate the object so that P will coincide with the origin: T(-tx, -ty)
  - Rotate the object: R(θ)
  - Translate the object back: T(tx,ty)

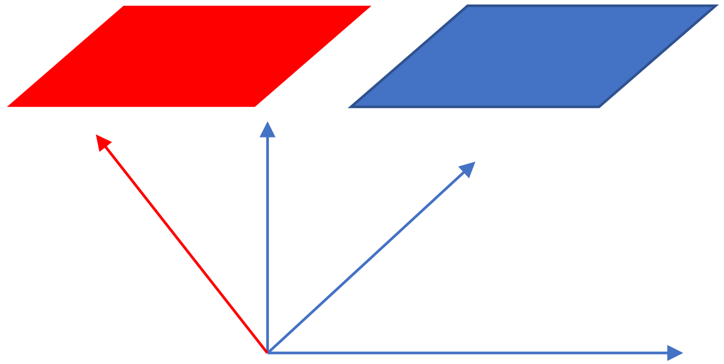
  - Put in matrix form:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Transformations in 2D

- **Rethinking 2D translation in Euclidian geometry**
  - All transformations in 2D can be regarded to "linear transform"
    - $\begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
  - Transformation is not linear transform → vector addition

- We can include translation into "linear transform" when using homogeneous coordinate
  - $\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} t_x \\ t_y \\ 1 \end{bmatrix}$



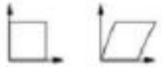
Transformations in 2D

- Affine transform
  - Translation, Scaling, Rotation, Shearing are all affine transformation
  - Affine transformation – transformed point P' (x',y') is a linear combination of the original point P (x,y), i.e.

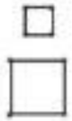
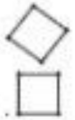
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

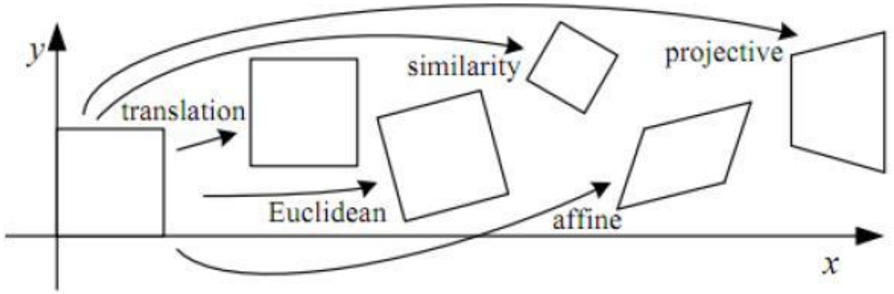
  - Any 2D affine transformation can be decomposed into a rotation, followed by a scaling, followed by a shearing, and followed by a translation. Affine matrix = translation x shearing x scaling x rotation

Transformations in 2D

2D Transformation	Figure	d. o. f.	H	H
Translation		2	$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} I & \boldsymbol{t} \\ \boldsymbol{0}^\mathsf{T} & 1 \end{bmatrix}$
Mirroring at y-axis		1	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} Z & \boldsymbol{0} \\ \boldsymbol{0}^\mathsf{T} & 1 \end{bmatrix}$
Rotation		1	$\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} R & \boldsymbol{0} \\ \boldsymbol{0}^\mathsf{T} & 1 \end{bmatrix}$
Motion		3	$\begin{bmatrix} \cos\varphi & -\sin\varphi & t_x \\ \sin\varphi & \cos\varphi & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} R & \boldsymbol{t} \\ \boldsymbol{0}^\mathsf{T} & 1 \end{bmatrix}$
Similarity		4	$\begin{bmatrix} a & -b & t_x \\ b & a & t_y \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \lambda R & \boldsymbol{t} \\ \boldsymbol{0}^\mathsf{T} & 1 \end{bmatrix}$
Scale difference		1	$\begin{bmatrix} 1+m/2 & 0 & 0 \\ 0 & 1-m/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} D & \boldsymbol{0} \\ \boldsymbol{0}^\mathsf{T} & 1 \end{bmatrix}$
Shear		1	$\begin{bmatrix} 1 & s/2 & 0 \\ s/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} S & \boldsymbol{0} \\ \boldsymbol{0}^\mathsf{T} & 1 \end{bmatrix}$
Asym. shear		1	$\begin{bmatrix} 1 & s' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} S' & \boldsymbol{0} \\ \boldsymbol{0}^\mathsf{T} & 1 \end{bmatrix}$
Affinity		6	$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} A & \boldsymbol{t} \\ \boldsymbol{0}^\mathsf{T} & 1 \end{bmatrix}$
Projectivity		8	$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$	$\begin{bmatrix} A & \boldsymbol{t} \\ \boldsymbol{p}^\mathsf{T} & 1/\lambda \end{bmatrix}$

Transformations in 2D

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$.
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area



Rigid body transform

- Rigid body rotation
  - Euclidian transform in 3D
  - Only <u>rotation (R)</u> and <u>translation (t)</u> are considered
  - To transform R, t in once, homogeneous coordinate (4D) is used!

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & t_1 \\ R_{21} & R_{22} & R_{23} & t_2 \\ R_{31} & R_{32} & R_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = x \begin{bmatrix} R_{11} \\ R_{12} \\ R_{13} \\ 0 \end{bmatrix} + y \begin{bmatrix} R_{12} \\ R_{22} \\ R_{23} \\ 0 \end{bmatrix} + z \begin{bmatrix} R_{13} \\ R_{23} \\ R_{33} \\ 0 \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ 1 \end{bmatrix}$$

Rigid body transform

- Rotation matrix
  - There are various methods to represent 3D rotational angle
    - Rotation matrix, RPY, Euler angle, Rodrigues, Quaternions, etc.
  - In 3D, rotation matrix is 3x3
  - Each column vector is orthogonal
  - Norm of the rotation matrix is 1 (preserve scale)
    - All basis are unit vector
    - Determinant is 1
    - $R^{-1}R = RR^{-1} = I$
  - Ex) Rotation 45deg along z-axis

$$\begin{bmatrix} cos45 & -sin45 & 0 \\ sin45 & cos45 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rigid body transform

- Rotation matrix example
  - 3D rotations along the main X axes

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \quad R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

  - Rotations are not commutative

$$R_x(\frac{\pi}{4}) \cdot R_y(\frac{\pi}{4}) = \begin{bmatrix} 0.707 & 0 & -0.707 \\ -0.5 & 0.707 & -0.5 \\ 0.5 & 0.707 & 0.5 \end{bmatrix}, \; R_x(\frac{\pi}{4}) \cdot R_y(\frac{\pi}{4}) \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1.414 \\ 0.586 \\ 3.414 \end{bmatrix}$$

$$R_y(\frac{\pi}{4}) \cdot R_x(\frac{\pi}{4}) = \begin{bmatrix} 0.707 & -0.5 & -0.5 \\ 0 & 0.707 & -0.707 \\ 0.707 & 0.5 & 0.5 \end{bmatrix}, \; R_y(\frac{\pi}{4}) \cdot R_x(\frac{\pi}{4}) \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1.793 \\ 0.707 \\ 3.207 \end{bmatrix}$$

Rigid body transform

- Translation
  - Translation vector

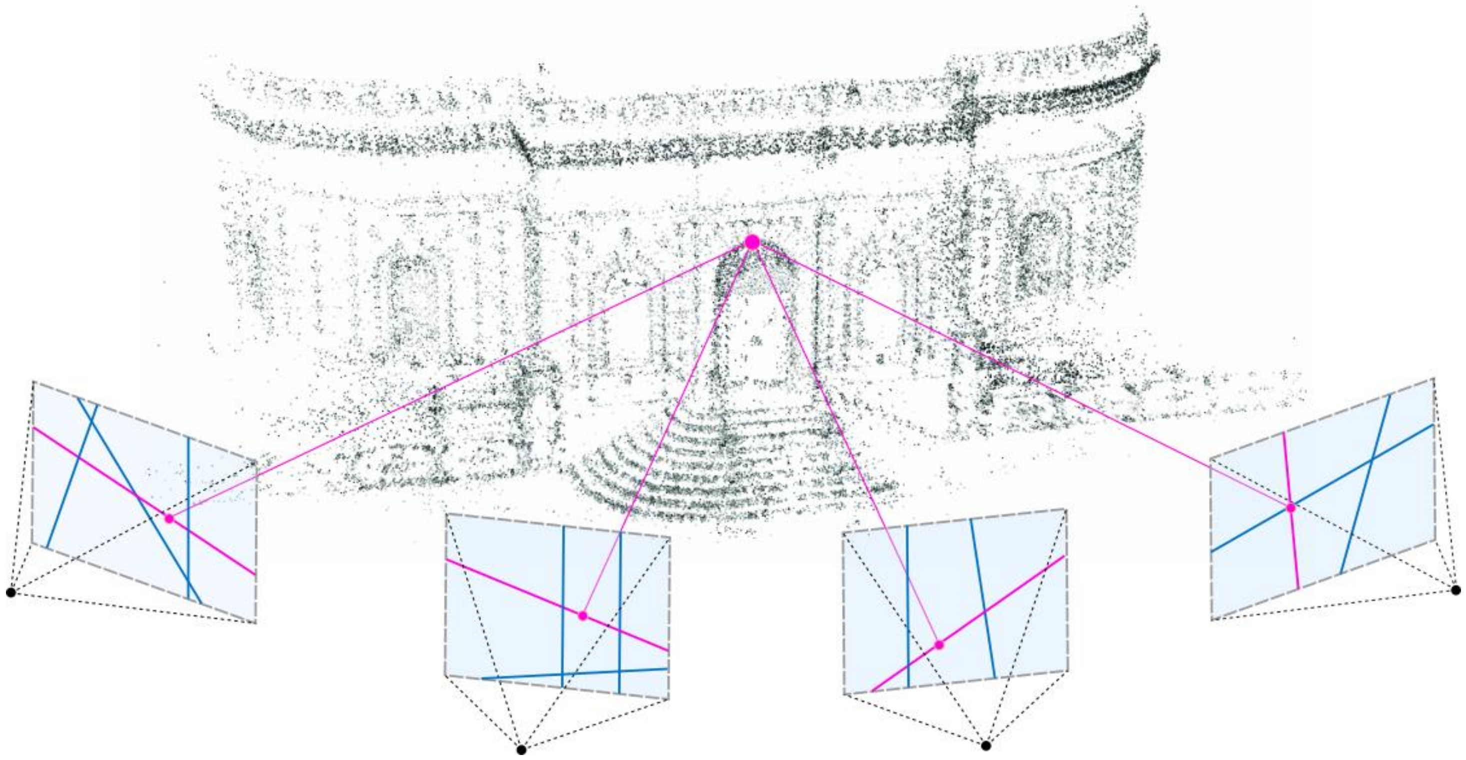
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \\ 1 \end{bmatrix}$$

- In homogeneous coordinate

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

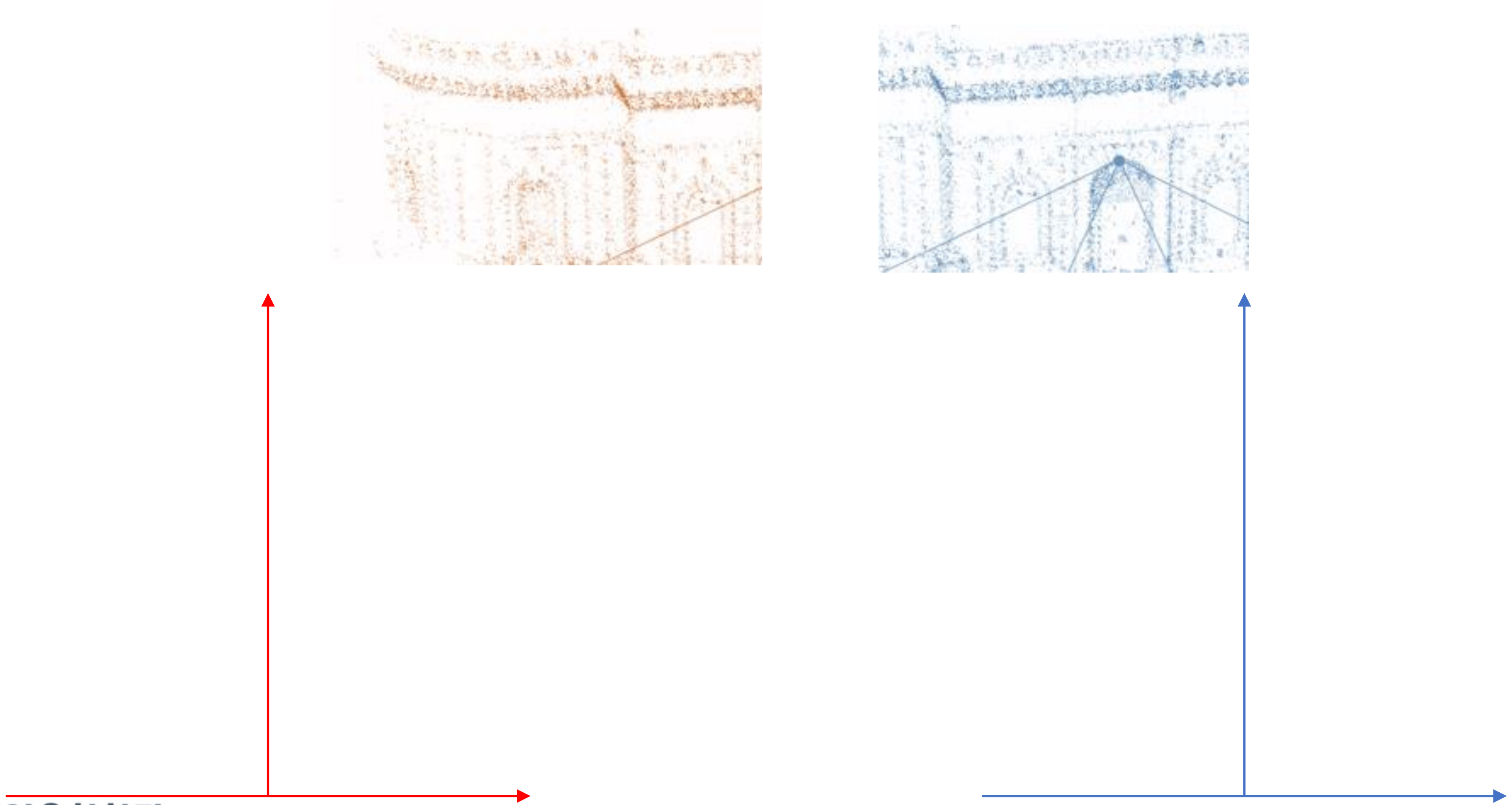
Rigid body transform

- Combining Transformations
  - If there are various coordinate systems, points or objects can be transformed using R, T between coordinates



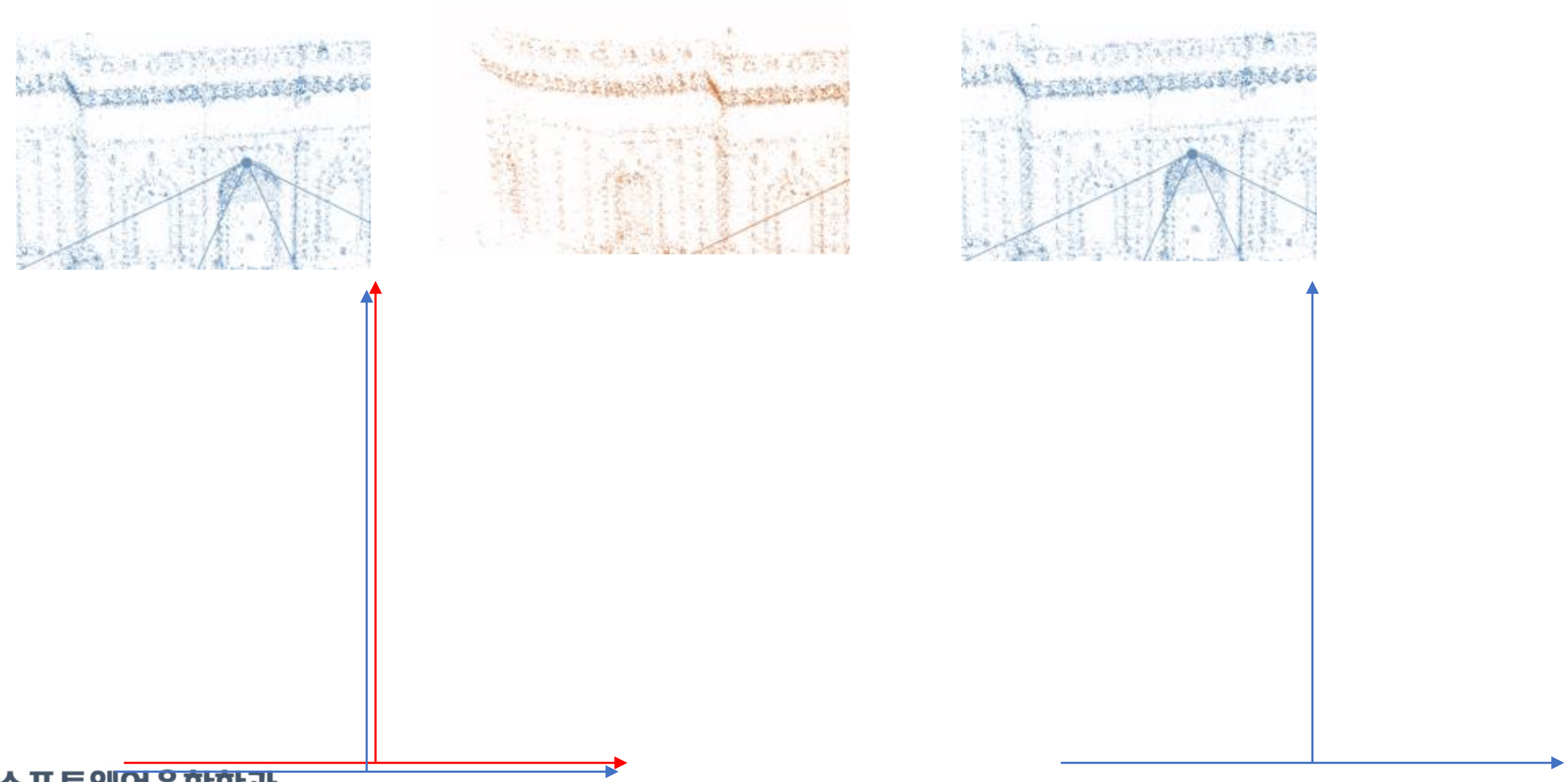
Rigid body transform

- Combining Transformations



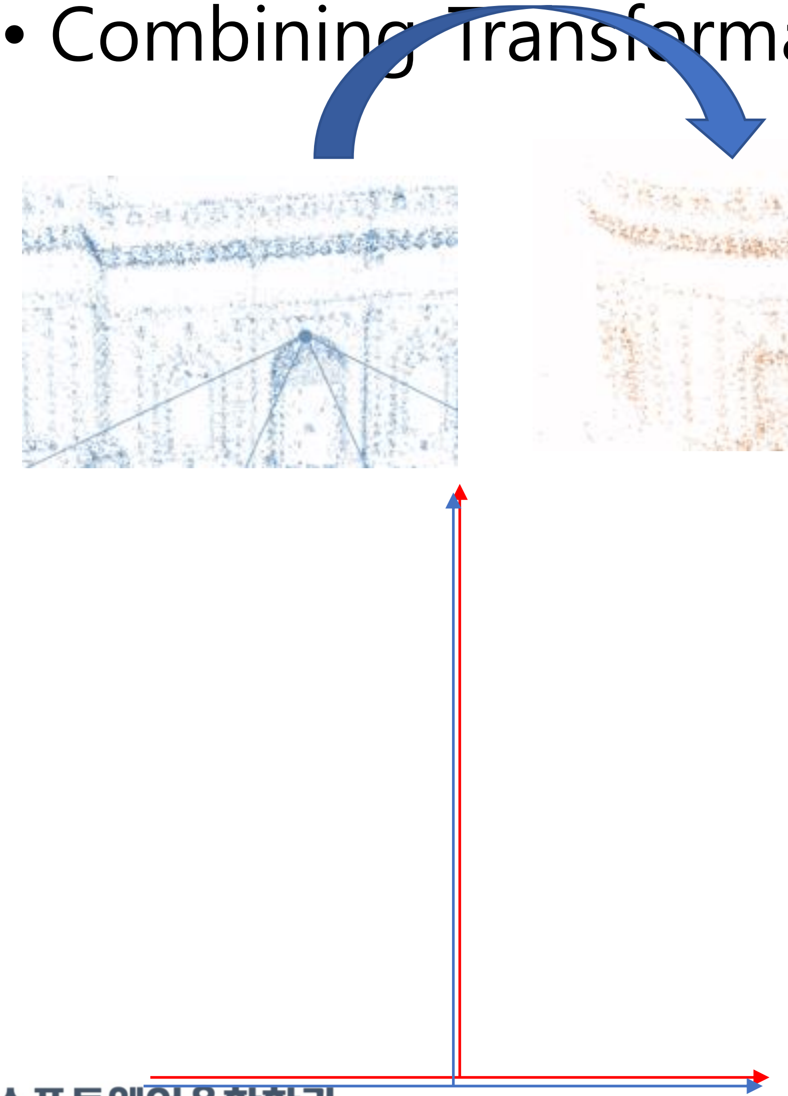
Rigid body transform

- Combining Transformations



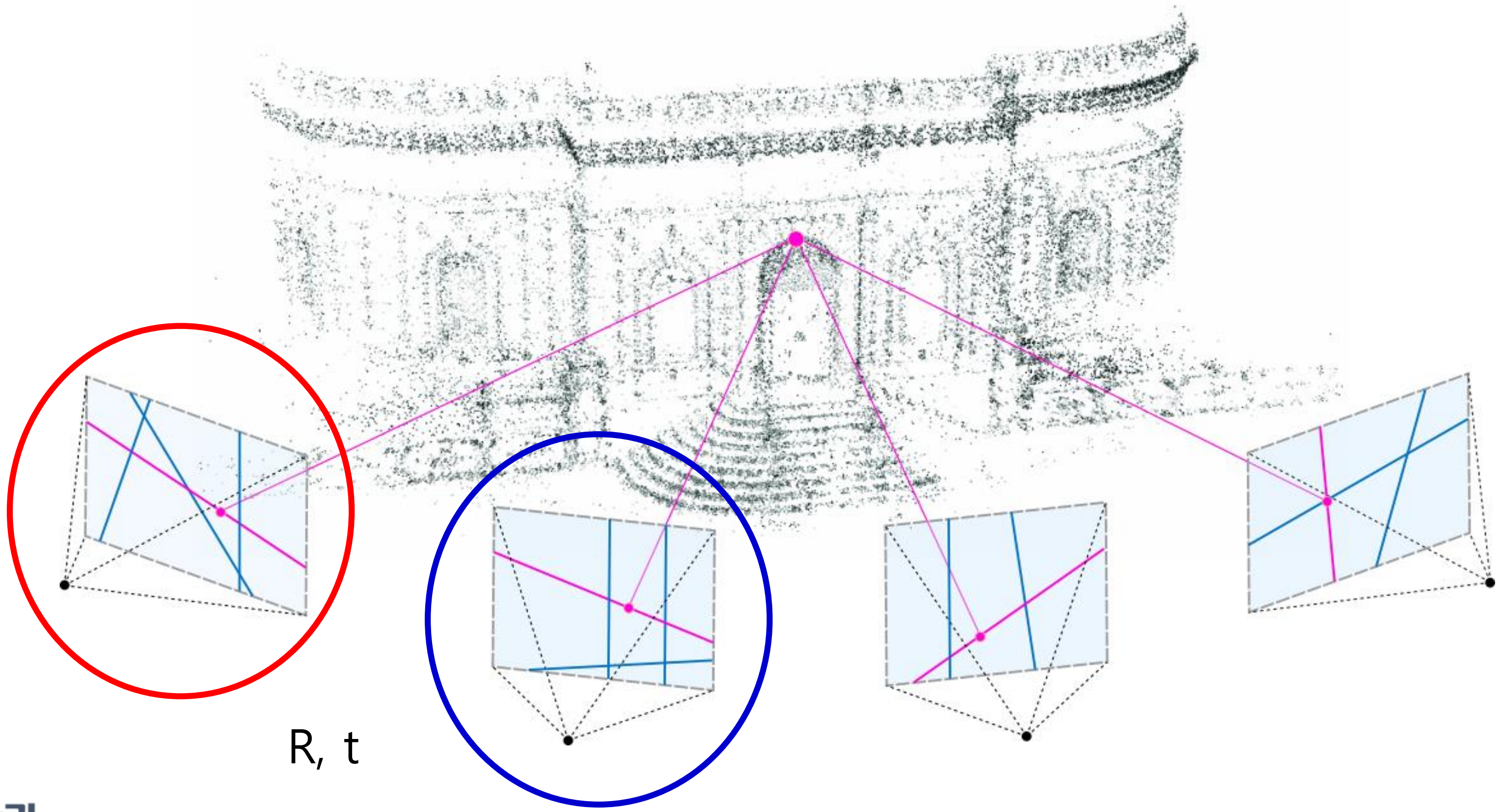
Rigid body transform

- Combining Transformations



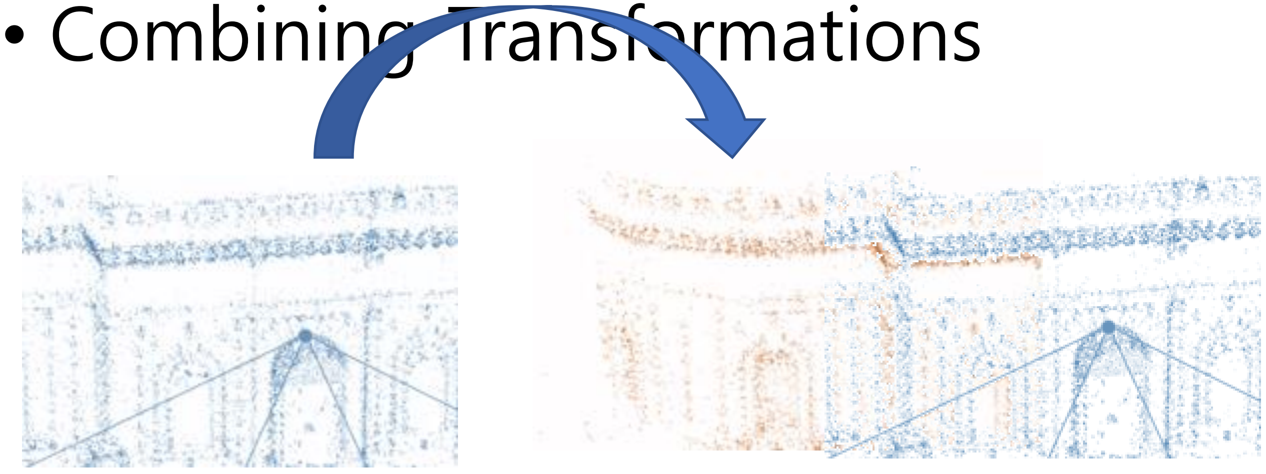
Rigid body transform

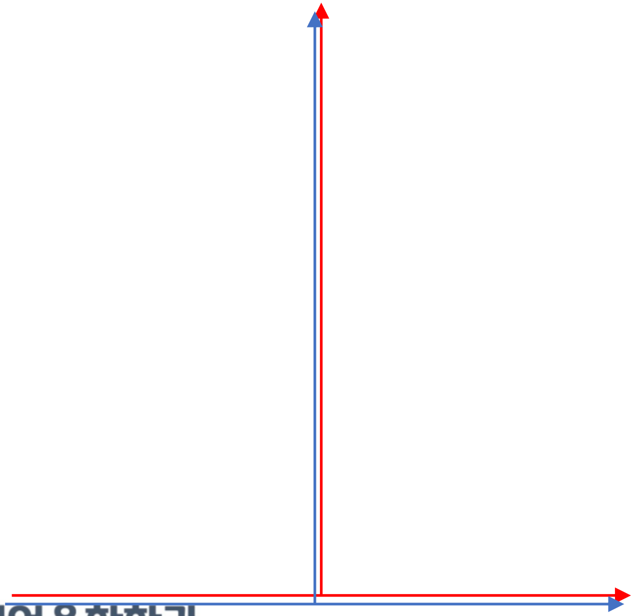
- Combining Transformations
  - If there are various coordinate systems, points or objects can be transformed using R, T between coordinates



Rigid body transform

- Combining Transformations

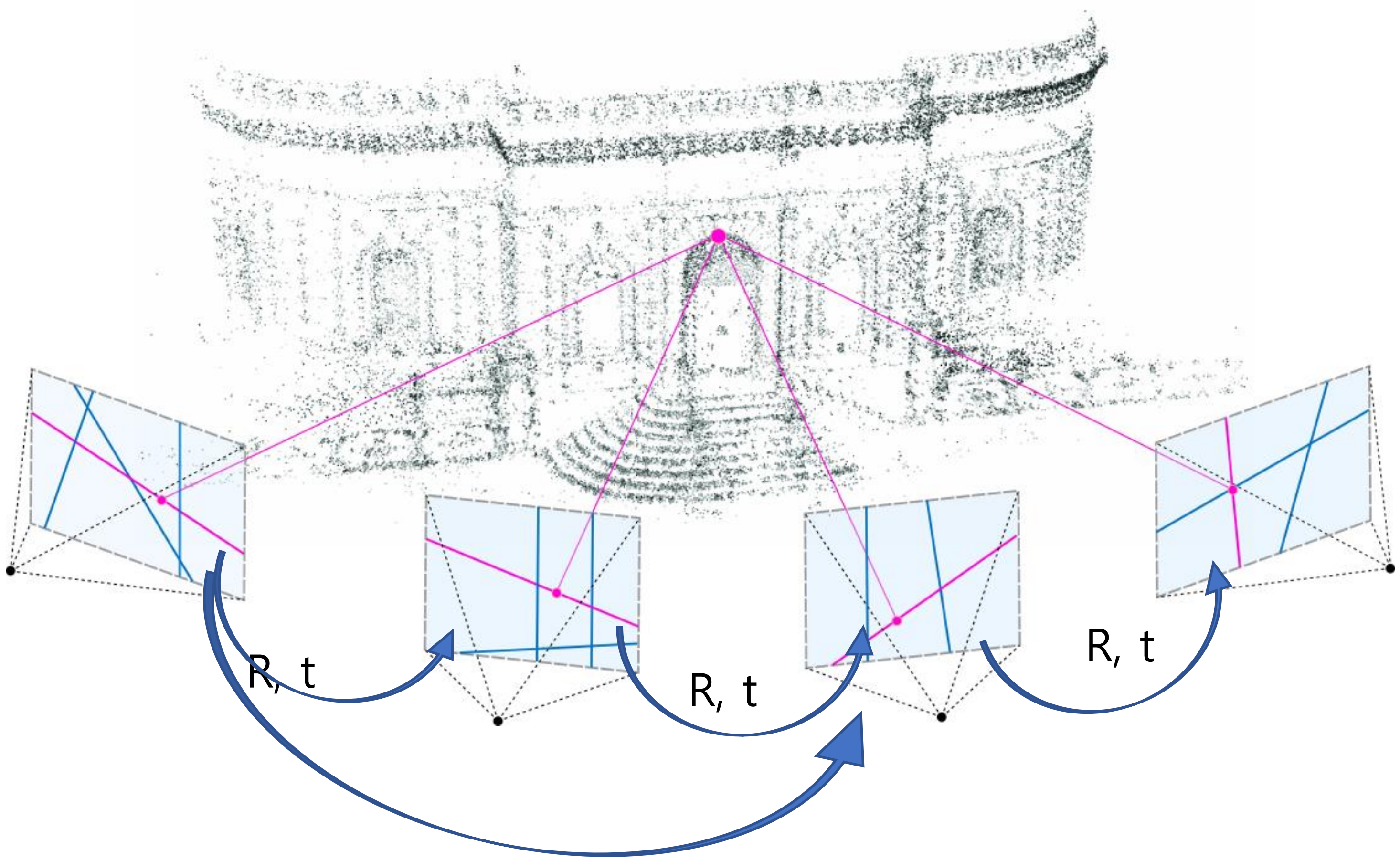




$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rigid body transform

- Combining Transformations
  - If there are various coordinate systems, points or objects can be transformed using R, T between coordinates



Appendix. Algebra and geometry

Q
$$y = -x - 1$$
$$y = x + 1$$

A

$$\left[\begin{array}{cc|c} 1 & 1 & -1 \\ -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & 0 & -2 \\ -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & -1 \\ -1 & 1 & 1 \end{array}\right]$$

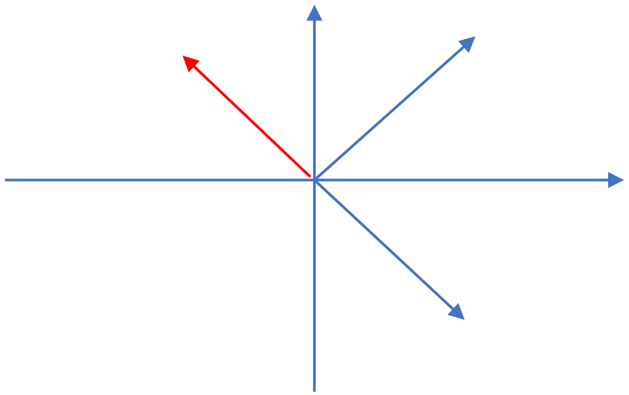
$$\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 0 \end{array}\right]$$

A

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

A

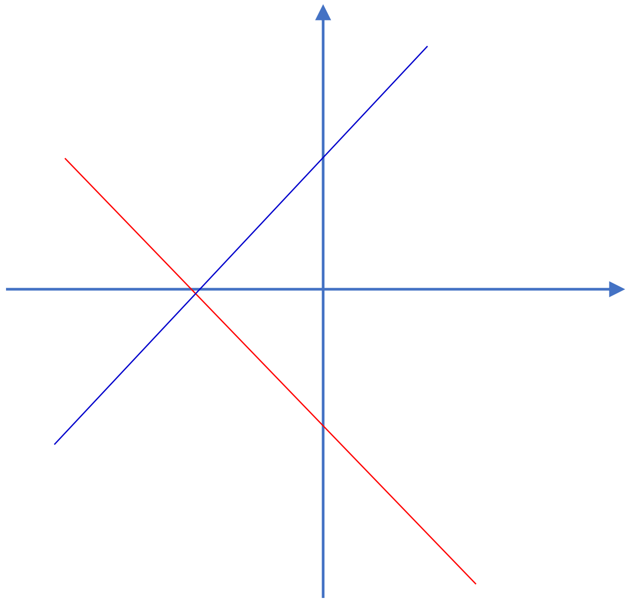


Appendix. Algebra and geometry

$$y = -x - 1$$

$$y = x + 1$$

A

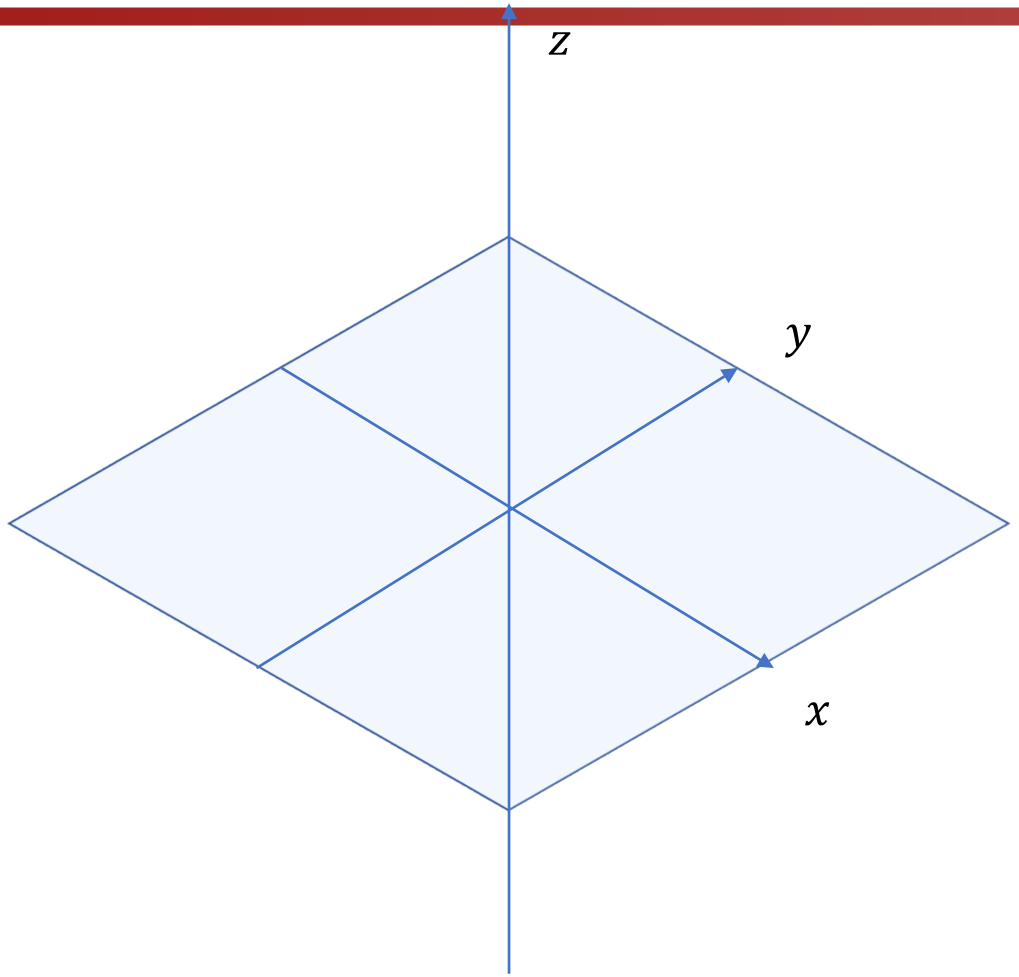


A

$$x + y + 1 = 0$$
$$-x + y - 1 = 0$$

$$\begin{bmatrix}1\\1\\1\end{bmatrix} \times \begin{bmatrix}-1\\1\\-1\end{bmatrix} = \begin{bmatrix}-2\\0\\2\end{bmatrix} = \begin{bmatrix}-2/2\\0\\2/2\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$$

Appendix.



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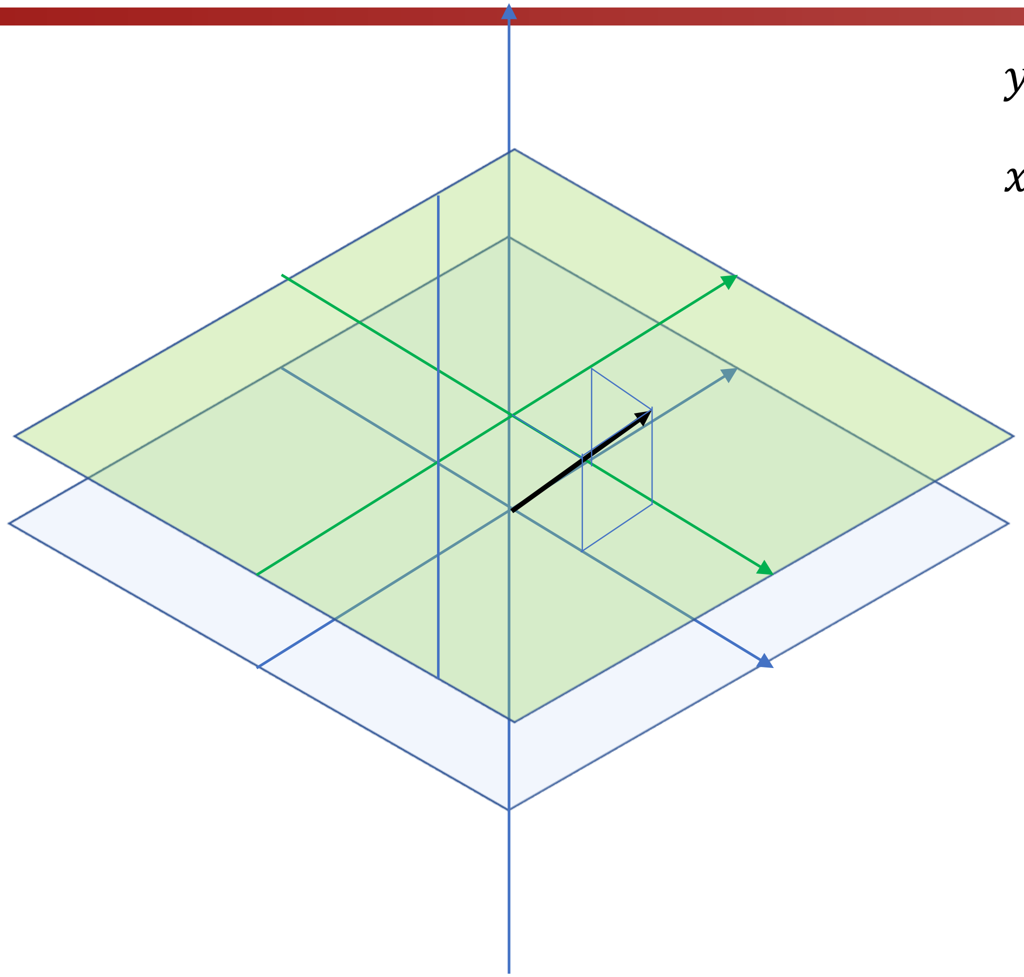


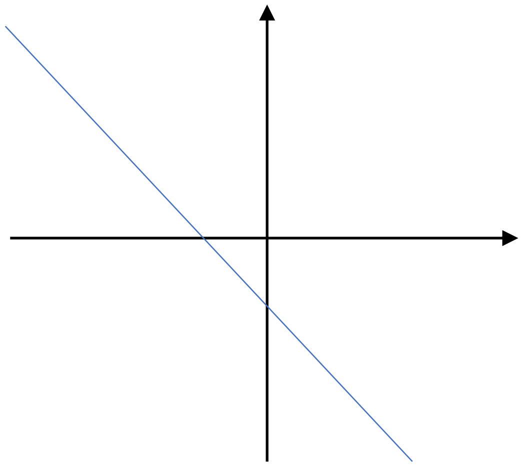
$$y = -x - 1$$

$$x + y + 1 = 0$$

$$[1\ 1\ 1]\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} kx \\ ky \\ k \end{bmatrix}$$



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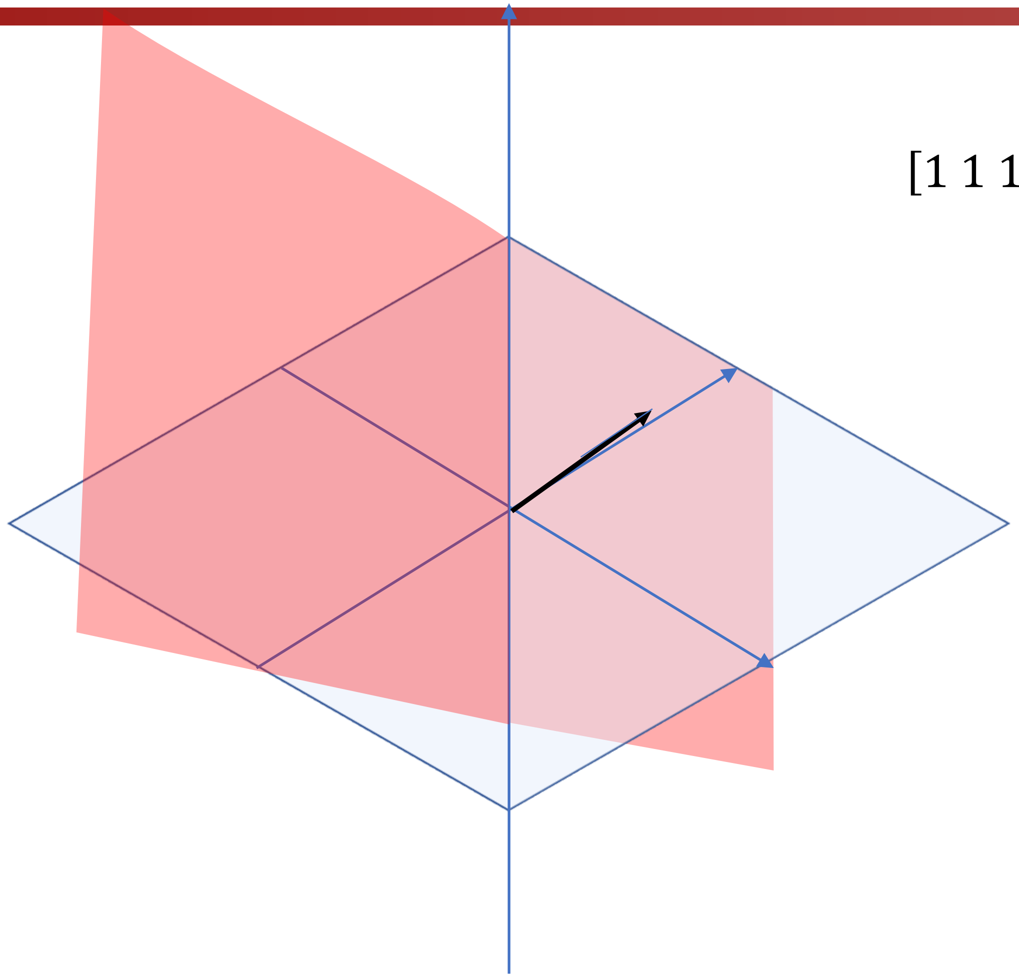
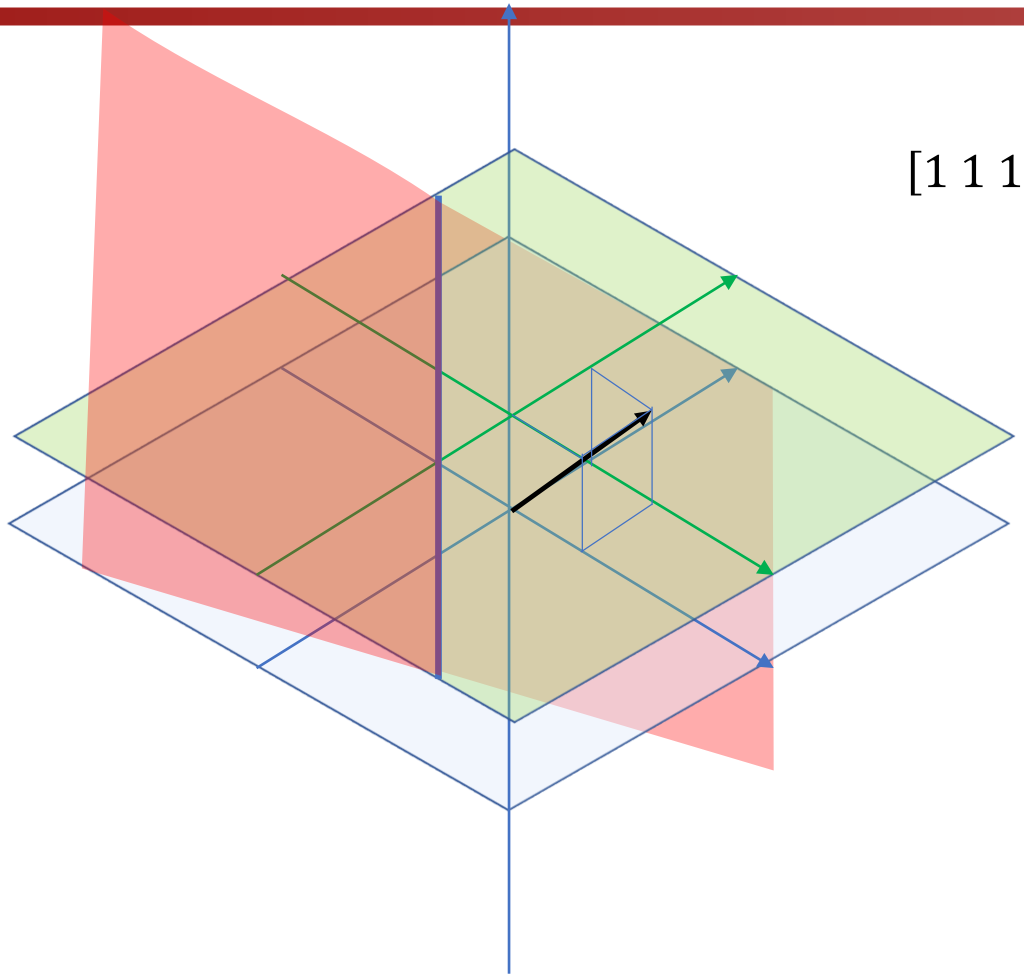
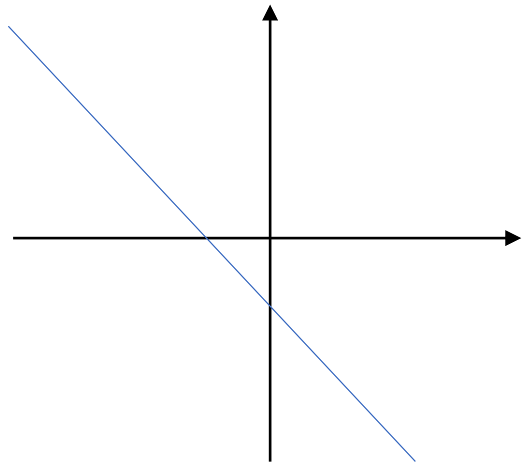


$$[1\ 1\ 1]\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}=0$$

Motions



$$[1\ 1\ 1]\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$



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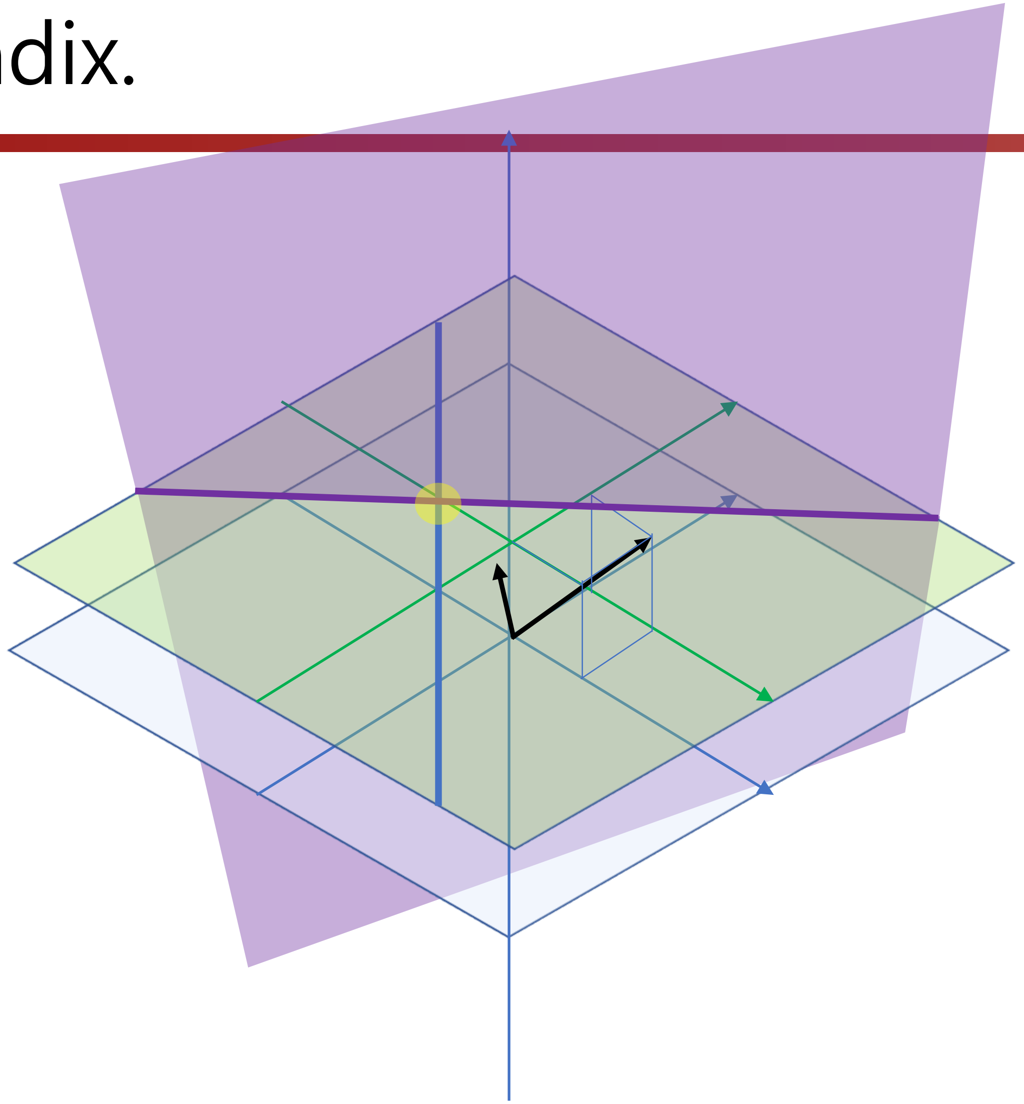
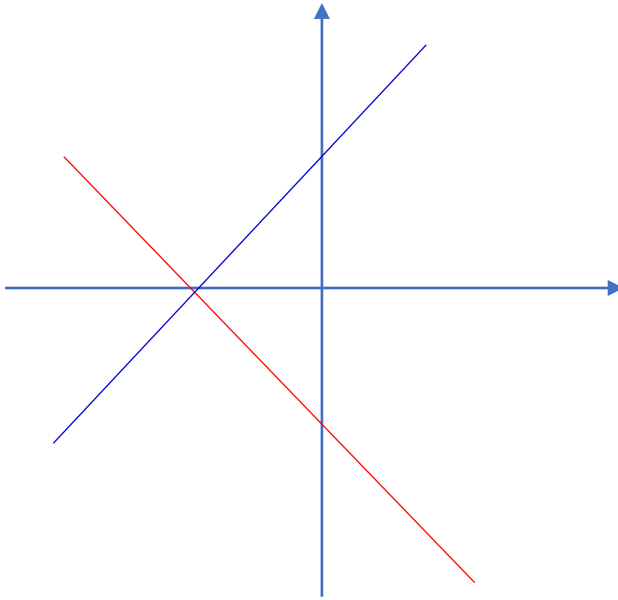


$y = -x - 1$

$x + y + 1 = 0$

$[1\ 1\ 1]\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$

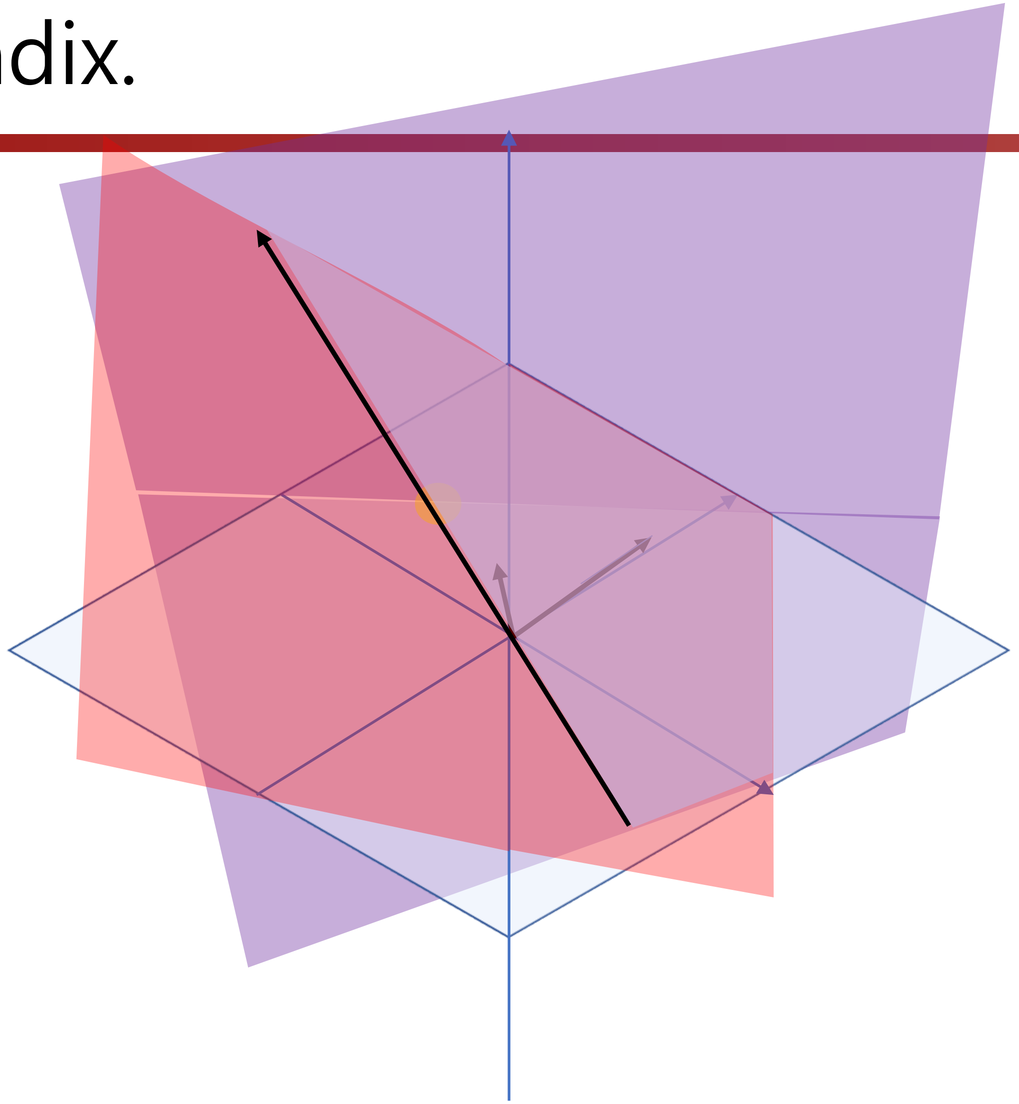


$y = x + 1$

$-x + y - 1 = 0$

$[-1\ 1\ \ -1]\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$

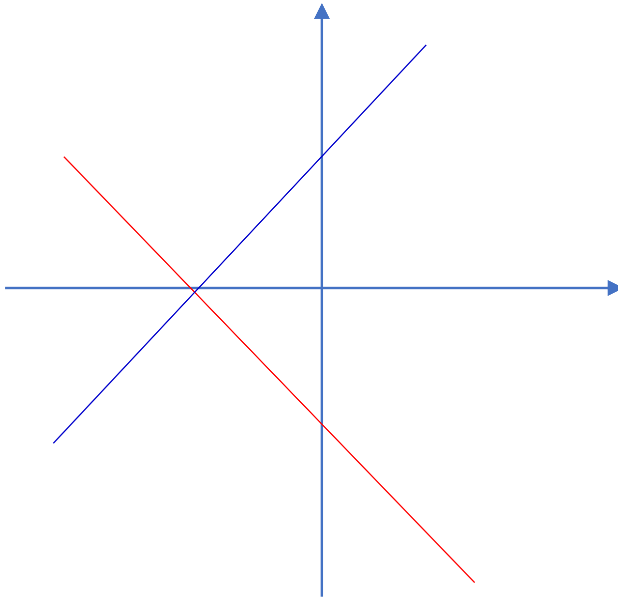
Appendix.



$y = -x - 1$

$x + y + 1 = 0$

$[1\ 1\ 1]\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$

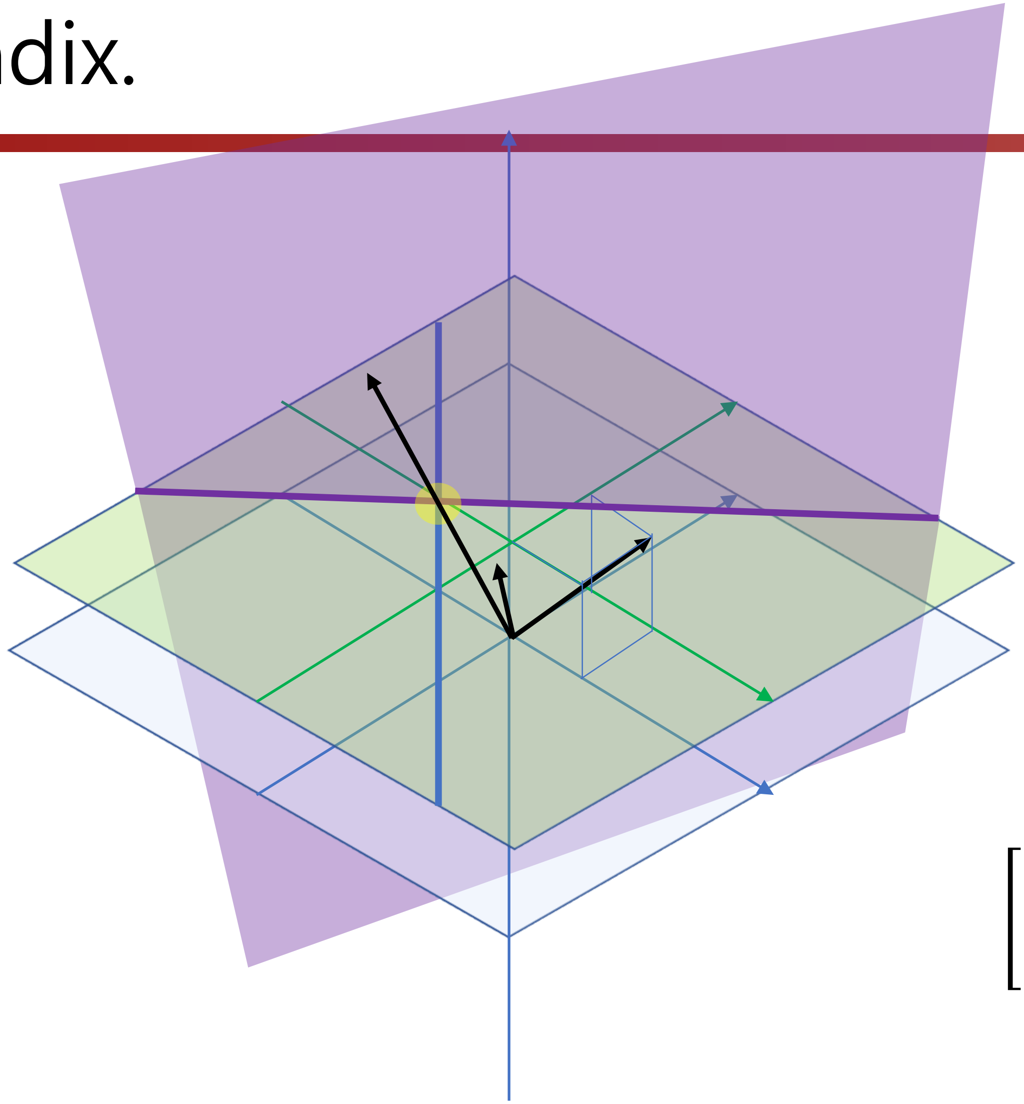


$y = x + 1$

$-x + y - 1 = 0$

$[-1\ 1\ -1]\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$

Appendix.



$$\begin{bmatrix}1\\1\\1\end{bmatrix} \times \begin{bmatrix}-1\\1\\-1\end{bmatrix} = \begin{bmatrix}-2\\0\\2\end{bmatrix} = 2\begin{bmatrix}-1\\0\\1\end{bmatrix}$$