1. Introduction to Inverse Kinematics

Game Engineering
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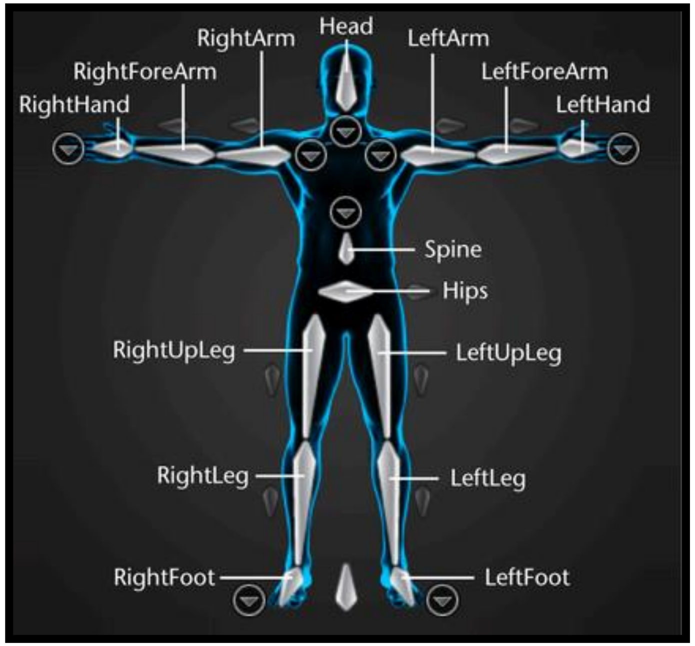
Introduction

Motivation

- A posture is defined as the skeletal configuration of a figure; for a realistic posture a set of criteria should be satisfied[1].
  - Character models have natural articulation limits.
  - Interpenetration of the body with other objects or themselves is not permitted.
  - Physical laws should be considered as well as numerous personal factors (height, length, etc.)

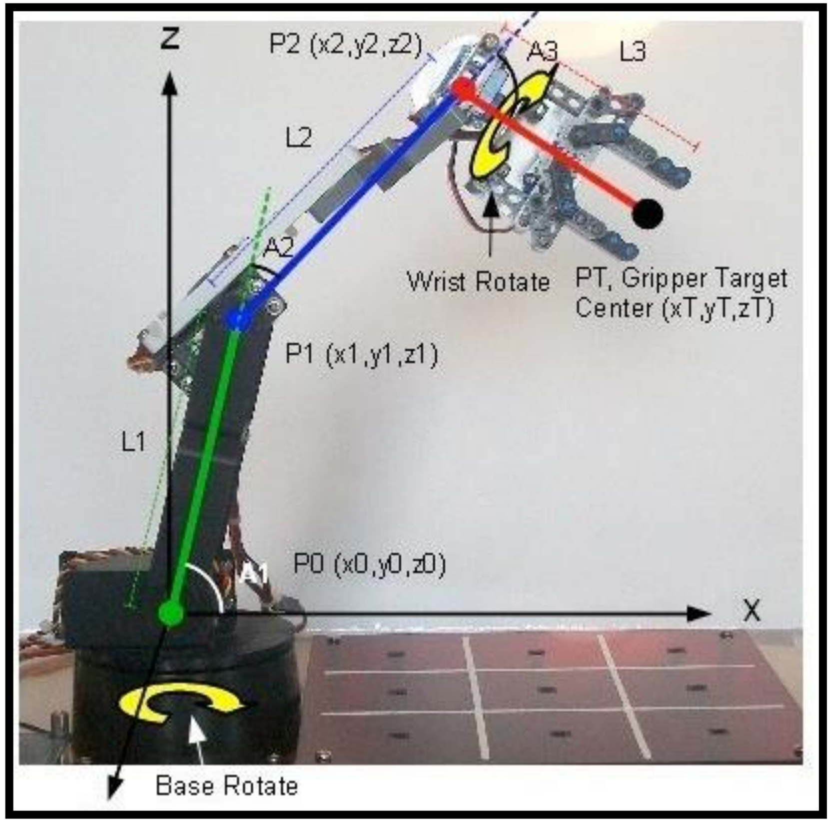
- General constraints can be applied to most character models, however special cases of posture control are needed when large number of degrees of freedom exist.



Introduction

Motivation

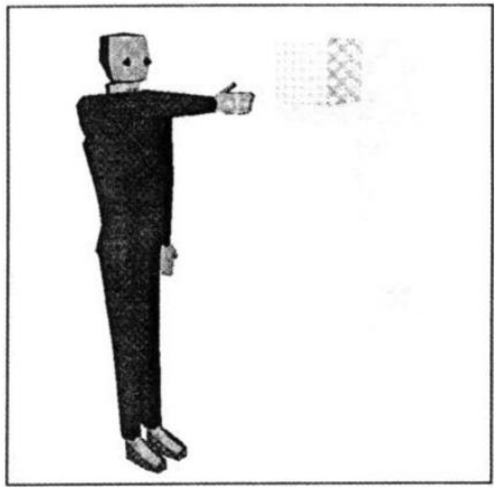
- Inverse Kinematics (IK) is a method for computing the posture via estimating each individual degree of freedom in order to satisfy a given task or criteria.
  - It plays an important role in many fields such as computer graphics, robotics, rehabilitation, etc.



Introduction

Brief literature review

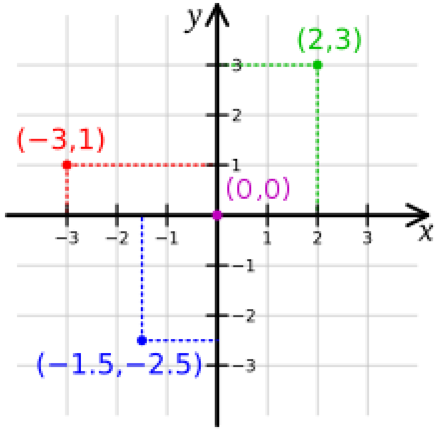
- "Inverse kinematics positioning using nonlinear programming for highly articulated figures, 1994" solves the IK task as a problem of finding a local minimum of a set of non-linear equations, defining Cartesian space constraints.



- A spatial constraints used in the paper involves two parts:
  - One constraint is a character hierarchy with the end-effector (hand).
  - The other constraint is the position of the goal.

Cartesian?

A Cartesian coordinate system in a plane is a coordinate system that specifies each point uniquely by a pair of numerical coordinates, which are the singed distances to the point from two fixed perpendicular oriented lines, measured in the same unit of length[2].



Introduction

Brief literature review

- In terms of IK problem, one of the popular numerical approaches is to use the Jacobian matrix to find a linear approximation to the solution.

- The Jacobian solutions linearly model the end effectors' movements relative to instantaneous system changes in link translation and joint angle.
  - To solve the linear problem, calculating or approximating the Jacobian inverse is required.
  - Methodologies: Jacobian Transpose, Damped Least Squares (DLS), Damped Least Squares with Singular Value Decomposition (SVD-DLS), Selectively Damped Least Squares (SDLS), etc.[3, 4, 5, 6, 7, 8]

Jacobian matrix?

The Jacobian matrix of a vector-valued function of several variables is the matrix of all its first-order partial derivatives. When this matrix is square (when the function takes the same number of variables as input as the number of vector components of its output), its determinant is referred to as the Jacobian determinant.

$$\mathbf{f}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \begin{bmatrix} x^2 y \\ 5x + \sin y \end{bmatrix}.$$

and the Jacobian matrix of $\mathbf{f}$ is

$$\mathbf{J}_{\mathbf{f}}(x,y) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\[2ex] \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy & x^2 \\ 5 & \cos y \end{bmatrix}$$

and the Jacobian determinant is

$$\det(\mathbf{J}_{\mathbf{f}}(x,y)) = 2xy\cos y - 5x^2.$$

Introduction

Brief literature review

- Jacobian-based approaches suffer from high computational cost, complex matrix calculations and singularity problems. (It occurs when *det*(J) = 0)

- An alternative approach is given by 'Inverse kinematic without matrix invertion, 2008' where the inverse kinematics problem is solved from a feedback loop. This approach is computationally more efficient than the pseudo-inverse based methods and does not suffer from singularity problems.

- The use of IK solvers based on Newton methods is also popular. These algorithms seek target configurations by solving unconstrained nonlinear optimization problems.
  - The most well known methods are Broyden's method, Powell's method, and the Broyden, Fletcher, Goldfarb and Shanno (BFGS) method[9].

- Recently, numerous way of solving IK have been proposed such as non-iterative technique (FTL, FABRIK), motion capture data based IK, and deep learning based IK.

The Articulated Body Model

The articulated body model

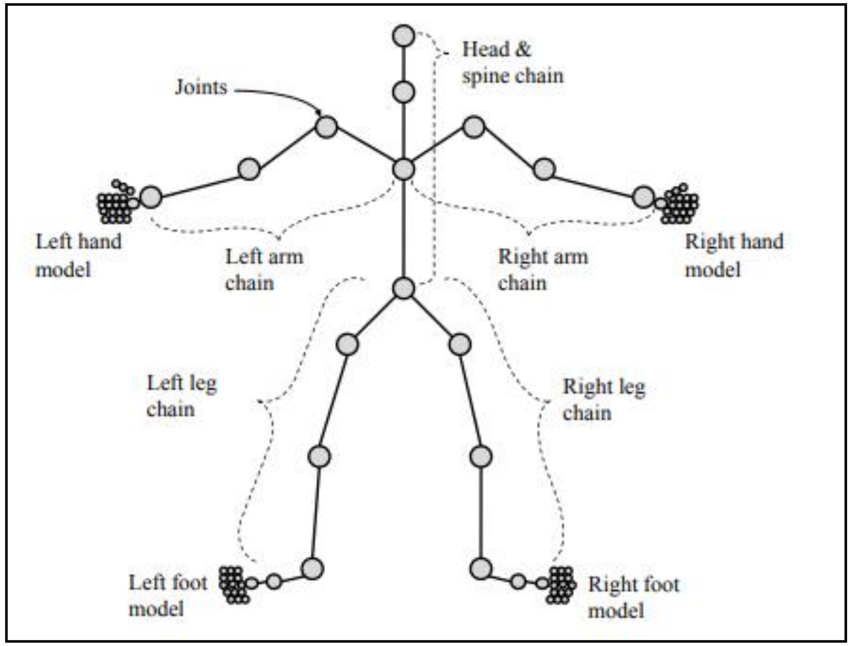
- In order to process the motion data, it requires to be preprocessed to ensure that the correct hierarchical connections and constraints are satisfied.

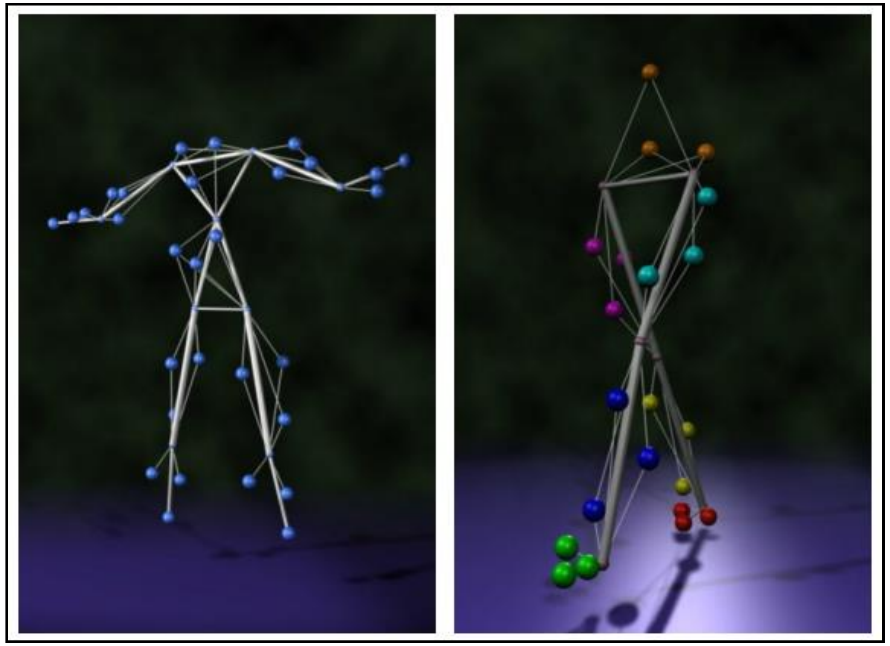
- In computer graphics applications, human body is defined by a complex hierarchical model consisting of many joints, each one having different degrees of freedom (DoF) and various possible restrictions.

The Articulated Body Model

Human body modelling

- A rigid multibody system consists of a set of rigid objects, called links, connected together by joints. A joint is the component concerned with motion; it permits some degree of relative motion between the connected segments.
- Virtual body modelling is important for human posture control. A well constrained model can restrict postures to a feasible set, therefore allowing a realistic motion. Most models assume that body parts are rigid, although this is just an assumption approximating reality.
- The skeletal structure is usually modeled as a hierarchy of rigid segments connected by joints, each defined by their length, shape, volume and mass properties. The skeletal structures are often defined using a parent-child system.

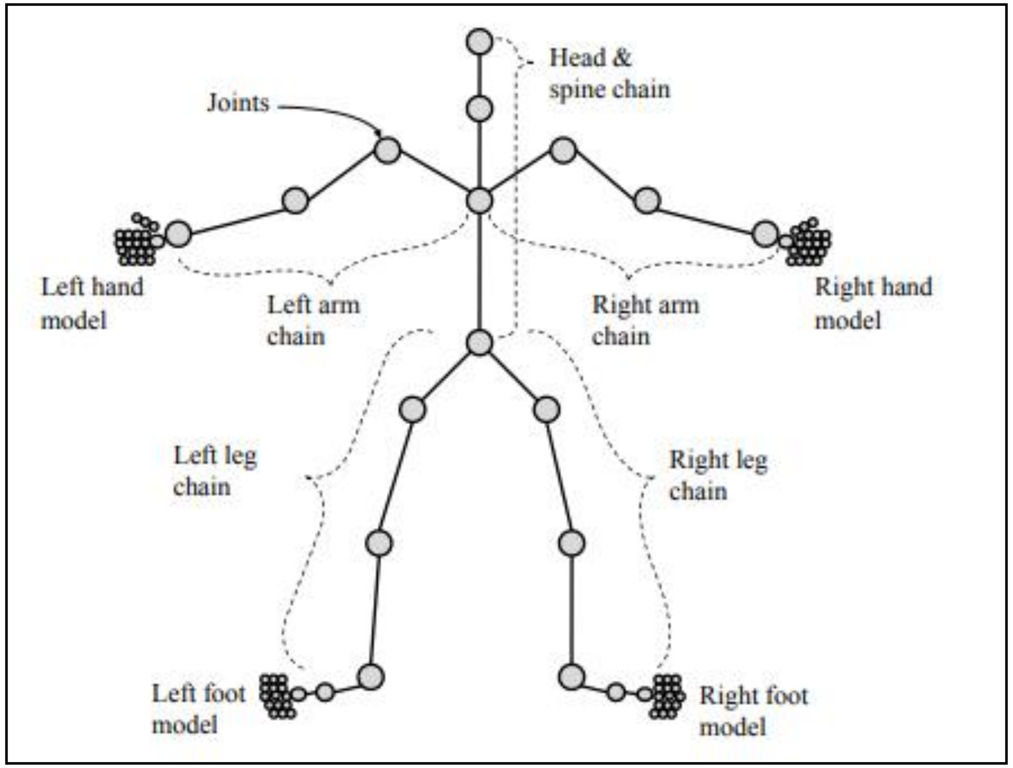




The Articulated Body Model

Human body modelling

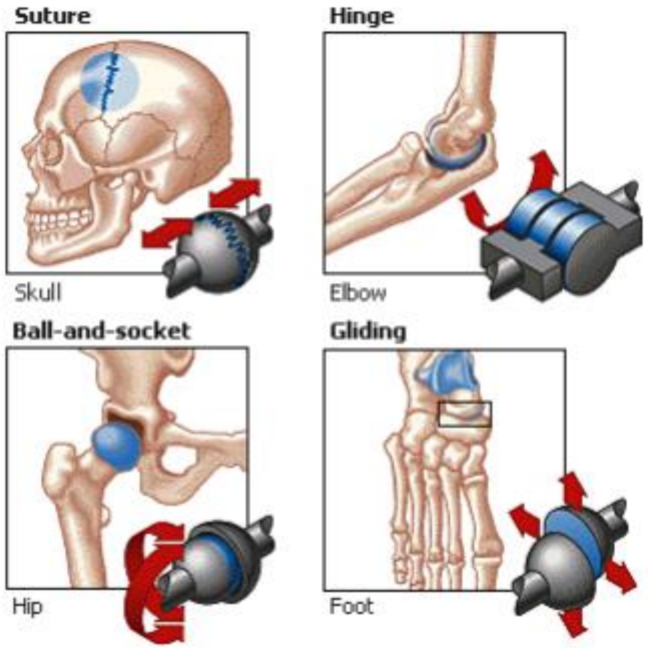
- A manipulator such as a robot arm or an animated graphics character is modeled as a *chain* composed of rigid links connected at their end by rotating joints.
- Any translation and rotation of the *i*-th joint affects the translation and rotation of any joint placed later in the chain.
- The chains are built under the assumption that all bones have at most one parent and any number of children.
- All bones (joints) with no children are marked as *end effectors*.
- A chain can be built for each end effector by moving back through the skeleton, going from parent to parent, until the root is reached.
- There are a variety of possible joint types and each joint provides a local rotation (and each bone a local translation) with different degrees of freedom (DoF).



The Articulated Body Model

Human body modelling

- The main human joint types are as follows:
  - **The suture joint model** (1 DoF): This is a fixed joint that allows very limited movement. Suture joints can be found in the skull.
  - **The hinge joint model** (1 DoF): The simplest type of joint; it can be found in the elbows, knees and the joints of the fingers and toes. Hinge joints allow movement in only one direction.
  - **The gliding joint model** (2 DoF): Gliding joints permit a wide range of mostly sideways movements as well as movements in one direction.
  - **The saddle joint model** (2 DoF): A saddle joint is more versatile than either a hinge joint or a gliding joint. It allows movement in two directions.
  - **The pivot joint model** (2 DoF): The pivot joint is a 2 degree of freedom joint and it can be found in the neck allowing a side to side turn of the head.
  - **The ball and socket joint model** (3 DoF): This is the most mobile type of joint in the human body; it allows 3 degrees of freedom. A limited (in the sense of restricted magnitude) version of the ball and socket joint is the Ellipsoidal joint.

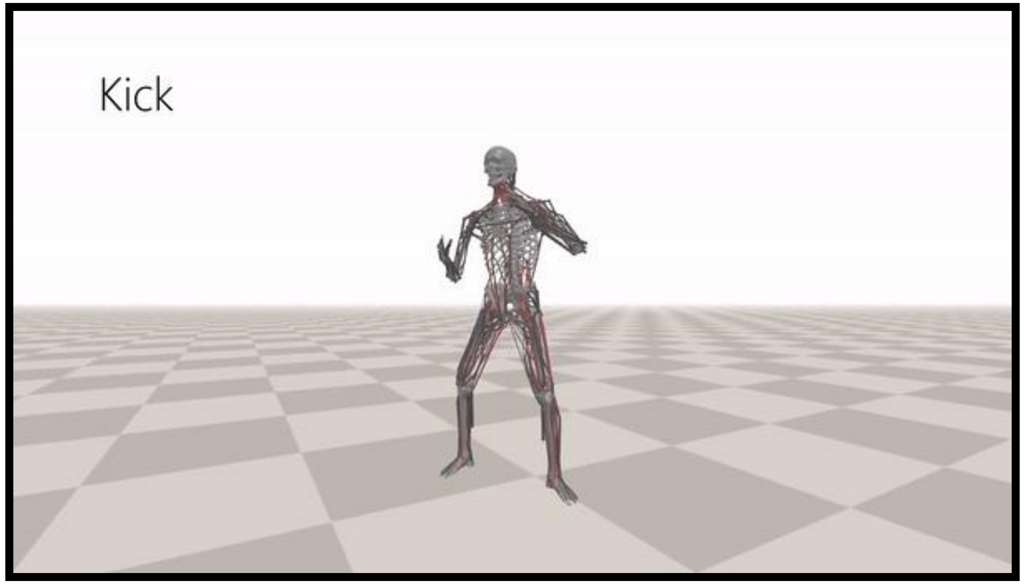


The Articulated Body Model

Motion

- Once a body model has been defined, it can then be animated, manipulated or simply used for simulation purposes.
- A motion can be achieved when a rotational or translational transformation has been applied in order to move the end effector(s) of a chain to a desired position.
- There are two types of kinematics strategy:
  - **Forward Kinematics** (FK): can be defined as the problem of locating the end effectors' positions after applying known transformations to the chain.
  - **Inverse Kinematics** (IK): is described as the problem of determining an appropriate joint configuration for which the end effectors move to desired positions, named target positions, as smoothly, rapidly, and as accurately as possible.

Forward Kinematics
Input: joint angles
Output: link positions and orientations, end effector position

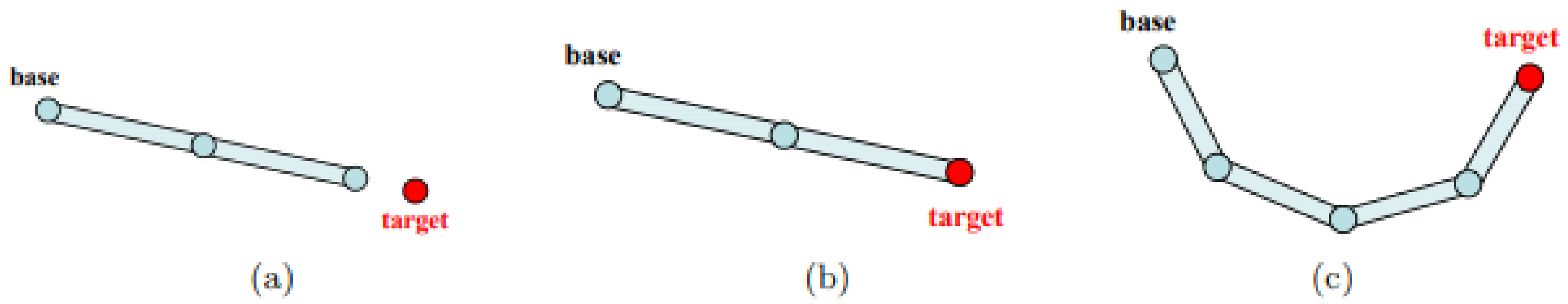


Inverse Kinematics
Input: end effector position
Output: joint angles

The Articulated Body Model

Motion

- The FK problem has a unique solution, and its success depends on whether the joints are allowed to do the desired transformation.
- In contrast, the IK problem is not always the case that a single solution can be achieved.
  - The goal can be unreachable. The targets which can be further than the chain can reach or can be at a point where no pivoting of links can bend the chain to reach. (Over-constrained problems)
  - More than one solution can exist. It is up to the IK method to choose the best solution and the IK solver's performance is ranked according to how realistic the solution is and the computational cost of choosing that solution.



(a) The target is unreachable, (b) One solution, (c) Many solutions

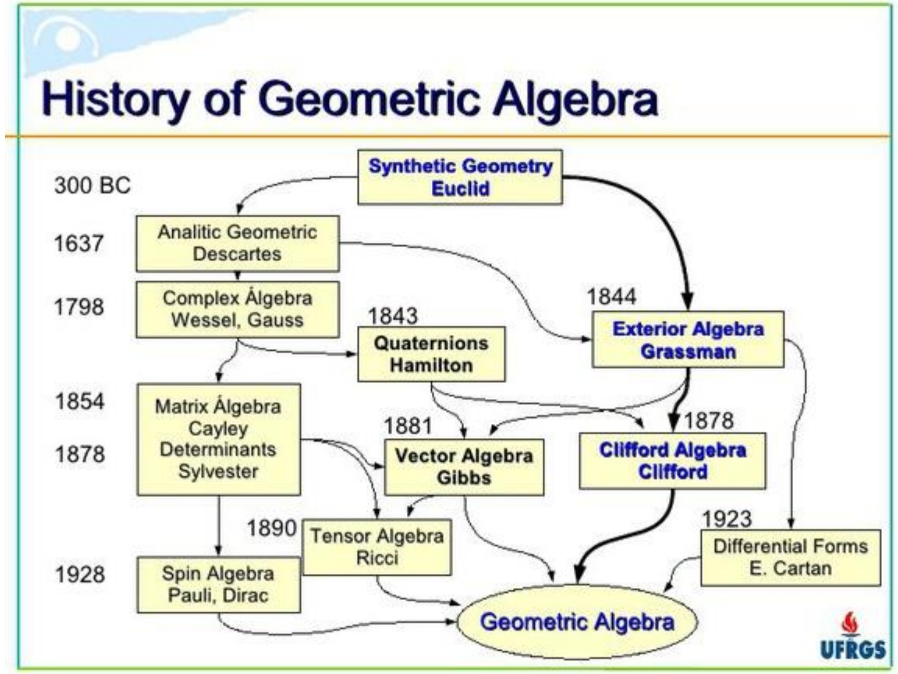
The Articulated Body Model

Orientations and rotations

- Most of the inverse kinematics are implemented using object orientations and rotations.
- Fortunately, Geometric Algebra provides a convenient mathematical notation for representing orientations and rotations of objects in three dimensions: Rotor (R)
- A rotor is an object in the Geometric Algebra (GA) of a vector space that represents a rotation about the origin.

Geometric algebra

- The algebraic properties of vector addition and scalar multiplication are insufficient to represent the geometric concept of a vector as a directed line segment.
  - How to represent a magnitude?
  - How to represent a relative direction?
- Geometric algebra is an extension of elementary algebra.



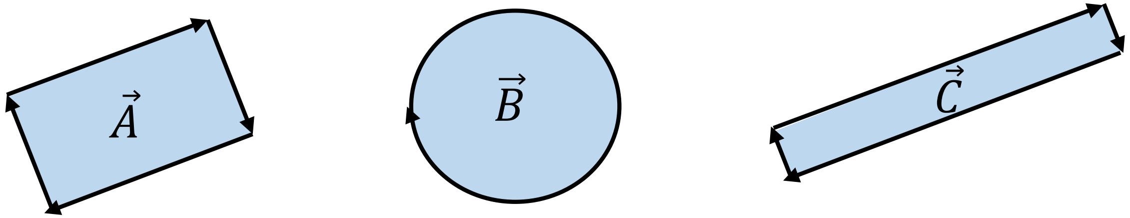
Geometric Primitive in GA

Bivector

- We call 0-dimensional object *scalar* and 1-dimensional object *vector*.

scalar *vector*

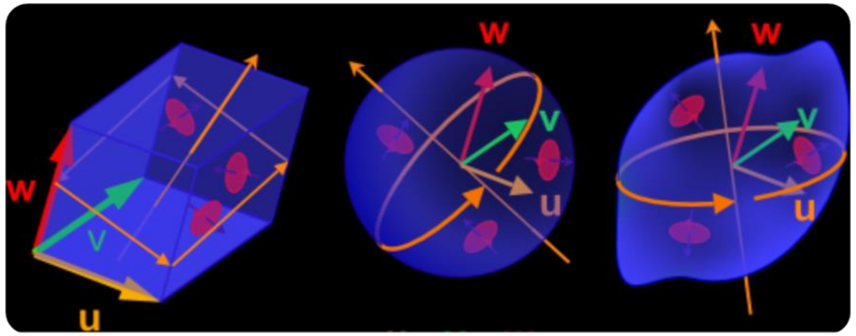
- Then what about 2-dimensional object?
  - We call this object *bivector*.



bivector

  - The magnitude of the bivector is its area.
  - If the areas occupied by $\vec{A}$, $\vec{B}$, and $\vec{C}$ are the same, $\vec{A} = \vec{B} = \vec{C}$.

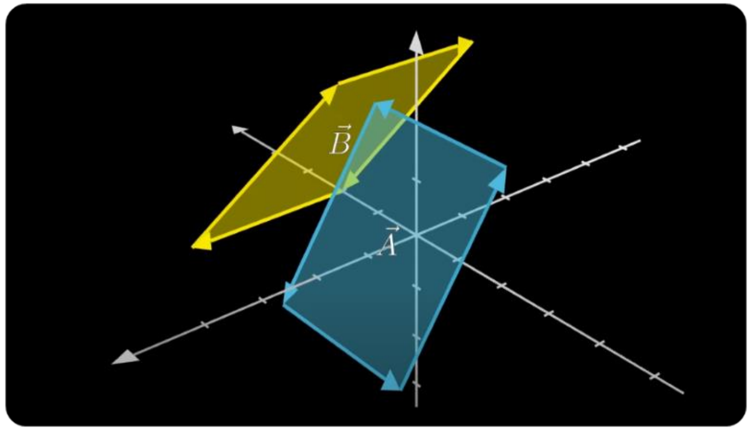
- Similarly, 3D object (trivector), 4D object, and *k*D object can be defined.



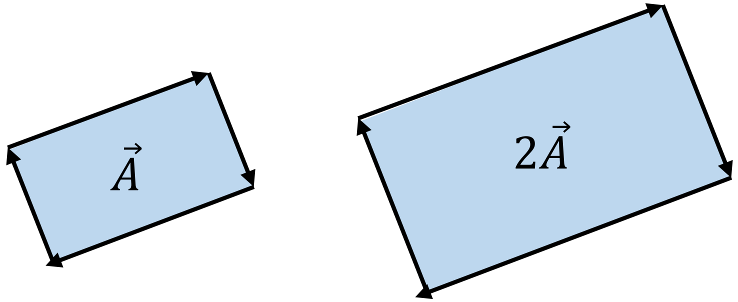
Geometric Primitive in GA

Bivector

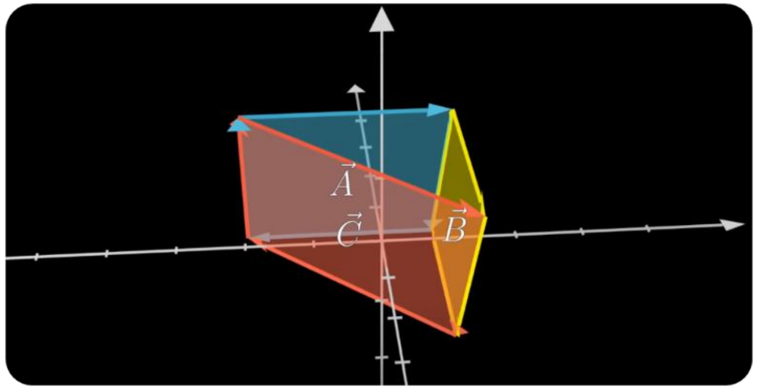
- In 3-d, even though two bivectors have the same magnitude, they are not equal because their orientation is not equal[bi].



- Multiply a bivector with a scalar increases the magnitude of the bivector.



- The geometric result of the addition of two bivectors are shown on the right.
  - To add the two bivectors, you can morph them into equivalent bivectors that share a side and put those sides together.
  - Figure on the right shows $\vec{A} + \vec{B} = \vec{C}$

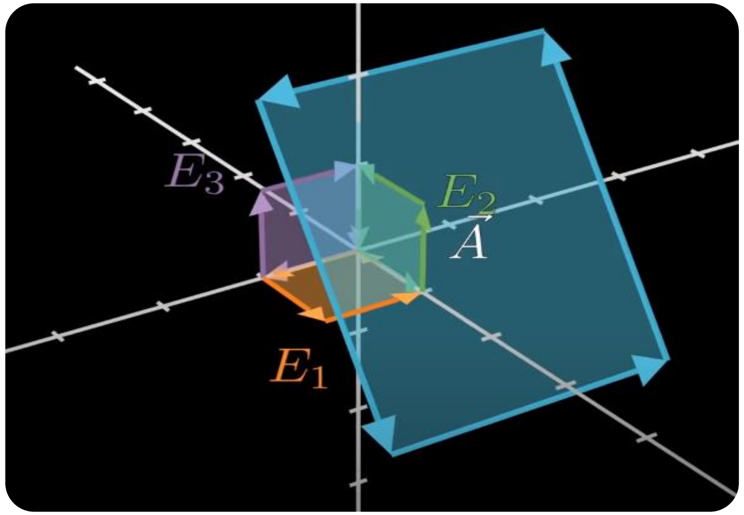


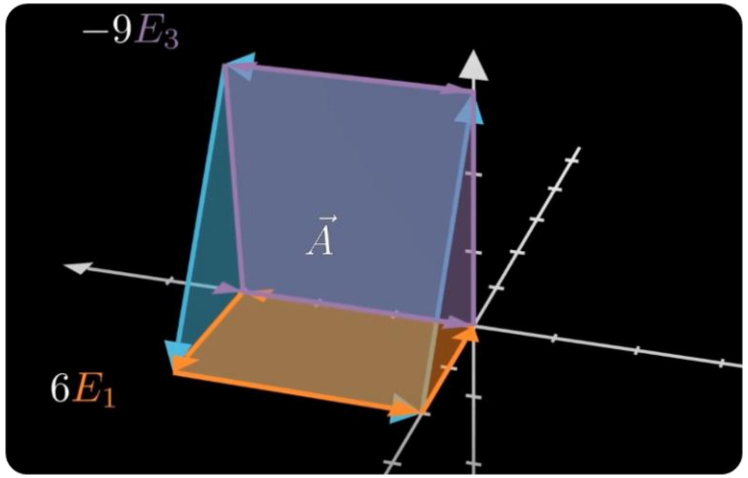
Geometric Primitive in GA

Bivector

- To help your understand, let's take a look at the bivector addition in terms of the basis calculation.

- In 3D coordinate, there are three bivectors each represents an orthogonal plane in 3D space (bases).
  - Then, $\vec{A}$ can be represented by the addition of two bivectors of E_3 and E_1.

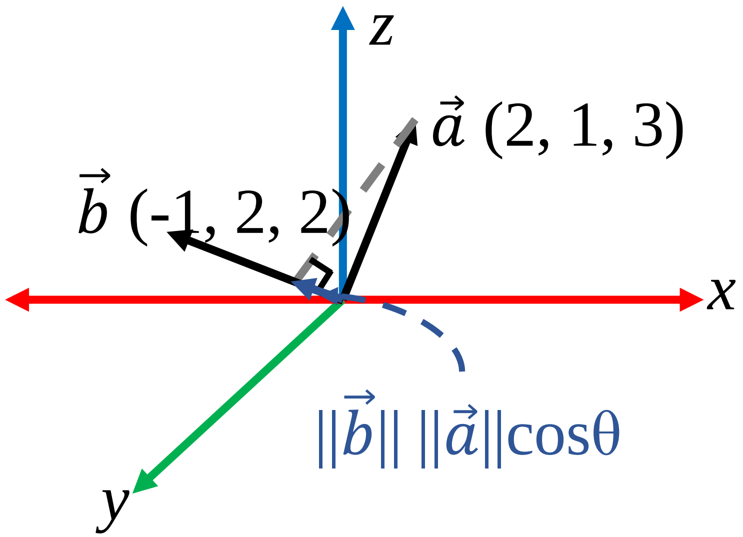




Products in GA

Inner product

- The inner product (or dot product) between two vectors are familiar to most CS students.
  - Inner product result = length of the projection of one vector onto the other vector = scalar.
  - This is also equal to the product of the cosine of the angle between them and the magnitude of two vectors.



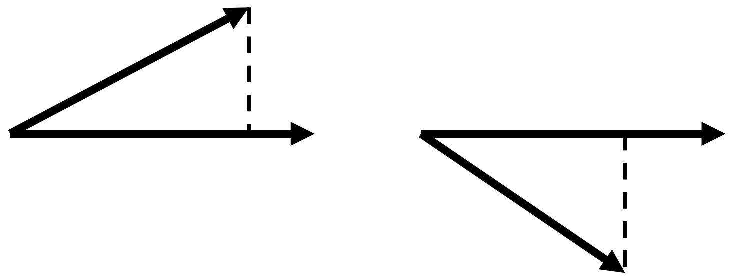
- The inner product of two vectors in the space is a scalar, often denoted with angle brackets such as in $\langle u, v\rangle$.

- For a real vector space, an inner product satisfies the following properties (Let *u*, *v*, and *w* be vectors and *a* be a scalar):
  - $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$
  - $\langle av, w\rangle = a\langle v, w\rangle$
  - $\langle v, w\rangle = \langle w, v\rangle$
  - if $v \neq 0$, $\langle v, v\rangle > 0$.

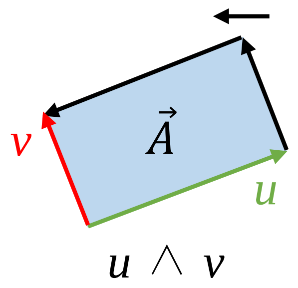
Products in GA

Outer product

- The problem of the inner product is that it does not give sufficient information about the original vectors.
  - For example, flipping one vector around the other vector does not change the inner vector = commutative.
  - In higher dimensions, there is even more freedom.



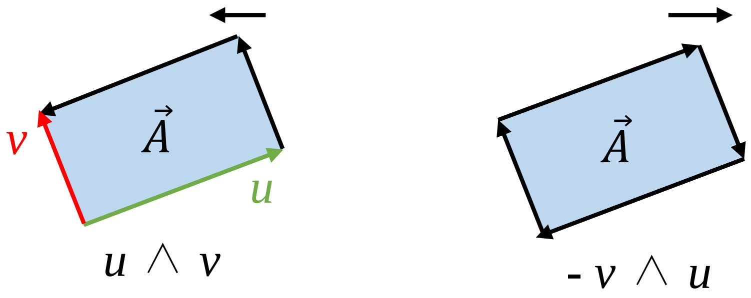
- To resolve this problem, outer product (wedge product, exterior product) is introduced.
  - The outer product of two vectors u and v, denoted by $u \wedge v$, increases vector dimension to a bivector.
  - The orientation of the bivector is along the direction of the first vector listed: u.



Products in GA

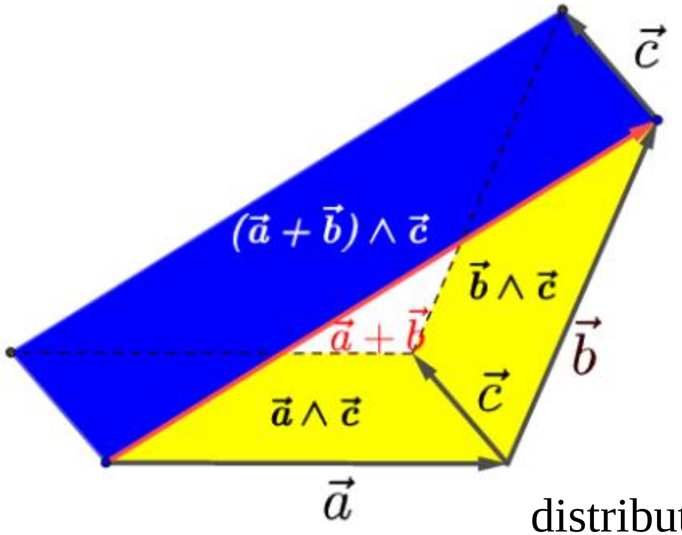
Outer product

- Anti-commutative
  - If the order of the outer product is reversed, the orientation of the bivector is also reversed = anti-commutative.
    - $u \wedge v = - v \wedge u$



- Associative
  - $(u \wedge v) \wedge w = u \wedge (v \wedge w)$

- Distributive
  - $(u \wedge w) + (v \wedge w) = (u + v) \wedge w$



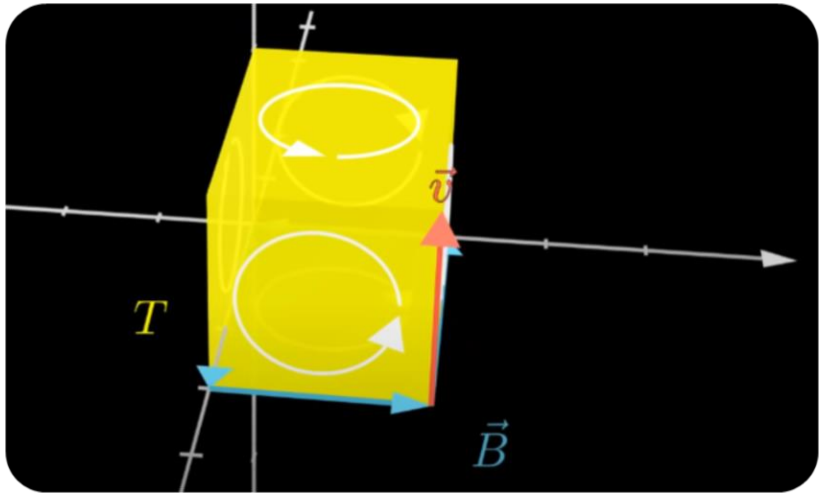
distributive law example[dis]

Products in GA

Outer product

- Outer product of two parallel vectors = no parallelogram = 0
  - Therefore, the outer product of any vector with itself is zero.

- The outer product between a bivector and a vector results a trivector



$$\vec{B} \wedge \vec{v} = T$$

Products in GA

Numerical explanation of the outer product

- The Cartesian plane R^2 is a real vector space equipped with a basis consisting of a pair of unit vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

- Suppose that $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix} = a\mathbf{e}_1 + b\mathbf{e}_2, \quad \mathbf{w} = \begin{bmatrix} c \\ d \end{bmatrix} = c\mathbf{e}_1 + d\mathbf{e}_2$ are a pair of given vectors in R^2.

- There is a unique parallelogram having *v* and *w* as two of its sides. The area of this parallelogram is given by the standard determinant formula:

$$\text{Area} = \left| \det \begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix} \right| = \left| \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right| = |ad - bc| .$$

- Consider now the exterior product of *v* and *w*:

$$\begin{aligned} \mathbf{v} \wedge \mathbf{w} &= (a\mathbf{e}_1 + b\mathbf{e}_2) \wedge (c\mathbf{e}_1 + d\mathbf{e}_2) \\ &= ac\mathbf{e}_1 \wedge \mathbf{e}_1 + ad\mathbf{e}_1 \wedge \mathbf{e}_2 + bc\mathbf{e}_2 \wedge \mathbf{e}_1 + bd\mathbf{e}_2 \wedge \mathbf{e}_2 \\ &= (ad - bc)\, \mathbf{e}_1 \wedge \mathbf{e}_2 \end{aligned}$$

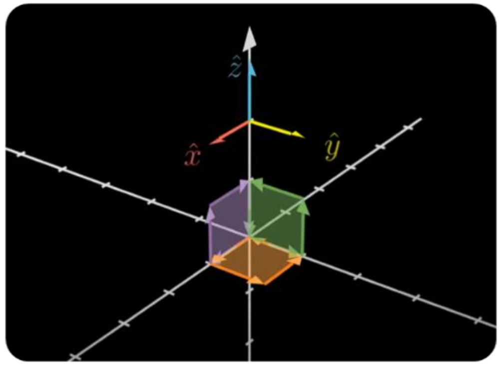
Products in GA

Numerical explanation of the outer product

- Let's take a look at the 3-dimensional oriented vector space.
- Using a standard basis (e_1, e_2, e_3), the outer product of a pair of vectors *u* and *v* is

$$\mathbf{u} \wedge \mathbf{v} = (u_1 v_2 - u_2 v_1)(\mathbf{e}_1 \wedge \mathbf{e}_2) + (u_2 v_3 - u_3 v_2)(\mathbf{e}_2 \wedge \mathbf{e}_3) + (u_3 v_1 - u_1 v_3)(\mathbf{e}_3 \wedge \mathbf{e}_1),$$

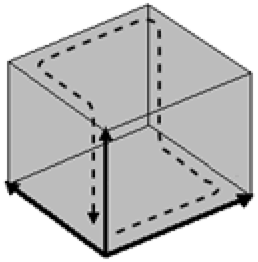
where ($e_1 \wedge e_2$, $e_2 \wedge e_3$, $e_3 \wedge e_1$) is a basis for the three-dimensional space $\wedge^2(R^3)$.



- The outer product of three vectors is

$$\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} = (u_1 v_2 w_3 + u_2 v_3 w_1 + u_3 v_1 w_2 - u_1 v_3 w_2 - u_2 v_1 w_3 - u_3 v_2 w_1)(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3)$$

where $e_1 \wedge e_2 \wedge e_3$ is the basis vector for the one-dimensional space $\wedge^3(R^3)$.

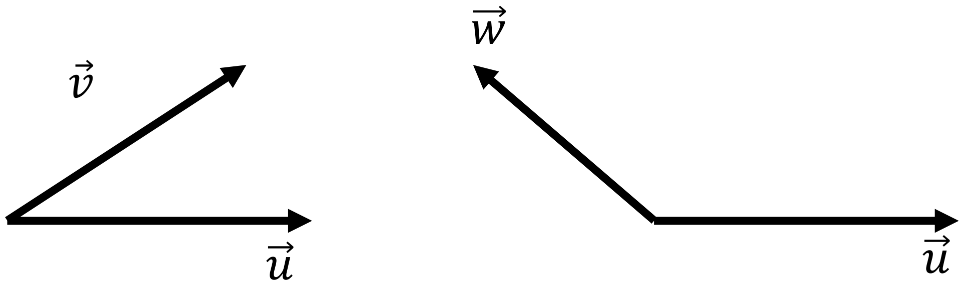


$e_i \wedge e_j \wedge e_k$

Geometric Product

Inner product and outer product

- Commutative vs. anti-commutative
  - Inner product of two vectors = $\|\vec{a}\|\ \|\vec{b}\|\ \cos\theta$
  - Outer product of two vectors = $\|\vec{a}\|\ \|\vec{b}\|\ \sin\theta$
- However, the outer product still has the same problem as the inner product: it does not provide sufficient information about the original vectors.



$$\vec{u} \wedge \vec{v} = \vec{u} \wedge \vec{w}$$

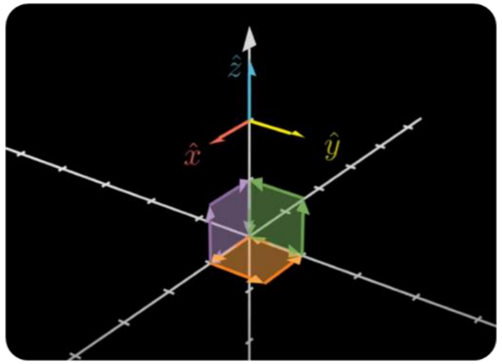
Geometric Product

Combining the inner product and outer product

- Adding the inner product and outer product.
  - Geometric product: $\vec{u}\vec{v} = \vec{u} \cdot \vec{v} + \vec{u} \wedge \vec{v} = ||\vec{u}||\ ||\vec{v}||\ \cos\theta + ||\vec{u}||\ ||\vec{v}||\ \sin\theta$
- Note that the inner product is a scalar and the outer product is a bivector = sounds impossible.
- To enable this, let's exploit the concept of 'complex number' = $a + bi$ = scalar + bivector (imaginary).

Understanding geometric product

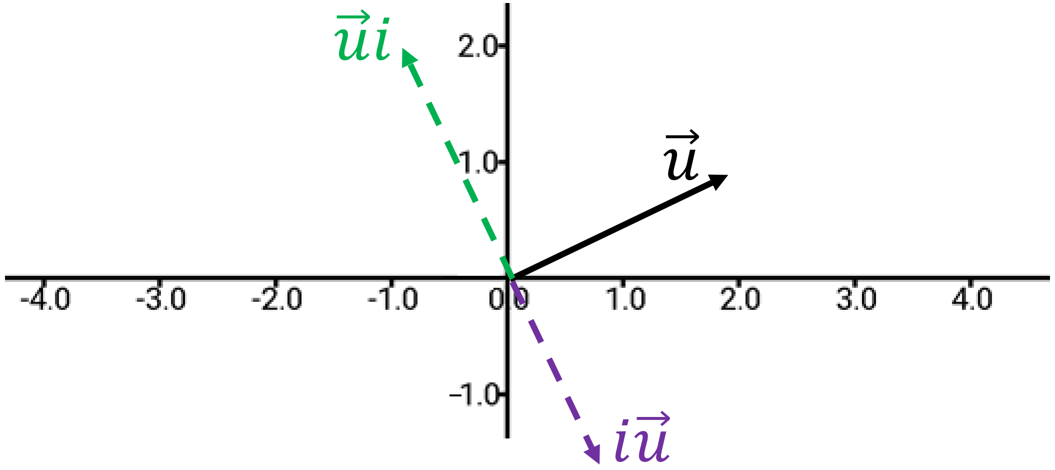
- To understand the product, let's consider the form with itself:
  - $\vec{u}\vec{u} = \vec{u} \cdot \vec{u} + \vec{u} \wedge \vec{u} = ||\vec{u}||^2 + 0.$
- The inverse of the vector can be represented by as follows:
  - $\frac{\vec{u}}{||\vec{u}||^2}\vec{u} = 1 \rightarrow \vec{u}^{-1} = \frac{\vec{u}}{||\vec{u}||^2}$
- The inner and outer products can be represented by the addition and subtraction of the geometric products:
  - $\vec{u}\vec{w} = \vec{u} \cdot \vec{w} + \vec{u} \wedge \vec{w}$
  - $\vec{w}\vec{u} = \vec{u} \cdot \vec{w} - \vec{u} \wedge \vec{w}$ (commutative and anti-commutative)
  - $\vec{u}\vec{w} + \vec{w}\vec{u} = 2\vec{u} \cdot \vec{w} \rightarrow \vec{u} \cdot \vec{w} = \frac{\vec{u}\vec{w} + \vec{w}\vec{u}}{2}$
  - $\vec{u}\vec{w} - \vec{w}\vec{u} = 2\vec{u} \wedge \vec{w} \rightarrow \vec{u} \wedge \vec{w} = \frac{\vec{u}\vec{w} - \vec{w}\vec{u}}{2}$
- Geometric product of two different basis vectors is simply their outer product:
  - $\{\hat{x} \wedge \hat{y}, \hat{y} \wedge \hat{z}, \hat{x} \wedge \hat{z}\} = \{\hat{x}\hat{y}, \hat{y}\hat{z}, \hat{x}\hat{z}\}$



Rotation with Geometric Algebra

2D geometric algebra

- In linear algebra in 2-dimensions, a vector is used and it has two components:
  - $\vec{u} = a\hat{x} + b\hat{y}$
- In geometric algebra in 2-dimensions, a multi-vector is used and it has four components:
  - $V = a + b\hat{x} + c\hat{y} + d\hat{x}\hat{y}$
- Exploiting the *pseudoscalar*, let's investigate the multiplication between the vector and bivector.
  - $\vec{u} = (2, 1) = 2\hat{x} + \hat{y}$
  - $\vec{u}i = 2\hat{x}\hat{x}\hat{y} + \hat{y}\hat{x}\hat{y} = 2\hat{y} - \hat{x}$ (let denote $\hat{x}\hat{y} = i$)
  - $i\vec{u} = -2\hat{y} + \hat{x}$
- The square of the pseudoscalar results in -1.
  - $(\hat{x}\hat{y})^2 = \hat{x}\hat{y}\hat{x}\hat{y} = -\hat{x}\hat{y}\hat{y}\hat{x} = -1$



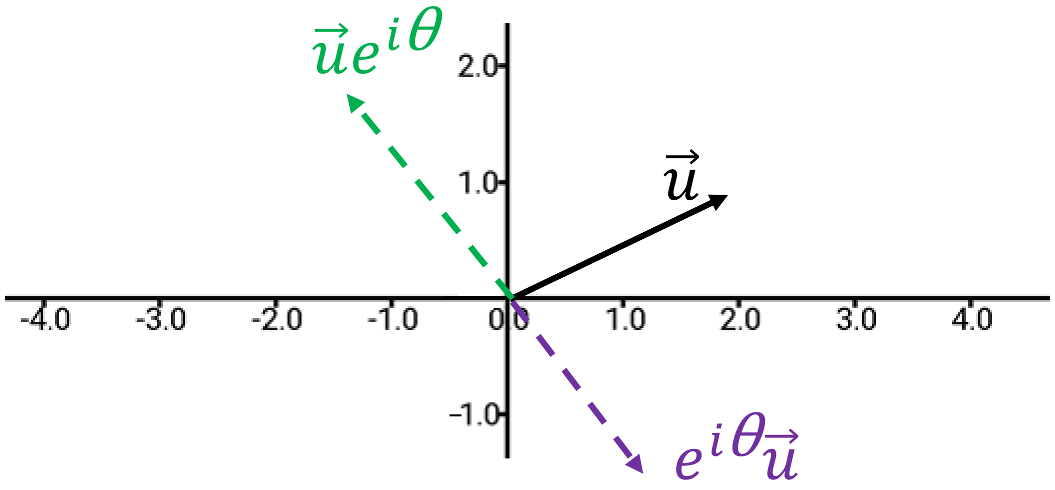
Pseudoscalar?

A pseudoscalar in a geometric algebra is a highest-grade element of the algebra. For example, in two dimensions there are two orthogonal basis vectors, $\{\hat{x}, \hat{y}\}$, and the associated highest-grade basis element is $\hat{x}\hat{y} = \widehat{xy}$. It behaves like the imaginary scalar i in the complex numbers ($\widehat{xy}^2 = -1$).

Rotation with Geometric Algebra

2D geometric algebra

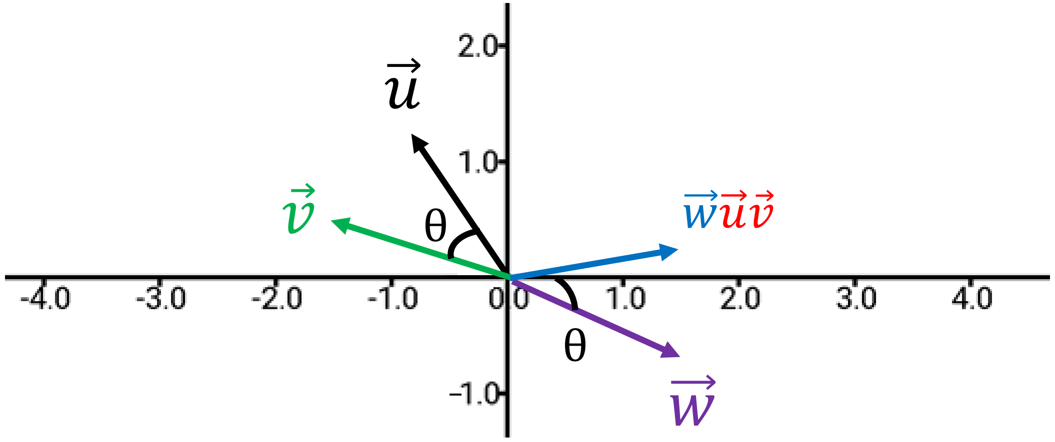
- Let's rotate a vector $\vec{u}$ by an angle θ.
  - By using a complex number that represents this rotation, we can find the result easily.
    - $a + bi$ - Cartesian form
    - $r(\cos\theta + i\sin\theta)$ - Polar form
  - In terms of unit complex number, the complex number can be written as the exponential form.
    - $\cos\theta + i\sin\theta = e^{i\theta}$ - Euler's formula



Rotation with Geometric Algebra

2D geometric algebra

- Geometric product = cos + sin:
  - $\vec{u}\vec{v} = \|\vec{u}\|\ \|\vec{v}\|\ \cos\theta + \|\vec{u}\|\ \|\vec{v}\|\ \sin\theta i = \|\vec{u}\|\ \|\vec{v}\|\ e^{i\theta}$
  - If $\vec{u}$ and $\vec{v}$ are normalized vector, $\vec{u}\vec{v} = e^{i\theta}$.
- Conjugate of the geometric product:
  - $\vec{v}\vec{u} = \vec{u} \cdot \vec{v} - \vec{u} \wedge \vec{v} \rightarrow (\vec{u}\vec{v})^* = \vec{v}\vec{u}$
  - $\vec{w}\vec{u}\vec{v} = (\vec{u}\vec{v})^*\vec{w}$

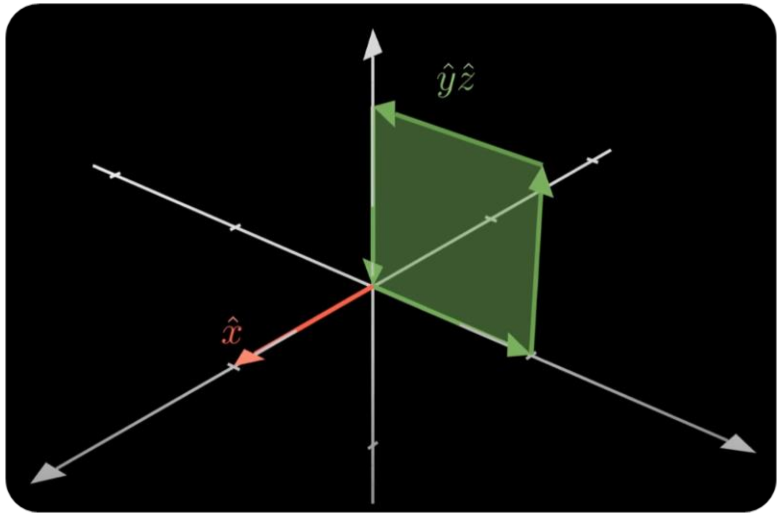


Rotation with Geometric Algebra

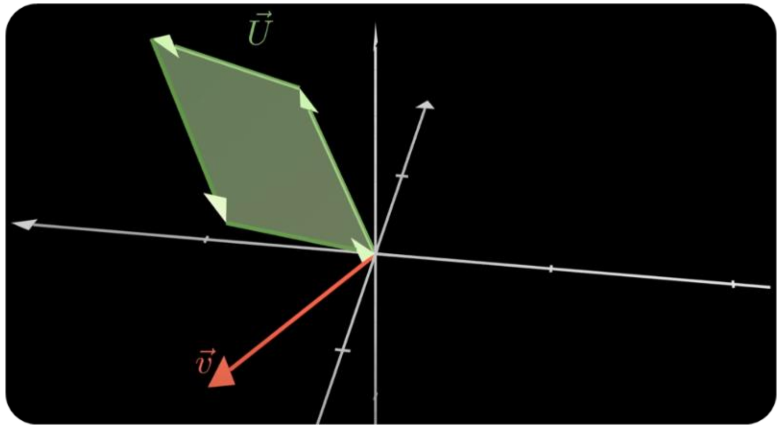
3D geometric algebra

- In geometric algebra in 3-dimensions, a multi-vector has eight components:
  - $V = a + b\hat{x} + c\hat{y} + d\hat{z} + e\hat{x}\hat{y} + f\hat{y}\hat{z} + g\hat{x}\hat{z} + h\hat{x}\hat{y}\hat{z}$
- In 3D, the *pseudoscalar* = $\hat{x}\hat{y}\hat{z} = i$.
  - $\hat{x}i = \hat{x}\hat{x}\hat{y}\hat{z} = \hat{y}\hat{z}$
  - $\vec{v}i = \vec{U}$ (note. $\vec{U}$ = bivector)
  - $-\vec{v} = \vec{U}i$
- Therefore, a general 3D multi-vector can be rewritten as follows:
  - $a + \vec{u} + \vec{v}i + bi$

- Outer product and cross product:
  - $\vec{u} = a_1\hat{x} + b_1\hat{y} + c_1\hat{z}$
  - $\vec{v} = a_2\hat{x} + b_2\hat{y} + c_2\hat{z}$
  - $\vec{u}\vec{v} = a_1a_2 + b_1b_2 + c_1c_2 + (a_1b_2 - b_1a_2)\hat{x}\hat{y} + (b_1c_2 - c_1b_2)\hat{y}\hat{z} + (a_1c_2 - c_1a_2)\hat{x}\hat{z}$
  - $\vec{u} \cdot \vec{v} = a_1a_2 + b_1b_2 + c_1c_2$
  - $\vec{u} \wedge \vec{v} = (a_1b_2 - b_1a_2)\hat{x}\hat{y} + (b_1c_2 - c_1b_2)\hat{y}\hat{z} + (a_1c_2 - c_1a_2)\hat{x}\hat{z}$
  - $\vec{u} \times \vec{v} = (a_1b_2 - b_1a_2)\hat{z} + (b_1c_2 - c_1b_2)\hat{x} + (a_1c_2 - c_1a_2)\hat{y}$
  - Therefore, $\vec{u} \wedge \vec{v} = i\vec{u} \times \vec{v}$



Curl your right hand fingers around the bivector, your thumb will point the direction of the vector

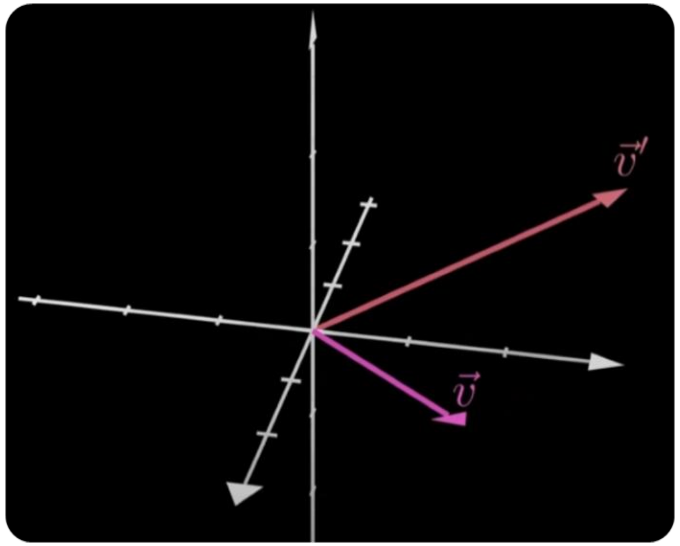


Due to the minus sign, this follows the left-hand rule.

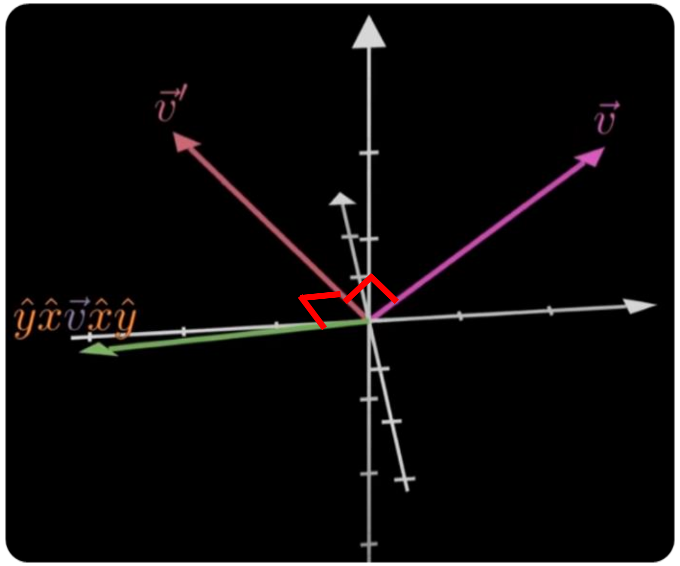
Rotation with Geometric Algebra

3D geometric algebra

- Let's rotate a vector $\vec{v}$ by 90 around the *z*-axis resulting in $\vec{v}$`:
  - $\vec{v} = 3\hat{x} + 2\hat{y} + \hat{z}$
  - $\vec{v}` = 3\hat{y} - 2\hat{x} + \hat{z}$
- As we did in 2D rotation, let's rotate the vector with a bivector $\hat{x}\hat{y}$:
  - $\vec{v}\hat{x}\hat{y} = (3\hat{x} + 2\hat{y} + \hat{z})\hat{x}\hat{y} = 3\hat{y} - 2\hat{x} + \hat{x}\hat{y}\hat{z}$ (Trivector remains..)
- Let's try another. Multiplying the complex conjugate to the left side:
  - $\vec{v}\hat{x}\hat{y} = (\hat{x}\hat{y})^*\vec{v} = \hat{y}\hat{x}\vec{v}$
  - $\hat{y}\hat{x}\vec{v} = \hat{y}\hat{x}(3\hat{x} + 2\hat{y} + \hat{z}) = 3\hat{y} - 2\hat{x} - \hat{x}\hat{y}\hat{z}$ (negative Trivector remains..)
- Then, let's combine them:
  - $\hat{y}\hat{x}\vec{v}\hat{x}\hat{y} = -3\hat{x} - 2\hat{y} + \hat{z}$ (Rotated too much..)
- Half the rotate angle:
  - $\hat{x}\hat{y} = e^{i\theta}$
  - $e^{-i\frac{\theta}{2}}\vec{v}e^{i\frac{\theta}{2}} = 3\hat{y} - 2\hat{x} + \hat{z} = \vec{v}$`
- In conclude, to rotate a vector $\vec{v}$ by an angle θ in the plane $\hat{I}$:
  - $e^{-\hat{I}\frac{\theta}{2}}\vec{v}e^{\hat{I}\frac{\theta}{2}}$
  - we call $e^{\hat{I}\frac{\theta}{2}}$ Rotor.



$\vec{v}$ and $\vec{v}$`



$\vec{v}$, $\vec{v}$`, and $\hat{y}\hat{x}\vec{v}\hat{x}\hat{y}$

Rotation with Quaternion

Quaternion

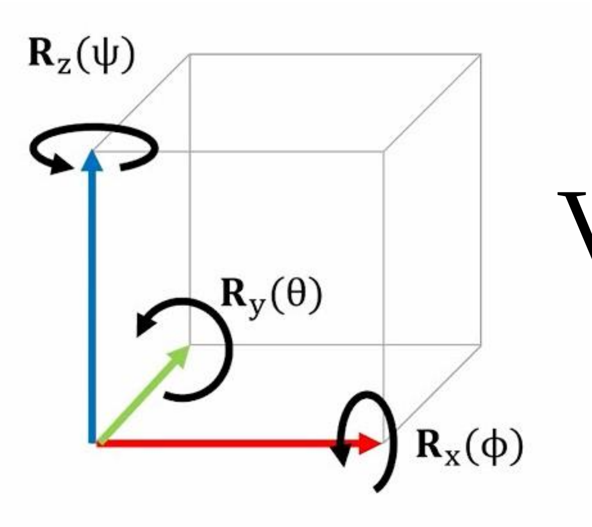
- Quaternions provide a convenient mathematical notation for representing spatial orientations and rotations of elements in three dimensional space.
  - Quaternions encode information about an axis-angle rotation about an arbitrary axis.
  - Compared to rotation matrices, quaternions are more compact, efficient, and numerically stable.
  - However, quaternions are not intuitive and easy to understand.

Rotation matrices

$$\boldsymbol{R}_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$$

$$\boldsymbol{R}_y(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\boldsymbol{R}_z(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

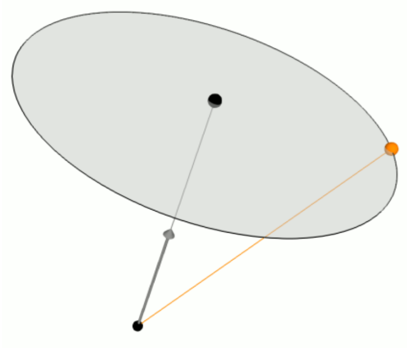


VS.

Quaternions

$$\mathbf{q} = (\cos(\theta/2), \sin(\theta/2)\vec{a}) = (w, (x, y, z))$$

$$R_{\mathbf{q}} = \begin{pmatrix} 1-2y^2-2z^2 & 2xy-2wz & 2xz+2wy & 0 \\ 2xy+2wz & 1-2x^2-2z^2 & 2yz-2wx & 0 \\ 2xz-2wy & 2yz+2wx & 1-2x^2-2y^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Rotation with Quaternion

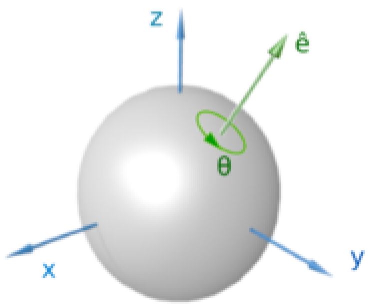
Quaternion

- In a 3-dimensional space, according to Euler's rotation theorem, any rotation of a rigid body about a fixed point is equivalent to a single rotation by a given angle θ about a fixed axis (called the Euler axis).

- The Euler axis is typically represented by a unit vector $\hat{e}$. Therefore, any rotation in three dimensions can be represented as a combination of a vector $\hat{e}$ and a scalar θ.

- Quaternions give a simple way to encode this axis-angle representation in four numbers, and can be used to apply the corresponding rotation to a position vector (x, y, z).

- Euclidean vectors such as (2, 3, 4) can be rewritten as $2i + 3j + 4k$, where i, j, k are unit vectors representing the three Cartesian axes (or $2e_1 + 3e_2 + 4e_3$).



a rotation about an Euler axis (ê) by and angle of θ

- A rotation of angle θ around the axis defined by the unit vector ($\vec{u} = (u_x, u_y, u_z) = u_x i + u_y j + u_z k$) can be represented by a quaternion using an extension of Euler's formula:

$$\mathbf{q} = e^{\frac{\theta}{2}(u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k})} = \cos\frac{\theta}{2} + (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})\sin\frac{\theta}{2}$$

Rotation with Quaternion

Quaternion

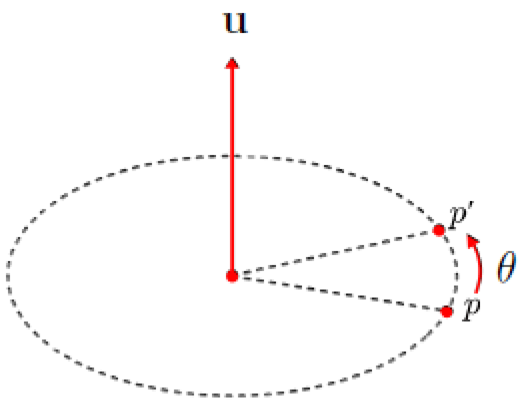
- Then, the desired rotation can be applied to an ordinary vector p = (p_x, p_y, p_z):

$$\mathbf{p}' = \mathbf{q}\mathbf{p}\mathbf{q}^{-1}$$

where p\` = ($p_x$\`, p_y\`, $p_z$\`) is the new position vector of the point after the rotation.

- A quaternion rotation p\` = $qpq^{-1}$ can be algebraically manipulated into a matrix rotation p\`= Rp, where R is the rotation matrix given by:

$$\mathbf{R} = \begin{bmatrix} 1-2s(q_j^2+q_k^2) & 2s(q_iq_j-q_kq_r) & 2s(q_iq_k+q_jq_r) \\ 2s(q_iq_j+q_kq_r) & 1-2s(q_i^2+q_k^2) & 2s(q_jq_k-q_iq_r) \\ 2s(q_iq_k-q_jq_r) & 2s(q_jq_k+q_iq_r) & 1-2s(q_i^2+q_j^2) \end{bmatrix}$$

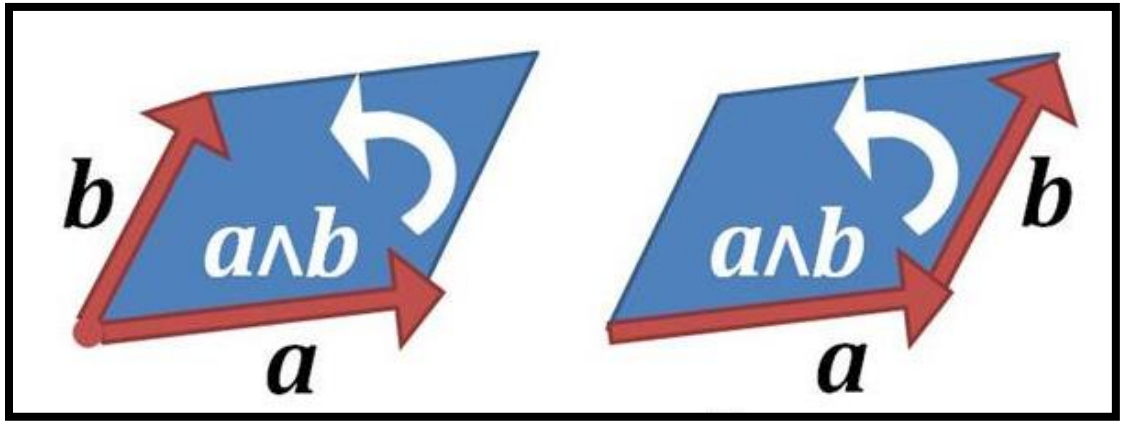


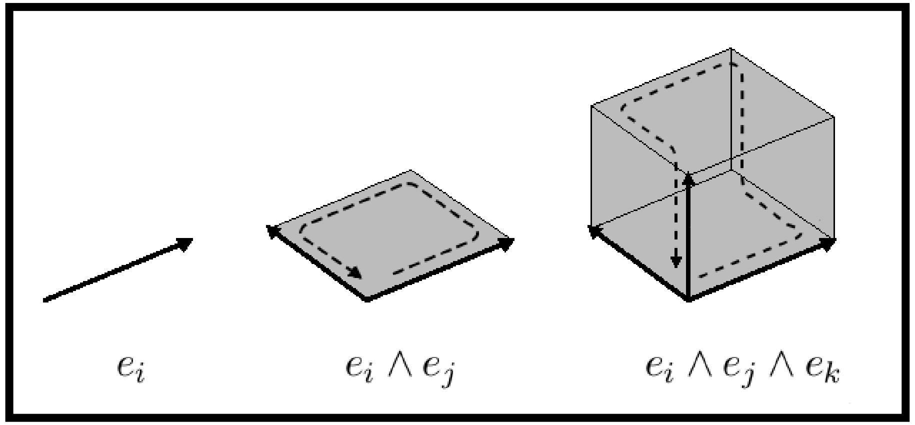
a rotation about an Euler axis (u) by and angle of θ

The Articulated Body Model

Orientations and rotations

- Most of the inverse kinematics are implemented using object orientations and rotations.
- Fortunately, Geometric Algebra provides a convenient mathematical notation for representing orientations and rotations of objects in three dimensions.
- The multi-vector orthonormal basis can be defined by multiple number of orthonormal basis vector (e_1, e_2, ..)
  - Bivector: a bivector or 2-vector is a quantity that extends the idea of scalars and vectors. If a scalar is considered a degree zero quantity, and a vector is a degree one quantity, then a bivector can be though of as being of degree two. Bivectors can be used to represent rotations in any number of dimensions.
  - Trivector: A multi-vector of grade three.
- The scalar plus bivector (known as a rotor R) represents a counter-clockwise rotation θ in a plane specified by the bivector B: R = $\exp(-B\frac{\theta}{2})$





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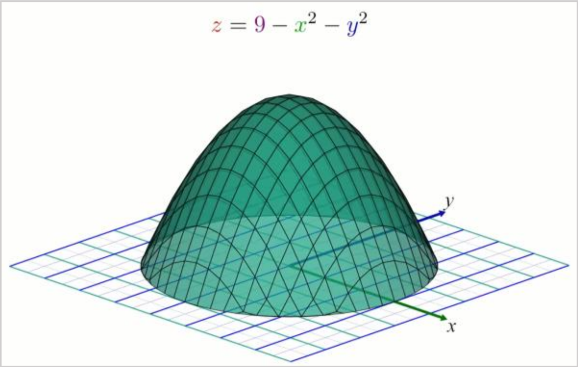
[dis] https://www.researchgate.net/publication/323355879_From_Vectors_to_Geometric_Algebra

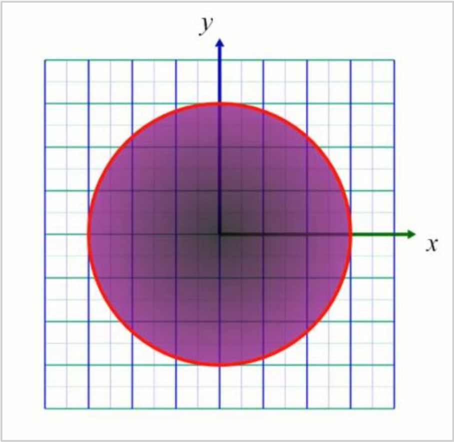
Jacobian - Practice

Q. In the polar coordinate system, find the area made of the following equation:

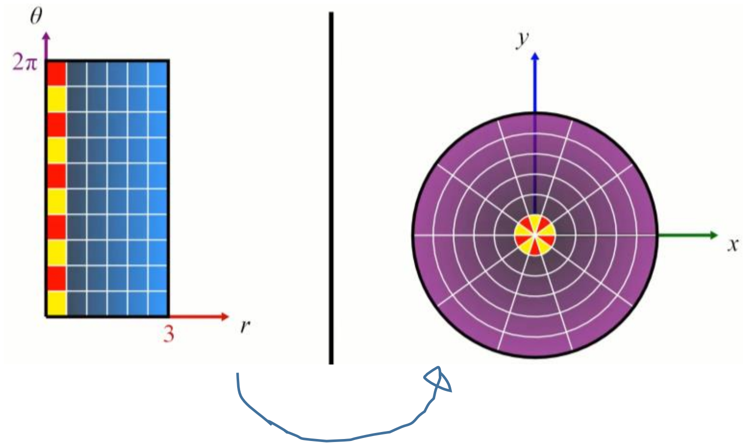
$z = 9 - x^2 - y^2$



In Polar Coordinates



Hint



$J(r, \theta) = r$

야코비안 행렬이 r, θ 좌표계와 x,y 좌표계(Polar coordinate) 사이 변환에 사용

1) https://angeloyeo.github.io/2020/07/24/Jacobian.html
2) https://www.youtube.com/watch?v=hhFzJvaY__U

Jacobian - Practice Solution

Q. In the polar coordinate system, obtain the area of the following equation:
$z = 9 - x^2 - y^2$

A. $\int_0^{2\pi}\int_0^3 (9 - x^2 - y^2)\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^3 (9 - r^2) J(r,\theta)\,dr\,d\theta$ $(x = r\cos\theta, y = r\sin\theta)$

$= \int_0^{2\pi}\int_0^3 (9 - r^2)\,r\,dr\,d\theta$

$= \frac{81}{2}\pi$

1) https://angeloyeo.github.io/2020/07/24/Jacobian.html
2) https://www.youtube.com/watch?v=hhFzJvaY__U

Vector - Practice

Q. There are two vectors $\vec{u} = (3i + 2j), \vec{v} = (-2i, 3j)$. Find the area made of these two vectors (Exploit the $\wedge$).

Vector – Practice Solution

Q. There are two vectors $\vec{u} = (3i + 2j), \vec{v} = (-2i, 3j)$. Find the area made of these two vectors (Exploit the $\wedge$).

A. Area $= (3i + 2j) \wedge (-2i + 3j)$

$= (3i) \wedge (-2i + 3j) + 2j \wedge (-2i + 3j)$

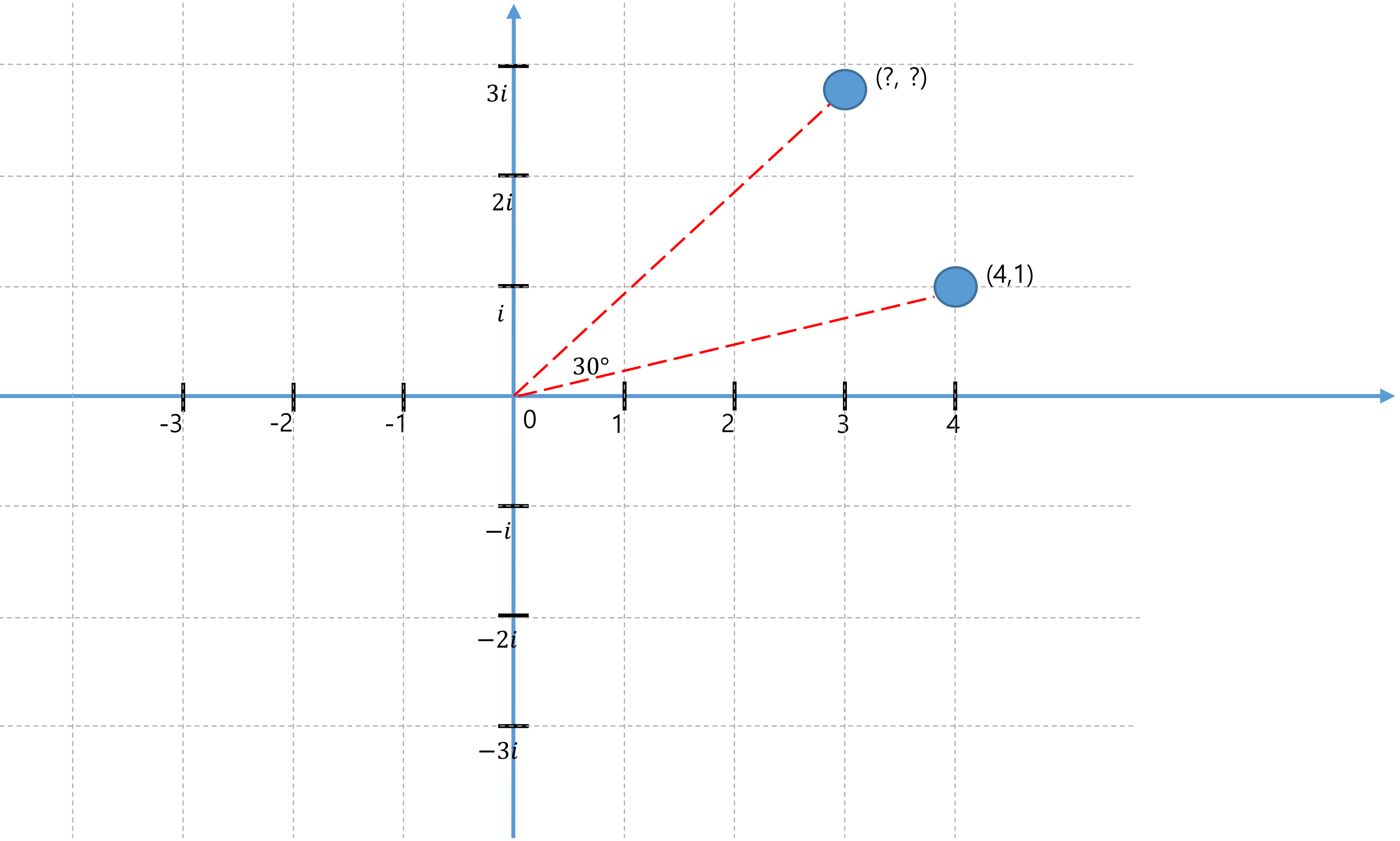
$= -6i \wedge i + 9\, i \wedge j - 4\, j \wedge i + 6j \wedge j$

$= 9\, i \wedge j + 4\, i \wedge j$

$= 13\, i \wedge j$

2D Rotation - Practice

Q. Find the angle when a point is rotated by 30 degrees. The point is in (4,1).



Hint : 2d rotation matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

출처 : https://www.youtube.com/watch?v=zjMuIxRvygQ – 3:51

2D Rotation – Practice Solution

Q. Find the angle when a point is rotated by 30 degrees. The point is in (4,1).