Regarding "Arbitrary" Elements...

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When writing proofs (particularly inductive ones), it is crucial to keep in mind which items are arbitrary and which items are chosen/constructed by you. In the example question below, we illustrate how mixing this up will lead to an incorrect proof. First, let us define our question.
Note: In the context of this document, 0 *is included in* $\mathbb{N}$.

> *Example Question*:
> For $n \in \mathbb{Z}^+$, let T_n denote the set of trees with vertex set $\{v \in \mathbb{N} | v \leq n\}$. Let $T = \bigcup_{n=1}^{\infty} T_n$.
> For $t \in T$, t is "happy" if $\exists v \in \mathbb{N}$ s.t. $\{v, v+1\} \in t$ $\big[$i.e., some edge $\{v, v+1\}$ is present in $t\big]$.
> For $n \in \mathbb{Z}^+$, let $P(n)$ denote that "Every tree in T_n is happy".
> Prove or disprove: $\forall n \in \mathbb{Z}^+$, $P(n)$ holds.

Here's an incorrect student solution. Try to locate the mistake while reading it.

> *(Incorrect) Student Solution*:
> Through simple induction, let's prove that $\forall n \in \mathbb{Z}^+$, $P(n)$ holds.
> **BASE CASE:** $P(1)$
> The only tree in T_1 is the tree consisting of vertices 0 and 1 and the edge $\{0, 1\}$. As the edge $\{0, 1\}$ is present in this tree, the tree is happy, so $P(1)$ holds.
>
> **INDUCTION STEP:** $\forall k \in \mathbb{Z}^+, P(k) \implies P(k+1)$
> Let $k \in \mathbb{Z}^+$ be given such that $P(k)$ holds. We will show $P(k+1)$ holds.
> Let $t \in T_k$ be given.
> Let an arbitrary $w \in \{v \in \mathbb{N} | v \leq k\}$ be given.
> Add the vertex $k+1$ to t, and add the edge $\{w, k+1\}$ to t. Call this new tree t'.
> As $P(k)$ holds, t must be happy, so there exists $h \in \{v \in \mathbb{N} | v \leq k\}$ such that $\{h, h+1\} \in t$.
> Note that t is a subtree of t', so $\{h, h+1\} \in t'$.
> As t' was constructed from an arbitrary w and an arbitrary t, t' is also arbitrary tree from T_{k+1}
> Thus, $P(k+1)$ holds.
>
> Thus, by induction, $\forall n \in \mathbb{Z}^+$, $P(n)$ holds.

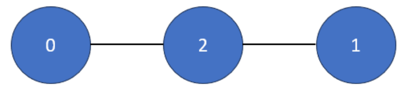
Did you find the mistake? If you'd like to find it for yourself, stop reading ahead for now.

The mistake is rather simple: t' is not an *arbitrary* tree from T_{k+1} despite being constructed by an arbitrary t from T_k and connecting the vertex $k+1$ to an arbitrary existing vertex. As the student was the one who made t', t' is a *construction of the student.*

Why does this matter?

For $k \in \mathbb{Z}^+$, let us denote the student's construction process of t' from an arbitrary $t \in T_k$ and an arbitrary $w \in \{v \in \mathbb{N} | v \leq k\}$ as $t' = c(t, w)$.
For $k \in \mathbb{Z}^+$, let $C_k = \big\{c(t, w) \big| t \in T_k,\, w \in \{v \in \mathbb{N} | v \leq k\}\big\}$.
For $k \in \mathbb{Z}^+$, we know for a fact that $C_k \subseteq T_{k+1}$, but is it necessarily true that $C_k = T_{k+1}$? If $C_k \neq T_{k+1}$, since we only showed that the trees in C_k are happy, there can easily be trees that are not happy in T_{k+1}, resulting in $P(k+1)$ not being true.

This, in fact, is the case with this proof. Briefly consider the case where $k = 1$. Then, the following tree is an element of T_{k+1}:

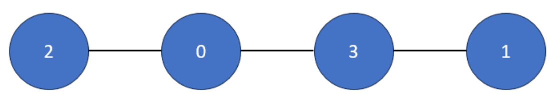


The procedure c can construct a tree in T_{k+1} that contains the edge $\{0, 2\}$ or $\{1, 2\}$, but it will not be able to construct a tree with both of these edges (convince yourself of this!). As such, this tree is not an element of C_k and so $C_k \neq T_{k+1}$.

This is the main issue of going bottom-up during your inductive step when dealing with sets: you need to show that you have considered every single possible case in the next step (i.e., showing that $C_k = T_{k+1}$ for all $k \in \mathbb{Z}^+$, where C_k and T_{k+1} are as defined in the previous paragraphs). Although this is easy to deal with when you have a recursively defined structure, when the structure is not recursively defined, *showing that this is true may not be trivial.* As such, whenever you write a proof by induction, for the inductive step, **please do not use an instance of the k-th step to generate an instance of the (k+1)-th step**. The result may not necessarily be an arbitrary instance of the (k+1)-th step. Instead, **please consider taking an arbitrary instance of the (k+1)-th step and consider deconstructing it to apply the induction hypothesis**. This way, the arbitrary instance of the (k+1)-th step is guaranteed to be arbitrary.

Still not convinced the student is wrong? Here's a correct solution.

Solution: Observe that the following tree is an element of T_3.



As this tree is not happy, $P(3)$ does not hold, so the statement must be false.