price of 2 apples and 3 bannanas is 8

" " 10 " " 16 " is 13

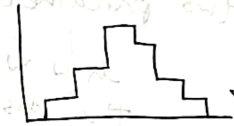
$$\left.\begin{aligned} 2a+3b&=8 \\ 10a+1b&=13 \end{aligned}\right\} \text{simultaneous equations}$$

a = price of one apple   b = price of 1 bannana

finding a, b $\longrightarrow$ $\boxed{\text{linear algebra problem}}$

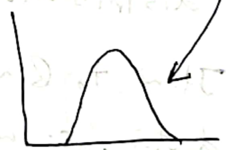
$$\underbrace{\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix}}_{\text{matrix}} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 8 \\ 13 \end{bmatrix}}_{\text{vector}}$$

suppose we have a histogram



a we want a eqnation that fits this data.

we need to find the values of parameters that best fits this data.



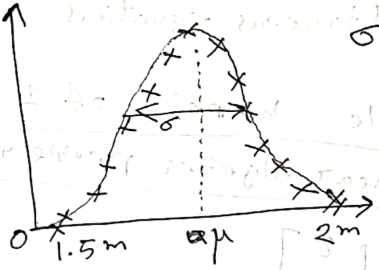
$\boxed{\text{optimization problem}}$

A example of curve fitting problem

μ = expected value / mean

σ = standard deviation

see wikipedia: Normal distribution



$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\}$$ normal / [gaussian distribution]

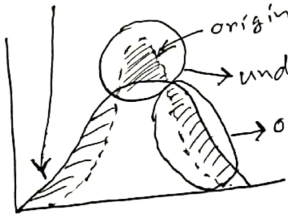
two parameters = σ, μ — center of the distribution

how wide the distribution is

area = 1, since 100% of people are in the distribution as a whole

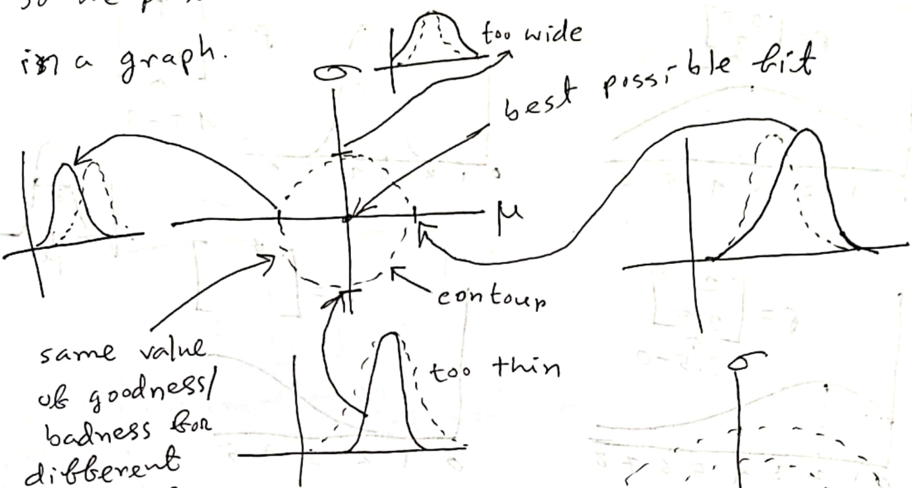
How to find σ, μ such that it best fits the data, as well as it is possible.

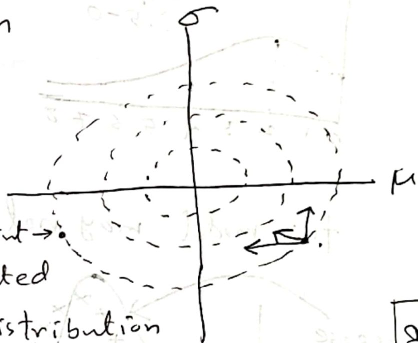
let's say we find this fit



original data
underestimation
overestimation

can be used to see how good or bad our fit is so we plot the sum of difference of values squared in a graph.





let's say we are at this point → and ~~we move along~~ it indicated that ~~the~~ we thought the distribution is shifted toward left and it is too thin. so we determine where to move in the parameter space by ~~seeing~~ using calculus

$$\underbrace{\begin{bmatrix} \mu \\ \sigma \end{bmatrix}}_{\text{vector}} \begin{bmatrix} \mu' \\ \sigma' \end{bmatrix}$$

↓

can be seen as a list

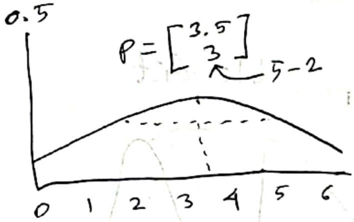
$\begin{bmatrix} \text{Nox} \\ \text{speed} \\ \text{tires} \\ \text{windows} \end{bmatrix}$

$\boxed{\text{different ways of seeing vectors}}$ → numerical, geometrical, algebraical

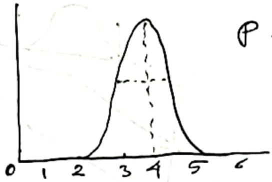
general relativity
↓
time is the 4th dimension

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} x & \text{meter} \\ y & \text{meter} \\ z & \text{meter} \\ s & \text{second} \end{bmatrix}$$

$P = \begin{bmatrix}\mu \\ \sigma\end{bmatrix}$

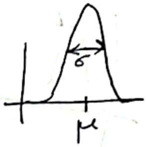


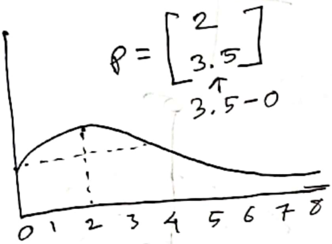
$P = \begin{bmatrix}3.5 \\ 3\end{bmatrix}$ ← 5−2



$P = \begin{bmatrix}4 \\ 1.5\end{bmatrix}$

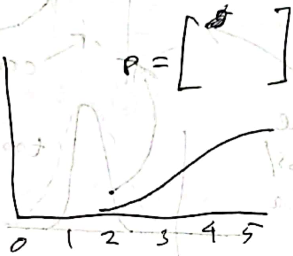
↑ 4.5−3





$P = \begin{bmatrix}2 \\ 3.5\end{bmatrix}$

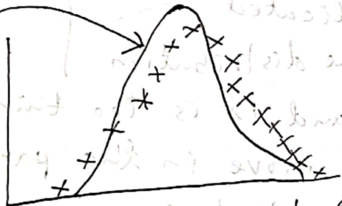
↑ 3.5−0



$P = \begin{bmatrix} \\ \end{bmatrix}$

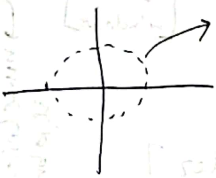
the model may not always fit the data.

Suppose this is the best curve a model can generate.

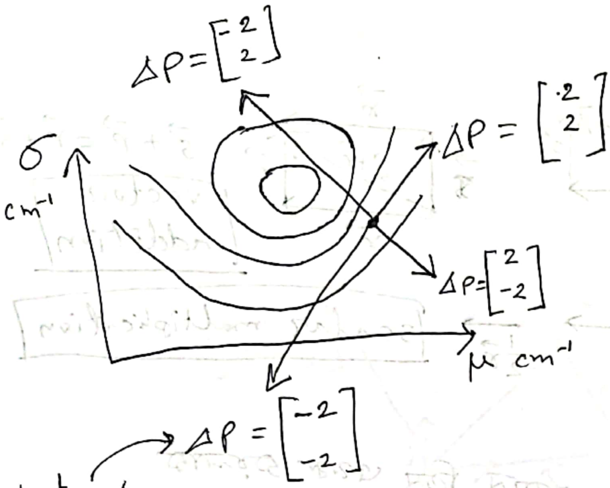


the data.

But it's not the best fit for the data.



the goodness is same for all points in the same contour line. But each point represents a different model



movement almost along a contour, so it will not improve the fit much

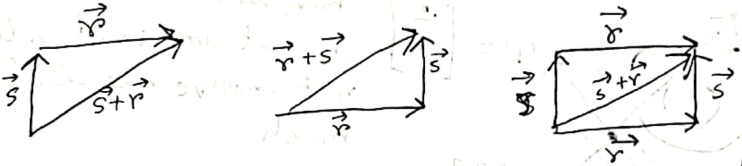
$|f - g_p|^2$

$\uparrow$

movement along contour, so it won't reduce SSR much

what change can be made to ΔP to better fit the data

$P' = P + \Delta P$

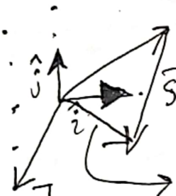


$\vec{s}+\vec{r}=\vec{r}+\vec{s}$

$\boxed{\text{vector addition}}$

$\boxed{\text{scalar multiplication}}$

ডানে তিন একক চললাম



$\vec{r}=\begin{bmatrix}3\\2\end{bmatrix}$

~~[illegible]~~ দুই একক সরলাম

$\vec{r}+\vec{s}=\begin{bmatrix}3-1\\2-2\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}$

$\vec{s}=\begin{bmatrix}-1\\-2\end{bmatrix}$

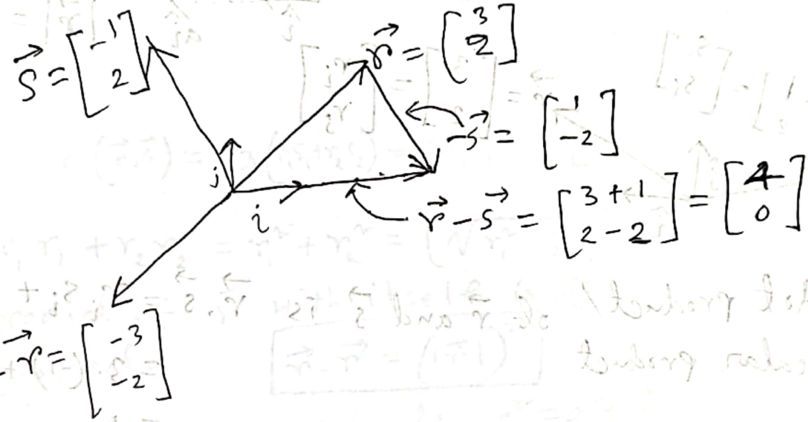
$\boxed{\text{associativity}}$ $\vec{r}+\vec{s}=\vec{s}+\vec{r}$ $(\vec{r}+\vec{s})+\vec{t}=\vec{r}+(\vec{s}+\vec{t})$

$\vec{r}=\begin{bmatrix}3\\2\end{bmatrix}$

$\vec{s}+\vec{r}=\begin{bmatrix}2\\0\end{bmatrix}$

$\begin{bmatrix}-1\\-2\end{bmatrix}=\vec{s}$

scalar multiplication



$$\vec{s}=\begin{bmatrix}-1\\2\end{bmatrix} \qquad \vec{r}=\begin{bmatrix}3\\2\end{bmatrix}$$

$$-\vec{s}=\begin{bmatrix}1\\-2\end{bmatrix}$$

$$\vec{r}-\vec{s}=\begin{bmatrix}3+1\\2-2\end{bmatrix}=\begin{bmatrix}4\\0\end{bmatrix}$$

$$-\vec{r}=\begin{bmatrix}-3\\-2\end{bmatrix}$$

$$\begin{bmatrix}120\\2\\1\\150\end{bmatrix}\begin{array}{l}\text{sqm}\\\text{beds}\\\text{bathrooms}\\\text{€,000}\end{array}$$

+

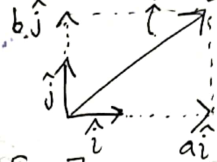
house

$$=2\begin{bmatrix}120\\2\\1\\150\end{bmatrix}=\begin{bmatrix}240\\4\\2\\300\end{bmatrix}$$ two houses

scaling information about an object

vector → length
→ direction

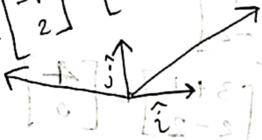
$\vec{r} = \begin{bmatrix} a \\ b \end{bmatrix} = a\hat{i} + b\hat{j}$



length/magnitude of $\vec{r}$ is

$|\vec{r}| = \sqrt{a^2+b^2}$

$s = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} s_i \\ s_j \end{bmatrix}$ $\quad \vec{r} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} r_i \\ r_j \end{bmatrix}$



dot product/ scalar product of $\vec{r}$ and $\vec{s}$ is $\vec{r}\cdot\vec{s} = r_i s_i + r_j s_j$

$= 3\cdot(-1) + 2\cdot 2$

$= 1$

dot product properties

→ commutative: $\vec{r}\cdot\vec{s} = \vec{s}\cdot\vec{r}$

→ distributive: $\vec{r}\cdot(\vec{s}+\vec{t}) = \vec{r}\cdot\vec{s} + \vec{r}\cdot\vec{t}$

over addition

Proof: $\vec{r} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$ $\quad \vec{s} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$ $\quad \vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix}$

$\vec{r}\cdot(\vec{s}+\vec{t}) = r_1(s_1+t_1) + r_2(s_2+t_2) + \dots + r_n(s_n+t_n)$

$= r_1 s_1 + r_1 t_1 + r_2 s_2 + r_2 t_2 + \dots + r_n s_n + r_n t_n$

$= r_1 s_1 + r_2 s_2 + \dots + r_n s_n + r_1 t_1 + r_2 t_2 + \dots + r_n t_n$

$= \vec{r}\cdot\vec{s} + \vec{r}\cdot\vec{t}$

$\boxed{\text{associative over scalar multiplication}}$ $\vec{r}\cdot(a\vec{s}) = a\cdot(\vec{r}\cdot\vec{s})$

proof: $\vec{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$ $\vec{s} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$

$\vec{r}\cdot(a\vec{s}) = \vec{r}\cdot\begin{bmatrix} as_1 \\ as_2 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}\cdot\begin{bmatrix} as_1 \\ as_2 \end{bmatrix}$

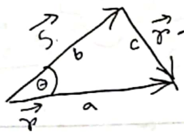
$\downarrow$

$a(\vec{r}\cdot\vec{s}) = a(r_1s_1+r_2s_2) = ar_1s_1 + ar_2s_2 = r_1as_1 + r_2as_2$

$\vec{r}\cdot\vec{r} = r_1r_1 + r_2r_2 = r_1^2 + r_2^2 = \left(\sqrt{r_1^2+r_2^2}\right)^2 = (|r|)^2$

the dot product of $\vec{r}$ with itself is equal to it's length squared. $\boxed{\vec{r}\cdot\vec{r} = (|\vec{r}|)^2}$

from cosine rule, $c^2 = a^2 + b^2 - 2ab\cos\theta$



$\Rightarrow |\vec{r}-\vec{s}|^2 = |\vec{r}|^2 + |\vec{s}|^2 - 2|\vec{r}||\vec{s}|\cos\theta$ ... (i)

By distributive property, $|\vec{r}-\vec{s}|^2 = (\vec{r}-\vec{s})(\vec{r}-\vec{s})$

$= (\vec{r}-\vec{s})\cdot(\vec{r}-\vec{s}) = r^2 - \vec{r}\cdot\vec{s} - \vec{r}\vec{s} + s^2 = r^2 - 2\vec{s}\vec{r} + s^2$

$\therefore |\vec{r}-\vec{s}| = r^2 - 2\vec{s}\cdot\vec{r} + s^2$ — (ii)

$\boxed{|\vec{r}| = r}$

$\vec{r}$ = vector

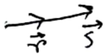
r = scalar

comparing (i) and (ii),

$r^2 + s^2 - 2rs\cos\theta = r^2 + s^2 - 2sr$

$\Rightarrow \boxed{\vec{r}\cdot\vec{s} = rs\cos\theta}$

if $\theta = 0°$ $\vec{r}\cdot\vec{s} = rs$

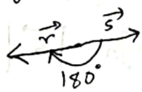


$\vec{r}, \vec{s}$ in same direction

if $\theta = 90°$ $\vec{r}\cdot\vec{s} = 0$

orthogonal

if $\theta = 180°$



$\vec{r}\cdot\vec{s} = -rs$

opposite

$\boxed{\text{scalar projection}}$ (লম্ব অভিক্ষেপ)

light source perpindicular to $\vec{r}$

$$\cos\theta = \frac{\text{সন্নিহিত বাহু}}{\text{অতিভুজ}} = \frac{\text{ভূমি}}{\text{অতিভুজ}}$$

$$= \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$\vec{r}$

$$\vec{r}\cdot\vec{s} = |\vec{r}|\,|\vec{s}|\cos\theta$$

$\boxed{|\vec{s}|\cos\theta}$

$$\Rightarrow \frac{\vec{r}\cdot\vec{s}}{|\vec{r}|} = |\vec{s}|\cos\theta$$

adjacent side of right triangle

projection of $\vec{s}$ over $\vec{r}$

scalar projection

shadow of $\vec{s}$ over $\vec{r}$ if a light source was there at 90° with $\vec{r}$

if $\vec{s}$ and $\vec{r}$ are orthogonal, scalar projection of $\vec{s}$ over $\vec{r}$ is 0

$$|\vec{s}|\cos\theta = 0$$

$\boxed{\text{vectore projection}}$

vector projection of $\vec{s}$ over $\vec{r}$ is

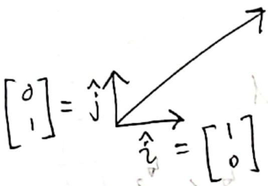
$$\boxed{\vec{r}\,\frac{\vec{r}\cdot\vec{s}}{|\vec{r}|\,|\vec{r}|}} = \vec{r}\,\frac{\vec{r}\cdot\vec{s}}{r.r} = \frac{\vec{r}}{r}\;\frac{\vec{r}\cdot\vec{s}}{r}$$

unit vector in the direction of r

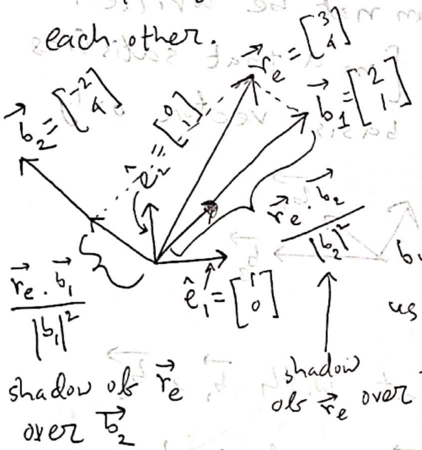
projection of $\vec{s}$ over $\vec{r}$

a vector in the direction of $\vec{r}$ of length $|\vec{s}|\cos\theta$

$\vec r = \begin{bmatrix}3\\4\end{bmatrix} = 3\hat i + 4\hat j$ $\hat i, \hat j$ form a basis for $\vec r$



the ~~chose~~ choice of the basis vectors is arbitrary. They can also be at any angle to each other.



$\vec r$ vector expressed in terms of the basis $\hat e_1, \hat e_2$ is

$$\vec r_e = \begin{bmatrix}3\\4\end{bmatrix}$$

but $\vec r$ can also be expressed using the basis vectors $\vec b_1, \vec b_2$

$$\vec r_b = \begin{bmatrix}\dfrac{\vec r_e \cdot \vec b_1}{|b_1|^2}\\[2ex] \dfrac{\vec r_e \cdot \vec b_2}{|b_2|^2}\end{bmatrix} = \begin{bmatrix}\dfrac{3\times 2 + 4\times 1}{2^2+1^2}\\[2ex] \dfrac{3\times -2 + 4\times 4}{(-2)^2+4^2}\end{bmatrix} = \begin{bmatrix}2\\ \frac{1}{2}\end{bmatrix}$$

now, $\vec b_1 \dfrac{\vec r_e \cdot \vec b_1}{|\vec b_1|^2} = \begin{bmatrix}2\\1\end{bmatrix} \cdot 2 = \begin{bmatrix}4\\2\end{bmatrix}$

and $\vec b_2 \dfrac{\vec r_e \cdot \vec b_2}{|\vec b_2|^2} = \begin{bmatrix}-2\\4\end{bmatrix} \frac{1}{2} = \begin{bmatrix}-1\\2\end{bmatrix}$

$$\begin{bmatrix}4\\2\end{bmatrix} + \begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix} = \vec r_e$$

we can change basis of $\vec r$ when the new basis vectors are orthogonal to each other n $\vec{b}_1, \vec{b}_2, \vec{b}_3$

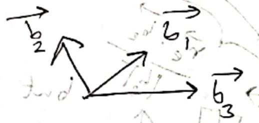
some vectors can form a basis for $\vec{r}$ iff

(i) $\vec{b}_1, \vec{b}_2, \vec{b}_3$ are linearly independent

(ii) $\vec{b}_1, \vec{b}_2, \vec{b}_3$ span the space

(iii) The space is $n=3$ dimentional

let's say we have two vectors b_1, b_2 in the xy plane and they are linearly independent, meaning b_1 can not be written as any combination of b_2 [no a that satisfies $\vec{b}_1 = a\vec{b}_2$] $\therefore$ $\vec{b}_3$ can be a basis vector only if



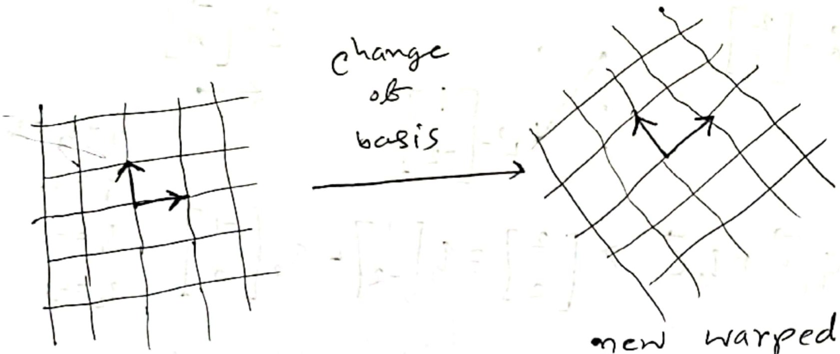
$$\vec{b}_3 \neq \underbrace{a\,\vec{b}_1 + c\,\vec{b}_2}_{\text{constants}}$$

i.e. $\vec{b}_3$ is linearly dependent with $\vec{b}_1, \vec{b}_2$

if $\vec{b}_3 = a_1\vec{b}_1 + a_2\vec{b}_2$, then $\vec{b}_3$ is linearly dependent on $\vec{b}_1, \vec{b}_2$ and it lies in the same plane as $\vec{b}_1, \vec{b}_2$

basis vectors don't have to be orthogonal or unit vectors, but it is convenient if they are.

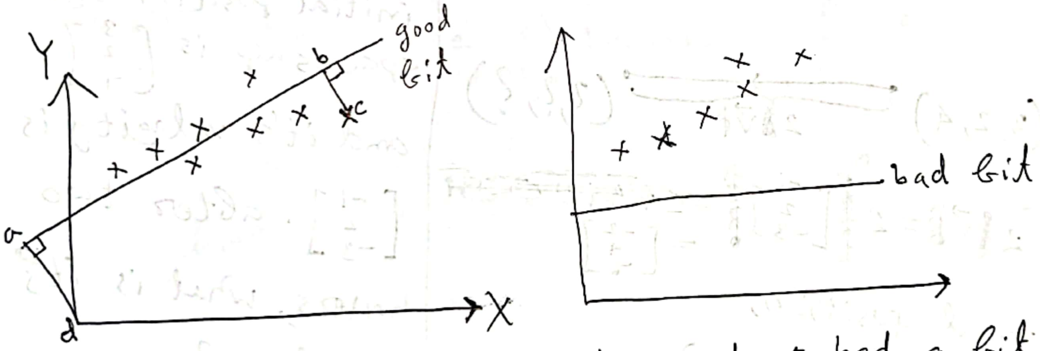
orthonormal = orthogonal + unit vectors



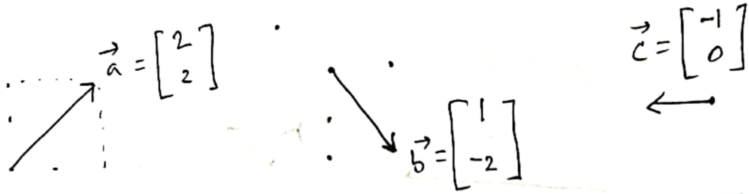
2D space

new warped space where vectors can be represented and vector addition, multiplication is possible.

for basis vectors which aren't at 90°, we can't use the dot product. we'll have to use matrix multiplication.



ab and bc help us measure how good or bad a fit is.
we can change the basis from XY plane to the $\vec{ab}$, $\vec{ad}$ basis vectors to get a better idea of the noise.



$$\vec{a} = q_1 \vec{b} + q_2 \vec{c} \Rightarrow \begin{bmatrix} 2 \\ 2 \end{bmatrix} = q_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + q_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$q_1 - q_2 = 2$
$-2q_1 = 2$
$q_1 = -1$
$q_2 = -3$

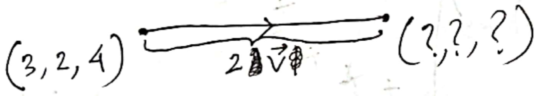
an n-dimensional space can have n linearly inpedendent vector.

$\vec{a}, \vec{b}, \vec{c}$ are linearly dependent if $\vec{a} = k_1 \vec{b} + k_2 \vec{c}$

or $\vec{b} = k_1 \vec{c} + k_2 \vec{a}$ or $\vec{c} = k_1 \vec{a} + k_2 \vec{b}$

we have two check all three conditions to tell whether $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent.

initial position of spaceship is $\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$ and it's velocity is $\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$. after two hours, what is it's location?



$$2\vec{v} = 2\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -6 \end{bmatrix}$$

final position,

$(3-2, 2+4, 4-6)$

$= \begin{bmatrix} 1 \\ 6 \\ -2 \end{bmatrix}$

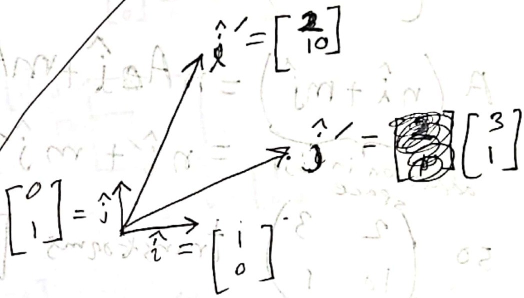
system of simultaneous equations $\begin{cases} 2a+3b=8 \\ 10a+b=13 \end{cases}$

in matrix form

$$\begin{pmatrix}2a+3b\\10a+1b\end{pmatrix}=\begin{pmatrix}8\\13\end{pmatrix} \xleftarrow[\text{multiplication}]{\text{matrix}} \underbrace{\begin{pmatrix}2&3\\10&1\end{pmatrix}}_{\text{matrix}}\underbrace{\begin{bmatrix}a\\b\end{bmatrix}}_{\text{vector}}=\underbrace{\begin{bmatrix}8\\13\end{bmatrix}}_{\text{vector}}$$

$$\begin{pmatrix}2&3\\10&1\end{pmatrix}\overbrace{\begin{bmatrix}1\\0\end{bmatrix}}^{\hat{i}}=\begin{bmatrix}2\\10\end{bmatrix}$$

$$\underbrace{\begin{pmatrix}2&3\\10&1\end{pmatrix}}_{\substack{\text{transforms}\\\text{a vector into}\\\text{another vector}}}\underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{\hat{j}}=\begin{bmatrix}3\\1\end{bmatrix}$$



$\begin{pmatrix}2&3\\10&1\end{pmatrix}$ transforms $\hat{i}$ to $\hat{i}'$

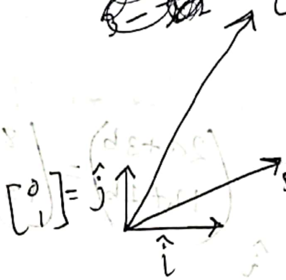
find a, b such that $\begin{pmatrix}2&3\\10&1\end{pmatrix}$ transforms $\begin{bmatrix}a\\b\end{bmatrix}$ to $\begin{bmatrix}8\\13\end{bmatrix}$

$$\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 8 \\ 13 \end{bmatrix}$$

$$A \quad \vec{r} = \vec{r}'$$

$$\hat{i}' = \begin{bmatrix} 2 \\ 10 \end{bmatrix} \qquad \hat{j}' = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$A(n\vec{r}) = nr'$$



$$A(\vec{r} + \vec{s}) = A\vec{r} + A\vec{s}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \hat{j} \qquad \hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A\underbrace{(n\hat{i} + m\hat{j})}_{\text{addition in } \hat{i},\hat{j} \text{ space}} = nA\hat{i} + mA\hat{j}$$

$$= n\hat{i}' + m\hat{j}' \rightarrow \text{addition in } \hat{i}', \hat{j}' \text{ space}$$

so $\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix}$ transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ from $\hat{i}, \hat{j}$ basis to a new space of basis $\hat{i}, \hat{j}$ where scalar multiplication and addition still holds.

$$\boxed{\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 32 \end{bmatrix}}$$

$$\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 32 \end{bmatrix} \Rightarrow \begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \left(3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

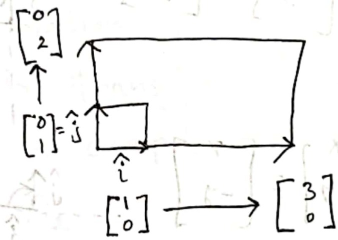
$$= 3\left(\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) + 2\left(\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = 3\begin{bmatrix} 2 \\ 10 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 12 \\ 32 \end{bmatrix}$$

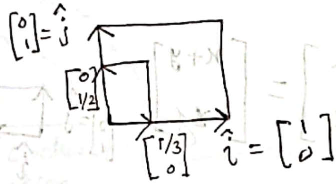
Identity matrix → matrix with basis vectors of the space

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$ (does nothing to a vector)

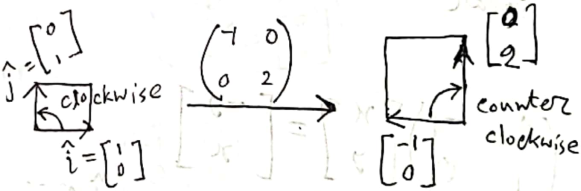
$$\underbrace{\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}}_{\text{expands space}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 2y \end{bmatrix}$$



$$\underbrace{\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}}_{\text{squishes space}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x/3 \\ y/2 \end{bmatrix}$$

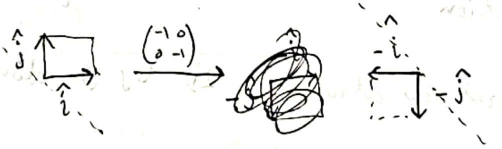


$$\underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}}_{\text{flips space}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ 2y \end{bmatrix}$$

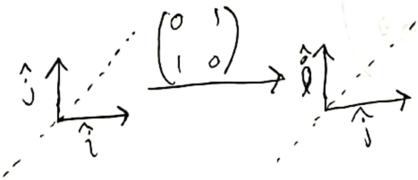


flipped in this mirror

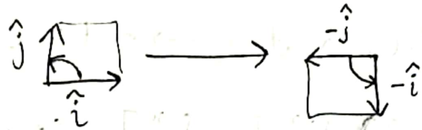
$$\underbrace{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{inverts both axis}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$



$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

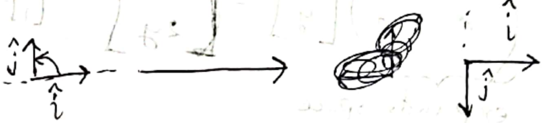


$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$$

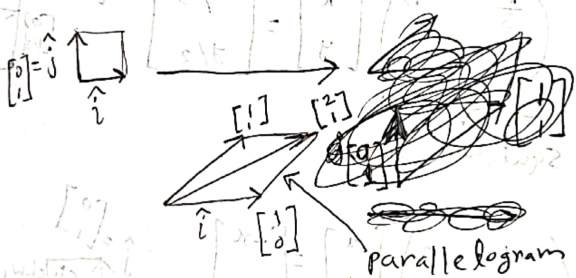


$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

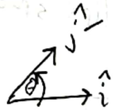


$$\underbrace{\begin{pmatrix} 1 & \cancel{0}1 \\ 0 & 1 \end{pmatrix}}_{\text{shear}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$$



$$\underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\text{rotation}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

general rotation in 2D by θ along the x axis:



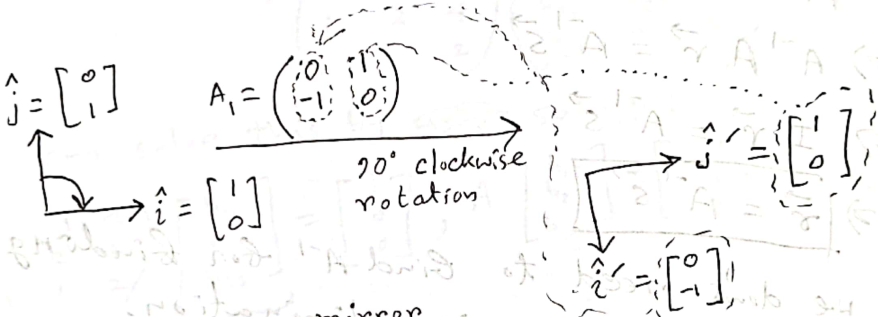
$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

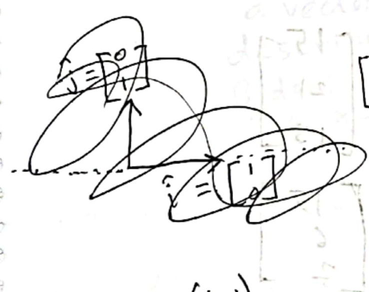
rotation along z axis (keeping z axis the same)

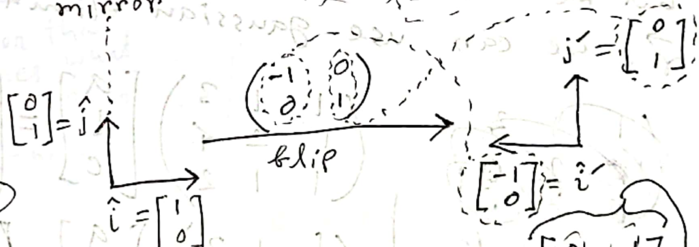
$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$A_2(A_1 r)$ → A_1 transformation applied to r, then result is transformed by A_2, so we have a composition of transformations A_1 and A_2 ~~matrix comp~~

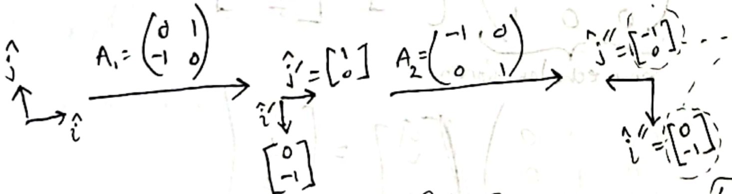
matrix composition
matrix multiplication







$$A_2 A_1 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$



notice that $A_1 A_2 \neq A_2 A_1$

$$A_2 A_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \cancel{\begin{pmatrix} 0\times -1 + 1\times 0 & 0\times 0 + 1\times 1 \\ -1\times -1 + 0\times 0 & -1\times 0 + 0\times 1 \end{pmatrix}} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$A_1A_2 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \neq A_2A_1$$

so matrix multiplication is not comutative.

but they are associative. $A_3(A_2A_1) = (A_3A_2)A_1$

$$\underbrace{\begin{pmatrix}2 & 3\\ 10 & 1\end{pmatrix}}_{A}\underbrace{\begin{bmatrix}a\\ b\end{bmatrix}}_{\vec{r}} = \underbrace{\begin{bmatrix}8\\ 13\end{bmatrix}}_{\vec{s}}$$

$A^{-1}A = I$

A^{-1} ← inverse matrix

$\Rightarrow A^{-1}A\vec{r} = A^{-1}\vec{s}$

$\Rightarrow I\vec{r} = A^{-1}\vec{s}$

$\Rightarrow \boxed{\vec{r} = A^{-1}\vec{s}}$

but we don't need to find A^{-1} for finding $\vec{r}$. we can use gaussian elimination.

$$\begin{pmatrix}1 & 1 & 3\\ 1 & 2 & 4\\ 1 & 1 & 2\end{pmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix} = \begin{bmatrix}15\\ 21\\ 13\end{bmatrix}$$

elimination ↓

$$\underbrace{\begin{pmatrix}1 & 1 & 3\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}}_{\text{row echelon form}}\begin{bmatrix}a\\ b\\ c\end{bmatrix} = \begin{bmatrix}15\\ 6\\ 2\end{bmatrix}$$

back substitution

$$\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix} = \begin{bmatrix}5\\ 4\\ 2\end{bmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{identity matrix}} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix} \quad \therefore a, b, c = 5, 4, 2$$

Finding A^{-1} using gaussian elimination

let $B = A^{-1}$ $\quad AB = I$

$$\Rightarrow \underbrace{\begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{pmatrix}}_{A} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we can solve this by ~~using~~ solving:

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{pmatrix} \underbrace{\begin{bmatrix} b_{11} \\ b_{22} \\ b_{32} \end{bmatrix}}_{\text{a vector that describes what B does to A in the x-axis}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad A \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

or we can use gaussian elemination directly to find A^{-1} in an efficient manner compared to finding co-factors.

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{pmatrix} B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} B = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B = \begin{pmatrix} -2 & 0 & 3 \\ -2 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B = \begin{pmatrix} 0 & -1 & 2 \\ -2 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \leftarrow A^{-1}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 15 \\ 28 \\ 23 \end{bmatrix}$$

$$\begin{array}{r} 45 \\ 28 \\ \hline 17 \end{array} \qquad \begin{array}{r} 30 \\ 23 \\ \hline 7 \end{array}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 15 \\ 17 \\ 7 \end{bmatrix} \quad r_2' = 3r_1 - r_2, \quad r_3' = 2r_1 - r_3$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 15 \\ 17 \\ 10 \end{bmatrix} \quad r_3' = r_2 - r_3$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 15 \\ 17 \\ 5 \end{bmatrix} \quad r_3' = r_3 / 2$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 15 \\ 17 \\ 5 \end{bmatrix}$$

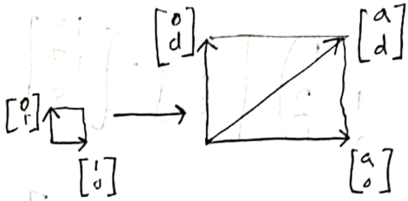
$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2 \\ 17 \\ 5 \end{bmatrix} \quad r_1' = r_1 - r_2$$

$r_1' = r_3 + r_1$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \\ 5 \end{bmatrix}$$

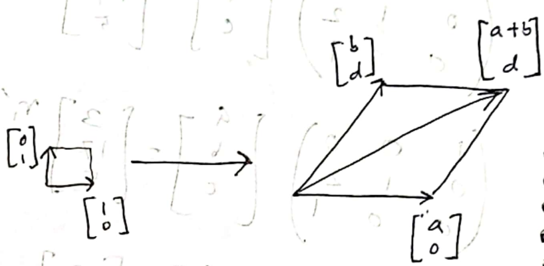
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix} \quad r_2' = r_2 - 2r_3$$

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ dy \end{bmatrix}$$



area = ad ← determinant

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ dy \end{bmatrix}$$

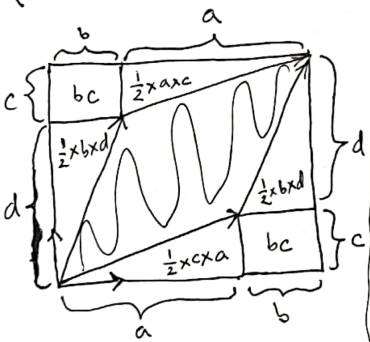


Parallelogram

area = ad ← determinant

---

let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ converts $\underset{\begin{bmatrix}1\\0\end{bmatrix}}{\hat{i}}, \underset{\begin{bmatrix}0\\1\end{bmatrix}}{\hat{j}}$ to $\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ c \end{bmatrix}$.



$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$|A|$

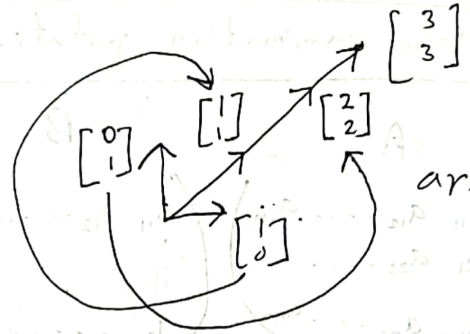
$$\Rightarrow \frac{1}{ad-bc}\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I}$$

A^{-1}

$$\text{[area]} = (a+b)(d+c) - bc - bc - \frac{ac}{2} - \frac{ac}{2} - \frac{bd}{2} - \frac{bd}{2}$$

$$= ad + \cancel{ac} + \cancel{bd} + bc - 2bc - \cancel{ac} - \cancel{bd} = ad - bc$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$



one column is a scalar multiple of another column, the columns aren't linearly independent.

area = 0

so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is transformed into a straight line

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 2 & 3 & 7 \end{pmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 12 \\ 17 \\ 29 \end{bmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 12 \\ 5 \\ 0 \end{bmatrix}$$

$0 \cdot c = 0$

here, $r_3 = r_1 + r_2$ $\quad$ det = 0

$c_3 = 2c_1 + c_2$

so the vectors are not linearly independent. and it won't transform from 3D to 3D. The transformation will collapse into 2D space

can't find unique solution to system of equation (it's true for all values of c) and the matrix is not inventible.

So we must check if the column vectors are linearly independent before transforming. otherwise, we cannot undo the transformation since we lost ~~a dimen~~ some values of a dimension, by transforming from 3D to 2D.

$\boxed{\text{Einstein summation notation}}$ → writing AB as a sum without Σ

$$\begin{array}{ccc} A & B & AB \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} & \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & & & \\ b_{n1} & \cdots & & b_{nn} \end{pmatrix} = & \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \end{array}$$

row column

element of AB in row 2, col 3

$$(ab)_{23} = \underbrace{a_{21} b_{13}}_{j} + \underbrace{a_{22} b_{23}}_{j} + \dots + \underbrace{a_{2n} b_{n3}}_{j}$$

$$\boxed{ab_{ik} = \sum a_{ij} b_{jk}}$$

$$\boxed{ab_{ik} = a_{ij} b_{jk}}$$ → useful in programming

helps us, better understand how the multiplication is done ~~better~~ compared to $C = AB$

$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \end{pmatrix}$

$B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

$C_{mn} = A_{mj} B_{jn}$

$C = AB$ $\qquad c_{ik} = a_{ij} b_{jk}$

$$\begin{pmatrix} & \\ & \end{pmatrix}_{m \times n} \begin{pmatrix} & \\ & \end{pmatrix}_{n \times k} = \begin{pmatrix} & \\ & \end{pmatrix}_{m \times k}$$

must be equal

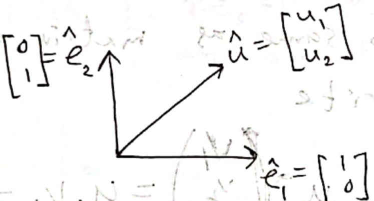
the dot product of two vectors $\vec{u}\cdot\vec{v}$ is the same as the matrix multiplication of $\vec{u}$ turned into a row matrix times v as a column matrix.

$$\underbrace{\begin{bmatrix} u_i \end{bmatrix}}_{\vec{u}} \cdot \underbrace{\begin{bmatrix} v_i \end{bmatrix}}_{\vec{v}} = (u_1 \cdots u_n)\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad \text{why?}$$

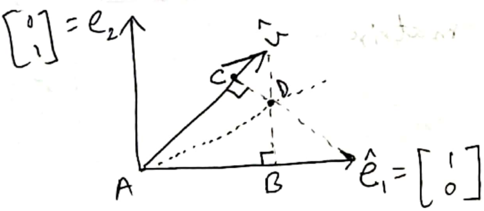
(geometrical) ← $\boxed{\text{dot product}}$ $\qquad$ $\boxed{\text{matrix multiplication}}$ (mathematical)

---

let's consider ~~the basis vectors~~ a unit vector $\hat{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ defined on the basis $\hat{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\hat{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



now we take projection of $\hat{u}$ in $\hat{e}_1$ as $AB = u_1$
and " " " $\hat{e}_1$ " $\hat{u}$ as AC
we can see that $\triangle ACD \cong \triangle ABD$; so $AC = AB = u_1$ which means $\hat{u}\cdot\hat{e}_1 = \hat{e}_1\hat{u}$ and similarly $\hat{u}\cdot\hat{e}_2 = \hat{e}_2\cdot\hat{u}$, so projection is symmetric and projection is the dot product.



so the matrix multiplication with a vector is a projection of that vector onto the ~~vectors~~ column vectors of the matrix.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix}$$

matrix vector

$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

einstein summation convension, $u_i v_i$ or $u_j v_j$

but not $u_i v_j$

the dot product is same as matrix multiplication here. so we write

$$\vec{u} \cdot \vec{v} = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = u_i v_i = \sum_{i=1}^{n} u_i v_i \text{ or}$$

$$\neq \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}_{n \times 1} \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}_{1 \times n}$$

because this multiplication gives $n \times n$ matrix

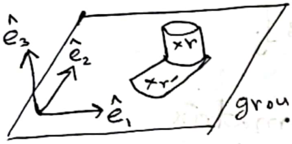
Example : using non-square matrices to do a projection

$\hat{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ unit vector $\quad \hat{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ unit vector $\quad \hat{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ unit vector

$\hat{s} = \begin{bmatrix}s_1\\s_2\\s_3\end{bmatrix} \qquad \vec{r} = \begin{bmatrix}r_1\\r_2\\r_3\end{bmatrix} \qquad \vec{r}' = \vec{r} + \lambda \hat{s}$ —— ①

$\vec{r}$: 3D vector $\qquad \vec{r}'$: 2D vector

$r_1, r_2, r_3, s_1, s_2, s_3, r_1', r_2', r_3'$ all are scalar quantities



ground $(r_3' = 0)$

$\Rightarrow \vec{r}' \cdot \hat{e}_3 = 0$ —— ②

$$\vec{r}' \cdot \hat{e}_3 = \begin{bmatrix}r_1'\\r_2'\\0\end{bmatrix} \cdot \begin{bmatrix}0\\0\\1\end{bmatrix} = 0$$

from eqⁿ ①, $\vec{r}' \cdot \hat{e}_3 = \vec{r} \cdot \hat{e}_3 + \lambda \vec{s} \cdot \hat{e}_3$ [taking dot product of $\hat{e}_3$ on both sides]

$\Rightarrow 0 = \vec{r} \cdot \hat{e}_3 + \lambda s_3$ [$\because \vec{r}' \cdot \hat{e}_3 = 0$ from ② and $\vec{s} \cdot \hat{e}_3 = \begin{bmatrix}s_1\\s_2\\s_3\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = s_1 \cdot 0 + s_2 \cdot 0 + s_3 \cdot 1 = s_3$]

$\Rightarrow \lambda s_3 = -\vec{r} \cdot \hat{e}_3$

$$\Rightarrow \boxed{\lambda = -\frac{\vec{r} \cdot \hat{e}_3}{s_3}}$$

scalar

now putting values of λ in ①, we get

$$\boxed{\vec{r}' = \vec{r} - \hat{s}\,\frac{\vec{r} \cdot \hat{e}_3}{s_3}} \text{ —— ③}$$

from ③, we get $\vec{r}' = \vec{r} - \hat{s}\,\dfrac{\vec{r} \cdot \hat{e}_3}{s_3} = \vec{r} - \hat{s}\,\dfrac{\begin{bmatrix}r_1\\r_2\\r_3\end{bmatrix} \cdot \begin{bmatrix}0\\0\\1\end{bmatrix}}{s_3}$

$$\Rightarrow \vec{r}' = \vec{r} - \hat{s}\,\frac{r_1 \cdot 0 + r_2 \cdot 0 + r_3 \cdot 1}{s_3} = \vec{r} - \hat{s}\,\frac{r_3}{s_3} = \begin{bmatrix}r_1\\r_2\\r_3\end{bmatrix} - \begin{bmatrix}s_1\\s_2\\s_3\end{bmatrix}\frac{r_3}{s_3}$$

$$\Rightarrow \vec{p}' = \begin{bmatrix}r_1\\r_2\\r_3\end{bmatrix} - \begin{bmatrix}\frac{s_1 r_3}{s_3}\\ s_2 r_3/s_3\\ s_3 s_3/s_3\end{bmatrix} = \begin{bmatrix}r_1 - s_1 \cdot \frac{r_3}{s_3}\\ r_2 - s_2\, r_3/s_3\\ r_3 - s_3\, r_3/s_3\end{bmatrix}$$

. so in einsum form,

$$\boxed{\vec{r}_i' = r_i - s_i \frac{r_3}{s_3}} \text{ —— ④}$$

from ③, $\vec{r}\,' = \vec{r} - \hat{s}\,\dfrac{\vec{r}\cdot\hat{e}_3}{s_3} = \vec{r} - \hat{s}\,\dfrac{\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}\cdot\begin{bmatrix} [\hat{e}_3]_1 \\ [\hat{e}_3]_2 \\ [\hat{e}_3]_3 \end{bmatrix}}{s_3}$

$\Rightarrow \vec{r}\,' = \vec{r} - \dfrac{\hat{s}}{s_3}\left(r_1[\hat{e}_3]_1 + r_2[\hat{e}_3]_2 + r_3[\hat{e}_3]_3\right)$

$\Rightarrow \vec{r}\,' = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} - \dfrac{1}{s_3}\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \sum_{j=1}^{3} r_j [\hat{e}_3]_j$

in Einstein summation notation,

$$\boxed{r'_i = r_i - \frac{s_i}{s_3} r_j [\hat{e}_3]_j} \longrightarrow ⑤$$

now, $\underset{\text{dot product}}{\vec{r}\cdot\hat{e}_3} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \overbrace{(r_1 \;\; r_2 \;\; r_3)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}^{\text{matrix multipulication}}$

$\Rightarrow \vec{r}\cdot\hat{e}_3 = r_1[\hat{e}_3]_1 + r_2[\hat{e}_3]_2 + r_3[\hat{e}_3]_3 = [\hat{e}_3]_j\, r_j$

$$\therefore \boxed{\vec{r}\cdot\hat{e}_3 = [\hat{e}_3]_j\, r_j} \longrightarrow ⑥$$

$\vec{r} = I\vec{r} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$ $\because$ Identity matrix does nothing to a vector

$\Rightarrow \vec{r} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$

$\therefore r_1 = I_{11} r_1 + I_{12} r_2 + I_{13} r_3$

$r_2 = I_{21} r_1 + I_{22} r_2 + I_{23} r_3$

$r_3 = I_{31} r_1 + I_{32} r_2 + I_{33} r_3$

$$\therefore r_i = \sum_{j=1}^{3} I_{ij} r_j = I_{ij} r_j \quad (\text{Einsum}) \text{ --- } 7$$

from 5, $r_i' = r_i - \dfrac{s_i [\hat{e}_3]_j r_j}{s_3}$

$$\Rightarrow r_i' = I_{ij} r_j - s_i \frac{[\hat{e}_3]_j r_j}{s_3} \quad [\text{from } 7]$$

$$\Rightarrow \boxed{r_i' = \left(I_{ij} - s_i \frac{[\hat{e}_3]_j}{s_3}\right) r_j} \text{ --- } 8$$

$$\boxed{[\hat{e}_a]_j = I_{aj}}$$

$$\cancel{\therefore r_i' = \left(I_{ij} r_j - \frac{s_i [\hat{e}_3]_j}{s_3}\right) r_j}$$

since $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\hat{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$; $[\hat{e}_3]_j = (0 \quad 0 \quad 1)$

so, $I_{3j} = [\hat{e}_3]_j$. putting this in 8, we get

$$\boxed{r_i' = \left(I_{ij} - \frac{s_i I_{3j}}{s_3}\right) r_j} \text{ --- } 9$$

from 9, comparing with $r' = Ar$, $A_1 = I_{1j} - \dfrac{s_1 I_{3j}}{s_3}$

$$\Rightarrow A_1 = (1 \quad 0 \quad 0) - \frac{s_1 (0 \quad 0 \quad 1)}{s_3} = (1 \quad 0 \quad 0) - \left(0 \quad 0 \quad \frac{s_1}{s_3}\right)$$

$$\Rightarrow A_1 = \left(1 \quad 0 \quad -\frac{s_1}{s_3}\right)$$

$$A_2 = I_{2j} - \frac{s_2 I_{3j}}{s_3} = (0 \quad 1 \quad 0) - \frac{s_2}{s_3}(0 \quad 0 \quad 1) = \left(0 \quad 1 \quad -\frac{s_2}{s_3}\right)$$

$$A_3 = I_{3j} - \frac{s_3 I_{3j}}{s_3} = (0 \quad 0 \quad 0) - (0 \quad 0 \quad 0) = (0 \quad 0 \quad 0)$$

$$\therefore A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -s_1/s_3 \\ 0 & 1 & -s_2/s_3 \\ 0 & 0 & 0 \end{pmatrix}$$

⊛ matrix can also be applied to a list of vectors.

$$\underset{\text{vector eq}^n}{\vec{r}' = A\vec{r}} \Rightarrow \begin{bmatrix} r_1' \\ r_2' \\ r_3' \end{bmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

$$\Rightarrow r_1' = A_{11} r_1 + A_{12} r_2 + A_{13} r_3$$

$$r_2' = A_{21} r_1 + A_{22} r_2 + A_{23} r_3$$

$$r_3' = A_{31} r_1 + A_{32} r_2 + A_{33} r_3 = 0 \quad [\text{since } r' \text{ is 2D}]$$

$$\therefore \boxed{r_i' = A_{ij} r_j} \quad (\text{in Einstein notation})$$

now, a matrix equation, $R' = AR$

$$\underbrace{\begin{bmatrix} r_1' & s_1' & t_1' & u_1' \cdots \\ r_2' & s_2' & t_2' & u_2' \cdots \end{bmatrix}}_{R' \text{matrix}} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \overbrace{\begin{pmatrix} r_1 & s_1 & t_1 \cdots \\ r_2 & s_2 & t_2 \cdots \\ r_3 & s_3 & t_3 \cdots \end{pmatrix}}^{R \text{ matrix}}$$

r' vector

r vector

$$R_{11}' = r_1' = A_{11} r_1 + A_{12} r_2 + A_{13} r_3 = A_{1j} r_j = A_{1j} R_{j1}$$

$$R_{12}' = A_{1j} R_{j2} \qquad R_{13}' = A_{1j} R_{j3} \qquad \therefore \boxed{R_{ia}' = A_{ij} R_{ja}}$$

1st row of A

3rd column of R

Matrices changing basis

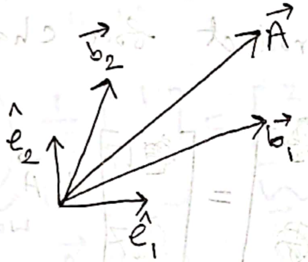
let $\hat{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\hat{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ defines my basis vectors for my world.

let $\underbrace{\hat{b}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \hat{b}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{in my world}}$ defines bear's basis vectors

~~where~~

but in bear's world, $\hat{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\hat{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

let a vector be $\vec{A} = \frac{1}{2}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ in bear's world.



$$\underbrace{\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}}_{\substack{\text{bear's basis} \\ \text{vectors in} \\ \text{my coordinates}}} \underbrace{\begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}}_{\substack{\vec{A} \text{ in} \\ \text{bear's basis}}} = \underbrace{\begin{bmatrix} 5 \\ 2 \end{bmatrix}}_{\vec{A} \text{ in my basis}}$$

How to go from $\vec{A}$ in my basis to $\vec{A}$ in bear's basis?

We use $\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{3 \cdot 1 - 1 \cdot 1}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}$

$$\underbrace{\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}}_{\text{my basis in bear's world}} \underbrace{\begin{bmatrix} 5 \\ 2 \end{bmatrix}}_{\vec{A}\text{ in my world}} = \underbrace{\begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}}_{\vec{A}\text{ in bear's world}}$$

so in bear's world, my basis is $\frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{e_1}$

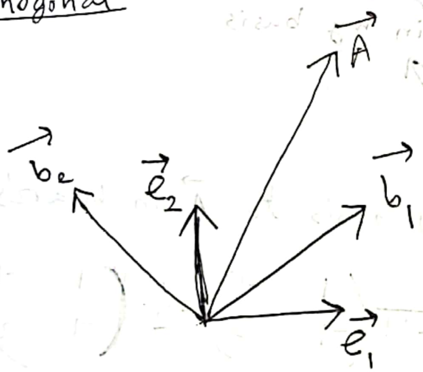
and $\vec{e_2} = \frac{1}{2}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$90°$

since bear's basis vectors aren't orthogonal, we can't use dot product for change in basis.

$$\begin{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 5 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 15 + 2 \cdot 1 \\ 5 + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 17 \\ 7 \end{bmatrix} \neq \vec{A} \text{ in bear's world}$$

another example where bear's basis vectors are orthogonal



$\vec{b_1} \perp \vec{b_2}$

$\vec{e_1} \perp \vec{e_2}$

$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

in my world

$\vec{b}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{b}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

in bear's world,

$\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{A} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{e}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\vec{e}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{\text{bear's basis in my coordinates}} \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\vec{A} \text{ in bear's world}} = \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{\vec{A} \text{ in my world}}$$

$$\underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}_{\text{my basis in bear's co-ordinates}} \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{\vec{A} \text{ in my world}} = \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\vec{A} \text{ in bear's basis}}$$

We can do the same using projection / dot product

$$\vec{A} \text{ in bear's basis} = \begin{bmatrix} \overbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}}^{\vec{A} \text{ in my world}} \cdot \overbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}^{\vec{b}_1 \text{ in my world}} \\ \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{\vec{A} \text{ in my world}} \cdot \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{b}_2 \text{ in my world}} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

remember to normalize (divide by length) if bear's basis vectors aren't unit vectors

Doing a transformation in changed basis

Let's say we have a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ defined in bear's basis $\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We want to rotate $\begin{bmatrix} x \\ y \end{bmatrix}$ by $45°$ in bear's world. How do we do it?

 $\begin{bmatrix} x \\ y \end{bmatrix}$ anti-clockwise

We already know how to roate by angle θ in the basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We use

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}$$

this transformation matrix. But we don't know how to write a transformation in bear's world for this rotation yet.

So we apply the transformation $\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$ to $\begin{bmatrix} x \\ y \end{bmatrix}$ for getting $\begin{bmatrix} x \\ y \end{bmatrix}$ in my world. ~~[illegible]~~

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

since we know how to rotate anti-clockwise in my world, we ~~rotate~~ apply the rotation transformation

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\overbrace{\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}}^{B}\begin{bmatrix} x \\ y \end{bmatrix}$$

so $\begin{bmatrix} x \\ y \end{bmatrix}$ is rotated now in my world. But we need $\begin{bmatrix} x \\ y \end{bmatrix}$ in bear's world. so we apply B^{-1} to transform back to bear's world.

$$\underbrace{\underbrace{\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}}_{B^{-1}}\underbrace{\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{R_{45^\circ}}\underbrace{\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}}_{B}\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{rotated vector in my frame}}}_{\text{rotated vector in Bear's frame}}$$

So the rotation matrix $\boxed{R_B = B^{-1} R B}$

therefore we should keep in mind that in non-standard basis, the transformations are different from the transformations in the standard basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

orthogonal matrices

transpose of a matrix

$$A^T_{ij} = A_{ji} \qquad \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^T = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}$$

let's define a transformation matrix A that defines a new space with basis vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3 \ldots \vec{a}_n$ which are perpendicular to each other and of unit length.

$A_{n\times n}$ square matrix

basis vector as column vector in matrix A

$$\begin{pmatrix} \begin{pmatrix} a_1 \end{pmatrix} & \begin{pmatrix} a_2 \end{pmatrix} & \begin{pmatrix} a_3 \end{pmatrix} & \cdots & \begin{pmatrix} a_n \end{pmatrix} \end{pmatrix}$$

$\perp$

$\vec{a}_i \cdot \vec{a}_j = 0$ if $i \neq j$ (orthogonal)

$\vec{a}_i \cdot \vec{a}_j = 1$ if $i = j$ (unit length)

$$\overbrace{\begin{pmatrix} (\; a_1 \;) \\ (\; a_2 \;) \\ (\; a_3 \;) \\ \vdots \\ (\; a_n \;) \end{pmatrix}}^{A^T} \overbrace{\begin{pmatrix} \begin{pmatrix} a_1 \end{pmatrix} & \begin{pmatrix} a_2 \end{pmatrix} & \begin{pmatrix} a_3 \end{pmatrix} & \cdots & \begin{pmatrix} a_n \end{pmatrix} \end{pmatrix}}^{A} = \underbrace{\begin{pmatrix} \overset{\vec{a}_1 \cdot \vec{a}_1}{1} & \overset{\vec{a}_1 \cdot \vec{a}_2}{0} & \overset{\vec{a}_1 \cdot \vec{a}_3}{0} & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 \end{pmatrix}}_{I}$$

$A^T A = I \quad AA^T = I$

which means $A^T = A^{-1}$

so a set of unit basis vectors that are all perpindicular to each other are called an $\boxed{\text{orthonormal basis set}}$. and the matrix composed of them (A) is an $\boxed{\text{orthogonal matrix}}$.

since all column vectors of orthogonal matrix are of unit length, it must scale space by a factor of 1.

so the determinant of $A = |A| = +1$ or -1

$|A| = -1$ means that A flips space.

the rows of the orthogonal matrix are orthonormal.

the rows of the orthogonal basis vector set.

A^T is also an orthogonal

In data science, we want to use an <u>orthonormal basis vector set</u> for transforming our data (vector). orthogonal matrix

because A^{-1} is easy to compute $(A^{-1} = A^T)$ and we can transform back and forth without losing data so transformation is reversible because it doesn't collapse space. the projection is just the dot product.

let $\hat{x}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$ $\hat{x}_2 = \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$ $\hat{x}_3 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

be orthogonal unit vectors.

$$\begin{array}{c} \left| \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & 1/\sqrt{3} \\ \frac{1}{\sqrt{2}} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} \right| = 1 \\ \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \end{array} \qquad \begin{array}{c} \begin{array}{ccc} x_2 & x_1 & x_3 \end{array} \\ \left| \begin{pmatrix} \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \right| = -1 \end{array}$$

$$\begin{array}{c} \left| \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -1/\sqrt{3} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix} \right| = 1 \\ \begin{array}{ccc} x_3 & x_1 & x_2 \end{array} \end{array} \qquad \begin{array}{c} \left| \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{pmatrix} \right| = -1 \\ \begin{array}{ccc} x_1 & x_3 & x_2 \end{array} \end{array}$$

Gram-Schmidt process

$\hat{e}_1, \hat{e}_2 \dots \hat{e}_n$

↳ constructs a list of orthonormal basis vectors from a list of linearly independent basis vectors.

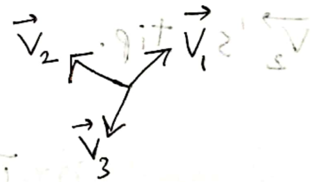
$\vec{v}_1, \vec{v}_2, \dots \vec{v}_n$

How to check for linear dependence:

construct a matrix of column vectors and calculate the determinant. if $\det \neq 0$, the vectors are linearly independent.

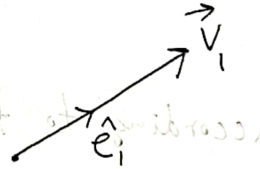
let we have a set of vectors (linearly independent)

$$V = \{\vec{v}_1, \vec{v}_2, \dots\dots, \vec{v}_n\}$$



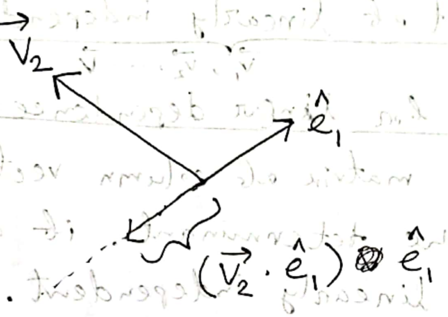
but the vectors $\vec{v}_1, \vec{v}_2, \dots \vec{v}_n$ aren't orthogonal to each other or of unit length.

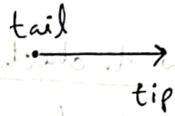
$$\vec{v}_1 \longrightarrow \hat{e}_1 = \frac{\vec{v}_1}{|\vec{v}_1|}$$



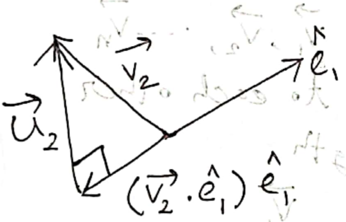
now we need to find $\hat{e}_2$ which will be on the same plane as $\hat{e}_1$ and $\vec{v}_1$ and $\vec{v}_2$ but it will be peppindicular to $\hat{e}_1$

let's take the projection of $\hat{e}_1$ onto $\vec{V}_2$ and make it a vector projection by multiplying by $\hat{e}_1$





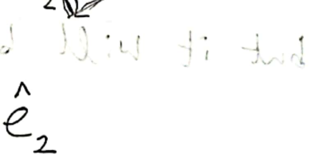
now we take a vector $\vec{u}_2$ ~~that is per~~ which has it's tail in the projection's tip and tip in $\vec{V}_2$'s tip.



according to the vector sum rule,

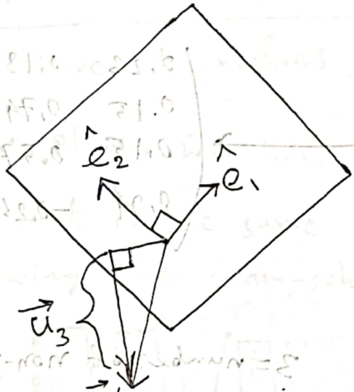
$$\vec{V}_2 = (\vec{V}_2 \cdot \hat{e}_1)\hat{e}_1 + \vec{u}_2$$

$$\Rightarrow \vec{u}_2 = \vec{V}_2 - (\vec{V}_2 \cdot \hat{e}_1)\hat{e}_1$$



$$\hat{e}_2 = \frac{\vec{u}_2}{|\vec{u}_2|} \qquad \hat{e}_1 \perp \hat{e}_2$$

since $\hat{e}_3$ has to be perpindicular to both $\hat{e}_1$ and $\hat{e}_2$, $\hat{e}_3$ cannot be on the same plane as $\hat{e}_1$ and $\hat{e}_2$. also $\vec{V}_3$ is not on the same plane as $\vec{V}_1$, $\vec{V}_2$ since $\vec{V}_1, \vec{V}_2, \vec{V}_3$ are linearly independent.



we take a vector $\vec{u}_3$ which is perpindicular to the plane of $\hat{e}_1$ and $\hat{e}_2$

$$\vec{u}_3 = \vec{V}_3 - \underbrace{(\vec{V}_3 \cdot \hat{e}_1)\hat{e}_1}_{\text{projection of } \vec{V}_3 \text{ on } \hat{e}_1} - \underbrace{(\vec{V}_3 \cdot \hat{e}_2)\hat{e}_2}_{\text{Projection of } \vec{V}_3 \text{ on } \hat{e}_2} \quad [\text{see } \vec{u}_2]$$

$$\hat{e}_3 = \frac{\vec{u}_3}{|u_3|}$$

so $$\hat{e}_r = \frac{\vec{V}_r - (\vec{V}_r \cdot \hat{e}_1)\hat{e}_1 - (\vec{V}_r \cdot \hat{e}_2)\hat{e}_2 \cdots - (\vec{V}_r \cdot \hat{e}_{r-1})\hat{e}_{r-1}}{|\vec{V}_r - (\vec{V}_r \cdot \hat{e}_1)\hat{e}_1 - (\vec{V}_r \cdot \hat{e}_2)\hat{e}_2 \cdots - (\vec{V}_r \cdot \hat{e}_{r-1})\hat{e}_{r-1}|}$$

number of vectors = 4 $\qquad\qquad$ new basis

$$\begin{pmatrix} 1 & 0 & 2 & 6 \\ 0 & 1 & 8 & 2 \\ 2 & 8 & 3 & 1 \\ 1 & -6 & 2 & 3 \end{pmatrix} \xrightarrow[\text{process}]{\text{Gram-schmidt}} \begin{pmatrix} 0.4 & -0.18 & 0.05 & 0.9 \\ 0 & 0.1 & 0.77 & -0.03 \\ 0.81 & 0.5 & -0.06 & -0.2 \\ 0.4 & -0.83 & 0.08 & -0.3 \end{pmatrix}$$

dimension = 4

$$\begin{pmatrix} 3 & 5 & -3 \\ 2 & 5 & -1 \\ 2 & 4 & 8 \\ 12 & 2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0.23 & 0.18 & 0.22 \\ 0.15 & 0.74 & -0.64 \\ 0.15 & 0.57 & 0.72 \\ 0.94 & -0.26 & -0.07 \end{pmatrix}$$

non-square matrix

3 = number of non-zero columns

$$\begin{pmatrix} 6 & 2 & 1 & 7 & 5 \\ 2 & 8 & 5 & 4 & 1 \\ 1 & -6 & 3 & 2 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 0.93 & -0.12 & -0.32 & 0 & 0 \\ 0.31 & 0.72 & 0.61 & 0 & 0 \\ 0.15 & -0.68 & 0.71 & 0 & 0 \end{pmatrix}$$

number of vectors = 5

but dimension = 3

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 1 & 0 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 0.7 & 0 & 0 \\ 0 & 1 & 0 \\ 0.7 & 0 & 0 \end{pmatrix}$$

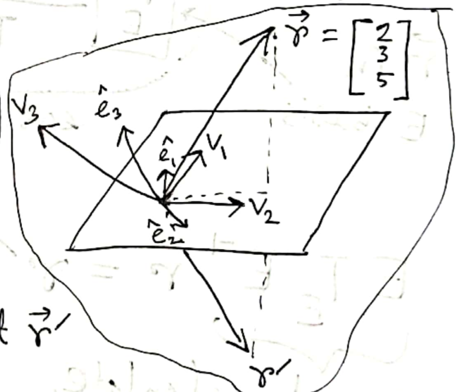
here, $\begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (-3)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

so the vectors are linearly dependent, so the orthonormal set isn't 3 dimensional and $\vec{v_1}, \vec{v_2}, \vec{v_3}$ don't span in R^3

---

let we have two vectors $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ in the same plane. we have another vector $\vec{v_3} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$ which is not in the same plane as $\vec{v_1}$ and $\vec{v_2}$ by following the gram-schmidt process, we get

$$E = \left(\begin{bmatrix} \hat{e}_1 \end{bmatrix} \begin{bmatrix} \hat{e}_2 \end{bmatrix} \begin{bmatrix} \hat{e}_3 \end{bmatrix} \right) = \left(\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \end{bmatrix} \right)$$

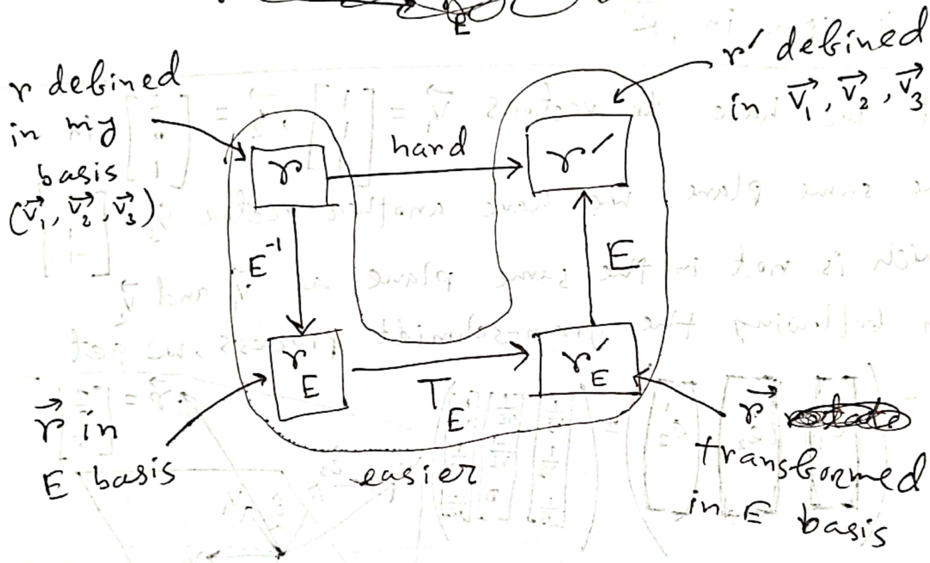


and we have $\vec{r}$ defined on basis $\vec{v}, \vec{v_2}, \vec{v_3}$ and we want to reflect it in the plane to get $\vec{r}'$

it is very easy to do in the basis $\hat{e}_1, \hat{e}_2, \hat{e}_3$ by using the transformation matrix

$$T_E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ \hat{e}_1 & \hat{e}_2 & -\hat{e}_3 \end{pmatrix}$$

but $\vec{r}$ is not defined in $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ yet. so we have to find $\vec{r}$ in E basis and then apply T_E. so $\vec{r} \longrightarrow \vec{r}'$ is hard.



$$\underbrace{E\, T_E}\, E^{-1}\, r = r'$$

easy to calculate, since $E^T = E^{-1}$

$r' =$

in the bear's case we used $B^{-1} R B$ but in this case we used $E T_E E^{-1}$ because we are transforming a vector defined in our basis whereas in bear's case we were operating on a vector defined in bear's space.

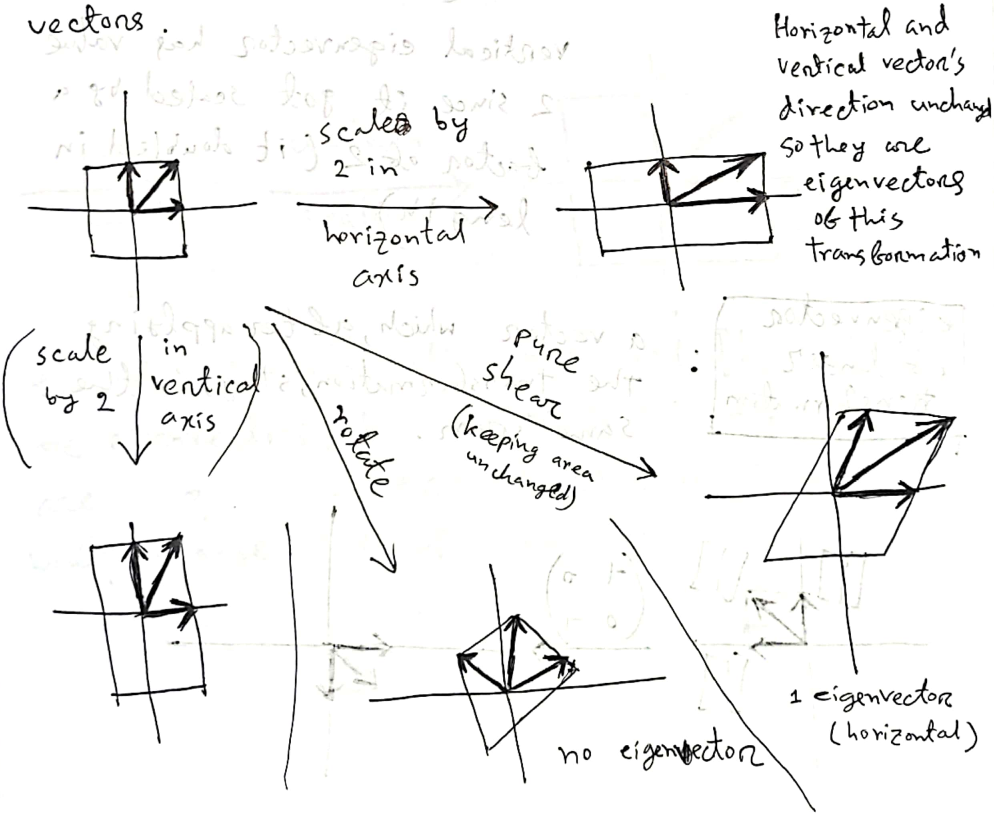
~~ ~~ So we can now reflect a point in space in a mirror. We can use this knowledge to transform images of faces and use neural network for facial recognition.

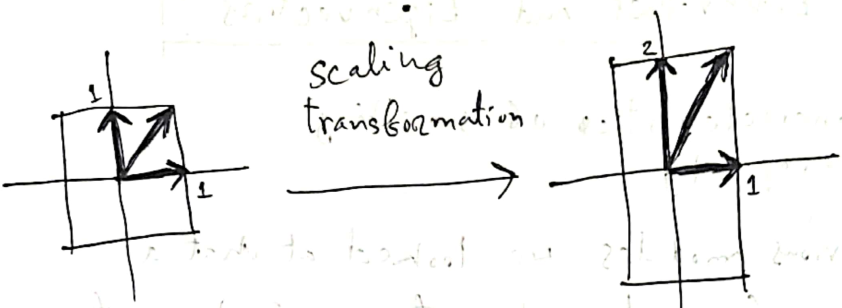
What are Eigenvalues and Eigenvectors?

Eigen = characteristics of something
property

in the previous modules we looked at what a matrix transformation does to a single vector.

now let's consider a unit square with 3 unit vectors.

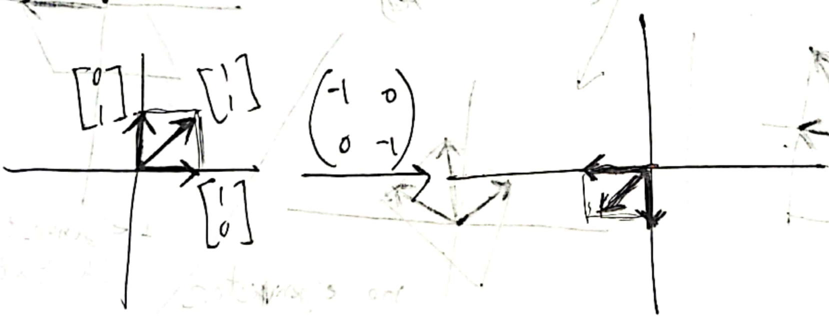




horizontal eigenvector has value 1 since it got scaled by a factor of 1 (unchanged).

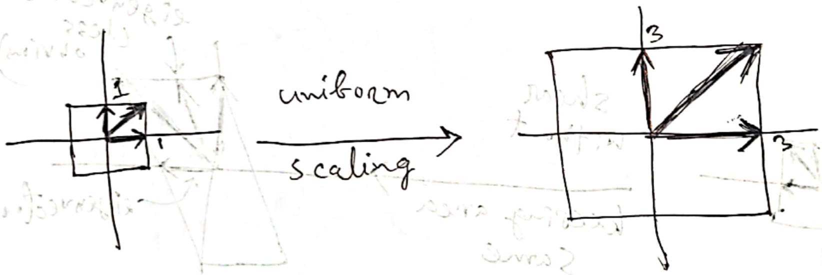
vertical eigenvector has value 2 since it got scaled by a factor of 2 (it doubled in length)

eigenvector of linear transformation : a vector which, after applying the transformation, stays in the same span.



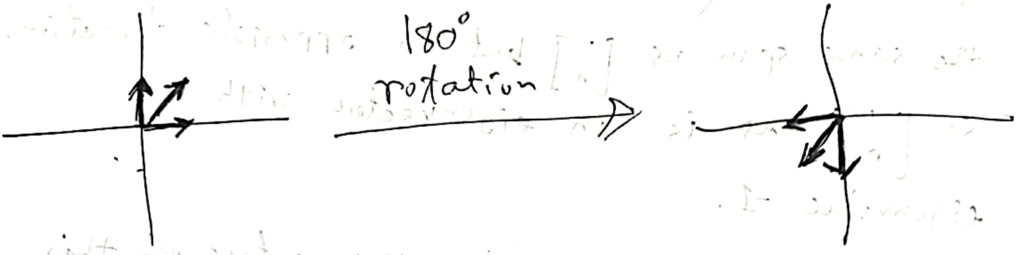
$\begin{bmatrix}1\\0\end{bmatrix}$ goes to $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$ which is in the same span as $\begin{bmatrix}1\\0\end{bmatrix}$ but in opposite direction. so $\begin{bmatrix}1\\0\end{bmatrix}$ is an eigenvector with eigenvalue -1.

similarly $\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$ are also eigenvectors of this transformation with eigenvalues -1, -1



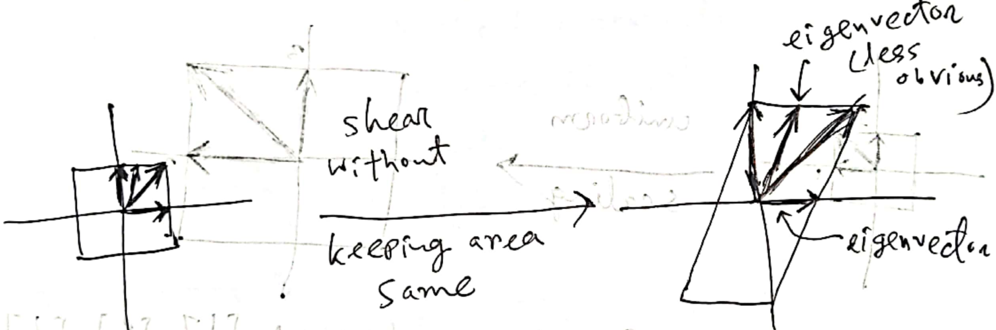
for this transformation, not just $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$ are eigenvectors, but all vectors in this space are ? . because this uniform scaling won't change the span of any vector.

rotations may have eigenvectors:

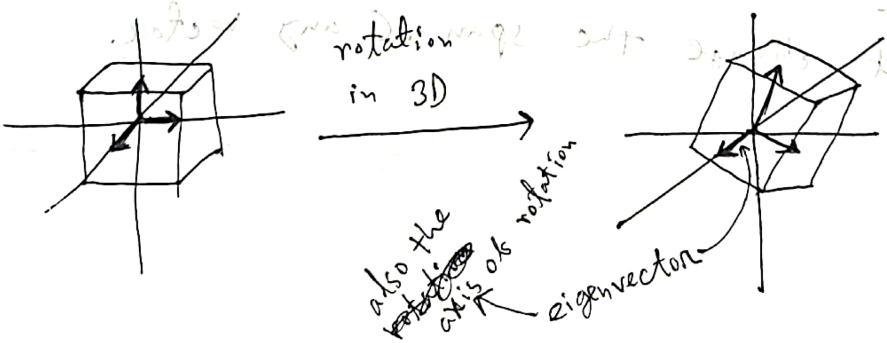


eigenvectors: $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$

eigenvalues: $-1, -1, -1$



so eigenvectors aren't always easy to spot.



n dimentional matrix transform $\rightarrow$ A; n dimentional vector $\rightarrow$ x

$$A x = \lambda x$$

λ $\leftarrow$ scalar

x $\leftarrow$ n dimentional vector

$$\Rightarrow A x - \lambda x = 0 \qquad I = \text{n dimensional identity matrix}$$

$$\Rightarrow (A - \lambda I) x = 0$$

either $A - \lambda I = 0$ or $\underbrace{x = 0}_{\text{trivial solution}}$

$$\therefore \det(A - \lambda I) = 0 \quad [\because \text{determinant of a matrix is a scalar}]$$

$$\Rightarrow \det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = 0$$

$$\Rightarrow \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0 \Rightarrow (a-\lambda)(d-\lambda) - bc = 0$$

$$\Rightarrow ad - a\lambda - \lambda d + \lambda^2 - bc = 0$$

$$\Rightarrow \lambda^2 - (a+d)\lambda + ad - bc = 0$$

solve for λ to get eigenvalues.

* Find eigenvalues and corresponding eigenvectors of the transformation $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$$Ax = \lambda x$$

$$\Rightarrow Ax - \lambda x = 0 \Rightarrow (A - I\lambda)x = 0$$

$$\Rightarrow \left| \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0 \Rightarrow \left| \begin{pmatrix} (1-\lambda) & 0 \\ 0 & (2-\lambda) \end{pmatrix} \right| = 0$$

$\Rightarrow (1-\lambda)(2-\lambda) = 0$ either $1-\lambda=0$ or $2-\lambda=0$

$\Rightarrow \lambda = 1$ $\quad \lambda = 2$

at $\lambda = 1$, $\begin{pmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$$\Rightarrow \begin{pmatrix} 1-1 & 0 \\ 0 & 2-1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_2 = 0$$

at $\lambda = 2$, $\begin{pmatrix} 1-2 & 0 \\ 0 & 2-2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$$\Rightarrow \begin{bmatrix} -x_1 \\ 0 \end{bmatrix} = 0 \Rightarrow x_1 = 0$$

at $\lambda = 1$, $x = \begin{bmatrix} t \\ 0 \end{bmatrix}$ t = arbitrary constant

$\lambda = 2$, $x = \begin{bmatrix} 0 \\ t \end{bmatrix}$

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ [90° rotation matrix]

char. polynomial : $\lambda^2 - (0+0)\lambda + (0+1) = 0$

$\Rightarrow \lambda^2 + 1 = 0$ [no real solution]

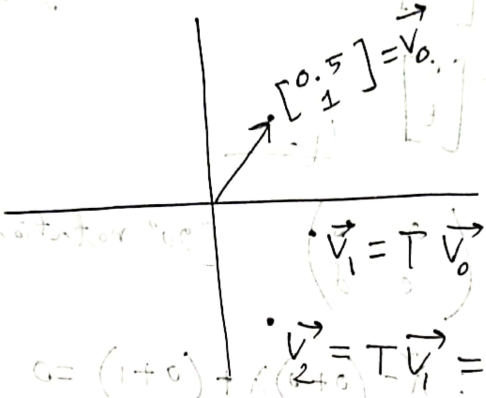
so $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has no real eigenvectors.

$\boxed{\text{changing to eigen basis}}$

sometimes we need to multiply a vector by a matrix many times.

for example, let $\vec{v_0}$ describe the initial position of a particle. the transformation

$T = \begin{pmatrix} 0.9 & 0.8 \\ -1 & 0.35 \end{pmatrix}$ ~~describes~~ gives the next position of the vector.



$\vec{v}_1 = T\vec{v}_0$

$\vec{v}_2 = T\vec{v}_1 = T(T\vec{v}_0)$

$= T^2\vec{v}_0$

$\vec{v}_n = T^n\vec{v}_0$

if we want to compute $\vec{v}_{10,000}$, it will take many matrix multiplications, so this won't be very efficient.

but it's very easy to ~~transform a~~ raise a diagonal matrix to some power n.

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}^n = \begin{bmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{bmatrix}$$

but T is not a diagonal matrix. so we can transform T to another basis where T is a diagonal matrix. then we raise T to the n. then we transform T back to our basis.

$$C = \left(\begin{bmatrix} x_1 \end{bmatrix} \begin{bmatrix} x_2 \end{bmatrix} \begin{bmatrix} x_3 \end{bmatrix} \right) \qquad D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

eigenvector of T

$\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of T

transforms T to eigenbasis

$$\therefore T = CDC^{-1} \qquad \therefore T^2 = CDC^{-1}CDC^{-1} = CDDC^{-1} = CD^2C^{-1}$$

$$\therefore \boxed{T^n = CD^nC^{-1}}$$

$\vec{V}$ in my basis — in my basis

$$\begin{tikzcd} \boxed{\vec{V}} \arrow[r, "T^n"] \arrow[d, "C^{-1}"'] & \boxed{T^n \vec{V}} \\ \boxed{[\vec{V}]_E} \arrow[r, "D^n"] & \boxed{[T^n \vec{V}]_E} \arrow[u, "C"'] \end{tikzcd}$$

$\vec{V}$ in eigenbasis — in eigenbasis

(T is now diagonalized)

$\boxed{\text{Example}}$

We have $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and we want to apply T^2 on $\vec{v}$.

Let's do it manually first.

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \quad \therefore T^2 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}$$

$$T^2\vec{v} = \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

now let's do it in eigenbasis approach.

$T = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ ~~has eigenvectors~~

eigenvalue $\lambda = 1$: eigenvector $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

" $\lambda = 2$: " $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$C = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \left(\begin{bmatrix} \vec{x}_1 \end{bmatrix} \begin{bmatrix} \vec{x}_2 \end{bmatrix} \right)$$

$$C^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$T^2 = CD^2C^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^2 \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \checkmark$$