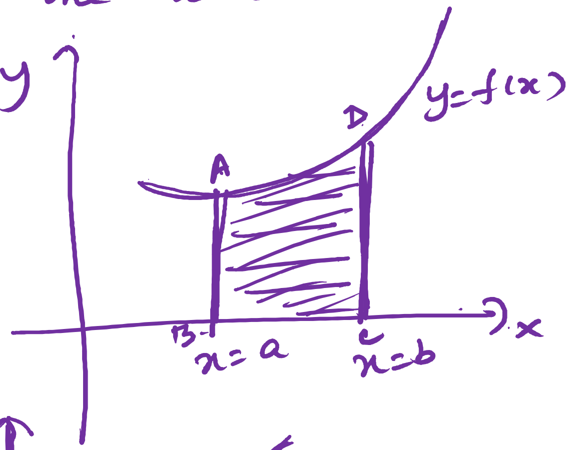
Areas of Cartesian Curves.

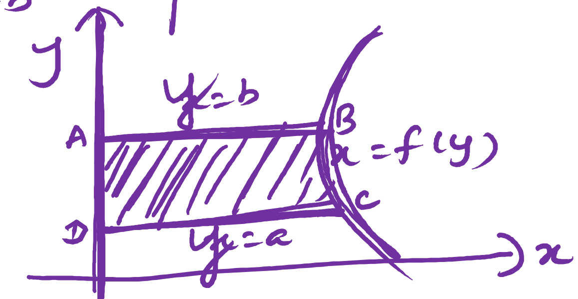
The **integral is** defined to be the (net signed) **area** under the curve

(1) The area bounded by the curve $y=f(x)$, the x-axis and the ordinates $x=a$, $x=b$ is $\int_{x=a}^{b} y\,dx$, $y=f(x)$



(2) The area bounded by the curve $x=f(y)$ the y axis, and the ordinates $y=a$, $y=b$

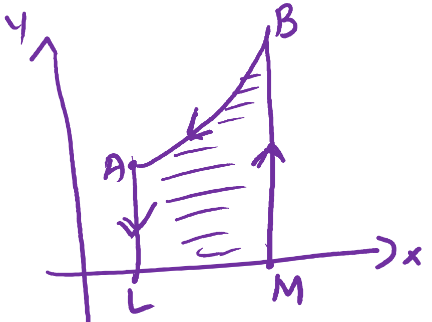
is $\int_{y=a}^{b} x\,dy$, $x=f(y)$



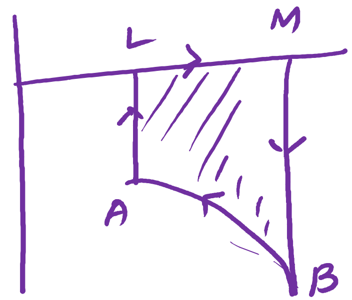
Note 1. The area bounded by a curve, the x-axis and the two ordinates is called area under the curve.

The process of finding the area of plane curves is often called quadrature.

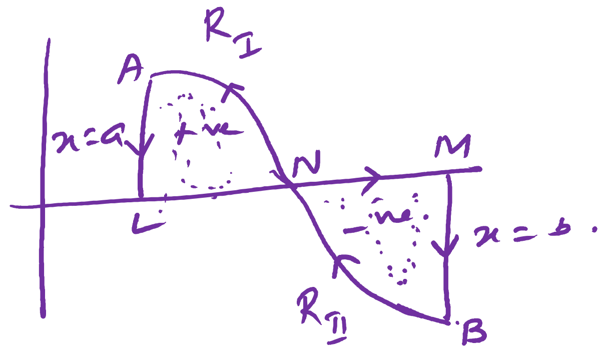
2. Sign of an Area An area whose boundary is described in the anticlockwise direction is considered positive and an area whose boundary is described in the clockwise direction is taken as negative.



+ve area



–ve area



Areas under the x-axis will come out negative and **areas** above the x-axis will be positive.

Problem:

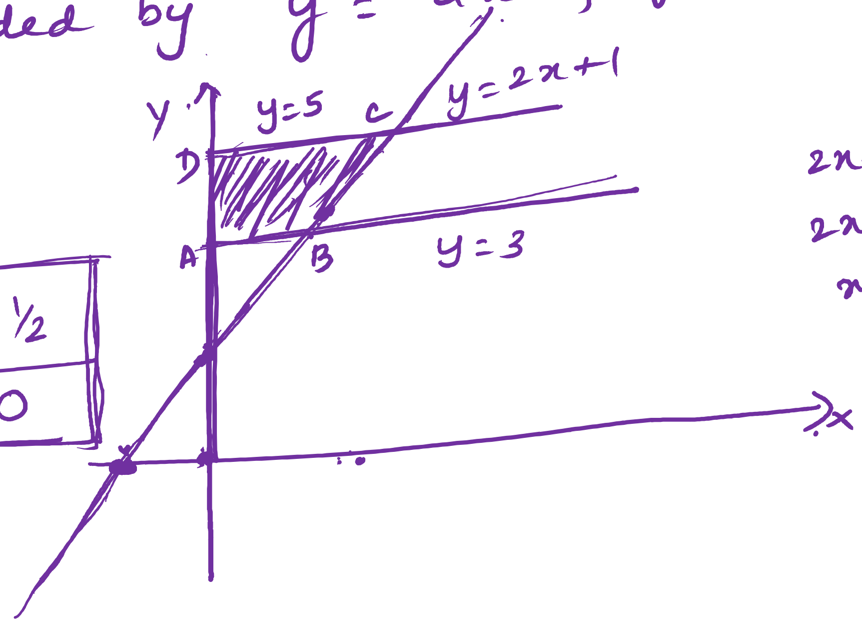
Find the area of the region bounded by $y = 2x+1$, $y=3$, $y=5$ and y-axis.

Soln

Required area is bdd by $y=2x+1$, $y=3$, $y=5$ & the y axis.

$y = 2x+1$

x	0	1	$-\frac{1}{2}$
y	1	3	0



$2x+1=0$

$2x=-1$

$x=-\frac{1}{2}$

Reqd area $= \int_{y=c}^{d} x\,dy$.

$y = 2x+1$

$2x = y-1$

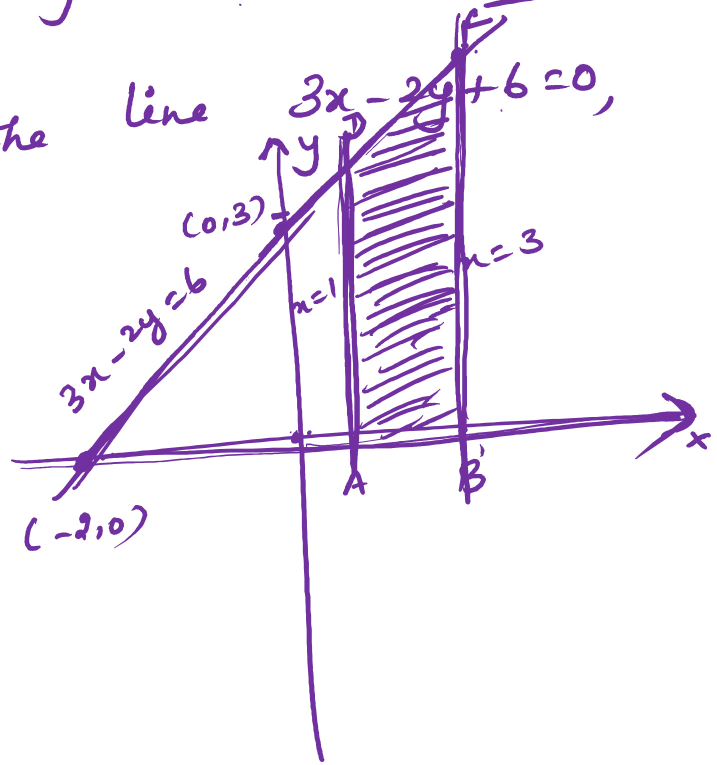
$x = \frac{y-1}{2}$

$= \int_{y=3}^{5} \left(\frac{y-1}{2}\right) dy = \frac{1}{2}\int_{y=3}^{5} (y-1)\,dy$

$\frac{1}{2}\left\{\left(\frac{25}{2} - \frac{9}{2}\right) - (5-3)\right\}$

$= \frac{1}{2}\left\{\frac{y^2}{2} - y\right\}_3^5 = \frac{1}{2}\left\{\left(\frac{25}{2} - 5\right) - \left(\frac{9}{2} - 3\right)\right\} = \frac{1}{2}(8-2) = 3$ sq.units

2) Find the area of the region bounded by the line $3x - 2y + 6 = 0$, $x = 1$, $x = 3$ and the x-axis.



Soln Area $A = \int_{x=a}^{b} y\,dx$.

$3x - 2y + 6 = 0$

x	0	-2	1
y	3	0	$\frac{9}{2}$

$3x - 2y + 6 = 0$

$2y = 3x + 6 \Rightarrow y = \frac{1}{2}(3x+6)$

Area $A = \int_1^3 \frac{1}{2}(3x+6)\,dx$

$= \frac{1}{2}\left(\frac{3x^2}{2} + 6x\right)_1^3 = \frac{1}{2}\left\{\left(\frac{3\times 9}{2} + 6\times 3\right) - \left(\frac{3}{2} + 6\right)\right\}$

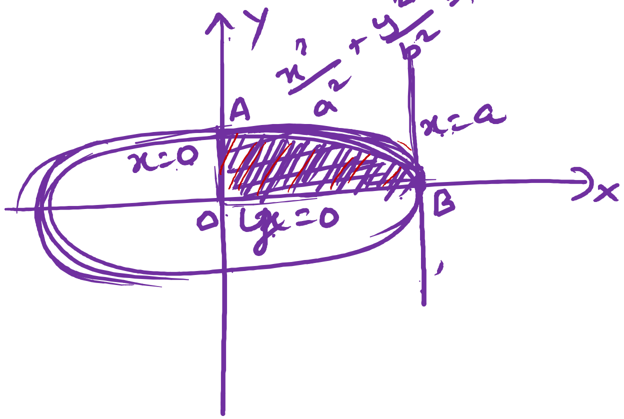
$= \frac{1}{2}\left[\left(\frac{27}{2} - \frac{3}{2}\right) + (18 - 6)\right]$

$= \frac{1}{2}(12 + 12)$

$= 12$ sq. units.

3) Find the area of the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Area of the ellipse $= 4 \times$ area of I quadrant.



Area $= 4 \times \int_0^a y\,dx.$

$= 4 \times \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx.$

$= \frac{4b}{a}\left\{\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right\}_0^a$

$= \frac{4b}{a}\left\{\left(0 + \frac{a^2}{2}\sin^{-1}(1)\right) - 0\right\}$

$= \frac{4b}{a} \cdot \frac{a^2}{2} \cdot \frac{\pi}{2}.$

$= \pi ab$ sq. units

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

$\frac{y^2}{b^2} = \frac{a^2-x^2}{a^2}$

$y^2 = \frac{b^2}{a^2}(a^2-x^2)$

$y = \frac{b}{a}\sqrt{a^2-x^2}$

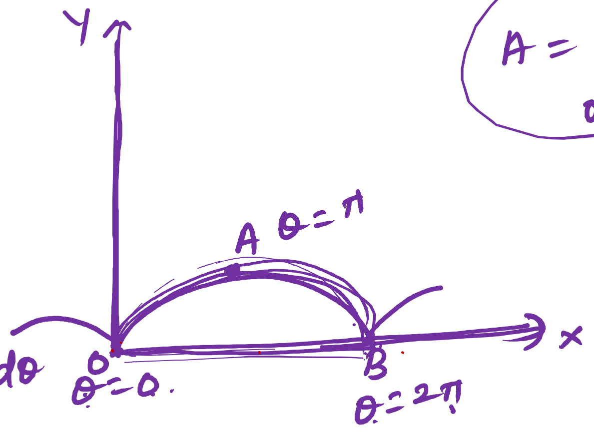
$\sin\left(\frac{\pi}{2}\right) = 1$

$\frac{\pi}{2} = \sin^{-1}(1)$

$\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

4) Find the area enclosed between one arch of the cycloid $x = a(\theta - \sin\theta)$; $y = a(1-\cos\theta)$ and its base.

$$A = \int_0^a y\,dx.$$



$$\text{Reqd area} = \int_{x=0}^{2a\pi} y\,dx. \quad , \quad \text{Where}$$

$$y = a(1-\cos\theta), \qquad x = a(\theta - \sin\theta)$$

$$dx = a(1-\cos\theta)\,d\theta$$

$$A = \int_{\theta=0}^{2\pi} a(1-\cos\theta)\,a(1-\cos\theta)\,d\theta$$

$$= a^2 \int_{\theta=0}^{2\pi} (1-\cos\theta)^2\,d\theta$$

$$= a^2 \int_{\theta=0}^{2\pi} \left(2\sin^2\frac{\theta}{2}\right)^2 d\theta.$$

$$1 - \cos 2\theta = 2\sin^2\theta$$

$$\therefore 1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$

$$= 4a^2 \int_{\theta=0}^{2\pi} \sin^4 \frac{\theta}{2}\, d\theta. \qquad 0 \text{ to } 2\pi \rightarrow 2(0 \text{ to } \pi)$$

$$= 4a^2 \times 2 \int_0^{\pi} \sin^4 \frac{\theta}{2}\, d\theta.$$

put $\frac{\theta}{2} = t \Rightarrow \theta = 2t$

$\Rightarrow d\theta = 2dt$

limits

when $\theta = 0$, $t = 0$

when $\theta = \pi$, $t = \frac{\pi}{2}$

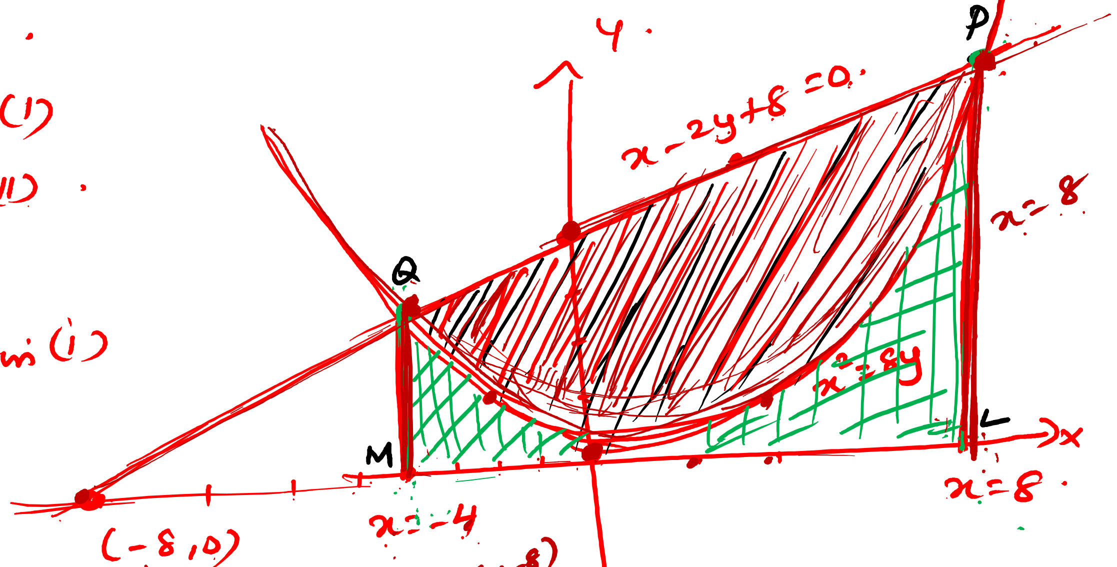
$$\therefore A = 8a^2 \int_0^{\pi/2} \sin^4 t\,(2dt)$$

$$= 16a^2 \int_0^{\pi/2} \sin^4 t\, dt \quad , \quad n = 4 \text{(even)} \quad \int_0^{\pi/2} \sin^n x\, dx, \ n \text{ is even} = \frac{(n-1)(n-3)\cdots 1}{n(n-2)\cdots 2} \cdot \frac{\pi}{2}$$

$$= 16a^2 \left\{ \frac{(4-1)(4-3)}{4(4-2)} \cdot \frac{\pi}{2} \right\}$$

$$= \cancel{16}a^2 \times \frac{3 \times 1}{\cancel{4} \times \cancel{2}} \times \frac{\pi}{\cancel{2}} = 3\pi a^2$$

5) Find the area of the ~~line tangent cut off from~~ the parabola $x^2 = 8y$ cut off by the line $x - 2y + 8 = 0$.



Soln Given parabola is $x^2 = 8y$ —(I)

St. line is $x - 2y + 8 = 0$ —(II).

$\Rightarrow 2y = x + 8$

$\Rightarrow y = \frac{x+8}{2}$ sub in (i)

$x^2 = 8y$

$\Rightarrow y = \frac{x^2}{8}$

x	0	2	-2
y	0	.5	.5

$x^2 = 8\left(\frac{x+8}{2}\right)$

$x^2 = 4(x+8) \Rightarrow x^2 = 4x + 32$

$\Rightarrow x^2 - 4x - \cancel{8}\,32 = 0$

$x^2 - 8x + 4x - 32 = 0 \Rightarrow x(x-8) + 4(x-8) = 0$

$\Rightarrow (x-8)(x+4) = 0$

$\Rightarrow x = 8, -4$.

-8, 4

~~x^2 4x~~

(i) & (II) intersect at P and Q

Where $x = 8$ and $x = -4$.

$x - 2y + 8 = 0$

x	0	-8	2
y	4	0	5

$-2y + 10 = 0$

$-2y = -10$

$y = 5$

$\therefore$ Reqd area POQ = area bounded by st. line (ii) and x axis from $x=-4$ to $x=8$ − area bdd by parabola (i) and x-axis from $x=-4$ to $x=8$.

(i) $x-2y+8=0$

$2y = x+8$

$y = \frac{x+8}{2}$

(ii) $x^2 = 8y$

$\Rightarrow y = \frac{x^2}{8}$

st. line / parabola

$$\text{Area} = \int_{-4}^{8} y\,dx \text{ from (ii)} - \int_{-4}^{8} y\,dx \text{ from (i)}$$

from (ii), $y = \frac{x^2}{8}$

From ① $y = \frac{x+8}{2}$

$$\therefore A = \int_{-4}^{8} \left(\frac{x+8}{2}\right) dx - \int_{-4}^{8} \frac{x^2}{8}\,dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2} + 8x\right)_{-4}^{8} - \left(\frac{x^3}{24}\right)_{-4}^{8}$$

$$= \frac{1}{2}\left\{\left(\frac{64}{2} + 64\right) - \left(\frac{16}{2} - 32\right)\right\} - \frac{1}{24}\left(8^3 - (-4)^3\right)$$

$$\frac{1}{2}\{96 + 24\} - \frac{1}{24}\left(512 - (-64)\right\}$$

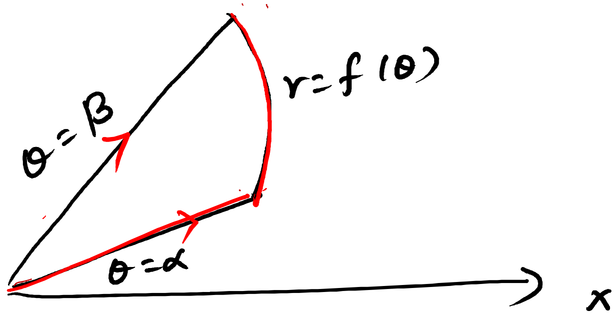
$$= 60 - \frac{\cancel{576}^{24}}{\cancel{24}} = 36$$

$$= \frac{1}{2}\left\{(32+64)-(-24)\right\} - \frac{1}{24}(512+64).$$

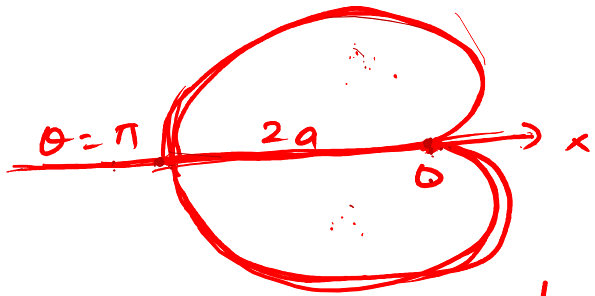
$$= \underline{\underline{36}}\,.$$

Area of polar curves.

Area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$ is $A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$.



1) Find the area of the cardiod $r = a(1-\cos\theta)$



Soln

upper half of the curve is traced from $\theta=0$ to $\theta=\pi$..

Area of the curve $= 2 \times$ Area of the upper half of the cardiod.

$$\frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$$

$$= 2 \times \frac{1}{2} \int_{\theta=\alpha}^{\beta} r^2 \, d\theta.$$

$$= \int_{\theta=0}^{\pi} a^2 (1-\cos\theta)^2 \, d\theta.$$

$1-\cos 2\theta = 2\sin^2\theta$

$\therefore 1-\cos\theta = \cancel{2\cos^2\theta}$ $2\sin^2\frac{\theta}{2}$

$$= a^2 \int_0^{\pi} \left(2\sin^2\frac{\theta}{2}\right)^2 d\theta. \quad = 4a^2 \int_{\theta=0}^{\pi} \sin^4\frac{\theta}{2} \, d\theta.$$

put $\frac{\theta}{2} = t \Rightarrow \theta = 2t \Rightarrow d\theta = 2\,dt$

When $\theta = 0$, $t = 0$; When $\theta = \pi$, $t = \frac{\pi}{2}$

$\therefore \theta : 0$ to $\pi/2$

$$\therefore \text{Area} = 4a^2 \int_0^{\pi/2} \sin^4 t\,(2\,dt) = 8a^2 \int_0^{\pi/2} \sin^4 t\,dt, \quad n = 4 \text{ is even}$$

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{(n-1)(n-3)\cdots}{n(n-2)(n-4)\cdots} \cdot \frac{\pi}{2}$$

$$= 8a^2 \cdot \frac{3 \times 1}{4 \times 2} \cdot \frac{\pi}{2} = \frac{3}{2}\pi a^2$$

$$= \frac{3\pi a^2}{2}$$

2) Find the area of a loop of the curve $r = a \sin 3\theta$.

$\boxed{r = a\sin 3\theta}$ $\qquad$ $r = a(1-\cos\theta)$

The curve consists of 3 loops.

put $r=0$, $\sin 3\theta = 0$ $\therefore 3\theta = 0$ or π.

i.e. $\theta = 0$ or $\pi/3$.

$\therefore$ limits for the first loop $\theta : 0$ to $\pi/3$.

$$\text{Area of a loop} = \frac{1}{2}\int_{\theta=\alpha}^{\beta} r^2\,d\theta = \frac{1}{2}\int_{\theta=0}^{\pi/3} a^2 \underline{\underline{\sin^2 3\theta}}\,d\theta.$$

$$= \frac{a^2}{2}\int_{\theta=0}^{\pi/3}\left(\frac{1-\cos 6\theta}{2}\right)d\theta.$$

$$= \frac{a^2}{4}\int_{\theta=0}^{\pi/3}(1-\cos 6\theta)\,d\theta = \frac{a^2}{4}\left[\theta - \frac{\sin 6\theta}{6}\right]_{\theta=0}^{\pi/3}$$

$$= \frac{a^2}{4}\left(\frac{\pi}{3} - 0\right)$$

$$= \frac{\pi a^2}{12} \;//.$$

$$\frac{\sin 6 \times \frac{\pi}{3}}{6} = \frac{0}{6} = 0$$

Note

$1 - \cos 2\theta = 2\sin^2\theta$

put $\theta = 3\theta$

$1 - \cos 2(3\theta) = 2\sin^2 3\theta$

$1 - \cos 6\theta = 2\sin^2 3\theta$

$\frac{1-\cos 6\theta}{2} = \sin^2 3\theta$