Module 2: Sequences and Series.

Defn. If $u_1, u_2, u_3 \ldots\ldots u_n, \ldots\ldots$ be an infinite sequence of real numbers, then $u_1 + u_2 + \ldots\ldots + u_n + \ldots \infty$ is called an infinite series.

An infinite series is denoted by Σu_n and the sum of its first 'n' terms is denoted by S_n.

Convergence, divergence and oscillation of a series.

Consider the infinite series $\Sigma u_n = u_1 + u_2 + \ldots + u_n + \ldots \infty$ and let the sum of the first 'n' terms be $S_n = u_1 + u_2 + \ldots + u_n$.

(i) If $S_n \rightarrow$ a finite limit as $n \rightarrow \infty$, the series Σu_n is said to be convergent.

(ii) If $S_n \rightarrow \pm\infty$, as $n \rightarrow \infty$, the series Σu_n is said to be divergent.

(iii) If S_n does not tend to a unique limit as $n \rightarrow \infty$, then the series Σu_n is said to be oscillatory or non-convergent.

* The Geometric Series $1+r+r^2+r^3+\dots\infty$ (i) converges if $|r|<1$

(II) diverges if $|r|\geq 1$ (III) Oscillates if $r\leq -1$.

Examine the following series for convergence.

1) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots\dots$ $(1+r+r^2+r^3+\dots$

Soln The given series is $1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\dots\dots$

$\therefore r=\frac{1}{2}$; $|r|=\frac{1}{2}<1$

$\therefore$ The given series is convergent.

2) $1-\frac{1}{3}+\frac{1}{3^2}-\frac{1}{3^3}+\frac{1}{3^4}-\dots\dots$

Soln. $1+\left(-\frac{1}{3}\right)+\left(-\frac{1}{3}\right)^2+\left(-\frac{1}{3}\right)^3+\dots\dots$, $r=-\frac{1}{3}$

$\therefore |r|=\left|-\frac{1}{3}\right|=\frac{1}{3}<1$. $\therefore$ The given series is convergent.

3) $1+2+3+\cdots\cdots\infty$

The sum of the first n terms $1+2+3+\cdots+n = \frac{n(n+1)}{2}$

ie $S_n = \frac{n(n+1)}{2}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$

$\therefore$ The given series is divergent.

(i) $1+2+4+8+16+\cdots\cdots$

$2^0+2^1+2^2+2^3+\cdots\cdots$

(ii) $1+3+9+27+\cdots\cdots$

$1+3^1+3^2+3^3+3^4+\cdots\cdots$

(i) $r = 2>1$, $|r|=2>1$ $\therefore$ (i) is dgt

(ii) $r=3$, $|r|=3>1$ $\therefore$ (ii) is dgt

4) $5-4-1+5-4-1+5-4-1+\cdots\cdots\infty$

$S_n = 5-4-1+5-4-1+5-4-1+\cdots n$

$S_n = 0, 5, 1$, according as the no. of terms $3m, 3m+1, 3m+2$

$\lim_{n\to\infty} S_n$ = 0, 5, 1, ~~uqu~~ unique value.

ie $\lim_{n\to\infty} S_n$ does not tend to a unique value.

Hence the series is oscillatory.

Test for comparison.

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots \infty$, (i) converges for $p > 1$

and (ii) diverges for $p \leq 1$.

Ex 1) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$

$p = 2 > 1$ $\therefore$ The series converges.

2) $\frac{1}{1^{-1}} + \frac{1}{2^{-1}} + \frac{1}{3^{-1}} + \frac{1}{4^{-1}} + \ldots$ $p = -1 < 1$ $\therefore$ The series diverges.

Comparison Tests

1. If two positive term of series $\sum U_n$ and $\sum V_n$ such that (i) $\sum V_n$ converges (ii) $U_n \leq V_n$ $\forall n$, then $\sum U_n$ also converges.

2. If two positive term series $\sum U_n$ and $\sum V_n$ such that (i) $\sum V_n$ diverges (ii) $U_n \geq V_n$ $\forall n$, then $\sum U_n$ also diverges.

$\boxed{\text{Note}}$ $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e = 2.719 > 1$

$\forall \rightarrow$ for all.

$\frac{U_n}{V_n} = 2$ $U_n = 2V_n$ $U_n \geq V_n$

Test for convergence, the series

$(2n-1) = n\left(2-\frac{1}{n}\right)$

(i) $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \ldots \infty$

Soln $U_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{\cancel{n}\left(2-\frac{1}{n}\right)}{\cancel{n}.\, n\left(1+\frac{1}{n}\right).\, n\left(1+\frac{2}{n}\right)}$

$= \frac{1}{n^2} \cdot \frac{\left(2-1/n\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$

$\lim_{n\to\infty} \frac{1}{n} = 0$

$\left(\frac{1}{\infty}\right) = 0$

$\times \frac{n^2}{1}$

$\frac{u_n}{v_n} = \frac{\left(2-\frac{1}{n}\right)}{\cancel{n^2}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$

$= \frac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$

Let $V_n = \frac{1}{n^2}$.

Now $\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = 2$, is finite.

$\therefore$ Both U_n & V_n converge or diverge together.

$\Sigma V_n = \Sigma \frac{1}{n^2}$ is convergent. $\longrightarrow$ $\left(\Sigma \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right.$

$p = 2 > 1$

$V_n = \Sigma \frac{1}{n^2}$ is cgt

$\therefore \Sigma U_n$ is also convergent.

2) $\frac{1}{4.7.10} + \frac{4}{7.10.13} + \frac{9}{10.13.16} + \cdots\cdots \infty.$

Soln

$$u_n = \frac{n^2}{(3n+1)(3n+4)(3n+7)} = \frac{\cancel{n^2}}{\cancel{n}(3+\frac{1}{n})\ \cancel{n}(3+\frac{4}{n})\ n(3+\frac{7}{n})}$$

$$= \frac{1}{n(3+\frac{1}{n})(3+\frac{4}{n})(3+\frac{7}{n})}$$

$$\therefore u_n = \frac{1}{n} \cdot \frac{1}{(3+\frac{1}{n})(3+\frac{4}{n})(3+\frac{7}{n})}$$

Let $v_n = \frac{1}{n}$.

$$\frac{u_n}{v_n} = \frac{\frac{1}{n(3+\frac{1}{n})(3+\frac{4}{n})(3+\frac{7}{n})}}{1/n} = \frac{1}{(3+\frac{1}{n})(3+\frac{4}{n})(3+\frac{7}{n})}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{(3+\frac{1}{n})(3+\frac{4}{n})(3+\frac{7}{n})} = \frac{1}{27} \neq 0. \left(\frac{1}{27} < 1\right)$$

$$\frac{1}{n} \to \frac{1}{\infty} = 0$$

Since Σv_n is divergent, $\therefore \Sigma u_n$ is also divergent.

$$\Sigma v_n = \Sigma \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \cdots\cdots$$

dgt

iii) $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \cdots\cdots$

Soln

Here $u_n = \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)\cdot(n+1)^n} = \frac{1}{n+1}\left(\frac{n}{n+1}\right)^n$

Let $v_n = \frac{1}{n}$

$\frac{u_n}{v_n} = \frac{1}{n+1}\left(\frac{n}{n+1}\right)^n \Big/ \frac{1}{n} = \frac{n}{n+1}\left(\frac{n}{n+1}\right)^n$

$\Rightarrow \frac{1}{n+1}\times\left(\frac{n}{n+1}\right)^n \times \frac{n}{1}$

$= \frac{n}{n+1}\cdot\left(\frac{n}{n+1}\right)^n$

$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n}{n+1} \cdot \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$

$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(1+\frac{1}{n})} \lim_{n\to\infty}\left(\frac{\cancel{n}}{\cancel{n}(1+\frac{1}{n})}\right)^n$

$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} \cdot \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = 1 \cdot \frac{1}{e} < 1$ & $\neq 0$.

Since $\sum v_n = \frac{1}{n}$ is divergent, $\sum u_n$ is also divergent.

D' Alembert's Ratio Test (DAR Test)

In a positive term series ΣU_n, if

$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lambda$, then the series converges for $\lambda < 1$

and diverges for $\lambda > 1$.

$\frac{U_{n+1}}{U_n} < 1$ converges

$\frac{U_{n+1}}{U_n} > 1$

In otherwards, If $\frac{U_n}{U_{n+1}} > 1$, then ΣU_n converges & $\frac{U_n}{U_{n+1}} < 1$, then ΣU_n diverges.

1) Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{n!}{(n^n)^2}$.

Soln. We have $U_n = \frac{n!}{(n^n)^2}$ and $U_{n+1} = \frac{(n+1)!}{((n+1)^{n+1})^2}$

i.e. $U_n = \frac{n!}{n^{2n}}$ and $U_{n+1} = \frac{(n+1)!}{(n+1)^{2(n+1)}}$

Now $\frac{u_n}{u_{n+1}} = \frac{n!}{n^{2n}} \times \frac{(n+1)^{2(n+1)}}{(n+1)!}$

$= \frac{\cancel{n!}}{n^{2n}} \cdot \frac{(n+1)^{2n} \cdot (n+1)^{\cancel{2}}}{\cancel{(n+1)} \cdot \cancel{n!}}$

$= \frac{(n+1)^{2n}}{n^{2n}} (n+1)$

$= \left(\frac{n+1}{n}\right)^{2n} \cdot (n+1)$

$\left\{ \frac{\cancel{n^{2n}} \left(\left(1 + \frac{1}{n} \right)^n \right)^2}{\cancel{n^{2n}}} \right\} (n+1)$

$(n+1)^{2(n+1)} = (n+1)^{2n+2}$

$= (n+1)^{2n} \cdot (n+1)^2$

$(n+1)! = \underbrace{1 \cdot 2 \cdot 3 \cdot 4 \cdots n} \cdot (n+1)$

$= n! \, (n+1)$

$\frac{\cancel{n^{2n}} \left[1 + \frac{1}{n} \right]^{2n}}{\cancel{n^{2n}}} \cdot (n+1)$

$\left\{ \left(1 + \frac{1}{n} \right)^n \right\}^2$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^n\right]^2 \cdot (n+1)$$

$$= e^2 \cdot \lim_{n\to\infty} (n+1) \longrightarrow \infty .$$

Hence the series is convergent.

2) Discuss the convergence of the series $1+\frac{2!}{2^2}+\frac{3!}{3^3}+\frac{4!}{4^4}+\cdots\infty.$

Soln The given Series is $\sum u_n = \sum \frac{n!}{n^n}$

Let $u_n = \frac{n!}{n^n}$, $u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$.

$$\therefore \frac{u_n}{u_{n+1}} = \frac{n!}{n^n} \times \frac{(n+1)^{n+1}}{(n+1)!} = \frac{n! \cdot (n+1)(n+1)^n}{n^n \cdot n!\,(n+1)}$$

$$= \frac{(n+1)^n}{n^n}$$

$$= \left(\frac{n+1}{n}\right)^n$$

$$= \left(1+\frac{1}{n}\right)^n$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1$$

Hence the given series is convergent.

3) Examine the convergence of the series

$$1+\left(\frac{a+1}{b+1}\right)+\frac{(a+1)(2a+1)}{(b+1)(2b+1)}+\frac{(a+1)(2a+1)(3a+1)}{(b+1)(2b+1)(3b+1)}+\cdots\cdots\infty.$$

Soln

$$u_{n+1}=1+\left(\frac{a+1}{b+1}\right)+\frac{(a+1)(2a+1)}{(b+1)(2b+1)}+\cdots+\frac{(a+1)(2a+1)+\cdots+(na+1)}{(b+1)(2b+1)+\cdots(nb+1)}$$

($\because$ the 1st term is 1)

$$u_{n+1}=u_n\cdot\frac{n.a+1}{n.b+1}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\cancel{u_n}\times\frac{nb+1}{na+1\cdot\cancel{u_n}}=\frac{\cancel{n}\left(b+\frac{1}{n}\right)}{\cancel{n}\left(a+\frac{1}{n}\right)}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{\left(b+\frac{1}{n}\right)}{a+\frac{1}{n}}=\frac{b}{a}$$

The series is cgt if $\frac{b}{a} > 1$ ie if $b > a$.

& the series is dgt if $\frac{b}{a} < 1$ ie if $b < a$.