Range and Kernel of a Linear Map. (Linear Transformation)

Range: Let $T: U \to V$ be a function.
The set $\{T(x) ; x \in U\}$ is called range of T, denoted by $R(T)$.

Kernel Let $T: U \to V$ be a linear map. The Kernel of T or null space of T is defined as

$$N(T) = \{u \in U \mid T(u) = 0\}$$

Ker T or

$T(x) = T(y) \Rightarrow x = y$

$T(u) = T(v) \Rightarrow u = v$

Note. A map $T: U \to V$ is said to be onto, if $R(T) = V$

Pbm. 1. If $T: V_3 \to V_3$ is defined by $T(x_1, x_2, x_3) = (x_1, x_2, 0)$ find the range and kernel of T.

Soln.

(i) $R(T) = \{T(x) : x \in U\}$

$\therefore R(T) = \{x_1, x_2, 0\}$. $[x_1 x_2\text{-plane}, \because x_3 = 0]$

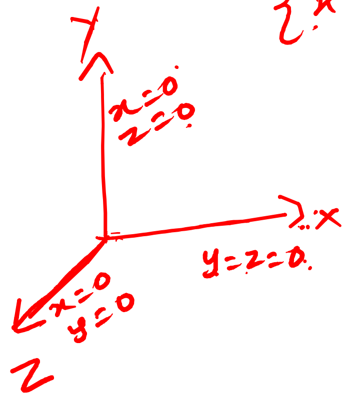
(ii) $\ker T = N(T) = \{u \in U \,/\, T(u) = 0\}$

ie $N(T) = T\{x_1, x_2, x_3\} = 0$

ie $(x_1, x_2, 0) = (0, 0, 0)$

$\Rightarrow x_1 = 0, \ x_2 = 0$

$= X_3$ axis.



$\{x, y, 0\}$

$\downarrow$

$z = 0$

$\downarrow$

Eqn of the xy plane

2) If $T : V_3 \to V_2$ is defined by $T(x_1, x_2, x_3) = (x_1 - x_2, \ x_1 + x_3)$

find the Range and Kernel of T.

Soln: $R(T) = \{T(x) : x \in U\}$

$= \{T(x_1, x_2, x_3) : x_i \in U, i = 1, 2, 3\}$

$= (x_1 - x_2, \quad x_1 + x_3) = (a, b)$

$R(T)$

$\Rightarrow (x_1 - x_2 = a, \quad x_1 + x_3 = b)$

$\Rightarrow x_2 = x_1 - a, \quad x_3 = b - x_1.$

$\therefore R(T) = \{x_1, \; x_1 - a, \; b - x_1\}$

$\ker T = N(T) = \{(x_1, x_2, x_3) \in V_3 / T(x_1, x_2, x_3) = 0\}$

$= (x_1 - x_2, \quad x_1 + x_3) = (0, 0)$

$\Rightarrow x_1 - x_2 = 0, \quad x_1 + x_3 = 0$

$\Rightarrow x_1 = x_2 \; ; \; x_1 = -x_3$

$\Rightarrow x_1 = x_2 = -x_3$

x_1

$x_2 = x_1$

$x_3 = -x_1$

ie all the vectors of the form $(x_1, x_1, -x_1)$ will be or $(1, 1, -1)$ mapped into zero.

$\therefore N(T) = \{x_1(1, 1, -1) \mid x_1, \text{any scalar}\} = [(1, +1, -1)]$, which is the subspace of V_3 generated by $(1, 1, -1)$

Defn: Let $T : U \to V$ be a linear map. Then

(a) If $R(T)$ is finite dimensional, the dimension of $R(T)$ is called the rank of T and is denoted by $r(T)$.

(b) If $N(T)$ (Ker T) is finite dimensional, the dimension of $N(T)$ is called the nullity of T and is denoted by $n(T)$.

$\dim R(T) = r(T) \rightarrow$ rank

$\dim N(T)$ or $\text{Ker}\, T$ $= n(T) \rightarrow$ nullity

Note: (Rank - Nullity Theorem) or Sylvester's Thm.

Let $T: U \to V$ be a linear map and U, a finite dimensional vector space. Then

$$\dim R(T) + \dim N(T) = \dim U.$$

$$r(T) + n(T) = \dim U.$$

(or)

i.e. rank + nullity = dimension of the domain space.

1) For $T: R^3 \to R^3$ given by $T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$ find rank and nullity of T.

Soln To find nullity, we find ker T, so image elt should be zero.

$N(T) = \ker T = T(x_1, x_2, x_3) = 0$

i.e. $(3x_1, x_1 - x_2, 2x_1 + x_2 + x_3) = (0, 0, 0)$

$$3x_1 = 0, \quad x_1 - x_2 = 0, \quad 2x_1 + x_2 + x_3 = 0$$

$\Rightarrow$

$\Rightarrow \ x_1 = 0, \ \ x_2 = 0, \ \ x_3 = 0.$

$\therefore \ \ker T = \{(0,0,0)\} \Rightarrow$ nullity $= \dim(\ker T) = 0.$

$n(T) = 0$

dim. of zero vector space $= 0$

By rank nullity thm,

$\dim(R^3) = \text{rank}(T) + \text{nullity}(T)$

$3 = r(T) + 0$

$\therefore \ r(T) = 3$

2) Find the range, rank, kernel and nullity of the L.T ($\because T : V_2 \to V_3$)

$T(x_1, x_2) = (x_1 + x_2, \ x_1 - x_2, \ x_2)$

Soln: Range $R(T) = \{T(x) : x \in U\}$

ie $(x_1 + x_2, x_1 - x_2, x_2) = x_1(1,1,0) + x_2(0,-1,1)$

$\therefore R(T)$ is spanned by $(1,1,0)$ & $(0,-1,1)$

$\therefore R(T) = \{(1,1,0), (1,-1,0)\}$

$r(T) = \dim R(T) = 2 \longrightarrow$ rank.

<u>kerT</u> $= N(T) = \{T(x) = 0 \,;\, x \in U\}$

$N(T) = T(x_1, x_2) = 0$

i.e. $(x_1 + x_2,\ x_1 - x_2,\ x_2) = (0,0,0)$

$\Rightarrow x_1 + x_2 = 0,\quad x_1 - x_2 = 0,\quad x_2 = 0$

$\Rightarrow x_1 = 0,\ x_2 = 0.$

$\therefore \ker T = \{(0,0)\}$ & $\dim \ker T = n(T) = 0.$

[dimension of a <u>zero</u> <u>vector space</u> is zero
all elts are zero]

3) Let $T: V_4 \to V_3$ be a linear map defined by $T(x_1, x_2, x_3, x_4) = (x_1 + x_2 + x_3 + x_4,\ x_1 - x_2,\ x_1 + x_2 + x_4)$. Verify that

$r(T) + n(T) = \dim U = 4$

$R(T),\ N(T)$

$R(T) = \{T(x) : x \in U\}$

$\therefore R(T) = \{x_1 + x_2 + x_3 + x_4,\ x_1 - x_2,\ x_1 + x_2 + x_4\}$

$= x_1(1,1,1) + x_2(1,-1,1) + x_3(1,0,0) + x_4\cancel{(1,0,1)}$

$R(T) = \{(1,1,1), (1,-1,1), (1,0,0), (1,0,1)\}$

We can discard $(1,0,1)$ $(\because \dim V_3 = 3)$

$\therefore$ In $R(T)$, 3 linearly independent vectors. $r(T) = \dim R(T) = 3$.

$\ker T = N(T) = \{T(u) = 0,\ u \in U\}$

$(x_1 + x_2 + x_3 + x_4,\ x_1 - x_2,\ x_1 + x_2 + x_4) = 0$

ie $(x_1 + x_2 + x_3 + x_4,\ x_1 - x_2,\ x_1 + x_2 + x_4 = 0$

$x_1 + x_2 + x_3 + x_4 = 0, \quad x_1 - x_2 = 0, \quad \underline{x_1 + x_2 + x_4} = 0$

$\Downarrow$ Sub $x_1 + x_2 + x_4 = 0$ $\Rightarrow x_3 = 0$

$\Downarrow$ $x_1 = x_2$

$\Downarrow$ $2x_1 + x_4 = 0$

$\Rightarrow$

Since $x_3 = 0$, $x_1 = x_2$

We have $x_1 + x_2 + x_3 + x_4 = 0 \Rightarrow 2x_1 + x_4 = 0$.

$x_1 + x_2 + x_4 = 0 \Rightarrow 2x_1 + x_4 = 0$

$\Rightarrow x_4 = -2x_1$

$\therefore N(T) = \{x_1, x_1, 0, -2x_1\} = \{x_1(1,1,0,-2)\}$, x_1 be any scalar.

$\therefore n(T) = \dim N(T) = 1$.

$\therefore r(T) + n(T) = 3 + 1 = 4 = \dim U$

$U \to V_4$