Subspace of a Vector Space.

Let W be the subset of a vector space V, then W is called subspace of V if and only if

(i) W is non-empty (or $0 \in W$)

(ii) $\forall\ u, v \in W$, $u+v \in W$

(iii) for $u \in W$ and α is a real number, $\alpha u \in W$ or $\alpha u + \beta v \in W$ for $u, v \in W$, α, β are real numbers.

Result (1) The intersection of any two subspaces W_1 and W_2 of a vector space V is also a subspace of V. [$W_1 \cap W_2$ is a subspace of V]

2. The union of two subspaces is a subspace iff one is contained in the other. [ie $W_1 \cup W_2$ is a subspace of V iff $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$]

Defn If $u_1, u_2 \dots u_n$ are 'n' vectors of a vector space V, then for any vector $v = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$, $\alpha_1, \alpha_2, \dots \alpha_n$ are scalars is called linear combination of the vectors $u_1, u_2 \dots u_n$.

The linear combination $\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$ is called a trivial linear combination, if all scalars $\alpha_1, \alpha_2 \dots \alpha_n$ are zero. Otherwise the linear combination is said to be non-trivial.

The linear combination.

$0u_1 + 0u_2 + \dots + 0u_n$ is a trivial linear combination.

$0u_1 + 0u_2 + \dots + 0u_{n-1} + 1u_n$ is a non trivial linear combination.

$1u_1 + 2u_2 + \dots + nu_n$ is also a non-trivial linear combination of any set of vectors

Note that, the trivial linear combination is always the zero vector for

$$0u_1 + 0u_2 + \dots + 0u_n = 0 + 0 + \dots + 0 = 0$$

Example-1) Let $(1,0,0)$, $(2,0,0)$ and $(0,0,1)$ be three Vectors in V_3.

Then We have

$$\underbrace{1}_{\alpha_1}(1,0,0) + \underbrace{\left(-\frac{1}{2}\right)}_{\alpha_2}(2,0,0) + \underbrace{0}_{\alpha_3}(0,0,1) = (0,0,0) = 0$$

or

$$-2(1,0,0) + 1(2,0,0) + 0(0,0,1) = 0.$$

$\therefore$ A non trivial linear combination may give the Zero vector.

2) Let $(1,0,0)$, $(0,1,0)$ & $(0,0,1)$ be 3 vectors of V_3.

Then $\alpha(1,0,0) + \beta(0,1,0) + \gamma(0,0,1) = 0$

$$\boxed{\text{ie } 0(1,0,0) + 0(0,1,0) + 0(0,0,1) = 0}$$

$\Rightarrow \alpha = \beta = \gamma = 0$.

$\therefore$ A trivial linear combination may give the Zero Vector.

$$\left[\begin{array}{ll} \alpha + 0\beta + 0\gamma = 0 & \Rightarrow \alpha = 0 \\ 0\alpha + \beta + 0\gamma = 0 & \Rightarrow \beta = 0 \\ 0\alpha + 0\beta + \gamma = 0 & \Rightarrow \gamma = 0 \end{array}\right.$$

3) Let $(1,1,1)$, $(1,1,0)$ and $(1,0,1)$ be three vectors.

Then $\alpha(1,1,1) + \beta(1,1,0) + \gamma(1,0,1) = 0$

$(\alpha+\beta+\gamma, \ \alpha+\beta, \ \alpha+\gamma) = (0,0,0)$

$\Rightarrow \underbrace{\alpha+\beta+\gamma = 0}_{①}, \ \underbrace{\alpha+\beta = 0}_{②}, \ \underbrace{\alpha+\gamma = 0}_{③}$. Sub in $\alpha+\beta+\gamma=0 \Rightarrow \boxed{\beta = 0}$ Sub in ② $\alpha+0=0 \Rightarrow \alpha = 0$ Sub in (III) $\alpha+\gamma=0$, $0+\gamma=0 \Rightarrow \gamma = 0$

$\Rightarrow \alpha = \beta = \gamma = 0$

$\therefore$ A trivial L.C may give the zero vector.

* Linear Dependence and Independence of Vectors.

A set $\{u_1, u_2, \ldots u_n\}$ of vectors is said to be linearly dependent, (LD), if there exists a non-trivial linear combination of $u_1, u_2, \ldots u_n$ that equals the zero vector.

non trivial L.C $\rightarrow$ Vectors are L.D

A set $\{u_1, u_2, \dots u_n\}$ of vectors is said to be linearly independent (LI), if the only linear combination of $u_1, u_2 \dots u_n$ equals the zero vector. (trivial linear combination) $\rightarrow$ L.I

Pbm 1. S.T the vectors $(1,1,0)$, $(2,1,1)$ and $(3,0,3)$ are linearly dependent.

Soln Let $\alpha, \beta, \gamma \in R$ are scalars such that

$$\alpha(1,1,0) + \beta(2,1,1) + \gamma(3,0,3) = (0,0,0)$$

$$\Rightarrow \alpha + 2\beta + 3\gamma = 0, \quad \alpha + \beta = 0, \quad \beta + 3\gamma = 0.$$

In matrix form

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

(OR)

$$\begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 0 & 3 \end{vmatrix}$$

$$= 1(3) - 1(6-3)$$

$$= 3 - 3$$

$$= 0$$

$\therefore$ The gn vectors are L.D

Now $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 0 & 1 & 3 \end{vmatrix} = 1(3) - 2(0) + 3(-1)$

$= 3 - 3 = 0$

$\Rightarrow$ The system posses a non zero soln.

$\therefore$ The given vectors are linearly dependent.

2) S.T the vectors $(1,0,1), (1,1,0)$ & $(0,1,1)$ in $\mathbb{R}^3$ are linearly independent.

Soln. $\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1(1) - 0 + 1(1)$

$= 2 \neq 0$

Note

1) If the determinant is 0, the vectors are L.D

2. If determinant $\neq 0$, the vectors are L.I.

$\therefore$ The given vectors are ~~not~~ linearly independent.

~~(linearly dependent)~~.

3) P.T the vectors $(1,0,1)$, $(1,1,0)$ & $(-1,0,-1)$ are L.D

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{vmatrix} = 1(-1) + 1(1) = 0$$

$\therefore$ The given vectors are L.D.

4) P.T the vectors $(1,0,1)$, $(1,1,0)$ & $(1,1,-1)$ are L.I.

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{vmatrix} = 1(-1) + 1(1-1) = -1 \neq 0$$

$\therefore$ The given vectors are L.I.

5) Check whether the following set of vectors are L.D or L.I

$\{(1,0,1), (1,1,0), (1,-1,1), (1,2,-3)\}$

Soln $\alpha(1,0,1) + \beta(1,1,0) + \gamma(1,-1,1) + \delta(1,2,-3) = 0$

$\alpha + \beta + \gamma + \delta = 0$ — (i)

$\beta - \gamma + 2\delta = 0$ — (ii)

$\alpha + \gamma - 3\delta = 0$ — (iii)

(iii) $\Rightarrow \alpha + \gamma = 3\delta$ sub in (i) $\beta + 3\delta + \delta = 0$

$\Rightarrow \boxed{\beta = -4\delta}$ sub in (ii)

$-4\delta - \gamma + 2\delta = 0 \Rightarrow -\gamma - 2\delta = 0 \Rightarrow \gamma = -2\delta$ sub in (iii)

$\alpha - 2\delta - 3\delta = 0 \Rightarrow \cancel{\alpha} = 5\delta$ $\alpha = 5\delta$

$\beta = -4\delta$
$\gamma = 5\delta$

$\gamma = -2$

Take $\delta = 1 \Rightarrow \alpha = 5, \beta = -4,$ & $\delta = 1$

$\therefore 5(1,0,1) - 4(1,1,0) - 2(1,-1,1) + 1(1,2,-3) = (0,0,0)$

$\therefore$ The given vectors are L.D.

for nontrivial L.C $\to$ L.Dep. vect

trivial L.C $\to$ L.I. vectors

Defn. 1: Given a vector $v \neq 0$, the set of all scalar multiples of v is called the line through v. (Geometrically v_1, v_2, v_3 is nothing but st. line thro' origin & v.)

Defn. 2 Two vectors v_1 and v_2 are collinear if one of them lies in the line through the other.

Clearly '0' is collinear with any non zero vector v.

Defn. 3 Two vectors v_1 & v_2 which are not collinear, their span namely $[v_1, v_2]$ is called the plane through v_1 & v_2.

Defn. 4 Three vectors v_1, v_2 & v_3 are coplanar, if one of them lies in the 'plane' through the other two.

Linear Span, denoted by $L(S)$ is the set of all linear combinations of finite number of elements of S.

Dimension and Basis.

Basis: A subset B of a vector space V is said to be a basis for V if (i) B is linearly independent.

(ii) $[B] = V$, i.e. B generates V, i.e. V is the set of all linear combinations of finite number of elements of B.

Defn: If a vector space V has a basis consisting of a finite no. of elements, the space is said to be finite dimensional, the no. of elements in a basis is called the dimension of the space and is written as $\dim V$. If $\dim V = n$, then V is said to be n-dimensional. If V is not finite dimensional, it is called infinite-dimensional.

Result: In an n-dimensional vector space V, any set of 'n' linearly independent vectors is a basis.

'n' linearly independent vectors is a basis.

Ex. In $V_3 \to 3$ L.I vectors is a basis.

$\{(1,1,1),(1,-1,1),(0,1,1)\}$ is a basis of V_3.

Pbm 1) P.T the set $\{(1,1,1),(1,-1,1),(0,1,1)\}$ is a basis of V_3.

To prove this, we have to prove that, the given set is L.I.

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1(-2) - 1(1) + 1(1)$$
$$= -2 \neq 0$$

$\therefore$ The given set is L.I & is therefore a basis for V_3.

2) Check whether $\{(1,2,5),(1,3,7),(1,-1,-1)\}$ is a basis of V_3.

$$\begin{vmatrix} 1 & 2 & 5 \\ 1 & 3 & 7 \\ 1 & -1 & -1 \end{vmatrix} = 1(-3+7) - 2(-1-7) + 5(-1-3)$$
$$4 + 16 - 20$$
$$= 0$$

$\therefore$ vectors are L.D

$\therefore$ The given set is not a basis for V_3