Integral Calculus.

Formulas.

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

2. $\int \frac{dx}{x} = \log x + c$

3. $\int e^x dx = e^x + c$

4. $\int \sin x \, dx = -\cos x + c$

5. $\int \cos x \, dx = \sin x + c$

6) $\int \tan x \, dx = \log \sec x + c$

7. $\int \cot x \, dx = \log \sin x + c$

$\int \sec^2 x \, dx = \tan x + c$

$\int \operatorname{cosec}^2 x \, dx = -\cot x + c$

$\int \sec x \tan x \, dx = \sec x + c$

$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$

$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \frac{x-a}{x+a} + c$

$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1} \frac{x}{a}$ (or) $\log\left(x + \sqrt{x^2+a^2}\right) + c$

$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left(x + \sqrt{x^2-a^2}\right) + c$

$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$

$\int 1 \cdot dx = x + c$

$\int k \, dx$, k is a constant $= kx + c$

$\int e^{ax} dx = \frac{e^{ax}}{a} + c$

$\int \sin ax \, dx = -\frac{\cos ax}{a} + c$

$\int \cos ax \, dx = \frac{\sin ax}{a} + c$

Simple problems

Integrate the following fns. (with respect to x)

1. x^{-4}

$$\int x^{-4}\,dx = \frac{x^{-4+1}}{-4+1} = \frac{x^{-3}}{-3} = -\frac{1}{3x^3} + c$$

2) $x^{3/2}$

$$\int x^{3/2}\,dx = \frac{x^{3/2+1}}{3/2+1} = \frac{x^{5/2}}{5/2} = \frac{2}{5}x^{5/2} + c$$

3) $ax + b/x^2$

$$\int (ax + b/x^2)\,dx = \int (ax + bx^{-2})\,dx$$

$$= \frac{ax^2}{2} + \frac{bx^{-2+1}}{-2+1} = \frac{ax^2}{2} + \frac{bx^{-1}}{-1} = \frac{ax^2}{2} - \frac{b}{x} + c$$

Rem

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

$$\frac{3}{2} + \frac{1}{1} = \frac{3+2}{2} = \frac{5}{2}$$

Note

$$\sqrt{x} = x^{1/2}$$

$$\sqrt[3]{x} = x^{1/3}$$

$$\frac{1}{x} = x^{-1}$$

$$\frac{1}{x^2} = x^{-2}$$

$$\frac{1}{x^n} = x^{-n}$$

$$x = x^1$$

4) ax^2+bx+c/x^3

$$\int\left(ax^2+bx+\frac{c}{x^3}\right)dx = \int(ax^2+bx+cx^{-3})dx$$

$$= \frac{ax^3}{3}+\frac{bx^2}{2}+\frac{cx^{-3+1}}{-3+1}+d$$

$$= \frac{ax^3}{3}+\frac{bx^2}{2}-\frac{cx^{-2}}{2}+d \quad \text{or} \quad \frac{ax^3}{3}+\frac{bx^2}{2}-\frac{c}{2x^2}+d.$$

Note

$(a-b)^2 = a^2-2ab+b^2$

$(a+b)^2 = a^2+2ab+b^2$

5) $x^2(1-x)^2$

$$\int x^2(1-x)^2\,dx = \int x^2(1-2x+x^2)dx = \int(x^2-2x^3+x^4)dx$$

$$= \frac{x^3}{3}-\frac{2x^4}{4}+\frac{x^5}{5}+c$$

Evaluate

1) $\int(x^{-2}+x^{-3}+x)\,dx$

2) $\int\left(\frac{1}{2x^4}+x^5\right)dx$

3) $\int\left(\sqrt{x}+x^{3/2}-x\right)dx$

4) $\int x(1+x)^2\,dx$

Integrate the following functions

1) $\int \left(x^2 + e^{2x} + \sin x\right) dx$.

Soln $\int \left(x^2 + e^{2x} + \sin x\right) dx = \frac{x^3}{3} + \frac{e^{2x}}{2} - \cos x . + c$

2) $\int \left(x^{-3} + e^{-3x} + \cos 2x\right) dx$.

Soln $\int \left(x^{-3} + e^{-3x} + \cos 2x\right) dx = \frac{x^{-3+1}}{-3+1} + \frac{e^{-3x}}{-3} + \frac{\sin 2x}{2} + c$

$= \frac{x^{-2}}{-2} - \frac{e^{-3x}}{3} + \frac{\sin 2x}{2} + c$

$= -\frac{1}{2x^2} - \frac{e^{-3x}}{3} + \frac{\sin 2x}{2}$.

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$\int e^{ax} dx = \frac{e^{ax}}{a}$

$\int \sin x dx = -\cos x$

$\int \cos ax dx = \frac{\sin ax}{a}$

1) $\int \left(x^{-4} + e^{-4x} + e^{2x} + \sin 3x\right) dx$

Ans $\frac{x^{-3}}{-3} - \frac{e^{-4x}}{4} + \frac{e^{2x}}{2} - \frac{\cos 3x}{3} + c$

2) $\int \left(\sqrt{x} + \cos 3x + \frac{1}{x}\right) dx$

Ans $\frac{x^{3/2}}{3/2} + \frac{\sin 3x}{3} + \log x + c$

$= \frac{2}{3} x^{3/2} + \frac{\sin 3x}{3} + \log x + c$

Ans (1) $-\frac{1}{3x^3} - \frac{e^{-4x}}{4} + \frac{e^{2x}}{2} - \frac{\cos 3x}{3} + c$

3) $\int \left(\frac{1}{x} + \tan x - \cot x\right) dx$

Note $\log a + \log b = \log ab$

$\log a - \log b = \log \frac{a}{b}$

Ans $I = \log x + \log \sec x - \log \sin x + c$

$I = \log (x \sec x) - \log \sin x + c$

$= \log \frac{x \sec x}{\sin x} + c$

1) $\int (\sec^2 2x + \cot 3x - e^{-3x}) dx$

2) $\int (\csc^2 2x - \tan 3x + e^{2x}) dx$

Ans $I =$

1) $\frac{\tan 2x}{2} + \frac{\log \sin 3x}{3} - \frac{e^{-3x}}{-3} + c$

2) $-\frac{\cot 2x}{2} - \frac{\log \sec 3x}{3} + \frac{e^{2x}}{2} + c$

4) $\int (\tan 2x + \cos 2x - x^4) dx$

Ans $I = \frac{\log \sec 2x}{2} + \frac{\sin 2x}{2} - \frac{x^5}{5} + c$

5) $\int (\sqrt{x} + \sin 2x - e^{4x}) dx$

$\sqrt{x} = x^{1/2}$

$I = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{\cos 2x}{2} - \frac{e^{4x}}{4} + c = \frac{x^{3/2}}{3/2} - \frac{\cos 2x}{2} - \frac{e^{4x}}{4} + c$

$= \frac{2}{3} x^{3/2} - \frac{\cos 2x}{2} - \frac{e^{4x}}{4} + c$

Definite Integrals

1. Evaluate $\int_1^2 \left(x^2 - 3x^{1/2} + \frac{1}{x^2}\right) dx$.

Soln.

$$I = \int_1^2 \left(x^2 - 3x^{1/2} + x^{-2}\right) dx.$$

$$= \left(\frac{x^{2+1}}{2+1} - \frac{3x^{1/2+1}}{1/2+1} + \frac{x^{-2+1}}{-2+1}\right)_1^2$$

$$= \left(\frac{x^3}{3} - \frac{3\,x^{3/2}}{3/2} + \frac{x^{-1}}{-1}\right)_1^2$$

$$= \left(\frac{x^3}{3} - \cancel{3}\times\frac{2}{\cancel{3}}\,x^{3/2} - x^{-1}\right)_1^2$$

$$= \left(\frac{x^3}{3} - 2x^{3/2} - x^{-1}\right)_{x=1}^2$$

$$\frac{8}{3} - \frac{1}{3} + 3 \qquad \frac{8-1+9}{3} = \frac{16}{3}$$

$$1^{3/2} \quad 2^{3/2} = 2^{1+\frac{3}{2}}$$

$$= \left\{\left(\frac{2^3}{3} - 2'(2)^{3/2} - 2^{-1}\right) - \left(\frac{1^3}{3} - 2(1)^{3/2} - 1^{-1}\right)\right\}$$

$$= \frac{8}{3} - 2^{5/2} - 2^{-1} - \frac{1}{3} + 2 + 1$$

$$\frac{16}{3} - 2^{5/2} - 2^{-1}$$

$$= \cancel{\frac{7}{3} - 2^{5/2} - \frac{1}{2} + 2^{3/2} + 1}$$

$$= \left(\frac{7}{3} - \frac{1}{2} + 1\right) - 2^{5/2} + 2^{3/2}$$

$$= \left(\frac{14 - 3 + 6}{6}\right) - 2^{5/2} + 2^{3/2}$$

$$= \frac{5}{6} - 2^{5/2} + 2^{3/2}$$

$$\frac{16}{3} - 2^{5/2} - \frac{1}{2}$$

Ans $I = \frac{29}{6} - 2^{5/2}$

2) Solve $\int_0^{\pi/2} \cos^2 x \, dx$

Note: $\cos^2 x = \frac{1+\cos 2x}{2}$ & $\sin^2 x = \frac{1-\cos 2x}{2}$

$$I = \int_0^{\pi/2} \left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx$$

$$= \left(\frac{x}{2} + \frac{1}{2}\cdot\frac{\sin 2x}{2}\right)_0^{\pi/2}$$

$$= \left\{\left(\frac{\pi/2}{2} + \frac{1}{2}\,\frac{\sin 2\frac{\pi}{2}}{2}\right) - \left(\frac{0}{2} + \frac{1}{2}\,\frac{\sin 0}{0}\right)\right\}$$

$$= \frac{\pi}{4} + \frac{1}{4}\sin\pi - 0$$

$(\because \sin\pi = 0,$ $\sin 0 = 0$

$$\therefore I = \pi/4$$

Note

$\sin 0 = 0$

$\sin\frac{\pi}{2} = 1$

$\sin\pi = 0$

$\sin n\pi = 0 \; \forall n$

$\cos 0 = 1$

$\cos\frac{\pi}{2} = 0$

$\cos\pi = -1$

$\cos 2\pi = 1$

$\vdots$

$\cos n\pi = -1$, if n is odd

$= 1$ if n is even

3) Evaluate $\int_0^{\pi/2} \sin^2 x \, dx$.

$$I = \int_0^{\pi/2} \left(\frac{1 - \cos 2x}{2} \right) dx$$

$$= \int_0^{\pi/2} \left(\frac{1}{2} - \frac{\cos 2x}{2} \right) dx$$

$$= \left(\frac{x}{2} - \frac{1}{2} \cdot \frac{\sin 2x}{2} \right)_0^{\pi/2}$$

$$= \left\{ \left(\frac{\pi/2}{2} - \frac{1}{2} \frac{\sin 2 \frac{\pi}{2}}{2} \right) - \left(0 - \frac{1}{2} \cdot \frac{\sin 0}{2} \right) \right\}$$

$$= \underline{\underline{\pi/4}} \qquad (\because \sin \pi = 0, \ \sin 0 = 0)$$

H.W 1) $\int (x^{-2} + x^{-3} + x)\, dx$.

Soln $I = \frac{x^{-2+1}}{-2+1} + \frac{x^{-3+1}}{-3+1} + \frac{x^{1+1}}{1+1} + c$

$= \frac{x^{-1}}{-1} + \frac{x^{-2}}{-2} + \frac{x^{2}}{2} + c$

Ans $I = -\frac{1}{x} - \frac{1}{2x^2} + \frac{x^2}{2} + c$

2) $\int \left(\frac{1}{2x^4} + x^5\right) dx$

Soln $I = \int \left(\frac{1}{2} x^{-4} + x^5\right) dx$

$= \frac{1}{2} \cdot \frac{x^{-4+1}}{-4+1} + \frac{x^{5+1}}{5+1} + c$

$= \frac{1}{2} \cdot \frac{x^{-3}}{-3} + \frac{x^6}{6} + c$

$= -\frac{1}{6x^3} + \frac{x^6}{6} + c$

3) $\int \left(\sqrt{x} + x^{3/2} - x\right) dx$

$I = \int \left(x^{1/2} + x^{3/2} - x^1\right) dx$

$= \frac{x^{1/2+1}}{1/2+1} + \frac{x^{3/2+1}}{3/2+1} - \frac{x^{1+1}}{1+1} + c$

$= \frac{x^{3/2}}{3/2} + \frac{x^{5/2}}{5/2} - \frac{x^2}{2} + c$

$= \frac{2}{3} x^{3/2} + \frac{2}{5} x^{5/2} - \frac{x^2}{2} + c$

4) $\int x(1+x)^2\,dx$

$(a+b)^2 = a^2+b^2+2ab$

$a=1,\quad b=x$

$(1+x)^2 = 1^2 + x^2 + 2\times1\times x$

$I = \int x(1+x^2+2x)\,dx$

$= \int (x+x^3+2x^2)\,dx.$

$= \frac{x^{1+1}}{1+1} + \frac{x^{3+1}}{3+1} + \frac{2x^{2+1}}{2+1} + c$

$= \frac{x^2}{2} + \frac{x^4}{4} + 2\frac{x^3}{3} + c$

Evaluate

1) $\int_1^2 (x^2 + x^3 + 3x^4)\,dx$.

2) $\int_0^1 (e^{2x} - e^{-3x})\,dx$

3) $\int_0^\pi (\sin x + \cos x)\,dx$.

4) $\int_0^\pi (\sin 2x - \cos 2x)\,dx$.

Ans 4) $I = \left(-\frac{\cos 2x}{2} - \frac{\sin 2x}{2}\right)_0^\pi$

$= \frac{1}{2}\left\{\left(-\cos 2\pi - \sin 2\pi\right) - \left(-\cos 0 - \sin 0\right)\right\}$

$= \frac{1}{2}\left\{-1+1\right\} = 0$

1) $I = \left(\frac{x^3}{3} + \frac{x^4}{4} + \frac{3x^5}{5}\right)_1^2 = \left\{\left(\frac{2^3}{3} + \frac{2^4}{4} + \frac{3\times 2^5}{5}\right) - \left(\frac{1^3}{3} + \frac{1^4}{4} + 3\frac{1^5}{5}\right)\right\}$

$= 24.6$

2) $I = \frac{e^{2x}}{2} - \frac{e^{-3x}}{-3} = \left(\frac{e^{2x}}{2} + \frac{e^{-3x}}{3}\right)_0^1$

$= \left(\frac{e^{2}}{2} + \frac{e^{-3}}{3}\right) - \left(\frac{1}{2} + \frac{1}{3}\right) \quad (\because e^0 = 1)$

$= \frac{e^2}{2} + \frac{e^{-3}}{3} - \frac{5}{6}$

3) $I = \left(-\cos x + \sin x\right)_0^\pi$

$= \left\{\left(-\cos\pi + \sin\pi\right) - \left(-\cos 0 + \sin 0\right)\right\}$

$= -(-1) + 1 = 2$