Module - 6

Graph Theory- Graphs and their properties, Degree, Sub Graph

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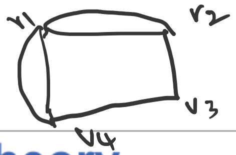
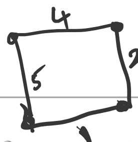
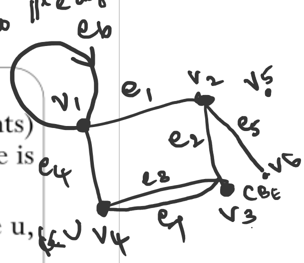




Simple
no loops, no || edges



Graph Theory

- A *graph* G=(V, E) consists of a non-empty set V called the set of vertices (nodes, points) and a set E of ordered or unordered pairs of elements of V called the edges such that there is a mapping from the set E to the set of ordered or unordered of elements of V.
- If an edge e in E is associated with an ordered pair (u,v) or an unordered pair (u,v) where u, v in V then e is said to connect or join the nodes u and v.
- The edge e that connects the nodes u and v is said to be incident on each of the nodes. The pair of nodes that are connected by an edge are called ***adjacent nodes.***
- A node of a graph which is not adjacent to any other node is called an ***Isolated node.***
- A graph containing only isolated nodes is called a ***null graph.*** Discrete graph
- An edge of a graph that joins a vertex to itself is called a ***loop.***
- If in a directed or undirected graph, certain pairs of vertices are joined by more that one edge such edges are called ***parallel edges***.
- A graph in which there is only one edge between a pair of vertices is called a ***simple graph***.
- A graph which contains some parallel edges is called a ***multigraph***.
- A graph in which loops and parallel edges are allowed is called a ***pseudograph.***
- Graph in which a number (weight) is assigned to each edge are called ***weighted graphs.***

$\nu : E \rightarrow V \times V$

$\nu(e_1) = (v_1, v_2)$

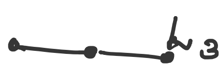
$\nu(e_2) = (v_3, v_2)$

ordered

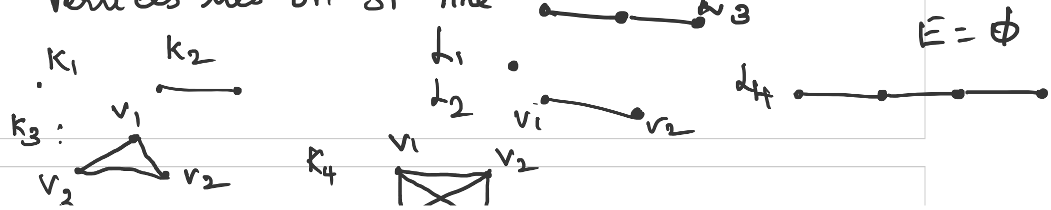
null graph

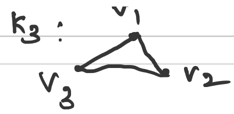
$E = \Phi$

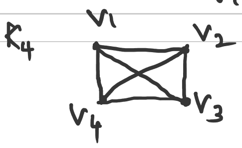
Linear graph : vertices lies on st. line



Complete graph :

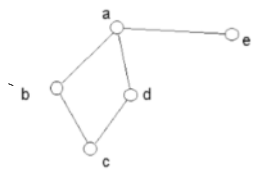






Degree of a Vertes

- The ***degree of a vertex*** in an undirected graph is the number of edges incident with it, with the exception that a loop at a vertex contributes twice to be degree of that vertex.
- The degree of a vertex v is denoted by deg(v).
- The degree of an isolated vertex is zero.
- If the degree of a vertex is one, it is called a ***pendant vertex***.



deg(a) = 3
deg(b) = 2
deg(c) = 2
deg(d) = 2
deg(e) = 1

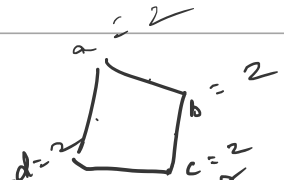
10

deg(a) = 4
deg(b) = 4
deg(c) = 2
deg(d) = 2
deg(e) = 2

out degree | Indegree
$deg^+(a) = 2$ | $deg^-(a) = 2$
$deg^+(b) = 2$ | $deg^-(b) = 2$
$deg^+(c) = 1$ | $deg^-(c) = 1$
$deg^+(d) = 1$ | $deg^-(d) = 1$
$deg^+(e) = 1$ | $deg^-(e) = 1$

$\Sigma deg^+(v) = \Sigma deg^-(v)$

$deg(a) = deg^+(a) + deg^-(a)$



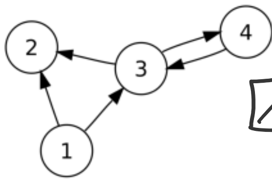
$2 \times 4 = 8$

$$\boxed{\Sigma deg(v) = 2e}$$

2×5

Graph Theory

- If G=(V, E) is an undirected graph with e edges, then Σ deg(v_i)= 2e.
- The number of vertices of odd degree in an undirected graph is even
- In a directed graph, the number of edges with v as their terminal vertex the number of edges that converge at v is called the ***indegree of v*** and is denoted as deg-(v).
- The number of edges with v as their initial vertex (viz the number of edges that emanate from v) is called the ***out degree of v*** and is denoted as deg+(v)
- A vertex with zero indegree is called a ***source*** and a vertex with zero out degree is called a ***sink.***

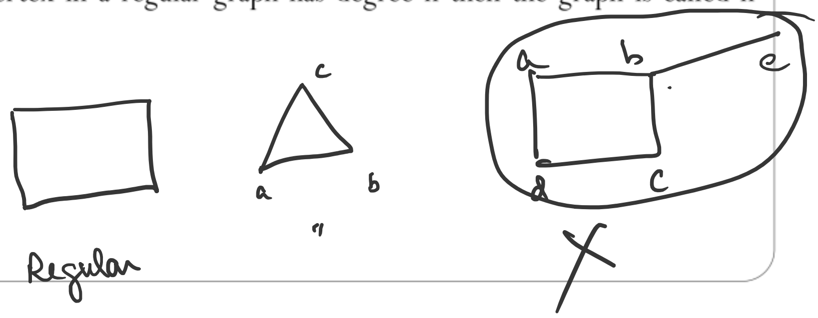


$deg^+(1) = 2$ | $deg^-(1) = 0$ (Source)
$\boxed{\text{Sink}}$ $deg^+(2) = 0$ | $deg^-(2) = 2$
$deg^+(3) = 2$ | $deg^-(3) = 2$
$deg^+(4) = 1$ | $deg^-(4) = 1$

Some Special simple Graphs

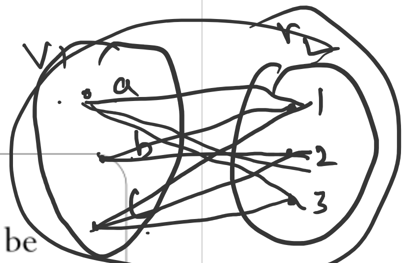
- ***Complete Graph:*** A simple graph in which there is exactly one edge between each pair of distinct vertices is called a complete graph.
- ***Note:*** The number of edges in K_n is nC2 or n(n-1)/2. Hence the maximum number of edges in a simple graph with n vertices is n)n-1)/2.
- ***Regular graph:*** If every vertex of a simple graph has the same degree then the graph is called a regular graph. If every vertex in a regular graph has degree n then the graph is called n-regular

$K_n = {}^{n}C_2 = \frac{n(n-1)}{2}$



$G = (V, E)$ $\quad V = V_1 \cup V_2$

$V_1 \cap V_2 = \phi$

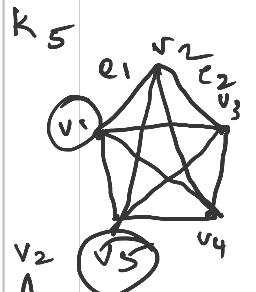


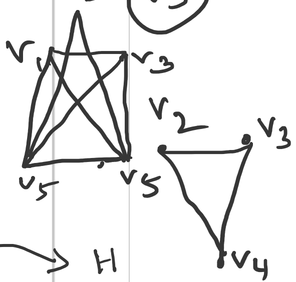
Some Special Simple Graphs

- ***Bipartite Graph:*** If the vertex set V of a simple graph G=(V,E) can be partitioned into two subsets V_1 and V_2 such that every edge of G connectsa vertex in V_1 and a vertex in V_2(so that no edge in G connects either two vertices in V_1 or two vertices in V_2) then G is called a bipartite graph.
- If each vertex of V_1 is connected with every vertex of V_2 by an edge then G is called ***Completely bipartite graph.*** If V_1 contains m vertices and V_2 contains n vertices the completely bipartite graph is denoted by K_(m,n)

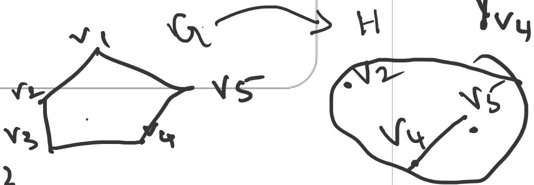
Some Special Simple Graphs

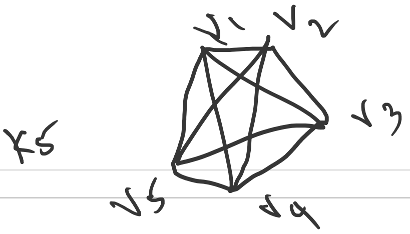
- ❖ ***Subgraphs:*** A graph H= (V', E') is called a subgraph of G=(V,E) if V'⊆V and E'⊆ E Then H is called a ***proper subgraph*** of G.
- ❖ If V' = V the H is called a ***spanning subgraph*** of G.
- ❖ If we delete a subset U of V and all the edges incident on the elements of U from a graph G=(V,E), then the subgraph (G-U) is called a ***vertex deleted subgraph*** of G.
- ❖ If we delete a subset F of E from a graph G, then the subgraph (G-F) is called a ***edge deleted subgraph*** of G.
- ❖ A subgraph H =(V', E') of G=(V,E) where V' ⊆V and E' consists of only those edges that are incident on the elements of V' is called an ***induced subgraph*** of G.





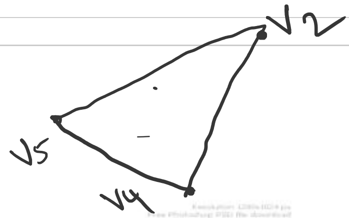
K5

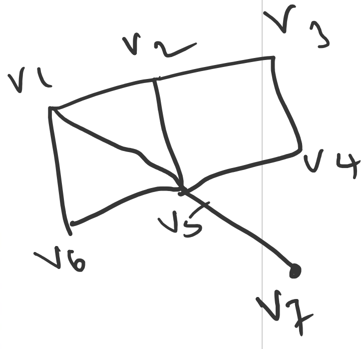




$V = \{v_1, v_2, v_3, v_4, v_5\}$

$V' = \{v_2, v_4, v_5\}$





$V = \{v_1, \ldots v_7\}$

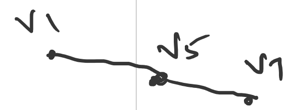
induced

$V' = \{v_1, v_7, v_3\}$

• v_3 • v_1 • v_7

9

$\{v_1, v_5, v_7\}$



Module - 6

Graphs and Trees– Isomorphism, Matrix representation of a graph

$v_1 \neq v_2 \Rightarrow f(v_1) \neq f(v_2)$ $a \in G'$ $f^{-1}(a) \in G$

morphism : mapping

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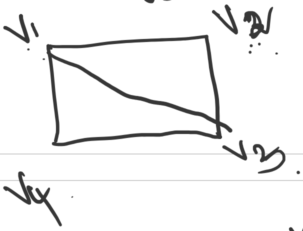
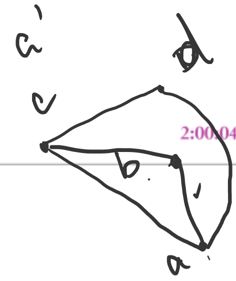
G



G'



$f : G \to G'$

$f(v_1) = a$

$f(v_2) = b$

$f(v_3) = c$

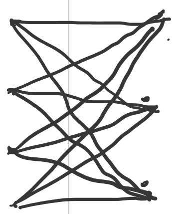
$f(v_4) = d$

iso

$$\begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 1 & 1 \\ v_2 & 1 & 0 & 1 & 0 \\ v_3 & 1 & 1 & 0 & 1 \\ v_4 & 1 & 0 & 1 & 0 \end{array}$$

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 1 & 1 \\ b & 1 & 0 & 1 & 0 \\ c & 1 & 1 & 0 & 1 \\ d & 1 & 0 & 1 & 0 \end{array}$$

4



Isometric Graphs

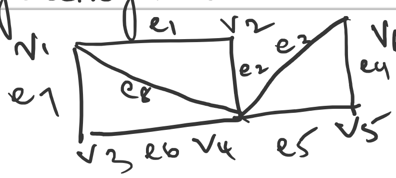
- Two graphs G1 and G2 ae said to be isomorphic to each other, if there exists a one-to-one correspondence between the vertex sets which preserves adjacency of the vertices.
- A graph G1=(V1, E1) is isomorphic to the graph G2=(V2,E2) if there is a one to one correspondence between the vertex sets V1 and V2 and between the edge sets E1 and E2 in such a way that if e1 is incident on u1 and v1 in G1 then the corresponding edge e2 in G2 is incident on u2 and v2 which corresponding to u1 and v1 respectively. Such a correspondence is called ***graph isomorphism.***
- The number of edges in a bipartite graph with n vertices is atmost $\boxed{n^2/2}$.

$\frac{7^2}{2} = \frac{49}{2}$

$12 < 24$

$v_1 v_3 = v_3 v_1$

Adjacency matrix



$$\begin{array}{c|cccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \hline v_1 & 0 & 1 & 1 & 1 & 0 & 0 \\ v_2 & 1 & 0 & 0 & 1 & 0 & 0 \\ v_3 & 1 & 0 & 0 & 1 & 0 & 0 \\ v_4 & 1 & 1 & 1 & 0 & 1 & 1 \\ v_5 & 0 & 0 & 0 & 1 & 0 & 1 \\ v_6 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

$$\begin{array}{c|cccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \hline e_1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}$$

Matrix Representation of Graphs

- When G is a simple graph with n vertices v1,v2,…,vn the matrix A=[aij] where

    aij = 1 if vivj is an edge in G

    0 otherwise is called the adjacency matrix of G.
- Note: Simce a simple graph has no loops, each diagonal entry of A is zero.
- The adjacency matrix of simple graph is symmetric ie aij = aji
- Degree of vi is equal to the total number of 1's in ith row or ith column
- ***Pseudo graph*** – an undirected graph with loops and parallel edges

	v_1	v_2	v_3	v_4	v_5	v_6
e_1	1	1	0	0	0	0
e_2	0	1	0	1	0	0
e_3	0	0	0	1	0	1
e_4	0	0	0	0	1	1
e_5	0	0	0	1	1	0
e_6	0	0	1	1	0	0
e_7	1	0	1	0	0	0
e_8	1	0	0	1	0	0
	3					

Incident Matrix

- If G=(V, E) is an undirected graph with n vertices v1,v2,…,vn and m edges e1,e2,…,em then the nXm matrix B=[bij] is defined as

    Where bij = 1 when edge ej is incident on vi

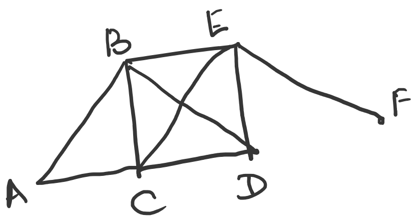
    0 otherwise is called the incidence matrix of G
- ***Note:*** Each column of B contains exactly two unit entries.
- A row with all 0 entries corresponds to an isolated vertex
- A row with a single unit entry corresponds to a pendant vertex
- Deg(Vi) is equal to number of 1's in the ith row.

Isomorphism and Adjacency Matrices

- Two graphs are isometric iff their vertices can be labelled in such a way that the corresponding adjacency matrices are equal.
- Two labelled graphs G1 and G2 with adjacency matrices A1 and A2 respectively are isomorphic iff there exists a permutation matrix P such that PA1P^T =A2

Example: Find the number of vertices, the number of edges and the degree if each vertex in the following undirected graphs. Verify also the hand shaking theorem in each case

$\sum \deg v_i = 2e$



No. of vertices - 6

No. of edges - 9

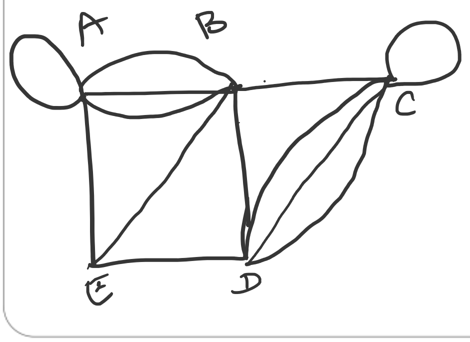
deg(A) = 2 deg(B) = 4 deg(c) = 4 deg(D) = 3

deg(e) = 4 deg(F) = 1

$\sum \deg(v_i) = 2 \cdot e$

$18 = 2 \times 9 = 18$ ✓

Example: Find the number of vertices, the number of edges and the degree if each vertex in the following undirected graphs. Verify also the hand shaking theorem in each case

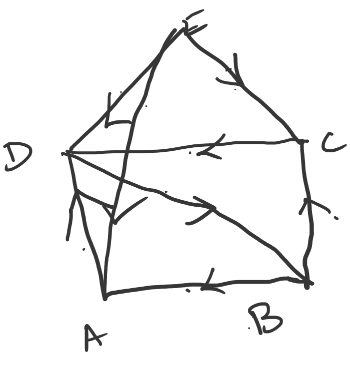


No. of vertices = 5
No. of edges = 13

deg(A) = 6 deg(B) = 6 deg(C) = 6
deg(E) = 3 deg(D) = 5

$\sum deg(v_i) = 26$ $\sum deg(v_i) = 2e$
No. of edges = 13
2x " = 2x13 = 26

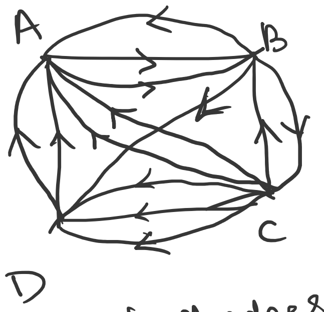
Find the indegree and outdegree of each vertex of each of the following ~~un~~directed graphs. Also verify that the sum of the indegrees (or the outdegrees) equals the number of edges.



$deg^-(A) = 2$ $deg^+(A) = 1$
$deg^-(B) = 1$ $deg^+(B) = 2$
$deg^-(C) = 2$ $deg^+(C) = 1$
$deg^-(D) = 3$ $deg^+(D) = 1$
$deg^-(E) = 0$ $deg^+(E) = 3$

No. of edges = 8 $\sum deg^-(E) = 8$

Find the indegree and outdegree of each vertex of each of the following undirected graphs. Also verify that the sum of the indegrees (or the outdegrees) equals the number of edges.



No. of edges = 13

Indegree

$deg^-(A) = 5$
$deg^-(B) = 3$
$deg^-(C) = 1$
$deg^-(D) = 4$

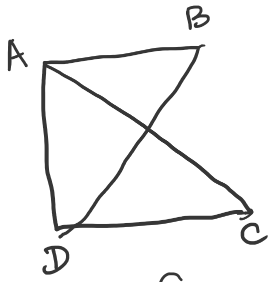
$\underline{13}$

outdegree

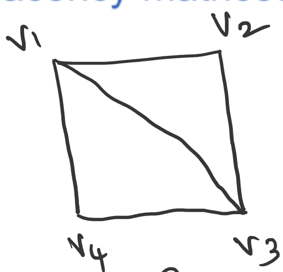
$deg^+(A) = 2$
$deg^+(B) = 3$
$deg^+(C) = 6$
$deg^+(D) = 2$

13

Establish the isomorphism of the two graphs given below by considering their adjacency matrices



G_1



G_2

$(2,3) = ?$

$$A_1 = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

$$A_2 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$$I_A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$4! = 24$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\boxed{PA_1P^T = A_2}$

$$P^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

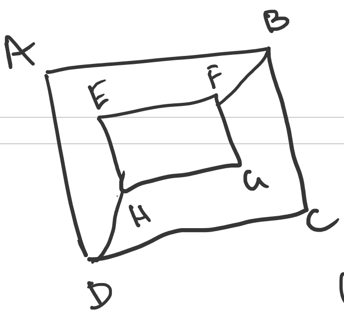
$$\begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$$

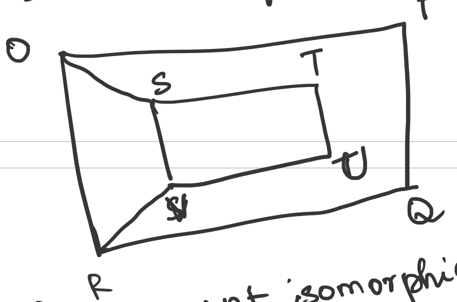
Establish the isomorphism of the two graphs given below by considering their adjacency matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

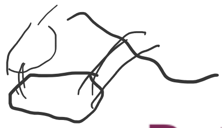
G_1 & G_2 are isomorphic

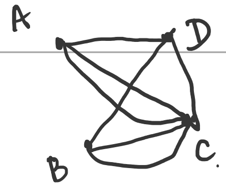


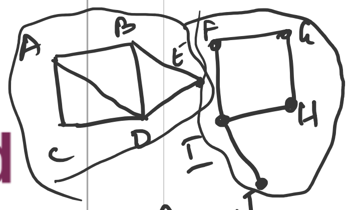


$f(A) = P$ ✓ X
$f(C) = Q$
$f(B) = O$ ✓
$f(D) = R$ X
$f(F) = S$
$f(H) = V$
$f(E) = T$
$f(G) = U$

G_1 & G_2 are not isomorphic







G2

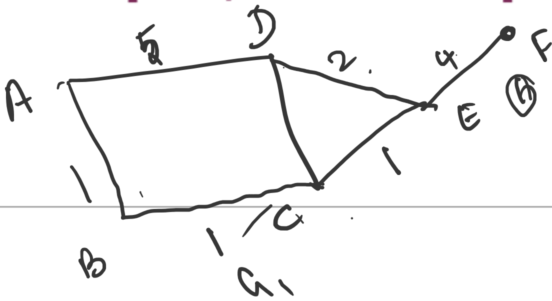
Disconnected

Module - 6

Graphs and Trees – Paths, Cycles and Connectivity, Eulerian and Hamiltonian Graphs, Shortest path problem

Dijkshtra's.

Warshall's



ADE – seq of edges

AD.DE

9:25:02 AM

2

ABCDE

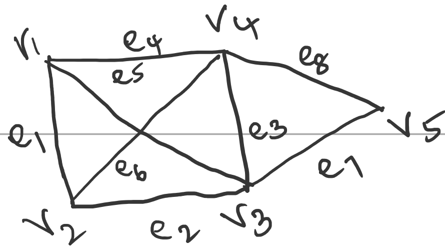
ABCDE CDA – circuit/cycle

CDEC

ABCDA

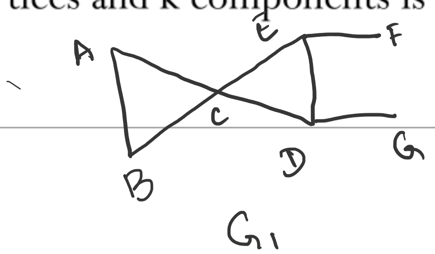
Paths, Cycles and Connectivity

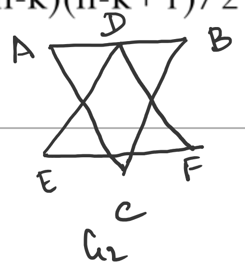
- A *path* is a finite alternating sequence of vertices and edges, beginning and ending with vertices such that each edge is incident on the vertices preceding and following it.
- If the edges in a path are distinct, it is called a ***simple path.***
- The number of edges in a path is called the ***length of the path***.
- A path which begins and ends at the same vertex is called ***circuit***.

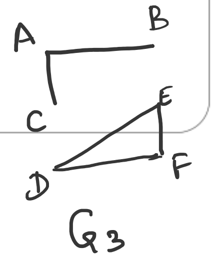


Connectedness in Undirected Graphs

- An undirected graph is said to be ***connected*** if a path between every pair of distinct vertices of the graph.
- A graph that is not connected is called ***disconnected.***
- A disconnected graph is the union of two or more connected subgraphs each pair of which has no vertex in common. These disjoint connected subgraphs are called ***connected components*** of the graph.
- The maximum number of edges in a simple disconnected graph G with n vertices and k components is (n-k)(n-k+1)/2







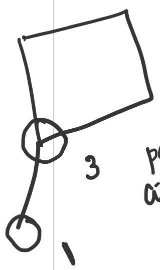
$(10-2)$

$(10-2+1)/2$

$\frac{\cancel{8}^{4} \times 9}{\cancel{2}} = 36$

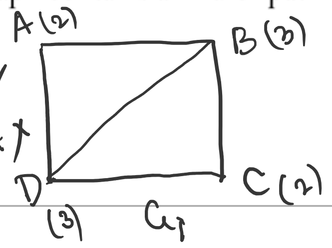
Eulerian and Hamiltonian Graphs

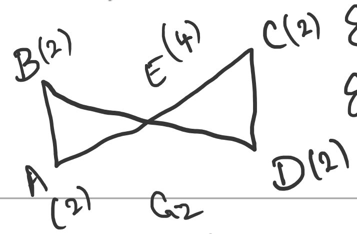
- A path of graph G is called an Eulerian path, if it includes each edge of G exactly once.
- A circuit of a graph G is called an Eulerian circuit if it includes each edge of G exactly once.
- A graph containing Eulerian circuit is called an Eulerian graph.
- The necessary and sufficient condition for existence of Euler circuit is as follows:
  - A connected graph contains an Euler circuit iff each of its vertices is of even degree.
  - A connected graph contains an Euler path iff it has exactly two vertices of odd degree



path ✓
circuit ✗

Euler path ✓
Euler circuit ✗





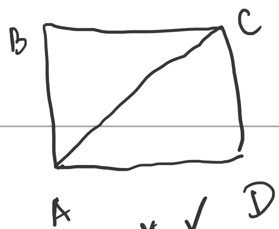
Euler path ✓
Euler circuit ✓

edge

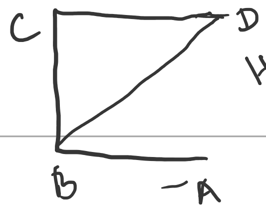
Euler - edge

Eulerian and Hamiltonian Graphs

- A path of a graph G is called a Hamiltonian path if it includes each vertex of G exactly once.
- A circuit of a graph G is called Hamiltonian circuit, if it includes each vertex of G exactly once except the starting and end vertices which appear twice.
- A graph containing Hamiltonian circuit is called Hamiltonian graph.
- A Hamiltonian circuit contains a Hamiltonian path but a graph containing a Hamiltonian path need not have a Hamiltonian circuit.
- A complete graph always have a Hamiltonian circuit.
- A given graph may contain more than one Hamiltonian circuit

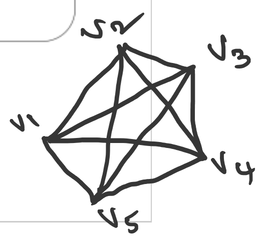


H . path ✓
H . circuit ✓



H . path ✓
H. circuit ✗

Complete
K_5



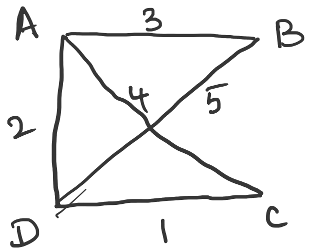
Shortest path Algorithm

- A graph in which each edge is assigned with a non negative real number w(e) is called a weighted graph. w(e) is called the weight of the edge e.
- A shortest path between two vertices in a weighted graph is a path of least weight. In an unweighted graph, a shortest path means one with the least number of edges

Warshall's Algorithm

- Warshall's Algorithm determines the shortest distances between all pair of vertices in a graph.
- The weight of matrix W=[wij] is defined as
  - wij =w(i,j) if there is an edge from vi to vj
  - = 0 if there is no edge
- The initial matrix L0 is same as the weight matrix w except that each non-diagonal 0 in W is replaced by ∝
- lr(i,j) the i,j th entry of the Lr is computed using the rule
  - lr(i,j) = min [lr-1(i,j), lr-1(i,k)+lr-1(k,j)]
- The final matrix Ln is the required shortest distance matrix L the ijth entrty of which gives the length of the shortest path between the certices vi and vj.

Example – Warshall's Algorithm



$$W = \begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 3 & 4 & 2 \\ B & 3 & 0 & 0 & 5 \\ C & 4 & 0 & 0 & 1 \\ D & 2 & 5 & 1 & 0 \end{array}$$

$$L_0 = \begin{pmatrix} 0 & 3 & 4 & 2 \\ 3 & 0 & \infty & 5 \\ 4 & \infty & 0 & 1 \\ 2 & 5 & 1 & 0 \end{pmatrix} \qquad L_1 = \begin{pmatrix} 0 & 3 & 4 & 2 \\ 3 & 0 & 7 & 5 \\ 4 & 7 & 0 & 1 \\ 2 & 5 & 1 & 0 \end{pmatrix}$$

$L_1(1,2) = \min(L_0(1,2),\ L_0(1,1) + L_0(1,2))$

$= \min(3,\ 0+3) = 3$

$L_1(3,2) = \min(L_0(3,2),$
$L_0(3,1) + L_0(1,2))$

$(\infty, 7) : 7$

$L_1(1,3) = \min(L_0(1,3),\ L_0(1,1) + L_0(1,3))$

$= \min(4,\ 0+4) = 4$

$L_1(2,1) = \min(L_0(2,1),\ L_0(2,1) + L_0(1,1))$

$= \min(3,\ 3)$

$(4,\ 2+3)$ $\quad 1,\ 4+2$

$(1,\ 2+4)$ $\quad 2,\ 2$

$L_1(2,3) = \min(L_0(2,3),\ L_0(2,1) + L_0(1,3))$

$= \min(\infty,\ 3+4) = 7$

$$\begin{pmatrix} 0 & 3 & 4 & 2 \\ 3 & 0 & 7 & 4 \\ 4 & 7 & 0 & 1 \\ 2 & 4 & 1 & 0 \end{pmatrix}$$

Example – Warshall's Algorithm

$$L_2 = \begin{pmatrix} 0 & 3 & 4 & 2 \\ 3 & 0 & 7 & 5 \\ 4 & 7 & 0 & 1 \\ 2 & 5 & 1 & 0 \end{pmatrix}$$

$1,\ 2 + 2,\ 4$

$L_2(1,2) = \min(L_1(1,2),\ L_1(1,2) + L_1(2,2))$

$4,2+2,2$

$2,\ L_1(4,2) + 2,1)$ $\quad 4,\ (4$

$4+3$ $\quad 4,3,\ 3,3$

$2;$

$2 \quad 5 \quad 1,\ (1$

$$L_3 = \begin{pmatrix} 0 & 3 & 4 & 2 \\ 3 & 0 & 7 & 5 \\ 4 & 7 & 0 & 1 \\ 2 & 5 & 1 & 0 \end{pmatrix}$$

$L_3(3,1) = \min(L_2(4,2),\ L_2(4,3) + L_2(3,2))$

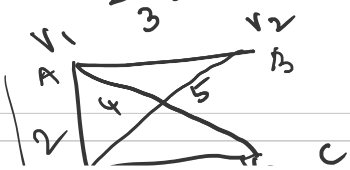
$= 5,\ 1+7$

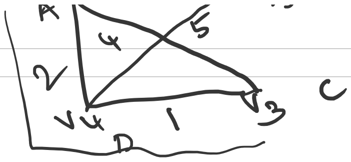
$L_4(1,2) = \min(L_3(3,4),\ L_3(3,4) + L_3(4,4))$

$= 5,\ 5+0$

$$L_4 = \begin{pmatrix} 0 & 3 & 3 & 2 \\ 3 & 0 & 6 & 5 \\ 3 & 6 & 0 & 1 \\ 2 & 5 & 1 & 0 \end{pmatrix}$$

$$\begin{array}{c|cccc} & A & B & C & D \\ \hline A & - & AB & ADC & AD \\ B & BA & - & BADC & BD \\ C & CDA & CDAB & - & CD \\ D & DA & DB & DC & - \end{array}$$





$$\begin{array}{c} C \\ D \end{array}\begin{pmatrix} CDA & CDAB & - & CD \\ DA & DB & DC & - \end{pmatrix}$$

Example – Warshall's Algorithm


Thank you!
12