Jacobians.

If u & v are functions of two independent variables x & y, then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ or $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$ is called the Jacobian of u, v w.r.to x & y and is written as $\frac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\frac{u,v}{x,y}\right)$

Similarly the Jacobian of u, v, w with respect to x, y, z is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \text{ or } \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}.$$

Properties of Jacobeans :

1. If $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$, Then $JJ' = 1$.

2) If u, v are functions of r, s & r, s are fns of x, y then

$$\boxed{\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}.}$$

Problem In polar co-ordinates (circular) $x = r\cos\theta$, $y = r\sin\theta$, S.T $\frac{\partial(x,y)}{\partial(r,\theta)} = 1$.

(or)

find the Jacobean of $x = r\cos\theta$, $y = r\sin\theta$.

Soln Given. $x = r\cos\theta$; $y = r\sin\theta$.

$\therefore \frac{\partial x}{\partial r} = \cos\theta$; $\frac{\partial x}{\partial \theta} = -r\sin\theta$. $\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$.

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}.$$

$J\left(\frac{x,y}{r,\theta}\right)$

$$= r\cos^2\theta - (-r\sin^2\theta) = r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta) \qquad (\because \cos^2\theta + \sin^2\theta = 1)$$

$$= \underline{\underline{r}}.$$

2) In cylinderical co-ordinates, $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$, S.T $\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \rho$.

<u>Soln</u>

$x = \rho\cos\phi$

$\frac{\partial x}{\partial \rho} = \cos\phi$.

$\frac{\partial x}{\partial \phi} = -\rho\sin\phi$.

$\frac{\partial x}{\partial z} = 0$

$y = \rho\sin\phi$

$\frac{\partial y}{\partial \rho} = \sin\phi$.

$\frac{\partial y}{\partial \phi} = \rho\cos\phi$

$\frac{\partial y}{\partial z} = 0$

$z = z$.

$\frac{\partial z}{\partial \rho} = 0$

$\frac{\partial z}{\partial \phi} = 0$.

$\frac{\partial z}{\partial z} = 1$.

$$\therefore \frac{\partial(x,y,z)}{\partial(\rho,\varphi,z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

$$J\left(\frac{x,y,z}{\rho,\varphi,z}\right) =$$

$$= \cos\varphi(\rho\cos\varphi) + \rho\sin\varphi(\sin\varphi)$$

$$= \rho\cos^2\varphi + \rho\sin^2\varphi.$$

$$= \rho(\cos^2\varphi + \sin^2\varphi)$$

$$= \underline{\underline{\rho}}.$$

3) In spherical co-ordinates, $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$,

S.T $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = r^2\sin\theta$.

Soln

$$\frac{\partial x}{\partial r} = \sin\theta\cos\varphi \;;\quad \frac{\partial x}{\partial \theta} = r\cos\theta\cos\varphi \;;\quad \frac{\partial x}{\partial \varphi} = -r\sin\theta\sin\varphi .$$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\varphi \;;\quad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\varphi \;;\quad \frac{\partial y}{\partial \varphi} = r\sin\theta\cos\varphi .$$

$$\frac{\partial z}{\partial r} = \cos\theta \;;\quad \frac{\partial z}{\partial \theta} = -r\sin\theta \;;\quad \frac{\partial z}{\partial \varphi} = 0 .$$

$$\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$x = r\sin\theta\cos\varphi$
$y = r\sin\theta\sin\varphi$
$z = r\cos\theta$

$$= \sin\theta\cos\varphi\left(0 + r^2\sin^2\theta\cos\varphi\right) - r\cos\theta\cos\varphi\left(0 - r\sin\theta\cos\theta\cos\varphi\right) + (-r\sin\theta\sin\varphi)\left(-r\sin^2\theta\sin\varphi - r\cos^2\theta\sin\varphi\right)$$

$$= r^2\sin^3\theta\cos^2\varphi + r^2\sin\theta\cos^2\theta\cos^2\varphi + r^2\sin^3\theta\sin^2\varphi + r^2\sin\theta\cos^2\theta\sin^2\varphi$$

$$= r^2\sin^3\theta\underbrace{(\cos^2\varphi + \sin^2\varphi)}_{1} + r^2\sin\theta\cos^2\theta\underbrace{(\cos^2\varphi + \sin^2\varphi)}_{1}$$

$$= r^2\sin^3\theta + r^2\sin\theta\cos^2\theta$$

$$= r^2\sin\theta\,(\underbrace{\sin^2\theta+\cos^2\theta}_{1})$$

$$= \underline{\underline{r^2\sin\theta}}.$$

4) If $u = x+3y^2-z^3$, $v = 4x^2yz$, $w = 2z^2-xy$, evaluate $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ at $(1,-1,0)$.

Soln

$u = x+3y^2-z^3$

$\dfrac{\partial u}{\partial x} = 1$, $\dfrac{\partial u}{\partial y} = 6y$, $\dfrac{\partial u}{\partial z} = -3z^2$

$v = 4x^2yz$

$\dfrac{\partial v}{\partial x} = 8xyz$, $\dfrac{\partial v}{\partial y} = 4x^2z$, $\dfrac{\partial v}{\partial z} = 4x^2y$.

$w = 2z^2-xy$

$\dfrac{\partial w}{\partial x} = -y$, $\dfrac{\partial w}{\partial y} = -x$, $\dfrac{\partial w}{\partial z} = 4z$.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 6y & -3\cancel{z}^2 \\ 8xy\cancel{z} & 4x^2\cancel{z} & 4x^2y \\ -y & -x & 4\cancel{z} \end{vmatrix}.$$

at $(1,-1,0)$ $= \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix}.$

$$= 1(0-4) + 6(0+4)$$

$$= -4+24$$

$$= \underline{\underline{20}}$$

5) If $u = x^2 - y^2$, $v = 2xy$ & $x = r\cos\theta$, $y = r\sin\theta$

$(u,v) \to (x,y) \to (r,\theta)$

find $\dfrac{\partial(u,v)}{\partial(r,\theta)}$.

Soln

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4x^2 - (-4y^2) = 4x^2 + 4y^2$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta - (-r\sin^2\theta)$$

$$= r(\underbrace{\cos^2\theta + \sin^2\theta}_{1}) = r$$

$$\therefore \frac{\partial(u,v)}{\partial(r,\theta)} = 4(x^2+y^2).r = 4(r^2\cos^2\theta + r^2\sin^2\theta).r = 4r^2(\underbrace{\cos^2\theta + \sin^2\theta}_{1}).r = \underline{\underline{4r^3}}$$

6) If $x = r\cos\theta,\ y = r\sin\theta$, P.T $\dfrac{\partial(r,\theta)}{\partial(x,y)} \cdot \dfrac{\partial(x,y)}{\partial(r,\theta)} = 1$.

Soln Given $x = r\cos\theta, \quad y = r\sin\theta$.

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta - (-r\sin^2\theta)$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= \underline{\underline{r}}.$$

$x = r\cos\theta, \quad y = r\sin\theta$.

$$\Rightarrow x^2 = r^2\cos^2\theta \ ; \quad y^2 = r^2\sin^2\theta$$

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta \Rightarrow x^2 + y^2 = r^2(\underbrace{\cos^2\theta + \sin^2\theta}_{1})$$

$\therefore \quad x^2+y^2 = r^2$

$\Rightarrow \quad r = \sqrt{x^2+y^2} \quad$ i.e. $\quad r = (x^2+y^2)^{1/2}$.

$$r = (x^2+y^2)^{1/2}$$

$$\frac{\partial r}{\partial x} = \frac{1}{\cancel{2}}(x^2+y^2)^{1/2-1} \cdot \cancel{2}x = x.(x^2+y^2)^{-1/2} = \frac{x}{(x^2+y^2)^{1/2}} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{1}{2}(x^2+y^2)^{1/2-1} \cdot 2y = y(x^2+y^2)^{-1/2} = \frac{y}{(x^2+y^2)^{1/2}} = \frac{y}{r}$$

$x = r\cos\theta \quad ; \quad y = r\sin\theta$

$$\frac{r\sin\theta}{r\cos\theta} = \frac{y}{x} \Rightarrow \tan\theta = \frac{y}{x}. \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

$$\therefore \frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{x.0 - y.1}{x^2} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \left(\frac{-y}{x^2}\right).$$

$$= \frac{1}{\frac{x^2+y^2}{x^2}} \cdot \left(\frac{-y}{x^2}\right), \quad = \frac{\cancel{x^2}}{x^2+y^2} \cdot \left(\frac{-y}{\cancel{x^2}}\right) = \frac{-y}{r^2}.$$

Ill ly

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{x.1 - y.0}{x^2}\right). = \frac{1}{\frac{x^2+y^2}{x^2}} \cdot \frac{1}{x}.$$

$$= \frac{1}{1 + \frac{y^2}{x^2}} \cdot \left(\frac{\cancel{x}}{x^{\cancel{2}}}\right)$$

$$= \frac{x^{\cancel{2}}}{x^2+y^2} \cdot \frac{1}{\cancel{x}} = \frac{x}{r^2}.$$

$$\therefore \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ -\frac{y}{r^2} & \frac{x}{r^2} \end{vmatrix} = \frac{x^2}{r^3} - \left(-\frac{y^2}{r^3}\right).$$

$$= \frac{x^2+y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}.$$

$$\therefore \frac{\partial(r,\theta)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} = \frac{1}{r} \cdot r = 1 \;//$$

Hence Proved.