**Raabe's Test** (When the Ratio test fails, we apply Raabe's Test)

In the positive term series $\sum u_n$, if $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = k$, then the series converges for $k>1$ and diverges for $k<1$ and the test fails for $k=1$.

1) Test for convergence the series $\sum \frac{4.7.\cdots(3n+1)}{1.2.\cdots n} . x^n$

$3(n+1)+1$

**Soln.** $u_n = \frac{4.7.\cdots(3n+1)}{1.2.3\cdots n} . x^n$ ; $u_{n+1} = \frac{4.7.\cdots(3n+1)(3n+4)}{1.2.\cdots n.(n+1)} x^{n+1}$

$$\frac{u_n}{u_{n+1}} = \frac{\cancel{4.7.\cdots(3n+1)}}{\cancel{1.2.3\cdots n}} . x^n \times \frac{\cancel{1.2.\cdots n}(n+1)}{\cancel{4.7.\cdots(3n+1)}(3n+4) . x^{n+1}}$$

$$= \frac{(n+1)}{(3n+4) . x} = \frac{1}{x}\left[\frac{\cancel{n}(1+1/n)}{\cancel{n}(3+\frac{4}{n})}\right]$$

$x^{n+1}$

$x^n . x^1$

$3(n+1)+1$

$3n+3+1$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1}{x}\left[\frac{1+\frac{1}{n}}{3+\frac{4}{n}}\right] = \frac{1}{3x}.$$

$\lim_{n\to\infty} \frac{1}{n} = 0$

$\left(\because \frac{1}{\infty} = 0\right)$

$\therefore$ By ratio Test, the series converges for $\frac{1}{3x} > 1$

i.e for $\frac{1}{x} > 3$ or $x < \frac{1}{3}$. and

diverges for $x > \frac{1}{3}$ & fails for $x = \frac{1}{3}$.

$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 \ldots\ldots$

By Rabee's Test, $(x = \frac{1}{3})$

$$\frac{u_n}{u_{n+1}} = \frac{1}{x}\frac{\left(1+\frac{1}{n}\right)}{\left(3+\frac{4}{n}\right)} = \frac{\cancel{1}}{\cancel{1/3}}\frac{\left(1+\frac{1}{n}\right)}{\cancel{3}\left(1+\frac{4}{3n}\right)} = \left(1+\frac{1}{n}\right)\left(1+\frac{4}{3n}\right)^{-1}$$

$$= \left(1+\frac{1}{n}\right)\left(1 - \frac{4}{3n} + \left(\frac{4}{3n}\right)^2 - \ldots\ldots\right)$$

$$= 1 - \frac{4}{3n} + \frac{16}{9n^2} + \frac{1}{n} - \frac{4}{3n^2} + \frac{16}{9n^3} + \ldots\ldots$$

$-\frac{4}{3n} + \frac{1}{n}$

$= \frac{1}{n}\left(1 - \frac{4}{3}\right) = -\frac{1}{3n}$

$$= 1 - \frac{1}{3n} + \frac{4}{9n^2} + \cdots\cdots$$

$$\frac{u_n}{u_{n+1}} - 1 = \left(\cancel{1} - \frac{1}{3n} + \frac{4}{9n^2} + \cdots\cdots\right) \cancel{-1}$$

$$= -\frac{1}{3n} + \frac{4}{9n^2} + \cdots\cdots$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = -\frac{1}{3} + \frac{4}{9n} + \cdots\cdots$$

$$\lim_{n\to\infty}\left(n\left(\frac{u_n}{u_{n+1}} - 1\right)\right) = \cancel{\lim_{n\to\infty}}\left(-\frac{1}{3}\right) \Rightarrow \frac{-1}{3} < 1$$

$\therefore$ By Raabe's test, the series diverges.

Hence the given Series converges for $x < \frac{1}{3}$ and diverges for $x \geq \frac{1}{3}$.

$$\frac{16}{9n^2} - \frac{4}{3n^2}$$

$$= \frac{1}{n^2}\left(\frac{16}{9} - \frac{4}{3}\right).$$

$$= \frac{1}{n^2}\left(\frac{16-12}{9}\right)$$

$$= \frac{4}{9n^2}$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right)$$

$$n \times -\frac{1}{3n}$$

$$\frac{4}{9n^2} \times \frac{n}{1}$$

2) Test for convergence the series $\sum \frac{(n!)^2}{(2n)!} x^{2n}$.

Soln

$$u_n = \frac{(n!)^2}{(2n)!} \cdot x^{2n} \quad ; \quad u_{n+1} = \frac{((n+1)!)^2}{(2(n+1))!} x^{2(n+1)}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n!)^2}{(2n)!} \cdot x^{2n} \times \frac{(2(n+1))!}{((n+1)!)^2 \cdot x^{2(n+1)}}$$

$$= \left(\frac{n!}{(n+1)!}\right)^2 \cdot \frac{(2(n+1))!}{(2n)!} \cdot \frac{\cancel{x^{2n}}}{\cancel{x^{2n}} \cdot x^2}$$

$$= \left(\frac{\cancel{n!}}{\cancel{n!}\,(n+1)}\right)^2 \cdot \frac{(2n+2)!}{(2n)!} \quad \frac{1}{x^2}.$$

$$= \frac{1}{(n+1)^2} \cdot \frac{(2n+2).(2n+1)!}{(2n)!} \cdot \frac{1}{x^2}$$

$$x^{2(n+1)} = x^{2n+2} = x^{2n} \cdot x^2$$

$$(n+1)! = \underbrace{1.2.3..n}_{n!}(n+1).$$

$$= n!\,(n+1)$$

$$\underbrace{1.2.3..2n(2n+1)}(2n+2)$$

$$(2n+1)!\,(2n+2)$$

$$= \frac{1}{(n+1)^2} \cdot \frac{(2n+2)(2n+1).\cancel{(2n)}!}{\cancel{(2n)}!} \cdot \frac{1}{x^2}$$

$$(2n+1)! = \underbrace{1.2.3\ldots2n}_{(2n)!}.(2n+1)$$

$$= \frac{1}{(n+1)^{\cancel{2}}} \cdot \frac{2\cancel{(n+1)}(2n+1)}{x^2}$$

$$\frac{u_n}{u_{n+1}} = \frac{2(2n+1)}{n+1} \cdot \frac{1}{x^2} = \frac{2\cancel{n}\left(2+\frac{1}{n}\right)}{\cancel{n}\left(1+\frac{1}{n}\right)} \cdot \frac{1}{x^2}$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{4}{x^2}.$$

$\therefore$ By ratio test, the series converges if $\frac{4}{x^2} > 1$

ie if $4 > x^2$ or $x^2 < 4$ and

diverges if $x^2 > 4$. But fails if $\underline{\underline{x^2 = 4}}$.

$$\frac{4}{x^2} > 1$$

$$\Rightarrow 4 > x^2$$

$$\text{or } x^2 < 4$$

When $x^2 = 4$,

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{2n+1}{2n+2} - 1\right) = n\left(\frac{2n+1-2n-2}{2n+2}\right) = \frac{-n}{2n+2}.$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty}\left(\frac{-\cancel{n}}{\cancel{n}\left(2+\frac{2}{n}\right)}\right) = -\frac{1}{2} < 1$$

$\therefore$ By Raabe's Test, the series diverges.

Hence the given series converges for $x^2 < 4$ and diverges for $x^2 \geq 4$.

$$\frac{u_n}{u_{n+1}} = \frac{\cancel{x}(2n+1)}{n+1} \times \frac{1}{\cancel{4}\,2} \quad (\because x^2 = 4$$

$$= \frac{2n+1}{2(n+1)} = \frac{2n+1}{2n+2}$$

## Cauchy's Root Test

In a positive series $\sum u_n$, if $\lim_{n\to\infty} (u_n)^{1/n} = \lambda$, then the series converges for $\lambda < 1$ and diverges for $\lambda > 1$.

Note: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

**Note**. Cauchy's Root Test fails when $\lambda = 1$.

1) Test for convergence the series $\sum \frac{n^3}{3^n}$.

**Soln**

$$u_n = \frac{n^3}{3^n} \;; \quad (u_n)^{1/n} = \left(\frac{n^3}{3^n}\right)^{1/n} = \frac{n^{3\times\frac{1}{n}}}{3^{n\times\frac{1}{n}}} = \frac{n^{\frac{3}{n}}}{3}$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty}\left(\frac{n^{\frac{3}{n}}}{3}\right) = \frac{n^0}{3} = \frac{1}{3} < 1.$$

Hence the given series converges by Cauchy's root test.

2) $\sum \frac{(n+1)^n x^n}{n^{n+1}}$

<u>Soln.</u> $U_n = \frac{(n+1)^n x^n}{n^{n+1}}$ ; $(U_n)^{1/n} = \left(\frac{(n+1)^n . x^n}{n^{n+1}}\right)^{1/n} = \frac{(n+1)^{n\times\frac{1}{n}} . x^{n\times\frac{1}{n}}}{(n+1)^{(n+1)\times\frac{1}{n}}}$

$\therefore (U_n)^{1/n} = \frac{(n+1) . x}{\left(n^{n+1}\right)^{\frac{1}{n}}} = \frac{(n+1) x}{n^{\frac{1}{n}(n+1)}} = \frac{(n+1) x}{n^{1+\frac{1}{n}}}$.

i.e. $(U_n)^{\frac{1}{n}} = \frac{(n+1) x}{n^1 . n^{\frac{1}{n}}} = \left(\frac{n+1}{n}\right) . \frac{x}{n^{1/n}}$. $= \frac{x(1+\frac{1}{n})}{n} . \frac{x}{n^{1/n}}$ $\left( n^{1+\frac{1}{n}} = n^1 . n^{\frac{1}{n}}\right.$

$\lim_{n\to\infty} (U_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right) . \frac{x}{n^{1/n}} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) . \frac{x}{n^{\frac{1}{\infty}}}$

$= x$.

$\therefore$ The Series converges for $x<1$ and diverges for $x>1$.

3) Discuss the nature of the series

$$\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \cdots\cdots \quad (x>0)$$

<u>Soln</u> After leaving the first term,

$$u_n = \left(\frac{n+1}{n+2}\right)^n \cdot x^n = \left(\frac{\cancel{n}(1+\frac{1}{n})}{\cancel{n}(1+\frac{2}{n})}\right)^n \cdot x^n.$$

$$\therefore (u_n)^{1/n} = \left[\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)^{\cancel{n}}\right]^{1/\cancel{n}} \cdot (x^{\cancel{n}})^{1/\cancel{n}} = \left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right) \cdot x.$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right) \cdot x = x$$

By Cauchy's Root test, the given series converges for $x<1$, diverges for $x>1$ and fails for $x=1$.

4) $\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \cdots \infty$

Soln

$$U_n = \left(\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right)^{-n}$$

$$\therefore (U_n)^{1/n} = \left(\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right)^{-1} = \left(\frac{n+1}{n}\right)^{-1}\left[\left(\frac{n+1}{n}\right)^{n} - 1\right]^{-1}$$

$$\frac{(n+1)^{n}\cdot(n+1)}{n^{n}\cdot n^{1}}$$

$$= \left(\frac{\cancel{n}(1+\frac{1}{n})}{\cancel{n}}\right)^{-1}\left[\left(\frac{\cancel{n}(1+\frac{1}{n})}{\cancel{n}}\right)^{n} - 1\right]^{-1}$$

$$= \left(1+\frac{1}{n}\right)^{-1}\left\{\left(1+\frac{1}{n}\right)^{n} - 1\right\}^{-1}$$

$$\lim_{n\to\infty} (U_n)^{1/n} = \lim_{n\to\infty} \underbrace{\left(1+\frac{1}{n}\right)^{-1}}_{1}\left\{\underbrace{\left(1+\frac{1}{n}\right)^{n}}_{e} - 1\right\}^{-1}$$

$$= 1\cdot(e-1)^{-1}$$

$$\lim_{n\to\infty} (U_n)^{1/n} = 1\cdot(e-1)^{-1} = \frac{1}{e-1} < 1$$

$\therefore$ The series converges.