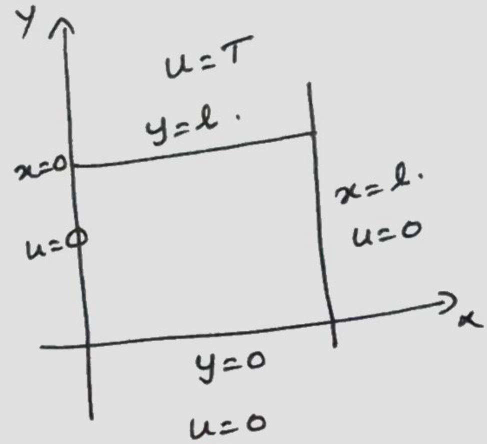
Type 2. Temperature distribution in finite plates.

Problem. Solve the Laplace eqn $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, subject to the conditions $u(0,y) = u(l,y) = u(x,0) = 0$ and $u(x,l) = T$.



Soln. The 2D heat eqn in the steady state is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

The boundary conditions are

(i) $u(0, y) = 0$

ii) $u(l, y) = 0$

iii) $u(x, 0) = 0.$

iv) $u(x, l) = T.$

The suitable soln of 2D heat flow in the steady state is

$$u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \quad —①.$$

Sub b.c (i) in ① i.e. put $x=0$ in ①.

$$0 = A(Ce^{py} + De^{-py}) \qquad (\because \cos 0 = 1,\ \sin 0 = 0.$$

$\Rightarrow \boxed{A = 0}$ Sub in ①.

$\therefore u(x,y) = B\sin px\,(Ce^{py} + De^{-py})$ —— (2)

Sub b.c (ii) in (2) i.e put $x = l$ in (2).

$0 = B\sin pl\,(Ce^{py} + De^{-py})$

$\Rightarrow \sin pl = 0 = \sin n\pi.$

$\Rightarrow pl = n\pi. \Rightarrow \boxed{p = \frac{n\pi}{l}}$ Sub in (2).

$u(x,y) = B\sin\frac{n\pi x}{l}\left(Ce^{\frac{n\pi y}{l}} + De^{-\frac{n\pi y}{l}}\right)$ —— (3).

Sub b.c (iii) in (1) i.e put $y = 0$ in (3).

$0 = B\sin\frac{n\pi x}{l}(C + D)$

$\therefore C + D = 0 \Rightarrow D = -C$ Sub in (3).

$u(x,y) = B\sin\frac{n\pi x}{l}\left(Ce^{\frac{n\pi y}{l}} - Ce^{-\frac{n\pi y}{l}}\right).$

$= BC\sin\frac{n\pi x}{l}\left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right).$

$= BC\sin\frac{n\pi x}{l}\cdot 2\sinh\frac{n\pi y}{l}.$ $\left(\because e^{\theta} - e^{-\theta} = 2\sinh\theta\right.$

Taking $2BC = B_n$, The most general soln is

$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cdot\sinh\frac{n\pi y}{l}.$ —— (4).

Sub b.c (IV) in ④, i.e. put $y = l$ in ④.

$$T = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} . \sinh\frac{n\pi l}{l}$$

i.e. $T = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} . \sinh n\pi$, H R F S. Series,

where $B_n \sinh n\pi = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l} dx .$

$$= \frac{2}{l}\int_0^l T \sin\frac{n\pi x}{l} dx .$$

$$= \frac{2T}{l}\left[-\frac{\cos\frac{n\pi x}{l}}{n\pi/l}\right]_0^l$$

$$= \frac{-2T}{l} \times \frac{l}{n\pi}\left[\cos\frac{n\pi x}{l}\right]_0^l .$$

$$= \frac{-2T}{n\pi}\left[\underset{(-1)^n}{\cos n\pi} - \underset{1}{\cos 0}\right] .$$

$$= \frac{2T}{n\pi}\left[1 - (-1)^n\right]$$

$$B_n \sinh n\pi = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4T}{n\pi} & \text{if } n \text{ is odd}, n = 1, 3, 5, \ldots \end{cases}$$

$$\therefore B_n = \frac{4T}{n\pi . \sinh n\pi} \quad \text{if } n = 1, 3, 5 \ldots$$

Sub in ④.

The reqd soln is

$$u(x, y) = \sum_{n=1,3,5\ldots}^{\infty} \frac{4T}{n\pi \sinh n\pi} . \sin\frac{n\pi x}{l} . \sinh\frac{n\pi y}{l} .$$