Module - 3

Basic Counting Techniques – Permutation and Combinations

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Permutation

- A permutation of a set of distinct objects is an ordered arrangement of these objects.
- Note: Permutation means selection and arrangement of factors.
- Notation: nP_r; P(n, r)
- r-Permutation: An r-permutation of n(distinct) elements x_1, x_2,…, x_n is an ordering of an r-element subset {x_1, x_2,…,x_n}. The number if r-permutation of a set of n distinct elements is denoted by P(n, r).

nP_r

Permutation

- **Theorem**: If n is a positive integer and r is an integer with 1 ≤r≤ n, then there are P(n,r)=n(n-1)(n-2)…(n-r+1) r – permutation of a set with n distinct elements.
- **Proof:** We apply the product rule to prove that the formula is correct.
- The first element of the permutation can be chosen in n ways since there are n elements in the set.
- The second element of the permutation can be chosen in n-1 ways since there are n-1 elements left in the set after using the element picked for the first position.
- Similarly there are n-2 ways to choose the third element, and so on, until there are exactly n-(r-1)=n-r+1 ways to choose the rth element

By the product rule nP1 = n; nP2 = n(n-1); nP3 = n(n-1)(n-2)……….

$nPr = n(n-1)(n-2)….(n-r+1) = \frac{n!}{(n-r)!}$

Product rule

n given to you

r

$(n-r+1)(n-r)$

$(n-r-1)(n-r-2)$

$(n-r+1)\cancel{(n-r)(n-r-1)}$

$n! = n(n-1) \cdot \cdot 3.2.1$

$(n-r)! = (n-r)(n-r-1)(n-r-2) \cdot \cdot 3.2.1$

$nP_r = n(n-1) \cdot \cdot (n-(r-1))$

$(n-r+1)$

1 2 3

$= n \cdot (n-1) \cdot \cdot (n-r+1)$

$= \frac{n!}{(n-r)!}$

20 × (20-1) × (20-2) × ……

$nP_r = \frac{n!}{(n-r)!} = n(n-1) \cdot \cdot (n-(r-1))$

(20-10)

Permutation - Results

- ❑ P(n, n) = n!✓
- ❑ P(n, r) = 0 if r > n
- ❑ P(n, 0)= 1 whenever n is a non negative integer since there is exactly one way to order zero elements.
- ❑ Cor:1 If n and r are integers with 0≤ r ≤ n, then $P(n,r) = \frac{n!}{(n-r)!}$ ✓
- ❑ Cor2: The number of permutations of n things taken all at a time is n!
- ❑ Note: The number of r-sequences of n objects (with repetition) is n^r.
- ❑ Let $P(n: n_1, n_2, \ldots, n_r) = \frac{n!}{n_1!, n_2!, \ldots, n_r!}$
- ❑ The number of ways in which n distinct objects can be arranged in a cycle without repetition is (n-1)!

$nP_r = n(n-1)(n-2) \cdot \cdot (n-r+1)$

$nP_r = \frac{n!}{(n-r)!}$

$\frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$

$\frac{n!}{(n-0)!} = \frac{\cancel{n!}}{\cancel{n!}} = 1$

$\binom{n}{r} = nC_r$

Combinations

- A combination is a selection of objects without regard to order (or) A combination is an unordered collection of distinct objects.
- Note: The number of r-combinations of n distinct objects is denoted by nCr or C(n,r) or $\binom{n}{r}$
- nCn = nC0 = 1
- nCr = nCn-r
- C(n,r) = P(n,r)/r!
- The number of r-combinations of a set with n elements, where n is a non-negative integer and r is an integer with 0≤ r ≤ n equals

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

- Let n and r be the non negative integers with r ≤ n. Then C(n, r) = C(n, n-r)

$nC_n = nC_0 = 1$

$nC_r = nC_{n-r}$

$nC_r = \frac{n!}{r!(n-r)!}$

$nC_r = \frac{nP_r}{r!}$

$nP_r = r!\, nC_r$

$nP_r = \frac{n!}{(n-r)!}$

$nC_r = \frac{n!}{r!\,(n-r)!}$

Combinations

- Example 1. Find the value of these
- P(6, 3) $= \frac{6!}{(6-3)!} = \frac{720}{6} = 120$
- P(8, 1) $= \frac{8!}{7!} = 8$
- P(8, 8) $= 8! =$
- C(5, 3) $= \frac{5!}{3!\,2!} = 10$
- C(8, 0) $= 1 = \frac{\cancel{8!}}{0!\,\cancel{(8-0)!}} = 1$

Combinations

- Determine the value of n if 4 (nP3)=(n+1)P3

$$4 \times \frac{n!}{(n-3)!} = \frac{(n+1)!}{(n+1-3)!}$$

$$^nP_r = \frac{n!}{(n-r)!}$$

$$3! = 3 \cdot 2!$$

$$4 \frac{n!}{(n-3)!} = \frac{(n+1) \cdot n!}{(n-2)!}$$

$$4 \frac{\cancel{n!}}{\cancel{(n-3)!}} = \frac{(n+1) \cdot \cancel{n!}}{(n-2)\cancel{(n-3)!}}$$

$$4(n-2) = n+1$$

$$4n - 8 = n+1 \Rightarrow 3n = 9$$

$$\boxed{n = 3}$$

Determine the value of n if 20Cn+2 = 20C2n-1

$$20C_{n+2} = 20C_{2n-1}$$

$$nC_x = nC_y$$

$$\Rightarrow x = y$$

$$n+2 = 2n-1$$

$$2+1 = 2n-n$$

$$\boxed{3 = n}$$

How many permutation of {a, b, c, d, e, f, g} (i) end with a (ii) begin with c (iii) begin with c and end with a (iv) c and a occupy the end places

(i) end with a $= 6! \cdot 1! = 720$

(ii) begin with c $= 1!\ 6! = 720$

(iii) begin with c + end with a $= 1!\ 5!\ 1! = 120$

(iv) c and a occupy end places $= 5!\ 2! = 240$

How many permutations of the letters A B C D E F G contain (i) the string BCD (ii) the string CFGA (iii) the strings BA and GF (iv) the strings ABC and DE (v) the strings ABC and CDE

(i) $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120$

(ii) $4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$

(iii) $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120$

(iv) $4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$

(v) $3 \cdot 2 \cdot 1 = 6$.

How many bit strings of length 10 contain (a) exactly four 1's (b) atmost four 1's (c) atleast four 1's (d) an equal number of 0's and 1's?

$n=10 \quad r=4$

(i) 10 bit string with exactly 4 1's $= 10C_4 = \frac{10!}{4!\,6!} = \frac{10\times 9\times \cancel{8}^{3}\times 7}{\cancel{4}\times 3\times \cancel{2}\times 1} = 210$

(ii) atmost 4 1's $\boxed{n < n}$ $= 10C_0 + 10C_1 + 10C_2 + 10C_3 + 10C_4$

$= 1 + 10 + \frac{\cancel{10}^{5}\times 9}{\cancel{2}} + \frac{10\times 9\times 8}{3\times 2} + \frac{10\times 9\times 8}{4\times 3\times 2\times 1}$

$= 11 + 45 + 120 + 210 = 386$

$nC_r = nC_{n-r}$

(iii) atleast 4 1's $n > n$ $= 10C_4 + 10C_5 + 10C_6 + 10C_7 + 10C_8 + 10C_9 + 10C_{10}$

$10C_2 + 10C_1 + 1$

$= 210 + \frac{\cancel{10}^{2}\times \cancel{9}^{3}\times \cancel{8}\times 7\times 6}{\cancel{5}\times 4\times \cancel{3}\times \cancel{2}\times 1} + 10C_4 + 10C_3 + 45 + 10 + 1$

$+ 252 + 210 + 120 +$

$= 848$

(iv) Equal no. of 0's + 1's $= 10C_5 = \frac{10!}{5!\,5!} = 252.$

Suppose that there are eight runners in a race. The winner receives first prize, the second-place finisher receives second prize and the third place finisher receives third prize. How many different ways are there to receive these prize, if all possible outcomes of the race can occur and there are no ties?

$nC_r = \frac{n!}{r!\,(n-r)!}$

$8C_3 = \frac{8!}{3!\,(8-3)!} = \frac{8!}{3!\,5!}$

$= \frac{8\times 7\times 6\times \cancel{5!}}{3!\times \cancel{5!}}$

$= \frac{8\times 7\times \cancel{6}}{\cancel{3}\times \cancel{2}} = 56$

In how many ways can a set of five letters to be selected from the English alphabet?

$$26C_5 = \frac{26 \times \cancel{25}^{5} \times \cancel{24} \times 23 \times 22}{\cancel{5} \times \cancel{4 \times 3 \times 2} \times 1} = \frac{26!}{5!\,(26-5)!}$$

$$= 65780$$

Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three ~~there~~ faculty members from the mathematics department and four from the computer science department?

$$9C_3 \cdot 11C_4 = \frac{\cancel{9}^{3} \times \cancel{8}^{4} \times 7}{\cancel{3} \times \cancel{2}} \times \frac{11 \times 10 \times \cancel{9}^{3} \times \cancel{8}}{\cancel{4} \times \cancel{3} \times \cancel{2} \times 1}$$

$$= 84 \times 330$$

$$= 27720$$

$$\begin{array}{r} 33 \\ 84\,(\\ \hline 132 \\ 264 \\ \hline 2772 \end{array}$$