The most general form of a PDE is

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + F = 0.$$

Where A, B, C, D, E, F are called Constants or fns of x & y.

This eqn is known as elliptic if $B^2 - 4AC < 0$

" " parabolic if $B^2 - 4AC = 0$

" " hyperbolic if $B^2 - 4AC > 0$.

Example 1

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

ie $$a^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$$

$A = a^2, \quad B = 0, \quad C = -1.$

$\therefore \quad B^2 - 4AC = 0 - 4a^2(-1) = 4a^2 > 0$

$\therefore$ The eqn is hyperbolic.

Example 2

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

ie $$a^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$$

$A = a^2, \quad B = 0, \quad C = 0$

$\therefore \; B^2 - 4AC = 0 \qquad \therefore$ It is parabolic.

Example 3 $\quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

$A = 1, \quad B = 0, \quad C = 1.$

$\therefore \; B^2 - 4AC = -4 < 0.$

$\therefore$ The eqn is elliptic.

4) $\frac{\partial^2 f}{\partial x^2} - 2\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} = 0.$

Here $A=1$, $B=-2$, $C=1$

$\therefore B^2 - 4AC = 4 - 4 = 0$

$\therefore$ The given eqn is parabolic.

5) $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} = 0.$

$A = B = C = 1$

$\therefore B^2 - 4AC = 1 - 4 = -3 < 0,$

$\therefore$ The gn eqn is elliptic.

Module - 2. Applications of Partial Differential Equations.

PDE of Engineering

A number of problems in Engineering give rise to the following well known partial diffl. eqns.

(i) One dimensional wave equation.

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad a^2 = \frac{T}{m}, \quad T \text{ - tension. } m \text{ - mass}$$

(ii) One dimensional heat flow equation.

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad \text{Where } a^2 = \frac{K}{\rho s}, \quad K \text{ - thermal conductivity}$$

ρ = density & s - Specific heat capacity of the Substance

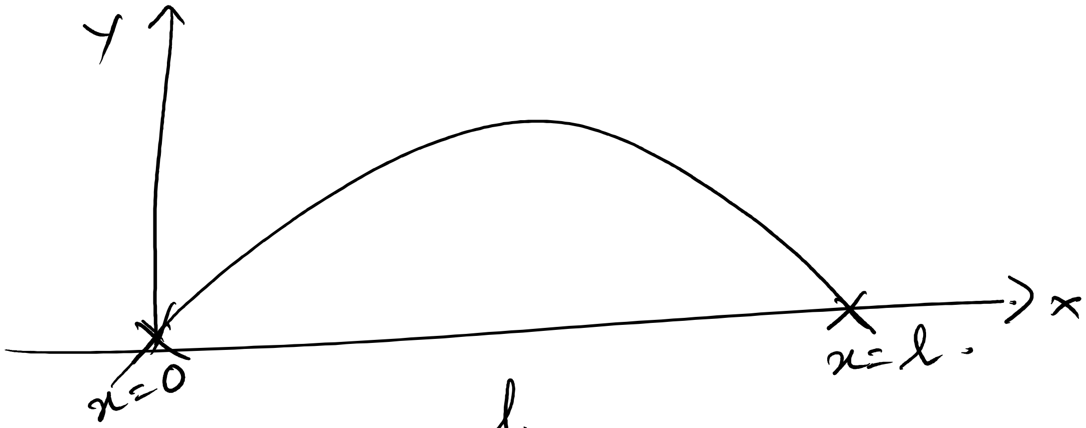
& $a^2 = \dfrac{k}{\rho c}$ is called diffusivity of the substance.
) (cm^2/sec)

(III) Two dimensional heat flow equation (or Laplace's Equation in Cartesian form)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Also PDE occur in Transmission line eqns, Vibrating membrane, theory of Elasticity and Hydraulics.

Vibrations of a stretched string – Wave Equation.



$u(x,t) \rightarrow$ displacement u at any pt x, at any time 't'.

$u(x,0) \rightarrow$ dispacement u at $t=0$ (initial displacement)

$u_t(x,0)$ or $\left(\frac{\partial u}{\partial t}\right)_{t=0} \rightarrow$ initial Velocity.

The 1-D wave eqn is $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$, $a^2 = \frac{T}{m}$

The various possible solutions of wave eqn are

(i) $u(x,t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{pat} + c_4 e^{-pat})$

(ii) $u(x,t) = (c_5 \cos px + c_6 \sin px)(c_7 \cos pat + c_8 \sin pat)$

(iii) $u(x,t) = (c_9 x + c_{10})(c_{11} t + c_{12})$

Of these 3 solutions, the most suitable soln is

$u(x,t) = (A \cos px + B \sin px)(C \cos pat + D \sin pat)$.

Since in vibration problems, displacement u must be a periodic fn of x and t.

(OR) $u(x,t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos pat + C_4 \sin pat)$

To solve a 1-D wave equation, we need 4 boundary conditions. (3 zero boundary conditions & 1 non zero boundary condition.

Type I

(i) $u(0,t) = 0, \quad t \geq 0$

(ii) $u(l,t) = 0, \quad t \geq 0.$

(iii) $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0, \quad 0 \leq x \leq l.$

(iv) $u(x,0) = f(x), \quad 0 \leq x \leq l.$

(Initial displacement is a fn of x)

Type II

(i) $u(0,t) = 0, \quad t \geq 0$

(ii) $u(l,t) = 0, \quad t \geq 0$

(iii) $u(x,0) = 0, \quad 0 \leq x \leq l.$

(iv) $u_t(x,0) = f(x), \quad 0 \leq x \leq l.$

(Initial velocity is a fn of x)