Full Range Fourier Series.

Full Range in π

$(0, 2\pi)$ or $0 \leq x \leq 2\pi$

The full Range F.S is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx.$$

Full Range in l

$(0, 2l)$ or $0 \leq x \leq 2l$.

The full range F.S is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l} \int_0^{2l} f(x)\, dx.$$

$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$b_n = \frac{1}{l} \int_0^{2l} f(x) \sin \frac{n\pi x}{l}\, dx$$

1) Find the fourier series of $f(x) = x^2$ in $(0, 2\pi)$

Soln $(0, 2\pi) \longrightarrow$ full range.

$\therefore$ The F.S is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{---} \quad (1)$$

Given $f(x) = x^2$.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx.$$

Note $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{3\pi}\left[(2\pi)^3 - 0^3\right]$$

$(2\pi)^3 = 2\pi \times 2\pi \times 2\pi$
$= 8\pi^3$

$$= \frac{8\pi^{\cancel{3}\,2}}{3\cancel{\pi}} = \frac{8\pi^2}{3}$$

$$a_0 = \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x).\cos nx\,dx. = \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx.$$

By Bernoulli's formula $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \dots$

$u = x^2$ $\quad v = \cos nx$

$u' = 2x$ $\quad v_1 = \frac{\sin nx}{n}$

$(-)$

$u'' = 2$ $\quad v_2 = -\frac{\cos nx}{n^2}$

$u''' = 0$ $\quad v_3 = -\frac{\sin nx}{n^3}$

$$\therefore a_n = \frac{1}{\pi}\left\{\frac{x^2\sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right\}_0^{2\pi}$$

$$a_n = \frac{1}{\pi}\left\{\left(\frac{4\pi^2}{n}0 + \frac{4\pi}{n^2}(1) - 0\right) - (0+0-0)\right\}$$

$$= \frac{1}{\pi}\times\frac{4\pi}{n^2}$$

$$\therefore a_n = \frac{4}{n^2}$$

Note $\cos n\pi = 1$, if n is even

$= -1$, n is odd

$\sin n\pi = 0 \ \forall n$, $\cos 0 = 1$

$\sin 0 = 0$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx.$$

$u = x^2 \qquad v = \sin nx$

$u' = 2x \qquad v_1 = -\frac{\cos nx}{n}$

$(-)$

$u'' = 2 \qquad v_2 = -\frac{\sin nx}{n^2}$

$u''' = 0 \qquad v_3 = \frac{\cos nx}{n^3}$

$$\therefore b_n = \frac{1}{\pi}\left\{-\frac{x^2}{n}\cos nx + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right\}_{x=0}^{2\pi}$$

$$b_n = \frac{1}{\pi}\left\{\left(-\frac{4\pi^2}{n}(1) + 0 + \cancel{\frac{2}{n^3}\cdot(1)}\right) - \left(0 + 0 + \cancel{\frac{2}{n^3}}\right)\right\}$$

$$\therefore b_n = \frac{1}{\pi} \times -\frac{4\pi^2}{n}$$

$$\text{i.e. } \boxed{b_n = -\frac{4\pi}{n}.}$$

Sub a_0, a_n, b_n in (1)

The reqd F.S is

$$f(x) = \frac{8\pi^2/3}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx + \left(-\frac{4\pi}{n}\right)\cdot\sin nx$$

$$\text{i.e. } \boxed{f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx - \sum_{n=1}^{\infty}\frac{4\pi}{n}\sin nx}$$

2) Find the F.S of $f(x) = x$ in $(0, 2\pi)$

Soln. $(0, 2\pi)$ — full range.

$\therefore$ The reqd F.S is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}.$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\,dx.$$

$$= \frac{1}{\pi} \int_0^{2\pi} x\,dx.$$

$$= \frac{1}{\pi} \left[\frac{x^2}{2} \right]_0^{2\pi}$$

$$= \frac{1}{2\pi} \left[(2\pi)^2 - 0 \right]$$

$$= \frac{4\pi^2}{2\cancel{\pi}}$$

$$\boxed{a_0 = 2\pi}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\,dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \cos nx\,dx.$$

$u = x \qquad v = \cos nx$

$u' = 1 \qquad v_1 = \frac{\sin nx}{n}$

$u'' = 0 \qquad (-) \qquad v_2 = -\frac{\cos nx}{n^2}$

$$a_n = \frac{1}{\pi}\left(\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right)_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{\left(0 + \frac{1}{n^2}\right) - \left(0 + \frac{1}{n^2}\right)\right\}$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx,$$

$u = x \qquad v = \sin nx.$

$u' = 1 \qquad v_1 = -\frac{\cos nx}{n}$

$u'' = 0 \qquad (-) \qquad v_2 = -\frac{\sin nx}{n^2}$

$$b_n = \frac{1}{\pi}\left\{-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right\}_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{\left(-\frac{2\pi}{n}\cdot 1 + 0\right) - (0 + 0)\right\}$$

$$= \frac{1}{\pi}\cdot\left(-\frac{2\pi}{n}\right) \Rightarrow b_n = -\frac{2}{n}$$

Sub a_0, a_n, b_n in (1)

$$f(x) = \frac{2\pi}{2} + \sum_{n=1}^{\infty} 0 + \sum_{n=1}^{\infty} -\frac{2}{n}\sin nx$$

$$\boxed{f(x) = \pi - \sum_{n=1}^{\infty}\frac{2}{n}\sin nx}$$

3) Find the fourier series of $f(x) = x(2l-x)$ in $(0,2l)$

Soln. Given range $(0,2l) \rightarrow$ full range.

The required fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{--- (1)}$$

Note

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

Given $f(x) = x(2l-x) = 2lx - x^2$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{l}\int_0^{2l} (2lx - x^2)\,dx$$

$$= \cancel{2}\frac{1}{l}\left[2l\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{2l}$$

$$= \frac{1}{l}\left\{l(2l)^2 - \frac{(2l)^3}{3}\right\}$$

$$= \frac{1}{l}\left\{4l^3 - \frac{8l^3}{3}\right\}$$

$$= \frac{1}{l}\left\{\frac{12l^3 - 8l^3}{3}\right\} = \frac{1}{l}\frac{4l^3}{3}$$

$$a_0 = 4l^2/3$$

$$a_n = \frac{1}{\ell}\int_0^{2\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx$$

$$= \frac{1}{\ell}\int_0^{\ell}(2\ell x - x^2)\cos\frac{n\pi x}{\ell}\,dx.$$

$u = 2\ell x - x^2$ $\qquad$ $v = \cos\frac{n\pi x}{\ell}$.

$u' = 2\ell - 2x$ $\qquad$ $v_1 = \dfrac{\sin\frac{n\pi x}{\ell}}{n\pi/\ell}$

$u'' = -2$ $\qquad$ $(-)$ $\qquad$ $= \frac{\ell}{n\pi}\sin\frac{n\pi x}{\ell}.$

$u''' = 0$

$v_2 = \frac{-\ell^2}{n^2\pi^2}\cos\frac{n\pi x}{\ell},$

$v_3 = -\frac{\ell^3}{n^3\pi^3}\sin\frac{n\pi x}{\ell}.$

$$a_n = \frac{1}{\ell}\left\{(2\ell x - x^2)\frac{\ell}{n\pi}\sin\frac{n\pi x}{\ell} + \frac{(2\ell - 2x)\ell^2}{n^2\pi^2}\cos\frac{n\pi x}{\ell} - \cancel{\frac{2\ell^3}{n^3\pi^3}} - \frac{2\ell^3}{n^3\pi^3}\sin\frac{n\pi x}{\ell}\right\}_0^{\ell}$$

$$= \frac{1}{\ell}\left\{\left(0 - \frac{2\ell\cdot\ell^{\cancel{2}}}{n^2\pi^2}\cdot\cos\frac{n\pi\cancel{2x}}{\cancel{\ell}} - 0\right.\right.$$

$$a_n = -\frac{2\ell^2}{n^2\pi^2}.$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

$$= \frac{1}{l}\int_0^{2l} (2lx - x^2)\sin\frac{n\pi x}{l}\,dx.$$

$u = 2lx - x^2$

$u' = 2l - 2x$

$u'' = -2$

$u''' = 0$

$V = \sin\frac{n\pi x}{l}$

$V_1 = -\frac{l}{n\pi}\cdot\cos\frac{n\pi x}{l}.$

$(-)$

$V_2 = -\frac{l^2}{n^2\pi^2}\cdot\sin\frac{n\pi x}{l}.$

$V_3 = \frac{l^3}{n^3\pi^3}\cdot\cos\frac{n\pi x}{l},$

$\cancel{b_n = \frac{1}{l}\left\{-(2lx - x^2)\frac{l^2}{n^2}\right.}$ $\cancel{n\pi}$

$\cancel{b_n = \frac{1}{l}\left\{(2lx - x^2)\frac{l}{n\pi}\right.}\cos$

$$= \frac{1}{l}\left\{-\left(\frac{2lx - x^2)\cdot l}{n\pi}\cdot \cos\frac{n\pi x}{l} + (2l-2x)\cdot\frac{l^2}{n^2\pi^2}\cdot \sin\frac{n\pi x}{l}\right.\right.$$

$$\left. - \frac{2l^3}{n^3\pi^3}\cdot \cos\frac{n\pi x}{l}\right\}_0^{2l}.$$

$$= \frac{1}{l}\left\{\left(0 + 0 - \frac{2l^3}{n^2\pi^3}\cdot \overset{0}{\cos\frac{n\pi}{2}}\right) - \left(\frac{-2l^3}{n^3\pi^3}\right)\right\}$$

$\left(\because \sin\frac{n\pi}{2} = 1, \cos\frac{n\pi}{2} = 0\right)$

$\cancel{b_n} = 0$

sub $b_n = \frac{1}{l}\cdot\frac{4l^3}{n^3\pi^3}$ i.e. $b_n = \frac{4l^2}{n^3\pi^3}$

We a_0, a_n & b_n in (1) The reqd F-S is

$$\cancel{f(x) = \frac{-\frac{4l^2}{n^2\pi^2}}{2} + \sum_{n=1}^{\infty}}$$

$$f(x) = \frac{\frac{4l^3}{3}}{2} + \sum_{n=1}^{\infty}\left(-\frac{2l^2}{n^2\pi^2}\right)\cdot \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty}\frac{4l^2}{n^3\pi^3}\cdot \sin\frac{n\pi x}{l}$$

$$f(x) = \frac{2l^3}{3} - \sum_{n=1}^{\infty}\frac{2l^2}{n^2\pi^2}\cdot \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty}\frac{4l^2}{n^3\pi^3}\cdot \sin\frac{n\pi x}{l}.$$

4) Find the F.S of $f(x) = e^x$ in $(0, 2l)$

Note: $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$$

F.S. is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$. — ①

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{1}{l}\int_0^{2l} e^x \, dx = \frac{1}{l}\left(e^x\right)_0^{2l}$$

$$= \frac{1}{l}\left\{e^{2l} - e^0\right\} = \frac{1}{l}\left(e^{2l} - 1\right) = a_0$$

$\because e^0 = 1$

$$a_n = \frac{1}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx.$$

$$= \frac{1}{l}\int_0^l e^x \cos\frac{n\pi x}{l}\,dx.$$

$$a=1,\quad b=\frac{n\pi}{l}$$

$$\therefore a_n = \frac{1}{l}\left[\frac{e^x}{1^2+\left(\frac{n\pi}{l}\right)^2}\left(1.\cos\frac{n\pi x}{l}+\frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)\right]_0^{2l}$$

$$= \frac{1}{l}\left\{\frac{e^{2l}}{1+\frac{n^2\pi^2}{l^2}}\left(\cos\frac{n\pi.2\cancel{l}}{\cancel{l}}+0\right)-\frac{1}{1+\frac{n^2\pi^2}{l^2}}.(1)\right\}$$

$$= \frac{1}{l}\left\{\frac{e^{2l}}{l^2+n^2\pi^2}\right.$$

$$=\frac{1}{l}\left\{\frac{l^2e^{2l}}{l^2+n^2\pi^2}(1)-\frac{l^2}{1+n^2\pi^2}\right.$$

$$=\cancel{\frac{l^2}{1+n^2\pi^2}.\frac{1}{l}\left\{e^{2l}-\right.}$$

$$=\frac{l^{\cancel{2}}}{1+n^2\pi^2}.\frac{1}{\cancel{l}}\left\{e^{2l}-1\right\}$$

$$a_n=\frac{(e^{2l}-1)l}{1+n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x) \sin\frac{n\pi x}{l}\, dx.$$

$$= \frac{1}{l}\int_0^{2l} e^x \sin\frac{n\pi x}{l}\, dx.$$

$a=1$, $b = \frac{n\pi x}{l}$.

$$= \frac{1}{l}\left\{ \frac{e^x}{1+\frac{n^2\pi^2}{l^2}}\left(1.\cancel{\sin\frac{n\pi x}{l}} - \frac{n\pi}{l}.\cos\frac{n\pi x}{l}\right)\right\}_0^{2l}.$$

$$= \frac{1}{l}\left\{ \frac{e^{2l}}{\frac{l^2+n^2\pi^2}{l^2}}.\left(-\frac{n\pi}{l}.(1)\right) - \frac{1}{\frac{l^2+n^2\pi^2}{l^2}}.\left(-\frac{n\pi}{l}\right)\right\}$$

$= \frac{1}{}$

H.W

F.S of $f(x) = e^x$ in $(0, 2\pi)$

$$= \frac{1}{l}\left\{ \frac{\cancel{l^2} e^{2l}}{1+n^2\pi^2} \cdot \left(-\frac{n\pi}{\cancel{l}}\right) + \frac{\cancel{l^2}}{1+n^2\pi^2}\left(\frac{n\pi}{\cancel{l}}\right) \right\}$$

$$= \frac{n\pi}{1+n^2\pi^2}\left(-e^{2l}+1\right)$$

$$b_n = \frac{n\pi\left(1-e^{2l}\right)}{1+n^2\pi^2}.$$

Sub a_0, a_n, b_n in 1

$$f(x) = \frac{\frac{1}{l}\left(e^{2l}-1\right)}{2} + \sum_{n=1}^{\infty}$$