Power Series:

Defn. A Series of the form $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$, where a's are independent of x is called a power series in x. Such a series may converge for some or all values of x.

Interval of Convergence

In the power series $u_n = a_n x^n$ & $u_{n+1} = a_{n+1} x^{n+1}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{a_{n+1} x^{n+1}}{a_n x^n} = \underbrace{\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)}_{l} x = lx$$

$x^{n+1} = x^n \cdot x^1$

$\frac{x^{n+1}}{x^n} = \frac{\cancel{x^n} \cdot x}{\cancel{x^n}}$

If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$, then the power series converges if $lx < 1$, i.e.

$x < \frac{1}{l}$

$|x| < \frac{1}{l}$

for other values.

when $|x| < 1/l$ and diverges for other values.

$\therefore$ The power series has an interval $-\frac{1}{l} < x < \frac{1}{l}$, within which it converges and diverges the values outside this interval.

$\left(-\frac{1}{l}, \frac{1}{l}\right)$

$|x| < \frac{1}{l} \Rightarrow$

$-\frac{1}{l} < x < \frac{1}{l}$

1) State the values of x for which the following series converge.

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\cdots$$

Soln

$$U_n = (-1)^{n-1} \cdot \frac{x^n}{n} \quad \text{and} \quad U_{n+1} = \frac{(-1)^n \cdot x^{n+1}}{n+1}$$

$\left|\frac{1}{x}\right| > 1$

$-x - x - = -$

$(-1)^{-1} = \frac{1}{(-1)} = -1$

$$\frac{U_n}{U_{n+1}} = \frac{(-1)^{n-1} \cdot x^n}{n} \cdot \frac{n+1}{(-1)^n \, x^{n+1}}$$

$\frac{n+1}{n} = \frac{n}{n} + \frac{1}{n}$

$$= \frac{(-1)^{\cancel{n}-1-\cancel{n}} \, \cancel{x^n} \cdot (n+1)}{n \cdot x^1 \cdot \cancel{x^n}} = -1 \, \cancel{\frac{n}{(n+1)x}} \quad -1 \, \frac{(n+1)}{n} \cdot \frac{1}{x}$$

$$= -\left(1 + \frac{1}{n}\right) \cdot \frac{1}{x} = -\cancel{\left(\frac{n}{n\left(1+\frac{1}{n}\right)} \cdot \frac{1}{x}\right)} = \cancel{\frac{-1}{1+\frac{1}{n}} \cdot \frac{1}{x}}$$

$\therefore \lim_{n\to\infty} \frac{1}{n} = 0.$

$= |-1| \left|\frac{1}{x}\right|$

$$\lim_{n\to\infty} \left|\frac{U_n}{U_{n+1}}\right| = \cancel{\lim_{n\to\infty} \left|\frac{-1}{x\left(1+\frac{1}{n}\right)}\right| = \left|\frac{1}{x}\right|} \quad \lim_{n\to\infty} \left|-\left(1+\frac{1}{n}\right) \cdot \frac{1}{x}\right| = \left|\frac{-1}{x}\right| \text{ or } |x| < 1$$

$\therefore$ By Ratio Test, the series converges for $\left|\frac{1}{x}\right| > 1$ & (or) $1 > |x|$

diverges for $\left|\frac{1}{x}\right| < 1$. $\longrightarrow$ or $|x| > 1$

Convergence of Exponential Series.

*: The Series $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \infty$ is convergent for all values of x.

$u_{n+1} = \frac{x^{n+1}}{n+1}$

$u_n = \frac{x^n}{n}$

$\frac{u_n}{u_{n+1}} = \frac{x^n}{n} \cdot \frac{n+1}{x^{n+1}}$

$= \underbrace{\left(1+\frac{1}{n}\right)}_{1} \cdot \frac{1}{x} = \frac{1}{x} > 1$

$x < 1$

Convergence of logarithmic Series.

The Series $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{(-1)^n x^n}{n} + \dots \infty$

is convergent for $-1 < x < 1$ or $|x| < 1$

* The series converges for $|x| < 1$ and diverges for $|x| > 1$

* When $x = 1$, the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \log 2$ is convergent.

* When $x = -1$, the series $-\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\right)$ is divergent.

Hence the series converges for $-1 < x < 1$ (or) interval of convergence $(-1, 1)$

$|x| < 1$

Convergence of Binomial Series.

The Series $1 + nx + \frac{n(n-1)}{2!} x^2 + \cdots\cdots + \frac{n(n-1)\cdots\overbrace{(n-(r-1))}^{n-r+1}}{r!} x^r + \cdots\cdots \infty$ $\downarrow$ $(1+x)^n$

converges for $|x| < 1$. $\rightarrow$ $-1 < x < 1$

Expansions of Functions.

* If $f(x)$ can be expanded as an infinite Series, then

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots\cdots \infty$$

is known as Maclaurin's Series. , $f', f'', f''', \ldots$ are known as derivatives of $f(x)$

* If $f(x)$ can be expressed as an infinite Series, then

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots \infty$$

is known as Taylor's Series, $f', f'', f''', \ldots$ are known as derivatives of $f(x)$

Taking $a=0$ in Taylor's series, we get the Maclaurin's Series.

Expansion by <u>use of known Series</u>. (Expansion of Trigonometric Series)

Trigonometric Series

1. $\sin\theta = \theta - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} - \dfrac{\theta^7}{7!} + \cdots$

2. $\cos\theta = 1 - \dfrac{\theta^2}{2!} + \dfrac{\theta^4}{4!} - \dfrac{\theta^6}{6!} + \cdots$

3. $\tan\theta = \theta + \dfrac{\theta^3}{3} + \dfrac{2\theta^5}{15} + \cdots$

hyperbolic function

4. $\sinh\theta = \theta + \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} + \dfrac{\theta^7}{7!} + \cdots$

5. $\cosh\theta = 1 + \dfrac{\theta^2}{2!} + \dfrac{\theta^4}{4!} + \dfrac{\theta^6}{6!} + \cdots$

$\theta = i\theta$:

$\sin i\theta = i\sinh\theta$

$\cos i\theta = \cosh\theta$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$i^5 = i$

6. $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots\cdots$

7. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\cdots$

8. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\cdots$

9. $\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\cdots\right)$

10. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots\cdots$

11) $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots\cdots$

Problems. Expand e^x about $\underline{x=0}$. (or) Find the Taylor's Series of $f(x) = e^x$ about the point $\underline{x=0}$ origin

$\downarrow$

Maclaurin's Series,

Soln Let

$f(x) = e^x \qquad f(0) = e^0 = 1$

$f'(x) = e^x \qquad f'(0) = e^0 = 1$

$f''(x) = e^x \qquad f''(0) = e^0 = 1$

$f'''(x) = e^x \qquad f'''(0) = e^0 = 1$

$f^{IV}(x) = e^x \qquad f^{IV}(0) = e^0 = 1$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \infty$$

$$e^x = 1 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(1) + \cdots \infty$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

2) Expand $\sin x$ in powers of $(x - \pi/2)$ or Find the Taylor's series of $f(x) = \sin x$ about the point $x = \frac{\pi}{2}$.

$\sin\frac{\pi}{2} = 1$

$\cos\frac{\pi}{2} = 0$

Soln.

$f(x) = \sin x$ | $f(\pi/2) = \sin\pi/2 = 1$

$f'(x) = \cos x$ | $f'(\pi/2) = \cos\pi/2 = 0$

$f''(x) = -\sin x$ | $f''(\pi/2) = -\sin\frac{\pi}{2} = -1$

$f'''(x) = -\cos x$ | $f'''(\pi/2) = -\cos\frac{\pi}{2} = 0$

$f^{IV}(x) = \sin x$ | $f^{IV}(\pi/2) = \sin\pi/2 = 1$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Here $a = \pi/2$

$$= 1 + (x-\pi/2)\cdot 0 + \frac{(x-\pi/2)^2}{2!}(-1) + \frac{(x-\pi/2)^3}{3!}(0) + \frac{(x-\pi/2)^4}{4!}(1) + \cdots$$

$$= 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} + \cdots$$

3) find the Taylor's series of $\cos x$ about $x = \pi/2$.

Soln

$f(x) = \cos x$ $\qquad f(\pi/2) = \cos \pi/2 = 0$

$f'(x) = -\sin x$ $\qquad f'(\pi/2) = -\sin \pi/2 = -1$

$f''(x) = -\cos x$ $\qquad f''(\pi/2) = -\cos \pi/2 = 0$

$f'''(x) = \sin x$ $\qquad f'''(\pi/2) = \sin \pi/2 = 1$

$f^{IV}(x) = \cos x$ $\qquad f^{IV}(\pi/2) = \cos \pi/2 = 0$.

Here $a = \frac{\pi}{2}$

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

$$\therefore f(x) = 0 + (x - \pi/2).(-1) + \frac{(x-\pi/2)^2}{2!}.0 + \frac{(x-\pi/2)^3}{3!}.(1) + \frac{(x-\pi/2)^4}{4!}.(0) + \cdots$$

$$\therefore \cos x = -(x - \pi/2) + \frac{(x-\pi/2)^3}{3!} + \cdots$$

4) Using Taylor's theorem express the polynomial $2x^3 + 7x^2 + x - 6$ in powers of $\underline{(x-1)}$. $\Longrightarrow$ The Series is about $x = 1$

Soln $f(x) = 2x^3 + 7x^2 + x - 6$.

$f'(x) = 6x^2 + 14x + 1$

$f''(x) = 12x + 14$

$f'''(x) = 12$

$f^{IV}(x) = 0$

$f(1) = 2 + 7 + 1 - 6 = 4$

$f'(1) = 21$

$f''(1) = 26$

$f'''(1) = 12$

$f^{IV}(1) = 0$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

Here $a = 1$

$$= 4 + (x-1)\,21 + \frac{(x-1)^2}{2!}\,26 + \frac{(x-1)^3}{6}\,.(12)$$

$$= 4 + 21(x-1) + 13(x-1)^2 + 2(x-1)^3$$

5) Expand $\log_e x$ in powers of $(x-1)$. $\longrightarrow$ about the point $x=1$

<u>Soln</u>

$f(x) = \log x$. $\qquad f(1) = \log 1 = 0$

$f'(x) = \frac{1}{x}$. $\qquad f'(1) = \frac{1}{1} = 1$.

$f''(x) = -\frac{1}{x^2}$ $\qquad f''(1) = -1$.

$f'''(x) = +\frac{2}{x^3}$ $\qquad f'''(1) = +2$

$f^{IV}(x) = -\frac{6}{x^4}$ $\qquad f^{IV}(1) = -6$

$f''(x) = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = (-1)\, x^{-1-1}$, nx^{n-1}

$= -1 . x^{-2}$

$= -\frac{1}{x^2}$

$f'''(x) = \frac{d}{dx}\left(-\frac{1}{x^2}\right)$

$= -\frac{d}{dx}(x^{-2})$, $n=-2$

$= -(-2).x^{-2-1}$

$= \frac{2}{x^3}$

$f^{IV} = \frac{d}{dx}\left(\frac{2}{x^3}\right)$

$= 2 . \left(\frac{d}{dx} x^{-3}\right)$

$2(-3). x^{-4} = \frac{-6}{x^4}$

$\therefore f(x) = f(a) + (x-a).f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$

Here $a=1$

$= 0 + (x-1).1 + \frac{(x-1)^2}{2}(-1) + \frac{(x-1)^3}{6}.(+2) + \frac{(x-1)^4}{24}.(-6) + \cdots$

$= (x-1) - \frac{(x-1)^2}{2} + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$

6) Evaluate $\lim_{x\to 0} \dfrac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)}$

Soln. $\lim_{x\to 0}\left(\dfrac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)}\right) = \dfrac{\left\{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)\cdot\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\cdots\right)\right\} - x - x^2}{x^2 + x\left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\cdots\right)}$

$$= \frac{\left\{\cancel{x} - \frac{x^3}{6} + \frac{x^5}{120} + x^2 - \frac{x^4}{6} + \frac{x^6}{120} + \frac{x^3}{2} - \frac{x^5}{12} + O(x^6)\cdots\right\} - \cancel{x} - x^2}{\cancel{x^2} - \left(\cancel{x^2} + \frac{x^3}{2} + \frac{x^4}{3} + \frac{x^5}{4} + \cdots\right)}$$

$$= \frac{\left(\cancel{x} + \cancel{x^2} + \frac{1}{3}x^3 - O(x^4) + \cdots\right) - \cancel{x} - \cancel{x^2}}{-\frac{1}{2}x^3 - \frac{1}{3}x^4 - \cdots\cdots}$$

$5! = 1\times 2\times 3\times 4\times 5$
$= 120$

$-\frac{1}{6}+\frac{1}{2}$
$= \frac{-1+3}{6}$

$-\frac{x^3}{6} + \frac{x^3}{2}$
$= \frac{-x^3 + 3x^3}{6}$
$= \frac{2x^3}{6} = \frac{1}{3}x^3$

$O(x^5) \to$ order of x^5

$$= \frac{\frac{1}{3}x^3 + O(x^4) + \cdots}{-\frac{1}{2}x^3 - \frac{1}{3}x^4 + \cdots}$$

$\div\ x^3$

$$= \frac{\frac{1}{3} + \cdots}{-\frac{1}{2} - \frac{1}{3}x + \cdots}$$

$$\lim_{x\to 0}\left(\frac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)}\right) = \lim_{x\to 0}\frac{\frac{1}{3} + \cdots}{-\frac{1}{2} - \frac{1}{3}x \cdots} = \frac{1/3}{-1/2} = -2/3.$$

7) H.W

$$\lim_{x\to 0}\frac{e^x + \sin x - 1}{\log(1+x)}.$$

$$\lim_{x\to 0}\left(\frac{e^x + \sin x - 1}{\log(1+x)}\right) = \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) - 1}{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots}$$