3) Find the evolute of the curve $x = a(\cos\theta + \theta\sin\theta)$; $y = a(\sin\theta - \theta\cos\theta)$

Soln Given $x = a(\cos\theta + \theta\sin\theta)$; $y = a(\sin\theta - \theta\cos\theta)$

$$\frac{dx}{d\theta} = a(-\sin\theta + (\theta.\cos\theta + \sin\theta.1))$$

$$= a(-\cancel{\sin}\theta + \theta\cos\theta + \cancel{\sin}\theta)$$

$$= a\theta\cos\theta.$$

$$\frac{dy}{d\theta} = a\left(\cos\theta - [\theta(-\sin\theta) + \cos\theta.1]\right)$$

$$= a(\cancel{\cos}\theta + \theta\sin\theta - \cancel{\cos}\theta)$$

$$= a\theta\sin\theta$$

$$\therefore \frac{dy}{dx} = y_1 = \frac{dy/d\theta}{dx/d\theta} = \frac{\cancel{a\theta}\sin\theta}{\cancel{a\theta}\cos\theta}.$$

i.e. $\boxed{y_1 = \tan\theta}$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{d\theta}\underbrace{\left(\frac{dy}{dx}\right)}_{y_1} \cdot \frac{d\theta}{dx} = \frac{d}{d\theta}(\tan\theta) \cdot \frac{1}{a\theta\cos\theta} = \frac{\sec^2\theta}{a\theta\cos\theta}$$

$$y_2 = \frac{1}{a\theta\cos\theta \cdot \cos^2\theta} = \frac{1}{a\theta \cdot \cos^3\theta} = \frac{1}{a\theta} \cdot \sec^3\theta$$

$$\bar{x} = x - \frac{y_1}{y_2}\left(1 + y_1^2\right)$$

$$= a(\cos\theta + \theta\sin\theta) - \frac{\tan\theta}{\frac{1}{a\theta} \cdot \sec^3\theta} \cdot (1 + \tan^2\theta)$$

$$= a(\cos\theta + \theta\sin\theta) - \frac{a\theta\tan\theta}{\sec^{\cancel{3}}\theta} \cdot \cancel{\sec^2}\theta$$

$$= a\cos\theta + a\theta\sin\theta - a\theta \cdot \frac{\sin\theta}{\cancel{\cos\theta}} \times \cancel{\cos\theta}$$

ie $\bar{x} = a\cos\theta$ —— (1)

$\sec\theta = \frac{1}{\cos\theta}$

$\sec^2\theta = \frac{1}{\cos^2\theta}$

$1 + \tan^2\theta = \sec^2\theta$

$\frac{1}{\sec\theta} = \cos\theta$

$$\bar{y} = y + \frac{1}{y_2}(1+y_1^2)$$

$$= a(\sin\theta - \theta\cos\theta) + \frac{1}{\frac{1}{a\theta}\sec^3\theta}(1+\tan^2\theta)$$

$$= a\sin\theta - a\theta\cos\theta + \frac{a\theta}{\sec^{\cancel{3}}\theta}\cdot\cancel{\sec^2}\theta$$

$$= a\sin\theta - a\theta\cos\theta + \frac{a\theta}{\sec\theta}.$$

$$= a\sin\theta - \cancel{a\theta\cos\theta} + \cancel{a\theta\cos\theta}$$

$$\therefore \boxed{\bar{y} = a\sin\theta} \quad \text{—② . Eliminating } \cos\theta \text{ \& } \sin\theta \text{ from ① \& ②}$$

From ① $\cos\theta = \frac{\bar{x}}{a}$ & from ② $\sin\theta = \frac{\bar{y}}{a}$.

Squaring & adding

$$\therefore \underbrace{\cos^2\theta + \sin^2\theta}_{1} = \frac{\bar{x}^2}{a^2} + \frac{\bar{y}^2}{a^2}$$

i.e $\frac{\bar{x}^2}{a^2} + \frac{\bar{y}^2}{a^2} = 1$

$\therefore$ locus of $(\bar{x}, \bar{y})$ is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$

i.e $\frac{x^2+y^2}{a^2} = 1$

$x^2+y^2 = a^2$, which is a circle.

$$\frac{a\theta}{\sec\theta} = a\theta\cos\theta$$

4) Find the evolute of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Soln The parametric form of the ellipse is

$x = a\cos\theta$; $y = b\sin\theta$.

$\frac{dx}{d\theta} = -a\sin\theta$; $\frac{dy}{d\theta} = b\cos\theta$.

$\frac{\cos\theta}{\sin\theta} = \cot\theta$

$\frac{\sin\theta}{\cos\theta} = \tan\theta$

$\frac{1}{\sin\theta} = \operatorname{cosec}\theta$

$\therefore y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$.

$y_2 = \frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\cdot\frac{d\theta}{dx} = \frac{d}{d\theta}\left(-\frac{b}{a}\cot\theta\right)\cdot\left(-\frac{1}{a\sin\theta}\right)$.

$\frac{dy}{dx} \downarrow y_1$

$= -\frac{b}{a}\cdot(-\operatorname{cosec}^2\theta)\cdot\left(-\frac{1}{a\sin\theta}\right)$

$y_2 = -\frac{b}{a^2}\operatorname{cosec}^3\theta \quad \left(\because \frac{1}{\sin\theta} = \operatorname{cosec}\theta\right)$

Now

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$$

$$= a\cos\theta - \frac{\left(-\frac{b}{a}\cot\theta\right)}{\left(-\frac{b}{a^2}\operatorname{cosec}^3\theta\right)}\cdot\left(1+\left(-\frac{b}{a}\cot\theta\right)^2\right)$$

$$= a\cos\theta - \frac{\cancel{b}}{\cancel{a}}\cdot\frac{\cos\theta}{\sin\theta}\times\frac{a^{\cancel{2}}}{\cancel{b}\operatorname{cosec}^3\theta}\left(1+\frac{b^2}{a^2}\cot^2\theta\right).$$

$$= a\cos\theta - \frac{a\cos\theta}{\cancel{\sin}\theta}\cdot\sin^{\cancel{3}\,2}\theta\left(\frac{a^2+b^2\cot^2\theta}{a^2}\right).$$

$$= a\cos\theta - \frac{\cancel{a}\cos\theta\sin^2\theta}{a^{\cancel{2}}}\left(a^2+\frac{b^2\cos^2\theta}{\sin^2\theta}\right)$$

$$= a\cos\theta - \frac{\cos\theta\sin^{\cancel{2}}\theta}{a}\left(\frac{a^2\sin^2\theta+b^2\cos^2\theta}{\sin^{\cancel{2}}\theta}\right)$$

$$= a\cos\theta - \frac{\cos\theta}{a}\left(a^2\sin^2\theta + b^2\cos^2\theta\right)$$

$$= a\cos\theta - \frac{\cancel{a^2}\sin^2\theta\cos\theta}{\cancel{a}} - \frac{b^2}{a}\cos^3\theta.$$

$\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta = 1 - \cos^2\theta$

$$= a\cos\theta - a\cos\theta(1 - \cos^2\theta) - \frac{b^2}{a}\cos^3\theta.$$

$$= a\cancel{\cos\theta} - a\cancel{\cos\theta} + a\cos^3\theta - \frac{b^2}{a}\cos^3\theta.$$

$$\bar{x} = \cos^3\theta\left(a - \frac{b^2}{a}\right)$$

$$\text{i.e. } \bar{x} = \cos^3\theta\left(\frac{a^2 - b^2}{a}\right) \Rightarrow \cos^3\theta = \frac{a\bar{x}}{a^2 - b^2}$$

$$\Rightarrow \cos\theta = \left(\frac{a\bar{x}}{a^2 - b^2}\right)^{1/3} \quad \text{—— } ①.$$

$$\bar{y} = y + \frac{1}{y_2}(1 + y_1^2)$$

$$= b\sin\theta + \frac{1}{-\frac{b}{a^2}\operatorname{cosec}^3\theta}\left(1 + \left(-\frac{b}{a}\cot\theta\right)^2\right)$$

$$= b\sin\theta - \frac{a^2}{b\operatorname{cosec}^3\theta}\left(1 + \frac{b^2}{a^2}\cot^2\theta\right)$$

$$= b\sin\theta - \frac{\cancel{a^2}}{b}\sin^3\theta\left(\frac{a^2 + b^2\cot^2\theta}{\cancel{a^2}}\right)$$

$$= b\sin\theta - \frac{\cancel{\sin}^3\theta}{b}\left(a^2 + \frac{b^2\cos^2\theta}{\cancel{\sin}^2\theta}\right)$$

$$= b\sin\theta - \frac{\sin^{\cancel{3}}\theta}{b}\left(\frac{a^2\sin^2\theta + b^2\cos^2\theta}{\cancel{\sin^2}\theta}\right)$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - \frac{\sin\theta}{\cancel{b}}\left(\cancel{b^2}\cos^2\theta\right)$$

$$\left(\cos^2\theta = 1 - \sin^2\theta\right)$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - b\sin\theta\left(1 - \sin^2\theta\right)$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - b\sin\theta + b\sin^3\theta.$$

$$= \cancel{b\sin\theta} - \frac{a^2}{b}\sin^3\theta - \cancel{b\sin\theta}$$

$$\bar{y} = \sin^3\theta\left(b - \frac{a^2}{b}\right) = \sin^3\theta\left(\frac{b^2 - a^2}{b}\right)$$

$$b^2 - a^2 = -(a^2 - b^2)$$

$$\bar{y}\,b = \sin^3\theta\,(b^2 - a^2)$$

$$\therefore \sin^3\theta = \frac{y\bar{b}}{b^2 - a^2} \Rightarrow \boxed{\sin^3\theta = \frac{-b\bar{y}}{a^2 - b^2}}$$

$$\therefore \sin\theta = \left(-\frac{y\bar{b}}{a^2 - b^2}\right)^{1/3} \quad \text{—— (2)}$$

Eliminating θ from (1) & (2)

Squaring & adding

$$\underbrace{\cos^2\theta + \sin^2\theta} = \left[\left(\frac{a\bar{x}}{a^2-b^2}\right)^{1/3}\right]^2 + \left[\left(\frac{-b\bar{y}}{a^2-b^2}\right)^{1/3}\right]^2$$

$$1 = \frac{(a\bar{x})^{2/3}}{(a^2-b^2)^{2/3}} + \frac{(b\bar{y})^{2/3}}{(a^2-b^2)^{2/3}} \Rightarrow \frac{(a\bar{x})^{2/3}+(b\bar{y})^{2/3}}{(a^2-b^2)^{2/3}} = 1$$

$$(a\bar{x})^{2/3} + (b\bar{y})^{2/3} = (a^2-b^2)^{2/3}$$

$\therefore$ The locus of centre of curvature is

$$(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}.$$

$(a-b)^2 = (b-a)^2$

$(a^2-b^2)^2 = (b^2-a^2)^2$

H.W.

Find the evolute of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$