Module 3 Functions of Several Variables.

Differentiation formulas.

1) $\frac{d}{dx}(a, \text{Constant}) = 0$

2) $\frac{d}{dx}(x^n) = n x^{n-1}$.

3) $\frac{d}{dx}(e^x) = e^x$.

4) $\frac{d}{dx}(e^{ax}) = e^{ax}.a$.

5) $\frac{d}{dx}(\sin x) = \cos x$.

6) $\frac{d}{dx}(\sin ax) = \cancel{\sin ax.(a)}$ $\cos ax.(a)$.
$= a\cos ax$

7) $\frac{d}{dx}(\log x) = \frac{1}{x}$. $\quad \| \quad \frac{d}{dx}(\log ax) = \frac{1}{ax}$.

8) $\frac{d}{dx}(\tan x) = \sec^2 x$.
$\frac{d}{dx}(\tan ax) = a.\sec^2 ax$

9) $\frac{d}{dx}(\sec x) = \sec x.\tan x$.

10) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

11) $\frac{d}{dx}(\cos x) = -\sin x$.
$\frac{d}{dx}(\cos ax) = -a\sin ax$

12) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x.\cot x$.
$\frac{d}{dx}(\operatorname{cosec} ax) = -a.\operatorname{cosec} ax.\cot ax$

Product rule of differentiation

$$\frac{d}{dx}(uv) = u.\frac{dv}{dx} + v.\frac{du}{dx}$$

Quotient rule of differentiation

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$$

Differentiate the following

1) $y = x^5$

$$\frac{dy}{dx} = 5x^{5-1} = 5x^4$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

2) $y = x$

$$\frac{dy}{dx} = 1$$

$y = x^1$

$\frac{dy}{dx} = 1x^{1-1} = 1x^0 =$

3) $y = \frac{1}{7x^2}$

Soln $y = \frac{1}{7}x^{-2}$

$$\frac{dy}{dx} = \frac{1}{7}(-2)x^{-2-1} = -\frac{2}{7}x^{-3} = -\frac{2}{7}$$

4) $y = 4x^2 - 3\cos x + e^x + 2\sin x$

$$\frac{dy}{dx} = 4.2x^{2-1} - 3(-\sin x) + e^x + 2(\cos x)$$

$$= 8x + 3\sin x + e^x + 2\cos x$$

5) $y = \cos 6x + \sin 4x$.

$$\frac{dy}{dx} = (-\sin 6x).6 + (\cos 4x).4.$$

$$= -6\sin 6x + 4\cos 4x.$$

6) $y = (x^2+5)^2$.

$$\frac{dy}{dx} = 2(x^2+5)^{2-1}.(2x+0)$$

$$= 4x\,(x^2+5)$$

7) $y = (3x^2+1)^3$.

$$\frac{dy}{dx} = 3(3x^2+1)^{3-1}.(6x+0)$$

$$= 18x\,(3x^2+1)^2 \;//.$$

8) $y = \frac{3}{\sqrt{x}}$.

Soln $y = \frac{3}{x^{1/2}} = 3x^{-1/2}$

$$\frac{dy}{dx} = 3.\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1} = -\frac{3}{2}x^{-\frac{3}{2}} = -\frac{3}{2\ldots}$$

9) $y = 3\underset{u}{x^4}\,\underset{v}{e^x}$.

$$\frac{dy}{dx} = 3\left[x^4.e^x + e^x.4x^3\right]$$

$$= 3e^x(x^4 + 4x^3)$$

$\frac{d}{dx}(uv) = u.\frac{dv}{dx}$ …

$v.\frac{d}{}$…

$\frac{d}{dx}(x^4)$ …

$4x$ …

$= 4$…

10) $y = \underbrace{(x^2+1)}_{u}\underbrace{(x^2+2)}_{v}$

$$\frac{dy}{dx} = (x^2+1)(2x) + (x^2+2)\,2x.$$

$$= 2x(x^2+1+x^2+2)$$

$$= 2x(2x^2+3)$$

11) $y = \underbrace{\cos 2x}_{u} \cdot \underbrace{e^{3x}}_{v}$.

$$\frac{dy}{dx} = \cos 2x \cdot \frac{d}{dx}(e^{3x}) + e^{3x} \cdot \frac{d}{dx}(\cos 2x)$$

$$= \cos 2x \cdot e^{3x} \cdot 3 + e^{3x}(-\sin 2x) \cdot 2.$$

$$= e^{3x}(3\cos 2x - 2\sin 2x)$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

12) $y = \frac{x^3 \,(u)}{3x-2 \,(v)}$

$$\frac{dy}{dx} = \frac{(3x-2) \cdot \frac{d}{dx}(x^3) - x^3 \cdot \frac{d}{dx}(3x-2)}{(3x-2)^2}$$

$$= \frac{(3x-2) \cdot 3x^2 - x^3 \cdot 3.}{(3x-2)^2}$$

$$= \frac{9x^3 - 6x^2 - 3x^3}{(3x-2)^2} = \frac{6x^3 - 6x^2}{(3x-2)^2} = \frac{6x^2(\ldots}{(3x\ldots}$$

13) $y = \frac{2x-3 \,(u)}{4x+5 \,(v)}$

$$\frac{dy}{dx} = \frac{\left\{(4x+5) \cdot \frac{d}{dx}(2x-3)\right\} - \left\{(2x-3)\frac{d}{dx}(4x+5)\right\}}{(4x+5)^2}$$

$$= \frac{(4x+5) \cdot 2 - (2x-3) \cdot 4}{(4x+5)^2}$$

$$= \frac{(\cancel{8x}+10) - (\cancel{8x}-12)}{(4x+5)^2} = \frac{10-(-12)}{(4x+5)^2} = \frac{22}{(4x\ldots}$$

Partial Differentiation

Sup. $z \to f(x, y)$

z ↓ dependent variable; x, y independent variables.

Partial derivatives $\to \dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y} \to$ first order partial derivatives.

$\dfrac{\partial^2 z}{\partial x^2}, \dfrac{\partial^2 z}{\partial y^2}, \dfrac{\partial^2 z}{\partial x \partial y} \to$ 2nd order partial derivatives.

Examples

1) $u = 2x + 3y$

$$\frac{\partial u}{\partial x} = u_x = 2$$

$$\frac{\partial^2 u}{\partial x^2} = 0 = u_{xx}$$

$$\frac{\partial u}{\partial y} = 3 = u_y$$

$$\frac{\partial^2 u}{\partial y^2} = u_{yy} = 0$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \text{ or } \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 0$$

2) $u = e^{2x-3y}$ find $u_x, u_y, u_{xx}, u_{yy}, u_{xy}$.

$u = e^{2x} \cdot e^{-3y}$.

$u_x = 2e^{2x} \cdot e^{-3y}$ or $2e^{2x-3y}$

$u_y = e^{2x} \cdot e^{-3y} \cdot (-3) = -3 \cdot e^{2x-3y}$

$u_{xx} = \frac{\partial}{\partial x}(u_x) = 2e^{-3y} \cdot e^{2x} \cdot 2 = 4e^{2x-3y}$

$u_{yy} = -3e^{2x} \cdot e^{-3y} \cdot (-3) = 9e^{2x-3y}$.

$u_{xy} = 2e^{2x}(e^{-3y} \cdot (-3)) = -6e^{2x-3y}$

$u_x \to \frac{\partial u}{\partial x}$, $u_y \to \frac{\partial u}{\partial y}$, $u_{xx} \to \frac{\partial^2 u}{\partial x^2}$

$u_{yy} \to \frac{\partial^2 u}{\partial y^2}$, $u_{xy} \to \frac{\partial^2 u}{\partial x \partial y}$.

$e^{m-n} = e^m \cdot e^{-n}$

3) $z = e^{x+4y}$ find z_x, z_y, z_{xx}, z_{yy} & z_{xy}.

4) $z = \sin(2x+3y)$ find $z_x, z_y, z_{xx}, z_{yy}, z_{xy}$.

H.W

1) $u = \cos(2x-3y)$

2) $u = x^2y + xy^2 - xy$,

find u_x, u_y, u_{xx}, u_{yy} & u_{x}

$z_x = \frac{\partial z}{\partial x}$ $z = e^x \cdot e^{4y}$ $\frac{\partial z}{\partial x} = e^{4y} \cdot e$

$= e$

H.W Answers.

1) $u = \cos(2x - 3y)$

$u_x = \dfrac{\partial u}{\partial x} = -\sin(2x-3y) \cdot 2$

$= -2\sin(2x-3y)$

$u_y = \dfrac{\partial u}{\partial y} = -\sin(2x-3y)(-3)$

$= 3\sin(2x-3y)$

$u_{xy} = \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right)$ or $\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)$

$= 3\cos(2x-3y) \cdot 2$

$= 6\cos(2x-3y)$

$u_{xx} = \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = -2\cos(2x-3y) \cdot 2$

$= -4\cos(2x-3y)$

$u_{yy} = \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right) = 3 \cdot \cos(2x-3y) \cdot (-3)$

$= -9\cos(2x-3y)$

2) $u = x^2y + xy^2 - xy$.

$u_x = 2xy + y^2 - y$.

$u_y = \dfrac{\partial u}{\partial y} = x^2 + x \cdot 2y - x \cdot 1 = x^2 + 2xy - x$

$u_{xy} = \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right)$ or $\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)$

$= 2x + 2y - 1$

$u_{xx} = \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right)$

$= 2y$.

$u_{yy} = \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(x^2 + 2xy - x\right)$

$= 2x$