Fourier Series of Odd & even functions

We can check whether the fn $f(x)$ is odd or even only in the interval $(-\pi, \pi)$ or $(-l, l)$

* When the fn $f(x)$ is odd, in the fourier series, the Euler constants $a_0 = a_n = 0$.
* When the fn $f(x)$ is even, in the fourier series, the Euler constant $b_n = 0$

Fourier Series of odd fn in $(-\pi, \pi)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad (\because a_0 = a_n = 0 \text{ for odd fn})$$

Where $b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x) \sin nx \, dx$. (?)

$$\left(\because b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\sin nx}_{\text{odd} \times \text{odd} = \text{even}} dx\right)$$

$$\therefore \int_{-\pi}^{\pi} = 2\int_0^{\pi}$$

Fourier Series of odd fn in $(-l, l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}.$$

Where $b_n = \dfrac{2}{l}\displaystyle\int_0^{l} f(x) \cdot \sin \frac{n\pi x}{l} \, dx.$

1) Find the fourier Series of $f(x) = x$ in $(-l, l)$

Soln

$f(x) = x$ is an odd fn $\left(\because f(-x) = -x = -f(x) \right)$

$\therefore f(x) = x$ is odd.

$\therefore$ In the Fourier series, $a_0 = a_n = 0$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \, dx \quad \text{———} \quad (1)$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx.$$

$$= \frac{2}{l} \int_0^l x \sin \frac{n\pi x}{l} \, dx.$$

Applying Bernoulli's formula,

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots\cdots$$

$u = x$ $\qquad v = \sin\frac{n\pi x}{l}$.

$v_1 = -\frac{l}{n\pi}\cdot\cos\frac{n\pi x}{l}$.

$u' = 1$ $\quad (-)$

$u'' = 0$ $\qquad v_2 = -\frac{l^2}{n^2\pi^2}\cdot\sin\frac{n\pi x}{l}$.

$\sin n\pi = 0$

$\sin 0 = 0$

$\cos n\pi = (-1)^n$

$$b_n = \frac{2}{l}\left\{-\frac{lx}{n\pi}\cdot\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}\cancel{\sin\frac{n\pi x}{l}}\right\}_0^l .$$

$\cancel{\sin\frac{n\pi x}{l}} \to 0$

$$= \frac{2}{\cancel{l}}\cdot\left(-\frac{l^{\cancel{2}}}{n\pi}\cos\frac{n\pi\cancel{l}}{\cancel{l}}\right) - 0$$

$$= \cancel{2l}\; b_n = -\frac{2l}{n\pi}\cdot(-1)^n .$$

$$\boxed{b_n = \frac{-2l}{n\pi}(-1)^n}$$

$((-1)^n = -1$, n is odd.
$= 1$ if n is even

$= \frac{-2l}{n\pi}(-1)$

$= +\frac{2l}{n\pi}$ if n is odd

$= -\frac{2l}{n\pi}$, if n is even

Sub b_n in (1)

The reqd fourier Series is

$$f(x) = \sum_{n=1}^{\infty} \frac{-2l}{n\pi}(-1)^n \cdot \sin\frac{n\pi x}{l}.$$

2) Find the fourier series of $f(x) = x$ in $(-\pi, \pi)$

Soln

$f(x) = x$ is an odd fn

$\therefore$ In the fourier Series, the Euler Constant ~~bn~~.

$$a_0 = a_n = 0$$

$\therefore$ The fourier series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{——— (1)}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx.$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx,$$

$=$

Applying Bernoulli's formula,

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \dots\dots$$

$u = x \qquad v = \sin nx$

$u' = 1 \qquad v_1 = -\frac{\cos nx}{n}$

$u'' = 0 \qquad (-) \qquad v_2 = -\frac{\sin nx}{n^2}$

Note

$\sin n\pi = 0$

$\cos n\pi = (-1)^n$

$\sin 0 = 0$

$\cos 0 = 1$

$$b_n = \frac{2}{\pi}\left[-\frac{x}{n}\cos nx + \cancel{\frac{\sin nx}{n^2}}\right]_0^\pi$$

$$b_n = \frac{2}{\cancel{\pi}}\left\{-\frac{\cancel{\pi}}{n}\cdot\cos n\pi - 0\right\}$$

$b_n = -\frac{2}{n}\cdot(-1)^n$. Sub in 1

$$\boxed{f(x) = \sum_{n=1}^{\infty} -\frac{2}{n}(-1)^n\cdot\sin nx}.$$

3) Find the F.S of $f(x) = x^2$ in $(-l, l)$

Soln $f(x) = x^2$ is an even fn $\left[\because f(-x) = (-x)^2 = x^2 = f(x) \quad \therefore f(x) = x^2 \text{ is even}\right]$

$\therefore$ In the F.S, the Euler Constant $b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad \text{——— (1)}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{l}\int_0^l x^2\,dx = \frac{2}{l}\left(\frac{x^3}{3}\right)_0^l$$

$$= \frac{2}{l}\cdot\frac{l^3}{3}$$

$$\boxed{a_0 = \frac{2l^2}{3}}$$

$$a_n = \frac{2}{l}\int_0^l f(x) \cos\frac{n\pi x}{l}\,dx.$$

$$= \frac{2}{l}\int_0^l x^2 \cos\frac{n\pi x}{l}\,dx$$; Applying Ber. formula,

$u = x^2$ $\quad$ $v = \cos\frac{n\pi x}{l}$.

$u' = 2x$ $\quad$ $v_1 = \dfrac{\sin\frac{n\pi x}{l}}{n\pi/l} = \frac{l}{n\pi}\cdot\sin\frac{n\pi x}{l}$.

$u'' = 2$ $\quad$ $(-)$ $\quad$ $v_2 = -\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}$,

$u''' = 0$ $\quad$ $v_3 = -\frac{l^3}{n^3\pi^3}\cdot\sin\frac{n\pi x}{l}$,

$$a_n = \frac{2}{l}\left\{ x^2 \cdot \frac{l}{n\pi} \cdot \cancel{\sin\frac{n\pi x}{l}}_{0} + \frac{2x\, l^2}{n^2\pi^2}\cos\frac{n\pi x}{l} - \cancel{\frac{2 l^3}{n^3\pi^3}\sin\frac{n\pi x}{l}}_{0} \right\}_0^l$$

$$= \frac{2}{l}\left\{ \frac{2l^3}{n^2\pi^2}\cdot \cos n\pi - 0 \right\}$$

$$a_n = \frac{4l^2}{n^2\pi^2}(-1)^n.$$

Sub a_0, a_n in (1)

The reqd Fourier Series is

$$f(x) = \frac{2l^2/3}{2} + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2}(-1)^n \cdot \cos\frac{n\pi x}{l}$$

ie $$f(x) = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2}(-1)^n \cdot \cos\frac{n\pi x}{l}.$$

4) Find the F.S of $f(x) = x^2$ in $(-\pi, \pi)$

Soln $f(x) = x^2$ is even fn.

$\therefore$ In the fourier series $b_n = 0$

$\therefore$ The fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{——— (1)}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx.$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\cdot\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{3\pi}(\pi^3 - 0)$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

$$\cancel{a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx}.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx,$$ Applying Bernoulli's formula,

$u = x^2 \qquad v = \cos nx$

$u' = 2x \qquad v_1 = \frac{\sin nx}{n}$

$u'' = 2 \qquad v_2 = -\frac{\cos nx}{n^2}$

$u''' = 0 \qquad v_3 = -\frac{\sin nx}{n^3}$

$$a_n = \frac{2}{\pi}\left\{\cancelto{0}{\frac{x^2\sin nx}{n}} + \frac{2x\cos nx}{n^2} - \cancelto{0}{\frac{2\sin nx}{n^3}}\right\}_{x=0}^{\pi}$$

$(\because \sin n\pi = 0,\ \sin 0 = 0)$

$$= \frac{2}{\cancel{\pi}}\left(\frac{2\cancel{\pi}\cos n\pi}{n^2} - 0\right)$$

$\sin n\pi = 0 \ \forall n$

$\cos n\pi = (-1)^n$.

$$\boxed{a_n = \frac{4}{n^2}(-1)^n}.$$

Sub a_0, a_n in (1)

The reqd fourier Series is

$$f(x) = \frac{\frac{2\pi^2}{3}}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx.$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx.$$