Differential Geometry

curvature is the amount by which a curve deviates from being a straight line, or a surface deviates from being a plane. For curves, the canonical example is that of a circle, which has a **curvature** equal to the reciprocal of its radius. Smaller circles bend more sharply, and hence have higher **curvature**.

Curvature is denoted by the symbol κ. (Kappa)

What is curvature?

- Refers to how much a geometric object deviates from being "flat" or "straight"
- The measure of the amount of curving
- The degree by which a non-linear or surface curves

Radius of Curvature

The reciprocal of the curvature of a curve at any point 'P' is called the radius of curvature, denoted by ρ.

Centre of Curvature

A point C on the normal at any point P of a curve, distance ρ from it is called centre of curvature.

Circle of Curvature

A circle with centre C (centre of curvature at P) and radius ρ is called the circle of curvature or osculating circle at P.

Note

* The radius of curvature of a circle is its radius.
* curvature of a straight line is zero. and radius of curvature of a st. line is '∞'.

* The locus of centre of curvature for a curve is called its evolute and the curve is called an involute of its evolute.

* Centre of curvature $(\bar{x}, \bar{y})$ at any point $P(x, y)$ is

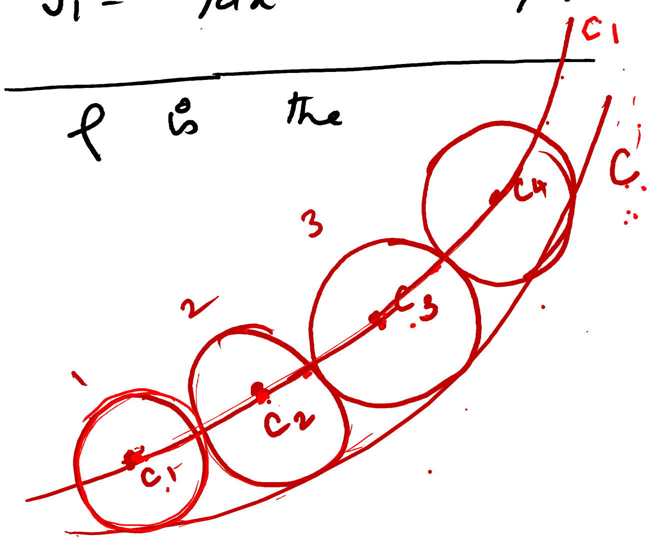
$$\bar{x} = x - \frac{y_1}{y_2}\left(1 + y_1^2\right) \; ; \; \bar{y} = y + \left(\frac{1 + y_1^2}{y_2}\right),$$

where $y_1 = dy/dx$ & $y_2 = d^2y/dx^2$

* Equation of circle of curvature is

$$(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2,$$

where ρ is the radius of curvature, $\rho = \dfrac{(1 + y_1^2)^{3/2}}{y_2}$.



Problem 1 Find the co-ordinates of the centre of curvature at any point of the parabola $y^2 = 4ax$ and hence show that its evolute is $27ay^2 = 4(x-2a)^3$.

Soln Given $y^2 = 4ax$. ——— (1).

Differentiating (1) w.r.to x.

$$2y.y_1 = 4a \Rightarrow y_1 = \frac{4a}{2y} = \frac{2a}{y}.$$

(2)

Diff (2) w.r.to x.

$$2(y.y_2 + y_1 y_1) = 0$$

$$\Rightarrow y y_2 + y_1^2 = 0$$

ie $y y_2 = -y_1^2$

$y^2 = \boxed{4ax}$

Diff w.r.to x

$2y \frac{dy}{dx} = 4a.1$

$2y.y_1 = 4a$

$y_1 = \frac{2\cancel{4}a}{\cancel{2}y} = \frac{2a}{y}$

Diff again w.r.to x

$$y_2 = \frac{y.0 - 2a.y_1}{y^2}$$

$$y_2 = -\frac{2ay_1}{y^2}$$

$$\Rightarrow y_2 = \frac{-y_1^2}{y} = -\frac{\left(\frac{2a}{y}\right)^2}{y}$$

$$= -\frac{4a^2}{y^2} \times \frac{1}{y}.$$

$$\text{i.e.} \quad \boxed{y_2 = -\frac{4a^2}{y^3}}$$

$$-\frac{4a^2}{y^2 \cdot y} = -\frac{\cancel{4}a^{\cancel{2}}}{\cancel{4}ax \cdot y}$$

$$y_2 = \frac{-2ay_1}{y^2}$$

$$= \frac{-2a \cdot \frac{2a}{y}}{4ax}$$

$$= -\frac{\cancel{4}a^{\cancel{2}}}{y} \times \frac{1}{\cancel{4}ax}$$

$$= -\frac{a}{xy}$$

Now

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2) = x - \frac{(2a/y)}{\left(-\frac{4a^2}{y^3}\right)}\left(1+\frac{4a^2}{y^2}\right)$$

$$= x + \frac{\cancel{2}a}{\cancel{y}} \cdot \frac{y^{\cancel{3}\,2}}{2\cancel{4}a^{\cancel{2}}}\left(1+\frac{4a^2}{y^2}\right)$$

$$= x + \frac{\cancel{y^2}}{2a}\left(\frac{y^2+4a^2}{\cancel{y^2}}\right) = x + \frac{1}{2a}(y^2+4a^2)$$

$$= x + \left(\frac{y^2 + 4a^2}{2a}\right)$$

$$= x + \left(\frac{4ax + 4a^2}{2a}\right)$$

$$= x + 2x + 2a \quad (3)$$

$$\boxed{\bar{x} = 3x + 2a}$$

$$\bar{y} = y + \left(\frac{1 + y_1^2}{y_2}\right)$$

$$= y + \left\{\frac{1 + (2a/y)^2}{-4a^2/y^3}\right\}$$

$$= y - \left\{\frac{1 + \frac{4a^2}{y^2}}{4a^2/y^3}\right\}$$

$$\left[\because y^2 = 4ax\right)$$

$$\frac{4ax}{2a} = 2x$$

$$\frac{2\cdot 4a}{4} = 2a$$

$$= y - \frac{y^3}{4a^2}\left(\frac{y^2 + 4a^2}{y^2}\right)$$

$$= \cancel{y} - \frac{y^3}{4a^2} - \frac{\cancel{y} \cdot 4a^2}{4a^2}$$

$$\bar{y} = -\frac{y^3}{4a^2} = -\frac{y^2 \cdot y}{4a^2}$$

$$\bar{y} = -\frac{4ax}{4a^2} \cdot y$$

$$y^2 = 4ax,$$
$$y = 2\sqrt{ax}$$

$$\bar{y} = -\frac{x \cdot 2\sqrt{ax}}{a} = -\frac{2x^{3/2} \cdot a^{1/2}}{a}$$

$$\boxed{\bar{y} = -\frac{2x^{3/2}}{\sqrt{a}}}$$

$$\sqrt{ax} = a^{1/2} \cdot x^{1/2}$$

$$\frac{a^{1/2}}{a} = \frac{1}{a^{1-\frac{1}{2}}}$$

To find evolute, we have to eliminate x & y between (3) & (4)

From (4), $\bar{y} = \dfrac{-2x^{3/2}}{\sqrt{a}}$

From (3), $\bar{x} = 3x + 2a$

$\therefore 3x = \bar{x} - 2a$

$x = \dfrac{\bar{x} - 2a}{3}$

Squaring, $\bar{y}^2 = \left(\dfrac{-2x^{3/2}}{\sqrt{a}}\right)^2 = \dfrac{4x^3}{a}$,

$\left(x^{3/2}\right)^2 = x^{3/2 \times 2} = x^3$

ie $\bar{y}^2 = \dfrac{4}{a}\left(\dfrac{\bar{x} - 2a}{3}\right)^3$

ie $\bar{y}^2 = \dfrac{4}{a} \cdot \dfrac{(\bar{x} - 2a)^3}{27}$

ie $27a\bar{y}^2 = 4(\bar{x} - 2a)^3$

The locus of $\bar{x}, \bar{y}$ is $27ay^2 = 4(x - 2a)^3$

Hence the evolute is $\underline{\underline{27ay^2 = 4(x - 2a)^3}}$.

2) Find the evolute of the parabola $x^2 = 4ay$.

Soln. Given $x^2 = 4ay$

$$\Rightarrow y = \frac{x^2}{4a}$$

diff y w.r.t x

$$y_1 = \frac{1}{4a}\cdot 2x = \frac{x}{2a}.$$

$$y_2 = \frac{1}{2a}\cdot(1) = \frac{1}{2a}.$$

$$\bar{x} = x - \frac{y_1}{y_2}\left(1+y_1^2\right)$$

$$= x - \frac{(x/2a)}{(1/2a)}\cdot\left(1+\frac{x^2}{4a^2}\right).$$

$$= \cancel{x} - \cancel{x} - \frac{x^3}{4a^2}$$

$$\boxed{\begin{aligned} x^2 &= 4ay \\ x &= \sqrt{4ay} \\ &= 2\sqrt{ay} \end{aligned}}$$

$$\bar{x} = -\frac{x^2\cdot x}{4a^2} = -\frac{\cancel{4}a\cancel{y}\cdot 2\sqrt{ay}}{\cancel{4}a^{\cancel{2}}}$$

$$= -\frac{2y\cdot a^{1/2}\cdot y^{1/2}}{a^1}$$

$$\boxed{\bar{x} = -\frac{2y^{3/2}}{a^{1/2}}} \quad \text{— } (1)$$

$$\bar{y} = y + \frac{1}{y_2}\left(1+y_1^2\right)$$

$$\bar{y} = y + 1\Big/\frac{1}{2a}\left(1+\frac{x^2}{4a^2}\right)$$

$$= y + 2a\left(1+\frac{\cancel{4}ay}{\cancel{4}a^{\cancel{2}}}\right)$$

$$= y + 2\cancel{a}\left(\frac{a+y}{\cancel{a}}\right)$$

$$= y + 2a + 2y$$

$$\boxed{\bar{y} = 3y + 2a} \quad \text{— } (2)$$

(1) $\Rightarrow$ $\bar{x} = \dfrac{-2y^{3/2}}{\sqrt{a}}$ $\Longrightarrow$ Squaring $(\bar{x})^2 = \left(\dfrac{-2y^{3/2}}{\sqrt{a}}\right)^2$

$\bar{x}^2 = \dfrac{4y^3}{a}$ — (3)

(2) $\Rightarrow$ $\bar{y} = 3y + 2a$

$\Rightarrow$ $3y = \bar{y} - 2a$

$\Rightarrow$ $y = \dfrac{\bar{y} - 2a}{3}$ — (4)

Sub (3) in (4) $\Longrightarrow$

$\bar{x}^2 = \dfrac{4}{a}\left(\dfrac{\bar{y} - 2a}{3}\right)^3$

$\bar{x}^2 = \dfrac{4}{a}\dfrac{(\bar{y} - 2a)^3}{27}$

$\Rightarrow$ $27a\bar{x}^2 = 4(\bar{y} - 2a)^3$

$\therefore$ locus of $\bar{x}, \bar{y}$ is $27ax^2 = 4(y - 2a)^3$, which is the evolute of the given curve.

Curve	Equation	Parametric form
Parabola	$y^2 = 4ax$	$x = at^2, \quad y = 2at$
Parabola	$x^2 = 4ay$	$x = 2at, \quad y = at^2$
ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$x = a\cos\theta, \quad y = b\sin\theta$
hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$x = a\sec\theta, \quad y = a\tan\theta$
Circle	$x^2 + y^2 = a^2$	$x = a\cos\theta, \quad y = a\sin\theta$

When the parametric form of the curve is given,

$$y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \text{ (or) } \frac{dy}{dt} \times \frac{dt}{dx} \text{ (or) } \frac{dy/d\theta}{dx/d\theta}.$$

$x = f(t)$
$y = f(t)$
$x = f(\theta)$
$y = f(\theta)$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx} \text{ (or) } \frac{d}{dt}(y_1) \cdot \frac{dt}{dx}$$

(or)

$$= \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \cdot \frac{d\theta}{dx}.$$

Another method

Find the evolute of $x^2 = 4ay$.

soln The parametric form is $x = 2at$; $y = at^2$

$$x = 2at \quad ; \quad y = at^2$$

$$\frac{dx}{dt} = 2a \quad ; \quad \frac{dy}{dt} = 2at$$

$$\therefore \quad y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2at}{2a} = t$$

$$y_2 = \frac{d}{dt}(y_1) \cdot \frac{dt}{dx} = \frac{d}{dt}(t) \cdot \frac{1}{2a} = \frac{1}{2a}$$

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$$

$$= 2at - \frac{t}{1/2a}(1+t^2)$$

$$= 2at - 2at(1+t^2)$$

$$= \cancel{2at} - \cancel{2at} - 2at^3$$

$$\bar{x} = -2at^3$$

$$\Rightarrow \boxed{t^3 = -\frac{\bar{x}}{2a}} \quad \text{— (1)}$$

$$\bar{y} = y + \frac{1}{y_2}(1+y_1^2)$$

$$= at^2 + \frac{1}{1/2a}(1+t^2)$$

$$= at^2 + 2a(1+t^2)$$

$$= at^2 + 2a + 2at^2$$

$$\bar{y} = 3at^2 + 2a$$

$$3at^2 = \bar{y} - 2a.$$

$$\Rightarrow t^2 = \frac{\bar{y} - 2a}{3a} \quad \text{— (2)}$$

Eliminating t from ① & ② will give the eqn of the evolute.

From ① $t^3 = -\dfrac{\bar{x}}{2a}$; from ②, $t^2 = \dfrac{\bar{y} - 2a}{3a}$.

Squaring $t^6 = \left(\dfrac{\bar{x}}{2a}\right)^2$ — ③ ; cubing $t^6 = \left(\dfrac{\bar{y} - 2a}{3a}\right)^3$ — ④

From (3) & (4)

$$\left(\frac{\bar{x}}{2a}\right)^2 = \left(\frac{\bar{y} - 2a}{3a}\right)^3 = \frac{\bar{x}^2}{4\cancel{a^2}} = \frac{(\bar{y} - 2a)^3}{27a^{\cancel{3}}}$$

$$\Rightarrow 4(\bar{y} - 2a)^3 = 27a\bar{x}^2$$

∴ locus of $(\bar{x}, \bar{y})$ is $4(y - 2a)^3 = 27ax^2$, which is the required evolute