One dimensional heat flow.

Empirical laws:

(i) Heat flows from higher to lower temperature.

(ii) The amount of heat required to produce a given temperature change in a body is proportional to the mass of the body and to the temperature change. This constant of proportionality is known as specific heat (s) of the conducting material.

(iii) The rate at which heat flows through an area is proportional to the area and to the temperature gradient normal to the area. This constant of proportionality is known as the thermal conductivity (K) of the material.

For one-dimensional heat flow, we consider a bar or rod of homogeneous material of density ρ (gr/cm^3) & having a constant cross sectional area A (cm^2). & the sides of the bar are <u>insulated</u>. → no heat inflow or out flow through the sids.

One dimensional heat flow -

The 1-D heat flow equation is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad \text{where} \quad a^2 = \frac{K}{\rho s}.$$

ρ is the density of the substance.

K " " thermal conductivity

s " " specific heat capacity -

$a^2 = \frac{\rho}{Ks}$ is called diffusivity (cm^2/sec) of the substance.

The three possible solutions of 1-D heat eqn are

(i) $u(x,t) = \left(c_1 e^{px} + c_2 e^{-px}\right) c_3 e^{a^2 p^2 t}$

(ii) $u(x,t) = \left(c_4 \cos px + c_5 \sin px\right) c_6 e^{-a^2 p^2 t}$

(iii) $u(x,t) = \left(c_7 x + c_8\right) c_9$.

Of these three solns, the most suitable soln, which suits the physical nature of the problem i.e. $u(x,t)$, (temperature) decreases as increase of time, is soln (ii)

i.e. $\boxed{u(x,t) = (A\cos px + B\sin px)\, e^{-a^2p^2t}}$ → only suitable soln of 1-D heat eqn.

$u(x,t)$ → temperature 'u' at any point x, at any time t.

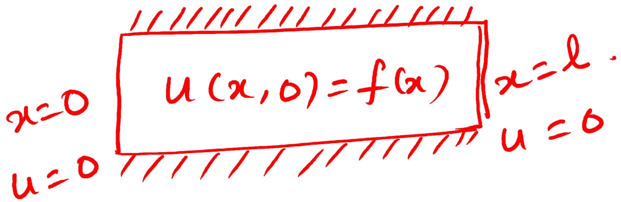
$u(0,t)$ → temperature at the end $x=0$.

$u(l,t)$ → " " $x=l$.

$u(x,0)$ → temperature at $t=0$ (initial temperature).

To solve 1-D heat eqn, 3 boundary conditions are needed.

Type 1 Both ends of the rod are kept at zero temperature, lateral surface area of the rod is insulated and the initial temperature $u(x,0)$ is given.



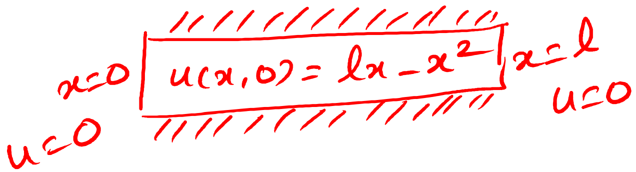
The boundary conditions are

(i) $u(0,t)=0$
(ii) $u(l,t)=0$ $\Big\}$ $t \geq 0$.

(iii) $u(x,0)=f(x)$, $0 \leq x \leq l$.

Pbm A rod l cm with insulated lateral surface is initially at temperature $u = lx - x^2$, at an inner point distant x cm from one end. If both the ends are kept at zero temperature, find the temperature at any point of the rod at any subsequent time.

Soln The eqn of 1-D heat flow is



$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

The boundary conditions are

$$\left.\begin{array}{ll} \text{(i)} & u(0,t) = 0 \\ \text{(ii)} & u(l,t) = 0 \end{array}\right\} t \geq 0.$$

(iii) $u(x,0) = lx - x^2$, $0 \leq x \leq l$.

The suitable soln of 1-D heat eqn is

$$u(x,t) = (A\cos px + B\sin px)\, e^{-a^2p^2t} \quad \text{——— (1)}$$

Sub b.c (i) in (1) ie put $x=0$ in (1)

$$0 = (A.\cos 0 + B\sin 0)\, e^{-a^2p^2t} \qquad \left(\because \cos 0 = 1,\ \sin 0 = 0.\right.$$

$$0 = A\left(e^{-a^2p^2t}\right) \Rightarrow \boxed{A=0}$$

sub in (1)

$$u(x,t) = B\sin px \,.\, e^{-\alpha^2p^2t} \quad \text{——— (2).}$$

Sub b.c (ii) in (2) ie put $x=l$ in (2)

$$0 = B\sin pl \,.\, e^{-\alpha^2p^2t}$$

$$\Rightarrow \sin pl = 0 = \sin n\pi \qquad (\because B \text{ Can not be zero})$$

$\therefore \sin pl = \sin n\pi \Rightarrow pl = n\pi \Rightarrow p = \dfrac{n\pi}{l}$. Sub in (2)

$$\therefore u(x,t) = B \sin \frac{n\pi x}{l} \cdot e^{-\frac{a^2 n^2 \pi^2}{l^2} t}$$

Taking $B = B_n$, the most general soln is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cdot e^{-\frac{a^2 n^2 \pi^2 t}{l}} \quad \text{— (3)}.$$

Sub b.c (iii) in (3) i.e. put $t = 0$ in (3).

$$lx - x^2 = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}, \quad (\because e^0 = 1)$$

Half range Fourier sine series, $B_n = \dfrac{2}{l} \displaystyle\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$.

$$\therefore B_n = \frac{2}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,dx.$$

Applying Bernoulli's formula, $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots\cdots$

$u = lx - x^2, \quad v = \sin\frac{n\pi x}{l}.$

$u' = l - 2x \quad v_1 = -\frac{l}{n\pi}\cdot\cos\frac{n\pi x}{l}.$

$u'' = -2 \quad v_2 = -\frac{l^2}{n^2\pi^2}\cdot\sin\frac{n\pi x}{l}$

$u''' = 0 \quad v_3 = \frac{l^3}{n^3\pi^3}\cdot\cos\frac{n\pi x}{l}.$

$$\therefore B_n = \frac{2}{l}\left\{(lx - x^2)\left(-\frac{l}{n\pi}\cdot\cos\frac{n\pi x}{l}\right) + (l-2x)\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l} - \frac{2l^3}{n^3\pi^3}\cdot\cos\frac{n\pi x}{l}\right\}_{x=0}^{l}.$$

$$= \frac{2}{l}\left\{\left(0 - 0 - \frac{2l^3}{n^3\pi^3}(-1)^n\right) - \left(0 - 0 - \frac{2l^3}{n^3\pi^3}(1)\right)\right\}$$

$$= \frac{2}{l}\cdot\frac{2l^3}{n^3\pi^3}\left(+1 - (-1)^n\right)$$

$$= \frac{4l^{2}}{n^3\pi^3}\left(\cancel{(-1)^n - 1}\right) \quad \frac{4l^2}{n^3\pi^3}\left(1 - (-1)^n\right)$$

$$= +\frac{8l^2}{n^3\pi^3} \text{ if } n \text{ is odd}$$

$$= 0 \text{ if } n \text{ is odd.}$$

$(-1)^n = 1$ if n is even

$\therefore (1 - (-1)^n) = 1 - 1 = 0$ if n is even.

$(-1)^n = -1$ if n is odd

$\therefore 1 - (-1)^n = 1 - (-1) = 2$ if n is odd.

$\therefore$ The soln is

$$u(x,t) = \sum_{n=1,3,5\cdots}^{\infty} +\frac{8l^2}{n^3\pi^3}\cdot \sin\frac{n\pi x}{l}\cdot e^{-\frac{a^2n^2\pi^2t}{l^2}}$$