Linear Equation and Solutions

A *linear equation* in unknowns $x_1, x_2, \ldots, x_n$ is an equation that can be put in the *standard form*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \qquad (1)$$

where $a_1, a_2, \ldots, a_n$, and b are constants. The constant a_k is called the *coefficient* of x_k, and b is called the *constant term* of the equation.

A solution of the linear equation (1) is a list of values for the unknowns or, equivalently, a vector u in K^n, say

$$x_1 = k_1, \quad x_2 = k_2, \quad \ldots, \quad x_n = k_n \qquad \text{or} \qquad u = (k_1, k_2, \ldots, k_n)$$

such that the following statement (obtained by substituting k_i for x_i in the equation) is true:

$$a_1k_1 + a_2k_2 + \cdots + a_nk_n = b$$

In such a case we say that u *satisfies* the equation.

Linear Systems of Equations

A linear system of m equations in n unknowns $x_1, x_2, \ldots, x_n$ is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \qquad (1)$$

where for $1 \leq i \leq n$, and $1 \leq j \leq m$; $a_{ij}, b_i \in \mathbb{R}$. Linear System (1) is called HOMOGENEOUS if $b_1 = 0 = b_2 = \cdots = b_m$ and NON-HOMOGENEOUS otherwise.

We rewrite the above equations in the form $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The matrix A is called the COEFFICIENT matrix and the block matrix $[A \ \ \mathbf{b}]$, is the AUGMENTED matrix of the linear system (1).

Example The linear system of equations $2x + 3y = 5$ and $3x + 2y = 5$ can be identified with the matrix $\begin{bmatrix} 2 & 3 & : & 5 \\ 3 & 2 & : & 5 \end{bmatrix}$.

Remarks:

1. Suppose $a, b \in \mathbb{R}$. Consider the system $ax = b$.

   (a) If $a \neq 0$ then the system has a UNIQUE SOLUTION $x = \frac{b}{a}$.

   (b) If $a = 0$ and

      i. $b \neq 0$ then the system has NO SOLUTION.

      ii. $b = 0$ then the system has INFINITE NUMBER OF SOLUTIONS, namely all $x \in \mathbb{R}$.

2. We now consider a system with 2 equations in 2 unknowns.

(a) Unique Solution

$x + 2y = 1$ and $x + 3y = 1$. The unique solution is $(x, y)^t = (1, 0)^t$.

(b) Infinite Number of Solutions

$x + 2y = 1$ and $2x + 4y = 2$.

The set of solutions is $(x, y)^t = (1 - 2y, y)^t = (1, 0)^t + y(-2, 1)^t$ with y arbitrary.

(c) No Solution

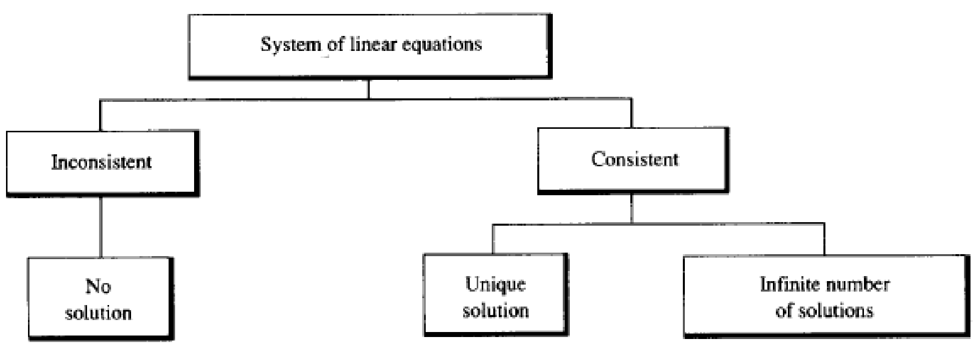
$x + 2u = 1$ and $2x + 4u = 3$.

The equations represent a pair of parallel lines and hence there is no point of intersection.

Definition (Consistent, Inconsistent) A linear system is called CONSISTENT if it admits a solution and is called INCONSISTENT if it admits no solution.

Remark

The system of linear equations has (i) a unique solution, (ii) no solution, or (iii) an infinite number of solutions. This situation is pictured



Examples:

1. Consider the system of linear equations: $x + y + 2z = 1$, $y - 3z = -3$, $5z = 10$

The system is consistent and has a solution. The solution $x = -6$, $y = 3$, $z = 2$.

2. Consider the system of linear equations: $x + 2y + z = 2$, $2y - 2z = 1$, $0 = 3$

The system is inconsistent and so has no solution.

Definition (Matrix) A rectangular array of numbers is called a matrix.

We shall mostly be concerned with matrices having real numbers as entries.

The horizontal arrays of a matrix are called its ROWS and the vertical arrays are called its COLUMNS.

A matrix having m rows and n columns is said to have the order $m \times n$.

A matrix A of ORDER $m \times n$ can be represented in the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

where a_{ij} is the entry at the intersection of the i^{th} row and j^{th} column.

In a more concise manner, we also denote the matrix A by $[a_{ij}]$ by suppressing its order.

Example

Let $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 5 & 6 \end{bmatrix}$. Then $a_{11} = 1,\ a_{12} = 3,\ a_{13} = 7,\ a_{21} = 4,\ a_{22} = 5,$ and $a_{23} = 6.$

Row Echelon Form:

Row Echelon form is any matrix with the following properties:

i. All zero rows (if any) belong at the bottom of the matrix.
ii. The leading entry in each nonzero row is 1(Note: some authors don't require that the leading coefficient is a 1; it could be any number.)
iii. Each leading entry (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
iv. All entries in a column below a leading entry are zero.

Note: The leading nonzero entry in the respective rows is called pivots of the matrix.

Example: The following examples are of matrices in echelon form:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The following examples are **not** in echelon form:

$$A\ \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad B\ \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

Reduced row echelon form or Row Canonical form:

Reduced Row Echelon form is any matrix with the following properties:

i. All zero rows (if any) belong at the bottom of the matrix.

ii. The leading entry in each nonzero row is 1

iii. Each leading entry (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.

iv. All entries in a column below and above a leading entry are zero.

Example: Which of the following matrices are in row echelon form? Which are in reduced row echelon form?

$$\begin{bmatrix}1&3&5\\2&3&0\\1&0&0\end{bmatrix}, \begin{bmatrix}1&0&0&0\\0&1&2&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}, \begin{bmatrix}1&0&2&3\\0&1&0&1\\0&1&2&0\end{bmatrix}, \begin{bmatrix}1&1&0&1\\0&0&1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}0&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}$$

The 2^{nd} and 5th are in row echelon form. The 2nd is the only one in reduced row echelon form.

Example: Are the following matrices in Row-Echelon form?

a) $\begin{bmatrix}1&-9&2&7\\0&0&1&4\\0&1&7&-6\end{bmatrix}$ b) $\begin{bmatrix}1&6&-8&11\\0&1&1&-3\\0&0&0&\mathbf{1}\end{bmatrix}$ c) $\begin{bmatrix}1&5&12\\0&1&-7\\0&0&1\end{bmatrix}$

Solution a): No, this matrix is not in Row-Echelon form since the leading entry in row three is in a column to the left of the leading entry in row two. **Please note:** If we swapped row two and row three, then the matrix would be in Row-Echelon form.

Solution b): Yes, this matrix is in Row-Echelon form as the leading entry in each row has 0's below, and the leading entry in each row is to the right of the leading entry in the row above. Notice the leading entry for row three is in column 4 not column 3. The leading entry **is** allowed to skip columns, but it **cannot** be to the left of the leading entry in any row above it.

Solution c): Yes, this matrix is in Row-Echelon form. Each leading entry in each row is to the right of the leading entry in the row above it, and each leading entry contains only 0's below it.

Problems: Transform the following matrix into row echelon form,

(a) $\begin{bmatrix}3&-2&4&7\\2&1&0&-3\\2&8&-8&2\end{bmatrix}$

$$\begin{bmatrix} 3 & -2 & 4 & 7 \\ 2 & 1 & 0 & -3 \\ 2 & 8 & -8 & 2 \end{bmatrix} \begin{matrix} r_1 - r_2 \to r_1 \\ r_3 - r_2 \to r_3 \end{matrix}$$

$$\begin{bmatrix} 1 & -3 & 4 & 10 \\ 2 & 1 & 0 & -3 \\ 0 & 7 & -8 & 5 \end{bmatrix} r_2 - 2r_1 \to r_1$$

$$\begin{bmatrix} 1 & -3 & 4 & 10 \\ 0 & 7 & -8 & -23 \\ 0 & 7 & -8 & 5 \end{bmatrix} \begin{matrix} r_3 - r_2 \to r_3 \\ \frac{1}{7}r_2 \to r_2 \end{matrix}$$

$$\begin{bmatrix} 1 & -3 & 4 & 10 \\ 0 & 1 & -\frac{8}{7} & -\frac{23}{7} \\ 0 & 0 & 0 & 28 \end{bmatrix} \frac{1}{28}r_3 \to r_3$$

$$\begin{bmatrix} 1 & -3 & 4 & 10 \\ 0 & 1 & -\frac{8}{7} & -\frac{23}{7} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) $\begin{bmatrix} 3 & -1 & 2 & 4 & 1 \\ 2 & 1 & 3 & -1 & 2 \\ 1 & 2 & 3 & -2 & 3 \end{bmatrix}$

$$\begin{bmatrix} 3 & -1 & 2 & 4 & 1 \\ 2 & 1 & 3 & -1 & 2 \\ 1 & 2 & 3 & -2 & 3 \end{bmatrix} r_1 \leftrightarrow r_3$$

$$\begin{bmatrix} 1 & 2 & 3 & -2 & 3 \\ 2 & 1 & 3 & -1 & 2 \\ 3 & -1 & 2 & 4 & 1 \end{bmatrix} \begin{matrix} r_2 - 2r_1 \to r_2 \\ r_3 - 3r_1 \to r_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & -2 & 3 \\ 0 & -3 & -3 & 3 & -4 \\ 0 & -7 & -7 & 10 & 8 \end{bmatrix} \quad -\frac{1}{3}r_2 \to r_2$$

$$\begin{bmatrix} 1 & 2 & 3 & -2 & 3 \\ 0 & 1 & 1 & -1 & \frac{4}{3} \\ 0 & -7 & -7 & 10 & 8 \end{bmatrix} r_3 + 7r_2 \to r_3$$

$$\left[\begin{array}{ccccc} 1 & 2 & 3 & -2 & 3 \\ 0 & 1 & 1 & -1 & \frac{4}{3} \\ 0 & 0 & 0 & 3 & \frac{4}{3} \end{array}\right] \frac{1}{3}r_3 \to r_3$$

$$\left[\begin{array}{ccccc} 1 & 2 & 3 & -2 & 3 \\ 0 & 1 & 1 & -1 & \frac{4}{3} \\ 0 & 0 & 0 & 1 & \frac{4}{9} \end{array}\right]$$

(c) solve $3x - 2y + z = -6$
$4x - 3y + 3z = 7$
$2x + y - z = -9$

We begin by finding the augmented matrix associated with this system of equations

$$\left[\begin{array}{ccc|c} 3 & -2 & 1 & -6 \\ 4 & -3 & 3 & 7 \\ 2 & 1 & -1 & -9 \end{array}\right] \begin{array}{c} r_2 - r_1 \to r_1 \\ 2r_3 - r_2 \to r_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 13 \\ 4 & -3 & 3 & 7 \\ 0 & 5 & -5 & -25 \end{array}\right] \begin{array}{c} r_2 - 4r_1 \to r_2 \\ \frac{1}{5}r_3 \to r_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 13 \\ 0 & -1 & 5 & 45 \\ 0 & 1 & -1 & -5 \end{array}\right] \begin{array}{c} r_1 + r_3 \to r_1 \\ r_3 + r_2 \to r_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & -1 & 5 & 45 \\ 0 & 0 & 4 & 40 \end{array}\right] \begin{array}{c} -r_2 \to r_2 \\ \frac{1}{4}r_3 \to r_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & 1 & -5 & -45 \\ 0 & 0 & 1 & 10 \end{array}\right]$$

Therefore, the unique solution is $x = -2$, $y = 5$, and $z = 10$

d. Determine if the following system of equations is consistent or inconsistent and state the solution

$$2x - 4y + z = 3$$
$$x - 3y + z = 5$$
$$3x - 7y + 2z = 12$$

Solution: First, create the augmented matrix.
$$\left[\begin{array}{ccc|c} 2 & -4 & 1 & 3 \\ 1 & -3 & 1 & 5 \\ 3 & -7 & 2 & 12 \end{array}\right]$$

Use the elementary row operations to obtain a Row-Echelon form.

$$\left[\begin{array}{ccc|c} 2 & -4 & 1 & 3 \\ 1 & -3 & 1 & 5 \\ 3 & -7 & 2 & 12 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 5 \\ 2 & -4 & 1 & 3 \\ 3 & -7 & 2 & 12 \end{array}\right] \xrightarrow{-2R_1 + R_2} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 5 \\ 0 & 2 & -1 & -7 \\ 3 & -7 & 2 & 12 \end{array}\right] \xrightarrow{-3R_1 + R_3}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 5 \\ 0 & 2 & -1 & -7 \\ 0 & 2 & -1 & -3 \end{array}\right] \xrightarrow{-1R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 5 \\ 0 & 2 & -1 & -7 \\ 0 & 0 & 0 & 4 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 5 \\ 0 & 1 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 0 & 0 & 4 \end{array}\right]$$

The last row indicates the system is inconsistent. This can most easily be seen if the last row is converted back to an equation.

$$0x + 0y + 0z = 4$$

According to this equation, there are not any values of x, y, or z that will make the above equation true. Therefore, the system has no solution