# Volumes of Revolution.

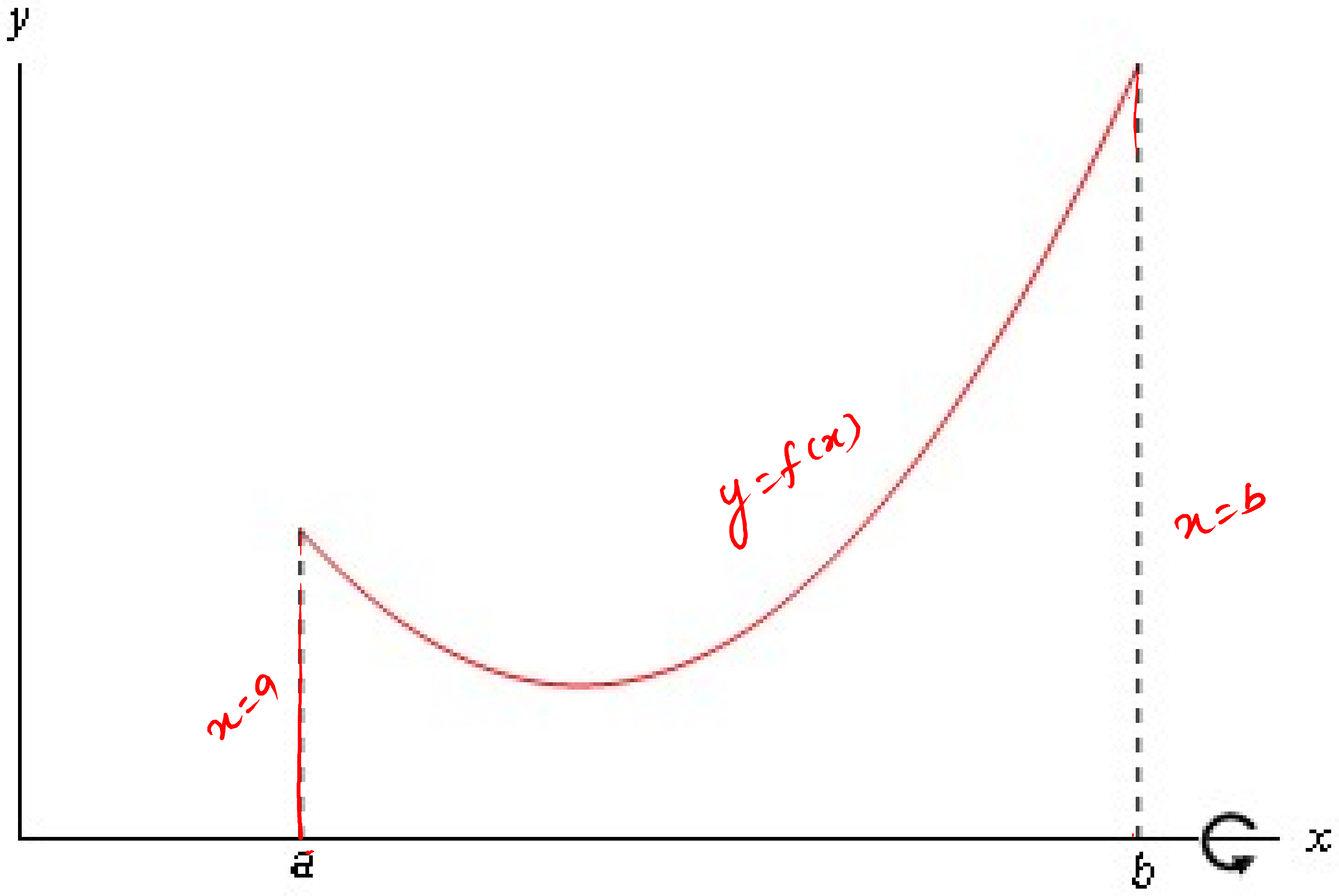
A plane area when revolved about a line lying in its own plane, generates a solid of revolution. The surface generated by the boundary of the plane area is called the surface of revolution.

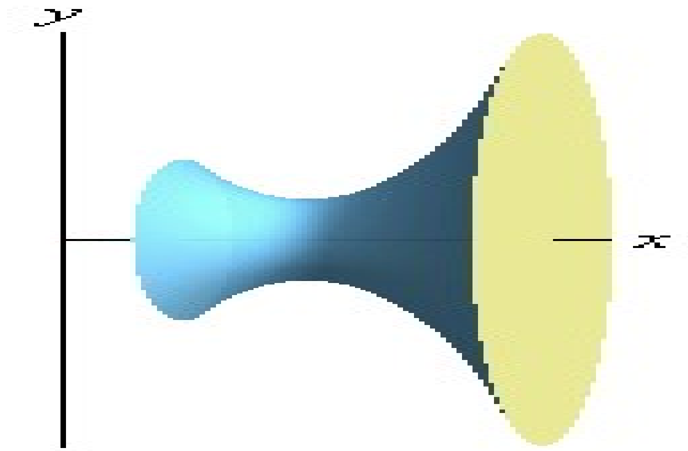
The fixed line is called axis of revolution.

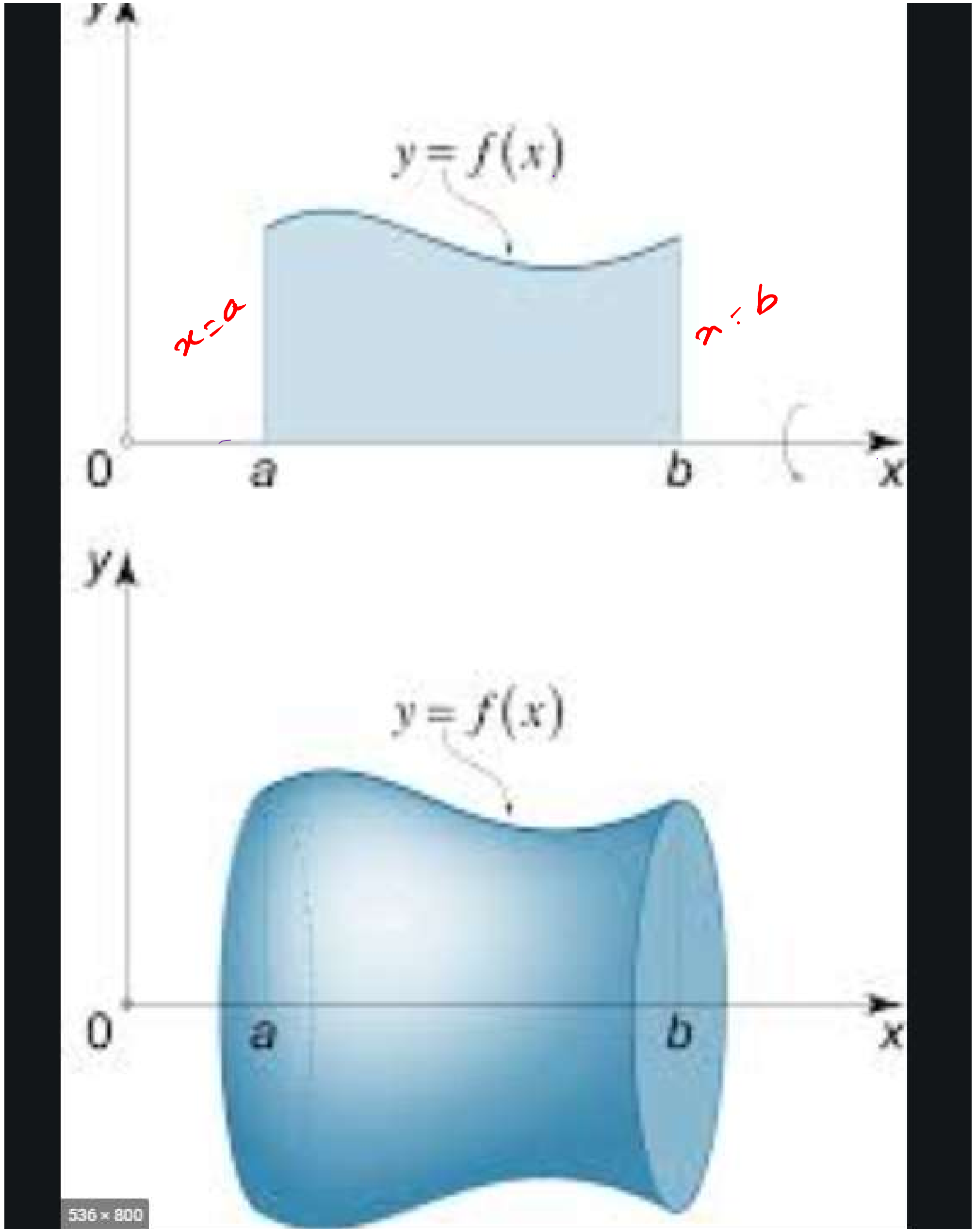
Ex.(I) A rectangle revolved about one of its sides generates a right circular cylinder.

(II) A right angled triangle revolved about one of its sides (along the right angle) generates a right circular cone.

(IV) A semicircle revolved about its diameter generates a sphere.


y
y=f(x)
x=a
x=b
a
b
x


y
x


y = f(x)
x=a
x=b
0
a
b
x
y
y = f(x)
0
a
b
x
536 × 800

## Revolution about x-axis.

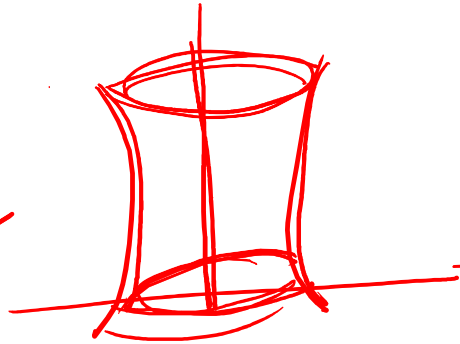
The volume of the solid generated by the revolution about the x-axis, of the area bounded by the curve $y=f(x)$, the x-axis and the ordinates $x=a$, $x=b$ is

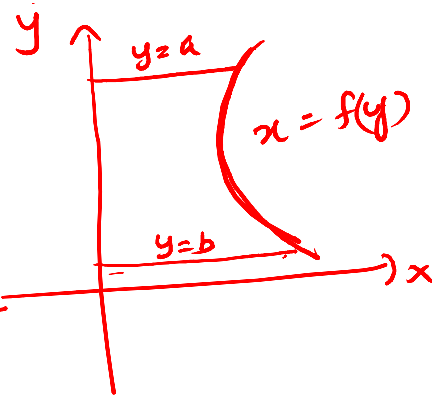
$$V = \int_a^b \pi y^2 \, dx.$$

## Revolution about the y-axis.

The volume of the solid generated by the revolution, about the y-axis, of the area bounded by the curve $x=f(y)$, the y-axis and the ordinates $y=a$, $y=b$ is

$$V = \int_a^b \pi x^2 \, dy.$$



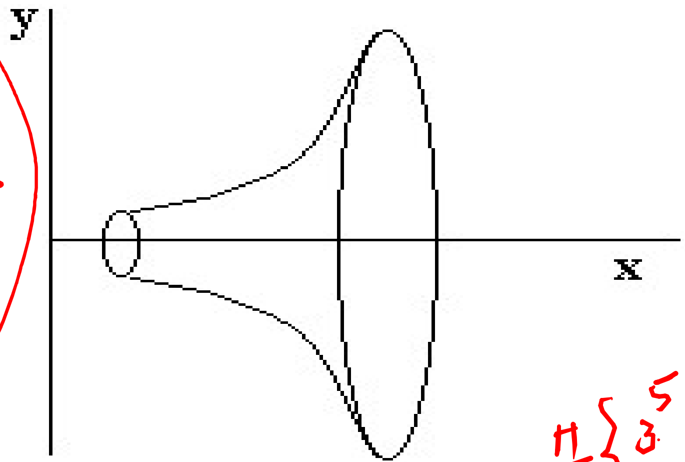


(1) volume = $\int_a^b \pi y^2\, dx$ (about x axis)

(2) $V = \int_a^b \pi x^2\, dy$ (about y axis)

Example: 1

The graph of $y = x^2$ between $x = 1$ and $x = 3$ is rotated completely around the x-axis. Find the volume generated.



$V = \int_{x=a}^{b} \pi y^2\, dx$

Gn $y = x^2$

$\therefore y^2 = (x^2)^2$

$= x^4$

$$\text{volume} = \int_1^3 \pi x^4\, dx$$

$= \pi \int_1^3 x^4\, dx$

$$= \left[\pi \frac{x^5}{5}\right]_1^3$$

$$= \frac{243\pi}{5} - \frac{\pi}{5}$$

$$= \underline{\underline{48.4\,\pi}}$$

$\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$

$\frac{\pi}{5}\left\{3^5 - 1^5\right\}$

$= \frac{\pi}{5}\left\{243 - 1\right\}$

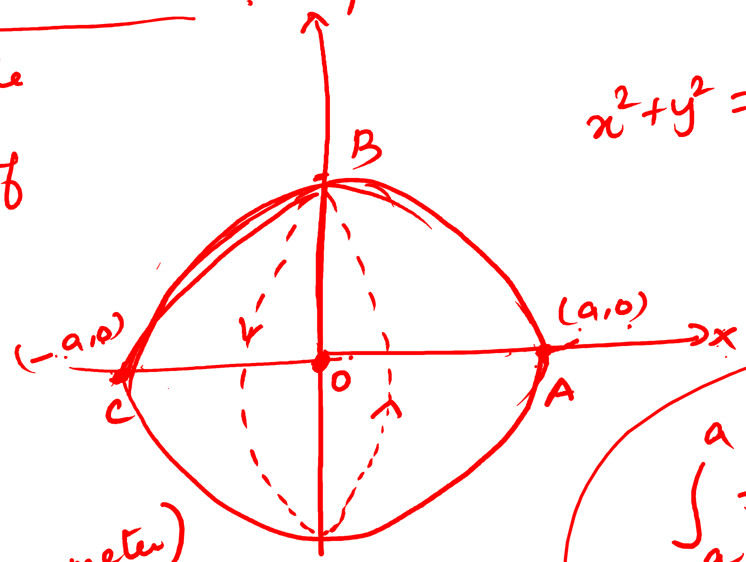
$= \frac{242\pi}{5}$

1 Example

$x = y^2$ $y = 2$ & $y = 4$ is what completely around the y axis

2) Find the volume of a sphere of radius 'a'.

Soln Let the sphere be generated by the revolution of the semicircle ABC of radius 'a' about its diameter CA.




$x^2+y^2=a^2$

Taking CA as the axis and its mid point O as the origin, the eqn of the circle ABC is $x^2+y^2=a^2$

Volume of the sphere generated by the revolution about x-axis (diameter) $= \int_{-a}^{a} \pi y^2 \, dx$.

$2\int_0^a \pi y^2 \, dx$

$= 2\pi \int_0^a y^2 \, dx$.

$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(x)$ is even

$$= 2\pi \int_0^a (a^2 - x^2)\, dx.$$

$$= 2\pi \left[ a^2 x - \frac{x^3}{3} \right]_0^a$$

$$= 2\pi \left[ a^3 - \frac{a^3}{3} \right] = 2\pi \left[ \frac{3a^3 - a^3}{3} \right]$$

$$= 2\pi \times \frac{2a^3}{3}$$

$$= \frac{4}{3} \pi a^3.$$

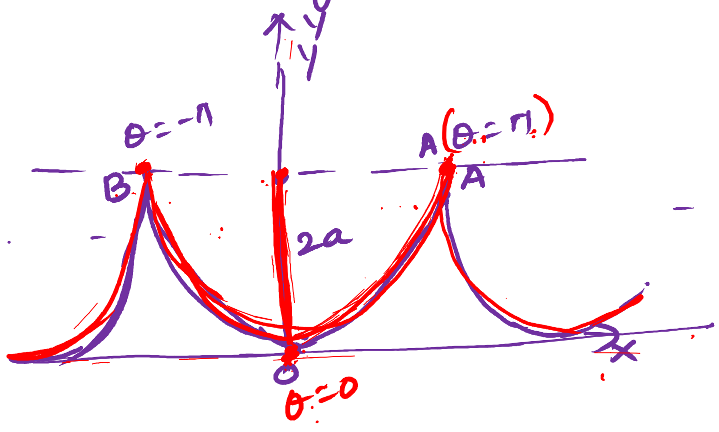
$$x^2 + y^2 = a^2$$

$$\therefore y^2 = a^2 - x^2$$

$$\int a^2\, dx = a^2 \int dx = a^2 x$$

$$\int x^2\, dx = \frac{x^{2+1}}{2+1} = \frac{x^3}{3}$$

3) P.T the volume of the reel formed by the revolution of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ about the tangent at the vertex is $\pi^2 a^3$.

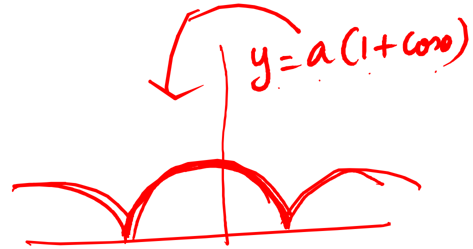



Soln. The arch AOB is symmetric about the y-axis & the tgt at the vertex is the x-axis. For half the cycloid, OA, $\theta$ varies from 0 to $\pi$.

$$\therefore \text{volume} = 2\int_0^{\pi} \pi y^2 dx = 2\pi \int_0^{\pi} a^2(1-\cos\theta)^2 . a(1+\cos\theta)\, d\theta$$

$x = a(\theta + \sin\theta)$

$dx = a(1+\cos\theta)\, d\theta$




$$= 2\pi a^3 \int_0^{\pi} (1-\cos\theta)^2 (1+\cos\theta)\, d\theta .$$

$y = a(1-\cos\theta)$

$y^2 = a^2(1-\cos\theta)^2$

$$= 2\pi a^3 \int_0^{\pi} \left(2\sin^2\frac{\theta}{2}\right)^2 . \, 2\cos^2\frac{\theta}{2}\, d\theta$$

$$= 16\pi a^3 \int_0^{\pi} \sin^4\frac{\theta}{2}\cos^2\frac{\theta}{2}\, d\theta.$$ ($m=4$, $n=2$ even)

put $\frac{\theta}{2} = \phi$.

$\theta = 2\phi \Rightarrow d\theta = 2d\phi$

when $\theta = 0$, $\phi = 0$

when $\theta = \pi$, $\phi = \frac{\pi}{2}$   $\phi : 0$ to $\frac{\pi}{2}$

$$\therefore V = 16\pi a^3 \int_0^{\pi/2} \sin^4\phi \cos^2\phi\, 2\, d\phi$$

$$= 32\pi a^3 \int_0^{\pi/2} \sin^4\phi \cos^2\phi\, d\phi$$

$$= 32\pi a^3 \cdot \frac{3\cdot 1\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2}$$

$$= \pi^2 a^3$$

$m=4$
$n=2$
$m+n=6$

Ans.

**Remember**

① $1+\cos 2\theta = 2\cos^2\theta$

$\therefore 1+\cos\theta = 2\cos^2\frac{\theta}{2}$

② $1-\cos 2\theta = 2\sin^2\theta$

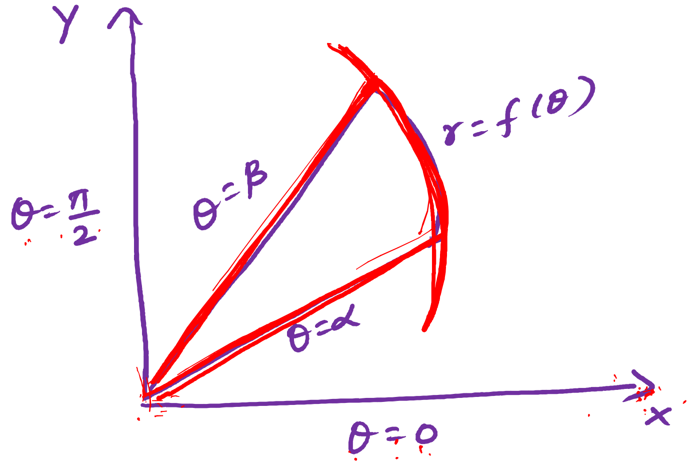
$\therefore 1-\cos\theta = 2\sin^2\frac{\theta}{2}$

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{(m-1)(m-3)\cdots(n-1)(n-3)\cdots}{(m+n)(m+n-2)\cdots}\cdot\frac{\pi}{2}$$

if $m$ & $n$ are even

$m=4$, $n=2$

## Volumes of revolution (polar curves)




The volume of the solid generated by the revolution of the area bounded by the curve $r=f(\theta)$ and the radii vectors $\theta=\alpha$, $\theta=\beta$

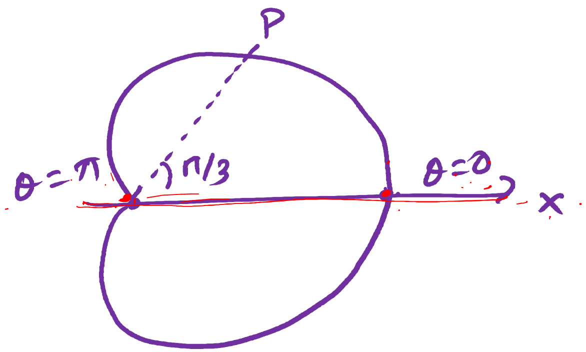
(a) about the initial line $OX$ $(\theta=0)$ $= \int_{\alpha}^{\beta} \frac{2\pi}{3} r^3 \sin\theta \, d\theta$.

(b) about the line $OY$ $(\theta=\frac{\pi}{2})$ $= \int_{\alpha}^{\beta} \frac{2\pi}{3} r^3 \cos\theta \, d\theta$.

1) Find the volume of the solid generated by the revolution of the cardioid $r = a(1+\cos\theta)$ about the initial line

Soln:

The cardioid is symmetric about the initial line and for its upper half, $\theta$ varies from 0 to $\pi$.




$\therefore$ Required volume $= \int_0^{\pi} \frac{2}{3}\pi r^3 \sin\theta \, d\theta$ , $r = a(1+\cos\theta)$

$$= \int_0^{\pi} \frac{2}{3}\pi \left(a(1+\cos\theta)\right)^3 \sin\theta \, d\theta.$$

$$= \frac{2}{3}\pi a^3 \int_0^{\pi} (1+\cos\theta)^3 \sin\theta . \, d\theta .$$

$\left(a(1+\cos\theta)\right)^3 = a^3 (1+\cos\theta)^3$

put $1+\cos\theta = t$

$\Rightarrow -\sin\theta \, d\theta = dt$.

$\therefore \sin\theta \, d\theta = -dt$.

When $\theta = 0$, $1+\cos 0 = t$,

$1+1 = t \Rightarrow t = 2$.

When $\theta = \pi$, $1+\cos\pi = t$

$1-1 = t \Rightarrow t = 0$

$\therefore t : 2$ to $0$

$\therefore$ Volume $= \frac{2}{3}\pi a^3 \int_2^0 t^3 \, (-dt) = -\int_0^2 t^3 \, (-dt)$

$= \frac{2}{3}\pi a^3 \int_0^2 t^3 \, dt = \frac{2}{3}\pi a^3 \left(\frac{t^4}{4}\right)_0^2$

$= \frac{2}{3}\pi a^3 \left(\frac{16}{4}\right)$

$= \frac{8}{3}\pi a^3$.

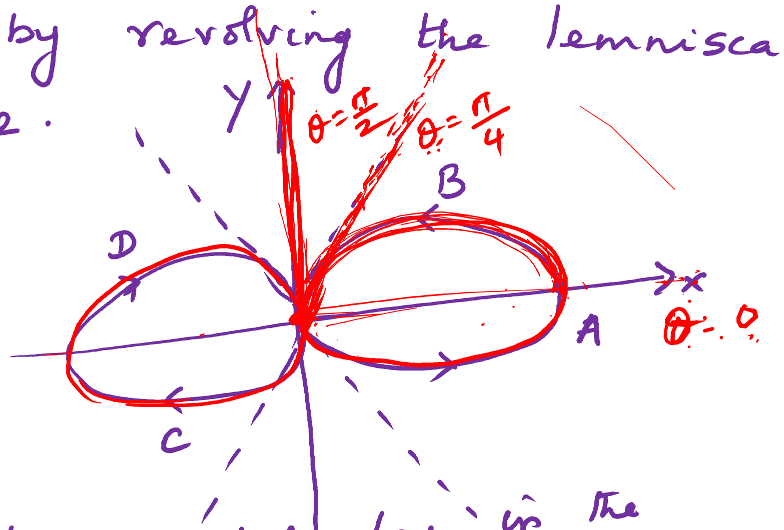
$\cos 0 = 1$

$\cos\pi = -1$

$\int_a^0 f(x)dx = -\int_0^a f(x)dx$

$\int x^n dx = \frac{x^{n+1}}{n+1}$

2) Find the volume of the solid generated by revolving the lemniscate $r^2 = a^2 \cos 2\theta$ about the line $\theta = \pi/2$.




Soln The curve is symmetrical about the pole. For the upper half of the RHS loop, $\theta$ varies from 0 to $\pi/4$

$\therefore$ Reqd volume = 2 (volume generated by the half loop in the first quadrant)

$$= 2 \int_0^{\pi/4} \frac{2}{3} \pi r^3 \cos\theta \, d\theta.$$

$$= \frac{4\pi}{3} \int_0^{\pi/4} a^3 (\cos 2\theta)^{3/2} \cos\theta \, d\theta$$

$$= \frac{4}{3} \pi a^3 \int_0^{\pi/4} (1 - 2\sin^2\theta)^{3/2} \cdot \cos\theta \, d\theta$$

Gn $r^2 = a^2 \cos 2\theta \Rightarrow r = \sqrt{a^2 \cos 2\theta}$

$r = a(\cos 2\theta)^{1/2}$

$r^3 = a^3 (\cos 2\theta)^{3/2}$

$1 - \cos 2\theta = 2\sin^2\theta$

$\therefore \cos 2\theta = 1 - 2\sin^2\theta$.

$r = a(\cos 2\theta)^{1/2}$

$r^3 = \left(a(\cos 2\theta)^{1/2}\right)^3$

$a^3 ((\cos 2\theta)^{1/2})^3$

$(a^m)^n = a^{mn}$

$$= \frac{4}{3}\pi a^3 \int_0^{\pi/4} \left(1 - (\sqrt{2}\sin\theta)^2\right)^{3/2} \cos\theta \, d\theta.$$

put $\sqrt{2}\sin\theta = \sin\varphi$

$\therefore \sqrt{2}\cos\theta \, d\theta = \cos\varphi \, d\varphi$ $\Rightarrow \cos\theta \, d\theta = \frac{1}{\sqrt{2}} \cos\varphi \, d\varphi$

When $\theta = 0$, $\varphi = 0$ $\boxed{\Rightarrow \sqrt{2}\sin 0 = \sin\varphi \quad 0 = \sin\varphi \Rightarrow \varphi = 0}$

When $\theta = \frac{\pi}{4}$, $\sqrt{2}.\sin\frac{\pi}{4} = \sin\varphi$

$\sqrt{2} \times \frac{1}{\sqrt{2}} = \sin\varphi$

$\sin\varphi = 1 \Rightarrow \varphi = \pi/2$

$\therefore \varphi : 0$ to $\pi/2$

$$\therefore V = \frac{4}{3}\pi a^3 \int_0^{\pi/2} (\underbrace{1 - \sin^2\varphi}) ^{3/2} \cdot \frac{1}{\sqrt{2}} \cos\varphi \, d\varphi$$

$$= \frac{4}{3\sqrt{2}}\pi a^3 \int_0^{\pi/2} (\cos^2\varphi)^{3/2} . \cos\varphi \, d\varphi.$$

$\to \cos^3\varphi$

$$= \frac{4}{3\sqrt{2}}\pi a^3 . \int_0^{\pi/2} \cos^4\varphi \, d\varphi.$$

$$= \frac{\cancel{4}}{\cancel{3}\sqrt{2}}\pi a^3 . \frac{\cancel{3} \times 1}{\cancel{4} \times 2} \times \frac{\pi}{2}$$

$$= \frac{\pi^2 a^3}{4\sqrt{2}}$$

// $\int_0^{\pi/2} \cos^n x \, dx$, $n$ is even $= \frac{(n-1)(n-3)\cdots 1}{n(n-2)(n-4)\cdots 2} \cdot \frac{\pi}{2}$

$(1 - 2\sin^2\theta)^{3/2}$

$(1 - (\sqrt{2}\sin\theta)^2)^{3/2}$

$(\cos^2\varphi)^{\frac{3}{2}} = \cos^{\cancel{2} \times \frac{3}{\cancel{2}}} \varphi$