Module 6 Fourier Series.

Note:

$\cos 0 = 1$

$\cos \pi = -1$

$\cos 2\pi = 1$

$\cos 3\pi = -1$

$\cos \frac{\pi}{2} = 0$

In general $\cos n\pi = (-1)^n$
$= 1$ if n is even
$= -1$ if n is odd

$\sin 0 = 0$

$\sin \pi = 0$

$\sin 2\pi = 0$

$\sin 3\pi = 0$

$\sin \frac{\pi}{2} = 1$

$\sin n\pi = 0 \quad \forall n.$

$e^0 = 1$

$e^{\infty} = \infty$

$e^{-\infty} = 0.$

$\cos(-x) = \cos x$

$\sin(-x) = -\sin x.$

Odd and even function. [in the interval $(-\infty, \infty)$]

A function $f(x)$ is said to be odd, if $f(-x) = -f(x) \; \forall x$

& a function $f(x)$ is said to be an even fn, if

$$f(-x) = f(x), \quad \forall x.$$

$f(x) = \cos x$

$f(-x) = \cos(-x) = \cos$

Examples.

1) $f(x) = x^2$

$f(-x) = (-x)^2 = x^2 = f(x)$ $\quad \therefore f(x) = x^2$ is an even fn.

$-x \times -x$

2) $f(x) = \cos x$

$f(-x) = \cos(-x) = \cos x = f(x)$ $\quad \therefore \cos x$ is even.

3) $f(x) = x$

$f(-x) = -x = -f(x)$ $\quad \therefore f(x) = x$ is odd.

4) $f(x) = \sin x$

$f(-x) = \sin(-x) = -\sin x = -f(x)$ $\quad \therefore f(x) = \sin x$ is odd.

Note $O \to$ odd fn $\quad E \to$ even fn.

$\left.\begin{array}{c} \text{Odd} \times \text{odd} \\ Ev \times Ev \\ \frac{O}{O} \\ E/E \end{array}\right\}$ even.

$\left.\begin{array}{c} O \times E \\ E \times O \\ O/E \\ E/O \end{array}\right\}$ odd.

4) $\int_{-\infty}^{\infty} (\text{Even fn})\,dx = 2\int_{0}^{\infty} (\text{Even fn})\,dx.$

For example

$\int_{-1}^{1} x^2\,dx = 2\int_{0}^{1} x^2\,dx$

$\int_{-\infty}^{\infty} E = 2\int_{0}^{\infty} E$

$\int_{-\infty}^{\infty} (\text{Odd fn})\,dx = 0.$

$\int_{-\infty}^{\infty} \text{Odd} = 0.$

$\rightarrow$ Ex $\int_{-1}^{1} x\,dx = 0 \rightarrow \left[\frac{x^2}{2}\right]_{-1}^{+1} = \frac{1}{2}[1-1] = 0$

Example ①

$\int_{-\infty}^{\infty} \frac{\overset{0}{\sin x}\ \overset{0}{\tan x}\cdot \overset{E}{x^2}}{\underset{0}{x^3}}\,dx.$

$\frac{E}{O} = O$

$= 0$

$\tan x = \frac{\overset{0}{\sin x}}{\underset{E}{\cos x}}$

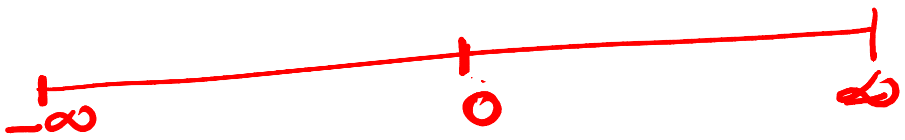
2) $\int_{-\infty}^{\infty} \underbrace{\overset{0}{x^3}\ \overset{e}{\cos 4x}}_{\text{odd}}\,dx.$ (or) $\int_{-\pi}^{\pi} x^3 \cos 4x\,dx$

$= 0$

Note

We can identify whether a fn is odd or even only in $(-\infty, \infty)$ or $(-l, l)$

$(-\pi, \pi)$



5) Bernoulli's formula

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots$$

$u', u'', u''' \dots$ are derivatives of u. (differentiation

$v_1 = \int v$, $v_2 = \int v_1$ or $\iint v$; $v_3 = \int v_2$ or $\iiint v, \dots$

are integrals of v.

b)
$$\int e^{ax} \cos bx\,dx = \left\{ \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx) \right\}$$

$$\int e^{ax} \sin bx\,dx = \left\{ \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx) \right\}.$$

Note $f(x)$ is defined as Even or odd only if the limits are $(-\pi, \pi)$ or $(-l, l)$.

Bernoulli's formula

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots\cdots$$

$u \rightarrow$ should be differentiated &

$v \rightarrow$ should be integrated.

Example ① $\int x e^x dx$

$u = x \qquad v = e^x$

$u' = 1 \quad (-) \quad v_1 = e^x$

$u'' = 0 \qquad v_2 = e^x$

$\therefore I = xe^x - e^x = e^x(x-1)$

② $\int x^2 e^{2x} dx$

$u = x^2 \qquad v = e^{2x}$

$u' = 2x \qquad v_1 = \frac{e^{2x}}{2}$

$u'' = 2 \quad (-) \quad v_2 = \frac{1}{2}\cdot\frac{e^{2x}}{2} = \frac{e^{2x}}{4}$

$u''' = 0 \qquad v_3 = \frac{e^{2x}}{8}$

$I = \frac{x^2 e^{2x}}{2} - 2x \times \frac{e^{2x}}{4} + 2\frac{e^{2x}}{8}$

3) $\int x \cos x \, dx$.

$u = x \qquad v = \cos x$

$u' = 1 \qquad v_1 = \sin x$

$(-)$

$u'' = 0 \qquad v_2 = -\cos x$

$I = x \sin x - (-\cos x)$

$= x \sin x + \cos x$

4) $\int x^2 \sin 2x \, dx$.

H.W

5) $\int x^2 e^{3x} \, dx$ $\quad$ ~~6) $\int x^3 e^{3x} d$~~

6) $\int x^3 e^{-3x} \, dx$

7) $\int x^2 \cos 2x \, dx$.

Answer,

8) $\int e^{x} \sin 3x \, dx$

9) $\int e^{-2x} \cos x \, dx$.

$$\int x^2 \sin 2x \, dx.$$

Bernoulli's formula

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \dots\dots$$

$u = x^2$ $\quad$ $v = \sin 2x.$

$u' = 2x$ $\quad$ $v_1 = -\dfrac{\cos 2x}{2}$

$u'' = 2$ $\quad$ $v_2 = -\dfrac{\sin 2x}{4}$

$u''' = 0$ $\quad$ $v_3 = \dfrac{\cos 2x}{8}$

$$\int x^2 \sin 2x \, dx = -\frac{x^2 \cos 2x}{2} + \frac{2x \sin 2x}{4} + 2\,\frac{\cos 2x}{8}$$

Evaluate $\int e^{2x} \cos 3x \, dx$

Note $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} [a \cos bx + b \sin bx]$

$a = 2 \quad b = 3$

$\therefore \int e^{2x} \cos 3x \, dx = \frac{e^{2x}}{2^2+3^2} (2 \cos 3x + 3 \sin 2x)$

$= \frac{e^{2x}}{13} (2 \cos 3x + 3 \sin 2x)$

Evaluate $\int e^{-x} \sin 2x \, dx$

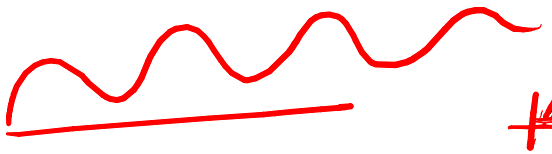
Note $\int e^{ax} \sin bx \, dx = \dfrac{e^{ax}}{a^2+b^2}\left(a \sin bx - b \cos bx\right)$

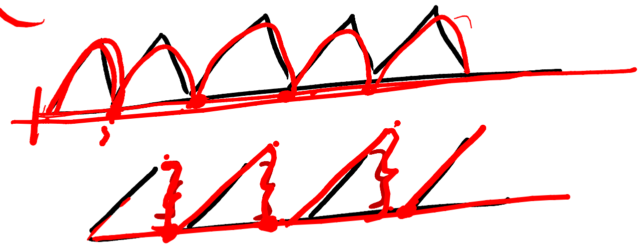
$a = -1, \quad b = 2$

$$\therefore \int e^{-x} \sin 2x \, dx = \frac{e^{-x}}{(-1)^2 + 2^2}\left[-1 \sin 2x - 2 \cos (-1) x\right]$$

$$= \frac{e^{-x}}{5}\left[-\sin 2x - 2 \cos x\right]$$

Defns: 1. A function which is continuous every where except for a finite number of jumps in a given interval is called piecewise continuous or sectionally continuous in that intervals.





Periodic function

A function $f(x)$ is said to be periodic if $f(x+2\pi) = f(x)$ $\forall x$ and 2π is said to be the period of the fn [In general $f(x+p) = f(x)$ $\forall x$ p is the period of the fn]

Ex $\sin(2\pi+x) = \sin x$, $\cos(2\pi+x) = \cos x$

Dirichlet conditions: $f(x)$ is a bounded fn of period 2π, which in any one period has atmost a finite no. of maxima and minima and a finite no. of points of discontinuity.

(3) If 'a' is a point of discontinuity, then

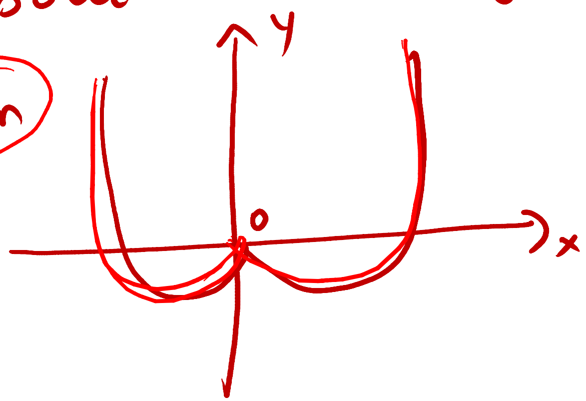
$$f(a) = \lim_{x \to a} f(x+a) - f(a)$$

Note: $\sin x$, $\cos x$ are periodic fns with period 2π & $\tan x$ is periodic with period π.

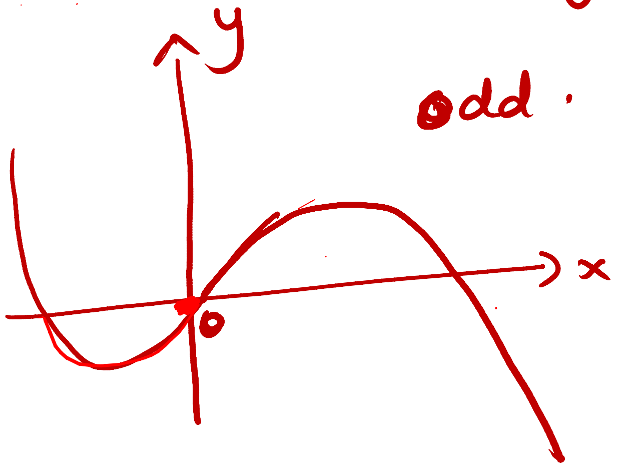
(4) The graph of an even function will be symmetric about the y-axis

(5) The graph of an odd function will be symmetric about the origin.

even



odd.



Fourier Series. A function $f(x)$ defined in the interval $c \le x \le c+2\pi$ i.e. $(c, c+2\pi)$ and satisfies Dirichlet's conditions can be expanded as a series of the form

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)}$$

is called the Fourier Series of the function $f(x)$.

where,

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx.$$

The constants a_0, a_n & b_n are called Euler constants or (or) Euler-Fourier formulas.

$c \le x \le c+2\pi$

$c = 0$

$(0, 2\pi)$

length 2π

$0 \le x \le 2\pi$

Note $f(x)$ is to be extendend in either $(-\pi$ to $\pi)$ or $(0, 2\pi)$, so that the value of c is either $-\pi$ or 0 in Euler's formulas.

~~$c \le x \le c+2$~~

$c \le x \le c+2\pi$

when $c = 0$ $(0, 2\pi)$

when $c = -\pi$, $(-\pi, \pi)$

	Full Range FS		Half Range FS	
In terms of π	$(c, c+2\pi)$ $(0, 2\pi)$ $(-\pi, \pi)$	2π	$(0, \pi)$	π
In terms of l	$(c, c+2l)$ $(0, 2l)$ $(-l, l)$	$2l$	$(0, l)$	l

$\rightarrow$ change of interval.

Full Range Fourier Series in $(c, c+2\pi)$, $(0, 2\pi)$, $(-\pi, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx\, dx$$

Full Range FS in $(c, c+2l)$, $(0, 2l)$, $(-l, l)$ [[Change of in te

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l} \int_c^{c+2l} f(x)\, dx.$$

$$a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$b_n = \frac{1}{l} \int_c^{c+2l} f(x) \sin \frac{n\pi x}{l}\, dx.$$

Even and odd function (F.S for odd & even fn)

We can check whether a function is odd or even only in the interval $(-\pi, \pi)$ & $(-l, l)$.

$$\int_{-\infty}^{\infty} (\text{Even fn}) = 2\int_{0}^{\infty} (\text{even function}) \quad [f(-x) = f(x)\ \forall x]$$

$$\int_{-\infty}^{\infty} (\text{odd function}) = 0. \quad [f(-x) = -f(x)\ \forall x]$$

When $f(x)$ is even in $(-\pi, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx.$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{e}.\underbrace{\cos nx}_{e}\,dx \;{\scriptstyle = e} = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx \quad [\because e \times e = e]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 \quad (\because e \times o = odd).$$

$$e \cdot o = 0$$

$\therefore$ When $f(x)$ is ~~odd~~ even, the fourier series is

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \cos nx}.$$

When f(x) is odd, the fourier Series is $(-\pi, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{odd}\, dx = 0$ $[\because f(x) \text{ is odd}]$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = 0 \quad [\because odd \times even = odd]$$

$o \cdot e = o$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\, dx$$

$o \quad o = e$ $\quad (\because o \times o = even)$

$\therefore$ When f(x) is odd in $(-\pi, \pi)$, the reqd F.S is

$f(x) = \sum_{n=}$

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin nx} \quad (\because a_0 = a_n = 0 \text{ when } f(x) \text{ is odd})$$

When $f(x)$ is even in $(-l, l)$.

The F.S is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad (\because b_n = 0)$$

Where,

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx \quad \&$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx.$$

When $f(x)$ is odd in $(-l, l)$,

The reqd F.S is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad \left(\because a_0 = a_n = 0 \right)$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx.$$