Linear Transformations.

Defn. Suppose U & V are vector spaces (real or complex)
Then the map $T: U \to V$ is said to be a linear map / Linear transformation,
if (i) $T(u_1+u_2) = T(u_1)+T(u_2)$ $\forall\, u_1, u_2 \in U$
(ii) $T(\alpha u) = \alpha T(u)$ $\forall\, u \in U$ & for all Scalars α.

Note A linear map $T: U \to U$ is also called a linear map on U.

Pbm 1. Define $T: V_3 \to V_3$ by the rule $T(x_1, x_2, x_3) = (x_1, x_2, 0)$ (Called projection of V_3 on the $x_1 x_2$ plane) $_{x_3=0}$. P.T T is a linear map.

Pf We have to S.T (i) $T(x+y) = T(x)+T(y)$ and $T(\alpha x) = \alpha T(x)$,
$\forall\, x, y \in V_3$ & $\forall$ Scalars α.

Let $x = (x_1, x_2, x_3)$ & $y = (y_1, y_2, y_3)$

$\left[\begin{array}{l} \underline{\text{defn}} \\ T(x_1, x_2, x_3) = (x_1, x_2, 0) \end{array}\right.$

Then $x+y = (x_1+y_1,\ x_2+y_2,\ x_3+y_3)$

$\alpha x = (\alpha x_1,\ \alpha x_2,\ \alpha x_3)$

Now $T(x+y) = T(x_1+y_1,\ x_2+y_2,\ x_3+y_3)$ — $\cancel{1}$

$= (x_1+y_1,\ x_2+y_2,\ 0)$ — ① (by the defn of T.

Now

$T(x) + T(y) = T(x_1, x_2, x_3) + T(y_1, y_2, y_3)$ (by the defn of T)

$= (x_1, x_2, 0) + (y_1, y_2, 0)$ — ②

$= (x_1+y_1,\ x_2+y_2,\ 0)$

From ① + ②, $T(x+y) = T(x) + T(y)$

(ii) $T(\alpha x) = T(\alpha x_1, \alpha x_2, \alpha x_3) = (\alpha x_1, \alpha x_2, 0) = \alpha(x_1, x_2, 0)$ — ③

$\alpha T(x) = \alpha T(x_1, x_2, x_3) = \alpha(x_1, x_2, 0)$ — ④

From (3) + (4), $T(\alpha x) = \alpha T(x)$.

$\therefore$ T is a linear map.

2) Define $T: V_3 \to V_2$ as $T(x_1, x_2, x_3) = (x_1 - x_2,\ x_1 + x_3)$

P.T T is a linear transformation.

<u>Soln</u> Let $x = (x_1, x_2, x_3)$ & $y = (y_1, y_2, y_3)$ | $x+y = (x_1+y_1, x_2+y_2, x_3+y_3)$

(i) $T(x+y) = T(x_1+y_1,\ x_2+y_2,\ x_3+y_3)$

$$= \left[\left((x_1+y_1) - (x_2+y_2)\right),\ \left((x_1+y_1) + (x_3+y_3)\right)\right]$$

$$= (x_1+y_1-x_2-y_2,\ x_1+y_1+x_3+y_3) \quad \text{—} \ (1)$$

$$T(x) + T(y) = T(x_1, x_2, x_3) + T(y_1, y_2, y_3)$$

$$= (x_1 - x_2,\ x_1 + x_3) + (y_1 - y_2,\ y_1 + y_3)$$

$$= (x_1+y_1-x_2-y_2,\ x_1+y_1+x_3+y_3) \quad \text{—} \ (2)$$

From (1) & (2)

$\therefore T(x+y) = T(x) + T(y)$.

(ii) $T(\alpha x) = T(\alpha x_1, \alpha x_2, \alpha x_3)$

$x = x_1, x_2, x_3$

$\alpha x = \alpha x_1, \alpha x_2, \alpha x_3$

$= (\alpha x_1 - \alpha x_2, \alpha x_1 + \alpha x_3)$

$= (\alpha(x_1 - x_2), \alpha(x_1 + x_3))$ — (3)

~~$= \alpha((x_1 - x_2), (x_1 + x_3))$~~

$\alpha T(x) = \alpha T(x_1, x_2, x_3)$

$= \alpha(x_1 - x_2, x_1 + x_3)$ — (4)

$= (\alpha(x_1 - x_2), \alpha(x_1 + x_3))$

From (3) & (4), $T(\alpha x) = \alpha T(x)$.

Both conditions for linearity of T are verified.

Hence T is linear.

3) Let $T : R^2 \to R^2$ be the translation mapping, defined by $T(x,y) = (x+3,\ y-2)$. S.T T is not linear.

Soln Given $T(x,y) = (x+3,\ y-2)$

$$T(0,0) = (0+3,\ 0-2)$$
$$= (3,\ -2)$$
$$\neq (0,0)$$

ie Zero vector is not mapped into zero vector. $\therefore T$ is not linear.

4) S.T the mapping $T : R^2 \to R^3$ defined by $T(x,y) = (x+y,\ x-y,\ y)$ is a linear Transformation.

Soln Let $x = (x_1, x_2)$; $y = (y_1, y_2)$

$$x+y = x_1 + y_1,\ x_2 + y_2$$

$$\therefore T(x+y) = \{x_1 + y_1 + x_2 + y_2,\ x_1 + y_1 - x_2 - y_2,\ x_2 + y_2\}$$

$= (x_1 + x_2, \; x_1 - x_2, \; x_2) + (y_1 + y_2, \; y_1 - y_2, \; y_2)$

$T(x+y) = T(x) + T(y)$.

$T(x,y) = (x+y, x-y, y)$

(ii) $\alpha x = (\alpha x_1, \alpha x_2)$

$T(\alpha x) = (\alpha x_1 + \alpha x_2, \; \alpha x_1 - \alpha x_2, \; \alpha x_2)$

$= \alpha (x_1 + x_2, \; x_1 - x_2, \; x_2)$

$= \alpha T(x)$.

$\therefore$ T is a linear transformation.

3) S.T $T: R^2 \to R$ defined by $T(x_1, x_2) = x_1^2 - x_2^2$ is not linear.

Soln Let $x = (1,0)$, $y = (1,0) \in R^2$.

$x + y = (2,0)$ $\quad T(2,0) =$

$T(x+y) = 2^2 - 0^2 = 4$

$T(x) = T(1,0) = 1^2 - 0^2 = 1$

$T(y) = T(1,0) = 1^2 - 0^2 = 1$

$T(x) + T(y) = 1 + 1 = 2$

$T(x+y) \neq T(x) + T(y)$

$$T(x)+T(y) = T(1,0) + T(1,0)$$
$$= 1^2 - 0 + 1^2 - 0$$
$$= 2$$
$$\Rightarrow T(x+y) \neq T(x) + T(y)$$

$\therefore$ T is not a L.T.

6) Consider $T: R^2 \to R^2$ defined by $T(x,y) = (x+y, x)$, S.T T is linear.

Let $x = (x_1, x_2)$; $y = (y_1, y_2)$

$$x+y = (x_1+y_1,\ x_2+y_2)$$
$$T(x+y) = T(x_1+y_1,\ x_2+y_2) = (x_1+y_1+x_2+y_2,\ x_1+y_1)$$
$$= (x_1+x_2,\ x_1) + (y_1+y_2,\ y_1)$$
$$= T(x) + T(y)$$

$$\alpha x = (\alpha x_1, \alpha x_2)$$

$$T(\alpha x) = T(\alpha x_1, \alpha x_2) = (\alpha x_1 + \alpha x_2, \ \alpha x_1)$$

$$= \alpha(x_1 + x_2, x_1)$$

$$= \alpha T(x)$$

Hence T is linear.

7) Find the L.T $T: R^2 \to R^2$ such that, $T(1,2) = (2,3)$ and $T(0,1) = (1,4)$

Soln We write (x,y) as linear combination of $(1,2)$ & $(0,1)$

ie $(x,y) = \alpha(1,2) + \beta(0,1) = (\alpha + 0, \ 2\alpha + \beta)$ — ①

$(x,y) = (\alpha, 2\alpha) + (0,\beta) = (\alpha, 2\alpha+\beta)$

$\Rightarrow x = \alpha, \ y = 2\alpha + \beta$ (or) $\alpha = x$ & $\beta = y - 2x$.

Sub in ①

$y = 2\alpha + \beta$
put $\alpha = x$
$y = 2x + \beta$
$\beta = y - 2x$

Now $T(x,y) = \alpha T(1,2) + \beta T(0,1)$

$= x(2,3) + (y-2x)(1,4) = (2x, 3x) + (y-2x, \ 4(y-2x))$

$= (2x + y - 2x, \ 3x + 4(y-2x))$

$T(x,y) = (y, \ -5x + 4y)$

8) Let $T: R^2 \to R^2$ be a L.T such that $T(1,1) = (1,3)$, $T(-1,1) = (3,1)$
find $T(1,2)$.

Soln. Write x, y as a L.C of $(1,1)$ & $(-1,1)$

$\therefore (x,y) = \alpha(1,1) + \beta(-1,1) = (\alpha - \beta, \alpha + \beta)$

$\Rightarrow x = \alpha - \beta$, $y = \alpha + \beta$

Adding, we get $2\alpha = x + y \Rightarrow \alpha = \frac{1}{2}(x+y)$

$y - x = \alpha + \beta - \alpha + \beta \Rightarrow 2\beta = y - x \Rightarrow \beta = \frac{1}{2}(y-x)$

$x = \alpha - \beta$
$y = \alpha + \beta$
solving

Now, $T(x,y) = \alpha T(1,1) + \beta T(-1,1)$

$= \frac{1}{2}(x+y)(1,3) + \frac{1}{2}(y-x)(3,1)$

$= \left(\frac{1}{2}(x+y), \frac{3}{2}(x+y)\right) + \left(\frac{3}{2}(y-x), \frac{1}{2}(y-x)\right)$

$= \left(\frac{x}{2} + \frac{y}{2} + \frac{3y}{2} - \frac{3x}{2}, \frac{3x}{2} + \frac{3y}{2} + \frac{y}{2} - \frac{x}{2}\right)$

$T(x,y) = (2y - x, x + 2y)$

$\therefore T(1,2) = (2\times 2 - 1, 1 + 2\times 2)$
$= (3, 5)$

9) Define $T: V_3 \to V_1$ as $T(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$.

This is not a linear map, because for $x = y = (1, 0, 0)$,

$$T(x+y) = (x_1+y_1)^2 + (x_2+y_2)^2 + (x_3+y_3)^2 = (1+1)^2 + 0^2 + 0^2 = 4$$

$$T(x) + T(y) = x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2 = 2$$

$x = (x_1, x_2, x_3) = (1, 0, 0)$

$y = (y_1, y_2, y_3) = (1, 0, 0)$

$$T(x+y) \neq T(x) + T(y)$$

$\therefore$ T is not linear.

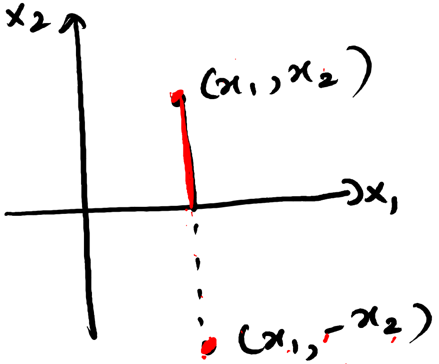
Defn. Define $T: U \to V$ (U & V - vector spaces) by the rule, $T(u) = 0_V$ $\forall u \in U$.

This is a linear map, called as <u>zero map</u>. because it maps every vector to the zero vector of V.

Defn: 2. Let U be a vector space. The identity map $I_U : U \to U$, defined by the rule $I_U(u) = u$ is also linear.

Defn: 3 Define $T : V_2 \to V_2$ by the rule $T(x_1, x_2) = (x_1, -x_2)$

This linear map is called reflection in the x_1-axis.



Defn. 4 Define $T : U \to U$ by the rule $T(x) = x + u_0 \quad \forall x \in U$.

T is not linear; because $T(x, y) = (x+y) + u_0 \neq (x + u_0) + (y + u_0) = T(x) + T(y)$

This map is called translation by the vector u_0.

Matrix associated with a linear map.
(Matrix transformation)

$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the jth column of the identity matrix in $\mathbb{R}^n$:

$$A = \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix} \tag{3}$$

$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & 1 \end{bmatrix}$

In R^2

$T(e_1) = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$T(e_2) = e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n) = x_1T(\mathbf{e}_1) + \cdots + x_nT(\mathbf{e}_n)$$

$$= \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A\mathbf{x}$$

The matrix A [redacted] is called the **standard matrix for the linear transformation** T.

We know now that every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be viewed as a matrix transformation, and vice versa. The term *linear transformation* focuses on a property of a mapping, while *matrix transformation* describes how such a mapping is implemented, [redacted].

EXAMPLE Find the standard matrix A for the dilation transformation $T(\mathbf{x}) = 3\mathbf{x}$, for $\mathbf{x}$ in $\mathbb{R}^2$.

Activate

$T(x_1, x_2) = (3x_1, 3x_2)$

SOLUTION Write

$$T(\mathbf{e}_1) = 3\mathbf{e}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = 3\mathbf{e}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

2) Find The std matrix A for the dilation transfor $T(\cancel{x}) = 2x$

$\Rightarrow A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$T(x) = kx$ in R^2

$A = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$