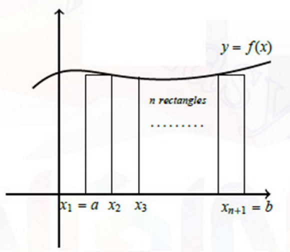
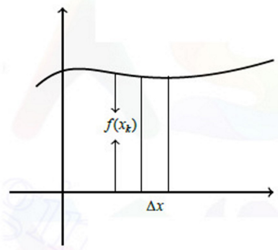
Integration

Friday, 27 November, 2020 10:55 AM

The definite integral

Integration as the limit of a sum





Consider the graph of the positive function $y = f(x)$. Let us find the area under the curve $y = f(x)$,between the x–axis and the ordinates $x = a$ and $x = b$. Divide this area into n rectangles of equal width $\Delta x = \dfrac{b-a}{n}$. Let the x– co ordinates at the left hand side of the rectangles be $x_1 = a, x_2, x_3, \ldots x_{n+1} = b$. Consider a typical rectangle, the k^{th} one with height $f(x_k)$. The area of this rectangle is $f(x_k)\Delta x$. The sum of all the areas of the n rectangles is

$$f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

In summation convention form, this can be written as $\sum\limits_{k=1}^{n} f(x_k)\Delta x$. This gives an estimate of the area under the curve but it is not exact. To improve the estimate we must take a large number of very thin rectangles. This can be achieved by allowing $n \to \infty$ and making $\Delta x \to 0$.

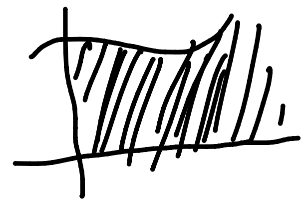
$\therefore$ Area under the curve = $\lim\limits_{n=\infty} \sum\limits_{k=1}^{n} f(x_k)\Delta x$.

The lower and upper limits on the sum correspond to the first and the last rectangle where $x = a$ and $x = b$ respectively. Hence the above limit can be written as $\lim\limits_{\Delta x \to 0} \sum\limits_{x=a}^{b} f(x)\Delta x$. Since the number of rectangles increase without bound, we drop the subsript k from x_k and write $f(x)$ which is the value of f at a typical value of x. If this can actually be found, it is called the definite integral of $f(x)$ from $x = a$ and $x = b$ and it is written as $\int\limits_a^b f(x)dx$.

$$\therefore \int\limits_a^b f(x)dx = \lim_{n=\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \lim_{\Delta x \to 0} \sum_{x=a}^{b} f(x)\Delta x.$$

Note. If we approximate each strip by a rectangle that has the same base as the strip and whose height is the same as the right edge of the strip; (i.e., x_k is taken at the right end points of the k^{th} rectangle and $f(x_k)$ as the height) then also the above result will be achieved.

$y=f(x)$



Definite

$\updownarrow$

Indefinite

Properties of definite integrals.

(i) $\int_b^a f(x)dx = -\int_a^b f(x)dx.$

(ii) $\int_a^a f(x)dx = 0.$

(iii) $\int_a^b c\,dx = c(b-a)$, where c is any constant.

(iv) $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$

(v) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, where c is any constant.

(vi) $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx.$

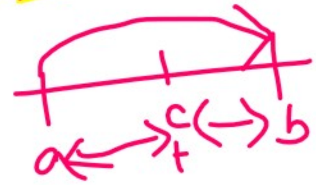
(vii) If $a < c < b$ then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$

$\int_a^b c\,dx$

$c\int_a^b dx = c[x]_a^b$

$= c[b-a]$

$a < c < b \leftarrow (a, b)$



Comparison properties of the integral.

(i) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0.$

(ii) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx.$

Basic results.

1. We have $\frac{d}{dx}(x^n) = nx^{n-1}.$

   (or) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n.$

   $\therefore \int x^n dx = \frac{x^{n+1}}{n+1} + c.$

2. $\frac{d}{dx}(\log x) = \frac{1}{x} \Rightarrow \int \frac{1}{x}dx = \log x + c.$

3. $\frac{d}{dx}(e^x) = e^x \Rightarrow \int e^x dx = e^x + c.$

4. $\int b^x dx = \frac{b^x}{\log b} + c.$ $\longrightarrow \int a^x dx = \frac{a^x}{\log a} + c$

5. $\int \sin x dx = -\cos x + c.$

6. $\int \cos x dx = \sin x + c.$

6. $\int \cos x dx = \sin x + c.$

7. $\int \sec^2 x dx = \tan x + c.$

8. $\int cosec^2 x dx = -\cot x + c.$

9. $\int \sec x \tan x dx = \sec x + c.$

10. $\int cosecx \cot x dx = -cosecx + c.$

11. $\int \frac{1}{\underline{\sqrt{1-x^2}}} dx = \sin^{-1} x + c.$

12. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c.$

13. $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c.$

14. $\int \sinh x dx = \cos hx + c.$

$\frac{d}{dx}(\sec x)$
$= \sec x \tan x$

$\int \sin x dx = -\cos x + C$

15. $\int \cosh x dx = \sin hx + c.$

16. $\int \frac{1}{\sqrt{x^2 - 1}} dx = \cos h^{-1} x + c$ or $\log(x + \sqrt{x^2 - 1}) + c.$

17. $\int \frac{1}{\sqrt{x^2 + 1}} dx = \sin h^{-1} x + c$ or $\log(x + \sqrt{x^2 + 1}) + c.$

Basic properties of indefinite integrals.

If $f(x)$ and $g(x)$ are functions of x and k is a constant, then the following properties are true.

(i) $\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx + c.$

(ii) $\int [f(x) - g(x)] dx = \int f(x) dx - \int g(x) dx + c.$

(iii) $\int k f(x) dx = k \int f(x) dx + c.$

Example Evaluate $\int (10x^4 - 2\sec^2 x) dx.$

Solution. $\int (10x^4 - 2\sec^2 x) dx = 10 \int x^4 dx - 2 \int \sec^2 x dx + c$

$$= 10 \cdot \frac{x^5}{5} - 2\tan x + c = 2x^5 - 2\tan x + c.$$

Example Evaluate $\int_1^2 \left(x^2 - 3\sqrt{x} + \frac{1}{x^2}\right) dx.$

Solution. $\int_1^2 \left(x^2 - 3\sqrt{x} + \frac{1}{x^2}\right) dx = \int_1^2 x^2 dx - 3 \int_1^2 x^{\frac{1}{2}} dx + \int_1^2 x^{-2} dx$

$$= \left(\frac{x^3}{3}\right)_1^2 - 3\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)_1^2 + \left(\frac{x^{-1}}{-1}\right)_1^2$$

$$= \frac{1}{3}[2^3 - 1^3] - 3 \times \frac{2}{3}[2^{\frac{3}{2}} - 1] - \left(\frac{1}{x}\right)_1^2$$

$$= \frac{1}{3}[8 - 1] - 2[2\sqrt{2} - 1] - [\frac{1}{2} - 1]$$

$$= \frac{7}{3} - 4\sqrt{2} + 2 + \frac{1}{2}$$

$$= \frac{21 + 12 + 3}{6} - 4\sqrt{2} = \frac{36}{6} - 4\sqrt{2} = 6 - 4\sqrt{2}.$$

Example Evaluate $\int_0^3 (x^3 - 6x)dx.$

Solution. $\int_0^3 (x^3 - 6x)dx = \int_0^3 x^3 dx - 6\int_0^3 x dx = \left(\frac{x^4}{4}\right)_0^3 - 6\left(\frac{x^2}{2}\right)_0^3$

$$= \frac{1}{4}[3^4 - 0] - 3[3^2 - 0]$$

$$= \frac{1}{4} \times 81 - 27 = \frac{81}{4} - 27 = \frac{81 - 108}{4} = -\frac{27}{4}.$$

Example Evaluate $\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right)dx.$

Solution. $\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right)dx = 2\int_0^2 x^3 dx - 6\int_0^2 x dx + 3\int_0^2 \frac{1}{x^2+1}dx$

$$= 2 \cdot \left(\frac{x^4}{4}\right)_0^2 - 6\left(\frac{x^2}{2}\right)_0^2 + 3\left(\tan^{-1} x\right)_0^2$$

$$= \frac{1}{2}[2^4 - 0] - 3[2^2 - 0] + 3\left(\tan^{-1} 2 - \tan^{-1} 0\right)$$

$$= 8 - 12 + 3\tan^{-1} 2 = 3\tan^{-1} 2 - 4.$$

Example Evaluate $\int_0^{\pi/6} \cos^2\left(\frac{x}{2}\right)dx.$

Solution. $\int_0^{\pi/6} \cos^2\left(\frac{x}{2}\right)dx = \frac{1}{2}\int_0^{\pi/6} (1 + \cos x)dx$

$$= \frac{1}{2}\left[\int_0^{\pi/6} dx + \int_0^{\pi/6} \cos x dx\right]$$

$$= \frac{1}{2}\left[(x)_0^{\pi/6} dx + \left(\sin x\right)_0^{\pi/6}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{6} - 0 + \sin\frac{\pi}{6} - \sin 0\right] = \frac{1}{2}\left[\frac{\pi}{6} + \frac{1}{2}\right] = \frac{\pi}{12} + \frac{1}{4}.$$

Example Evaluate $\int_1^9 \frac{2x^2 + x^2\sqrt{x} - 1}{x^2}dx.$

Solution.

$$\int_1^9 \frac{2x^2 + x^2\sqrt{x} - 1}{x^2}dx = \int_1^9 \left(\frac{2x^2}{x^2} + \frac{x^2\sqrt{x}}{x^2} - \frac{1}{x^2}\right)dx$$

$$= \int_1^9 \left(2 + x^{\frac{1}{2}} - x^{-2}\right)dx$$

$$= 2\int_1^9 dx + \int_1^9 x^{\frac{1}{2}}dx - \int_1^9 x^{-2}dx$$

$$= 2(x)_1^9 + \left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)_1^9 - \left(\frac{x^{-1}}{-1}\right)_1^9$$

$$= 2[9 - 1] + \frac{2}{3}\left(9^{\frac{3}{2}} - 1\right) + \left(\frac{1}{9} - 1\right)$$

$$= 16 + \frac{2}{3} \times 26 - \frac{8}{9}$$

$$= 16 + \frac{52}{3} - \frac{8}{9} = \frac{144 + 156 - 8}{9} = \frac{292}{9}.$$

Example Evaluate $\int \frac{1}{\sin^2 x \cos^2 x}dx$

Solution.

$$\int \frac{1}{\sin^2 x \cos^2 x}dx = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}dx$$

$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x}dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x}dx$$

$$= \int \sec^2 x dx + \int \mathrm{cosec}^2 x dx$$

$$= \tan x - \cot x + c.$$

Example Evaluate $\int \sin^2 3x dx.$

Solution.

$$\int \sin^2 3x dx = \int \frac{1}{2}(1 - \cos 6x)dx$$

$$= \frac{1}{2}\left[\int dx - \int \cos 6x dx\right]$$

$$= \frac{1}{2}\left[x - \frac{\sin 6x}{6}\right] + c$$

$$= \frac{x}{2} - \frac{\sin 6x}{12} + c.$$

Example Evaluate $\int \cos^3 2x dx$.

Solution.
$$\int \cos^3 2x dx = \int \left(\frac{1}{4}\cos 6x + \frac{3}{4}\cos 2x\right) dx$$
$$= \frac{1}{4}\int \cos 6x dx + \frac{3}{4}\int \cos 2x dx$$
$$= \frac{1}{4}\cdot\frac{\sin 6x}{6} + \frac{3}{4}\cdot\frac{\sin 2x}{2} + c$$
$$= \frac{\sin 6x}{24} + 3\frac{\sin 2x}{8} + c.$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$
$$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$$
$$\cos^3 2x = \frac{1}{4}\cos 6x + \frac{3}{4}\cos 2x.$$

Example Evaluate $\int \sin^4 x dx$.

Solution.
$$\int \sin^4 x dx = \int \left(\sin^2 x\right)^2 dx$$
$$= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx.$$
$$= \frac{1}{4}\int (1 + \cos^2 2x - 2\cos 2x) dx$$
$$= \frac{1}{4}\int \left(1 + \frac{1+\cos 4x}{2} - 2\cos 2x\right) dx$$
$$= \frac{1}{4}\int \left(1 + \frac{1}{2} + \frac{1}{2}\cos 4x - 2\cos 2x\right) dx$$
$$= \frac{1}{4}\int \left(\frac{3}{2} + \frac{1}{2}\cos 4x - 2\cos 2x\right) dx$$
$$= \frac{1}{4}\left[\frac{3}{2}\int dx + \frac{1}{2}\int \cos 4x dx - 2\int \cos 2x dx\right]$$
$$= \frac{1}{4}\left[\frac{3}{2} + \frac{1}{2}\frac{\sin 4x}{4} - 2\frac{\sin 2x}{2}\right] + c = \frac{3}{8}x + \frac{\sin 4x}{32} - \frac{\sin 2x}{4} + c.$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$
$$\cos^2 2x = \frac{1+\cos 4x}{2}.$$

Example Evaluate $\int \sin^2 4x dx$.

Solution.
$$\int \sin^2 4x dx = \int \left(\frac{1-\cos 8x}{2}\right) dx.$$
$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 8x\right) dx$$
$$= \frac{1}{2}\left[\int dx - \int \cos 8x dx\right]$$
$$= \frac{1}{2}\left[x - \frac{\sin 8x}{8}\right] + c = \frac{x}{2} - \frac{\sin 8x}{16} + c.$$

$$2\sin^2 x = 1 - \cos 2x$$
$$2\sin^2 4x = 1 - \cos 8x$$
$$\sin^2 4x = \frac{1-\cos 8x}{2}$$

The Substitution Rule

Integrals of functions of the form $\int f(ax+b)dx$.

This type of integrals can be evaluated by making a substitution $ax + b = t$.

For example consider $\int (2x+3)^2 dx$.

Let $t = 2x + 3$

$dt = 2dx$

$\therefore dx = \dfrac{dt}{2}$

$$\text{Now, } \int (2x+3)^2 dx = \int t^2 \cdot \frac{dt}{2} = \frac{1}{2}\int t^2 dt = \frac{1}{2}\cdot\frac{t^3}{3} + c = \frac{1}{6}\cdot(2x+3)^3 + c.$$

Standard results.

1. Evaluate $\int (ax+b)^n dx$.

Solution. Let $ax + b = t$.

$$adx = dt \Rightarrow dx = \frac{dt}{a}.$$

$$\int (ax+b)^n dx = \int t^n \frac{dt}{a} = \frac{1}{a}\cdot\frac{t^{n+1}}{n+1} + c = \frac{(ax+b)^{n+1}}{a(n+1)} + c.$$

In a similar way, the following results can be easily derived.

2. $\displaystyle\int \frac{1}{ax+b}dx = \frac{\log(ax+b)}{a} + c.$

3. $\displaystyle\int e^{ax+b}dx = \frac{e^{ax+b}}{a} + c.$

4. $\displaystyle\int \sin(ax+b)dx = -\frac{\cos(ax+b)}{a} + c.$

5. $\displaystyle\int \cos(ax+b)dx = \frac{\sin(ax+b)}{a} + c.$

6. $\displaystyle\int \sec^2(ax+b)dx = \frac{\tan(ax+b)}{a} + c.$

7. $\displaystyle\int \operatorname{cosec}^2(ax+b)dx = -\frac{\operatorname{cosec}(ax+b)}{a} + c.$

8. $\displaystyle\int \sec(ax+b)\tan(ax+b)dx = \frac{\sec(ax+b)}{a} + c.$

9. $\displaystyle\int \operatorname{cosec}(ax+b)\cot(ax+b)dx = -\frac{\cos(ax+b)}{a} + c.$

Example Evaluate $\int \frac{x^2}{(ax+b)^3}dx.$

Solution. Let $ax + b = t \Rightarrow ax = t - b \Rightarrow x = \frac{t-b}{a}$

$$adx = dt \quad \Rightarrow dx = \frac{dt}{a}.$$

$$\int \frac{x^2}{(ax+b)^3}dx = \int \frac{\left(t-\frac{b}{a}\right)^2}{t^3}\frac{dt}{a}$$

$$= \frac{1}{a}\int \frac{t^2 + \frac{b^2}{a^2} - \frac{2b}{a}t}{t^3}dt$$

$$= \frac{1}{a}\int \left(\frac{1}{t} + \frac{b^2}{a^2}\cdot\frac{1}{t^3} - \frac{2b}{a}\frac{1}{t^2}\right)dt$$

$$= \frac{1}{a}\left[\int \frac{1}{t}dt + \frac{b^2}{a^2}\int t^{-3}dt - \frac{2b}{a}\int t^{-2}dt\right]$$

$$= \frac{1}{a}\left[\log t + \frac{b^2}{a^2}\cdot\frac{t^{-2}}{(-2)} - \frac{2b}{a}\times\frac{t^{-1}}{(-1)}\right] + c$$

$$= \frac{1}{a}\left[\log t - \frac{b^2}{2a^2}\cdot\frac{1}{t^2} + \frac{2b}{a}\frac{1}{t}\right] + c$$

$$= \frac{1}{a}\left[\log(ax+b) - \frac{b^2}{a^2}\cdot\frac{1}{(ax+b)^2} + \frac{2b}{a}\frac{1}{(ax+b)}\right] + c$$

$$= \frac{1}{a}\log(ax+b) - \frac{b^2}{2a^3}\cdot\frac{1}{(ax+b)^2} + \frac{2b}{a^2}\frac{1}{(ax+b)} + c$$

Evaluation of integrals of the form $\int f(g(x))g'(x)dx$.

Method. Let $u = g(x) \quad du = g'(x)dx.$

Now, $\int f(g(x))g'(x)dx = \int f(u)du.$

This can be evaluated by earlier methods.

Example Evaluate $\int 2x\sqrt{1+x^2}dx.$

Solution. Let $1 + x^2 = u.$

$$2xdx = du.$$

$$\therefore \int 2x\sqrt{1+x^2}dx = \int \sqrt{u}du = \int u^{\frac{1}{2}}du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}(1+x^2)^{\frac{3}{2}} + c.$$

Example Evaluate $\int x^2\sqrt{1-4x^3}dx$.

Solution. Let $u = 1-4x^3$.

$$du = -4\times 3x^2dx.$$

$$x^2dx = -\frac{du}{12}$$

$$\therefore \int x^2\sqrt{1-4x^3}dx = \int \sqrt{u}\left(\frac{-du}{12}\right)$$

$$= -\frac{1}{12}\int u^{\frac{1}{2}}du$$

$$= -\frac{1}{12}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= -\frac{1}{12}\frac{2}{3}(1-4x^3)^{\frac{3}{2}} + c = -\frac{1}{18}(1-4x^2)^{\frac{3}{2}} + c.$$

Example Evaluate $\int \frac{1}{(1+e^x)(1+e^{-x})}dx$.

Solution. $\int \frac{1}{(1+e^x)(1+e^{-x})}dx = \int \frac{1}{(1+e^x)(1+\frac{1}{e^x})}dx$

$$= \int \frac{e^x}{(1+e^x)(1+e^x)}dx$$

$$= \int \frac{e^x}{(1+e^x)^2}dx \qquad 1+e^x = t$$

$$= \int \frac{dt}{t^2} \qquad e^xdx = dt$$

$$= \int t^{-2}dt = \frac{t^{-1}}{-1} + c = -\frac{1}{t} + c = -\frac{1}{1+e^x} + c.$$

Example Evaluate $\int x^3\cos(x^4+2)dx$.

Solution. Let $x^4+2 = t$.

$$4x^3dx = dt$$

$$x^3dx = \frac{dt}{4}$$

$$\therefore \int x^3\cos(x^4+2)dx = \int \cos t\frac{dt}{4}$$

$$= \frac{1}{4}\int \cos t dt = \frac{1}{4}\sin t + c = \frac{\sin(x^4+2)}{4} + c.$$

Example Evaluate $\int \frac{x}{\sqrt{1-4x^2}}dx$.

Solution. Let $1-4x^2 = t$.

$$-8xdx = dt$$

$$xdx = -\frac{dt}{8}$$

$$\therefore \int \frac{x}{\sqrt{1-4x^2}}dx = \int \frac{1}{\sqrt{t}}\left(-\frac{dt}{8}\right)$$

$$= -\frac{1}{8}\int t^{-\frac{1}{2}}dt$$

$$= -\frac{1}{8}\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c = -\frac{1}{8}\times 2\sqrt{t} + c = -\frac{1}{4}\sqrt{1-4x^2} + c.$$

Example Evaluate $\int_0^4 \sqrt{2x+1}dx$.

Solution. Let $2x+1 = t$.

$$2dx = dt \Rightarrow dx = \frac{dt}{2}.$$

When $x = 0, t = 1$.

When $x = 4, t = 9$.

$$\therefore \int_0^4 \sqrt{2x+1}dx = \int_1^9 \sqrt{t}\frac{dt}{2}$$

$$= \frac{1}{2}\int_1^9 t^{\frac{1}{2}}dt$$

$$= \frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right)_1^9 = \frac{1}{2}\times\frac{2}{3}[9^{\frac{3}{2}}-1] = \frac{1}{3}[27-1] = \frac{1}{3}\times 26 = \frac{26}{3}.$$

Example Evaluate $\int_1^2 \frac{1}{(3-5x)^2} dx$.

Solution. Let $3 - 5x = t$.

$$-5dx = dt \Rightarrow dx = -\frac{dt}{5}.$$

When $x = 0, t = -2$.

When $x = 2, t = -7$.

$$\therefore \int_1^2 \frac{1}{(3-5x)^2} = \int_{-2}^{-7} \frac{1}{t^2}\left(-\frac{dt}{5}\right)$$

$$= -\frac{1}{5}\int_{-2}^{-7} t^{-2} dt$$

$$= -\frac{1}{5}\left(\frac{t^{-1}}{-1}\right)_{-2}^{-7}$$

$$= \frac{1}{5}\left(\frac{1}{t}\right)_{-2}^{-7} = \frac{1}{5}\left(\frac{-1}{7} + \frac{1}{2}\right) = \frac{1}{5}\left[\frac{-2+7}{14}\right] = \frac{1}{5} \cdot \frac{5}{14} = \frac{1}{14}.$$

Integrals of symmetric functions

Theorem. Suppose f is continuous on $[-a, a]$

(i) If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.

(ii) If f is odd, then $\int_{-a}^{a} f(x)dx = 0$.

Example Evaluate $\int_{-2}^{2} (x^6 + 1)dx$.

Solution. Let $f(x) = x^6 + 1$.

$$f(-x) = (-x)^6 + 1 = x^6 + 1.$$

$\therefore$ f(x) is even.

By the properties of definite integrals

$$\therefore \int_{-2}^{2} f(x)dx = 2\int_{0}^{2} f(x)dx$$

$$= 2\int_{0}^{2} (x^6 + 1)dx$$

$$= 2\left[\int_{0}^{2} x^6 dx + \int_{0}^{2} dx\right]$$

$$= 2\left[\left(\frac{x^7}{7}\right)_0^2 + (x)_0^2\right] = 2\left[\frac{2^7}{7} - 0 + 2 - 0\right]$$

$$= 2\left[\frac{128}{7} + 2\right] = 2\left[\frac{128 + 14}{7}\right] = \frac{2 \times 142}{7} = \frac{284}{7}.$$

Example Evaluate $\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4} dx.$

Solution. Let $f(x) = \frac{\tan x}{1+x^2+x^4}.$

$$f(-x) = \frac{\tan(-x)}{1+(-x)^2+(-x)^4}$$

$$= \frac{-\tan x}{1+x^2+x^4} = -f(x).$$

$\therefore f(x)$ is odd.

By the properties of definite integrals

$$\int_{-1}^{1} f(x)dx = 0$$

$$\text{i.e,} \int_{-1}^{1} \frac{\tan x}{1+x^2+x^4} dx = 0.$$

Integration by parts

Every differentiation rule has a corresponding integration rule. The rule that correspond to the product for differentiation is called the integration by parts.

From differential calculus we have

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides with respect to x

$$\int \frac{d}{dx}(uv)dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

$$uv = \int udv + \int vdu$$

$$\int udv = uv - \int vdu.$$

Example Evaluate $\int x \sin x dx.$

Solution. Let $u = x$ $\qquad dv = \sin x dx$

$$du = dx \qquad \int dv = \int \sin x dx \Rightarrow v = -\cos x.$$

Applying integration by parts

$$\int x \sin x dx = \int udv = uv - \int vdu$$

$$= -x \cdot \cos x - \int -\cos x dx = -x\cos x + \sin x + c.$$

Example Evaluate $\int xe^x dx$.

Solution. Let $u = x$ $\qquad dv = e^x dx$

$$du = dx \qquad \int dv = \int e^x dx \Rightarrow v = e^x.$$

Applying integration by parts

$$\int xe^x dx = \int u dv = uv - \int v du$$

$$= x \cdot e^x - \int e^x dx$$

$$= xe^x - e^x + c = e^x(x-1) + c.$$

Example Evaluate $\int \log x dx$.

Solution. Let $u = \log x$ $\qquad dv = dx$

$$du = \frac{1}{x}dx \qquad \int dv = \int dx \Rightarrow v = x.$$

Applying integration by parts

$$\int \log x dx = \int u dv = uv - \int v du$$

$$= x \cdot \log x - \int x \cdot \frac{1}{x} dx$$

$$= x \log x - \int dx + c = x \log x - x + c = x(\log x - 1) + c.$$

Example Evaluate $\int x^n \log x dx$.

Solution. Let $u = \log x$ $\qquad dv = x^n dx$

$$du = \frac{1}{x}dx \qquad \int dv = \int x^n dx \Rightarrow v = \frac{x^{n+1}}{n+1}.$$

Applying integration by parts

$$\int x^n \log x dx = \int u dv = uv - \int v du$$

$$= \frac{x^{n+1}}{n+1} \cdot \log x - \int \frac{x^{n+1}}{n+1} \cdot \frac{1}{x} dx$$

$$= \frac{x^{n+1}}{n+1} \log x - \frac{1}{n+1} \int x^n dx + c$$

$$= \frac{x^{n+1}}{n+1} \log x - \frac{1}{n+1} \cdot \frac{x^{n+1}}{n+1} + c$$

$$= \frac{x^{n+1}}{n+1} \left(\log x - \frac{1}{n+1} \right) + c.$$

Example Evaluate $\int x^2 e^x dx.$

Solution. Let $u = x^2$ $\quad dv = e^x dx$

$$du = 2x dx \qquad \int dv = \int e^x dx \Rightarrow v = e^x.$$

Applying integration by parts

$$\int x^2 e^x dx = \int u dv = uv - \int v du$$

$$= x^2 \cdot e^x - \int e^x \cdot 2x\, dx$$

$$= x^2 e^x - 2\int x e^x\, dx$$

$$= x^2 e^x - 2\,[x e^x - e^x] + c$$

$$= x^2 e^x - 2x e^x + 2e^x + c = e^x(x^2 - 2x + 2) + c$$

Example Evaluate $\int x^2 \tan^{-1} x dx.$

Solution. Let $u = \tan^{-1} x$ $\quad dv = x^2 dx$

$$du = \frac{1}{1+x^2}dx \qquad \int dv = \int x^2 dx \Rightarrow v = \frac{x^3}{3}.$$

Applying integration by parts

$$\int x^2 \tan^{-1} x dx = \int u dv = uv - \int v du$$

$$= \frac{x^3}{3}\tan^{-1} x - \int \frac{x^3}{3} \cdot \frac{1}{1+x^2}dx$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int \frac{x^3}{1+x^2}dx \qquad \text{Let } 1 + x^2 = t$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int \frac{x^2}{1+x^2}\cdot x dx \qquad 2x dx = dt$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int \frac{t-1}{t}\cdot \frac{dt}{2} \qquad x dx = \frac{dt}{2}.$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{6}\int \left(1 - \frac{1}{t}\right) dt$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{6}\int dt + \frac{1}{6}\int \frac{1}{t}dt$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{6}\cdot t + \frac{1}{6}\cdot \log t + c = \frac{x^3}{3}\tan^{-1} x - \frac{1+x^2}{6} + \frac{1}{6}\log(1+x^2) + c.$$