Note In the steady state, the temperature at any particular point does not vary with time. That is temp. u depends only on x and not on time 't'.

Hence the PDE in the steady state becomes

$$\frac{d^2u}{dx^2} = 0. , \quad \text{Solving.}$$

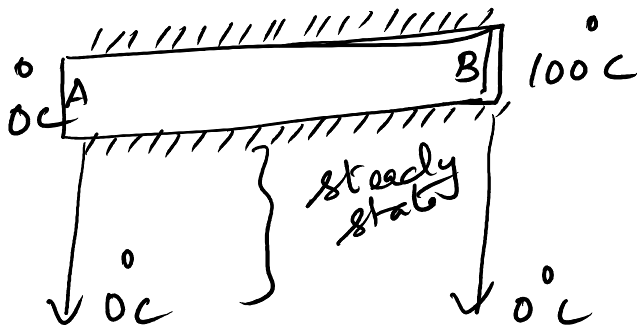
$$\frac{du}{dx} = a \quad \text{(one time integrating w. r. to } x\text{)}$$

Again integrate w. r. to x.

$$u = ax + b.$$

$\therefore$ In the steady state, temperature is $u(x) = ax+b$.

Problem. An insulated rod of length 'l' has its ends A & B maintained at $0^\circ C$ & $100^\circ C$ respectively until steady state conditions prevail. If B is suddenly reduced to $0^\circ C$ & A is maintained at $0^\circ C$, find the temperature at a distance x from A at time 't'.



Soln. The eqn of 1-D heat flow is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

In the steady state, 1-D heat eqn is

$$\frac{d^2 u}{dx^2} = 0.$$

The solution of this eqn is $u = ax + b$.

u is 0 at $x = 0$. $(u = ax + b)$

$\therefore$ $b = 0$.

u is 100 at $x = l$.

$\therefore$ $100 = al + b$, since $b = 0$,

$100 = al \Rightarrow a = \frac{100}{l}$.

$\therefore$ $\boxed{u = \frac{100x}{l}}$ $\rightarrow$ This is the initial temp $u(x,0)$ of the transient state (no more steady state)

$\therefore$ The boundary conditions are

(i) $u(0,t) = 0$

(ii) $u(l,t) = 0$ $\Bigg\}$ $t \geq 0$

(iii) $u(x,0) = \frac{100x}{l}$, $0 \leq x \leq l$.

The suitable soln of 1-D heat flow eqn is

$$u(x,t) = (A\cos px + B\sin px)\, e^{-a^2p^2t} \quad \text{—— (1)}$$

Sub b.c (i) in (1) i.e put $x=0$ in (1)

$$0 = A \cdot e^{-a^2p^2t} \quad (\because \sin 0 = 0,\ \cos 0 = 1)$$

$$\Rightarrow \boxed{A=0} \quad \text{Sub in (1).}$$

$$u(x,t) = B\sin px \cdot e^{-a^2p^2t} \quad \text{—— (2).}$$

Sub b.c (ii) in (1) i.e put $x=l$ in (2)

$$0 = B\sin pl \cdot e^{-a^2p^2t}.$$

$$\Rightarrow \sin pl = 0 = \sin n\pi.$$

$$\Rightarrow pl = n\pi$$

$$\Rightarrow p = \frac{n\pi}{l}. \quad \text{Sub in (2).}$$

$$u(x,t) = B \sin \frac{n\pi x}{l} e^{-\frac{a^2 n^2 \pi^2 t}{l^2}}$$, Generalising, (The most general Soln is)

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2 t a^2}{l^2}} \quad \text{——— (3)}.$$

Sub b.c (iii) in (3), $(\because e^0 = 1)$, Half range fourier Sine Series,

$$\frac{100x}{l} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}.$$

Where $B_n = \frac{2}{l} \int_0^l f(x) \cdot \sin \frac{n\pi x}{l} \, dx.$

$$B_n = \frac{2}{l} \int_0^l \frac{100x}{l} \cdot \sin \frac{n\pi x}{l} \, dx.$$

$$= \frac{200}{l^2} \int_0^l x \cdot \sin \frac{n\pi x}{l} \, dx.$$

Applying Bernoulli's Integral formula, $\int uv\,dx = uv_1 - u'v_2 + u''v_3 \cdots$

$u = x \quad (+) \quad v = \sin\frac{n\pi x}{l}$

$u' = 1 \quad (-) \quad v_1 = -\frac{l}{n\pi}\cos\frac{n\pi x}{l}$

$u'' = 0 \quad v_2 = -\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}$

$$\therefore B_n = \frac{200}{l^2}\left[-\frac{xl}{n\pi}\cos\frac{n\pi x}{l} + \frac{l^2}{n^2}\sin\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{200}{l^2}\left[\left(\frac{-l^2}{n\pi}\cdot(-1)^n + 0\right) - (0+0)\right] = \frac{200}{n\pi}\left(-(-1)^n\right)$$

$$B_n = \frac{200}{n\pi}(-1)^{n+1}.$$

sub in (3) The reqd soln is

$$u(x,t) = \sum_{n=1}^{\infty}\frac{200}{n\pi}(-1)^{n+1}\cdot\sin\frac{n\pi x}{l}\cdot e^{-\frac{a^2n^2\pi^2t}{l^2}} \;//.$$