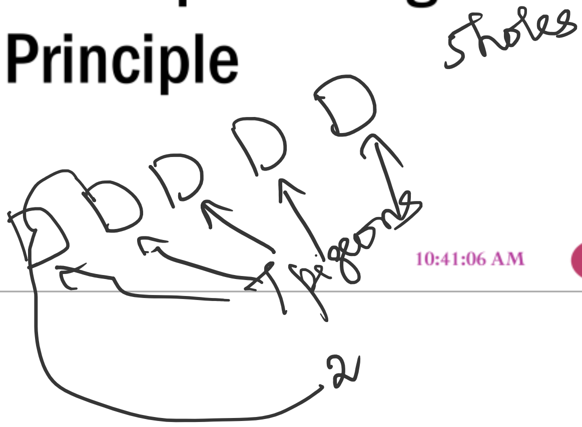
Module - 3

Basic Counting Techniques – Pigeonhole Principle



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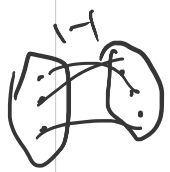
Pigeonhole Principle



- If k is a positive integer and k+1 or more objects are placed into k boxes then there is at least one box containing two or more of the objects.
- Cor: A function f from a set with k+1 or more elements to a set with k elements is not one to one.
- If n pigeons are assigned to m pigeonholes and m<n, then at least one pigeonhole contains two or more pigeons.
- If k pigeons are assigned to n pigeonholes then one of the pigeon holes must contain at least $\left\lfloor\frac{k-1}{n}\right\rfloor + 1$ pigeons.
- If n objects are placed into k boxes, then there are n pigeons, at least one box containing one box containing at least $\left\lceil\frac{n}{k}\right\rceil$ objects

Pigeonhole Principle - Examples

- Among any group of 367 people, there must be atleast two with the same birthday, because there are only 366 maximum possible birth days.
- In any group of 27 English words, there must be at least two that starts with the same letter, since there are 26 letters in English alphabet.
- Show that among 13 children, there are at least two children who were born in the same month.
  - Soln: Let us assume that 13 children as pigeons and the 12 months as the pigeon holes then by the pigeonhole principle there will be at least two children who were born in the same month.

k

$\boxed{4}$ $\boxed{3}$ $\boxed{3}$

$k+1 >$

$k = 10$

$n=3$

$\left\lceil \frac{k-1}{n} \right\rceil + 1$

$\frac{10-1}{3} + 1$

$3+1 = 4$

Pigeonhole Principle - Examples

- Show that if any four numbers from 1 to 6 are chosen then two of them will add to 7.
  - Soln: Let us form 3 sets containing two numbers whose sum is 7. A ={1,6}, B={2,5}, C={3,4}. The four numbers that will be chosen to the set that contains it. As there are only 3 sets, two numbers that there chosen is from the set whose sum is 7.
- Show that among any group of five (not necessarily consecutive) integers, there are two with the same reminder when divided by 4.
  - Soln: Take any group of five integers. When these are divided by 4 each have some remainder. Since there are five integers and four possible remainders when an integer is divided by 4, the pigeonhole principle implies that given five integers, at least two have the same remainder.

5, 6, 7, 13, 11

1 2 3 1 3

Pigeonhole Principle - Examples

- A bag contains 12 pairs of socks(each pair is in different color). If a person draws the socks one by one at random, determine almost how many draws are required to get atleast one pair of matched socks.
  - Soln: Let n denote the number of the draw. For n≤ 12, it is possible that the socks drawn are of different colors, since there are 12 colors. For n=13, all socks cannot have different colors at least two must have the same color. Here 13 as the number of pigeons and 12 colors as 12 pigeonholes. Thus, at most 13 draws are required to have at least one pair of socks of the same color.

Pigeonhole Principle - Examples

- Show that if seven colors are used to paint 50 cars atleast eight cars will have the same color.

  $k=50$ $n=7$

  - Soln: Assume that 50 cars (pigeons) are assign 7 colors (pigeonholes). Hence, by the generalized pigeonhole principle, at least [50-1/7]+1=8 cars will have the same color.

  $\left\lceil \frac{k-1}{n} \right\rceil + 1$

  $\frac{49}{7}+1$

  $7+1=8$

- Seven members of a family have total Rs. 2,886 in their pockets. Show that at least one of them must have at least Rs 413 in his pocket.
  - Let us assume members --→ pigeonholes; Rupees --→ Pigeons. Now 2886 pigeons are to be assigned to 7 pigeonholes. Using the extended pigeonhole principle. (i.e) k-1/n +1 where k=2886, n=7; 2886-1/7+1= 413. Hence there are 413 rupees in one member's pocket.

  $n=7$ $k=2886$

  $\frac{2885}{7}+1$ $\left\lceil \frac{2886-1}{7} \right\rceil + 1$ $\left\lceil \frac{k-1}{n} \right\rceil + 1$

  41

Pigeonhole Principle - Examples

- If 9 books are to be kept in 4 shelves, there must be atleast one shelf which contain atleast 3 books.
  - Soln: Let us assume books --→ pigeons; shelves -- → pigeonholes. Now 9 pigeons are to be assigned to 4 pigeonholes. Using the extended pigeonhole principle (k-1/n+1) when k = 9, n=4; 9-1/4 +1=8/4+1=3. Hence there are 3 books in one shelf at least.
- How many people must you have to guarantee that at least 9 of them will have birthdays in the same day of the week.
  - Let us assume that, the days week --→ pigeonholes; people --→ Pigeons. Now 7 pigeonholes and we have to find pigeons. Using the extended pigeonhole principle, (k-1/n)+1 where n=7 to find k. (k-1/7)+1=9; k+6=63; k=63-6=57. Thus, there must be 57 people to guarantee that at least 9 of them will have birthdays in the same day of the week.

$k+6=63$

$k=63-6=\underline{\underline{57}}$

$\left\lceil \frac{k-1}{7} \right\rceil + 1 = 9$

$k - 9$

$n - 4$

$\left\lceil \frac{k-1}{n} \right\rceil + 1$

$\left\lceil \frac{9-1}{4} \right\rceil + 1$

$\frac{8}{4} + 1 = 2+1$

$= \underline{\underline{3}}$

7 day = n

k = ?

$\frac{k-1+7}{7} = 9$

Pigeonhole Principle - Examples

- Show that if 30 dictionaries in a library contain a total of 61327 pages, then one of the dictionaries must have at least 2045 pages.
  - Let us assume that pages --→ pigeons; dictionaries --→ pigeonholes. Assign each page to the dictionary in which it appears. Then by extended pigeonhole principle, one dictionary must contain at least (k-1/n)+1 pages, when k=61327, n=30 61327-1/30 +1=2045 pages.

$k = 61327$

$n = 30$

$\left\lceil \frac{k-1}{n} \right\rceil + 1$

$\left\lceil \frac{61327-1}{30} \right\rceil + 1$

$\frac{61326}{30} + 1$

$2044.2 + 1$

$\simeq 2045$