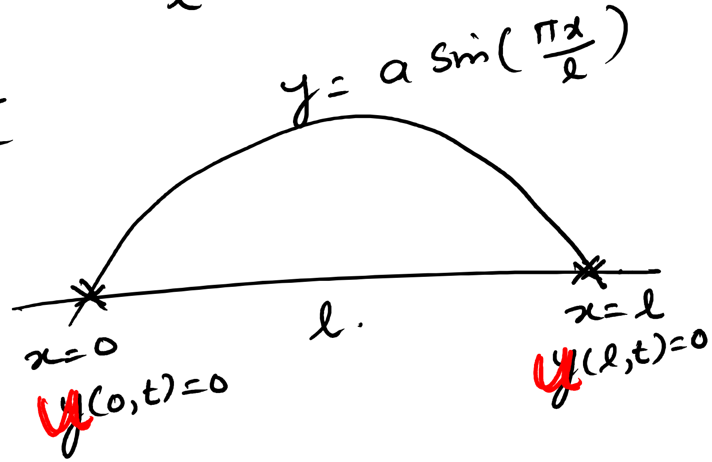
Problem: A string is stretched and fastened to two points 'l' apart. Motion is started by displacing the string in the form $u\ \cancel{y} = a\sin\left(\frac{\pi x}{l}\right)$, from which it is realeased at time $t=0$. S.T the displacement of at any point at a distance 'x' from one end at time 't' is given by $u(x,t)\ \cancel{y(x,t)} = a\sin\left(\frac{\pi x}{l}\right) \cdot \cos\left(\frac{\pi a t}{l}\right)$.



Soln The one-dim. wave eqn is $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$

The boundary conditions are

(i) $u(0,t) = 0, \quad t \geq 0$

(ii) $u(l,t) = 0, \quad t \geq 0$

(iii) $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0, \quad 0 \leq x \leq l$

(iv) $u\ \cancel{y}(x,0) = a\sin\frac{\pi x}{l}, \quad 0 \leq x \leq l.$

The suitable soln of 1-D Wave eqn is

$$u\cancel{y}(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \quad \text{—— } (1)$$

Substitute the first boundary condition (bc) in (1) ie put $x=0$ in (1).

$$0 = (A\cos 0 + B\sin 0)(C\cos pat + D\sin pat)$$

$$\Rightarrow A(C\cos pat + D\sin pat) = 0$$

$$\Rightarrow \boxed{A = 0}.$$ Sub in (1).

$$(1) \Rightarrow u\cancel{y}(x,t) = B\sin px\,(C\cos pat + D\sin pat) \quad \text{—— } (2).$$

Sub (II) b.c in (2) ie put $x=l$ in (2).

$$0 = B\sin pl\,(C\cos pat + D\sin pat)$$

$$\Rightarrow B\sin pl = 0$$

$$\Rightarrow \sin pl = 0 = \sin n\pi.$$

$$\Rightarrow pl = n\pi. \quad \Rightarrow \boxed{p = \frac{n\pi}{l}}.$$ Sub in (2).

$$\therefore u(x,t) = B \sin \frac{n\pi x}{l} \left(C \cos \frac{n\pi a t}{l} + D \sin \frac{n\pi a t}{l} \right) \quad \text{——} (3).$$

Diff. (3) partially w.r. to t.

$$\left(\frac{\partial u}{\partial t}\right)(x,t) = B. \sin \frac{n\pi x}{l}. \left(-C. \sin \frac{n\pi a t}{l}. \left(\frac{n\pi a}{l}\right) + D. \cos \frac{n\pi a t}{l}. \left(\frac{n\pi a}{l}\right)\right) \quad (3a)$$

Sub (III) b.c in (3a) i.e put $t=0$ in (3a).

$$\text{i.e} \quad 0 = B \sin \frac{n\pi x}{l} \left(0 + D. \cos 0. \left(\frac{n\pi a}{l}\right)\right)$$

$\Rightarrow D = 0$ ($\because$ B can not be equal to 0). Sub in (3).

$$\therefore u(x,t) = B \sin \frac{n\pi x}{l}. C \cos \frac{n\pi a t}{l}.$$

$$\text{i.e} \quad u(x,t) = BC \sin \frac{n\pi x}{l}. \cos \frac{n\pi a t}{l}.,$$ Taking $BC = B_n$,

The most general solution is,

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) \cdot \cos\left(\frac{n\pi a t}{l}\right) \quad \text{—— (4)}$$

Sub (IV) b.c in (4) i.e. put $t=0$ in (4)

$$a \sin\frac{\pi x}{l} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cdot \cos 0 \xrightarrow{} 1.$$

$$\therefore a \sin\frac{\pi x}{l} = B_1 \sin\frac{\pi x}{l} + B_2 \sin\frac{2\pi x}{l} + B_3 \sin\frac{3\pi x}{l} + \cdots\cdots$$

Equating the like terms on bothsides.

$\boxed{B_1 = a}$. $\therefore$ We have the soln only for $n=1$

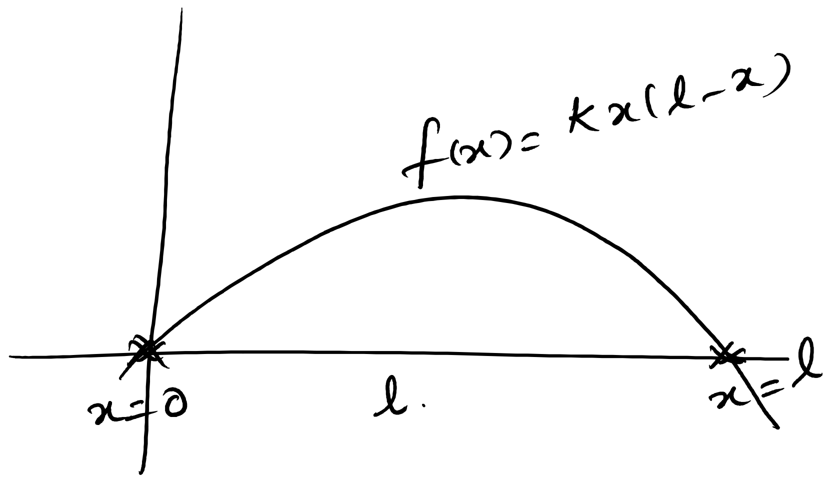
$$\therefore (4) \Rightarrow y(x,t) = B_1 \sin\frac{\pi x}{l} \cdot \cos\frac{\pi a t}{l}. \quad (\text{put } B_1 = a)$$

The reqd displacement is

$$y(x,t) = a \sin\frac{\pi x}{l} \cdot \cos\frac{\pi a t}{l}. //$$

2) A tightly stretched flexible string has its ends fixed at $x=0$ & $x=l$. At time $t=0$, the string is given a shape defined by $F(x) = Kx(l-x)$, Where K is a constant & then released. Find the displacement at any point x of the string at any time $t>0$.

Soln. The 1-D Wave eqn is $\dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}$.



The boundary conditions are.

$$\left.\begin{aligned} &(i)\ u(0,t) = 0 \\ &(ii)\ u(l,t) = 0 \end{aligned}\right\}\ t \geq 0.$$

$$\left.\begin{aligned} &(iii)\ \left(\frac{\partial u}{\partial t}\right)_{t=0} = 0 \\ &(iv)\ u(x,0) = Kx(l-x) \end{aligned}\right\}\ 0 \leq x \leq l.$$

The suitable soln of 1-D wave eqn is

$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$ —— ①.

Sub (i) b.c in ① i.e put $x=0$ in ①.

$0 = A(C\cos pat + D\sin pat)$

$\Rightarrow \boxed{A=0}$ Sub in ①.

$y(x,t) = B\sin px(C\cos pat + D\sin pat)$ —— ②

Sub (ii) b.c in ② i.e put $x=l$ in ②.

$0 = B\sin pl(C\cos pat + D\sin pat)$

$\Rightarrow \sin pl = 0$ ($B \neq 0$, if $B=0$, the soln is trivial)

$\therefore \sin pl = 0 = \sin n\pi$.

$\therefore pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$ Sub in ②.

$$y(x,t) = B \sin \frac{n\pi x}{l} \left(C \cos \frac{n\pi a t}{l} + D \sin \frac{n\pi a t}{l} \right) \quad \text{—— (3)}$$

Differentiating (3) partially w. r. to 't'.

$$\left(\frac{\partial y}{\partial t}\right)(x,t) = B \sin \frac{n\pi x}{l} \left(-C \sin \frac{n\pi a t}{l} \left(\frac{n\pi a}{l}\right) + D . \cos \frac{n\pi a t}{l} . \left(\frac{n\pi a}{l}\right)\right) \quad \text{—— (3a)}.$$

Sub (iii) b.c in (3a) i.e put $t=0$ in (3a).

$$0 = B \sin \frac{n\pi x}{l} \left(0 + D . \frac{n\pi a}{l} \right) \quad \left(\because \begin{array}{l} \sin 0 = 0 \\ \cos 0 = 1 \end{array} \right.$$

$$\Rightarrow \boxed{D = 0} \quad \text{Sub in (3)}.$$

$$y(x,t) = B \sin \frac{n\pi x}{l} . C \cos \frac{n\pi a t}{l}, \quad \text{Taking } BC = B_n,$$

The most general soln is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cdot \cos\frac{n\pi at}{l} \quad \text{—— (4)}$$

Sub (IV) in (4) i.e. put $t=0$ in (4)

$$Kx(l-x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}, \quad (\because \cos 0) = 1,$$ which is a half range fourier sine series where $B_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$.

$$\text{i.e. } B_n = \frac{2}{l}\int_0^l Kx(l-x) \sin\frac{n\pi x}{l}\,dx.$$

$$B_n = \frac{2K}{l}\int_0^l (xl - x^2) \sin\frac{n\pi x}{l}\,dx.$$

Applying Bernoulli's integral formula,

$u = lx - x^2$ $\qquad v = \sin \frac{n\pi x}{l}$.

$u' = l - 2x$ $\qquad v_1 = \frac{-\cos\left(\frac{n\pi x}{l}\right)}{n\pi/l} = -\frac{l}{n\pi} \cdot \cos \frac{n\pi x}{l}$.

$u'' = -2$ $\qquad v_2 = -\frac{l^2}{n^2\pi^2} \cdot \sin \frac{n\pi x}{l}$.

$u''' = 0$

$\qquad v_3 = \frac{l^3}{n^3\pi^3} \cdot \cos \frac{n\pi x}{l}$.

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

$$\therefore B_n = \frac{2k}{l}\left[-(lx - x^2)\frac{l}{n\pi}\cos\frac{n\pi x}{l} + (l-2x)\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l} - 2 \cdot \frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]_0^l.$$

$$= \frac{2k}{l}\left[\left(0 + 0 - \frac{2l^3}{n^3\pi^3}(-1)^n\right) - \left(0 + 0 - \frac{2l^3}{n^3\pi^3}\right)\right]$$

$\left(\because \cos 0 = 1,\ \sin 0 = 0,\ \cos n\pi = (-1)^n,\ \sin n\pi = 0\right)$

$$= \frac{2k}{\cancel{l}} \cdot \frac{2l^{\cancel{3}\,2}}{n^3\pi^3}\left(-(-1)^n + 1\right)$$

i.e. $B_n = \frac{4kl^2}{n^3\pi^3}\left(1-(-1)^n\right)$. Sub in (4).

The reqd soln is

$$y(x,t) = \sum_{n=1}^{\infty} \frac{4kl^2}{n^3\pi^3}\left(1-(-1)^n\right) \cdot \sin\frac{n\pi x}{l} \cdot \cos\frac{n\pi at}{l}.$$

$\left(1-(-1)^n\right) = 0$ if n is even.

$1-(-1)^n = 2$ if n is odd.

$$\therefore y(x,t) = \sum_{n=1,3,5\ldots}^{\infty} \frac{8kl^2}{n^3\pi^3} \cdot \sin\frac{n\pi x}{l} \cdot \cos\frac{n\pi at}{l}.$$

<u>Note</u>

$(-1)^n = 1$ if n is even.

$\therefore \left(1-(-1)^n\right) = 0$ if n is even.

$(-1)^n = -1$ if n is odd

$\therefore 1-(-1)^n = 1-(-1) = 2$ if n is odd,