Reduction formula

A reduction formula connects an integral with another of the same type but of lower order.

Wallis's formula

✓ (i) $\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx = \frac{(n-1)(n-3)(n-5)\cdots 1}{n(n-2)(n-4)\cdots 2} \times \frac{\pi}{2}$ if n is even.

✓ (ii) $\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx = \frac{(n-1)(n-3)(n-5)\cdots 2}{n(n-2)(n-4)\cdots 3} \times 1$ if n is odd

A **reduction formula** is one that enables us to solve an integral problem by **reducing it** to a problem of solving an easier integral problem, and then **reducing** that to the problem of solving an easier problem, and so on.

Problems. (i) $\int_0^{\pi/2} \sin^4 x\, dx$ (ii) $\int_0^{\pi/2} \sin^5 x\, dx$ (iii) $\int_0^{\pi/2} \cos^6 x\, dx$ (iv) $\int_0^{\pi/2} \cos^7 x\, dx$.

Rem: $\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx = \frac{(n-1)(n-3)(n-5)\cdots 1}{n(n-2)(n-4)\cdots 2} \times \frac{\pi}{2}$ if n is even

(i) $\int_0^{\pi/2} \sin^4 x\, dx$; $n = 4$ (even)

$$I = \frac{(4-1)(4-3)}{4 \times (4-2)} \times \frac{\pi}{2}$$

$$= \frac{3 \times 1}{4 \times 2} \times \frac{\pi}{2} = \frac{3\pi}{16}.$$

2) $\int_0^{\pi/2} \sin^5 x\, dx$; $n = 5$ (odd)

$$I = \frac{(5-1)(5-3)}{5(5-2)} \cdot 1$$

$$= \frac{4 \times 2}{5 \times 3} \times 1 = \frac{8}{15}$$

$$I = \int_0^{\pi/2} \cos^6 x\, dx \quad n = 6, \text{ even}$$

$$\therefore I = \frac{(6-1)(6-3)(6-5)}{6 \cdot (6-2)(6-4)} \cdot \frac{\pi}{2}$$

$$= \frac{5 \times 3 \times 1}{6 \times 4 \times 2} \times \frac{\pi}{2} = \frac{15\pi}{\cancel{4}96.} = \frac{5\pi}{32}$$

4) $I = \int_0^{\pi/2} \cos^7 x\, dx$, $n = 7$ (odd)

$$= \frac{(7-1)(7-3)(7-5)}{7(7-2)(7-4)} \cdot 1$$

$$= \frac{\overset{2}{\cancel{6}} \times \cancel{4} \times 2}{7 \times 5 \times \cancel{3}} \times 1 = \frac{16}{35}$$

5) Evaluate $\int_0^a \frac{x^7}{\sqrt{a^2-x^2}}\,dx$.

Soln: put $x = a\sin\theta$

$dx = a\cos\theta\,.\,d\theta$

When $x=0$, $a\sin\theta = 0 \Rightarrow \sin\theta = \frac{0}{a} = 0$ $(\because \sin 0^\circ = 0)$

$\therefore \boxed{\theta = 0}$

When $x=a$, $a\sin\theta = a \Rightarrow \sin\theta = 1$ $(\sin 90^\circ = 1)$

$\Rightarrow \theta = \pi/2$

$\therefore \theta : 0$ to $\pi/2$

$$\therefore \int_0^a \frac{x^7}{\sqrt{a^2-x^2}}\,dx = \int_0^{\pi/2} \frac{(a\sin\theta)^7 .\, a\cos\theta\,d\theta}{\sqrt{a^2 - a^2\sin^2\theta}} = \int_0^{\pi/2} \frac{a^8 .\sin^7\theta\cos\theta\,d\theta}{\sqrt{a^2(1-\sin^2\theta)}}$$

$(a\sin\theta)^7 = a^7\sin^7\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\therefore 1 - \sin^2\theta = \cos^2\theta$

$$= \frac{a^8}{a}\int_0^{\pi/2} \frac{\sin^7\theta\,\cancel{\cos\theta}}{\sqrt{\cos^2\theta}}\,d\theta$$

$\sqrt{\cos^2\theta} \rightarrow \cancel{\cos\theta}$

$$= a^7\int_0^{\pi/2} \sin^7\theta\,d\theta = \frac{6\times4\times2}{7\times5\times3}\times 1 = \frac{16}{35}$$

$(\because n$ is odd$)$

b) $$\int_0^\infty \frac{dx}{(a^2+x^2)^4}$$

put $x = a\tan\theta$.

$dx = a\sec^2\theta\, d\theta$

When $x=0$, $a\tan\theta = 0 \Rightarrow \tan\theta = 0$

$\Rightarrow \theta = 0$.

When $x=\infty$, $a\tan\theta = \infty$ $\tan\theta = \infty$

$\Rightarrow$ $(\theta = 90^\circ)$

$\Rightarrow \theta = \pi/2$

$\therefore \theta : 0$ to $\pi/2$

$$\therefore \int_0^\infty \frac{dx}{(a^2+x^2)^4} = \int_0^{\pi/2} \frac{a\sec^2\theta\, d\theta}{(a^2+a^2\tan^2\theta)^4} = \int_0^{\pi/2} \frac{a\sec^2\theta\, d\theta}{(a^2)^4(1+\tan^2\theta)^4}$$

$$= \frac{a}{a^8}\int_0^{\pi/2} \frac{\sec^2\theta}{(\sec^2\theta)^4}\, d\theta = \frac{1}{a^7}\int_0^{\pi/2} \frac{\sec^2\theta}{\sec^8\theta}\, d\theta$$

$$= \frac{1}{a^7}\int_0^{\pi/2} \frac{1}{\sec^6\theta}\, d\theta$$

$\sqrt{a^2-x^2} \rightarrow$ put $x = a\sin\theta$.

$a^2+x^2 \rightarrow x = a\tan\theta$.

$(a^m)^n = a^{mn}$

$\frac{1}{\sec\theta} = \cos\theta$

$$= \frac{1}{a^7}\int_0^{\pi/2} \cos^6\theta \, d\theta .$$

$n = 6$, even

$$\int_0^{\pi/2} \cos^n x \, dx = \frac{(n-1)(n-3)(n-5)\ldots 1}{n(n-2)(n-4)\ldots\ldots 2} \cdot \frac{\pi}{2}$$

$$= \frac{1}{a^7} \frac{5 \times 3 \times 1}{6 \times 4 \times 2} \times \frac{\pi}{2}$$

$$= \frac{15\pi}{96 a^7} \quad \text{or} \quad \frac{5\pi}{32}$$

$\cancel{a\cos\theta \, d\theta}$..

$$I = \int_0^{\pi/2} \frac{(a\sin\theta)^5}{\sqrt{a^2 - a^2\sin^2\theta}} .$$

H.W

1) $$\int_0^a \frac{x^5}{\sqrt{a^2 - x^2}} dx .$$

$$\frac{8a^5}{15}$$

put $x = a\sin\theta$

$dx = a\cos\theta \, d\theta$

limits

When $x = 0$, $a\sin\theta = 0$

$\Rightarrow \theta = 0$

When $x = a$, $a\sin\theta = a$

$\Rightarrow \sin\theta = 1$

$\theta = \pi/2$

$\theta : 0$ to $\pi/2$

2) $$\int_0^\infty \frac{dx}{(a^2 + x^2)^5}$$

$$\frac{105\pi a^9}{768}$$

Formulae.

1) $$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{(m-1)(m-3)(m-5)\cdots\cdots(n-1)(n-3)\cdots\cdots}{(m+n)(m+n-2)(m+n-4)\cdots\cdots} \times \frac{\pi}{2}$$ if

both m & n are even.

2) $$\int_0^{\pi/2} \sin^m x \cdot \cos^n x \, dx = \frac{(m-1)(m-3)\cdots\cdots(n-1)(n-3)\cdots\cdots}{(m+n)(m+n-2)(m+n-4)\cdots\cdots} \times 1$$ if

both m & n are odd

<u>Problem</u> Integrate (i) $\int_0^{\pi/2} \sin^4 x \cos^2 x \, dx$.

<u>Rem</u>: $\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{(m-1)(m-3)\ldots(n-1)(n-3)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots} \cdot \frac{\pi}{2}$ if m, n are even.

$m = 4, \; n = 2$ (even)

$$\therefore \int_0^{\pi/2} \sin^4 x \cos^2 x \, dx = \frac{(4-1)(4-3)\cdot(2-1)}{6\cdot(6-2)(6-4)} \cdot \frac{\pi}{2}$$

$$= \frac{\cancel{3} \times 1 \times 1}{2\cancel{6} \times 4 \times 2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi}{32}.$$

2) $\int_0^{\pi/2} \sin^7 x \cos^5 x \, dx$.

$m = 7, \quad n = 5 \quad (\text{odd})$

$$\therefore I_{m,n} = \frac{(7-1)(7-3)(7-5) \cdot (5-1)(5-3)}{12 \cdot (12-2)(12-4)(12-6)(12-8)(12-10)} \times 1.$$

$$= \frac{6 \times 4 \times 2 \times 4 \times \cancel{2}}{12 \times 10 \times \cancel{8} \times 6 \times \cancel{4} \times 2} \cdot 1$$

$$= \frac{1}{120}.$$

Note

$$\int_0^{\pi/2} \sin^m x \cdot \cos^n x \, dx = \frac{(m-1)(m-3) \cdots (n-1)(n-3) \cdots}{(m+n)(m+n-2)(m+n-4) + \cdots} \times 1$$

if both m & n are odd.

3) Evaluate $\int_0^{\pi/6} \cos^4 3\theta \cdot \sin^3 6\theta \, d\theta$.

Soln

$$I_{m,n} = \int_0^{\pi/6} \cos^4 3\theta \cdot \sin^3 2(3\theta)\, d\theta.$$

$$= \int_0^{\pi/6} \cos^4 3\theta \left(2\sin 3\theta \cos 3\theta\right)^3 d\theta.$$

$$= 8\int_0^{\pi/6} \cos^4 3\theta \sin^3 3\theta \cos^3 3\theta \, d\theta.$$

$$= 8\int_0^{\pi/6} \sin^3 3\theta \cdot \cos^7 3\theta \, d\theta.$$

put $3\theta = x$

$\therefore 3\,d\theta = dx$

$d\theta = \frac{dx}{3}$

$\sin 6\theta = \sin 2(3\theta)$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin 2(3\theta) = 2\sin 3\theta \cos 3\theta$

$\sin^3 6\theta = \left(2\sin 3\theta \cos 3\theta\right)^3$

$= 2^3 \sin^3 3\theta \cdot \cos^3 3\theta$.

$a^m \cdot a^n = a^{m+n}$

$\boxed{3\theta = x}$

When $\theta = 0$, $3(0) = x$

$\Rightarrow x = 0$

When $\theta = \frac{\pi}{6}$, $3 \times \frac{\pi}{6} = x$

$x = \pi/2$

$\therefore x : 0$ to $\pi/2$

$3\theta = x$,

When $\theta = 0$, $x = 0$

When $\theta = \frac{\pi}{6}$, $x = 3 \times \frac{\pi}{6} = \frac{\pi}{2}$

$\therefore \theta : 0$ to $\frac{\pi}{2}$

$$\therefore I_{m,n} = 8\int_0^{\pi/2} \sin^3 x \cos^7 x \cdot \frac{dx}{3}$$

$$= \frac{8}{3}\int_0^{\pi/2} \sin^3 x \cos^7 x \, dx$$

$$= \frac{8}{3}\left\{ \frac{(3-1)\ (7-1)(7-3)(7-5)\cdot 1}{10\,(10-2)(10-4)(10-6)(10-8)} \right\}$$

$$= \frac{8}{3} \times \frac{2\times 6\times 4\times 2}{10\times 8\times 6\times 4\times 2} = \frac{1}{15}.$$

$m=3$, $n=7$

(both m & n are odd)

$$I_{m,n} = \frac{(m-1)(m-3)\cdots(n-1)(n-3)\cdots 1}{(m+n)(m+n-2)(m+n-4)\cdots}$$

2) $\int_0^1 x^4 (1-x^2)^{3/2}\, dx.$

put $x = \sin\theta$

$\therefore dx = \cos\theta\, d\theta.$

When $x=0$, $\sin\theta = 0 \Rightarrow \theta = 0$

When $x=1$, $\sin\theta = 1 \Rightarrow \theta = \pi/2$

$\therefore \theta : 0$ to $\pi/2$

$a^2 - x^2 \to$ put $x = a\sin\theta$

$1 - x^2 \to$ put $x = \sin\theta$

$a^2 + x^2 \to x = a\tan\theta$

$1 + x^2 \to x = \tan\theta.$

$(a^m)^n = a^{mn}$

$(\cos^2\theta)^{3/2} = \cos^{\cancel{2} \times \frac{3}{\cancel{2}}}\theta = \cos^3\theta$

$$\therefore I_{m,n} = \int_0^{\pi/2} \sin^4\theta\, (1 - \sin^2\theta)^{3/2} \cdot \cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^4\theta\, \underbrace{(\cos^2\theta)^{3/2}}_{\cos^3\theta.} \cdot \cos\theta\, d\theta.$$

$$= \int_0^{\pi/2} \sin^4\theta \cdot \cos^4\theta\, d\theta. = \frac{(4-1).(4-3).(4-1).(4-3)}{8.(8-2)(8-4)(8-6)} \cdot \frac{\pi}{2}$$

$$= \frac{3\pi}{256}$$

$\sin^2\theta + \cos^2\theta = 1$

$1 - \sin^2\theta = \cos^2\theta$

Evaluate $\int_0^{\infty} \frac{x^2}{(1+x^2)^{7/2}}\, dx$.

Soln: $I_{m,n} = \int_0^{\infty} \frac{x^2}{(1+x^2)^{7/2}}\, dx$.

put $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$.

When $x=0$, $\tan\theta = 0 \Rightarrow \theta = 0$

When $x=\infty$, $\tan\theta=\infty \Rightarrow \theta = \frac{\pi}{2}$

θ : 0 to $\pi/2$

$\therefore I_{m,n} = \int_0^{\pi/2} \frac{\tan^2\theta}{(1+\underline{\tan^2\theta})^{7/2}} \cdot \sec^2\theta\, d\theta$.

$= \int_0^{\pi/2} \frac{\tan^2\theta}{(\sec^2\theta)^{\frac{7}{2}}} \cdot \sec^2\theta\, d\theta$

Rem

$\sec^2\theta - \tan^2\theta = 1$

$\therefore 1+\tan^2\theta = \sec^2\theta$.

$(a^m)^n = a^{mn}$

$(\sec^2\theta)^{7/2} = (\sec\theta)^{\cancel{2} \times \frac{7}{\cancel{2}}}$

$= \sec^7\theta$

$$= \int_0^{\pi/2} \frac{\tan^2\theta \sec^2\theta}{\sec^7\theta}\, d\theta.$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

$$= \int_0^{\pi/2} \frac{\sin^2\theta}{\cos^2\theta \cdot \sec^5\theta}\, d\theta. = \int_0^{\pi/2} \frac{\sin^2\theta}{\cos^2\theta} \cdot \cos^5\theta\, d\theta$$

$$\frac{1}{\sec\theta} = \cos\theta$$

$$= \int_0^{\pi/2} \sin^2\theta \cos^3\theta\, d\theta.$$

$$= \frac{(2-1)\,(3-1)}{5\,(5-2)\,(5-4)} = \frac{1\times 2}{5\times 3}$$

$$= \frac{2}{15}.$$

H.W

$$\int_0^{\infty} \frac{t^6}{(1+t^2)^7}\, dt$$

$$\int_0^\infty \frac{t^6}{(1+t^2)^7}\,dt$$

put $t = \tan\theta$.

$dt = \sec^2\theta\, d\theta$.

When $t=0$, $\tan\theta = 0 \Rightarrow \theta = 0$.

When $t=\infty$, $\tan\theta = \infty \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore I_{min} = \int_0^{\pi/2} \frac{\tan^6\theta}{(1+\tan^2\theta)^7}\cdot \sec^2\theta\, d\theta.$$

$$= \int_0^{\pi/2} \frac{\tan^6\theta}{(\sec^2\theta)^7}\cdot \sec^2\theta\, d\theta.$$

$$= \int_0^{\pi/2} \frac{\sin^6\theta}{\cos^6\theta \times \sec^{14}\theta}\cdot \sec^2\theta\, d\theta.$$

$$= \int_0^{\pi/2} \frac{\sin^6\theta}{\cos^6\theta\cdot \sec^{12}\theta}\, d\theta$$

$$= \int_0^{\pi/2} \frac{\sin^6\theta\cdot \cos^{12}\theta}{\cos^6\theta}\, d\theta$$

$$= \int_0^{\pi/2} \sin^6\theta\cdot \cos^6\theta\, d\theta.$$

$$= \frac{(6-1)(6-3)(6-5)\cdot(6-1)(6-3)(6-5)}{12(12-2)(12-4)(12-6)(12-8)(12-10)}\cdot\frac{\pi}{2}$$

$$= \frac{5\times 3\times 1\times 5\times 3\times 1}{12\times 10\times 8\times 6\times 4\times 2}\times\frac{\pi}{2}$$

$$= \frac{5\pi}{2048} \;//$$