Jacobian and its Properties

Definition. If u and v are continuous functions of two independent variables x and y having first order partial derivatives, then the Jacobian determinant or the Jacobian of u and v is defined by

$$\frac{\partial(u,v)}{\partial(x,y)} \text{ or } J\left(\frac{u,v}{x,y}\right) \text{ or } J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}.$$

If u, v, w are continuous functions of three independent variables x, y, z having first order partial derivatives, then the Jacobian of u, v, w w.r.t. x, y, z is defined as

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}.$$

Example If $x = r\cos\theta, y = r\sin\theta$, find the Jacobian of x and y w.r.t r and θ.

Solution.

$$x = r\cos\theta \Longrightarrow \frac{\partial x}{\partial r} = \cos\theta, \frac{\partial x}{\partial \theta} = -r\sin\theta.$$

$$y = r\sin\theta \Longrightarrow \frac{\partial u}{\partial r} = \sin\theta, \frac{\partial u}{\partial \theta} = r\cos\theta.$$

$$\text{Now, } \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Example In cylindrical polar coordinates, $x = \rho\cos\phi, y = \rho\sin\phi, z = z$. Show that $\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \rho$.

Solution.

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1(\rho\cos^2\phi + \rho\sin^2\phi) = \rho.$$

Properties of Jacobians

Property I. If u and v are functions of r and s where r and s are functions of x, y then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)}\frac{\partial(r,s)}{\partial(x,y)}$

Example If $u = 2xy, v = x^2 - y^2, x = r\cos\theta, y = r\sin\theta$, evaluate $\frac{\partial(u,v)}{\partial(r,\theta)}$ without actual substitution.

Solution. Given that u and v are functions of x and y. x and y are functions of r and θ.

$\therefore$ By property (1) of the Jacobians we have

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(r,\theta)}.$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix}$$

$$= -4y^2 - 4x^2 = -4(x^2 + y^2) = -4r^2.$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = -4(r^2)r = -4r^3.$$

Property II. If J_1 is the Jacobian of u, v with respect to x, y and J_2 is the Jacobian of x, y w.r.t. u, v then $J_1 J_2 = 1$. i.e., $\frac{\partial(u,v)}{\partial(x,y)} \bullet \frac{\partial(x,y)}{\partial(u,v)} = 1.$

Example If $x = u(1-v), y = uv$ then compute J_1 and J_2 and prove that $J_1 J_2 = 1$.

Solution. We have

$$J_1 = \frac{\partial(x,y)}{\partial(u,v)}, J_2 = \frac{\partial(u,v)}{\partial(x,y)}$$

$$x = u - uv, y = uv.$$

$$J_1 = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - uv + uv = u.$$

We shall express u and v interms of x and y.

$$x = u - uv = u - y \Rightarrow x + y = u$$

$$y = uv \Rightarrow v = \frac{y}{u} = \frac{y}{x+y}.$$

$$\text{we get } J_2 = \frac{1}{u}.$$

$$\text{Now } J_1 J_2 = u\frac{1}{u} = 1.$$

Property III. If the functions u, v, w of three independent variables x, y, z are not independent, then the Jacobian of u, v, w with respect to x, y, z vanishes.

Example If $u = x + 2y + z, v = x - 2y + 3z$ and $w = 2xy - xz + 4yz - 2z^2$, show that they are not independent. Find the relation between u, v and w.

Solution. Given: $u = x + 2y + z$.

$$\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 2, \frac{\partial u}{\partial z} = 1.$$

$$v = x - 2y + 3z.$$

$$\frac{\partial v}{\partial x} = 1, \frac{\partial v}{\partial y} = -2, \frac{\partial v}{\partial z} = 3.$$

$$w = 2xy - xz + 4yz - 2z^2.$$

$$\frac{\partial w}{\partial x} = 2y - z, \frac{\partial w}{\partial y} = 2x + 4z, \frac{\partial w}{\partial z} = -x + 4y - 4z.$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 2y - z & 2x + 4z & -x + 4y - 4z \end{vmatrix}$$

$$= 1(-2(-x + 4y - 4z) - 3(2x + 4z)) - 2(-x + 4y - 4z - 3(2y - z))$$
$$+ 1(2x + 4z + 2(2y - z))$$
$$= 2x - 8y + 8z - 6x - 12z + 2x - 8y + 8z + 12y - 6z + 2x + 4z + 4y - 2z = 0.$$

Hence, u, v, w are not independent.

Now $u + v = 2x + 4z, u - v = 4y - 2z.$

$$(u + v)(u - v) = 2(x + 2z).2(2y - z)$$

$$u^2 - v^2 = 4(2xy - xz + 4yz - 2z^2)$$

$$u^2 - v^2 = 4w.$$