Half Range Fourier Cosine Series & Sine Series.

(HRFCS & HRFSS)

	Interval (Range) $(0, \pi)$	$(0, l)$
Half Range Fourier Cosine Series (HRFCS)	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$,
Half Range Fourier Sine Series.	$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$	$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$.

Euler Constants.

	Range (0, π)	(0, l)
HRFCS	$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$ $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$	$a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx$ $a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx$
HRFSS	$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$	$b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx$

Note

HRFCS = F.S of even fn

HRFSS = F.S of odd fn.

1) Find the HRFCS & HRFSS for $f(x)=x$ in $(0,\pi)$

Soln

HRFCS.

The HRFCS is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

Given $f(x) = x$

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$

$= \frac{2}{\pi}\int_0^{\pi} x\,dx$

$= \frac{\cancel{2}}{\pi} \cdot \left(\frac{x^2}{\cancel{2}}\right)_0^{\pi}$

$= \pi^2/\pi \Rightarrow \boxed{a_0 = \pi}$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\,dx$

$= \frac{2}{\pi}\int_0^{\pi} x \cos nx\,dx$

$v = \cos nx$.

$u = x \qquad v_1 = \frac{\sin nx}{n}$

$u' = 1 \quad (-) \quad v_2 = -\frac{\cos nx}{n^2}$

$u'' = 0$

$a_n = \frac{2}{\pi}\left(\cancel{\frac{x \sin nx}{n}} + \frac{\cos nx}{n^2}\right)_0^{\pi}$

$$a_n = \frac{2}{\pi}\left(\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right)$$

$$= \frac{2\left((-1)^n - 1\right)}{\pi n^2} \quad \text{sub in } ①$$

The HRFCS is

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2\left((-1)^n - 1\right)}{n^2\pi} \cdot \cos nx .$$

(II) The HRFSS is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{——} ②$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx .$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx.$$

$v = \sin nx.$

$u = x \qquad v_1 = -\frac{\cos nx}{n}$

$u' = 1 \quad (-) \qquad v_2 = -\frac{\sin nx}{n^2}$

$u'' = 0$

$$b_n = \frac{2}{\pi}\left\{-\frac{x\cos nx}{n} + \cancel{\frac{\sin nx}{n^2}}^{\,0}\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\cdot\left(-\frac{\pi \cdot \cos n\pi}{n}\right) - 0$$

$$= -\frac{2(-1)^n}{n}.$$

Note

$-(-1)^n = (-1)^1 \times (-1)^n$

$= (-1)^{n+1}$

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\cdot \cancel{\sin nx}\ \sin nx$$

Subs in (2), The reqd HRFSS is

$$f(x) = \sum_{n=1}^{\infty} -\frac{2(-1)^n}{n}\cdot \sin nx.$$

2) Find the HRFCS & HRFSS of $f(x) = \underline{\underline{K}}$ (Constant) in $(0, \pi)$

To find HRFCS.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- (1)}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx.$$

$$= \frac{2}{\pi} \int_0^{\pi} K\,dx$$

$$= \frac{2K}{\pi} \int_0^{\pi} dx$$

$$= \frac{2K}{\pi} (x)_0^{\pi}$$

$$= \frac{2K\cancel{\pi}}{\cancel{\pi}}$$

$$a_0 = 2K$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx.$$

$$= \frac{2}{\pi} \int_0^{\pi} K \cos nx\,dx$$

$$= \frac{2K}{\pi} \left[\frac{\sin nx}{n}\right]_0^{\pi}$$

$$= \frac{2K}{\pi n} \left[\sin n\pi - \sin 0\right]$$

$$= 0$$

$\therefore$ Sub a_0 & a_n in (1).

$\therefore$ The HRFCS is

$$f(x) = \frac{2K}{2} + 0.$$

i.e. $f(x) = K.$

The HRFSS is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—} (2)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} K \sin nx\, dx.$$

$$= \frac{2K}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{-2K}{n\pi}\left[\cos n\pi - \cos 0\right]$$

$$= \frac{-2K}{n\pi}\left[(-1)^n - 1\right] \quad \text{Sub in } (2)$$

$\therefore$ The HRFSS is

$$f(x) = \sum_{n=1}^{\infty} \frac{-2K}{n\pi}\left((-1)^n - 1\right)\sin nx$$

(OR)

$$f(x) = \sum_{n=1}^{\infty} \frac{2K}{n\pi}\left(1 - (-1)^n\right)\sin nx.$$

Note $1 - (-1)^n = 2$ if ~~n is 0~~ n is odd

$\therefore (-1)^n = -1$, n is odd.

$(-1)^n = 1$ if n is even

$\therefore (1 - (-1)^n) = 0$, if n is even.

3) Find the HRFCS & HRFSS of $f(x) = c$, a constant in $(0, l)$

HRFCS

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \quad \text{—①}$$

$$a_0 = \frac{2}{l}\int_0^l f(x) \, \cancel{\cos x} \, dx$$

$$= \frac{2}{l}\int_0^l c \, dx.$$

$$= \frac{2}{l}(cx)_0^l$$

$$= \frac{2}{\cancel{l}} \cdot c\cancel{l}$$

$$a_0 = 2c$$

$$a_n = \frac{2}{l}\int_0^l f(x) \cos\frac{n\pi x}{l} \, dx.$$

$$= \frac{2}{l}\int_0^l c \cos\frac{n\pi x}{l} \cdot dx.$$

$$= \frac{2c}{\cancel{l}}\left(\sin\frac{n\pi x}{l}\right)_0^l \cdot \frac{\cancel{l}}{n\pi}$$

$$= \frac{2c}{n\pi} \cdot \left\{\underbrace{\sin\frac{n\pi \cancel{l}}{\cancel{l}}}_{\to 0} - \underbrace{\sin 0}_{\to 0}\right\}$$

$$\boxed{a_n = 0}$$

sub in ①

The HRFCS is

$$f(x) = \frac{2c}{2}$$

ie $\boxed{f(x) = c}$

HRFSS. $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$.

$$b_n = \frac{2}{l}\int_0^l \underbrace{f(x)}_{c} \sin \frac{n\pi x}{l}\, dx.$$

$$b_n = \frac{2c}{l}\int_0^l \sin \frac{n\pi x}{l}\, dx.$$

$$= \frac{2c}{l}\cdot\left[-\cos\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)\right]_0^l.$$

$$= -\frac{2c}{\cancel{l}}\cdot\frac{\cancel{l}}{n\pi}\left[\cos\frac{n\pi \cancel{l}}{\cancel{l}} - \cos 0\right]$$

$$= -\frac{2c}{n\pi}\cdot\left[(-1)^n - 1\right]$$

$$b_n = \frac{2c}{n\pi}\left(1-(-1)^n\right).$$

$\therefore$ HRFSS is

$$f(x) = \sum_{n=1}^{\infty} \frac{2c}{n\pi}\left(1-(-1)^n\right)\cdot \sin\frac{n\pi x}{l}$$

$(1-(-1)^n) = 0$ if n is even

$= 2$ if n is odd.

Note:

(i) ~~I H~~ The HRFCS of a Constant is the same constant

Ex HRFCS of $f(x) = 5$ is, ~~$\not 5$~~ 5

HRFCS of $f(x) = 1$ is 1

$a_0 = 2$ times the constant & $a_n = 0$.

(ii) Also, Fourier Series of a constant K is K.

Here $a_0 = 2K$, $a_n = b_n = 0$.

4) Find the HRFCS & HRFSS of $f(x)=x^2$ in $0<x<1$

Soln Given range is $(0,\pi) \rightarrow$ Half range

$\downarrow$

$(0,l)$

here $l=1$

(i) HRFCS is $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$, put $l=1$ — (1)

i.e. $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos n\pi x$

$a_0=\frac{2}{1}\int_0^1 f(x)\,dx = 2\int_0^1 x^2\,dx = 2\left(\frac{x^3}{3}\right)_0^1$

$(\because l=1)$

$a_0 = 2/3$

$$a_n = \frac{2}{1}\int_0^1 f(x)\cos\frac{n\pi x}{1}\,dx$$

$$= 2\int_0^1 x^2\cos n\pi x\,dx.$$

$u = x^2$ $\quad v = \cos n\pi x$

$v_1 = \frac{\sin n\pi x}{n\pi}$

$u' = 2x \quad (-) \quad v_2 = -\frac{\cos n\pi x}{n^2\pi^2}$

$u'' = 2 \quad v_3 = -\frac{\sin n\pi x}{n^3\pi^3}$

$u''' = 0$

$$a_n = 2\left\{ \cancel{\frac{x^2\sin n\pi x}{n\pi}} + \frac{2x\cos n\pi x}{n^2\pi^2} - \cancel{\frac{2\sin n\pi x}{n^3\pi^3}} \right\}_0^1$$

0

$$a_n = 2\left\{\frac{2\cos n\pi}{n^2\pi^2} - 0\right\}$$

$$a_n = \frac{4(-1)^n}{n^2\pi^2}$$ Sub in ①

The HRF.CS. is

$$f(x) = \frac{2/3}{2} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2\pi^2}\cdot\cos n\pi x$$

$$= \frac{1}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2\pi^2}\cos n\pi x$$

(ii) HRFSS

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{1}$$

$$\text{i.e. } f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x \quad \text{--- (2)}$$

$$b_n = \frac{2}{1}\int_0^1 f(x) \sin \frac{n\pi x}{1}\, dx$$

$$= 2\int_0^1 x^2 \sin n\pi x \, dx.$$

$u = x^2$ $\quad v = \sin n\pi x$

$u' = 2x$ $\quad v_1 = -\frac{\cos n\pi x}{n\pi}$

$u'' = 2$ $\quad (-)$ $\quad v_2 = -\frac{\sin n\pi x}{n^2\pi^2}$

$u''' = 0$ $\quad v_3 = \frac{\cos n\pi x}{n^3\pi^3}$

$$b_n = 2\left\{ -\frac{x^2}{n\pi}\cos n\pi x + \cancel{2x\sin} \; \cancel{2x\frac{\sin n\pi x}{n^2\pi^2}} + \frac{2\cos n\pi x}{n^3\pi^3} \right\}_0^1$$

$$= 2\left\{\left(-\frac{1}{n\pi}(-1)^n + \frac{2}{n^3\pi^3}(-1)^n\right) - \left(0 + \frac{2}{n^3\pi^3}\right)\right\}$$

$$b_n = 2\left(-\frac{(-1)^n}{n\pi} + \frac{2(-1)^n}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right)$$

Sub in (2)

$$f(x) = 2\left(-\frac{(-1)^n}{n\pi} + \frac{2(-1)^n}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right). \sin n\pi x$$

* Root Mean Square Value (RMS value)

The RMS value of a fn $f(x)$ in (a, b) is defined as $\bar{y} = \sqrt{\dfrac{\int_a^b (f(x))^2 dx}{b-a}}$.

Ex 1.

Find the RMS values of $f(x) = x$ in $(0, l)$

Soln

RMS value $\bar{y} = \sqrt{\dfrac{\int_0^l x^2 dx}{l-0}} = \sqrt{\dfrac{\left(\frac{x^3}{3}\right)_0^l}{l}}$

$= \sqrt{\dfrac{l^3/3}{l}} = \sqrt{\dfrac{l^{\cancel{3}2}}{3} \times \dfrac{1}{\cancel{l}}} = \sqrt{\dfrac{l^2}{3}} = \dfrac{l}{\sqrt{3}}$

2) Find the RMS Value of $f(x) = x^2$ in $(0, 2\pi)$

soln

$$\bar{y} = \sqrt{\frac{\int_a^b (f(x))^2 dx}{b-a}} \quad , \quad f(x) = x^2$$

$$= \sqrt{\frac{\int_0^{2\pi} x^4 \, dx}{2\pi - 0}} = \sqrt{\frac{\left(\frac{x^5}{5}\right)_0^{2\pi}}{2\pi}} = \sqrt{\frac{\frac{(2\pi)^5}{5}}{2\pi}}$$

$$= \sqrt{\frac{2\pi \times 2\pi \times 2\pi \times 2\pi \times \cancel{2\pi}}{5} \times \frac{1}{\cancel{2\pi}}} = \sqrt{\frac{16\pi^4}{5}}$$

$$= \frac{\sqrt{16}\sqrt{\pi^4}}{\sqrt{5}} = \frac{4\pi^2}{\sqrt{5}} \quad //$$

Parsevell's Identity (RMS value interms of Euler/Fourier Constants).

$$\bar{y}^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

H.W (1) Find HRFCS & HRFSS of

(i) $f(x) = 5$ in $(0, \pi)$

(ii) $f(x) = x^2$ in $(0, \pi)$