Inverse of a Linear Transformation.

* A linear map $T: U \to V$ is said to be non-singular if it is one-to-one and onto. Such a map is called as isomorphism.
* Any function has an inverse iff it is 1-1 and onto.
* A linear transformation is non-singular iff it has an inverse.

Pbm Let $T: V_2 \to V_2$ defined by $T(x_1, x_2) = (x_1, -x_2)$. find T^{-1}.

Soln. To prove (i) T is 1-1 (ii) T is onto.

$$N(T) = T(x_1, x_2) = (0, 0)$$

$$\Rightarrow (x_1, -x_2) = (0, 0)$$

$$\Rightarrow x_1 = 0, -x_2 = 0 \Rightarrow (x_1, x_2) = (0, 0)$$

$$\Rightarrow x_2 = 0 \therefore T \text{ is } 1-1.$$

$T(x_1) = T(x_2)$
$\Rightarrow x_1 = x_2$

$u \in U$, $v \in V$
$T(u) = T(v) \Rightarrow$
$u = v$.
T is 1-1
If $R(T) = V$, T is said to be onto.

$R(T) = \{T(x) ; x \in V_2\}$

$= \{T(x_1, x_2) ; x_1, x_2 \in V_2\}$

$= (x_1, -x_2)$ i.e. $R(T) = V_2$

$\therefore T$ is onto.

$T(x_1, x_2) = (x_1, -x_2)$

To find $T^{-1}(y_1, y_2)$

Top $T^{-1}(y_1, y_2) = (y_1, -y_2)$

i.e. $T(y_1, -y_2) = (y_1, -(-y_2)) = (y_1, y_2)$ $\Rightarrow T(y_1, y_2) = (y_1, y_2)$

$T^{-1}(y_1, y_2) = (y_1, -y_2)$

$T(y_1, y_2) = (y_1, y_2)$

$\Rightarrow (y_1, -y_2) = T^{-1}(y_1, y_2)$

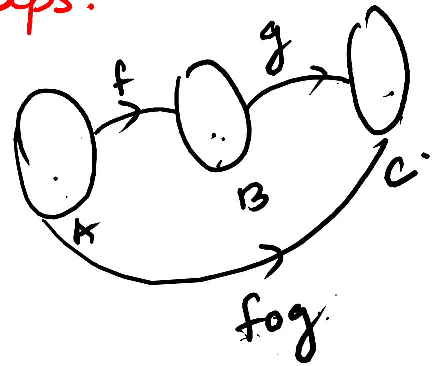
Thus $T^{-1} : V_2 \to V_2$ is defined by

$T^{-1}(y_1, y_2) = (y_1, -y_2)$

Composition of Linear Maps.

Defn. Let $T : U \to V$ and $S : V \to W$ be two linear maps. The composition $S \circ T : U \to W$ is defined as

$$S \circ T\,(u) = S(T(u)) \quad \forall u \in U.$$



Symbolically $U \xrightarrow{S \circ T} W = U \xrightarrow{T} V \xrightarrow{S} W$

To prove $S \circ T$ is linear.

$$\begin{aligned} S \circ T\,(u_1 + u_2) &= S(T(u_1 + u_2)) \\ &= S(T(u_1) + T(u_2)) \\ &= S(T(u_1)) + S(T(u_2)) \\ &= S \circ T\,(u_1) + S \circ T\,(u_2) \end{aligned}$$

$$\begin{aligned} S \circ T\,(\alpha u) &= S(T(\alpha u)) \\ &= S(\alpha T(u)) \\ &= \alpha S(T(u)) \\ &= \alpha (S \circ T)(u), \quad u \in U, \end{aligned}$$

α, scalar.

Hence $S \circ T$ is linear.

Pbm: Let a linear map $T: V_3 \to V_4$ be defined by $T(e_1) = (1,1,0,0)$, $T(e_2) = (1,-1,1,0)$, $T(e_3) = (0,-1,1,1)$, where $\{e_1, e_2, e_3\}$ is the standard basis of V_3 & let $S: V_4 \to V_2$ is linear, defined by $S(f_1) = (1,0)$, $S(f_2) = (1,1)$, $S(f_3) = (1,-1)$, $S(f_4) = (0,1)$, where $\{f_1, f_2, f_3, f_4\}$ is the std. basis for V_4. find $S \circ T(e_1)$, $S \circ T(e_2)$, $S \circ T(e_3)$

$e_1 = \{1,0,0)$
$e_2 = (0,1,0)$
$e_3 = (0,0,1)$

soln. Let $f_1 \to (1,0,00)$, $f_2 \to (0,1,0,0)$
$f_3 \to (0,0,1,0)$; $f_4 \to (0,0,0,1)$

(i) $S \circ T(e_1) = S(T(e_1)) = S(1,1,0,0) = S(f_1 + f_2)$
$= S(f_1) + S(f_2)$
$= (1,0) + (1,1)$.
$= (2,1)$

ii) $S \circ T(e_2) = S(T(e_2))$

$= S(1,-1,1,0)$

$= S(f_1 - f_2 + f_3)$

$= S(f_1) - S(f_2) + S(f_3)$

$= (1,0) - (1,1) + (1,-1)$

$= (1,-2)$.

iii) $S \circ T(e_3) = S(T(e_3)) = S(0,-1,1,1) = S(-f_2 + f_3 + f_4)$

$= -S(f_2) + S(f_3) + S(f_4)$

$= -(1,1) + (1,-1) + (0,1)$

$= (0,-1)$

$f_1 = (1,0,0,0)$
$f_2 = (0,1,0,0)$
$f_3 = (0,0,1,0)$
$f_4 = (0,0,0,1)$
$f_1 - f_2 + f_3$