Continuity

Definition. A function f is continuous at a number a, if $\lim_{x \to a} f(x) = f(a)$.
f is discontinuous at a if it is not continuous at a.

Note. For proving a function f to be continuous, we have to prove the following.

1. $f(a)$ must be defined.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

Definition. A function f is said to be continuous from the right at a number a if $\lim_{x \to a+} f(x) = f(a)$, and f is said to be continuous from the left at a if $\lim_{x \to a-} f(x) = f(a)$.

Definition. A function f is continuous on an interval if it is continuous at every number in the interval.

Theorem. If f and g are continuous at a and c is a constant, then the following functions are also continuous at a.

1. $f + g$ 2. $f - g$ 3. cf 4. fg 5. $\dfrac{f}{g}$ $g(a) \neq 0$.

Example 1 Find the domain where the fnction f is continuous. Also find the numbers at which the function f is discontinous where

$$f(x) = \begin{cases} 1 + x^2 & , \text{if } x \leq 0 \\ 2 - x & , \text{if } 0 < x \leq 2 \\ (x-2)^2 x & , \text{if } x > 2 \end{cases}$$

Solution. The function f changes its value at $x = 0$, and $x = 2$.

$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} (1 + x^2) = 1.$
$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} (2 - x) = 2.$
Since, $\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x)$, f is not continuous at $x = 0$.
$\lim_{x \to 2-} f(x) = \lim_{x \to 2-} (2 - x) = 0.$
$\lim_{x \to 2+} f(x) = \lim_{x \to 2+} (x - 2)^2 = 0.$
Since, $\lim_{x \to 2-} f(x) = \lim_{x \to 2+} f(x) = 0 = f(2)$, $f(x)$ is continuous at $x = 2$.
$\therefore$ The only number at which the function is discontinuous is at $x = 0$.

The domain of continuity of f is $\{(-\infty, 0) \cup (0, \infty)$.

Derivatives and differentiation rules

Definition. The derivative of a function f at a number a, denoted by $f'(a)$ is defined as $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$, if the limit exists.

Note. Let $x = a+h$. As $h \to 0$, $x \to a$. The equivalent definition for the derivative is $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$.

Derivative of a function. Let $f(x)$ be a given function. The derivative of $f(x)$ at any variable point x is defined by $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

Worked Examples

Derivatives of simple functions

1. If c is a constant, prove that $\frac{d}{dx}(c) = 0$.

   Proof. Let $f(x) = c$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} \frac{0}{h} = 0. \end{aligned}$$

Derivatives of trigonometric functions

1. Prove that $\frac{d}{dx}(\sin x) = \cos x$.

   Proof. Let $f(x) = \sin(x)$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h} \\ &= \lim_{h\to 0} \left(\frac{\sin(x)(\cos(h)-1)}{h} + \frac{\cos(x)\sin(h)}{h}\right) \\ &= \lim_{h\to 0} \sin(x)\left(\frac{(\cos(h)-1)}{h} + \lim_{h\to 0} \cos(x)\frac{\sin(h)}{h}\right) \\ &= \sin(x)\lim_{h\to 0}\left(\frac{(\cos(h)-1)}{h} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}\right) \\ &= \sin(x)\times 0 + \cos(x)\times 1 = 0 + \cos(x) = \cos(x). \end{aligned}$$

2. Prove that $\frac{d}{dx}(\cos x) = -\sin x.$

Proof. Let $f(x) = \cos(x)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h}$$
$$= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$
$$= \lim_{h\to 0} \frac{\cos(x)(\cos(h) - 1) - \sin(x)\sin(h)}{h}$$
$$= \lim_{h\to 0} \left(\frac{\cos(x)(\cos(h) - 1)}{h} - \frac{\sin(x)\sin(h)}{h}\right)$$
$$= \lim_{h\to 0} \cos(x)\left(\frac{(\cos(h) - 1)}{h} - \lim_{h\to 0} \sin(x)\frac{\sin(h)}{h}\right)$$
$$= \cos(x) \times 0 - \sin(x) \times 1 = -\sin(x).$$

$$f'(x) = -\sin x.$$

3. Prove that $\frac{d}{dx}(\tan x) = \sec^2 x.$

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$$
$$= \frac{\cos x \times \cos x - \sin x \times (-\sin x)}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

4. Prove that $\frac{d}{dx}(cosec x) = -cosec x \cot x.$

Proof. $$\frac{d}{dx}(cosec x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{\sin x \frac{d}{dx}(1) - 1 \times \frac{d}{dx}(\sin x)}{\sin^2 x}$$
$$= \frac{\sin x \times 0 - 1 \times (\cos x)}{\sin^2 x}$$
$$= \frac{-\cos x}{\sin^2 x} = \frac{-1}{\sin x} \times \frac{\cos x}{\sin x} = -cosec x \cot x.$$

5. Prove that $\frac{d}{dx}(\sec x) = \sec x \tan x.$

Proof. $$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(1) - 1 \times \frac{d}{dx}(\cos x)}{\cos^2 x}$$
$$= \frac{\cos x \times 0 - 1 \times (-\sin x)}{\cos^2 x}$$
$$= \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x} \times \frac{1}{\cos x} = \sec x \tan x.$$

6. Prove that $\frac{d}{dx}(\cot x) = -cosec^2 x$.

Proof. $\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{\sin x \frac{d}{dx}(\cos x) - \cos x \times \frac{d}{dx}(\sin x)}{\sin^2 x}$

$$= \frac{\sin x \times (-\sin x) - \cos x \times (\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = -\frac{1}{\sin^2 x} = -cosec^2 x.$$

Rules on differentiation

1. **The sum rule.** If f and g are both differentiable, then $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$.

2. **Constant multiple rule.** If c is a constant and f is a differentiable function, then $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$.

3. **The difference rule.** If f and g are both differentiable, then $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$.

4. **The product rule.** If f and g are both differentiable, then $\frac{d}{dx}[f(x) \times g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$.

5. **Quotient rule.** If f and g are both differentiable, then $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$.

Example 1 If $y = x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5$, find $\frac{dy}{dx}$.

Solution. Given, $y = x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5\right)$$

$$= \frac{d}{dx}(x^8) + \frac{d}{dx}(12x^5) - \frac{d}{dx}(4x^4) + \frac{d}{dx}(10x^3) - \frac{d}{dx}(6x) + \frac{d}{dx}(5)$$

$$= 8x^{8-1} + 12\frac{d}{dx}(x^5) - 4\frac{d}{dx}(x^4) + 10\frac{d}{dx}(x^3) - 6\frac{d}{dx}(x) + 0$$

$$= 8x^7 + 12 \times 5(x^{5-1}) - 4 \times 4(x^{4-1}) + 10 \times 3(x^{3-1}) - 6 \times 1$$

$$= 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6.$$

Example If $y = e^x - x$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Solution. Given $y = e^x - x$.

$$\frac{dy}{dx} = e^x - 1$$
$$\frac{d^2y}{dx^2} = e^x.$$

Example If $y = x^2 \sin x$, find $\frac{dy}{dx}$.

Solution. Given $y = x^2 \sin x$.

Applying the product rule we obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(x^2 \sin x\right)\\ &= x^2\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}(x^2)\\ &= x^2 \times (\cos x) + \sin x \times (2x)\\ &= x^2 \cos x + 2x \sin x.\end{aligned}$$

Example If $y = \dfrac{x^2 + x - 2}{x^3 + 6}$, find $\dfrac{dy}{dx}$.

Solution. Given $y = \dfrac{x^2 + x - 2}{x^3 + 6}$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^3+6)\frac{d}{dx}(x^2+x-2) - (x^2+x-2)\frac{d}{dx}(x^3+6)}{(x^3+6)^2}\\ &= \frac{(x^3+6)(2x+1) - (x^2+x-2)(3x^2)}{(x^3+6)^2}\\ &= \frac{2x^4 + x^3 + 12x + 6 - 3x^4 - 3x^3 + 6x^2}{(x^3+6)^2}\\ &= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3+6)^2}.\end{aligned}$$

The chain rule. If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = fog$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product $F'(x) = f'(g(x)) \times g'(x)$.

Example Differentiate the following functions

(i) $y = (x^3 - 1)^{100}$.

(ii) $y = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}$.

(iii) $y = (2x + 1)^5(x^3 - x + 1)^4$.

(iv) $y = e^{\sin x}$.

(v) $y = \sin(\cos(\tan x))$.

(vi) $y = e^{\sec 3x}$.

(vii) $y = \left(\dfrac{x - 2}{2x + 1}\right)^9$.

Solution. (i) Given $y = (x^3 - 1)^{100}$

Let $u = x^3 - 1$. Then $y = u^{100}$.

$$\text{Now} \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

$$= 100u^{99} \times 3x^2 = 300x^2(x^3 - 1)^{99}.$$

(ii) Given $y = \frac{1}{\sqrt[3]{x^2+x+1}}$

Let $u = x^2 + x + 1$. Then $y = u^{\frac{-1}{3}}$.

$$\text{Now} \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

$$= \frac{-1}{3} u^{\frac{-1}{3} - 1} \times (2x + 1)$$

$$= \frac{-1}{3} u^{\frac{-4}{3}} \times (2x + 1) = \frac{-1}{3u^{\frac{4}{3}}} \times (2x + 1) = \frac{-(2x + 1)}{3(x^2 + x + 1)^{\frac{4}{3}}}.$$

(iii) Given $y = (2x + 1)^5(x^3 - x + 1)^4$

Let $u = x^2 + x + 1$. Then $y = u^{\frac{-1}{3}}$.

$$\text{Now} \frac{dy}{dx} = \frac{d}{dx}\left((2x + 1)^5(x^3 - x + 1)^4\right)$$

$$= (2x + 1)^5 \frac{d}{dx}\left((x^3 - x + 1)^4\right) + (x^3 - x + 1)^4 \frac{d}{dx}\left((2x + 1)^5\right)$$

$$= (2x + 1)^5 \left(4(x^3 - x + 1)^3(3x^2 - 1)\right) + (x^3 - x + 1)^4 \left(5(2x + 1)^4 \times 2\right)$$

$$= 4(2x + 1)^5(x^3 - x + 1)^3(3x^2 - 1) + 10(x^3 - x + 1)^4(2x + 1)^4$$

$$= 2(2x + 1)^4(x^3 - x + 1)^3\{2(2x + 1)(3x^2 - 1) + 5(x^3 - x + 1)\}$$

$$= 2(2x + 1)^4(x^3 - x + 1)^3\{12x^3 - 4x + 6x^2 - 2 + 5x^3 - 5x + 5)\}$$

$$= 2(2x + 1)^4(x^3 - x + 1)^3\{17x^3 + 6x^2 - 9x + 3)\}.$$

(iv) Given $y = e^{\sin x}$

$$\frac{dy}{dx} = e^{\sin x} \cos x.$$

(v) Given $y = \sin(\cos(\tan x))$.

$$\frac{dy}{dx} = \cos(\cos(\tan x))(-\sin(\tan x))\sec^2 x$$

$$= -\cos(\cos(\tan x))\sin(\tan x)\sec^2 x.$$

(vi) Given $y = e^{\sec 3x}$.

$$\frac{dy}{dx} = e^{\sec 3x} \times \sec 3x \times \tan 3x \times 3 = 3\sec 3x \tan 3x e^{\sec 3x}.$$

(vii) Given $y = \left(\frac{x-2}{2x+1}\right)^9$.

$$\frac{dy}{dx} = 9\left(\frac{x-2}{2x+1}\right)^8 \left(\frac{(2x+1)\times 1 - (x-2)\times 2}{(2x+1)^2}\right)$$

$$= 9\left(\frac{x-2}{2x+1}\right)^8 \left(\frac{2x+1-2x+4}{(2x+1)^2}\right) = 45\frac{(x-2)^8}{(2x+1)^{10}}.$$

Example If $y = \sqrt{x + \sqrt{x + \sqrt{x + \cdots \infty}}}$, find $\frac{dy}{dx}$.

Solution. Given $y = \sqrt{x + \sqrt{x + \sqrt{x + \cdots \infty}}}$.

$$y = \sqrt{x+y}$$

$$y^2 = x + y.$$

Differentiating w.r.to x we get

$$2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx}(2y-1) = 1$$

$$\frac{dy}{dx} = \frac{1}{(2y-1)}.$$

Example If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \cdots \infty}}}$, find $\frac{dy}{dx}$.

Solution. Given $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \cdots \infty}}}$.

$$y = \sqrt{\sin x + y}.$$

Squaring on both sides we get

$$y^2 = \sin x + y.$$

Differentiating w.r.to x we get

$$2y\frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$2y\frac{dy}{dx} - \frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx}(2y-1) = \cos x$$

$$\frac{dy}{dx} = \frac{\cos x}{(2y-1)}.$$

Example Find y'' if $x^4 + y^4 = 16$.

Solution. Given $x^4 + y^4 = 16$

Differentiating w.r.to x we get

$$4x^3 + 4y^3\frac{dy}{dx} = 0$$

$$x^3 + y^3\frac{dy}{dx} = 0$$

$$y^3\frac{dy}{dx} = -x^3$$

$$\frac{dy}{dx} = -\frac{x^3}{y^3}.$$

Again differentiating w.r.to x

$$\frac{d^2y}{dx^2} = -\frac{y^3 \times 3x^2 - x^3 \times 3y^2\frac{dy}{dx}}{y^6}$$

$$= \frac{-3x^2y^2\left(y - x\frac{dy}{dx}\right)}{y^6} = \frac{-3x^2y^2\left(y - x\frac{dy}{dx}\right)}{y^6}$$

$$= \frac{-3x^2\left(y - x\left(\frac{-x^3}{y^3}\right)\right)}{y^4} = \frac{-3x^2\left(y^4 + x^4\right)}{y^7} = \frac{-3x^2 \times 16}{y^7} = \frac{-48x^2}{y^7}.$$