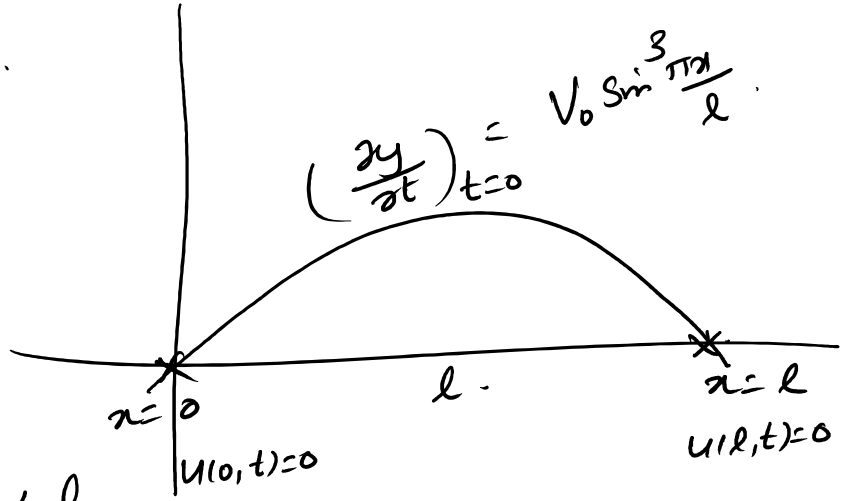
Type 2. (Initial velocity $\left(\frac{\partial u}{\partial t}\right)_{t=0} = f(x)$).

Problem 1 - A tightly stretched string of length 'l' with fixed ends is initially in equilibrium position. It is set vibrating by giving each point a velocity $V_0 \sin^3 \frac{\pi x}{l}$. Find the displacement $y(x,t)$.

Soln. The 1-D wave eqn is $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$.



The boundary conditions are

$$\left.\begin{array}{ll}(i) & u(0,t) = 0 \\ (ii) & u(l,t) = 0\end{array}\right\} t \geq 0$$

$$\left.\begin{array}{ll}(iii) & u(x,0) = 0 \\ (iv) & \left(\frac{\partial u}{\partial t}\right)_{t=0} = V_0 \sin^3 \frac{\pi x}{l}\end{array}\right\} 0 \leq x \leq l.$$

The suitable soln of wave eqn is

$u(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$ —— ①

Sub bc(i) in ① i.e put $x=0$ in ①.

$0 = A(C\cos pat + D\sin pat)$

$\Rightarrow \boxed{A=0}$ Sub in ①.

$\therefore u(x,t) = B\sin px\,(C\cos pat + D\sin pat)$ —— ②

Sub b.c (ii) in ② i.e put $x=l$ in ②

$0 = B\sin pl\,(C\cos pat + D\sin pat)$

$\Rightarrow \sin pl = 0 = \sin n\pi \quad (\because B \neq 0,)$

$\Rightarrow pl = n\pi.$

$\Rightarrow \boxed{p = \frac{n\pi}{l}}$ Sub in ②

$$u(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \quad \text{--- } (3)$$

Sub (III) b.c in (3) i.e. put $t=0$ in (3).

$$0 = B\sin\frac{n\pi x}{l}\left(c.1 + 0\right) \quad (\because \sin 0 = 0, \ \cos 0 = 1$$

$$\Rightarrow \boxed{c = 0}. \quad \text{Sub in } (3).$$

$$u(x,t) = B\sin\frac{n\pi x}{l} \cdot D\sin\frac{n\pi at}{l} = BD\sin\frac{n\pi x}{l} \cdot \sin\frac{n\pi at}{l}.$$

Taking $BD = B_n$, the most general soln is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cdot \sin\frac{n\pi at}{l} \quad \text{--- } (4).$$

Diff. (4) partially w.r.to t.

$$\left(\frac{\partial u}{\partial t}\right)(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cdot \cos\frac{n\pi a t}{l} \cdot \left(\frac{n\pi a}{l}\right) \quad \text{— (4a)}$$

Sub b.c (IV) in (4a) ie put $t=0$ in (4a).

$$V_0 \sin^3\frac{\pi x}{l} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cdot \frac{n\pi a}{l} \quad (\because \cos 0 = 1.$$

expanding RHS, ie put $n = 1, 2, 3 \ldots$

LHS: $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

$$\therefore \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta.$$

$$V_0\left[\frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}\right] = \frac{\pi a}{l} \cdot B_1 \sin\frac{\pi x}{l} + \frac{2\pi a}{l} B_2 \sin\frac{2\pi x}{l} +$$

$$\frac{3\pi a}{l} B_3 \sin\frac{3\pi x}{l} + \frac{4\pi a}{l} \cdot \sin\frac{4\pi x}{l} + \cdots$$

Equating the coeff of like terms on both sides.

$$\frac{3V_0}{4} = \frac{\pi a}{l} \cdot B_1 \quad \Rightarrow B_1 = \frac{3V_0 l}{4\pi a}.$$

$$0 = \frac{2\pi a}{l} \cdot B_2 , \quad -\frac{V_0}{4} = \frac{3\pi a}{l} \cdot B_3 \Rightarrow B_3 = \frac{-V_0 l}{12\pi a}.$$

$B_2 = 0, \quad B_4 = 0, \quad \ldots$

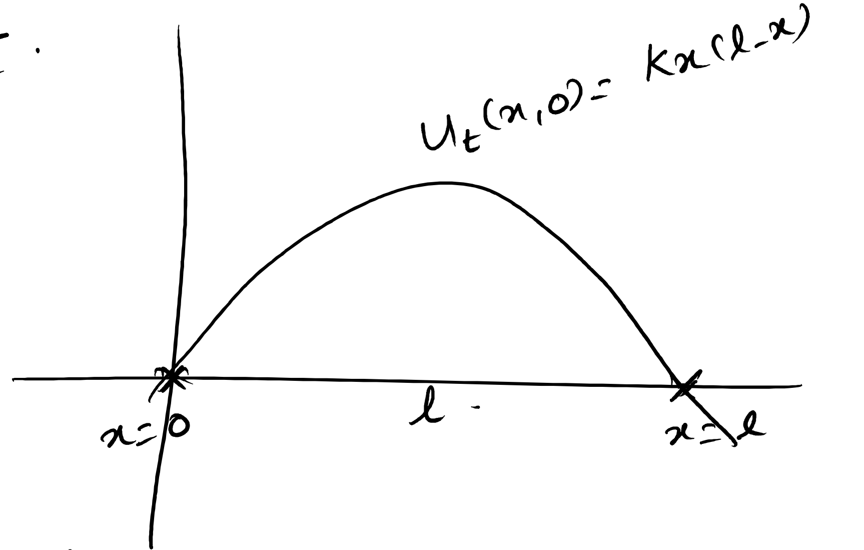
$$\therefore u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cdot \sin\frac{n\pi at}{l}.$$

$$= B_1 \sin\frac{\pi x}{l} \sin\frac{\pi at}{l} + B_2 \sin\frac{2\pi x}{l} \cdot \sin\frac{2\pi at}{l} + B_3 \sin\frac{3\pi x}{l} \cdot \sin\frac{3\pi at}{l} + \ldots$$

$$= \frac{3V_0 l}{4\pi a}\left(\sin\frac{\pi x}{l} \cdot \sin\frac{\pi at}{l}\right) - \frac{V_0 l}{12\pi a}\left(\sin\frac{3\pi x}{l} \cdot \sin\frac{3\pi at}{l}\right).$$

2) A tightly stretched string with fixed end points $x=0$ & $x=l$ is initially at rest in its equilibrium position. If each of its points is given a velocity $Kx(l-x)$, find the displacement of the string at any distance 'x' from one end at any time 't'.

<u>Soln.</u> The 1-D Wave eqn is $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$.



The boundary conditions are

$$\left.\begin{array}{l}\text{(i) } u(0,t) = 0 \\ \text{(ii) } u(l,t) = 0\end{array}\right\} t \geq 0$$

$$\left.\begin{array}{l}\text{(iii) } u(x,0) = 0 \\ \text{(iv) } \left(\frac{\partial u}{\partial t}\right)_{t=0} \text{ or } u_t(x,0) = Kx(l-x)\end{array}\right\}, 0 \leq x \leq l.$$

The suitable soln of 1-D Wave eqn is

$$u(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \quad \text{--- (1)}$$

Sub b.c (i) in (1) i.e. put $x=0$ in (1)

$$0 = A(C\cos pat + D\sin pat)$$

$$\Rightarrow \boxed{A=0} \text{ sub in (1)}$$

$$u(x,t) = B\sin px\,(C\cos pat + D\sin pat) \quad \text{--- (2)}$$

Sub b.c (ii) in (2) i.e. put $x=l$ in (2)

$$0 = B\sin pl\,(C\cos pat + D\sin pat)$$

$$\Rightarrow \sin pl = 0 = \sin n\pi \quad (\text{Since } B \neq 0 \text{ cannot be})$$

$$\Rightarrow \sin pl = \sin n\pi \Rightarrow pl = n\pi$$

$$\Rightarrow \boxed{p = \frac{n\pi}{l}} \text{ Sub in (2)}$$

$$u(x,t) = B \sin \frac{n\pi x}{l} \left(C \cos \frac{n\pi a t}{l} + D \sin \frac{n\pi a t}{l} \right) \quad \text{--- (3)}$$

Sub b.c (iii) in (3) i.e put $t=0$ in (3)

$$0 = B \sin \frac{n\pi x}{l} \cdot (C + D.0) \quad (\because \cos 0 = 1, \ \sin 0 = 0$$

$$\Rightarrow \boxed{C=0} \quad \text{sub in (3).}$$

$$u(x,t) = B \sin \frac{n\pi x}{l} \cdot D \sin \frac{n\pi a t}{l} = BD \sin \frac{n\pi x}{l} \cdot \sin \frac{n\pi a t}{l}.$$

Taking $BD = B_n$, The most general soln is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cdot \sin \frac{n\pi a t}{l}. \quad \text{--- (4)}$$

Diff (4) partially w.r. to 't'.

$$\left(\frac{\partial u}{\partial t}\right)_{(x,t)} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi a t}{l} \cdot \left(\frac{n\pi a}{l}\right) \quad \text{--- (4a)}$$

Sub b.c (iv) in (4 a) ie put $t=0$ in (4a)

$$Kx(l-x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$, Which is a Half range fourier

Sine series, where $B_n = \frac{2}{l}\int_0^l f(x) \cdot \sin\frac{n\pi x}{l}\, dx$.

ie $B_n = \frac{2}{l}\int_0^l K(xl - x^2) \sin\frac{n\pi x}{l}\, dx$. Applying Bernoulli's integral formula,

$u = xl - x^2$ $\quad V = \sin\frac{n\pi x}{l}$.

$u' = l - 2x$ $\quad V_1 = \frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} = -\frac{l}{n\pi} \cdot \cos\frac{n\pi x}{l}$.

$u'' = -2$ $\quad V_2 = -\frac{l^2}{n^2\pi^2} \cdot \sin\frac{n\pi x}{l}$.

$u''' = 0$ $\quad V_3 = \frac{l^3}{n^3\pi^3} \cdot \cos\frac{n\pi x}{l}$.

$$\therefore B_n = \frac{2k}{l}\left\{\left[(lx - x^2)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (l-2x).\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right) + (-2).\left(\frac{l^3}{n^3\pi^3}.\cos\frac{n\pi x}{l}\right)\right\}_{x=0}^{l}$$

$$= \frac{2k}{l}\left[\left(0 - 0 - \frac{2l^3}{n^3\pi^3}(-1)^n\right) - \left(0 - 0 - \frac{2l^3}{n^3\pi^3}\right)\right]$$

$$= \frac{2k}{l}.\frac{2l^3}{n^3\pi^3}\left(1 - (-1)^n\right) = \frac{4kl^2}{n^3\pi^3}\left(1 - (-1)^n\right)$$

$$= \frac{8kl^2}{n^3\pi^3} \text{ if } n \text{ is odd } (\because (-1)^n = -1$$

$= 0$ if n is even.

$\therefore$ The reqd soln is

$$u(x,t) = \sum_{n=1,3,5\ldots}^{\infty} \frac{8kl^2}{n^3\pi^3}.\sin\frac{n\pi x}{l}.\sin\frac{n\pi at}{l}.$$