Definition. Let c be a number in the domain $\mathbb{D}$ of a function f. Then $f(c)$ is the absolute maximum value of f on $\mathbb{D}$ if $f(c) \geq f(x)$ for all x in $\mathbb{D}$. $f(c)$ is the absolute minimum value of f on D if $f(c) \leq x$ for all x in $\mathbb{D}$.

Maximum and minimum values of f are called extreme values of f.

Definition. The number $f(c)$ is a local maximum value of f if $f(c) \geq f(x)$ when x is near c. $f(c)$ is the local minimum value of f if $f(c) \leq f(x)$ when x is near c.

Increasing and decreasing test

(i) If $f'(x) > 0$, on an interval, then f is increasing on that interval.

(ii) If $f'(x) < 0$, on an interval, then f is decreasing on that interval.

Fermat's theorem. If f has a local maximum or minimum at c, and $f'(c)$ exits, then $f'(c) = 0$.

Critical number. A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Result. If f has a local maximum or minimum at c, then c is a Critical number of f.

First derivative test. Suppose that c is a critical number of a continuous function f.

(i) If f' changes from positive to negative at c, then f has a local maximum at c.

(ii) If f' changes from negative to positive at c, then f has a local minimum at c.

(iii) If f' is positive to the left and right of c or negative to the left and right of c, then f has no local maximum or minimum at c.

Second derivative test. Suppose f'' is continuous near c.

(i) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(ii) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Example Find the intervals on which the function $f(x) = \sin x + \cos x$ is increasing or decreasing.

Solution. $f(x) = \sin x + \cos x.$

$f'(x) = \cos x - \sin x.$

$f''(x) = -\sin x - \cos x.$

At critical points $f'(x) = 0$

i.e., $\cos x - \sin x = 0.$

i.e., $\sin x = \cos x.$

i.e., $\tan x = 1.$

$\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}.$

Let us evaluate the intervals at which f decrease or increases.

Interval	$f'(x)$	f
$0 < x < \frac{\pi}{4}$	+	increases in $(0, \frac{\pi}{4})$
$\frac{\pi}{4} < x < \frac{5\pi}{4}$	-	decreases in $(\frac{\pi}{4}, \frac{5\pi}{4})$
$\frac{5\pi}{4} < x < 2\pi$	+	increases in $(\frac{5\pi}{4}, 2\pi)$

Since f' changes from positive to negative at $x = \frac{\pi}{4}$,

f has a maximum at $x = \frac{\pi}{4}$.

Maximum value of $f = f(\frac{\pi}{4}) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}.$

Also f' changes from negative to positive at $x = \frac{5\pi}{4}$.

Minimum value of $f = f(\frac{5\pi}{4}) = \sin\frac{5\pi}{4} + \cos\frac{5\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}.$

Example Find the intervals on which the function $f(x) = x^4 - 2x^2 + 3$ is increasing or decreasing. Also find the local maximum and minimum values of f by using first derivative test.

Solution. $f(x) = x^4 - 2x^2 + 3.$

$f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1).$

Critical numbers occur at points where $f'(x) = 0$

i.e., $4x(x-1)(x+1) = 0.$

$x = -1, x = 0, x = 1.$

The critical numbers are $x = -1, x = 0, x = 1.$

Let us evaluate the intervals at which f decreases or increases.

Interval	$4x$	$x-1$	$x+1$	$f'(x)$	f
$x<-1$	-	-	-	-	decreases in $(-\infty,-1)$
$-1<x<0$	-	-	+	+	increases in $(-1,0)$
$0<x<1$	+	-	+	-	decreases in $(0,1)$
$x>1$	+	+	+	+	increases in $(1,\infty)$

Since f' changes from negative to positive at $x=-1$,

f has a local minimum at $x=-1$.

Minimum value of $f = f(-1) = 1-2+3 = 1$.

Also f' changes from positive to negative at $x=0$.

$\therefore f$ has a maximum at $x=0$.

Maximum value of $f = f(0) = 3$.

Again f' changes from negative to positive at $x=1$.

$\therefore f$ has a local minimum at $x=1$.

Minimum value of $f = 1-2+3 = 2$.

Example Using first derivative test, examine for maximum and minimum of the function $f(x) = x^3 - 3x + 3,\ x \in \mathbb{R}$.

Solution. $f(x) = x^3 - 3x + 3$.

$f'(x) = 3x^2 - 3 = 3(x^2-1) = 3(x-1)(x+1)$.

Critical numbers occur at points where $f'(x) = 0$.

i.e., $3(x-1)(x+1) = 0$.

i.e., $x=-1,\ x=1$.

The critical numbers are $x=-1, 1$.

Let us evaluate the intervals at which f decreases or increases.

Interval	$x-1$	$x+1$	$f'(x)$	f
$x<-1$	-	-	+	increases in $(-\infty,-1)$
$-1<x<1$	-	+	-	decreases in $(-1,1)$
$x>1$	+	+	-	increases in $(1,\infty)$

Since f' changes from positive to negative at $x=-1$.

f has a local maximum at $x = -1$.

$\therefore$ Maximum value of $f = f(-1) = (-1)^3 - 3(-1) + 3 = -1 + 3 + 3 = 5$.

Also f' changes from negative to positive at $x = 1$.

$\therefore$ f has a local minimum at $x = 1$.

$\therefore$ Minimum value of $f = f(1) = 1^3 - 1 \times 3 + 3 = 1 - 3 + 3 = 1$.

Example Find the maximum and minimum of the function $f(x) = 2x^3 - 3x^2 - 36x + 10$. by using **Second derivative test.**

Solution. $f(x) = 2x^3 - 3x^2 - 36x + 10$.

$f'(x) = 6x^2 - 6x - 36$.

$f''(x) = 12x - 6$.

Critical pints occur at $f'(x) = 0$.

i.e., $6x^2 - 6x - 36 = 0$.

$\therefore x^2 - x - 6 = 0$.

$\therefore (x - 3)(x + 2) = 0$.

$\therefore x = -2, x = 3$.

The critical points are at $x = -2, x = 3$.

At $x = -2$, $f''(x) = 12 \times (-2) - 6 = -24 - 6 = -30 = -ve$

$\therefore$ f is maximum at $x = -2$.

Maximum value of

$$f = 2(-2)^3 - 3(-2)^2 - 36(-2) + 10 = -16 - 12 + 72 + 10 = 54.$$

At $x = 3$, $f''(x) = 12 \times 3 - 6 = 36 - 6 = 30 = +ve$.

$\therefore$ f has minimum at $x = 3$.

$\therefore$ Minimum value of

$$f = f(3) = 2 \times 3^3 - 3 \times 3^2 - 36 \times 3 + 10 = 54 - 27 - 108 + 10 = -71.$$