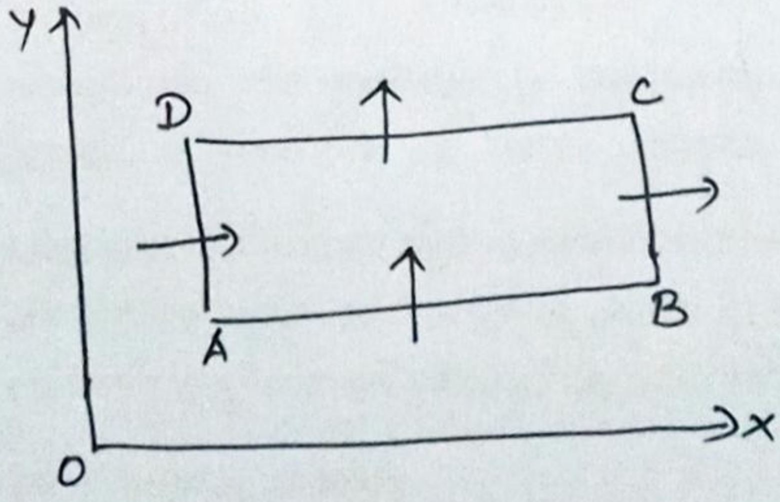
Two Dimensional heat flow.

* When the heat flow is along curves, instead of along straight lines, all the curves lying in the parallel planes, then the flow is called two dimensional.
* For 2D heat flow, we consider, flow of heat in a metal plate in the xoy plane. The plate be of uniform thickness h, density ρ, thermal conductivity k & the specific heat capacity 's'.
* The heat flow lies in the xoy-plane (xy-plane) and is zero direction normal to the xoy-plane.



Equation of 2-D heat flow

The PDE of 2-D heat flow is

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad \text{--- (1)}$$ where $a^2 = \frac{k}{\rho s}$,

Called diffusivity. (This eqn is the temp. distribution of the plate in the transient state).

In the <u>steady state</u>, u is independent of 't'. ie 'u' depends on x & y only. ie $\frac{\partial u}{\partial t} = 0$.

$\therefore$ In the <u>steady state</u>, 2-D heat eqn is

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0} \quad \text{--- (2)}.$$

This also known as Laplace's eqn in two dimensions. (<u>cartesian form</u>) ie $\nabla^2 u = 0$ (or) $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

The <u>3 possible</u> soln of eqn (2) are.

(i) $u(x, y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$.

(ii) $u(x, y) = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py})$

(iii) $u(x, y) = (c_9 x + c_{10})(c_{11} y + c_{12})$

Of these solns, we have to select a soln, to suit the boundary conditions.

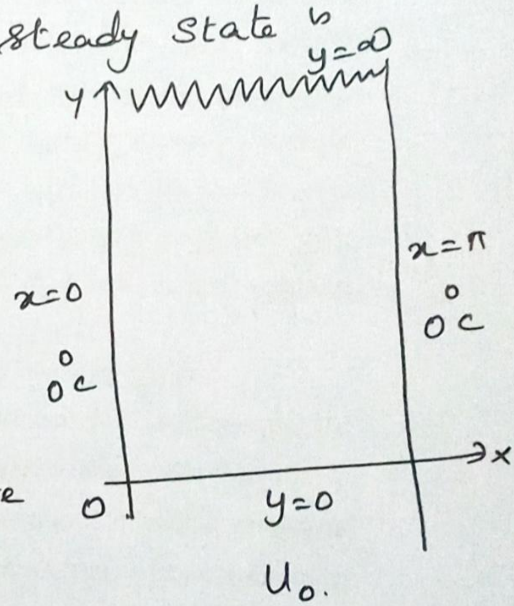
Type I. Temperature distribution in long plates.

Problem 1. An infinitely long plane uniform plate is bounded by the two parallel edges and an edge end at right angles to them. The breadth is π; this end is maintained at a temperature u_0 at all points and other edges are at zero temperature. Determine the temperature at any point of the plate in the steady state.

Soln.

The 2D heat eqn in the steady state is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$



The boundary conditions are

Note. $u(x,y)$ denotes the temp. at any pt of the plate.

The boundary conditions are

(i) $u(0,y) = 0$
(ii) $u(\pi, y) = 0$ } for all values of y

(iii) $u(x,\infty) = 0$
(iv) $u(x,0) = u_0$. } $0 \le x \le \pi$.

Note In the non zero b.c, variable x is present.
$\therefore$ Choose the soln, where x is in the trig. fn.

$\therefore$ The suitable soln is

$$u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \quad \text{--- (1)}.$$

Sub b.c (i) in (1) i.e put $x = 0$ in (1).

$$0 = A(Ce^{py} + De^{-py})$$

$$\boxed{A = 0} \quad \text{Sub in (1)}.$$

$$\Rightarrow \quad Ce^{py} + De^{-py}) \quad \text{--- (2)}.$$

$\therefore u(x,y) = B\sin px\,(Ce^{py} + De^{-py})$

Sub b.c (ii) in (2). i.e put $x = \pi$ in (2)

$$0 = B\sin p\pi\,(Ce^{py} + De^{-py}).$$

$$\Rightarrow \quad \sin p\pi = 0 \quad = \sin n\pi.$$

$$\Rightarrow \quad p\pi = n\pi \quad \Rightarrow \boxed{p = n} \quad \text{Sub in (2)}.$$

$$\therefore u(x,y) = B\sin nx\,(Ce^{ny} + De^{-ny}) \quad \text{--- (3)}.$$

Sub b.c (iii) in (3) i.e put $y = \infty$ in (3).

$$0 = B\sin nx\,(Ce^{\infty} + \cancel{De^{-\infty}}_{\to 0}) \quad (\because e^{-\infty} = 0.$$

$$\Rightarrow \quad \boxed{C = 0.} \quad \text{Sub in (3)}.$$

$$u(x,y) = B\sin nx \cdot De^{-ny}. \quad = BD\sin nx\, e^{-ny}$$

Taking $BD = B_n$ & Generalising,

The most general soln is

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin nx \, e^{-ny}. \quad \text{--- } (4).$$

Sub b.c (IV) in (4) i.e put $y=0$ in (4).

$$u_0 = \sum_{n=1}^{\infty} B_n \sin nx \quad (\because e^0 = 1)$$ which is a Half range Fourier Sine series.

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx.$$

$$B_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin nx \, dx.$$

$$= \frac{2u_0}{\pi}\left(-\frac{\cos nx}{n}\right)_0^{\pi}.$$

$$= -\frac{2u_0}{n\pi}\left[\cos n\pi - \cos 0\right]$$

$$= -\frac{2u_0}{n\pi}\left((-1)^n - 1\right).$$

$$B_n = \frac{2u_0}{n\pi}\left(1 - (-1)^n\right).$$

$$\therefore B_n = \frac{4u_0}{n\pi} \text{ if } n \text{ is odd} \quad n = 1, 3, 5, \ldots$$

$(-1)^n = 1$ if n is even
$\therefore (1 - (-1)^n) = 1 - 1 = 0$ if n is even

$(-1)^n = -1$ if n is odd
$\therefore 1 - (-1)^n = 1 - (-1) = 2$ if n is odd.

Sub in (4).

The reqd soln is

$$u(x,y) = \sum_{n=1,3,5\ldots}^{\infty} \frac{4u_0}{n\pi} . \sin nx . e^{-ny}. \quad //.$$