Module - 3

Vector Spaces

Vectors in R^n

- An ordered *n*-tuple:

  a sequence of *n* real number $(x_1, x_2, \cdots, x_n)$

- *n*-space: R^n

  the set of all ordered n-tuple

▪ Example:

$n = 1 \quad R^1 = 1\text{-space}$

$= \text{set of all real number}$

$n = 2 \quad R^2 = 2\text{-space}$

$= \text{set of all ordered pair of real numbers}$ (x_1, x_2)

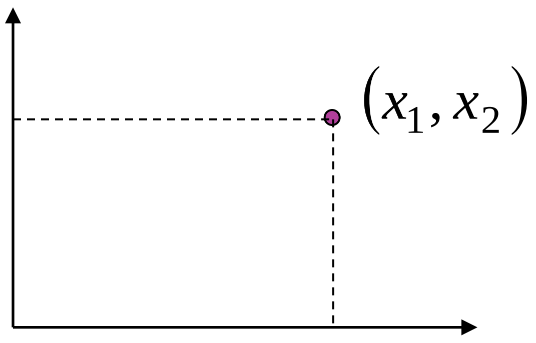
$n = 3 \quad R^3 = 3\text{-space}$

$= \text{set of all ordered triple of real numbers}$ (x_1, x_2, x_3)

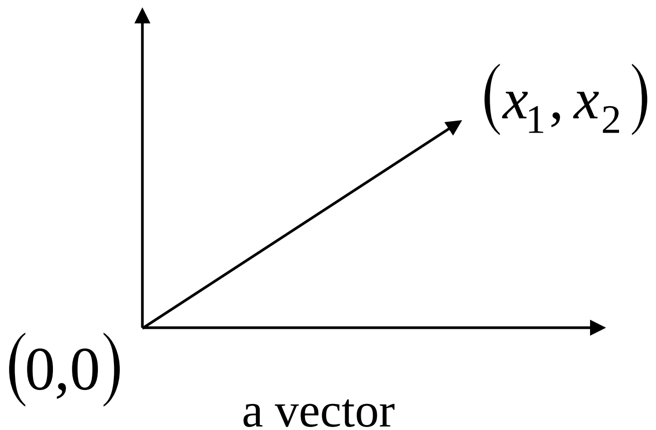
- **Notes:**

(1) An n-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as <u>a point</u> in R^n with the x_i's as its coordinates.

(2) An n-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as <u>a vector</u> $x = (x_1, x_2, \cdots, x_n)$ in R^n with the x_i's as its components.



a point



a vector

$$\mathbf{u}=(u_1,u_2,\cdots,u_n),\ \mathbf{v}=(v_1,v_2,\cdots,v_n)$$ (two vectors in R^n)

- Equal:

$\mathbf{u}=\mathbf{v}$ if and only if $u_1=v_1,\ u_2=v_2,\cdots,u_n=v_n$

- Vector addition (the sum of **u** and **v**):

$$\mathbf{u}+\mathbf{v}=(u_1+v_1,u_2+v_2,\cdots,u_n+v_n)$$

- Scalar multiplication (the scalar multiple of **u** by *c*):

$$c\mathbf{u}=(cu_1,cu_2,\cdots,cu_n)$$

- Notes:

The sum of two vectors and the scalar multiple of a vector in R^n are called the standard operations in R^n.

- Negative:

$$-\mathbf{u} = (-u_1, -u_2, -u_3, \ldots, -u_n)$$

- Difference:

$$\mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2, u_3 - v_3, \ldots, u_n - v_n)$$

- Zero vector:

$$\mathbf{0} = (0, 0, \ldots, 0)$$

- Notes:

(1) The zero vector **0** in R^n is called the **additive identity** in R^n.

(2) The vector $-\mathbf{v}$ is called the **additive inverse** of **v**.

- **Thm : (Properties of vector addition and scalar multiplication)**

Let u, v, and w be vectors in R^n, and let c and d be scalars.

(1) $\mathbf{u}+\mathbf{v}$ is a vector in R^n

(2) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$

(3) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$

(4) $\mathbf{u}+\mathbf{0} = \mathbf{u}$

(5) $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$

(6) $c\mathbf{u}$ is a vector in R^n

(7) $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v}$

(8) $(c+d)\mathbf{u} = c\mathbf{u}+d\mathbf{u}$

(9) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(10) $1(\mathbf{u}) = \mathbf{u}$

- **Ex 5: (Vector operations in R^4)**

Let $\mathbf{u}=(2, -1, 5, 0)$, $\mathbf{v}=(4, 3, 1, -1)$, and $\mathbf{w}=(-6, 2, 0, 3)$ be vectors in R^4. Solve $\mathbf{x}$ for x in each of the following.

(a) $\mathbf{x} = 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w})$

(b) $3(\mathbf{x}+\mathbf{w}) = 2\mathbf{u} - \mathbf{v}+\mathbf{x}$

Sol: (a)

$$
\begin{aligned}
\mathbf{x} &= 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w}) \\
&= 2\mathbf{u} - \mathbf{v} - 3\mathbf{w} \\
&= (4, -2, 10, 0) - (4, 3, 1, -1) - (-18, 6, 0, 9) \\
&= (4 - 4 + 18, -2 - 3 - 6, 10 - 1 - 0, 0 + 1 - 9) \\
&= (18, -11, 9, -8).
\end{aligned}
$$

(b)

$$3(\mathbf{x}+\mathbf{w}) = 2\mathbf{u}-\mathbf{v}+\mathbf{x}$$

$$3\mathbf{x}+3\mathbf{w} = 2\mathbf{u}-\mathbf{v}+\mathbf{x}$$

$$3\mathbf{x}-\mathbf{x} = 2\mathbf{u}-\mathbf{v}-3\mathbf{w}$$

$$2\mathbf{x} = 2\mathbf{u}-\mathbf{v}-3\mathbf{w}$$

$$\mathbf{x} = \mathbf{u}-\tfrac{1}{2}\mathbf{v}-\tfrac{3}{2}\mathbf{w}$$

$$= (2,1,5,0)+\left(-2,\tfrac{-3}{2},\tfrac{-1}{2},\tfrac{1}{2}\right)+\left(9,-3,0,\tfrac{-9}{2}\right)$$

$$= \left(9,\tfrac{-11}{2},\tfrac{9}{2},-4\right)$$

- Thm : (Properties of additive identity and additive inverse)

Let **v** be a vector in R^n and c be a scalar. Then the following is true.

(1) The additive identity is unique. That is, if $\mathbf{u}+\mathbf{v}=\mathbf{v}$, then $\mathbf{u} = \mathbf{0}$

(2) The additive inverse of $\mathbf{v}$ is unique. That is, if $\mathbf{v}+\mathbf{u}=\mathbf{0}$, then $\mathbf{u} = -\mathbf{v}$

(3) $0\mathbf{v}=\mathbf{0}$

(4) $c\mathbf{0}=\mathbf{0}$

(5) If $c\mathbf{v}=\mathbf{0}$, then $c=0$ or $\mathbf{v}=\mathbf{0}$

(6) $-(-\mathbf{v}) = \mathbf{v}$

- Linear combination:

The vector **x** is called a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$

if it can be expressed in the form

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \qquad c_1, c_2, \cdots, c_n : \text{scalar}$$

- Ex 6:

Given **x** = (– 1, – 2, – 2), **u** = (0,1,4), **v** = (– 1,1,2), and **w** = (3,1,2) in R^3, find *a*, *b*, and *c* such that **x** = *a***u**+*b***v**+*c***w**.

Sol:

$$\begin{array}{rcrcrcr} & & -b & + & 3c & = & -1 \\ a & + & b & + & c & = & -2 \\ 4a & + & 2b & + & 2c & = & -2 \end{array}$$

$$\Rightarrow a = 1,\ b = -2,\ c = -1$$

Thus $\mathbf{x} = \mathbf{u} - 2\mathbf{v} - \mathbf{w}$

- **Notes:**

A vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ in R^n can be viewed as:

a $1\times n$ row matrix (row vector): $\mathbf{u} = [u_1,\ u_2, \cdots, u_n]$

a $n\times 1$ column matrix (column vector): $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$

Vector addition

$$\mathbf{u}+\mathbf{v}=(u_1,u_2,\cdots,u_n)+(v_1,v_2,\cdots,v_n)$$
$$=(u_1+v_1,u_2+v_2,\cdots,u_n+v_n)$$

$$\mathbf{u}+\mathbf{v}=[u_1,u_2,\cdots,u_n]+[v_1,v_2,\cdots,v_n]$$
$$=[u_1+v_1,u_2+v_2,\cdots,u_n+v_n]$$

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}+\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=\begin{bmatrix}u_1+v_1\\u_2+v_2\\\vdots\\u_n+v_n\end{bmatrix}$$

Scalar multiplication

$$c\mathbf{u}=c(u_1,\ u_2,\cdots,\ u_n)$$
$$=(cu_1,cu_2,\cdots,\ cu_n)$$

$$c\mathbf{u}=c[u_1,\ u_2,\cdots,\ u_n]$$
$$=[cu_1,cu_2,\cdots,\ cu_n]$$

$$c\mathbf{u}=c\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}=\begin{bmatrix}cu_1\\cu_2\\\vdots\\cu_n\end{bmatrix}$$

Group

Group.

A non-empty set S, with binary operation $*$ i.e. $(S,*)$ is said to be a group if the following properties are true.

(i) Closure property : $\forall\, a,b \in S \Rightarrow a*b \in S$.

(ii) Associative property : $\forall\, a,b,c \in S \Rightarrow a*(b*c) = (a*b)*c$

(iii) Identity property : $\forall\, a \in S$, $\exists\, e \in S$ such that $a*e = e*a = e$.

(iv) Inverse property : $\forall\, a \in S$, $\exists\, a^{-1} \in S$ such that $a*a^{-1} = a^{-1}*a = e$

Note $(S,*)$ is said to be an abelian group if $\forall\, a,b \in S$, $a*b = b*a$. (Commutative property).

Group

Example. $(\mathbb{Z}, +)$ is a group.

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\} = \{\dots -3, -2, -1, 0, 1, 2, 3 \dots\}$

(i) $1, 2 \in \mathbb{Z} \Rightarrow 1+2 = 3 \in \mathbb{Z}$ (closure property true)

(ii) $2, 3, 4 \in \mathbb{Z} \Rightarrow 2+(3+4) = (2+3)+4$ (associative property true)

(iii) $\forall a \in \mathbb{Z}$, $\exists\, 0 \in \mathbb{Z}$ such that $a + 0 = 0$

$2 \in \mathbb{Z}$, $\exists\, 0 \in \mathbb{Z}$ such that $2 + 0 = 2$

$\therefore$ 0 is the identity element.

(iv) $\forall a \in \mathbb{Z}$, $\exists\, -a \in \mathbb{Z}$ such that $a + (-a) = 0$.

$3 \in \mathbb{Z}$, $\exists\, -3 \in \mathbb{Z}$ such that $3 + (-3) = 0$

-3 is the inverse of A.

$\therefore$ Prop (i), (ii), (iii) are true. $\therefore$ ~~G~~ $(\mathbb{Z}, +)$ is a group.

Vector spaces

Vector Spaces

A non empty set V is called a real vector space if the following axioms are satisfied.

VS.1. There is a binary operation '+' defined on V called addition.

VS.2. There is a scalar multiplication '$\cdot$' defined on V

VS-3 Addition & multiplication satisfy the following.

(i) $(V,+)$ is a commutative abelian group for addition: (I) + is closed. (II) associative. (III) identity elt. (IV) inverse elt. (V) commutative law $\forall a, b \in V$, $a+b=b+a$

$(V,+)$ is an abelian group.

(ii) $\alpha\cdot(u+v) = \alpha u + \alpha v$ and

$(\alpha+\beta)\cdot u = \alpha u + \beta u$ $\forall$ real numbers $\alpha+\beta$ & $\forall u, v \in V$

(iii) $\alpha(\beta u) = (\alpha\beta)u = \beta(\alpha u)$ $\forall$ real numbers α, β & $\forall u \in V$

(iv) $1\cdot u = u$ $\forall u \in V$

$(V,+,\cdot)$ is said to be a vector space.

Vector Spaces

- Vector spaces:

Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the following axioms are satisfied for every **u**, **v**, and **w** in V and every scalar (real number) c and d, then V is called a **vector space**.

Addition:

(1) $\mathbf{u}+\mathbf{v}$ is in V

(2) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(3) $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$

(4) V has a zero vector $\mathbf{0}$ such that for every $\mathbf{u}$ in V, $\mathbf{u}+\mathbf{0}=\mathbf{u}$

(5) For every $\mathbf{u}$ in V, there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$

Scalar multiplication:

(6) $c\mathbf{u}$ is in V.

(7) $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(8) $(c+d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(9) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(10) $1(\mathbf{u}) = \mathbf{u}$

Example 1 Let V_n be the set of all ordered n-tuples of real numbers.

$V_n = \{(x_1, x_2 \ldots x_n) / x_i \in R, i = 1, 2, \ldots n\}$

Let $u = (x_1, x_2, \ldots x_n)$ & $v = (y_1, y_2 \ldots y_n) \in V_n$ & $\alpha \in R$.

$u + v = (x_1 + y_1, x_2 + y_2, \ldots x_n + y_n)$

$\alpha . u = (\alpha x_1, \alpha x_2, \ldots \alpha x_n)$

To prove $(V_n, +, \cdot)$ is a real vector space.

Note.
real vector space $\alpha, \beta \in R$
cpx vector space $\alpha, \beta \in C$.

Soln First let us prove $(V_n, +)$ is an abelian group.

(i) **Closure property**

Let $u = (x_1, x_2 \ldots x_n)$; $v = (y_1, y_2, \ldots y_n) \in V_n$.

$u + v = (x_1 + y_1, x_2 + y_2, \ldots x_n + y_n) \in R$ $(\because x_i, y_i \in R \Rightarrow x_i + y_i \in R$

$\therefore$ '+' is closed in V_n.

(ii) **Associative property**: Let $u = (x_1, x_2 \ldots x_n)$; $v = (y_1, y_2 \ldots y_n)$; $w = (z_1, z_2, \ldots z_n) \in V_n$

$$u+(v+w) = (x_1, x_2 \ldots x_n) + [(y_1, y_2 \ldots y_n) + (z_1, z_2, \ldots z_n)]$$
$$= (x_1, x_2 \ldots x_n) + [y_1+z_1, y_2+z_2, \ldots y_n+z_n]$$
$$= [x_1+(y_1+z_1), x_2+(y_2+z_2), \ldots, x_n+(y_n+z_n)]$$
$$= [(x_1+y_1)+z_1, (x_2+y_2)+z_2, \ldots, (x_n+y_n)+z_n]$$
$$= [x_1+y_1, x_2+y_2, \ldots, x_n+y_n] + (z_1, z_2 \ldots z_n)$$
$$= [(x_1, x_2 \ldots x_n) + (y_1, y_2 \ldots y_n)] + (z_1, z_2 \ldots z_n)$$
$$= (u+v) + w.$$

$2+(3+4) = (2+3)+4$

$(u+(v+w) = (u+v)+w$

$\therefore$ '$+$' is associative in V_n.

(III) Identity property

Let $0 = (0, 0, \ldots 0) \in V_n$

Let $u = (x_1, x_2 \ldots x_n) \in V_n$.

$u + 0 = 0 + u = u \quad \forall u \in V_n \Rightarrow 0$ is identity elt in V_n.

$\exists\, e \in G$, such that $a * e = e * a = a \ \forall a \in G$.

(IV) Inverse property $\quad 0 \in V_n$ is the identity element

$u = (x_1, x_2 \ldots x_n) \in V_n$, $\quad -u = (-x_1, -x_2, \ldots, -x_n) \in V_n$

$u + (-u) = -u + u = 0 \quad \therefore$ for each $u \in V_n$, $-u \in V_n$ is its inverse.

($-u$ is the inverse of u)

$a * a^{-1} = a^{-1} * a = e$

V) Commutative property : Let $u = (x_1, x_2 \ldots x_n)$; $v = (y_1, y_2, \ldots y_n) \in V_n$.

$$\therefore u + v = (x_1, x_2, \ldots x_n) + (y_1, y_2 \ldots y_n)$$
$$= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$
$$= (y_1 + x_1, y_2 + x_2 \ldots\ldots, y_n + x_n) = v + u$$

$\Rightarrow +$ is commutative in V_n

$\Rightarrow (V_n, +)$ is an abelian group

(ii) $\alpha(u+v) = \alpha u + \alpha v \quad \forall \alpha \in R, \ \forall u \in V_n.$

Let $\alpha \in R$, $u = (x_1, x_2 \dots x_n)$; $v = (y_1, y_2 \dots y_n) \in V_n$

$$\alpha\cdot(u+v) = \alpha\cdot(x_1+y_1, x_2+y_2, \dots x_n+y_n) = (\alpha(x_1+y_1), \alpha(x_2+y_2), \dots \alpha(x_n+y_n))$$

$$= (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2, \dots \alpha x_n + \alpha y_n)$$

$$= (\alpha x_1, \alpha x_2, \dots \alpha x_n) + (\alpha y_1, \alpha y_2, \dots \alpha y_n)$$

$$= \alpha(x_1, x_2, \dots x_n) + \alpha(y_1, y_2 \dots y_n)$$

$$= \alpha u + \alpha v \quad \forall \alpha \in R \quad \forall u, v \in V_n$$

(iii) IIIly we can prove that $(\alpha+\beta)u = \alpha u + \beta v$

$(\alpha\beta)u = \alpha(\beta u) = \beta(\alpha u)$

(iv) $1.u = u \quad \forall u \in V_n$

Hence $(V_n, +, \cdot)$ is a vector space.

2) Let $V = \{(a_1, a_2) ; a_1, a_2 \in R\}$; define $(a_1, a_2) + (b_1, b_2) = (a_1+b_1, a_2+b_2)$,

$\alpha \cdot (a_1, a_2) = (\alpha a_1, a_2)$, $\alpha \in R$. Then S.T V is not a vector space over R.

Soln Let $\alpha, \beta \in R$ & $u = (a_1, a_2) \in V$ then,
$v = (b_1, b_2) \in V$

$(\alpha+\beta) v = (\alpha+\beta)(a_1, a_2) = ((\alpha+\beta)a_1, a_2)$

$= (\alpha a_1 + \beta a_1, a_2)$

$\neq (\alpha a_1, a_2) + (\beta a_1, a_2)$

$u \neq \alpha v + \beta v$

$\therefore (\alpha+\beta) v \neq \alpha v + \beta v \Rightarrow V$ is not a vector space over R.

$\alpha \cdot (u+v) = \alpha \cdot (a_1+b_1, a_2+b_2)$

$= \alpha \cdot (a_1+b_1, a_2+b_2)$

$= (\alpha(a_1+b_1), a_2+b_2)$

$= (\alpha a_1 + \alpha b_1, a_2+b_2)$

$\neq \alpha u + \alpha v$

3) Show that $V = R^2$ is not a vector space over R, with respect to the operations; $(a, b) + (c, d) = (a, b)$, $K(a, b) = (Ka, Kb)$

Soln Let $u = (1,2)$, $v = (3,4)$

$u + v = (1,2) + (3,4) = (1,2)$

$v + u = (3,4) + (1,2) = (3,4)$

$\left.\right\}$ $\left(\because \cancel{a+b}\ (a,b) + (c,d) = (a,b) \right)$

$\therefore u + v \neq v + u$ $\Rightarrow \therefore V$ is not a Vector space.

$(V,+)$ is not an abelian group

4) S.T the set $M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; a,b,c,d \in R \right\}$ is a vector space over R w.r.to matrix addition and multiplication of a matrix by a scalar.

Soln. (i) Let $A, B \in M \Rightarrow A + B \in M \rightarrow$ closure property is true.

(ii) Let $A, B, C \in M$, then $(A+B)+C = A + (B+C)$.

$\therefore$ Associative property is true.

(iii) There exist zero matrix such that $\forall A \in M$, $A + 0 = 0 + A = A$. also true

(iv) for each $A \in M$, $\exists\ -A \in M$ s.t $A + (-A) = 0$ $\therefore$ Inverse prop is also true

(v) matrix addition is commutative

i.e $\cancel{AB = BA}$ $A+B = B+A$ $\forall A, B \in M$

$\therefore (M,+)$ is an abelian group

Zero matrix $= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$

$e = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}$

$-A = \begin{pmatrix} -a_1 & -a_2 \\ -a_3 & -a_4 \end{pmatrix}$

v) $\forall A, B \in M$, $A+B = C \in M$ (obviously)

vi) Let $\alpha, \beta \in R$ and $A, B \in M$, then $\alpha . A \in M$ and

$$\alpha(A+B) = \alpha A + \alpha B.$$

can be verified

vii) $(\alpha+\beta)A = \alpha A + \beta A.$

viii) $\alpha(\beta A) = (\alpha\beta)A. = \beta(\alpha A)$

ix) $1.A = A.$

Hence M is a vector space over R. (or) $(M, +, .)$ is a vector space over R.

- Examples of vector spaces:

(1) n-tuple space: R^n

$$(u_1,u_2,\cdots,u_n)+(v_1,v_2,\cdots,v_n)=(u_1+v_1,u_2+v_2,\cdots,u_n+v_n)$$ vector addition

$$k(u_1,u_2,\cdots,u_n)=(ku_1,ku_2,\cdots,ku_n)$$ scalar multiplication

(2) Matrix space: $V = M_{m\times n}$ (the set of all $m\times n$ matrices with real values)

Ex: ：$(m = n = 2)$

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}+\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}=\begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$ vector addition

$$k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}=\begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$ scalar multiplication

(3) *n*-th degree polynomial space: $V = P_n(x)$

(the set of all real polynomials of degree *n* or less)

$$p(x)+q(x)=(a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n$$

$$kp(x)=ka_0+ka_1x+\cdots+ka_nx^n$$

(4) Function space: $V = c(-\infty,\infty)$ (the set of all real-valued continuous functions defined on the entire real line.)

$$(f+g)(x)=f(x)+g(x)$$

$$(kf)(x)=kf(x)$$

- Notes: To show that a set is not a vector space, you need only find one axiom that is not satisfied.

  - Ex The set of all integer is not a vector space.

$$1 \in V, \tfrac{1}{2} \in R$$

$$\underset{\text{scalar}}{\left(\tfrac{1}{2}\right)}\underset{\text{integer}}{(1)} = \underset{\text{noninteger}}{\tfrac{1}{2}} \notin V$$ (it is not closed under scalar multiplication)

- Ex : The set of all second-degree polynomials is not a vector space.

Let $p(x) = x^2$ and $q(x) = -x^2 + x + 1$

$$\Rightarrow p(x) + q(x) = x + 1 \notin V$$

(it is not closed under vector addition)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Thank you!