Lectures 14: Ratio Test and Root Test

For using the comparison test and the limit comparison test, the given series needs to be compared with a series whose behavior is already known. In many cases, it is difficult to apply these tests. For instance, consider the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$. In this example, the factorial makes it difficult for employing the tests mentioned above. In the ratio test and the root test, we will decide the convergence/divergence of a given series $\sum_{n=1}^{\infty} a_n$ by looking into the behaviors of the ratio $|\frac{a_{n+1}}{a_n}|$ (when $a_n \neq 0$ for all n) and the root $|a_n|^{1/n}$ respectively.

Ratio test

We have already seen in Lecture 12 that if $\sum_{n=1}^{\infty} a_n$ converges then $a_n \to 0$ but the converse need not be true. But if the terms $a_n's$ get smaller such as $|a_n| \leq r^n$ for some $r \in (0,1)$, then by the comparison test, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. The next result explains that under certain condition on $|\frac{a_{n+1}}{a_n}|$ the terms of the series can get smaller (as given above) so that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

The following result is a consequence of the comparison test.

Theorem 14.1. *Consider the series* $\sum_{n=1}^{\infty} a_n,\ a_n \neq 0$ *for all* n. *Suppose there exists* $r \in (0,1)$ *and* $N \in \mathbb{N}$ *such that* $|\frac{a_{n+1}}{a_n}| \leq r$ *for all* $n \geq N$. *Then* $\sum_{n=1}^{\infty} |a_n|$ *converges.*

Proof. Note that $|a_{n+1}| \leq r \mid a_n \mid$ for all $n \geq N$. Hence

$$a_{N+2} \leq r|a_{N+1}| < r \cdot r|a_N|.$$

Continue this process and obtain that $|a_{N+k}| \leq r^k |a_N|$ for all $k \geq 1$. Since N is constant, by the the comparison test, $\sum_{k=1}^{\infty} a_{N+k}$ converges which implies that $\sum_{n=1}^{\infty} a_n$ converges. □

Example 14.1. Consider the series $\sum_{n=1}^{\infty} a_n$ where $a_{2n-1} = \frac{1}{4^{n-1}}$ and $a_{2n} = \frac{1}{3 \times 4^{n-1}}$ for all $n \in \mathbb{N}$. Since $\frac{a_{2n}}{a_{2n-1}} = \frac{1}{3}$ and $\frac{a_{2n+1}}{a_{2n}} = \frac{3}{4}$, we have $|\frac{a_{n+1}}{a_n}| \leq \frac{3}{4}$ for all $n \in \mathbb{N}$. Hence, by Theorem 14.1, $\sum_{n=1}^{\infty} a_n$ converges. Alternatively, $\sum_{n=1}^{\infty} a_n$ can be considered as sum of two convergent series and shown that it converges.

Employing Theorem 14.1 for testing the convergence of a given series is not an easy task because of the difficulty involved in finding an upper bound r, satisfying $|\frac{a_{n+1}}{a_n}| \leq r$ for all $n \geq N$. The following result, which is easier to use, is a consequence of Theorem 14.1.

Theorem 14.2 (Ratio test). *Suppose* $a_n \neq 0$ *for all* n *and* $|\frac{a_{n+1}}{a_n}| \to L$ *for some* L.

(1) *If* $L < 1$ *then* $\sum_{n=1}^{\infty} \mid a_n \mid$ *converges.*

(2) *If* $L > 1$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Proof. (1) Since $|\frac{a_{n+1}}{a_n}| \to L$ and $L < 1$, there exists $N \in \mathbb{N}$ such that $\mid \frac{a_{n+1}}{a_n} \mid < L + \frac{(1-L)}{2}$ for all $n \geq N$. Denote $L + \frac{(1-L)}{2}$ by r and note that $r \in (0,1)$. By Theorem 14.1, $\sum_{n=1}^{\infty} \mid a_n \mid$ converges.

(2) Since $|\frac{a_{n+1}}{a_n}| \to L$ and $L > 1$, there exists $N \in \mathbb{N}$ such that $|\frac{a_{n+1}}{a_n}| > L - \frac{(L-1)}{2}$ for all $n \geq N$. This shows that $|a_{n+1}| > |a_n|$ for all $n \geq N$ which implies that $a_n \nrightarrow 0$. Hence $\sum_{n=1}^{\infty} a_n$ diverges. □

Example 14.2. 1. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges because $\frac{a_{n+1}}{a_n} \to 0$.

2. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges because $\frac{a_{n+1}}{a_n} = (1+\frac{1}{n})^n \to e > 1$ whereas $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

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3. $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$ converges whereas $\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$ diverges.

4. We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. However, in both these cases $\frac{a_{n+1}}{a_n} \to 1$. This demonstrates that if $L = 1$ in the ratio test then the test is inconclusive, i.e., the series could either converge or diverge

Root test

We will see that the root test, which will be stated, is suitable in many cases for determining the convergence/divergence of series compared to the ratio test. The following result is analogous to Theorem 14.1.

Theorem 14.3. *Consider the series $\sum_{n=1}^{\infty} a_n$. Suppose there exists $r \in (0, 1)$ and $N \in \mathbb{N}$ such that $|a_n|^{1/n} \leq r$ for all $n \geq N$. Then $\sum_{n=1}^{\infty} |a_n|$ converges.*

Proof. If $|a_n|^{1/n} \leq r$ for all $n \geq N$, then $|a_n| \leq r^n$ for all $n \geq N$. Hence, by the the comparison test, $\sum_{k=1}^{\infty} |a_{N+k}|$ converges which implies that $\sum_{n=1}^{\infty} |a_n|$ converges. $\square$

The following result, which is analogous to Theorem 14.2, is a consequence of Theorem 14.3.

Theorem 14.4 (Root test). *Consider the series $\sum_{n=1}^{\infty} a_n$. Suppose $|a_n|^{1/n} \to L$ for some L.*

(1) *If $L < 1$ then $\sum_{n=1}^{\infty} |a_n|$ converges.*

(2) *If $L > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.*

Proof. The proof is similar to proof of Theorem 14.2.

(1) Since $|a_n|^{1/n} \to L$ and $L < 1$, there exists $N \in \mathbb{N}$ such that $|a_n|^{1/n} < L + \frac{(1-L)}{2}$ for all $n \geq N$. Denote $L + \frac{(1-L)}{2}$ by r and note that $r \in (0, 1)$. By Theorem 14.3, $\sum_{n=1}^{\infty} \mid a_n \mid$ converges.

(2) Observe that if $L > 1$, then $a_n \nrightarrow 0$. Hence $\sum_{n=1}^{\infty} a_n$ diverges. $\square$

Example 14.3.

1. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$ converges because $a_n^{1/n} = \frac{1}{\ln n} \to 0$.

2. $\sum_{n=1}^{\infty} (n^{1/n} - 1)^n$ converges as $a_n^{1/n} = n^{1/n} - 1 \to 0$. However, $\sum_{n=1}^{\infty} (3n^{1/n} - 1)^n$ diverges.

3. $\sum_{n=1}^{\infty} (\frac{n}{n+1})^{n^2}$ converges because $a_n^{1/n} = \frac{1}{(1+\frac{1}{n})^n} \to \frac{1}{e} < 1$.

4. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. However, in both these cases, $a_n^{1/n} \to 1$.