Part VI

Looking into the future

Previous lectures

- Classically, binary numbers $\mathbf{0}$ and $\mathbf{1}$ are represented by off $|\mathbf{0}\rangle$ and on $|\mathbf{1}\rangle$ state of a system. These are known as **bits** and could be high and low voltages on a system.
- Quantum-mechanically binary numbers $\mathbf{0}$ and $\mathbf{1}$ are represented by states $|\mathbf{0}\rangle$ and $|\mathbf{1}\rangle$ of two level quantum systems. These are known as **qubits**.
- A major difference between classical and quantum realization of $\mathbf{0}$ and $\mathbf{1}$ is that in quantum case, a system can be in the superposition of the two states $\boldsymbol{\alpha}|\mathbf{0}\rangle + \boldsymbol{\beta}|\mathbf{1}\rangle$.
- Examples of two-level quantum systems are polarization of a photon and spin states of an electron.
- Superposition state is not possible classically where the system is either off or on but can certainly not be simultaneously in off and on state.

Previous lectures

- In two level systems the two states are represented by column vectors of order $\mathbf{2} \times \mathbf{1}$.
- The basis for representing a general wavefunction is $|\mathbf{0}\rangle = \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}$ and $|\mathbf{1}\rangle = \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix}$.
- A general qubit is then given as $\boldsymbol{\alpha}|\mathbf{0}\rangle + \boldsymbol{\beta}|\mathbf{1}\rangle = \boldsymbol{\alpha}\begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix} + \boldsymbol{\beta}\begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}$.
- The dual state of $|\mathbf{0}\rangle$ is $\langle\mathbf{0}| = (\mathbf{1} \quad \mathbf{0})$ and of $|\mathbf{1}\rangle$ is $\langle\mathbf{1}| = (\mathbf{0} \quad \mathbf{1})$ such that the scalar product $\left(\equiv \int \boldsymbol{\psi}^*(\boldsymbol{x})\boldsymbol{\psi}(\boldsymbol{x})\boldsymbol{dx}\right)$ of these states is

$$\langle\mathbf{0}|\mathbf{0}\rangle = (\mathbf{1} \quad \mathbf{0})\begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix} = \mathbf{1} \qquad \langle\mathbf{1}|\mathbf{1}\rangle = (\mathbf{0} \quad \mathbf{1})\begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix} = \mathbf{1}$$

$$\langle\mathbf{0}|\mathbf{1}\rangle = (\mathbf{1} \quad \mathbf{0})\begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix} = \mathbf{0} \qquad \langle\mathbf{1}|\mathbf{0}\rangle = (\mathbf{0} \quad \mathbf{1})\begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix} = \mathbf{0}$$

Previous lectures

- Dual of $\alpha|0\rangle + \beta|1\rangle$ is $\alpha^*(1 \quad 0)^* + \beta^*(0 \quad 1)^* = (\alpha^* \quad \beta^*) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}^\dagger$, known as the adjoint of $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$.

- Scalar product $\left(\equiv \int \psi^*(x)\psi(x)dx\right)$ of this state is

$$(\alpha^* \quad \beta^*)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = |\alpha|^2 + |\beta|^2.$$

Previous lectures

- Operators for two-level systems are given by $\mathbf{2} \times \mathbf{2}$ matrices.
- These could be any four linearly independent matrices like

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \quad \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

- We work in terms of Pauli matrices and the unit matrix

$$\boldsymbol{\sigma_x} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \quad \boldsymbol{\sigma_y} = \begin{pmatrix} \mathbf{0} & -\boldsymbol{i} \\ \boldsymbol{i} & \mathbf{0} \end{pmatrix} \quad \boldsymbol{\sigma_z} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \quad \mathbf{I} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

- This is because they are related to a physical observable (spin of an electron) and therefore satisfy certain desirable mathematical property (unitarity to be explained later in this lecture).

Lecture 36

Introduction to Quantum Computing-I
(Quantum Gates)

February 13-20, 2023 issue of TIME magazine

What is a Quantum Computer and how does it work?

[How Does a Quantum Computer Work? - Scientific American]

Mathematical operations on classical bits
&
Their Quantum computation counterparts

- Classical computations are done using binary numbers 0 and 1.
- Mathematical operations on them are performed using different Gates.
- Quantum computation are done representing 0 and 1 by quantum states $|\mathbf{0}\rangle = \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}$ and $|\mathbf{1}\rangle = \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix}$ known as **qubits**.
- We need to develop Quantum Gates to perform mathematical operations on these qubits. These operations are represented by operators, expressed by matrices.

Writing matrix for an operator acting on a single qubit

- Suppose an operator acting on the basis vectors gives

$$\hat{O}|0\rangle = \alpha_0|0\rangle + \beta_0|1\rangle$$
$$\hat{O}|1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$$

- Then the matrix for operator $\hat{O}$ is

$$\begin{pmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \end{pmatrix} = \begin{pmatrix} \alpha_0 & \beta_0 \\ \alpha_1 & \beta_1 \end{pmatrix}^T$$

- That is the coefficients of $|0\rangle$ and $|1\rangle$ when $\hat{O}$ operates on $|0\rangle$ are written as the first column of the matrix and when $\hat{O}$ operates on $|1\rangle$ are written as the second column.

An operator acting on a superposition of single qubits

- Suppose an operator acts on a superposition of single-qubits as

$$\boldsymbol{\hat{O}(\alpha|0\rangle + \beta|1\rangle)}$$

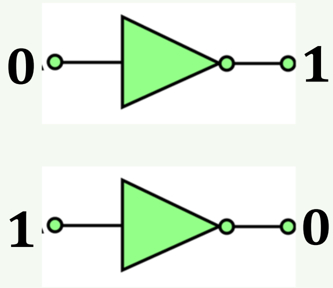
- The result of this is the superposition (with the same coefficients) of the states obtained by the operator acting on each quantum bit, because quantum mechanics is a linear theory and the operators are linear.
- Thus

$$\boldsymbol{\hat{O}(\alpha|0\rangle + \beta|1\rangle) = \alpha\hat{O}|0\rangle + \beta\hat{O}|1\rangle}$$

Single bit/qubit operations

Classical NOT gate

INPUT	OUTPUT
0	1
1	0



Quantum NOT gate

$|0\rangle \rightarrow |1\rangle$ and $|1\rangle \rightarrow |0\rangle$

In polarization

Matrix Representation $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$$X = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \sigma_x$$

More single qubit Operations in Quantum computing

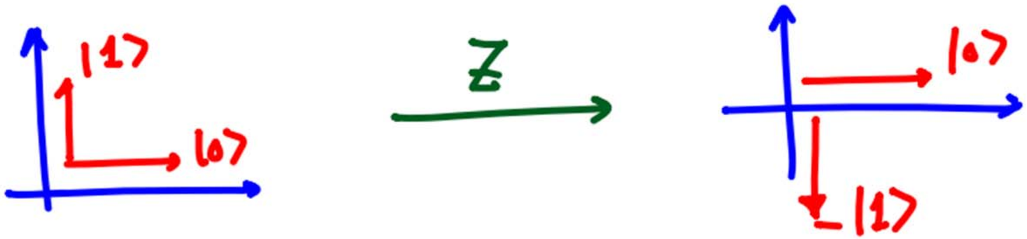
Z gate

Z gate : Z

$|0\rangle \to |0\rangle$

$|1\rangle \to -|1\rangle$

Polarization



Matrix Representation $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

More single qubit Operations in Quantum computing

Hadamard (pronounced Adamar) gate

Hadamard gate $\mathcal{H}$

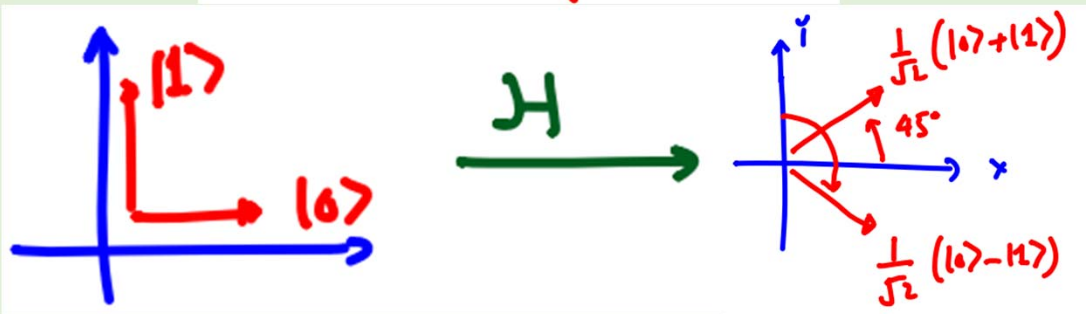
$$\mathcal{H}\,|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$\mathcal{H}\,|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

Matrix Representation : $\mathcal{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$= \frac{1}{\sqrt{2}}\left(\sigma_x + \sigma_z\right)$$

Polarization Representation



Writing matrix for an operator acting on a single qubit

(continuing from slide 10)

- NOT ($\hat{\mathrm{X}}$) and Hadamard ($\hat{\mathrm{H}}$) operators give

$$\hat{\mathbf{X}}|\mathbf{0}\rangle = |\mathbf{1}\rangle = \mathbf{0}|\mathbf{0}\rangle + \mathbf{1}|\mathbf{1}\rangle \qquad \hat{\mathbf{H}}|\mathbf{0}\rangle = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(|\mathbf{0}\rangle + |\mathbf{1}\rangle)$$

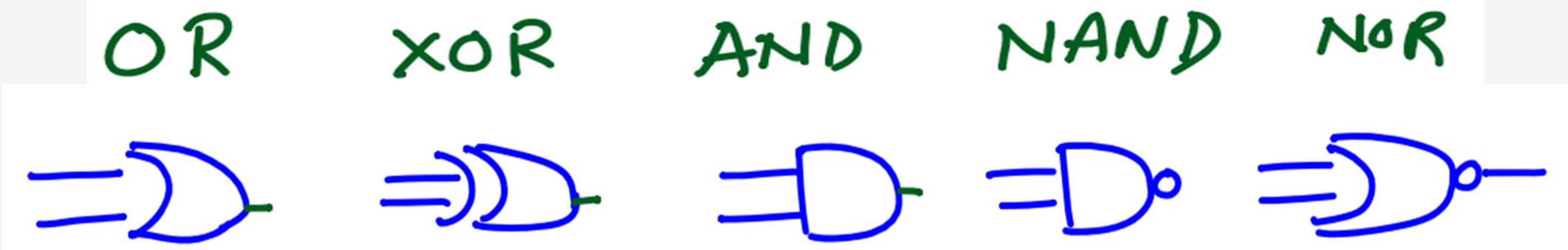
$$\hat{\mathbf{X}}|\mathbf{1}\rangle = |\mathbf{0}\rangle = \mathbf{1}|\mathbf{0}\rangle + \mathbf{0}|\mathbf{1}\rangle \qquad \hat{\mathbf{H}}|\mathbf{0}\rangle = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(|\mathbf{0}\rangle - |\mathbf{1}\rangle)$$

- So the corresponding matrices are

$$\mathbf{X} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \text{ and } \mathbf{H} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{pmatrix}$$

Next we consider operations on two bits/qubits

Classical two bit gates



OR

Input	Output
0 0	0
0 1	1
1 0	1
1 1	1

XOR

Input	Output
0 0	0
0 1	1
1 0	1
1 1	0

AND

Input	Output
0 0	0
0 1	0
1 0	0
1 1	1

NAND

Input	Output
0 0	1
0 1	1
1 0	1
1 1	0

NOR

Input	Output
0 0	1
0 1	0
1 0	0
1 1	0

Classical gates are irreversible

Introducing mathematical operation $\oplus$

$$\oplus \equiv add\ (mod\ 2)$$

$$x \oplus y = (x + y)(mod\ 2)$$

So

$$0 \oplus 0 = 0\ (mod\ 2) = 0$$

$$0 \oplus 1 = 1\ (mod\ 2) = 1$$

$$1 \oplus 0 = 1\ (mod\ 2) = 1$$

$$1 \oplus 1 = 2\ (mod\ 2) = 0$$

Representing classical XOR and NAND gate as add mod2 operation

XOR

x	y	output
0	0	0
0	1	1
1	0	1
1	1	0

OUTPUT $= x \oplus y$

NAND

x	y	output
0	0	1
0	1	1
1	0	1
1	1	0

OUTPUT $= 1 \oplus xy$

NAND gate is a universal gate. That means all other gates can be made using NAND gates.

Working with two qubits

What are two-qubit states?

- Two-qubit states are like the spin-states of two electrons taken together. Thus their number is four.
- These are the eigenstates of the sum $S_{z1} + S_{z2}$ of the z-components of spin of two electrons.
- By separation of variables these can then be written as the product of spin quantum state of each electron.
- This gives states $|\mathbf{0}\rangle\,|\mathbf{0}\rangle$, $|\mathbf{0}\rangle\,|\mathbf{1}\rangle$, $|\mathbf{1}\rangle\,|\mathbf{0}\rangle$ and $|\mathbf{1}\rangle\,|\mathbf{1}\rangle$ which are written simply as $|\mathbf{00}\rangle$, $|\mathbf{01}\rangle$, $|\mathbf{10}\rangle$ and $|\mathbf{11}\rangle$.

Two-qubit states……

- In matrix form the two qubit states $|\mathbf{00}\rangle$, $|\mathbf{01}\rangle$, $|\mathbf{10}\rangle$ and $|\mathbf{11}\rangle$ are written as

$$|\mathbf{00}\rangle = \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \qquad |\mathbf{01}\rangle = \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \qquad |\mathbf{10}\rangle = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{pmatrix} \qquad |\mathbf{11}\rangle = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix}$$

Note that writing the states sequentially, we start with 1 in the first row and then keep shifting it to the lower row for the next state

- Their order is the same as the decimal numbers 0, 1, 2 and 3 that the binary representation 00, 01, 10 and 11 give.
- The states form the basis for working with two-qubit states.
- Operations on two-qubit states are represented by $\mathbf{4} \times \mathbf{4}$ matrices.

Writing matrix for an operator acting on a two-qubit states

- Suppose an operator acting on the basis vectors gives

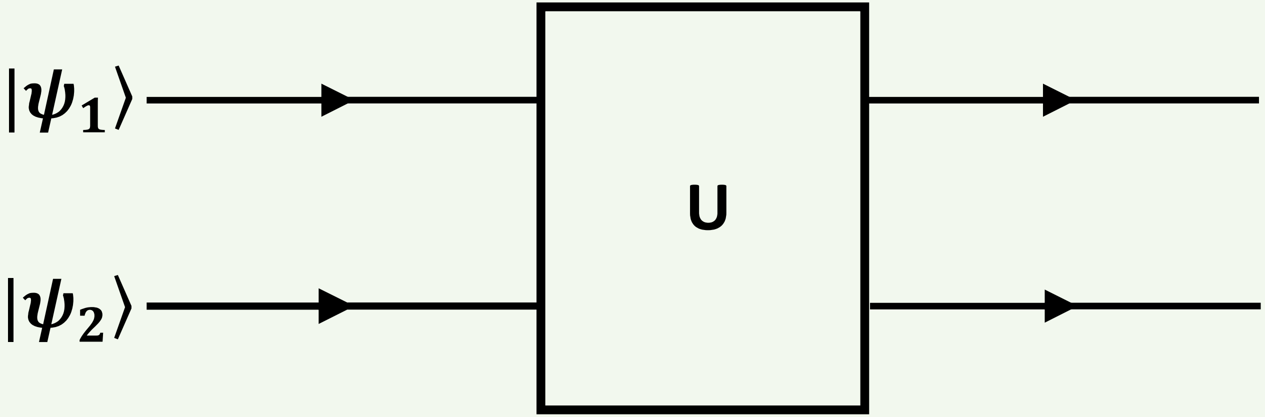
$$\hat{O}|00\rangle = \alpha_0|00\rangle + \beta_0|01\rangle + \gamma_0|10\rangle + \delta_0|11\rangle$$
$$\hat{O}|01\rangle = \alpha_1|00\rangle + \beta_1|01\rangle + \gamma_1|10\rangle + \delta_1|11\rangle$$
$$\hat{O}|10\rangle = \alpha_2|00\rangle + \beta_2|01\rangle + \gamma_2|10\rangle + \delta_2|11\rangle$$
$$\hat{O}|11\rangle = \alpha_3|00\rangle + \beta_3|01\rangle + \gamma_3|10\rangle + \delta_3|11\rangle$$

- Then the matrix for operator $\hat{O}$ is

$$\begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \\ \delta_0 & \delta_1 & \delta_2 & \delta_3 \end{pmatrix} = \begin{pmatrix} \alpha_0 & \beta_0 & \gamma_0 & \delta_0 \\ \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \delta_3 \end{pmatrix}^T$$

Two qubit gates

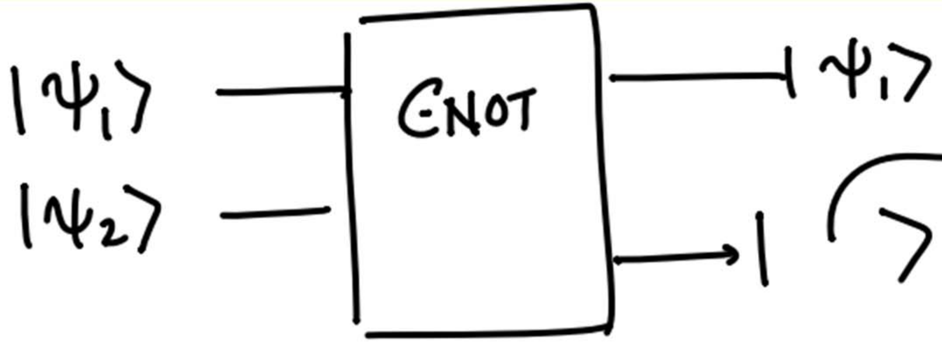
Two qubit gates have two inputs (one qubit each) and two outputs (one qubit each)



Two qubit gates are reversible

We now introduce C-NOT gate that is a universal quantum gate. That means all other quantum gates can be made using C-NOT gates.

Controlled NOT (C-NOT) gate



output depends on $|\psi_1\rangle$

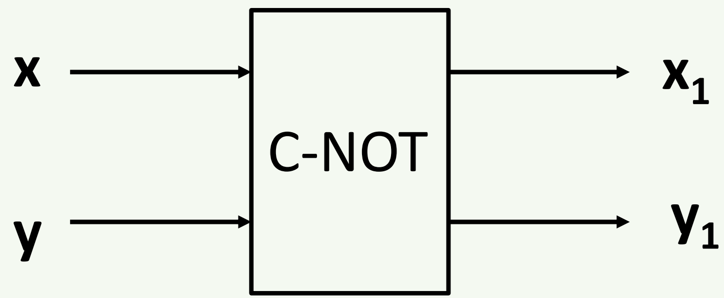
By $|\psi_1\rangle$ one is controlling the NOT operator for $|\psi_2\rangle$

If $|\psi_1\rangle = |0\rangle$ then $|\psi_2\rangle = |\psi_2\rangle$

If $|\psi_1\rangle = |1\rangle$ then $|\psi_2\rangle = |\overline{\psi_2}\rangle$

or $|\psi_2\rangle = X|\psi_2\rangle$

Representing Quantum C-NOT gate as add mod2 operation

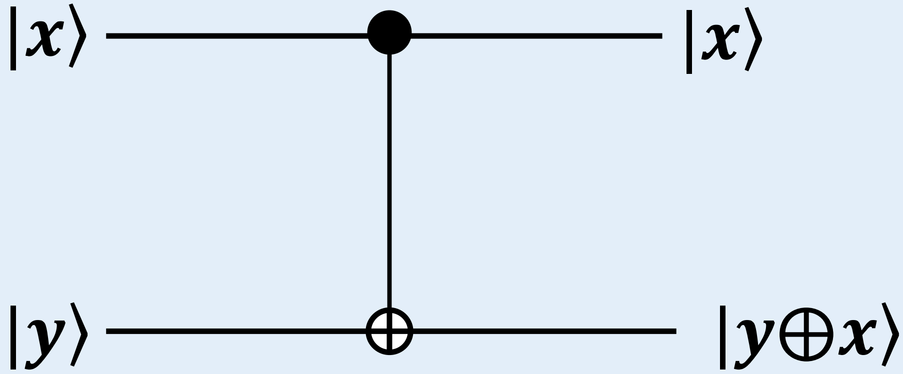


C-NOT

$\lvert xy\rangle$	x	y	x_1	y_1	$\lvert x_1y_1\rangle$
$\lvert 00\rangle$	0	0	0	0	$\lvert 00\rangle$
$\lvert 01\rangle$	0	1	0	1	$\lvert 01\rangle$
$\lvert 10\rangle$	1	0	1	1	$\lvert 11\rangle$
$\lvert 11\rangle$	1	1	1	0	$\lvert 10\rangle$

OUTPUT $= x \quad y \oplus x$ OR $\lvert x\, y \oplus x\rangle$

Symbol for C-NOT gate



Matrix for C-NOT gate

$$\hat{O}|00\rangle = 1|00\rangle + 0|01\rangle + 0|10\rangle + 0|11\rangle$$

$$\hat{O}|01\rangle = 0|00\rangle + 1|01\rangle + 0|10\rangle + 0|11\rangle$$

$$\hat{O}|10\rangle = 0|00\rangle + 0|01\rangle + 0|10\rangle + 1|11\rangle$$

$$\hat{O}|11\rangle = 0|00\rangle + 0|01\rangle + 1|10\rangle + 0|11\rangle$$

So the matrix for C-NOT gate is

$$\begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \\ \delta_0 & \delta_1 & \delta_2 & \delta_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} I_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & \sigma_{x_{2\times2}} \end{pmatrix}$$