Lecture - 3 Blackbody 08/01/2024

#3 (1/8)

Density of Modes Jeans Cube

$$A \sin(kx - \omega t) + A \sin(kx + \omega t)$$

$$= 2A \sin kx \cos(\omega t)$$

At $x = L$ it is 0. $k_x L = n_x \pi$

$$\Rightarrow \left(\frac{2\pi}{\lambda}\right) L = n\pi$$

$$k_y = n_y \frac{\pi}{L}, \quad k_z = n_z \frac{\pi}{L}, \quad k_x = n_x \frac{\pi}{L}$$

n_x, n_y, n_z positive integers.

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{c}\nu$$

$$dk = \frac{2\pi}{c} d\nu$$

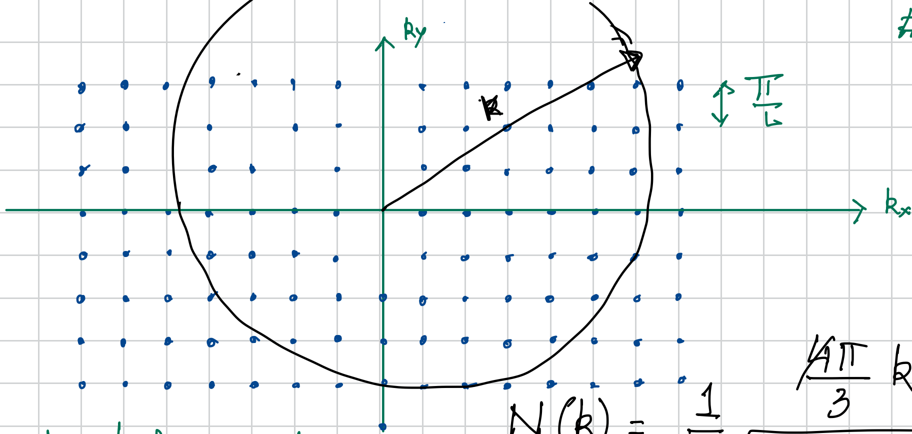
$$g(k)\,dk = \frac{1}{\pi^2}\left(\frac{2\pi}{c}\nu\right)^2 \cdot \frac{2\pi}{c} d\nu$$

$$\boxed{g(\nu)\,d\nu = \frac{8\pi}{c^3}\nu^2\,d\nu}$$

$$u_\nu\,d\nu = \underline{\langle\epsilon\rangle}\frac{1}{\pi^2}\frac{8\pi}{c^3}\,d\nu$$

$$= \boxed{\frac{8\pi}{c^3} k_B T}$$

Rayleigh-Jeans



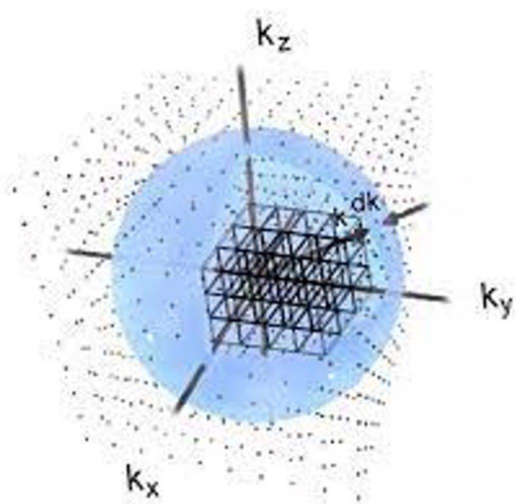
Each state 2 polarization.

1-state/$\left(\frac{\pi}{L}\right)^3$ in k-space

Count only positive quadrant.

$$N(k) = \frac{1}{\cancel{8}} \frac{\frac{\cancel{4}\pi}{3} k^3}{\left(\frac{\pi}{L}\right)^3} \cdot \cancel{2}$$

$$\frac{N(k)}{L^3} = \frac{k^3}{3\pi^2}$$



$$g(k)\,dk = \frac{d}{dk}\left(\frac{k^3}{3\pi^2}\right) dk$$

$$g(k)\,dk = \frac{k^2}{\pi^2}\,dk$$

$$\langle E\rangle = \frac{\cancel{c}\int_0^\infty E\, e^{-E/k_BT}\, dE}{\cancel{c}\int_0^\infty e^{-E/k_BT}\, dE}$$

$$= \frac{1}{k_BT}\int_0^\infty E\, e^{-E/k_BT}\, dE$$

$$= \frac{(k_BT)^2}{k_BT} = \boxed{k_BT}$$

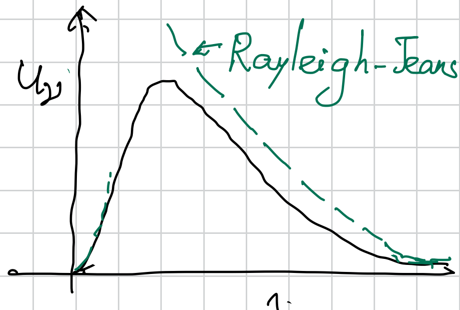
$$\boxed{f(E) = C\, e^{-E/k_BT}}$$

Normalize.

$$\int_0^\infty f(E)\, dE = 1$$

$$\Rightarrow \quad C = \frac{1}{k_BT}$$

$$u_\nu = \frac{8\pi}{c^3}\nu^2\, k_BT$$



Ultra-violet catastrop

Planck's Proposal

$$E_n = \underset{=}{n}\, \underline{\underline{h\nu}}$$

Discrete

Integer, h a fitting parameter.

Need to find. $\langle E_n\rangle$ Average Energy

Modes in a cavity:

- Only standing waves possible in eqlbrm.

$$y_{net} = A\sin(kx-\omega t) + A\sin(kx+\omega t)$$

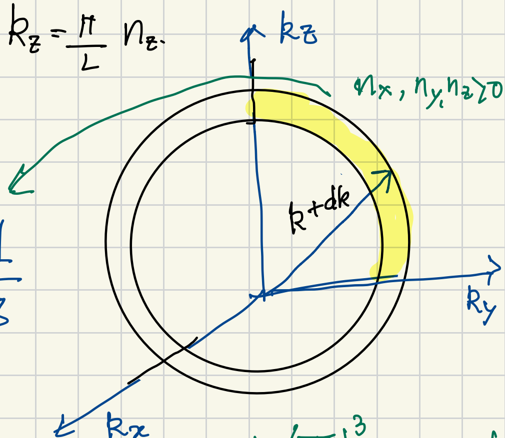
$$= 2A\sin(kx)\cos\omega t$$

At edges it vanishes $\therefore$ $k_x L = n_x \pi$

$$\Rightarrow k_x = \frac{\pi}{L} n_x, \quad n_x \text{ +ve integ.}$$

Similarly $k_y = \frac{\pi}{L} n_y$, $k_z = \frac{\pi}{L} n_z$.

Alternative:



$$dN = \frac{4\pi k^2 dk}{\left(\frac{\pi}{L}\right)^3} \cdot 2 \cdot \frac{1}{8}$$

Each $\left(\frac{\pi}{L}\right)^3 \to 1$ mode with two polarization.

$$= V \frac{k^2}{\pi^2} dk$$

$$\frac{dN}{V} = \frac{k^2}{\pi^2} dk$$

$$g(k)\,dk = \frac{k^2}{\pi^2} dk$$

$$= \frac{1}{\pi^2}\left(\frac{2\pi}{c}\right)^2 \nu^2 \cdot \left(\frac{2\pi}{c}\right) d\nu$$

$$= \frac{8\pi}{c^3} \nu^2 d\nu$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{c}\nu$$

$$dk = \frac{2\pi}{c} d\nu$$

No. of modes between ν and $\nu + d\nu$

$$= \frac{8\pi}{c^3} \nu^2 \, d\nu$$

Average energy of oscillators.

$$\langle \epsilon \rangle = \frac{\int c \, \epsilon \, e^{-\epsilon / k_B T} \, d\epsilon}{\int c \, e^{-\epsilon / k_B T} \, d\epsilon}$$

$$= \frac{(k_B T)^2}{k_B T} = k_B T$$

- Justified on the basis of Equipart

$$\boxed{u(\nu) = \frac{8\pi}{c^3} \nu^2 (k_B T)}$$

Rayleigh & Jeans.

Note

- u_ν diverges as $\nu \to \infty$
- OK for long-wavelength limit

Planck's proposal

$$\epsilon_n = n h\nu$$

f_n: Probability of occupation.

$$\langle \epsilon \rangle = \sum \epsilon_n f_n$$ ⟵ Average Energy

$$f_n = c' e^{-nh\nu/k_B T}$$

$$\sum f_n = 1 \Rightarrow c'$$ Normalization

$$x = \frac{h\nu}{k_B T}$$

$$\sum f_n = c' \sum e^{-nx}$$

$$= c' \left[1 + e^{-x} + (e^{-x})^2 + (e^{-x})^3 + \cdots \right]$$

$$= \frac{c'}{1 - e^{-x}} \Rightarrow c' = \left(1 - e^{-h\nu/k_B T}\right)$$

$$\langle \epsilon \rangle = \sum \epsilon_n f_n = \left(1 - e^{-h\nu/k_BT}\right) h\nu \sum n e^{-nx}$$

$$= \left(1 - e^{-h\nu/k_BT}\right) h\nu \left[e^{-x} + 2\left(e^{-x}\right)^2 + 3\left(e^{-x}\right)^3 + \cdots \right]$$

$$= \left(1 - e^{-h\nu/k_BT}\right) h\nu \left\{ -\frac{d}{dx}\left[e^{-x} + \left(e^{-x}\right)^2 + \left(e^{-x}\right)^3 + \cdots \right] \right.$$

$$= \left[\frac{d}{dx} \left(1 - e^{-x}\right)^{-1} \right]$$

$$= \frac{h\nu}{e^{h\nu/k_BT} - 1}$$

$$\boxed{u_\nu = \frac{8\pi}{c^3} \nu^2 \frac{h\nu}{e^{h\nu/k_BT} - 1}}$$

Excellent Agreement

- For ν small i.e. $h\nu \ll k_BT$ quanta inconsequential. number of states within k_BT large $\Rightarrow$ continuum approx. ok.
- If $h\nu \gg k_BT$: spacing is crucial

Example:

$h\nu = 6 k_B T$

$n=1$

$n=0$

$$e^{-6} \simeq \frac{1}{403}$$

At constant T, excitation probability diminishes exponentially with $\nu \uparrow$

In general:
- if $h\nu \ll k_B T$ Classical
- $h\nu \gg k_B T$ Quantum.
- or if $h \to 0$, $\sum \longrightarrow \int$

Back to slides.

h is the quantum of Action!

- Quantum age was born. (14 Dec. 1900)
- Significance of h.
- Indicates a new statistics for Thermodynamics.

Attached: Class Slides.

8/1/24