Lecture - 9 PHY114

23/01/2024

L9: 1/12 ynm

Illustration of Event Amplitudes & their Sum.
(using slit interference)

Preliminary Intro to $\hat{x}$, $\hat{p}$, $|\Psi\rangle$

Expectation Value of Operators

1-D Schrödinger Equation

Stationary States

L9: 2/12 ynm

What does Ψ stand for?

Born's Statistical interpretation

Probability density $\Psi^* \Psi = \rho$

Ex. Prob. of Finding in dx :

$$\Psi^* \Psi \, dx$$

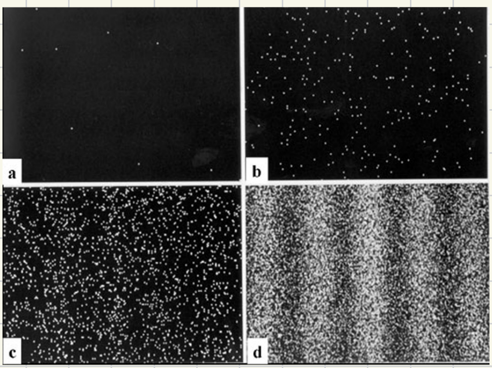


Fig. 2 Single electron events build up to from an interference pattern in the double-slit experiments.

- The number of electron accumulated on the screen.

(a) 8 electrons;

(b) 270 electrons;

(c) 2000 electrons;

(d) 160,000.

The total exposure time from the beginning to the stage (d) is 20 min.

Hitachi Ltd, 1994,2024

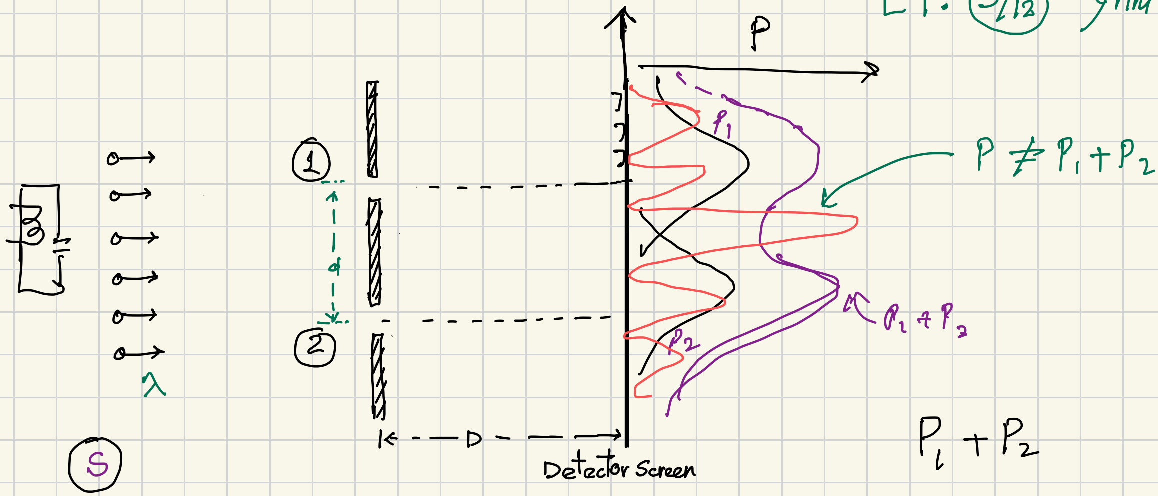
Double slit experiment with electrons

Occurrence of each dot on screen has probility associated with it.

Doing large number of expeiments the probability distribution of the observable emerges as we observe in this case.

The 'intensity pattern' or square of the amplitudes for each occurence is measured.

The spectrum of the observable (in this case, position on the screen emerges.)



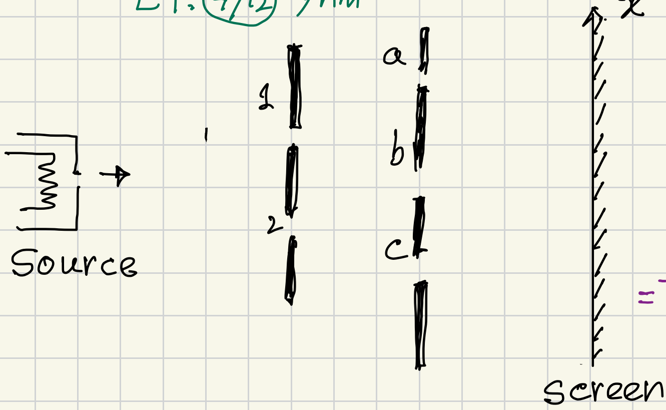
Probability Amplitude $\Rightarrow$ Probability density
(a complex number) (square of the amplitude) can be measured

Prob. amplitude for e leaving source and arriving at x : $\langle x | S \rangle$ ← a complex no.

Amplitude for: Leaving s, arriving at 1, arriving at x

$$\underbrace{\underbrace{\langle x | 1 \rangle \langle 1 | S \rangle}_{\text{product of sequential amplitudes of an event}} + \langle x | 2 \rangle \langle 2 | S \rangle}_{\text{Amplitude of Sum of all possible events (Interference occurs because of this.)}}$$

L9: 4/12 ynm



All possible indpdt. amplitude.

$$\langle x|a\rangle\langle a|1\rangle\langle 1|s\rangle + \langle x|a\rangle\langle a|2\rangle\langle 2|s\rangle + \langle x|b\rangle\langle b|1\rangle\langle 1|s\rangle + \langle x|b\rangle\langle b|2\rangle\langle 2|s\rangle + \cdots + \langle x|c\rangle\langle c|2\rangle\langle 2|s\rangle$$

$$= \text{Total Amplitude} = \sum_{\substack{i=1,2 \\ \alpha=a,b,c}} \langle x|\alpha\rangle\langle\alpha|i\rangle\langle i|s\rangle$$

Typical amplitudes involved in this case

$$\langle \vec{r}_2|\vec{r}_1\rangle \sim \frac{e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}_{12}}}{|\vec{r}_{12}|},$$

where $p = \sqrt{2mE}$ for non-relativistic case or, $p^2c^2 = E^2 - (m_0c^2)^2$ for relativistic case.

If it is probabilistic theory, outcomes have probabilities associated with it.

How does it connect with experiment?

bra vector $\rightarrow \langle\Psi|$

$|\Psi\rangle \leftarrow$ A ket vector (independent of representation)

$\downarrow$ Measurement

Physical Observables.

Ex., $p, E, L_z, S_z \ldots\ldots$

$$\langle\Psi|\Psi\rangle = 1 = \int_{-\infty}^{\infty}\Psi^*\Psi\,dx$$

With probabilities one can find Expectation Values

Ket-bra notation will be discussed in more detail later. Do not worry about it right now

- We measure one outcome from all possible outcomes at any instance.

- Measurement Outcome has a probability associated with.

  By making many measurements, we are measuring probabity of all possible outcomes.

- $|\Psi\rangle$ State can be a superposition of possible states.

  It is not a superposition of probabities.

  But amplitudes — and complex amplitudes

$$|\Psi\rangle \quad \text{a state vector}$$

$\hat{A}$.

Results in $a_1, a_2, a_3, \ldots\ldots$ can be continuous or discrete

$|a_1\rangle, |a_2\rangle, |a_3\rangle$ finite or infinite.

$$|\Psi\rangle = \sum_i a_i \, |a_i\rangle$$

{ Superposition of all possible states (pure, or base states) corresponding to the observable 'a' and its operator $\hat{A}$. (Fourier Expansion Series !!)

- You measure physical quantities:
  there is probability associated with it.

Ex. $P(x)\,dx \; : \; \Psi^* \Psi \, dx$

Prob. x and $x+dx$ [e^- landing on screen]

We measure $P(x)$, but events have amplitude.

Prob. is square of amplitude. (Like intensity is square of field amplitude).

- Events may happen. $\exists$ an Amplitude for it

Ex. $\underbrace{\langle x | \psi \rangle}$ ← complex amplitude for ψ at x.

- Reverse Amplitude is complex conjugate.

Ex. $\langle x | \psi \rangle = \langle \psi | x \rangle^*$

Prob. density $\rho = |\langle \varphi | \psi \rangle|^2$

Three Important Operators (in Position Space)

$$\hat{x}\,\psi(x) = x\,\psi(x)$$

$$\hat{p}_x\,|\psi\rangle = -i\hbar\frac{\partial}{\partial x}|\psi\rangle$$

$$\langle x\rangle = \int \psi^*\,\hat{x}\,\psi\,dx$$

In Bra-Ket notation the expection value of any operator $\hat{A}$ is given as:

$$\langle\psi|\hat{A}|\psi\rangle$$

$$\langle f|g\rangle = \int f^* g\,dx$$

Inner Product

These will be discussed in more detail next week. Do not worry about them right now.

But, would $\langle p_x\rangle = \int \psi^*\,\cancel{p_x(x)}\,\psi\,dx$ work?

$$= \int \psi^* \left(-i\hbar\frac{\partial}{\partial x}\right)\psi\,dx$$

$$\langle E\rangle = \int \psi^* \left(i\hbar\frac{\partial}{\partial t}\right)\psi\,dx$$

In general, expectation value of an operator $\hat{A}$

$$\langle\hat{A}\rangle = \int \psi^*\,\hat{A}\,\psi\,dx$$

$$\langle\hat{A}^2\rangle = \int \psi^*\,\hat{A}\left(\hat{A}\,\psi\right)dx.$$

$$\Delta\hat{A} = \hat{A} - \langle\hat{A}\rangle \quad \checkmark$$

$$\sigma_A = \sqrt{\langle\Delta\hat{A}^2\rangle} = \sqrt{\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2}$$

$$E = \frac{p^2}{2m} + V(\hat{x}, t)$$

1-dim.

$$i\hbar \frac{\partial}{\partial t} \Psi(x,t) = \frac{1}{2m}\left[-i\hbar\frac{\partial}{\partial x}\left(-i\hbar\frac{\partial}{\partial x}\right) + V(x,t)\right]\Psi(x,t)$$

$$\boxed{i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x,t)\,\Psi}$$

Prob. conservation — Q.M. — Kinetic Energy — Potential Energy

Note: 2^{nd} Order in space, but 1^{st} order in time.

$i \rightarrow$ Complex solutions are allowed for Probability amplitudes. Needed for Probability conservation.

$\hbar \rightarrow$ signature of Q.M.; units consistency

Following is the generalized, respresention indpt. form of SE.

$$\boxed{i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \hat{H}\,|\Psi\rangle}$$

$\hat{H}$ is Hamiltonian operator.

- Contrast with classical wave Eqn.

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

* 2^{nd} order both in space and time.

* Real solutions, Complex exponentials are used for convenience.

- Other Partial Diff. Equations of Physics & Engg.

$$\frac{\partial^2 T}{\partial x^2} = k\frac{\partial T}{\partial t}$$

$k = \frac{K}{c\rho}$, Real positive

$\boxed{u_t = \alpha u_{xx}}$ $\alpha > 0$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{D}\frac{\partial u}{\partial t}$$

| $u(x,t)$ is the conc. of the diffusing material.

Such 'heat' or 'diffusion' equation are widely used in science and engg.

Notice absence of 'i' and ħ.

The solutions of these equations subjected to necessary boundary conditions must be <u>real</u>.

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x,t)\,\Psi$$

If V is NOT time-dependent

try separation of variables

$$\Psi(x,t) = \psi(x)\,\phi(t)$$

$$i\hbar\,\psi(x)\frac{\partial \phi(t)}{\partial t} = -\frac{\hbar^2}{2m}\phi(t)\frac{\partial^2 \psi(x)}{\partial x^2} + V(x)\,\psi(x)\,\phi(t)$$

Dividing by $\Psi(x,t) = \phi(t)\,\psi(x)$

$$\underbrace{i\hbar \frac{1}{\phi}\frac{\partial \phi}{\partial t}}_{\text{only depends on } t} = \underbrace{-\frac{\hbar^2}{2m}\frac{1}{\psi}\frac{\partial^2 \psi}{\partial x^2} + V(x)}_{\text{only depends on } x} = E$$

Must be a CONSTANT

$$\hat{H}\psi = E\psi \qquad \text{(TISE)}$$

Time Independent Schrödinger Eqn.

$$i\hbar \frac{1}{\phi}\frac{d\phi}{dt} = E$$

$$\frac{d\phi}{dt} = -\left(i\frac{E}{\hbar}\right)\phi$$

$$\Rightarrow \quad \phi(t) = e^{-i\frac{E}{\hbar}t}$$

Stationary States :

$$\Psi(x,t) = \psi(x)\, e^{-i\frac{E}{\hbar}t}$$

Note $\Psi^*\Psi$ is t-independent | only if $V(x)$ is

To get Ψ : First solve for $\psi(x)$ using TISE and simply multiply $\phi(t)$, since you would have solved for E already.

$$\Psi(x,t) = \psi(x)\,\phi(t)$$

$$= \psi(x)\, e^{-i\frac{E}{\hbar}t}$$

Conditions on Ψ : $\psi(x)$ is solⁿ of TISE.

- $\int_{-\infty}^{\infty} \Psi^* \Psi \, dx = 1$ Normalizable

  $\psi(x) \to 0, \quad x \to \infty$

- $\frac{\partial^2 \psi}{\partial x^2}$ must be finite unless $V(x)$ is singular.

- $\psi(x)$ must be <u>Continuous.</u>

  $\frac{\partial \psi}{\partial x}$ " " " .

Most problems in Introductory QM

→ Writing general solⁿ of 2^{nd} diff. eq. with requiste no. of constants

& → Applying boundary conditions ψ and $\frac{\partial \psi}{\partial x}$ being continuous to fix the constants.

Practice this skill. in HA #4