Practice Problems 8: Mean Value Theorem, Cauchy Mean Value Theorem, L'Hospital's Rule

1. Establish the following inequalities using the MVT.

   (a) $\frac{1}{2\sqrt{n+1}} < \sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}}$ for all $n \in \mathbb{N}$.

   (b) $\frac{x-1}{x} < \log x < x - 1$ for $x > 1$.

   (c) $e^x \geq ex$ for all $x \in \mathbb{R}$.

2. Does there exist a differentiable function $f : [0, 2] \to \mathbb{R}$ satisfying $f(0) = -1, f(2) = 4$ and $f'(x) \leq 2$ for all $x \in [0, 2]$?

3. Let $f$ be twice differentiable on $[0, 2]$. Suppose that $f(0) = 0, f(1) = 2$ and $f(2) = 4$. Show that there is $x_0 \in (0, 2)$ such that $f''(x_0) = 0$.

4. Let $a > 0$ and $f : [-a, a] \to \mathbb{R}$ be differentiable. Suppose that $f'(x) \leq 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, then show that $f(x) = x$ for every $x \in (-a, a)$.

5. Let $f : [0, 1] \to \mathbb{R}$ be twice differentiable. Suppose that the line segment joining the points $(0, f(0))$ and $(1, f(1))$ intersect the graph of $f$ at a point $(a, f(a))$ where $0 < a < 1$. Show that there exists $x_0 \in [0, 1]$ such that $f''(x_0) = 0$.

6. Let $f : [0, 1] \to \mathbb{R}$ be continuous. Suppose that $f$ is differentiable on $(0, 1)$ and $\lim_{x\to 0} f'(x) = \alpha$ for some $\alpha \in \mathbb{R}$. Show that $f'(0)$ exists and $f'(0) = \alpha$.

7. Let $f : [0, 1] \to \mathbb{R}$ be differentiable and $f(0) = 0$. Suppose that $|f'(x)| \leq |f(x)|$ for all $x \in [0, 1]$. Show that $f(x) = 0$ for all $x \in [0, 1]$.

8. Let $f : [0, \infty) \to \mathbb{R}$ be continuous and $f(0) = 0$. Suppose that $f'(x)$ exists for all $x \in (0, \infty)$ and $f'$ is increasing on $(0, \infty)$. Show that the function $g(x) = \frac{f(x)}{x}$ is increasing on $(0, \infty)$.

9. Establish the following inequalities.

   (a) For $\alpha > 1$, $(1 + x)^\alpha \geq 1 + \alpha x$ for all $x > -1$.

   (b) For $x > 0$, $e \log x \leq x$.

10. Let $a > 0$ and $f : [a, b] \to \mathbb{R}$ be differentiable. Show that there exists $c \in (a, b)$ such that $\frac{bf(a)-af(b)}{b-a} = f(c) - cf'(c)$.

11. Let $f : [a, b] \to \mathbb{R}$ be differentiable and $a \geq 0$. Using the Cauchy mean value theorem, show that there exist $c_1, c_2 \in (a, b)$ such that $\frac{f'(c_1)}{a+b} = \frac{f'(c_2)}{2c_2}$.

12. Evaluate the following limits using L'Hospital's Rule.

    (a) $\lim\limits_{x\to 0^+} (\frac{1}{\sin x} - \frac{1}{x})$.

    (b) $\lim\limits_{x\to\infty} (e^x + x)^{1/x}$.

    (c) $\lim\limits_{x\to\infty} (\log x - x)$.

    (d) $\lim\limits_{x\to 0^+} (\cos x)^{1/x}$.

    (e) $\lim\limits_{x\to 0^+} (\sin x)^{\sqrt{x}}$.

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13. Let $f : (0,\infty) \to [1,\infty)$ be differentiable. Suppose that $\lim_{x\to\infty}(f(x) + f'(x)) = \ell$ for some $\ell \in \mathbb{R}$. Using L'Hospital's rule, show that $\lim_{x\to\infty} f(x) = \ell$.

14. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f''(c)$ exists at some $c \in \mathbb{R}$. Using L'Hospital's Rule, show that

$$\lim_{h\to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c).$$

Give an example of a function $f$ and a point $c$ such that the above limit exists but $f$ is not twice differentiable at $c$.

15. **(*)** Let $f : [a,b] \to \mathbb{R}$ be differentiable. If $f'(x) \neq 0$ for all $x \in [a,b]$, then show that either $f'(x) > 0$ for all $x \in [a,b]$ or $f'(x) < 0$ for all $x \in [a,b]$.

16. **(*)** Let $f : [a,b] \to \mathbb{R}$ be such that $f'(x) \neq 0$ for all $x \in [a,b]$ and $J = \{f(x) : x \in [a,b]\}$. Show that $f^{-1} : J \to [a,b]$ is differentiable and $(f^{-1})'(f(x)) = \frac{1}{f'(x)}$ for all $x \in [a,b]$.

Practice Problems 8: Hints/Solutions

1. (a) By the MVT, there exists $c \in (n, n+1)$ such that $\sqrt{n+1} - \sqrt{n} = \frac{1}{2\sqrt{c}}$.
   (b) By the MVT, there exists $c \in (1, x)$ such that $\log x - \ln 1 = \frac{1}{c}(x-1)$.
   (c) By the MVT (see Application 7.3), $e^x \geq 1+x$ for all $x \in \mathbb{R}$. That is, $e^{x-1} \geq 1+(x-1)$.

2. If so, then by the MVT there exits $c \in (0,2)$ such that $5 = f(2) - f(0) = 2f'(c)$.

3. By the MVT there exist $x_1 \in (0,1)$ and $x_2 \in (1,2)$ such that $f'(x_1) = f(1) - f(0) = 2$ and $f'(x_2) = f(2) - f(1) = 2$. Apply Rolle's theorem for $f'$ on $[x_1, x_2]$

4. Let $g(x) = f(x) - x$ for all $x \in [-a, a]$. Note that $g'(x) \leq 0$ on $(-a, a)$. Therefore, $g$ is decreasing. Since $g(a) = g(-a) = 0$, we have $g = 0$.

5. Using the MVT on $[0, a]$ and $[a, 1]$, obtain $b \in (0, a)$ and $c \in (a, 1)$ such that $\frac{f(a)-f(0)}{a-0} = f'(b)$ and $\frac{f(1)-f(a)}{1-a} = f'(c)$. Note that $f'(b) = f'(c)$ because they are slopes of the same chord. By Rolle's theorem there exists $x_0 \in (b, c)$ such that $f''(x_0) = 0$.

6. Let $x \in (0, 1]$. By the MVT, there exists $c_x \in (0, x)$ such that $\frac{f(x)-f(0)}{x} = f'(c_x)$. Now $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x} = \lim_{x\to 0} f'(c_x) = \lim_{c_x \to 0} f'(c_x) = \alpha$.

7. For $x \in (0, 1)$, by the MVT, there exists $x_1$ such that $0 < x_1 < x$ and $f(x) = f'(x_1)x$. This implies that $|f(x)| \leq x|f(x_1)|$. Similarly there exists $x_2$ such that $0 < x_2 < x_1$ and $|f(x_1)| \leq x_1|f(x_2)|$. Therefore $|f(x)| \leq x^2|f(x_2)|$. Find a sequence $(x_n)$ in $(0, 1)$ such that $|f(x)| \leq x^n|f(x_n)|$. Since $f$ is bounded on $[0, 1]$, $x^n|f(x_n)| \to 0$. Hence $f(x) = 0$.

8. Note that $g'(x) = \frac{xf'(x)-f(x)}{x^2} = \frac{f'(x)-\frac{f(x)}{x}}{x}$. Observe that, by the MVT, $\frac{f(x)}{x} = f'(c_x)$ for some $c_x \in (0, x)$. Since $f'$ is increasing, $g'(x) \geq 0$. Hence $g$ is increasing.

9. (a) Let $\alpha > 1$ and $f(x) = (1+x)^\alpha - (1+\alpha x)$ on $(-1, \infty)$. Then $f'(x) \leq 0$ on $(-1, 0]$ and $f'(x) \geq 0$ on $[0, \infty)$. Hence $f(x) \geq f(0) = 0$ on $(-1, 0]$ and $f(x) \geq f(0) = 0$ on $[0, \infty)$. Therefore $f(x) \geq 0$ on $(-1, \infty)$.

   (b) Define $f(x) = x - e\ \log x$ on $(0, \infty)$. Then $f'(x) = \frac{x-e}{x}$. Therefore $f'(x) > 0$ on $(e, \infty)$ and $f'(x) < 0$ on $(0, e)$. Hence $f(x) > f(e)$ for all $x \in (0, \infty)$ and $x \neq e$.

10. Observe that $\frac{bf(a)-af(b)}{b-a} = \frac{\frac{f(b)}{b} - \frac{f(a)}{a}}{\frac{1}{b} - \frac{1}{a}}$. Apply the CMVT to $\frac{f(x)}{x}$ and $\frac{1}{x}$.

11. Apply the CMVT to $f(x)$ and $g_1(x) = x$. Again apply to $f(x)$ and $g_2(x) = x^2$.

12. (a) We have $\lim\limits_{x\to 0^+} (\frac{1}{\sin x} - \frac{1}{x}) = \lim\limits_{x\to 0^+} \frac{x-\sin x}{x \sin x} = \lim\limits_{x\to 0^+} \frac{1-\cos x}{\sin x + x\cos x} = \lim\limits_{x\to 0^+} \frac{\sin x}{2\cos x - x\sin x} = 0$.

    (b) Note that $\log(\lim\limits_{x\to\infty} (e^x + x)^{1/x}) = \lim\limits_{x\to\infty} \log(e^x + x)^{1/x} = \lim\limits_{x\to\infty} \frac{\log(e^x+x)}{x} = 1$. Thus $\lim\limits_{x\to\infty} (e^x + x)^{1/x} = e$.

    (c) Observe that $\log x - x = \log(xe^{-x})$ and $\lim\limits_{x\to\infty} \frac{x}{e^x} = 0$. Thus $\lim\limits_{x\to\infty} (\log x - x) = -\infty$.

    (d) Since $\log(\lim\limits_{x\to^+} (\cos x)^{\frac{1}{x}}) = \lim\limits_{x\to 0^+} \log((\cos x)^{\frac{1}{x}}) = \lim\limits_{x\to 0^+} \frac{\log(\cos x)}{x} = \lim\limits_{x\to 0^+} (-\tan x) = 0$, $\lim\limits_{x\to^+} (\cos x)^{\frac{1}{x}} = e^0 = 1$.

    (e) Since $\log(\lim\limits_{x\to 0^+} (\sin x)^{\sqrt{x}}) = \lim\limits_{x\to 0^+} \frac{\log(\sin x)}{1/\sqrt{x}} = -2 \lim\limits_{x\to 0^+} \frac{x^{\frac{3}{2}} \cos x}{\sin x}$ and $\lim\limits_{x\to 0^+} \frac{x^{\frac{3}{2}} \cos x}{\sin x} = (\lim\limits_{x\to 0^+} \sqrt{x}\cos x)(\lim\limits_{x\to 0^+} \frac{x}{\sin x}) = 0$, $\lim\limits_{x\to 0^+} (\sin x)^{\sqrt{x}} = e^0 = 1$.

13. Observe that $f(x) = \frac{e^x f(x)}{e^x}$. Apply L'Hospital's Rule.

14. Since $f''(c)$ exists, there exists a $\delta > 0$ such that $f'(x)$ exists on $(c-\delta, c+\delta)$. Therefore, by L'Hospital's Rule, the given limit is equal to $\lim_{h\to 0} \frac{f'(c+h)-f'(c-h)}{2h}$ if it exists. But $\lim_{h\to 0} \frac{f'(c+h)-f'(c-h)}{2h} = \frac{1}{2}\left[\lim_{h\to 0} \frac{f'(c+h)-f'(c)}{h} + \lim_{h\to 0} \frac{f'(c-h)-f'(c)}{-h}\right] = \frac{1}{2}\left[f''(c) + f''(c)\right]$.

    Let $f(x) = 1$ on $(0, \infty), f(0) = 0$ and $f(x) = -1$ on $(-\infty, 0)$. Then $f$ is not continuous at 0 hence $f''(0)$ does not exist. It can be easily verified that the limit given in the question exists.

15. Since $f$ is one-one (see Application 7.2), it is either strictly increasing or strictly decreasing (see Problem 20 of PP 6). Hence either $f'(x) \geq 0$ for all $x \in [a, b]$ or $f'(x) \leq 0$ for all $x \in [a, b]$. This problem can also be solved using Problem 18 in PP 7.

16. First note that $f$ is one-one as $f'(x) \neq 0$ for all $x \in [a, b]$ (See Application 7.2). Let $y_0 \in J$ and $y_0 = f(x_0)$ for some $x_0 \in [a, b]$. Let $(y_n)$ be any arbitrary sequence in $J$ such that $y_n \neq y_0$ for all $n$, $y_n \to y_0$ and $y_n = f(x_n)$ for some $x_n \in [a, b]$. Since $f^{-1}$ is continuous (see Problem 16 in PP5) and $f^{-1}$ is also one-one, we have $x_n \to x_0$ and $x_n \neq x_0$ for all $n$. Now

$$\lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} = \lim_{n\to\infty} \frac{x_n - x_0}{f(x_n) - f(x_0)} = \lim_{n\to\infty} \frac{1}{\frac{f(x_n)-f(x_0)}{x_n - x_0}} = \frac{1}{f'(x_0)}.$$