Lecture 4: Cauchy Criterion, Bolzano-Weierstrass Theorem

In this lecture, we state and proof the Bolzano-Weierstrass theorem which is an important result in calculus. We will see several applications of this result. Further, we will use this result to proof that if a sequence satisfies the Cauchy criterion then it converges.

The following remark is useful for understanding the proof of the Bolzano-Weierstrass Theorem.

Remark 4.1. If a sequence (x_n) is given, then the set $\{x_n : n \in \mathbb{N}\}$ may have finite or infinite number of elements. For instance, if $x_n = (-1)^n$, for all $n \in \mathbb{N}$, then the set $\{x_n : n \in \mathbb{N}\}$ has only two elements. However, the sequence (x_n) has infinite number of terms. That is, for each n, we have the nth term x_n.

Theorem 4.1 (***Bolzano-Weierstrass theorem***)**.** *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Proof (*). Let (x_n) be a bounded sequence. Then there exists an interval $[a_1, b_1]$ such that $a_1 \leq x_n \leq b_1$ for all $n \in \mathbb{N}$. Note that at least one of the intervals $[a_1, \frac{a_1+b_1}{2}]$ or $[\frac{a_1+b_1}{2}, b_1]$ contains infinite number of terms of (x_n). If $[a_1, \frac{a_1+b_1}{2}]$ has infinite number of terms of (x_n) then let $[a_2, b_2] = [a_1, \frac{a_1+b_1}{2}]$. Otherwise let $[a_2, b_2] = [\frac{a_1+b_1}{2}, b_1]$. Again, at least one of the intervals $[a_2, \frac{a_2+b_2}{2}]$ or $[\frac{a_2+b_2}{2}, b_2]$ contains infinite number of terms of (x_n). Let $[a_3, b_3]$ be one of the intervals $[a_2, \frac{a_2+b_2}{2}]$ or $[\frac{a_2+b_2}{2}, b_2]$ which contains infinite number of terms of (x_n). If we proceed, then for each $n \in \mathbb{N}$, we obtain $[a_n, b_n]$ such that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all $n \in \mathbb{N}$ and $b_n - a_n \to 0$. Hence, by the nested interval theorem, there exists $x_0 \in \mathbb{R}$ such that $\bigcap\limits_{n=1}^{\infty} [a_n, b_n] = \{x_0\}$, $a_n \to x_0$ and $b_n \to x_0$.

We now construct a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to x_0$ as $k \to \infty$. Note that as per our construction, for each $n \in \mathbb{N}$, the interval $[a_n, b_n]$ has infinite number of terms of (x_n). If we let $n_1 = 1$ then $x_{n_1} \in [a_1, b_1]$. Since $[a_2, b_2]$ has infinite number of terms, find $n_2 > n_1$ such that $x_{n_2} \in [a_2, b_2]$. Similarly, find $n_3 > n_2$ such that $x_{n_3} \in [a_3, b_3]$. Proceed to generate (x_{n_k}) such that $x_{n_k} \in [a_k, b_k]$ for all $k \in \mathbb{N}$. Since $a_k \leq x_{n_k} \leq b_k$ for all $k \in \mathbb{N}$, by the sandwich theorem, $x_{n_k} \to x_0$ as $k \to \infty$. □

Theorem 4.2. *If a sequence (x_n) satisfies the Cauchy criterion then (x_n) converges.*

Proof (*). Let (x_n) satisfy the Cauchy criterion. Since (x_n) is bounded, by Theorem 4.1, there exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to x_0$ for some x_0. We now show that $x_n \to x_0$. Let $\epsilon > 0$ be given. Since (x_n) satisfies the Cauchy criterion,

$$\text{there exists } N_1 \text{ such that } \mid x_n - x_m \mid < \epsilon/2 \quad \text{for all } n, m \geq N_1. \tag{1}$$

Since $x_{n_k} \to x_0$,

$$\text{there exists } N_2 \text{ such that } \mid x_{n_k} - x_0 \mid < \epsilon/2 \text{ for all } k \geq N_2 \tag{2}$$

(see Remark 3.2). Let $N = \max\{N_1, N_2\}$ and $n \geq N$. Choose some $k > n$. Then $n_k \geq k > n$ and hence $n_k \geq N$. Therefore $n, n_k \geq N_1$ and $k \geq N_2$. By (1) and (2) we have

$$\mid x_n - x_0 \mid \leq \mid x_n - x_{n_k} \mid + \mid x_{n_k} - x_0 \mid < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $|x_n - x_0| < \epsilon$ for all $n \geq N$. This proves that $x_n \to x_0$. □

Please write to psraj@iitk.ac.in if any typos/mistakes are found in these notes.

For a given sequence, if the limit theorem, sandwich theorem, ratio test and monotone criterion cannot be applied for determining its convergence, then an available option is to check whether the sequence satisfies the Cauchy criterion. Verifying the Cauchy criterion directly from the definition can be very difficult. The following result, in which we deal only with the consecutive terms of a given sequence, is useful to check whether the given sequence satisfies the Cauchy criterion.

Proposition 4.3. *Let $0 < \alpha < 1$. Suppose that (x_n) satisfis the* ***contractive condition****:*

$$|x_{n+2} - x_{n+1}| \;\leq\; \alpha|x_{n+1} - x_n| \;\; \textit{for all } n \in \mathbb{N}.$$

Then (x_n) satisfies the Cauchy criterion.

Proof (*). For $n \in \mathbb{N}$,

$$|x_{n+2} - x_{n+1}| \leq \alpha|x_{n+1} - x_n| \leq \alpha^2|x_n - x_{n-1}| \leq \cdots \leq \alpha^n|x_2 - x_1|.$$

Let $n, m \in \mathbb{N}$ be such that $n > m$. Since $|x_n - x_m| \leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m|$,

$$|x_n - x_m| \leq (\alpha^{n-2} + \alpha^{n-3} + \cdots + \alpha^{m-1})|x_2 - x_1| \leq \frac{\alpha^{m-1}}{1-\alpha}|x_2 - x_1|.$$

Since $\alpha^m \to 0$ as $m \to \infty$, (x_n) satisfies the Cauchy criterion. □

Example 3.5. 1. Let $x_1 = 1$ and $x_{n+1} = \frac{1}{2+x_n}$ for $n \in \mathbb{N}$. Then

$$|\, x_{n+2} - x_{n+1} \,| = \frac{1}{(2 + x_{n+1})(2 + x_n)} \,|\, x_n - x_{n+1} \,| < \frac{1}{4} \,|\, x_n - x_{n+1} \,| \,.$$

Therefore (x_n) satisfies the contractive condition with $\alpha = 1/4 < 1$ and hence it satisfies the Cauchy criterion. Therefore it converges by Theorem 4.2. Suppose $x_n \to l$. Then $l = \frac{1}{2+l}$ and hence $l = \sqrt{2} - 1$.

2. Consider the well known **Fibonacci sequence** (see Wikipedia) which is defined inductively by $x_1 = 1$, $x_2 = 1$ and $x_{n+2} = x_n + x_{n+1}$ for $n \in \mathbb{N}$. We now show a well known result which says that the sequence $(\frac{x_{n+1}}{x_n})$ converges to the **golden ratio** $\frac{1+\sqrt{5}}{2}$. Let $y_n = \frac{x_{n+1}}{x_n}$ for all $n \in \mathbb{N}$. Then, $y_1 = 1$ and for $n \in \mathbb{N}$,

$$y_{n+1} = \frac{x_{n+2}}{x_{n+1}} = \frac{x_n + x_{n+1}}{x_{n+1}} = 1 + \frac{1}{y_n}.$$

We now show that (y_n) satisfies the contractive condition. Note that for all $n \geq 2$,

$$|y_{n+1} - y_n| = |\frac{1}{y_n} - \frac{1}{y_{n-1}}| = |\frac{y_{n-1} - y_n}{y_n y_{n-1}}|$$

and $|y_n y_{n-1}| = |(1 + \frac{1}{y_{n-1}})y_{n-1}| = |y_{n-1} + 1| \geq 2$. This implies that $|y_{n+1} - y_n| \leq \frac{1}{2}|x_n - y_{n-1}|$ for all $n \geq 2$. Hence (y_n) satisfies the contractive condition and therefore it satisfies the Cauchy criterion. Hence (y_n) converges. Showing that $y_n \to \frac{1+\sqrt{5}}{2}$ is routine.

3. Recall the sequence (x_n) defined inductively by $x_1 = 1, x_2 = 2$ and for $n \in \mathbb{N}$, $x_{n+2} = \frac{x_{n+1}+x_n}{2}$. Now $|x_{n+2} - x_{n+1}| = \frac{1}{2}|x_{n+1} - x_n|$ for $n \in \mathbb{N}$. Since (x_n) satisfies the contractive condition, it converges. For finding the limit of (x_n), observe that $x_{n+1} + \frac{1}{2}x_n = x_n + \frac{1}{2}x_{n-1} = \cdots = x_2 + \frac{1}{2}x_1$ for all $n \in \mathbb{N}$. Hence if $x_n \to \ell$, then ℓ satisfies $\ell + \frac{1}{2}\ell = x_2 + \frac{1}{2}x_1 = \frac{5}{2}$. Therefore $\ell = \frac{5}{3}$. The convergence of (x_n) can also be directly determined from the nested interval theorem (see Problem 8 in PP4).

Remark 4.2: 1. Whenever we use Proposition 4.3, we should make sure that the number α that we get is a constant and satisfies $0 < \alpha < 1$. If (x_n) satisfies $|x_{n+2} - x_{n+1}| < |x_{n+1} - x_n|$ for all $n \in \mathbb{N}$, then the sequence (x_n) need not satisfy the Cauchy criterion (see Problem 3 in PP4).

2. Proposition 4.3 says that if (x_n) satisfies the contractive condition then it converges. The converse need not be true. That is, if a sequence converges, then it need not satisfy the contractive condition (see Problem 3 in PP4).

In the last four lectures, we discussed several results which basically deal with various properties of $\mathbb{R}$. These properties will be used in the subsequent lectures in which we deal with the real valued functions.