<u>On Action for a charged particle moving in a magnetic field</u>

We have learnt that for a charge q of mass m moving in a magnetic field B in a circle of radius R, the "momentum" is taken as

$$\vec{p} = m\vec{v} + \frac{q\Phi}{2\pi R}\hat{\phi} \ ,$$

where $\Phi = \pi R^2 B$ is the flux passing through the circle and the direction $\hat{\phi}$ is given by the right-hand rule. This translates into the action being

$$A = 2\pi Rmv - \pi|q|R^2 B \ .$$

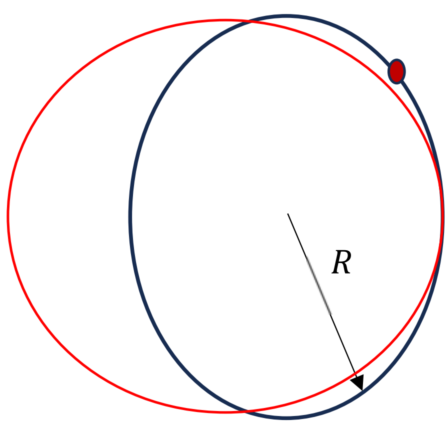
Question arises why the second term is there. We now show that if the principle of least action is to give the correct equation of motion for the charged particle, the second term must be a part of the action. For this let us first state the principle of least action once more.

If the action is calculated for any path close to the true path of a particle, the first order difference of action for the true path and for the nearby paths with the same end points vanishes if the energy of the particle is kept the same on the two paths. In other words, $\delta A = 0$.

OR EQUIVALENTLY

Satisfaction of the equation of motion implies that action is stationary on the true path (I have not stated this in my lecture, but this is another way of expressing the principle). Here stationarity means what has been stated in the statement above.

We now show this for a charge q of mass m moving in a magnetic field B in a circle of radius R. This is shown in the figure below.



The equation satisfied by the charged particle is

$$mv = |q|BR \ .$$

Now we take the nearby path to be an ellipse with any of its common point with the circle taken as the starting and the end point. Although shown to be quite large for clarity, the ellipse is very close to the circle with its major and minor axes being

$$a = R + \delta a \ (\delta a \to 0 \ )\text{and} \ \ b = R + \delta b \ (\delta b \to 0) \ .$$

The speed of the particle on the ellipse is the same as that on the circle since the energy of the particle is to be kept unchanged. Now we calculate the change in action. It will be

$$\delta A = mv\delta l - |q|B\delta S \ ,$$

where δl is the change in the path length and δS is the change in the area enclosed by the paths. For an ellipse that is nearly a circle, the perimeter is $\pi(a + b)$ (*note that this is an approximate formula true only for the kind of ellipse we are taking*). Its area is πab. Therefore, to the first order in changes of the parameters of the path,

$$\delta l = \pi(\delta a + \delta b) \ and \ \delta S = \pi R(\delta a + \delta b) \ .$$

This gives the change in action to be

$$\delta A = \pi(\delta a + \delta b)(mv - |q|BR) \ .$$

Now if we demand that $\delta A = 0$ for all arbitrary δa and δb, it follows that

$$mv - |q|BR = 0 \ ,$$

which is the true equation of motion. Alternatively, if we take the equation of motion to be correct, $\delta A = 0$.

Notice that if we had action only as $2\pi Rmv$, principle of least action will not be followed. So, to be consistent with the equation of motion, we have action

$$A = 2\pi Rmv - \pi|q|R^2B \ .$$

And it is this quantity that should be equal to nh in the quantum condition.