Lecture 11: Fixed Point Iteration Method, Newton's Method

In Lecture 7, we have seen some applications of the MVT. In this lecture, we will see that some important results which deal with some numerical methods are proved using the MVT.

In this lecture, we discuss the problem of finding approximate solutions to the equation

$$f(x) = 0 \tag{1}$$

for a given f. In many cases, it is not possible to find the exact solutions to equation (1). Even if $f(x)$ is a quadratic or cubic polynomial, the (real) solutions to the equation (1) could be irrationals. Therefore, it is natural to look for approximate solutions using some (numerical) methods. Here, we discuss a method called fixed point iteration method and a particular case of this method, called Newton's method.

Fixed Point Iteration Method (or Picard's Method)

In this method, we first change the equation (1) to a form called fixed point form

$$x = g(x) \tag{2}$$

in such a way that any solution to equation (2) is a solution to equation (1).

Let us see some examples of g for a given f. Consider the equation $f(x) = 0$ where $f(x) = x^3 + 7x - 2$. We can change the equation $f(x) = 0$ to a fixed point form $x = g(x)$ in many ways such as:

1. $x = g_1(x) = x - x^3 - 7x + 2 = 2 - 6x - x^3$
2. $x = g_2(x) = \frac{1}{7}(2 - x^3)$
3. $x = g_3(x) = x - e^x(x^3 + 7x - 2)$

For finding approximate solutions to equation (2), we consider the following method.

Picard's Method: Start from an initial point x_0 and consider the recursive process

$$x_{n+1} = g(x_n), \quad n = 0, 1, 2, ... \tag{3}$$

Such a sequence (x_n) is called a Picard sequence (for g and x_0). Observe that if a Picard sequence (x_n) converges to some ℓ_0 and g is continuous at ℓ_0 then $g(x_n) = x_{n+1} \to \ell_0$ and $g(x_n) \to g(\ell_0)$. Therefore $g(\ell_0) = \ell_0$, that is, ℓ_0 is a fixed point of g. Hence ℓ_0 is a solution to equation (1).

From the above discussion, we conclude that if a Picard sequence (x_n) for g mentioned in (2) converges and g is continuous, then x_n (for a large n) can be considered as an approximate solution to equation (1). But, why should (x_n) converge? In this regard, let us see some examples below.

Consider the equation $f(x) = x^3 + 7x - 2 = 0$ as mentioned above. Observing that f is strictly increasing and using the IVT, it is easy to verify that the equation $f(x) = 0$ has a unique (real) solution and the same lies in $[0, 1]$. If we consider $x = g_2(x) = \frac{1}{7}(2 - x^3)$, $x_0 \in [0, 1]$ and $x_{n+1} = g_2(x_n)$ for $n = 0, 1, 2, ...$, then (x_n) converges which can be verified as follows. Note that since $x_0 \in [0, 1]$,

$$|x_{n+2} - x_{n+1}| = \frac{1}{7}|x_{n+1}^3 - x_n^3| = \frac{1}{7}|x_{n+1}^2 + x_{n+1}x_n + x_n^2||x_{n+1} - x_n| \leq \frac{3}{7}|x_{n+1} - x_n|.$$

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This shows that (x_n) satisfies the Cauchy criterion and hence it converges. Whereas, if we start with, for example, $x_0 = 10$ then the sequence (x_n) does not converge. If we consider $x = g_1(x) = 2 - 6x - x^3$ and $x_{n+1} = g_1(x_n)$ for $n = 0, 1, 2, ...$, then (x_n) diverges for the initial value $x_0 = 0$ or $x_0 = 1$.

It is clear from the above examples that the convergence of a Picard sequence for g and x_0 depends on g and the starting point x_0. Moreover, in general, showing the convergence of a Picard sequence is not easy. The following result which is a consequence of the MVT gives sufficient conditions for the convergence of the Picard sequences.

Theorem 11.1 (Picard's Theorem). *Let $g : [a, b] \to [a, b]$ be a differentiable function such that*

$$|g'(x)| \ \leq \alpha < 1 \text{ for all } x \in [a, b]. \tag{4}$$

Then g has exactly one fixed point l_0 in $[a, b]$ and the Picard sequence for g with an initial point $x_0 \in [a, b]$ converges to l_0.

Proof. Let (x_n) be the Picard's sequence for g with the intial point x_0. Then, by the MVT, there exists c between x_n and x_{n+1} such that,

$$|x_{n+2} - x_{n+1}| = |g(x_{n+1}) - g(x_n)| = |g'(c)||x_{n+1} - x_n| < \alpha|x_{n+1} - x_n|.$$

Since (x_n) satisfies the Cauchy criterion, let $x_n \to \ell_0$ for some $\ell_0 \in [a, b]$. By the continuity of g, ℓ_0 is a fixed point of g.

If $g(\ell) = \ell$ for some $\ell \in [a, b]$ and $\ell \neq \ell_0$. Then, again by the MVT, there exists c_1 between ℓ and ℓ_0 such that

$$|\ell_0 - \ell| = |g(\ell) - g(\ell_0)| = |g'(c_1)||\ell_0 - \ell| < |\ell_0 - \ell|$$

which is a contradiction. Therefore, g has exactly one fixed point in $[a, b]$. $\square$

Remark 11.1. The assumption that $|g'(x)| \ \leq \alpha < 1$ for all $x \in [a, b]$ appearing in Picard's theorem can be replaced by the weaker assumption that $|g'(x)| \ < 1$ for all $x \in (a, b)$. A proof of Picard's theorem with the weaker assumption is outlined in Problem 8 in PP 11.

The convergence of a Picard sequence is illustrated in Figure.

Example 11.1. (i) For $g_2(x) = \frac{1}{7}(2 - x^3)$, it is clear that $g_2 : [0, 1] \to [0, 1]$ and $|g'(x)| \ < \frac{3}{7}$ for all $x \in [0, 1]$. Hence by Theorem 11.1, g_2 has a unique fixed point in $[0, 1]$. Further, the Picard sequence for g_2 and any $x_0 \in [0, 1]$ converges to the fixed point of g_2 in $[0, 1]$. The first five terms of the Picard sequence for g_2 and $x_0 = \frac{3}{4}$, are the following

$$x_1 = 0.2254, \quad x_2 = 0.2841, \quad x_3 = 0.2824, \quad x_4 = 0.2825, \quad x_5 = 0.2825.$$

Since the fixed point of g_2 in $[0, 1]$ is the solution to $x^3 - 7x + 2 = 0$, $x_5 = 0.2825$ can be considered as an approximate value of the solution to the equation $x^3 - 7x + 2 = 0$. Here it is natural to consider the interval $[0, 1]$, because, the solution to $x^3 - 7x + 2 = 0$ lies in $[0, 1]$.

(ii) For $g_1(x) = 2 - 6x - x^3$, we see that g_1 does not map $[0, 1]$ into itself and further $|g'(x)| > 1$ for all $x \in [0, 1]$. Hence Theorem 11.1 cannot be applied in this case. Although Theorem 11.1 does not say that a Picard sequence for g_1 does not converge, there is no guarantee that it can converge. In this sense, the choice of g_2 is preferable for finding approximate solutions to the equation $x^3 - 7x + 2 = 0$.

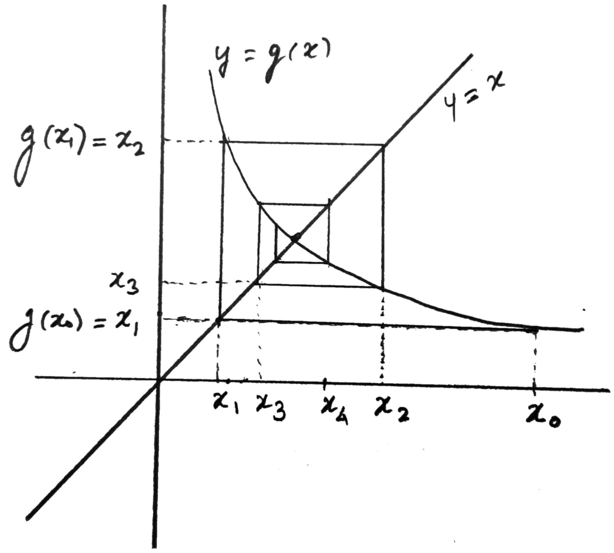

Figure 1: Picard sequence

(iii) Let $g : [0,1] \to [0,1]$ be defined by $g(x) = x^2$ for all $x \in [0,1]$. Note that 0 and 1 are fixed points of g. In this case $|g'(x) \nleq 1$ for all $x \in [0,1]$. However, every Picard sequence corresponding to g and any $x \in [0,1]$ converges to a fixed point of g.

(iv) Consider $g(x) = \frac{x^2+2}{4}$ for all $x \in [0,1]$. Since $g(0) = \frac{2}{4}$, $g(1) = \frac{3}{4}$ and g is increasing, $g : [0,1] \to [0,1]$. Observe that $|g'(x)| \leq \frac{1}{4}$ for all $x \in [0,1]$. By Theorem 11.1, g has a unique fixed point. Further, the Picard sequence for g and any $x \in [0,1]$ converges to the fixed point of g in $[0,1]$. For finding the fixed point of g in $[0,1]$, let $x_0 \in [0,1]$ be the fixed point of g. Then $x_0 = \frac{x_0^2+2}{4}$ which implies that $x_0 = 2 - \sqrt{2}$.

If a function g satisfies the conditions stated in Theorem 11.1, then the initial point x_0 can be chosen in $[a,b]$ so that the corresponding Picard sequence converges. In many cases, it is difficult to find a and b such that $\{g(x) : x \in [a,b]\} \subseteq [a,b]$ as stated in Theorem 11.1. In such cases, how do we pick an initial point? Can we at least expect that if the initial point x_0 is closer to a fixed point, then the corresponding Picard sequence converges? The following result which is also a consequence of the MVT addresses the proceeding question.

Theorem 11.2. *Let l_0 be a fixed point of g. Suppose that g' is continuous in some open interval containing l_0. If $|g'(l_0)| < 1$ then there exists $\epsilon > 0$ such that $g : [l_0 - \epsilon, l_0 + \epsilon] \to [l_0 - \epsilon, l_0 + \epsilon]$. Further, the Picard sequence for g with any initial point $x_0 \in [l_0 - \varepsilon, l_0 + \varepsilon]$ converges to l_0.*

Proof (*). Since $|g'(l_0)| < 1$, find some $\alpha > 0$ such that $|g'(l_0)| < \alpha < 1$. By the continuity of g' we find $\epsilon > 0$ (see Problem 10 in PP 5) such that $|g'(x)| < \alpha$ for all $x \in [l_0 - \epsilon, l_0 + \epsilon]$. Let $x \in [l_0 - \epsilon, l_0 + \epsilon]$ and $x \neq l_0$. Then by the MVT there exists c between l_0 and x such that

$$|g(x) - l_0| = |g(x) - g(l_0)| = |g'(c)||x - l_0| < \alpha\epsilon < \epsilon.$$

This shows that $\{g(x) : x \in [l_0 - \epsilon, l_0 + \epsilon]\} \subseteq [l_0 - \varepsilon, l_0 + \epsilon]$. The rest of the proof follows from Theorem 11.1. $\square$

Newton's Method (or Newton-Raphson Method)

Newton's method is used for finding approximate solutions to the equation $f(x) = 0$ for a given f. We have already seen that Picard's method can be used for such purpose. However, in order to use Picard's method, we need to convert the equation $f(x) = 0$ to a fixed point form $x = g(x)$. We have also seen that all fixed point forms may not be suitable for applying Picard's method. We will see that Newton's method uses a particular fixed point form $x = g(x)$ for a given f. Hence, Newton's method is a particular case of Picard's method.

Newton's iterative method: For a given f, start from an initial point x_0 such that $f'(x_0) \neq 0$ and consider the recursive process

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, ... \tag{5}$$

when $f'(x_n) \neq 0$. The sequence (x_n) generated by the process (5) is called a Newton sequence (for f and x_0).

Remark 11.2. Suppose that a Newton sequence (x_n) converges to some l_0 and $(f'(x_n))$ is bounded. Then $f(x_n) = f'(x_n)(x_n - x_{n+1}) \to 0$ and $f(x_n) \to f(l_0)$. Therefore $f(l_0) = 0$. Such x_n (for a large n) can be considered as an approximate solution to the equation $f(x) = 0$.

Observe that a sequence (x_n) is a Newton sequence for f if and only if (x_n) is a Picard sequence for g where $g(x) = x - \frac{f(x)}{f'(x)}$. Hence the sufficient conditions for the convergence of the Picard sequences given in Theorem 11.1 and Remark 11.1 can be used for the convergence of the Newton sequences (see Problem 7 in PP 11).

Example 11.2. Suppose that $f(x) = x^2 - 2$ and we look for the positive root of $f(x) = 0$. Since $f'(x) = 2x$, the iterative process of Newton's method is $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n}), n = 0, 1, 2, ...$ The convergence of (x_n) to $\sqrt{2}$ is discussed in Problem 5 in PP3 The convergence of (x_n) can also be derived from Theorem 11.1 by taking $g(x) = \frac{1}{2}(x + \frac{2}{x})$, $x \in [1, 2]$ (see Problem 7 in PP 11). The first four terms of the Newton sequence for f and $x_0 = 2$, are the following

$$x_1 = 1.5, \quad x_2 = 1.4166, \quad x_3 = 1.4142, \quad x_4 = 1.4142.$$

Hence 1.4142 is considered as an approximate value of $\sqrt{2}$.

Geometric interpretation of the iterative process of Newton's method: Suppose that we have found x_n and $f'(x_n) \neq 0$. To find x_{n+1}, we approximate the graph of $y = f(x)$ near the point $(x_n, f(x_n))$ by the tangent: $y - f(x_n) = f'(x_n)(x - x_n)$. Observe that x_{n+1} is the point of intersection of the x-axis and the tangent to the graph of $y = f(x)$ at $(x_n, f(x_n))$. See the following figure.