Projection Operator

$$\hat{P}^\dagger = \hat{P} \quad \& \quad \hat{P}^2 = \hat{I}$$

Claim:

$|\psi\rangle\langle\psi|$ is a projection operator. if $|\psi\rangle$ is normalized

$$\left(|\psi\rangle\langle\psi|\right)^\dagger = \langle\psi|\psi\rangle \quad \leftarrow \text{Hermitian}$$

$$\left(|\psi\rangle\langle\psi|\right)^2 = \left(|\psi\rangle\langle\psi|\right)\left(|\psi\rangle\langle\psi|\right)$$

$$= |\psi\rangle \underline{\langle\psi|\psi\rangle} \langle\psi| = |\psi\rangle\langle\psi|$$

Meaning of $\hat{P}$

$$|\psi\rangle \underbrace{\langle\psi|\phi\rangle}$$

What fraction of $|\phi\rangle$ is along $|\psi\rangle$

$$|\psi\rangle = \sum_{i=1}^{N} c_i |\phi_i\rangle$$

$$\langle\phi_i|\psi\rangle = \sum_{j=1}^{N} \langle\phi_i| c_j |\phi_j\rangle = \sum c_j \underbrace{\langle\phi_j|\phi_i\rangle}_{\delta_{ij}}$$

$$= c_i$$

$$|\psi\rangle = \sum_{i=1}^{N} |\phi_i\rangle\langle\phi_i|\psi\rangle$$

$\leftarrow$ Identity $\sum_i |\phi_i\rangle\langle\phi_i|$

Example 13.1. Amplitude of Reverse Events $\langle p|x\rangle$?

$$\langle p|x\rangle = \langle x|p\rangle^*$$

A key axiom of QM.
Interchanging bra & ket is equivalent to complex conjugate.

Prob. density 'to-find-x-given-p'
$\equiv$ 'to-find-p-given x'

For PoR problem: $\langle x|p\rangle = \frac{1}{\sqrt{L}} e^{i\frac{p}{\hbar}x}$

then $\langle p|x\rangle = \frac{1}{\sqrt{L}} e^{-ix\frac{p}{\hbar}}$

PoR Example: $p_n = \frac{nh}{L} = \frac{nh}{2\pi R}$, $n = 0, \pm 1, \pm 2, \ldots$

State you prepare can be a superposition of states.

$$|\psi\rangle = \sum_p |p\rangle\langle p|\psi\rangle$$

$$= \sum_x |x\rangle\langle x|\psi\rangle$$

$$\rightarrow \int dx\, |x\rangle\langle x|\psi\rangle$$

same state

- The KET-BRA SUM is always ONE

$$\sum_\alpha |\alpha\rangle\langle\alpha| = 1$$

Often called The Completeness Thm.

So, whenever required you can replace

$$1, \; \mathbb{I}, \; \mathbb{1} \quad \text{by} \quad \sum_\alpha |\alpha\rangle\langle\alpha|$$

- KET-BRA SUM $\equiv$ FOURIER EXPANSION

$$\langle x|\psi\rangle$$

Replace this by KET-BRA SUM.

$$\langle x|\psi\rangle = \sum_p \langle x|p\rangle\langle p|\psi\rangle$$

Initially I had $|\psi\rangle = \sum |p\rangle\langle p|\psi\rangle$ Expressed in $|p\rangle$ space.

To cast it in x-space apply $\langle x|$ to all from the left.

$$|\psi\rangle = \sum |p\rangle\langle p|\psi\rangle$$

$$\langle x|\psi\rangle = \sum_p \langle x|p\rangle \langle p|\psi\rangle$$

For PoR example, let us rewrite

$$\psi(x) = \sum_p \left(\frac{1}{\sqrt{L}}\right) e^{i\frac{p}{\hbar}x} \langle p|\psi\rangle \quad \text{— (1)} \qquad \Bigg| \quad p = \hbar k = \frac{nh}{L}$$

Compare :

$$f(x) = \sum_k c_k \, e^{ikx} \quad \text{— (2)} \qquad \Bigg| \quad kL = 2\pi n, \quad c_k = \frac{1}{L}\int_0^L dx \, e^{-ikx} f(x)$$

Comparing (1) and (2)

$$c_k \sqrt{L} = \langle p|\psi\rangle$$

call it c_p or c_n, n is the index.

Physical Meaning: These co-efficients are event amplitudes.

Given $|\psi\rangle$, an instaneous measurement of p, $\sqrt{L}\, c_k$ is the amplitude to find the particle in its n^{th} momentum state $\left(p = \hbar k = \frac{nh}{L}\right)$

i.e. Fourier Co-efficients are event amplitudes.

Look at inverse transformation : $c_k = \frac{1}{L}\int_0^L dx \, e^{-ikx} f(x)$

In Dirac notation it is $\boxed{\sum_x |x\rangle\langle x|}$ i.e. ket-bra position space sum.

$$\langle p|\psi\rangle = \sum_x \langle p|x\rangle \underbrace{\langle x|\psi\rangle}_{\psi(x) \text{ or } f(x)}$$

i.e. Fourier expansions are examples of 'ket-bra sum' Thm.

$$|\psi\rangle \nearrow \psi(x), \quad |\psi\rangle \searrow \{C_k\}$$

Both are representations of same thing.

$$\psi(x) \longleftrightarrow \phi(p)$$

Same $|\psi\rangle$, $\quad \langle x|\psi\rangle \longleftrightarrow \langle p|\psi\rangle$

Dirac notation makes this sameness evident.

- The basis space is the 'language' to express it.

$$\boxed{\langle q|r\rangle}$$

The **amplitude** of the event 'to find q when I know r'.

The **representation** of state $|r\rangle$ in the language of q, that describe $|r\rangle$.

The **transformation matrix elements** between the two bases r and q; dictionary entry to translate from r-language to q-lang.

Similarly, for operators:

$\hat{A}|\delta\rangle$: Instruction to act on $|\delta\rangle$

$\langle q|\hat{A}$: Instruction in the language of q.

- Amplitude of a state in its own space

$$\langle p|p_0\rangle = \delta_{pp_0} = \delta_{mn}$$

$$\langle x|x_0\rangle = \delta(x-x_0)$$

Example: Translation Operator

Given

$$\hat{T}(a)\,|x\rangle = |x+a\rangle \qquad \text{Find } \hat{T}_a\,|\psi\rangle .$$

$$\psi' = \hat{T}_a\,|\psi\rangle$$

$$\hat{T}_a\,|\psi\rangle = \hat{T}_a \int dx'\, |x'\rangle\langle x'|\psi\rangle$$

$$= \int dx'\, |x'+a\rangle\,\langle x'|\psi\rangle$$

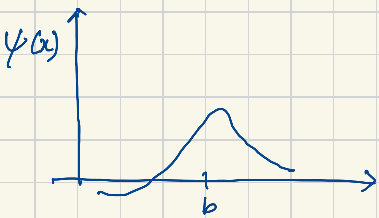
$$\psi'(x) = \langle x|\psi'\rangle = \langle x|\hat{T}_a|\psi\rangle$$

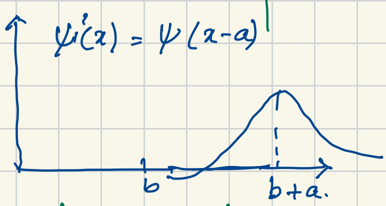
$$= \int dx'\, \langle x|x'+a\rangle\,\langle x'|\psi\rangle$$

$$= \int dx'\, \delta\left[x-(x'+a)\right]\,\langle x'|\psi\rangle$$

$$= \langle x-a|\psi\rangle = \psi(x-a)$$

Not $\psi' \neq \psi(x+a)$





The state has been translated in +ve x-direction.

Now, find: $\langle x|\hat{T}_a = ?$

$$\langle x|\hat{T}_a|\psi\rangle = \langle x-a|\psi = \psi(x-a)$$

Commutator

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

anti-comm.
$\hat{A}\hat{B} + \hat{B}\hat{A}$

Two operators $\hat{A}$ and $\hat{B}$ commute if

$$[\hat{A}, \hat{B}] = 0 \quad \text{i.e.} \quad \hat{A}\hat{B} = \hat{B}\hat{A}$$

<u>Claim:</u> If the product of two Hermitian operators is Hermitian, they commute.

<u>Example:</u>

$$[x, p]\, f(x) = \left[x \frac{\hbar}{i} \frac{d}{dx} f - \frac{\hbar}{i} \frac{d}{dx}(xf) \right]$$

$$= \frac{\hbar}{i} \left(x \frac{df}{dx} - x \frac{df}{dx} - f \right)$$

$$= i\hbar\, f(x)$$

$$\boxed{[\hat{x}, \hat{p}] = i\hbar}$$

Generalized Uncertainty Principle

$$\sigma_A^2 = \langle (\hat{A} - \langle A \rangle)\Psi \mid (\hat{A} - \langle A \rangle)\psi \rangle$$

$$= \langle f | f \rangle$$

$$f \equiv (\hat{A} - \langle \hat{A} \rangle)\Psi$$

$$\sigma_B^2 = \langle g | g \rangle$$

$$g \equiv (\hat{B} - \langle \hat{B} \rangle)\Psi$$

Schwarz Inequality

$$\sigma_A^2 \sigma_B^2 = \langle f | f \rangle \langle g | g \rangle \geqslant |\langle f | g \rangle|^2$$

Recall : $|\vec{A}|^2 |\vec{B}|^2 \geqslant |\vec{A} \cdot \vec{B}|^2$

For any complex number $z = \langle f | g \rangle$

$z = a + ib.$

$$|z|^2 = a^2 + b^2 \geqslant b^2 = \left[\frac{1}{2i}(z - z^*)\right]$$

$$\sigma_A^2 \sigma_B^2 \geqslant \left(\frac{1}{2i}\left[\langle f | g \rangle - \langle g | f \rangle\right]\right)^2$$

$$\langle f | g \rangle = \langle (\hat{A} - \langle A \rangle)\Psi \mid (\hat{B} - \langle B \rangle)\Psi \rangle$$

$$= \langle \psi | (\hat{A} - \langle A \rangle)(\hat{B} - \langle B \rangle) | \psi \rangle$$

$$= \langle \psi | \hat{A}\hat{B} | \psi \rangle - \langle B \rangle \langle \Psi | \hat{A}\psi \rangle - \langle A \rangle \langle \Psi | \hat{B}\Psi \rangle + \langle A \rangle \langle B \rangle \langle \psi | \psi \rangle$$

$$= \langle \hat{A}\hat{B} \rangle - \langle B \rangle \langle A \rangle - \langle A \rangle \langle B \rangle + \langle A \rangle \langle B \rangle$$

$$= \langle \hat{A}\hat{B} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle$$

i.e. $\langle f | g \rangle = \langle \Delta\hat{A} | \Delta\hat{B} \rangle = \langle \hat{A}\hat{B} \rangle - \langle A \rangle \langle B \rangle$

S'lly, $\langle g | f \rangle = \langle \Delta\hat{B} | \Delta\hat{A} \rangle = \langle \hat{B}\hat{A} \rangle - \langle A \rangle \langle B \rangle$

So, $\langle f | g \rangle - \langle g | f \rangle = \langle \hat{A}\hat{B} \rangle - \langle \hat{B}\hat{A} \rangle$

$$= \langle [\hat{A}, \hat{B}] \rangle$$

We have already shown above

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} [\langle f | g \rangle - \langle g | f \rangle \right)^2$$

i.e. $\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$ | $[\hat{A}, \hat{B}]$ carries i being Hermitian.

Example: $[\hat{x}, \hat{p}] = i\hbar$

$$\Rightarrow \quad \sigma_x^2 \sigma_p^2 \geq \left(\frac{1}{2i} i\hbar \right)^2 = \left(\frac{\hbar}{2} \right)^2$$

$$\Rightarrow \quad \sigma_x \sigma_p \geq \frac{\hbar}{2}$$

| σ_A, σ_B are positive by definition

- HUP for every pair of observables that do not commute.

  They are INCOMPATIBLE.

- Non-commuting observables cannot have a shared complete set of eigenfunctions.

i.e. If $\hat{A}$ and $\hat{B}$ have a complete set of common eigenfunctions, then $[\hat{A},\hat{B}]|\psi\rangle = 0$.

Let $|n\rangle$ be eigenfunction of both $\hat{A}$ and $\hat{B}$,

$$\hat{A}|n\rangle = a_n|n\rangle, \qquad \hat{B}|n\rangle = b_n|n\rangle$$

$$\begin{aligned}
[\hat{A},\hat{B}]|n\rangle &= (\hat{A}\hat{B}-\hat{B}\hat{A})\sum c_n|n\rangle \\
&= \hat{A}\left(\sum c_n b_n|n\rangle\right) - \hat{B}\left(\sum c_n a_n|n\rangle\right) \\
&= \sum a_n b_n c_n|n\rangle - \sum a_n b_n c_n|n\rangle = 0
\end{aligned}$$

Since this is true for any $|n\rangle$, it follows that

$$[\hat{A},\hat{B}] = 0$$

- Experimentally, observation of one loses information about the other.