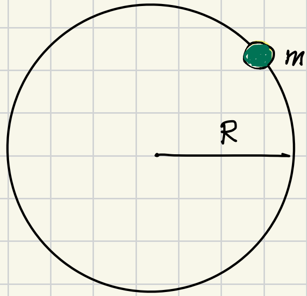
Lecture-7: Wavepackets-2

16 JAN 2024

- Toy model : Particle on a Ring (PoR)

$$\Psi(x,0) = \psi(x)$$



Freely moving on a ring :

What are the allowed ps?

Periodic Boundary condition ! $\psi(x+2\pi R) = \psi(x)$

$$\psi(x) = A\, e^{ik(x+2\pi R)} = e^{ikx}\, e^{ik(2\pi R)}$$

$$\Rightarrow k \cdot 2\pi R = 2\pi \cdot n \qquad \text{i.e.} \qquad \boxed{k_n = \frac{n}{R}} \quad n \text{ intg}$$

$$\psi(x) = \sum_n A_n\, e^{i\frac{n}{R}x}$$

$$A_n = \frac{1}{2\pi R}\int_{-\pi R}^{+\pi R} \psi(x)\, e^{-i\left(\frac{n}{R}\right)x}\, dx$$

Comments:

- Confinement leads to discrete p & E spectrum disc. and continu.
- Periodic B.C. is the source of quantizati..
- e^{ik_nx} forms base states.

How do I get A_n in the expansion?

$$\psi(x) = \sum_{n=-\infty}^{\infty} A_n \, e^{i\frac{n}{R}x}$$

Multiply both sides by $e^{-i\frac{m}{R}x}$ and Integrate (m all integers).

$$\int_{-\pi R}^{+\pi R} e^{-i\frac{m}{R}x}\psi(x)\,dx = \sum_{n=-\infty}^{\infty} \underbrace{\int_{-\pi R}^{+\pi R} A_n \, e^{i\left(\frac{n-m}{R}\right)x}\,dx}$$

If $n-m \neq 0$, integral is 0.

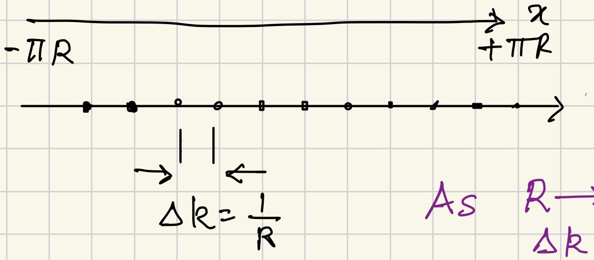
For $n=m$, integral is $2\pi R$

$$= A_n \, 2\pi R$$

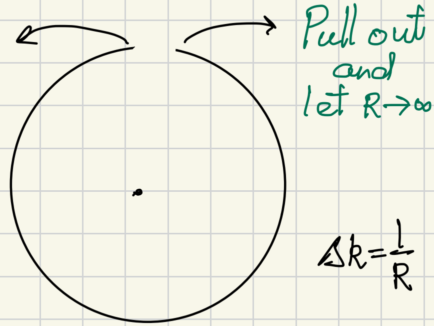
$$\Rightarrow \quad A_m = \frac{1}{2\pi R}\int_{-\pi R}^{+\pi R} \psi(x)\, e^{-i\frac{m}{R}x}\,dx$$

$$\boxed{A_n = \frac{1}{2\pi R}\int_{-\pi R}^{+\pi R} \psi(x)\, e^{-i\frac{n}{R}x}\,dx}$$

How do I convert it to continuous case?



As $R \to \infty$
$\Delta k \to dk$



$$\psi(x) = \sum \left(A_n \frac{\Delta k}{\sqrt{2\pi}} \right) e^{ikx}$$

$$\frac{1}{\sqrt{2\pi}\,\cancel{R}} A_n = \frac{1}{2\pi \cancel{R}} \int_{-\pi R}^{+\pi R} \psi(x)\, e^{-i\frac{n}{R}x}\, dx$$

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{ikx}\, dk$$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} \psi(x)\, e^{-ikx}\, dx.$$

Can I write $\phi(k)$ in terms of $\phi(p)$, p is the momentum.

$\phi(k) \to \phi(p)$ is only a change in variable $\quad p = \hbar k$, $dp = \hbar\, dk$

Note prob. of finding within dk or dp is same.

$|\phi(k)|^2 dk = |\phi(p)|^2 dp \Rightarrow |\phi(k)|^2 = |\phi(p)|^2 \frac{dp}{dk} = |\phi(p)|^2 \hbar$

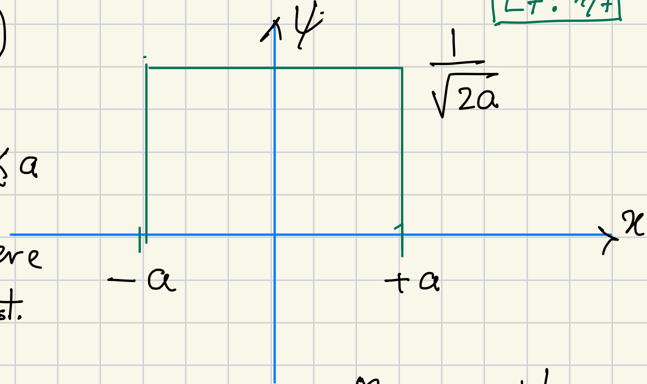
$\Rightarrow \phi(k)$ can be replaced by $\boxed{\frac{\phi(p)}{\sqrt{\hbar}}}$.

$$\boxed{\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int \psi(x)\, e^{-i\frac{p}{\hbar}x}\, dx}$$

Example: $\psi(x)$

$$\psi(x) = \begin{cases} A, & -a \leq x \leq a \\ 0, & \text{elsewhere} \end{cases}$$

A is a constant. Need to normalize first.



Obtain $\phi(k)$.

$\psi(x) = \frac{1}{\sqrt{2\pi}}$ within $[-a, a]$.

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x)\, e^{-ikx}\, dx$$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \frac{1}{\sqrt{2a}}\, e^{-ikx}\, dx$$

$$= \frac{1}{2} \frac{1}{\sqrt{\pi a}} \int_{-a}^{a} e^{-ikx}\, dx$$

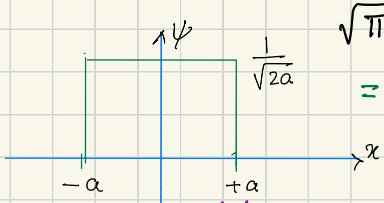
$$= \left. \frac{e^{-ikx}}{-ik} \right|_{-a}^{a}$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{\pi a}} \frac{\left[-e^{-ika} + e^{+ika} \right]}{ik}$$

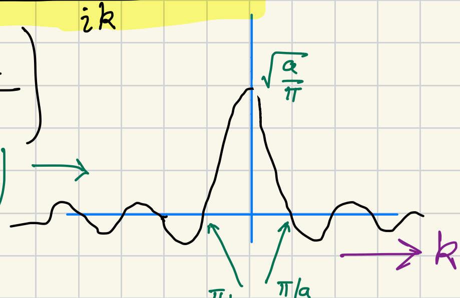
$2\,\frac{\sin ka}{k}$

$$= \frac{1}{\sqrt{\pi a}} \left(\frac{\sin ka}{k} \right)$$

$$= \sqrt{\frac{a}{\pi}} \cdot \left(\frac{\sin ka}{k} \right)$$



Real space representation



k-space (or, momentum space) representation.

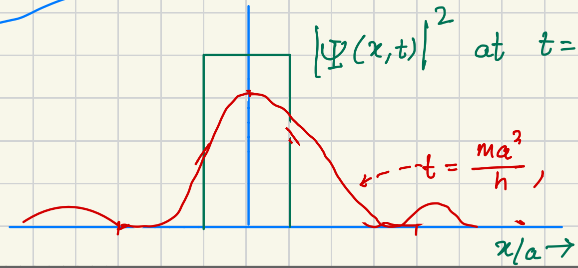
$\Delta x = 2a \qquad \Delta k = \frac{2\pi}{a}$

$\Delta p = \hbar \frac{2\pi}{a}$

$\Delta x \cdot \Delta p = 2h \leftarrow$ Heisenburg Uncertainity Product here.

$$\Psi(x,t) = \frac{1}{\pi\sqrt{2a}} \int_{-\infty}^{\infty} \frac{\sin ka}{k} \, e^{i\left(k - \frac{\hbar k^2}{2m} t\right)} \, dk$$

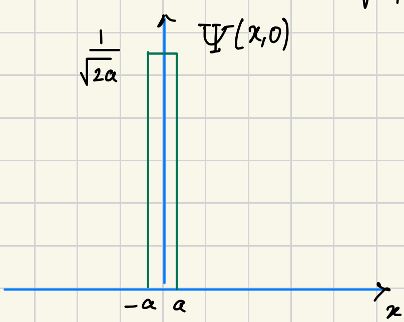
Needs numerical evaluation.

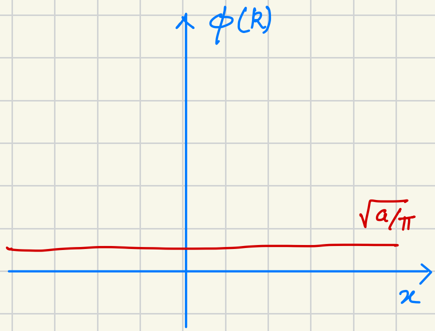


But, we can discuss limiting cases.

→ a very small i.e. localized spike

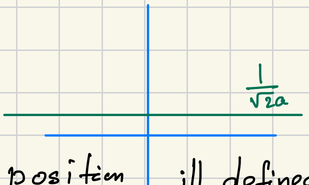
$$\phi(k) \approx \sqrt{\frac{a}{\pi}}$$

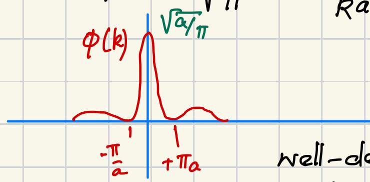




– a very large

$$\phi(k) = \sqrt{\frac{a}{\pi}} \, \frac{\sin(ka)}{ka}$$





well-defined momentum.

$$\psi_0 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{ikx}\, dk$$

identical in structure

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{ik(x - v_0 t)}\, dk$$

$$= \Psi_0(x - v_0 t)$$

If $\omega = v_0 k$, the wave packet travels to right with v_0, without distortion.

In general, ω need not be linear in k (eg. free particle).
We can Taylor expand around k_0.

$$\omega(k) = \omega(k_0) + \left.\frac{d\omega}{dk}\right|_{k=k_0} (k-k_0) + \frac{1}{2} \left.\frac{d^2\omega}{dk^2}\right|_{k=k_0} (k-k_0)^2 + \dots$$

For wavepacket $\Delta k = k - k_0$, $\phi(k)$ peaks at k_0

$$\omega(k) = \omega(k_0) + v_g (k-k_0) + \alpha (k-k_0)^2 + \dots$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\, e^{ik_0(x - v_{ph} t)} \int_{-\infty}^{\infty} g(k-k_0)\, e^{i(k-k_0)(x - v_g t)}\, e^{-i(k-k_0)^2 \alpha t + \dots}\, dk$$

$$v_g = \frac{d\omega}{dk} \qquad v_{ph} = \frac{\omega(k)}{k}$$

- Home Assignment: Show $v_g = v_{ph} + p \dfrac{dv_{ph}}{dp}$

How does the wavepacket evolve with time?

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} e^{i k_0 (x - v_{ph} t)} \int_{-\infty}^{\infty} g(k-k_0)\, e^{i(k-k_0)(x-v_g t)}\, e^{-i(k-k_0)^2 \alpha t + \cdots}\, dk$$

Linear approx. : neglect k^2 term i.e. $\boxed{(k-k_0)^2 \alpha \ll 1}$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} e^{i(k_0 - v_{ph} t)} \int_{-\infty}^{\infty} g(k-k_0)\, e^{i(k-k_0)(x-v_g t)}\, dk$$

$$\psi_0(x - v_g t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k')\, e^{i(x-v_g t)\Delta k + i k_0 (x - v_g t)}\, dk'$$

modulated amplitude

where $k' = k - k_0$

$$\Rightarrow \quad |\psi(x,t)|^2 = |\psi_0(x - v_g t)|^2$$

→ wavepacket v_g is particle v

→ Size of the wavepacket?

Travelling to right without distortion in the linear approximation.

→ No distortion as long as $\alpha(k-k_0)^2$ is negligible.

Question: In case quadratic term is there, what is time required for 'significant' distortion.

Let us think of <u>significant</u> distortion as spreading the width becomes equal to the width original, say $t=0$.