Lecture 5: Continuity, Existence of maximum and minimum points

We first define discontinuity of a function at a point. Let $f : \mathbb{R} \to \mathbb{R}$. Note that the graph of f is defined as the subset $\{(x, f(x)) : x \in \mathbb{R}\}$ of the plane $\mathbb{R}^2$. Let $x_0 \in \mathbb{R}$. Intuitively, we think that if f is discontinuous at x_0, then the graph of f is broken at the point $(x_0, f(x_0))$. For understanding, let us assume that the graph of f is broken as shown in the following figure:

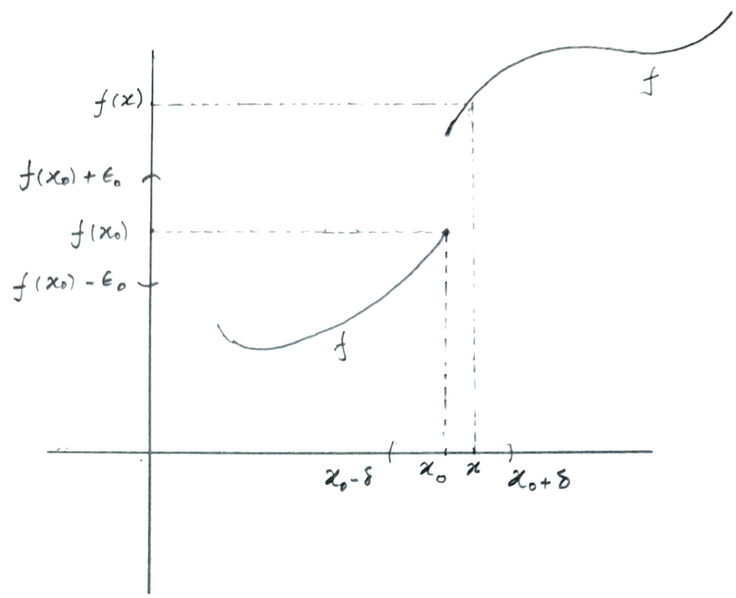

Figure 1: Discontinuous Graph

Observe in the figure that we can choose some ϵ_0-neighbourhood $(f(x_0) - \epsilon_0, f(x_0) + \epsilon_0)$ of $f(x_0)$, such that if we consider **any** $\delta-$neighbourhood $(x_0 - \delta, x_0 + \delta)$ of x_0, then we can find **at least one element** x in $(x_0 - \delta, x_0 + \delta)$ such that $f(x) \notin (f(x_0) - \epsilon_0, f(x_0) + \epsilon_0)$.

We use the above observation for defining the discontinuity of a function at a point.

Definition 5.1. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be discontinuous at a point $x_0 \in \mathbb{R}$ if there exists $\epsilon_0 > 0$ such that for every $\delta > 0$ there exists $x \in (x_0 - \delta, x_0 + \delta)$ but $f(x) \notin (f(x_0) - \epsilon_0, f(x_0) + \epsilon_0)$.

Let us see some examples to understand the "ϵ-δ language" used in Definition 5.1.

Example 5.1. 1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$ if $x \leq 2$ and $f(x) = \frac{9}{2}$ if $x > 2$. Let $x_0 = 2$. We can guess that f is discontinuous at x_0. We show it using Definition 5.1. By observing the gap in the range of f, choose, for instance, $\epsilon_0 = \frac{1}{4}$. Let $\delta > 0$ be given. Then define $x = x_0 + \frac{\delta}{2} = 2 + \frac{\delta}{2}$. Note that $x \in (x_0 - \delta, x_0 + \delta)$ but $f(x) = \frac{9}{2} \notin (f(x_0) - \epsilon_0, f(x_0) + \epsilon_0) = (4 - \frac{1}{4}, 4 + \frac{1}{4})$. Hence f is discontinuous at x_0.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 0$ if x is rational and $f(x) = 1$ if x is irrational. Here we guess that f is discontinuous at every element of $\mathbb{R}$. We prove it using Definition 5.1. Let $x_0 \in \mathbb{R}$ and assume that x_0 is rational. Then choose $\epsilon_0 = \frac{1}{2}$. Let $\delta > 0$ be given. Then using Corollary 1.1, find some irrational $x \in (x_0 - \delta, x_0 + \delta)$. Then $f(x) = 1 \notin (f(x_0) - \epsilon_0, f(x_0 + \epsilon_0) = (-\frac{1}{2}, \frac{1}{2})$. If x_0 is irrational then we proceed with the same argument and in this case we choose x to be rational.

Please write to psraj@iitk.ac.in if any typos/mistakes are found in these notes.

We now use the contrapositive of the statement of Definition 5.1, for giving a formal definition of continuity of a function at a point.

Definition 5.2. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be continuous at a point $x_0 \in \mathbb{R}$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $f(x) \in (f(x_0 - \epsilon, f(x_0 + \epsilon)$ whenever $x \in (x_0 - \delta, x_0 + \delta)$.

Note that in Definition 1.2, the condition "$f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$ whenever $x \in (x_0 - \delta, x_0 + \delta)$" can be replaced by "$|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$". Observe that the value of δ depends on the choice of ϵ. In general, the smaller the value of ϵ, the smaller δ must be. This fact is illustrated in the following example.

Example 5.2. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2x \sin(\frac{1}{x})$ when $x \neq 0$ and $f(0) = 0$. We show that f is continuous at 0 using Definition 5.2. Remember that for given $\epsilon > 0$, we have to find $\delta > 0$ (not the other way!). Note that here $x_0 = 0$ and

$$\mid f(x) - f(x_0) \mid = \mid 2x \sin(\frac{1}{x}) - 0 \mid \leq \mid 2x \mid = 2 \mid x - x_0 \mid .$$

Let ϵ be given. Choose any $\delta > 0$ such that $\delta \leq \frac{\epsilon}{2}$. Then we have

$$\mid f(x) - f(x_0)| < \epsilon \text{ whenever } \quad \mid x - x_0 \mid < \delta.$$

This shows that f is continuous at $x_0 = 0$.

We now characterize the continuity of a function at a point in terms of convergent sequences. This will help in two ways. The characterization can be used to verify the continuity instead of using the definition. Next, since the sequences are involved in the characterization, we can make use of the results proved in the previous lectures to derive certain properties of continuity.

Theorem 5.1. *Let $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$. Then f is continuous at $x_0 \in \mathbb{R}$ if and only if $(f(x_n))$ converges to $f(x_0)$ whenever (x_n) converges to x_0.*

Proof. Suppose f is continuous at x_0 and $x_n \to x_0$. Let us show that $f(x_n) \to f(x_0)$. Let $\epsilon > 0$ be given. We must find N such that $|f(x_n) - f(x_0)| < \epsilon$ for all $n \geq N$. Since f is continuous at x_0, there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. Since $x_n \to x_0$, there exists $N \in \mathbb{N}$ such that $|x_n - x_0| < \delta$ for all $n \geq N$. This N serves our purpose.

To prove the converse, let us assume that $f(x_n) \to f(x_0)$ whenever $x_n \to x_0$. Suppose on the contrary, f is not continuous at x_0. Then by Definition 5.1, there exists $\epsilon_0 > 0$ such that for each n, there is an element $x_n \in (x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$ but $|f(x_n) - f(x_0)| \geq \epsilon_0$. This implies that $x_n \to x_0$ by the sandwich theorem but $(f(x_n))$ does not converge to $f(x_0)$ which contradicts our assumption. $\square$

We now see some examples in which we use Theorem 5.1 and verify the continuity.

Example 5.3. 1. Consider the function f which is defined in Example 5.2 and let $x_0 = 0$. We have already shown using Definition 5.2, that f is continuous at x_0. Let us use Theorem 5.1 to verify the same. It is already noted in Example 5.2 that $0 \leq |f(x) - f(x_0)| \leq 2|x - x_0|$. If $x_n \to x_0$, then by sandwich theorem $f(x_n) \to f(x_0)$. Hence f is continuous at x_0.

2. Let $f(x) = \sin(1/x)$ for all $x \neq 0$ and $f(0) = 0$. We show that f is not continuous at 0. To show the discontinuity at 0, we produce **one sequence** (x_n) such that $x_n \to 0$ but $f(x_n) \nrightarrow f(0)$, i.e, $(f(x_n))$ does not converge to $f(0)$. Let $x_n = 2/\{\pi(2n+1)\}$ for $n = 1, 2, \ldots$. Then $x_n \to 0$ and $f(x_n) = (-1)^n$ for every $n \in \mathbb{N}$. Note that $f(x_n) \nrightarrow f(0)$. Hence f is not continuous at 0.

3. Let $f(x) = 0$ when x is rational and $f(x) = x$ when x is irrational. We will see that this function is continuous only at 0. Let (x_n) be any sequence such that $x_n \to 0$. Since $\mid f(x_n) \mid \leq \mid x_n \mid$ for all $n \in \mathbb{N}$, $f(x_n) \to 0 = f(0)$. Therefore by Theorem 5.1, f is continuous at 0. Suppose $x_0 \neq 0$ and it is rational. We show that f is not continuous at x_0. Choose **one sequence** (x_n) such that x_n is an irrational number for every n and $x_n \to x_0$ (see Problem 8 in PP2). Observe that $f(x_n) = x_n \to x_0 \neq f(x_0)$. This shows that f is not continuous at x_0. When x_0 is irrational, the proof is similar.

Remark 5.1. In order to show that a function f is not continuous at a point x_0 it is sufficient to produce one sequence (x_n) such that $x_n \to x_0$ but $f(x_n) \nrightarrow f(x_0)$. However, to show a function is continuous at x_0, we have to show that $f(x_n) \to f(x_0)$ whenever $x_n \to x_0$ i.e, for every (x_n) such that $x_n \to x_0$.

It follows from Theorem 5.1 and Theorem 2.1 that if $f$and g are continuous at some $x_0 \in \mathbb{R}$ then $(f + g)$ and (fg) are continuous at x_0.

Continuous function on a subset of $\mathbb{R}$

Let S be a subset of $\mathbb{R}$ and $x_0 \in S$. Suppose $f : S \to \mathbb{R}$. We say that f is continuous at x_0, if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $x \in S$ and $|x - x_0| < \delta$. By repeating the proof of Theorem 5.1, we see that f is continuous at x_0 if and only if $f(x_n) \to f(x_0)$ whenever $x_n \to x_0$ and $x_n \in S$ for all $n \in \mathbb{N}$. If f is continuous at every $x \in S$, then we say that f is continuous on S.

Example 5.4. Let $f : (0, \infty) \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$ for all $x \in (0, \infty)$. Then f is continuous on $(0, \infty)$. To verify this, take some $x_0 \in (0, \infty)$ and a sequence (x_n) in $(0, \infty)$ such that $x_n \to x_0$. We know that $\frac{1}{x_n} \to \frac{1}{x_0}$. Hence f is continuous at x_0. Since x_0 is an arbitrary element in $(0, \infty)$, f is continuous at every element of $(0, \infty)$ and therefore f is continuous on $(0, \infty)$.

Continuity properties $(f \circ g)$ and $(\frac{f}{g})$ are discussed in Problems 3 and 10 in PP5.

Continuous functions on closed bounded intervals

We will see that if a continuous function is defined on a closed bounded interval then it has some interesting and important properties. Such properties will be applied later.

Definition 5.4. Let $S \subseteq \mathbb{R}$ and $f : S \to \mathbb{R}$. We say that f is bounded on S if the set $\{f(x) : x \in S\}$ is a bounded subset of $\mathbb{R}$.

Theorem 5.2. *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is bounded on $[a, b]$.*

Proof. Suppose that f is not bounded on $[a, b]$. Then for each $n \in \mathbb{N}$ there is a point $x_n \in [a, b]$ such that $|f(x_n)| > n$. Since (x_n) is a bounded sequence, by the Bolzano-Weierstrass theorem, it has a convergent subsequence (x_{n_k}). Suppose $x_{n_k} \to x_0$ for some x_0. Since $x_{n_k} \in [a, b]$ for every $k \in \mathbb{N}$, $x_0 \in [a, b]$. By the continuity of f at x_0, we have $f(x_{n_k}) \to f(x_0)$. Since $|f(x_n)| > n$ for all n, $f(x_{n_k}) \to \infty$ as $k \to \infty$. Hence there is a contradiction. Therefore f is bounded on $[a, b]$. $\square$

For $a, b \in \mathbb{R}$, we let $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ and $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$. We remark that if a function is continuous on an open interval (a, b) or on a semi-open interval of the type $(a, b]$ or $[a, b)$, then it is not necessary that the function has to be bounded. For example, consider the continuous function $f : (0, 1] \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ for every $x \in (0, 1]$.

Definition 5.5. Let $S \subseteq \mathbb{R}$ and $f : S \to \mathbb{R}$. An element $x_0 \in S$ is called a *point of maximum* for f on S if $f(x_0) \geq f(x)$ for all $x \in S$. *Point of minimum* for f on S is defined similarly.

The following theorem which will be used later is a consequence of the Bolzano-Weierstrass theorem and is also an important result in calculus.

Theorem 5.3. *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there exist $x_0, y_0 \in [a, b]$ such that x_0 is a point of maximum for f on $[a, b]$ and y_0 is a point of minimum for f on $[a, b]$.*

Proof (*). By Theorem 5.2, f is bounded on $[a, b]$. Let $M = \sup\{f(x) : x \in [a, b]\}$. Then there exists a sequence $(f(x_n))$ in $\{f(x) : x \in [a, b]\}$ such that $f(x_n) \to M$ (see Problem 9 of PP2). Since (x_n) is a sequence in $[a, b]$, by the Bolzano Weierstrass theorem, it has a convergent subsequence (x_{n_k}). Suppose $x_{n_k} \to x_0$ for some x_0. Then $x_0 \in [a, b]$. By the continuity of f at x_0, $f(x_{n_k}) \to f(x_0)$. Since, we also have $f(x_{n_k}) \to M$, we get $f(x_0) = M$. Hence x_0 is a point of maximum for f on $[a, b]$. The proof for the existence of a point of minimum is similar. $\square$