Lecture 3 : Monotone and Cauchy criteria, subsequences

In Lecture 2, we used the limit theorem, sandwich theorem and ratio test for determining the convergence of certain sequences as well as their limits. These results are not applicable in many instances. Hence we look for some sufficient conditions (also called criteria) which ensure at least the convergence of a sequence (without having any knowledge of its limit). Once the convergence is established, finding the limit can be considered as a different task. We will discuss two criteria. One is called monotone criterion and the other Cauchy criterion.

Before presenting a criterion (a sufficient condition), let us see a necessary condition for the convergence of a sequence. For stating a necessary condition, we need the following definition.

Definition 3.1. Let A be a subset of $\mathbb{R}$. We say that A is bounded if it is both bounded above and bounded below. A sequence (x_n) is said to be bounded if the set $\{x_n : n \in \mathbb{N}\}$ is bounded.

Let A be a subset of $\mathbb{R}$. It is easy to verify that A is a bounded set if and only if there exits $M > 0$ such that $|x| \leq M$ for all $x \in A$.

Theorem 3.1. *Every convergent sequence is a bounded sequence.*

Proof. Let $x_n \to x$. Then, for $\epsilon = 1$, there exist $N \in N$ such that

$$|x_n - x| \leq 1 \quad \text{for all} \;\; n \geq N.$$

This implies that $|x_n| \leq |x| + 1$ for all $n \geq N$. If we let $L = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|\}$, then $|x_n| \leq L + |x| + 1$ for all $n \in \mathbb{N}$. Therefore (x_n) is a bounded sequence. □

Remark 3.1. Theorem 3.1 says that if a sequence is convergent then it is necessary that the sequence has to be bounded. Whereas bounded sequence need not converge. For example, the sequence $((-1)^n)$ is a bounded sequence but it does not converge. The necessary condition stated in Theorem 3.1 can be used to establish the divergence of certain sequences. For example, the sequences (n), $(\sqrt{n})$ and $((-1)^n n)$ diverge, because, they are not bounded.

One naturally asks the following question: *Can an additional condition on a bounded sequence ensure the convergence?* As an answer to this question, we show that if the terms of a bounded sequence either increases or decreases then the sequence converges.

Monotone Sequences

Definition 3.2. We say that a sequence (x_n) is increasing if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. Similarly we define decreasing sequence. Sequences which are either increasing or decreasing are called monotone.

Theorem 3.2. *If a sequence is bounded and monotone then it converges.*

Proof. Suppose that (x_n) is a bounded and increasing sequence. Observe that an obvious candidate for the limit of (x_n) is $\beta = \sup\,\{x_n : n \in \mathbb{N}\}$. Hence we claim that $x_n \to \beta$. Let $\epsilon > 0$ be given. Since $\beta - \epsilon$ is not an upper bound of $\{x_n : n \in \mathbb{N}\}$, there exists $N \in \mathbb{N}$ such that $\beta - \epsilon < x_N$. Since (x_n) is increasing, we have $x_N \leq x_n$ for all $n \geq N$. This implies that

$$\beta - \epsilon < x_n \leq \beta \leq \beta + \epsilon \;\; \text{for all } n \geq N.$$

This shows that $x_n \to \beta$.

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Similarly, we can show that if (x_n) is bounded and decreasing, then $x_n \to \inf\{x_n : n \in \mathbb{N}\}$. $\square$

In light of Theorem 3.2, we say that a sequence satisfies monotone criterion if it is both monotone and bounded.

Examples 3.1. 1. Let $x_1 = 4$, $x_n = \frac{n+1}{n-1}$ for $n \geq 2$ and $y_n = \frac{n+2}{n+1}$ for $n \in \mathbb{N}$. Then $x_{n+1} - x_n < 0$ and $y_{n+1} - y_n > 0$ for $n \in \mathbb{N}$. Hence (x_n) is decreasing and (y_n) is increasing.

2. Let (x_n) be defined inductively by $x_1 = \sqrt{2}$ and $x_n = \sqrt{2 + x_{n-1}}$ for $n > 1$. We use Theorem 3.2 and show that (x_n) converges. Note that $0 < x_1 < x_2 < 2$. Now we use induction and show that $x_n \leq 2$ and $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$. These facts are already verified for $n = 1$. Suppose $x_k \leq 2$ for some $k > 1$. Then $x_{k+1} = \sqrt{2 + x_k} \leq 2$. Hence by induction $x_n \leq 2$ for all $n \geq 1$. Next we show that (x_n) is increasing using induction. Suppose $x_{k-1} \leq x_k$ for some $k > 2$. Then $x_k = \sqrt{2 + x_{k-1}} \leq \sqrt{2 + x_k} = x_{k+1}$. Hence by induction $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. Therefore, by Theorem 3.2, (x_n) converges. Let us find the limit of (x_n). Suppose $x_n \to \lambda$. Then using the fact that $x_n \to \lambda$ on the left and right hand sides of the expression $x_n = \sqrt{2 + x_{n-1}}$, we obtain that $\lambda = \sqrt{2 + \lambda}$. This implies that λ must satisfy the equation $\lambda^2 - \lambda - 2 = 0$. Hence either $\lambda = 2$ or $\lambda = -1$. Since $x_n > 0$ for all $n \in \mathbb{N}$, we must have $\lambda \geq 0$. Therefore we conclude that $x_n \to 2$.

3. Let $x_1 = 8$ and $x_{n+1} = \frac{1}{2}x_n + 2$ for $n \geq 1$. We show that (x_n) converges as follows. Note that $x_2 = 6$ and $x_3 = 5$. If $x_n \to \lambda$, then λ must satisfy $\lambda = \frac{\lambda}{2} + 2$ and therefore, $\lambda = 4$. Further, if (x_n) is deceasing and bounded, then $\lambda = \inf\{x_n : n \in \mathbb{N}\}$. From this information we guess that (x_n) is decreasing and $x_n \geq 4$ for all $n \in \mathbb{N}$. First show that $x_n \geq 4$ by induction. Then it is easy to verify that $\frac{x_{n+1}}{x_n} \leq 1$ for all $n \in \mathbb{N}$. Hence the sequence is decreasing and bounded. Therefore (x_n) converges and its limit is 4.

Cauchy Criterion

Theorem 3.2 is applicable only to sequences that are monotone. Consider the sequence (x_n) which is defined inductively by $x_1 = 1, x_2 = 2$ and $x_{n+2} = \frac{x_n + x_{n+1}}{2}$ for $n \geq 1$. The sequence (x_n) is bounded but not monotone. However, there is an impression that this sequence might converge and, in fact, it does which will be seen. At first look, guessing a candidate for the limit of this sequence does not seem to be obvious. Hence it appears that there is no possibility of applying the sandwich theorem or ratio test for (x_n). Therefore we look for a sufficient condition, which is different from the monotone criterion, on a bounded sequence that can ensure the convergence.

Let us start with a convergent sequence and find out a necessary condition which does not involve the limit of the sequence. Suppose that a sequence (x_n) converges to x. Let $\epsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon/2$ for all $n \geq N$. Hence for $n, m \geq N$ we have

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x - x_m| < \epsilon.$$

Thus we arrive at the following conclusion:

*"If a sequence (x_n) converges then it is necessary that it satisfies the **Cauchy criterion** which is stated in the following definition".*

Definition 3.3. We say that a sequence (x_n) satisfies the Cauchy criterion if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon$ whenever $n, m \geq N$.

In case $x_n \to x_0$ we can, roughly, say that $|x_n - x_0|$ moves "closer" to 0 when n becomes larger. Whereas, if (x_n) satisfies the Cauchy criterion, then $|x_n - x_m|$ gets closer to 0 when both n and m become larger. To understand Definition 3.3, let us see some examples.

Example 3.2. 1. Let $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ for all $n \in \mathbb{N}$. Let us verify that (x_n) does not satisfy the Cauchy criterion. Let $n \in \mathbb{N}$. Then $x_{2n} - x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geq n\frac{1}{2n} = \frac{1}{2}$. This shows that (x_n) does not satisfy the Cauchy criterion.

2. Using Definition 3.3, we show that the sequence $(\frac{n^2-1}{n^2})$ satisfies the Cauchy criterion. Let $\epsilon > 0$ be given. We have to find $N \in \mathbb{N}$ such that $|x_n - x_m| = |\frac{1}{m^2} - \frac{1}{n^2}| < \epsilon$ whenever $m, n \geq N$. Observe that if $m, n \geq N$ for some $N \in \mathbb{N}$, then $|\frac{1}{n^2} - \frac{1}{m^2}| < \frac{1}{n^2} + \frac{1}{m^2} \leq \frac{2}{N^2}$. Therefore choose N such that $N > \sqrt{\frac{2}{\epsilon}}$ so that we get $|x_n - x_m| < \epsilon$ for all $n, m \geq N$.

Note that a sequence satisfying the Cauchy criterion is a bounded sequence (see Problem 2 in PP4) with some additional property. Therefore it is natural to ask whether a sequence satisfying the Cauchy criterion will converge. *We will show that a sequence satisfying Cauchy criterion does converge.* We need some results to prove this.

If $a, b \in \mathbb{R}$ and $a < b$ then the closed bounded interval from a to b is the set $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$.

Theorem 3.3 ***(Nested interval Theorem).*** *Let $I_n = [a_n, b_n]$ be such that $I_n \supseteq I_{n+1}$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty}(b_n - a_n) = 0$. Then there exists $x_0 \in \mathbb{R}$ such that $\bigcap\limits_{n=1}^{\infty} I_n = \{x_0\}, a_n \to x_0$ and $b_n \to x_0$.*

Proof (*). Note that the sequences (a_n) and (b_n) are respectively increasing and decreasing. Moreover both are bounded. Hence, by Theorem 3.2 both converge, say $a_n \to a$ and $b_n \to b$. It follows from the proof of Theorem 3.2 that $a = \sup\ \{a_n : n \in \mathbb{N}\}$ and $b = \inf\ \{b_n : n \in \mathbb{N}\}$. Therefore $a_n \leq a$ and $b \leq b_n$ for all $n \in N$. Since $b - a = \lim\limits_{n\to\infty}(b_n - a_n) = 0$, $a = b$. Thus we have $a_n \leq a = b \leq b_n$ for all n. Therefore $a \in \bigcap\limits_{n=1}^{\infty} I_n$. Suppose $c \in \bigcap\limits_{n=1}^{\infty} I_n$ for some $c \in \mathbb{R}$. Then $|c - a| \leq b_n - a_n$ for all $n \in \mathbb{N}$. Since $b_n - a_n \to 0, a = c$. This proves the result. $\square$

We need a notion called subsequence which is defined below.

Subsequences

Definition 3.4. Let (x_n) be a sequence and let (n_k) be any sequence of positive integers such that $n_1 < n_2 < n_3 < \ldots$. The sequence (x_{n_k}) is called a subsequence of (x_n).

We can, roughly, say that a subsequence is formed by deleting some of the terms of the sequence and retaining the remaining in the same order. Note that in Definition 3.4, k varies from 1 to ∞ and $n_k \geq k$ for all $k \in \mathbb{N}$.

Example 3.3. 1. Sequences $(1, 1, 1, \ldots)$ and $(0, 0, 0, \ldots)$ are both subsequences of $(1, 0, 1, 0, \ldots)$.

2. The sequence $(\frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{6} \ldots)$ is not a subsequence of $(\frac{1}{n})$, because, here $n_1 = 4, n_2 = 2, n_3 = 8, \ldots$.

3. The sequences $(\frac{1}{k^2})$ and $(\frac{1}{2^k})$ are subsequences of $(\frac{1}{n})$.

Theorem 3.4. *Let (x_{n_k}) be a subsequence of (x_n). If $x_n \to x_0$ then $x_{n_k} \to x_0$ as $k \to \infty$.*

Proof. For every $k \in \mathbb{N}$, let $y_k = x_{n_k}$. It is enough to show that $y_k \to x_0$ as $k \to \infty$. Let $\epsilon > 0$ be given. Since $x_n \to x_0$, there exists $N \in \mathbb{N}$, such that $|x_k - x_0| < \epsilon$ for all $k \geq N$. Observe that if $k \geq N$ then $n_k \geq k \geq N$. Therefore, for all $k \geq N$, $|x_{n_k} - x_0| < \epsilon$ and hence $|y_k - x_0| < \epsilon$. This shows that $y_k \to x_0$ as $k \to \infty$. $\square$

Remark 3.2. From the proof of Theorem 3.4, we conclude that a subsequence (x_{n_k}) converges to some x_0 if and only if for every given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_{n_k} - x_0| < \epsilon$ whenever $k \geq N$.

We see from Example 3.3 that a given sequence may have convergent subsequences but the sequence itself may not converge. It may happen that a sequence may not have a convergent subsequence at all, for instance, take (n). In the next lecture we will address the question: *under what condition on a given sequence, we can extract a convergent subsequence from the sequence?*