Assignment 3 : Functions of several variables (Continuity and Differentiability)

1. Identify the points, if any, where the following functions fail to be continuous:

   (a) $f(x,y) = \begin{cases} xy & \text{if } xy \geq 0 \\ -xy & \text{if } xy < 0 \end{cases}$

   (b) $f(x,y) = \begin{cases} xy & \text{if } xy \text{ is rationnal} \\ -xy & \text{if } xy \text{ is irrational.} \end{cases}$

2. Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2y^2+(x-y)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

   Show that the function satisfy the following:

   (a) The iterated limits $\lim\limits_{x \longrightarrow 0} \left[\lim\limits_{y \longrightarrow 0} f(x,y) \right]$ and $\lim\limits_{y \longrightarrow 0} \left[\lim\limits_{x \longrightarrow 0} f(x,y) \right]$ exist and equals 0;

   (b) The limit $\lim\limits_{(x,y) \longrightarrow (0,0)} f(x,y)$ does not exist;

   (c) The function $f(x,y)$ is not continuous at $(0,0)$;

   (d) The partial derivatives of f exist at $(0,0)$.

3. Let

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   Show that f is differentiable at every point of $\mathbb{R}^2$ but the partial derivatives are not continuous at $(0,0)$.

4. Let $f(x,y) = |xy|$ for all $(x,y) \in \mathbb{R}^2$. Show that

   (a) f is differentiable at $(0, 0.)$

   (b) $f_x(0, y_0)$ does not exist if $y_0 \neq 0$.

5. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a function with $f_x(x,y) = f_y(x,y) = 0$ for all (x,y). Then show that $f(x,y) = c$, a constant.

Assignment 4: Directional derivatives, Maxima, Minima

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by
$$f(x, y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$
Show that the directional derivative of f at $(0, 0)$ in all directions exist but f is not differentiable at $(0, 0)$.

2. Let $f(x, y) = x^2e^y + \cos(xy)$. Find the directional derivative of f at $(1, 2)$ in the direction $(\frac{3}{5}, \frac{4}{5})$.

3. Find the equation of the surface generated by the normals to the surface $x + 2yz + xyz^2 = 0$ at all points on the z-axis.

4. Examine the following functions for local maxima, local minima and saddle points:
   (a) $4xy - x^4 - y^4$
   (b) $x^3 - 3xy^2$