

# Why Proofs?

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Alexander Shen

LIRMM

# Outline

**Proofs? Really?**

Proof by example

Impossibility proof

Impossibility Proof, II

Conclusion

# Do We Need Proofs?

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- programmers: more skeptical
  - correctness proof of a video game? or search engine?
  - but real-time operating system or cryptographic protocol is a different thing

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- not at all: an argument that is so convincing that you are ready to use it to convince other people
- a by-product and a sign of understanding
- believe or not: proofs are fun

# Course Objectives

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Intended audience: beginners

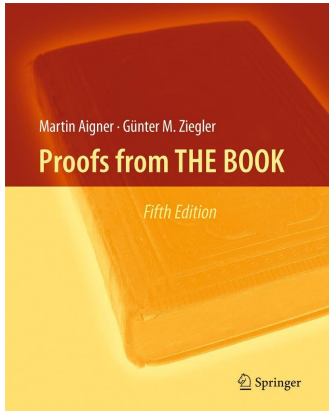
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Prerequisites: curiosity

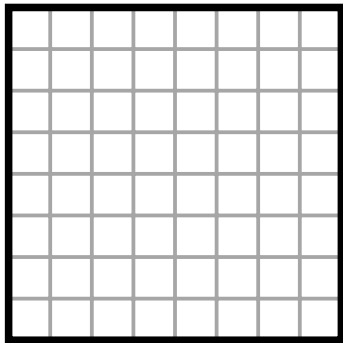
# Paul Erdős and The Book



[Google Books and Wikipedia images]

## To Begin With...

Can a chessboard  $8 \times 8$  be tiled (no overlaps or empty space) by domino  $1 \times 2$ -tiles?



# Outline

Proofs? Really?

**Proof by example**

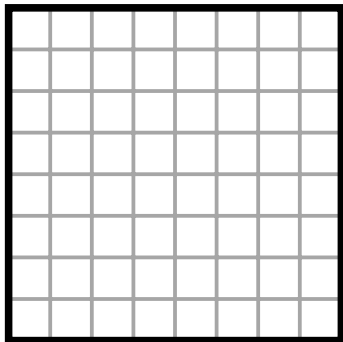
Impossibility proof

Impossibility Proof, II

Conclusion

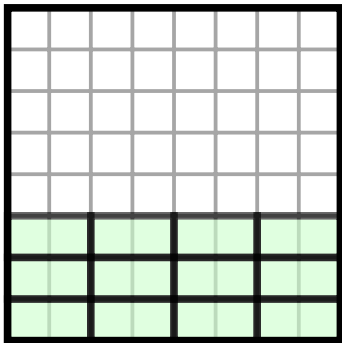
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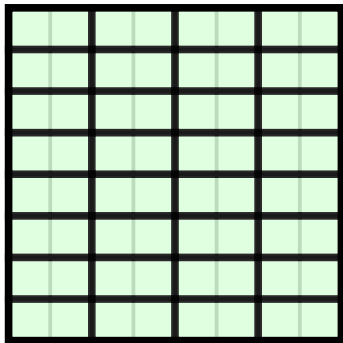
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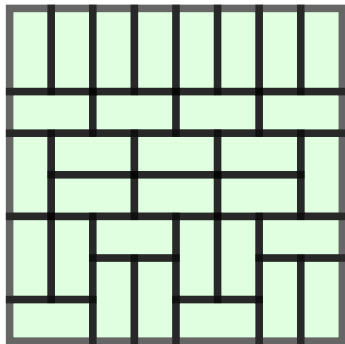
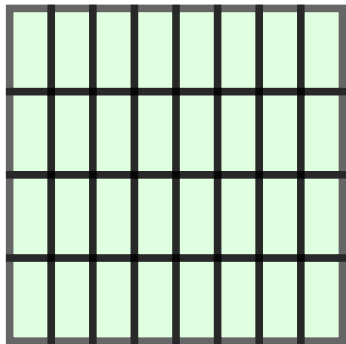
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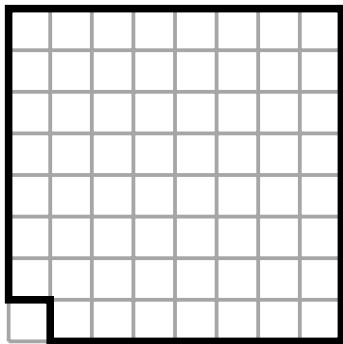


# Other Tilings

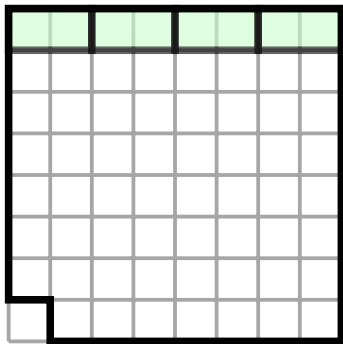
One tiling is enough for the proof. But there are others:



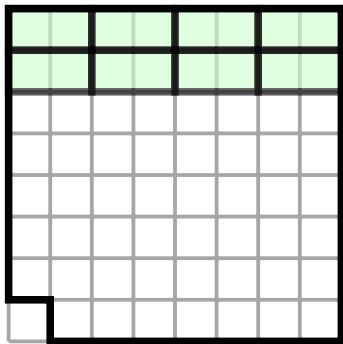
# Can We Tile This Board?



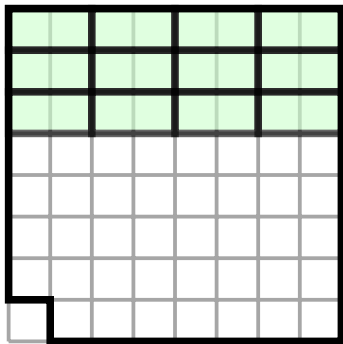
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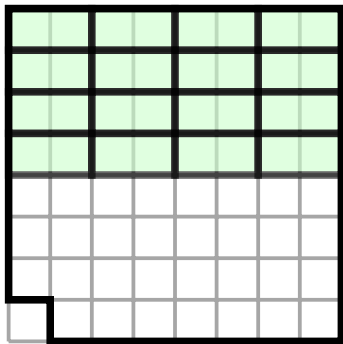
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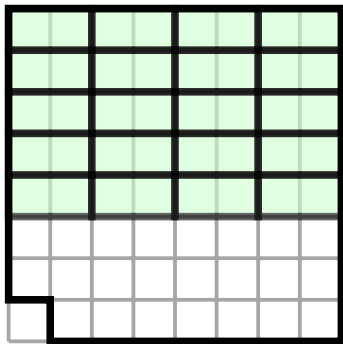
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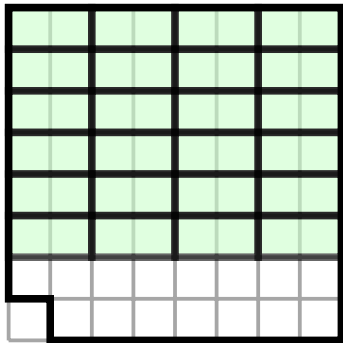
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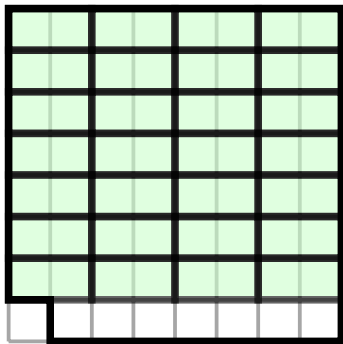


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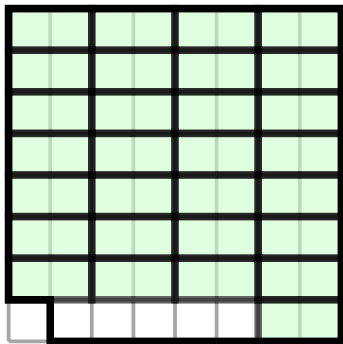




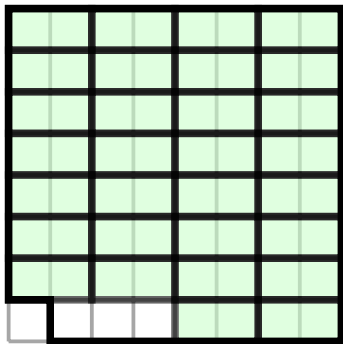
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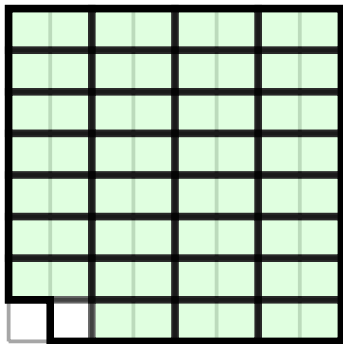
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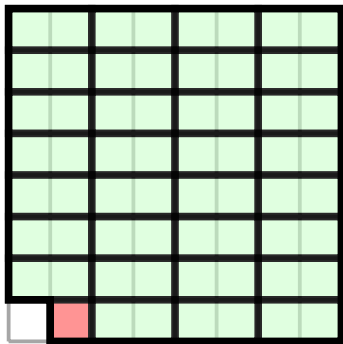
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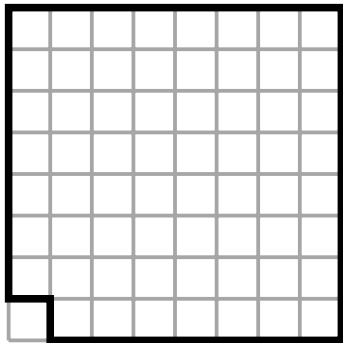
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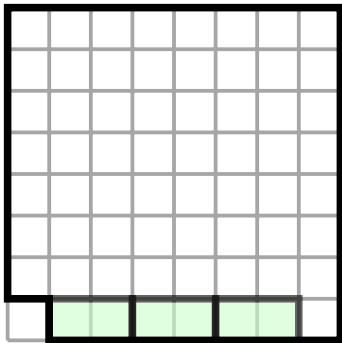
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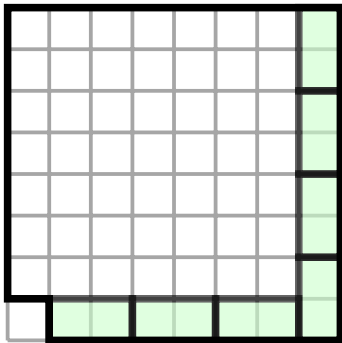
# Trying Again



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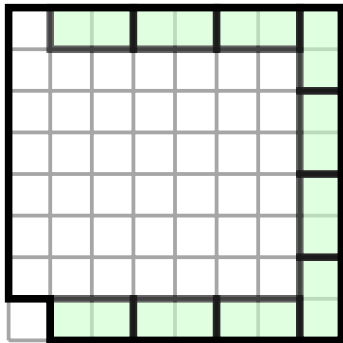


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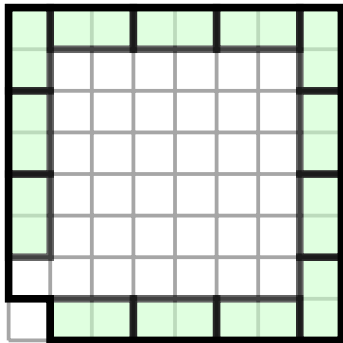




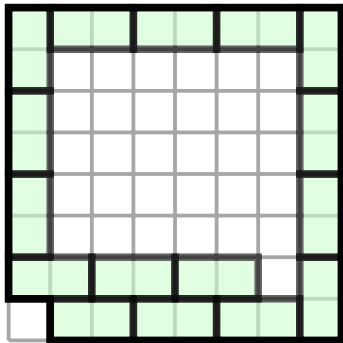
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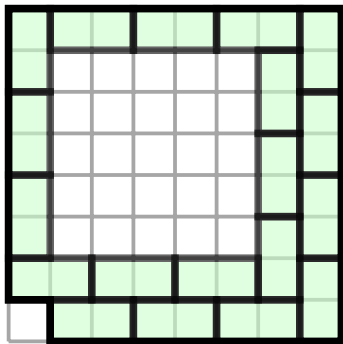
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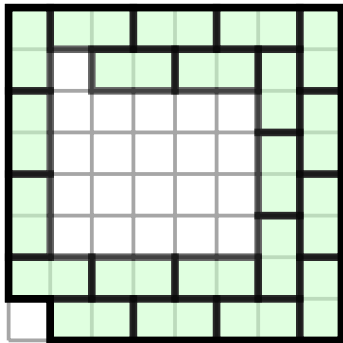
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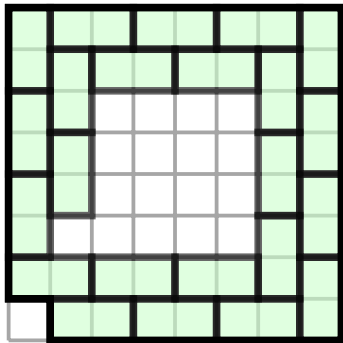
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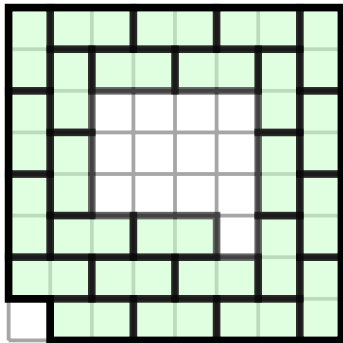
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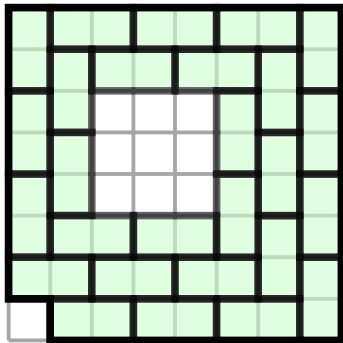
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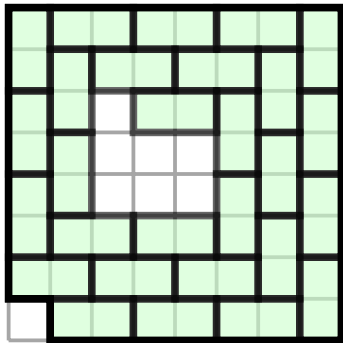


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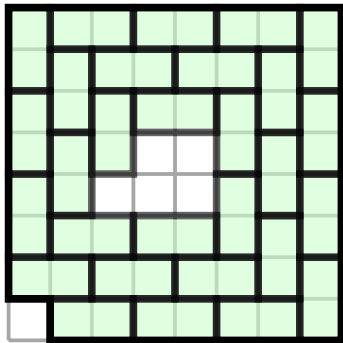




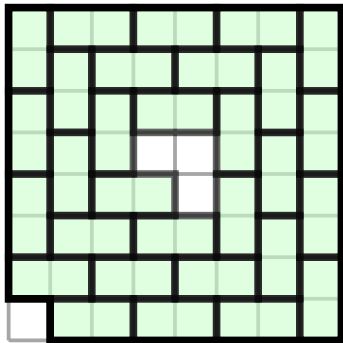
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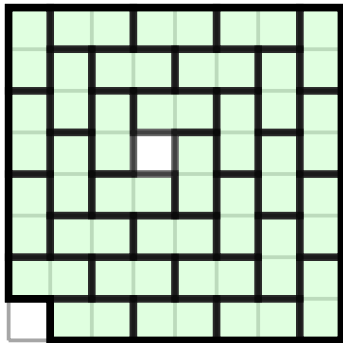
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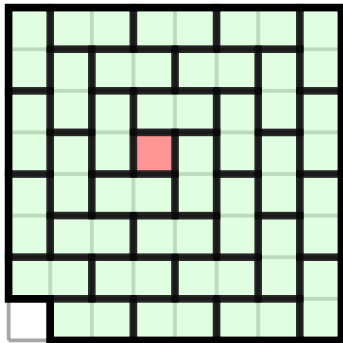
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- did we prove that tiling does not exist? No.
- can we do one of these two proofs?

# Outline

Proofs? Really?

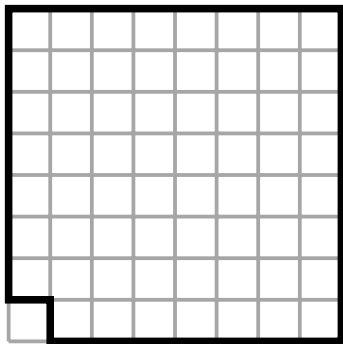
Proof by example

**Impossibility proof**

Impossibility Proof, II

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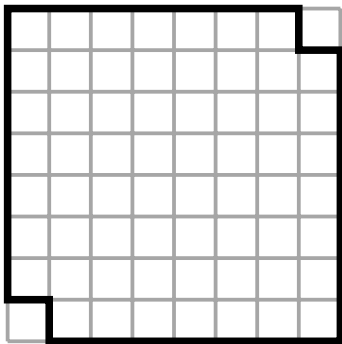
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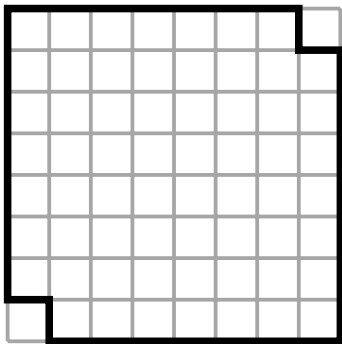
- one cell will always remain, because...
- there are  $63 = 8 \times 8 - 1$  cells, an odd number
- 31 tiles cover 62 cells, one remains
- mission *provably* impossible

# Two Corners Cut



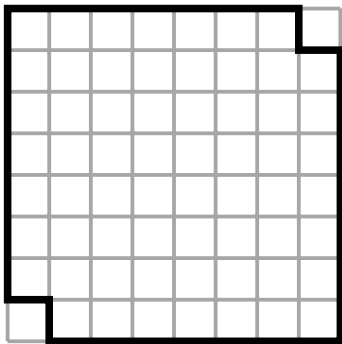
## Two Corners Cut

$64 - 2 = 62$  cells,  $62/2 = 31$  tiles. So the tiling is possible...



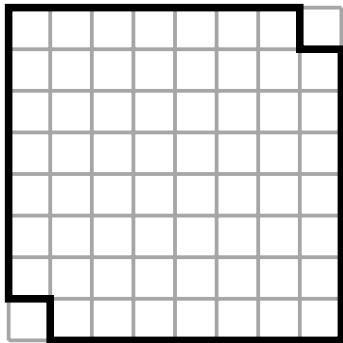
## Two Corners Cut

$64 - 2 = 62$  cells,  $62/2 = 31$  tiles. So the tiling is possible... is this argument OK?

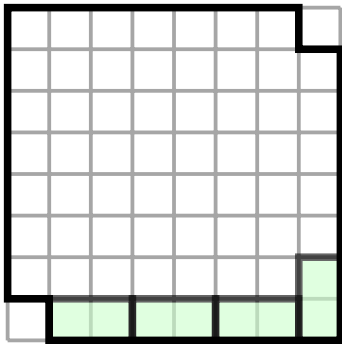


**Trying...**

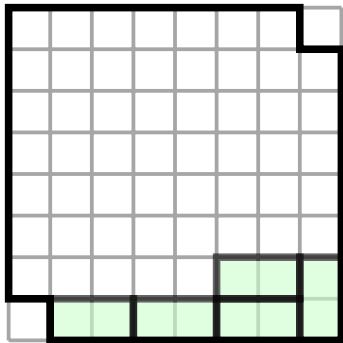
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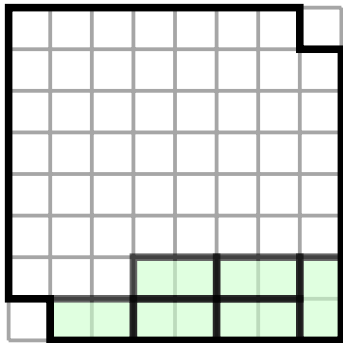


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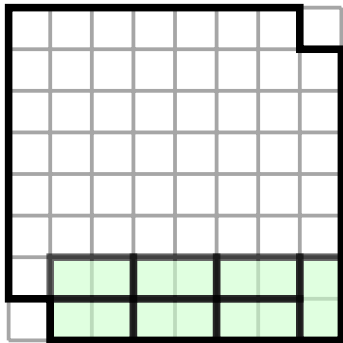




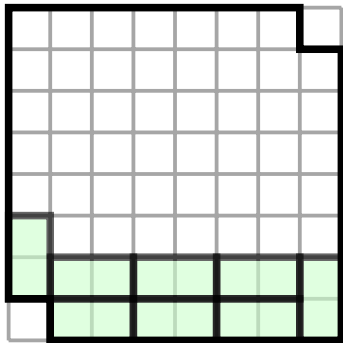
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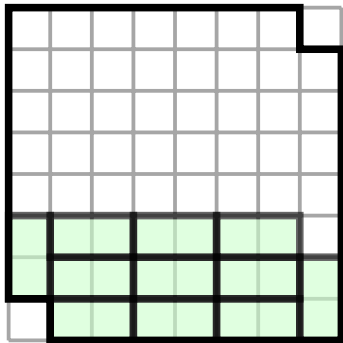
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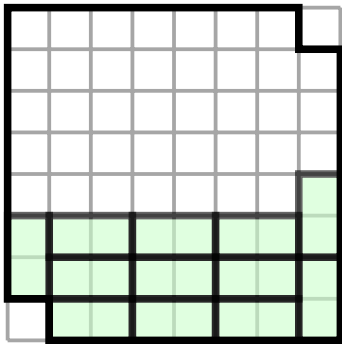
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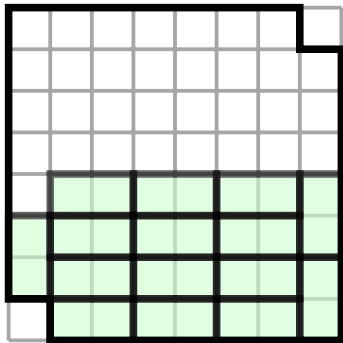
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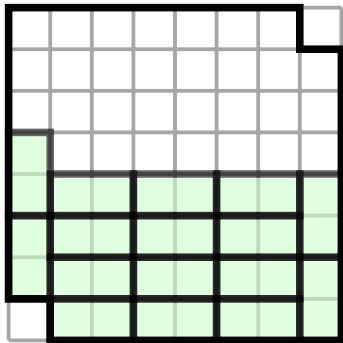
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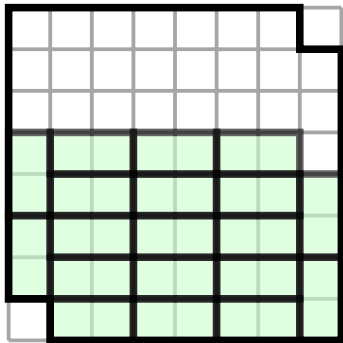
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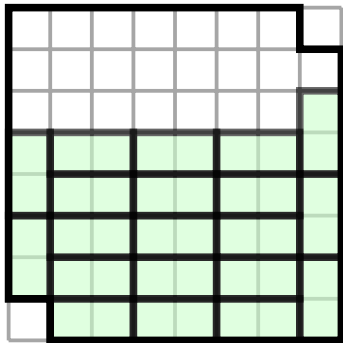


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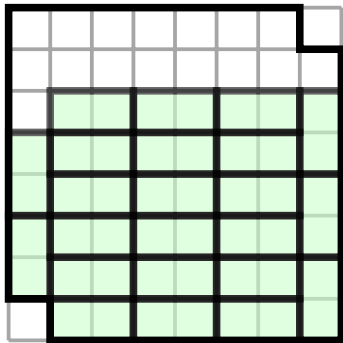




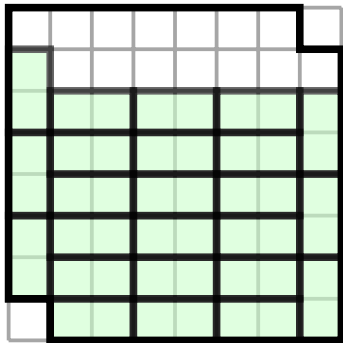
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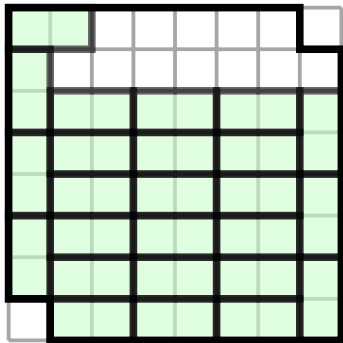
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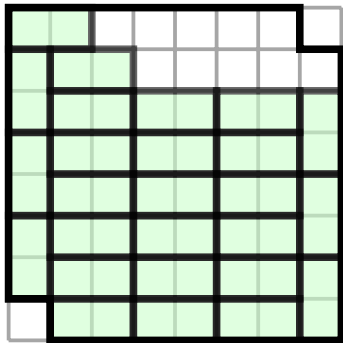
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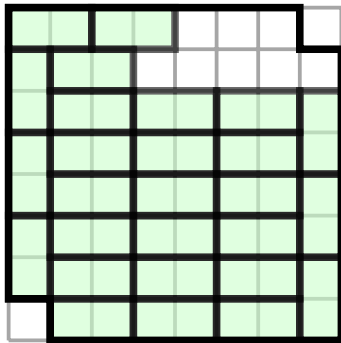
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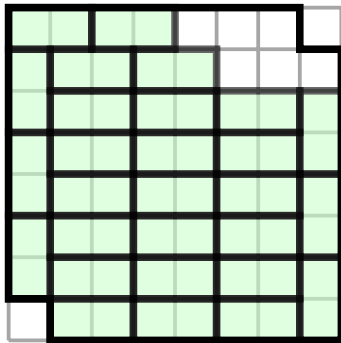
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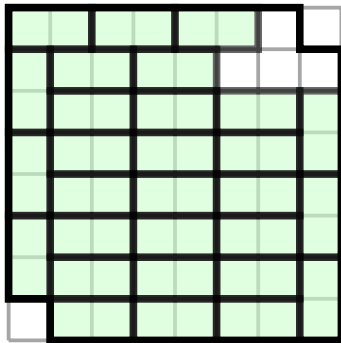
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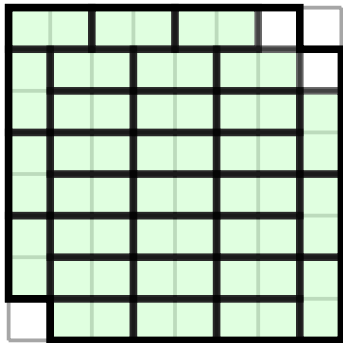


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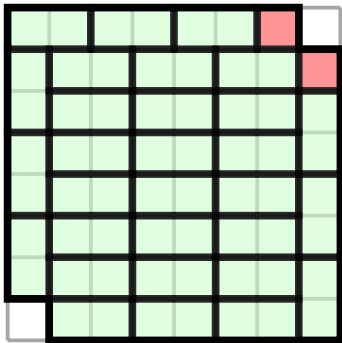




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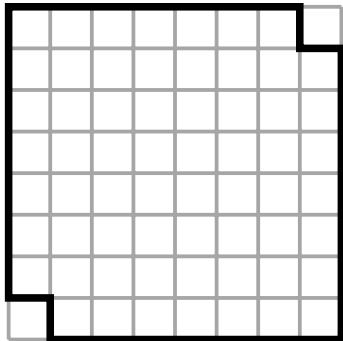


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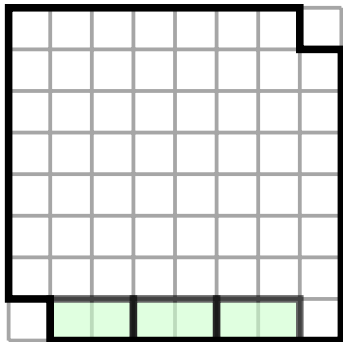


**One More Attempt**

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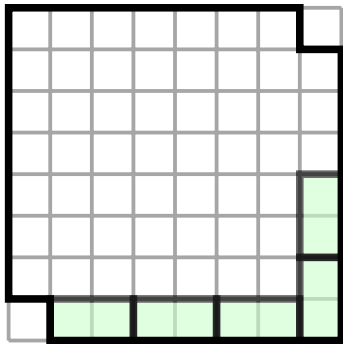


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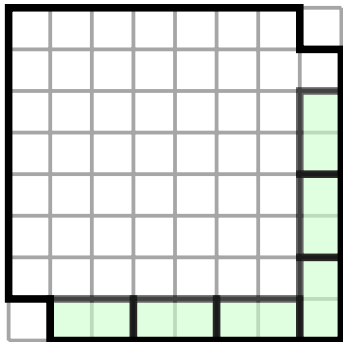




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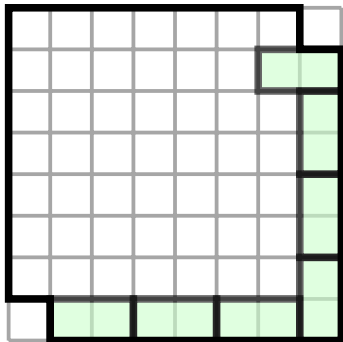


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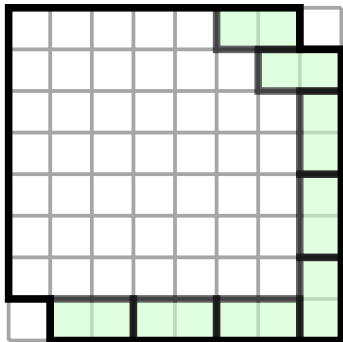




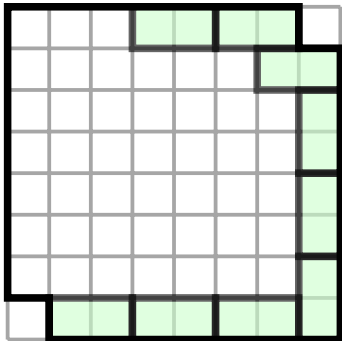
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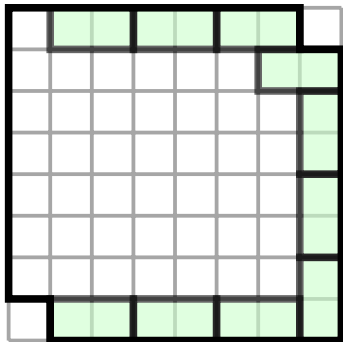
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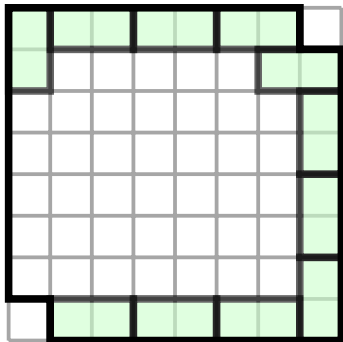
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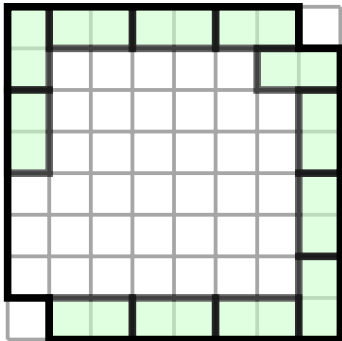
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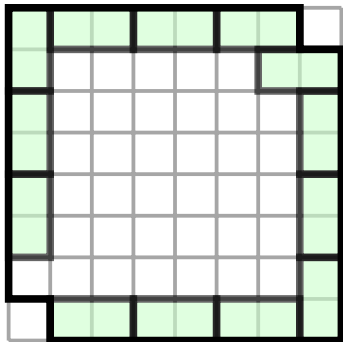
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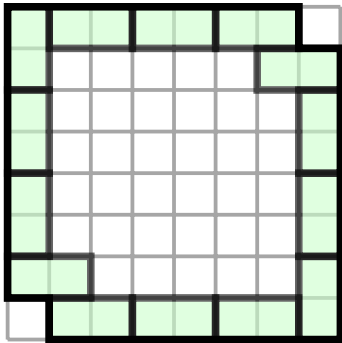
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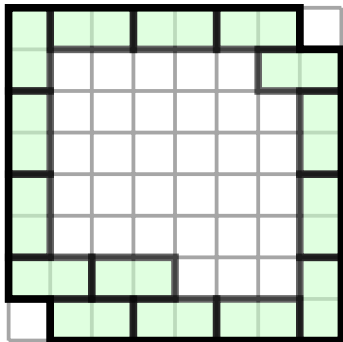


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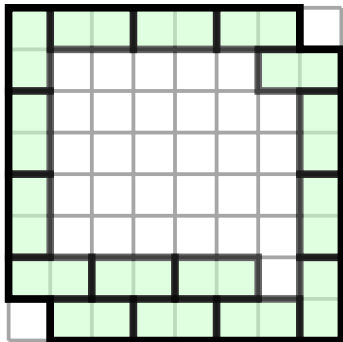




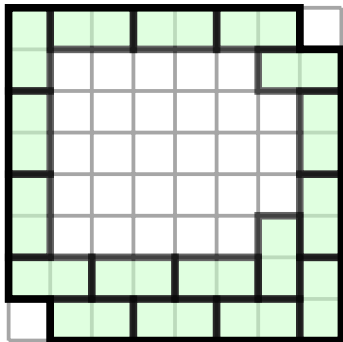
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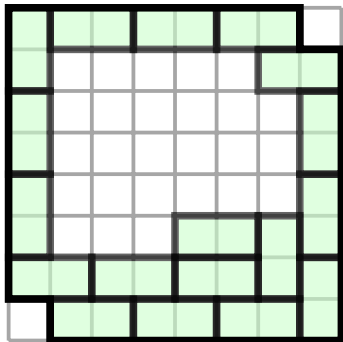
# One More Attempt



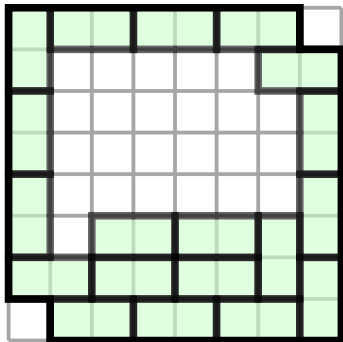
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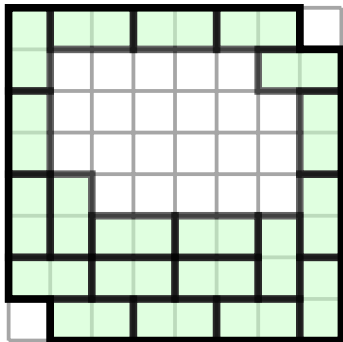
# One More Attempt



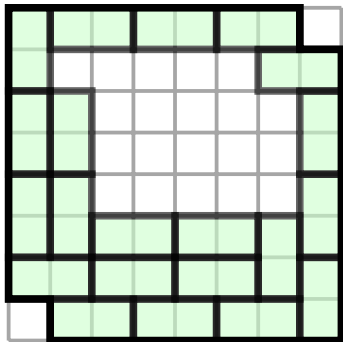
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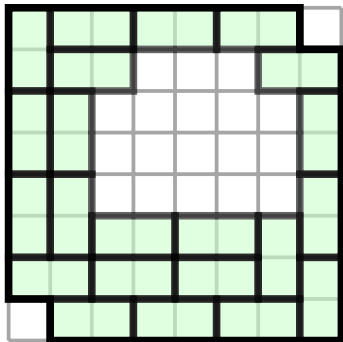
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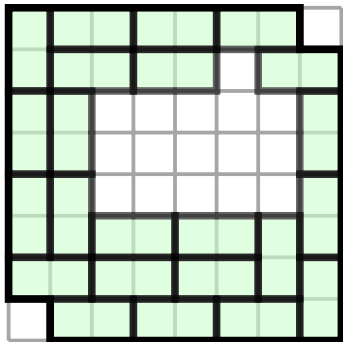


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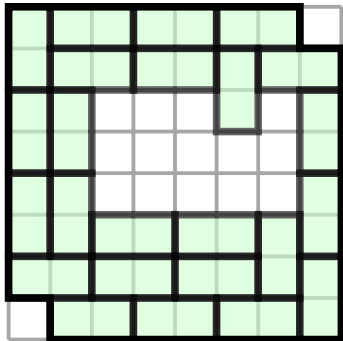




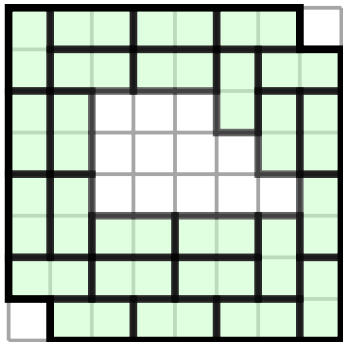
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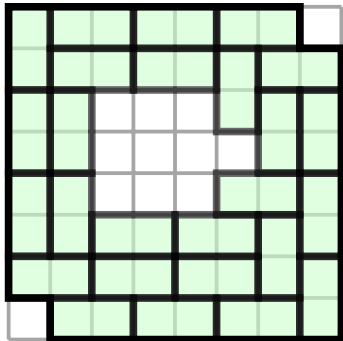
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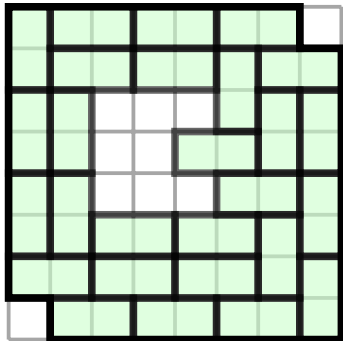
# One More Attempt



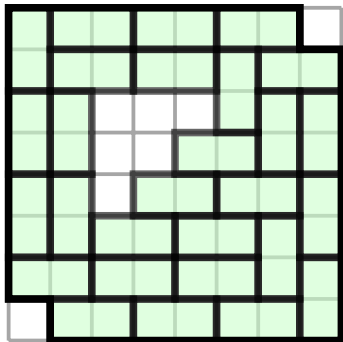
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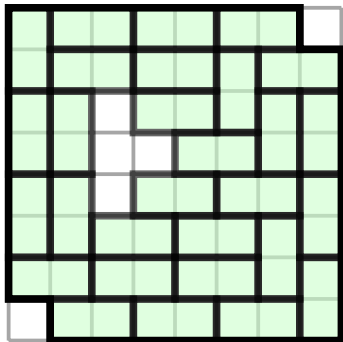
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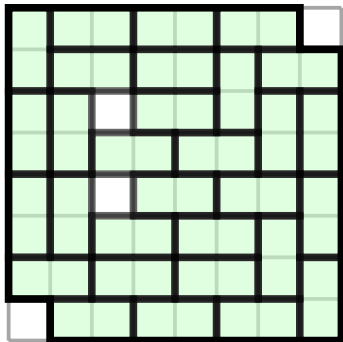
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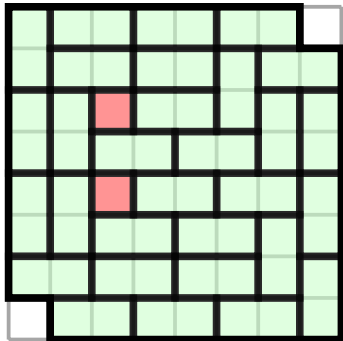


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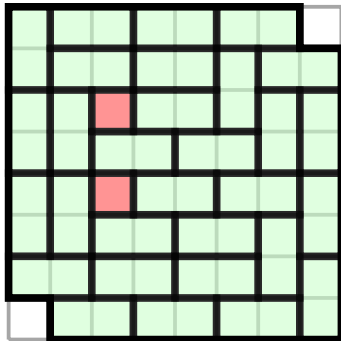


## One More Attempt



# One More Attempt

should we try harder?



# Outline

Proofs? Really?

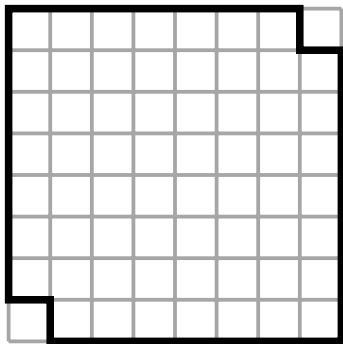
Proof by example

Impossibility proof

**Impossibility Proof, II**

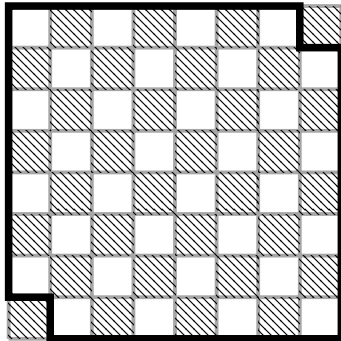
Conclusion

# Can We Tile This Board?

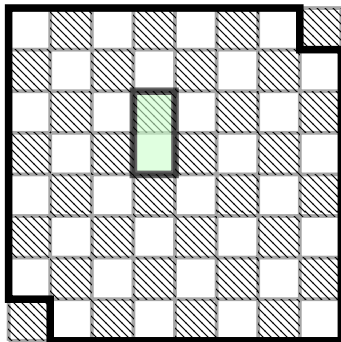


# Spoiler

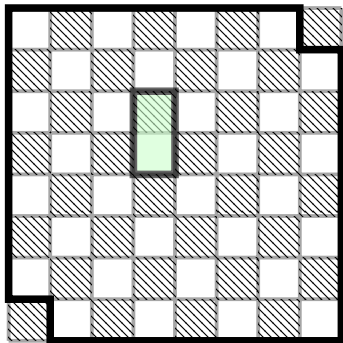
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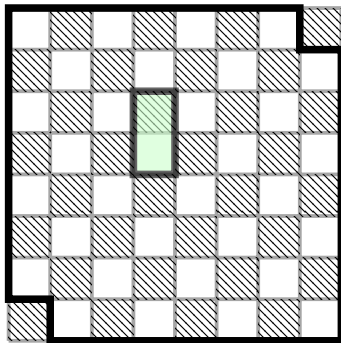
# Spoiler



32 white, 30 black



# Spoiler



32 white, 30 black  $\Rightarrow$  two whites remain

## Recap

**Theorem.** A chess board  $8 \times 8$  without two opposite corners cannot be tiled by  $1 \times 2$  dominos.

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- quod erat demonstrandum

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Proofs? Really?

Proof by example

Impossibility proof

Impossibility Proof, II

**Conclusion**

# Take-home Message and Questions

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- Proofs can be really convincing
- What if we cut two non-opposite corners?  
Can we tile the rest?
- Bonus problem: what if we cut any two cells of different colors?

# Spoiler

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