Unit 3
Logic and Proof Methods

3.1 Logic

The rules of logic give precise meaning to mathematical statements. These rules are used to distinguish between valid and invalid mathematical arguments.

Propositions

A proposition is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

EXAMPLE 1: All the following declarative sentences are propositions.

1. Kathmandu is the capital of Nepal.
2. Toronto is the capital of Canada.
3. 1 + 1 = 2.
4. 2 + 2 = 3.

Propositions 1 and 3 are true, whereas 2 and 4 are false.

EXAMPLE 2: Consider the following sentences.

1. What time is it?
2. Read this carefully.
3. x + 1 = 2.
4. x + y = z.

Sentences 1 and 2 are not propositions because they are not declarative sentences. Sentences 3 and 4 are not propositions because they are neither true nor false. Note that each of sentences 3 and 4 can be turned into a proposition if we assign values to the variables.

Some Terminology Related to Propositions

- **Propositional variables** (or **statement variables**), that is, variables that represent propositions, just as letters are used to denote numerical variables. The conventional letters used for propositional variables are p, q, r, s . . . The truth value of a proposition is true, denoted by T, if it is a true proposition, and the truth value of a proposition is false, denoted by F, if it is a false proposition.
- The area of logic that deals with propositions is called the **propositional calculus** or **propositional logic**.
- New propositions, called **compound propositions**, are formed from existing propositions using logical operators.

Logical Operators

The *logical operators* that are used to form new propositions from two or more existing propositions. These logical operators are also called **connectives.**

Negation

Let p be a proposition. The *negation* of p, denoted by ¬p (also denoted by $\bar{p}$), is the statement

"It is not the case that p."

Truth Table

P	***¬p***
T	F
F	F

Table: The truth table for the Negation of a Proposition

EXAMPLE 1: Find the negation of the proposition

"Michael's PC runs Linux"

and express this in simple English.

Solution: The negation is

"It is not the case that Michael's PC runs Linux."

This negation can be more simply expressed as

"Michael's PC does not run Linux."

EXAMPLE 2: Find the negation of the proposition

"Vandana's smartphone has at least 32GB of memory"

and express this in simple English.

Solution: The negation is

"It is not the case that Vandana's smartphone has at least 32GB of memory."

This negation can also be expressed as

"Vandana's smartphone does not have at least 32GB of memory"

or even more simply as

"Vandana's smartphone has less than 32GB of memory."

Conjunction

- Let p and q be propositions. The *conjunction* of p and q, denoted by p ∧ q, is the proposition "p and q."
- The conjunction p ∧ q is true when both p and q are true and is false otherwise.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table: The truth table for the Conjunction of two Proposition

Disjunction

- Let p and q be propositions. The disjunction of p and q, denoted by p ∨ q, is the proposition "p or q".
- The disjunction p ∨ q is false when both p and q are false and is true otherwise.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table: The truth table for the Disjunction of two Proposition

Exclusive OR

Let p and q be propositions. The exclusive or of p and q, denoted by p ⊕ q, is the proposition that is true when exactly one of p and q is true and is false otherwise.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Table: The truth table for the Exclusive OR of two Proposition

Conditional Statements

- Let p and q be propositions. The conditional statement p → q is the proposition "if p, then q."
- The conditional statement p → q is false when p is true and q is false, and true otherwise.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Table: The truth table for the Conditional Statements of two Proposition

EXAMPLE: Let p be the statement "Maria learns discrete mathematics" and q the statement "Maria will find a good job." Express the statement p → q as a statement in English.

Solution: From the definition of conditional statements, we see that when p is the statement "Maria learns discrete mathematics" and q is the statement "Maria will find a good job," p → q represents the statement

"If Maria learns discrete mathematics, then she will find a good job."

There are many other ways to express this conditional statement in English. Among the most natural of these are:

"Maria will find a good job when she learns discrete mathematics."

"For Maria to get a good job, it is sufficient for her to learn discrete mathematics."

and

"Maria will find a good job unless she does not learn discrete mathematics."

Converse, Contrapositive, and Inverse

Let p and q be propositions.

- The proposition q → p is called the *converse* of p → q.
- The *contrapositive* of p → q is the proposition ¬q → ¬p.
- The proposition ¬p → ¬q is called the *inverse* of p → q.

EXAMPLE: What are the contrapositive, the converse, and the inverse of the conditional statement "The home team wins whenever it is raining?"

Solution: Because "q whenever p" is one of the ways to express the conditional statement p → q, the original statement can be rewritten as

"If it is raining, then the home team wins."

Consequently, the contrapositive of this conditional statement is

"If the home team does not win, then it is not raining."

The converse is

"If the home team wins, then it is raining."

The inverse is

"If it is not raining, then the home team does not win."

Only the contrapositive is equivalent to the original statement.

Biconditionals

- Let p and q be propositions.
- The biconditional statement p $\leftrightarrow$ q is the proposition "p if and only if q."
- The biconditional statement p $\leftrightarrow$ q is true when p and q have the same truth values, and is false otherwise.
- Biconditional statements are also called bi-implications.
- The truth table for p $\leftrightarrow$ q is shown in Table below.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

TABLE: The Truth Table for the Biconditional p $\leftrightarrow$ q.

EXAMPLE: Let p be the statement "You can take the flight," and let q be the statement "You buy a ticket." Then p $\leftrightarrow$ q is the statement

"You can take the flight if and only if you buy a ticket."

Precedence of Logical Operators

Operator	*Precedence*
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

TABLE: Precedence of Logical Operators.

Tautology, Contradiction and Contingency

- A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a *tautology*.
- A compound proposition that is always false is called a *contradiction*.
- A compound proposition that is neither a tautology nor a contradiction is called a *contingency*.

EXAMPLE:

- Consider the truth tables of p ∨ ¬p and p ∧ ¬p, shown in Table below.
- Because p ∨ ¬p is always true, it is a tautology.
- Because p ∧ ¬p is always false, it is a contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

TABLE: Examples of a Tautology and a Contradiction.

Logical Equivalences

- Compound propositions that have the same truth values in all possible cases are called logically equivalent.
- The compound propositions p and q are called logically equivalent if p ↔ q is a tautology.
- The notation p ≡ q denotes that p and q are logically equivalent.

Proving Logical Equivalences Using Truth Tables

EXAMPLE 1: Show that ¬(p ∨ q) and ¬p ∧ ¬q are logically equivalent.

Solution: The truth tables for these compound propositions are displayed in Table below.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

TABLE: Truth Tables for ¬(p ∨q) and ¬p ∧¬q.

EXAMPLE 2: Show that p → q and ¬p ∨ q are logically equivalent.

Solution: We construct the truth table for these compound propositions in Table below.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

TABLE: Truth Tables for ¬p ∨q and p → q.

EXAMPLE 3: Show that p ∨ (q ∧ r) and (p ∨ q) ∧ (p ∨ r) are logically equivalent. This is the distributive law of disjunction over conjunction.

Solution: We construct the truth table for these compound propositions in Table below.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

TABLE: A Demonstration That p ∨(q ∧r) and (p ∨q) ∧(p ∨r) Are Logically Equivalent.

Proving Logical Equivalences Using Symbolic Derivation

Some Useful Results Related to Logical Equivalences

Equivalence	***Name***
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

TABLE: Logical Equivalences.

$$\begin{array}{l}
p \to q \equiv \neg p \vee q \\
p \to q \equiv \neg q \to \neg p \\
p \vee q \equiv \neg p \to q \\
p \wedge q \equiv \neg(p \to \neg q) \\
\neg(p \to q) \equiv p \wedge \neg q \\
(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r) \\
(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r \\
(p \to q) \vee (p \to r) \equiv p \to (q \vee r) \\
(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r
\end{array}$$

TABLE: Logical Equivalences Involving Conditional Statements.

$$\begin{array}{l}
p \leftrightarrow q \equiv (p \to q) \wedge (q \to p) \\
p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q \\
p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q) \\
\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q
\end{array}$$

TABLE: Logical Equivalences Involving Biconditional Statements.

Some Examples

EXAMPLE 1: Show that $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent.

Solution:

$\neg(p \to q) \equiv \neg(\neg p \vee q)$	by implication
$\equiv \neg(\neg p) \wedge \neg q$	by the second De Morgan law
$\equiv p \wedge \neg q$	by the double negation law

EXAMPLE 2: Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Solution:

$$\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law}\\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law}\\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law}\\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law}\\
&\equiv \mathbf{F} \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv \mathbf{F}\\
&\equiv (\neg p \wedge \neg q) \vee \mathbf{F} && \text{by the commutative law for disjunction}\\
&\equiv \neg p \wedge \neg q && \text{by the identity law for } \mathbf{F}
\end{aligned}$$

Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

EXAMPLE 3: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to T.

$$\begin{aligned}
(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{(by Implication)}\\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{(by the first De Morgan law)}\\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{(by the associative and commutative laws for disjunction)}\\
&\equiv T \vee T && \text{(by the commutative law for disjunction)}\\
&\equiv T && \text{(by the domination law)}
\end{aligned}$$

Predicates and Quantifiers

Predicates

- A predicate is an expression of one or more variables defined on some specific domain.
- A predicate with variables can be made a proposition by either assigning a value to the variable or by quantifying the variable.
- The following are some examples of predicates:
    - i) $x > 3$
    - ii) $x = y + 3$
    - iii) $x + y = z$
    - iv) computer x is under attack by an intruder
- We can denote the statement “x is greater than 3” by P (x), where P denotes the predicate “is greater than 3” and x is the variable.
- In general, a statement involving the n variables $x_1, x_2,...,x_n$ can be denoted by

$$P\ (x_1, x_2,...,x_n)$$

EXAMPLE 1: Let P (x) denote the statement “x > 3.” What are the truth values of P (4) and P (2)?

Solution: We obtain the statement P (4) by setting x = 4 in the statement “x > 3.” Hence, P (4), which is the statement “4 > 3,” is true. However, P (2), which is the statement “2 > 3,” is false.

EXAMPLE 2: Let Q(x, y) denote the statement “x = y + 3.” What are the truth values of the propositions Q(1, 2) and Q(3, 0)?

Solution: To obtain Q(1, 2), set x = 1 and y = 2 in the statement Q(x, y). Hence, Q(1, 2) is the statement “1 = 2 + 3,” which is false. The statement Q(3, 0) is the proposition “3 = 0 + 3,” which is true.

Quantifiers

- Quantification expresses the extent to which a predicate is true over a range of elements.
- In English, the words all, some, many, none, and few are used in quantifications.

Universal Quantification

- The universal quantification of P (x) is the statement

  “P (x) for all values of x in the domain.”

- The notation $\forall_x P(x)$ denotes the universal quantification of P(x).
- Here ∀ is called the universal quantifier.
- Read $\forall_x P(x)$ as “for all x P(x)” or “for every x P(x).”
- An element for which P(x) is false is called a counterexample of ∀xP(x).

EXAMPLE 1: Let P(x) be the statement “x + 1 > x.” What is the truth value of the quantification $\forall_x P(x)$, where the domain consists of all real numbers?

Solution: Because P(x) is true for all real numbers x, the quantification $\forall_x P(x)$ is true.

EXAMPLE 2: Let Q(x) be the statement “x < 2.” What is the truth value of the quantification $\forall_x Q(x)$, where the domain consists of all real numbers?

Solution: Q(x) is not true for every real number x, because, for instance, Q(3) is false. That is, x = 3 is a counterexample for the statement $\forall_x Q(x)$. Thus $\forall_x Q(x)$ is false.

Existential Quantification

- The existential quantification of P(x) is the proposition

  “There exists an element x in the domain such that P(x).”

- We use the notation $\exists_x P(x)$ for the existential quantification of P(x).
- Here ∃ is called the existential quantifier.
- The existential quantification $\exists_x P(x)$ is read as

  “There is an x such that P(x),”

  “There is at least one x such that P(x),” or

"For some x P(x)."

EXAMPLE 1: Let P(x) denote the statement "x > 3." What is the truth value of the quantification $\exists_x P(x)$, where the domain consists of all real numbers?

Solution: Because "x > 3" is sometimes true—for instance, when x = 4—the existential quantification of P(x), which is $\exists_x P(x)$, is true.

EXAMPLE 2: Let Q(x) denote the statement "x = x + 1." What is the truth value of the quantification $\exists_x Q(x)$, where the domain consists of all real numbers?

Solution: Because Q(x) is false for every real number x, the existential quantification of Q(x), which is $\exists_x Q(x)$, is false.

Uniqueness Quantifier

- The uniqueness quantifier, denoted by $\exists!$ or $\exists 1$.
- The notation $\exists!_x P(x)$ [or $\exists 1_x P(x)$] states "There exists a unique x such that P (x) is true.
- For instance, $\exists!_x(x - 1 = 0)$, where the domain is the set of real numbers, states that there is a unique real number x such that $x - 1 = 0$.
- This is a true statement, as $x = 1$ is the unique real number such that $x - 1 = 0$.

Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus.
- For example, $\forall x P(x) \lor Q(x)$ is the disjunction of $\forall x P(x)$ and Q(x). In other words, it means $(\forall x P(x)) \lor Q(x)$ rather than $\forall x(P(x) \lor Q(x))$.

Binding Variables

- When a quantifier is used on the variable x, we say that this occurrence of the variable is bound.
- An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be free.

EXAMPLE: In the statement $\exists x(x + y = 1)$, the variable x is bound by the existential quantification $\exists x$, but the variable y is free because it is not bound by a quantifier and no value is assigned to this variable. This illustrates that in the statement $\exists x(x + y = 1)$, x is bound, but y is free.

Logical Equivalences Involving Quantifiers

- Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions.
- We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

Negating Quantified Expressions (De Morgan's laws for quantifiers)

$\neg\forall xP(x) \equiv \exists x\, \neg P(x)$

$\neg\exists xQ(x) \equiv \forall x\, \neg Q(x)$

Translating from English into Logical Expressions

Translating sentences in English (or other natural languages) into logical expressions is a crucial task in mathematics, logic programming, artificial intelligence, software engineering, and many other disciplines.

EXAMPLE: Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.

Solution: Let us write the given sentence as:

"For every student in this class, that student has studied calculus."

First approach:

"For every student x in this class, x has studied calculus."

⇨ $\forall x(S(x)$

Second Approach:

"For every person x, if person x is a student in this class then x has studied calculus."

⇨ $\forall x(S(x) \rightarrow C(x))$

Nested Quantifiers

One quantifier is within the scope of another is called nested quantifiers.

EXAMPLE: Assume that the domain for the variables x and y consists of all real numbers. The statement

$\forall x\forall y(x + y = y + x)$

says that $x + y = y + x$ for all real numbers x and y. This is the commutative law for addition of real numbers. Likewise, the statement

$$\forall x \exists y(x + y = 0)$$

says that for every real number x there is a real number y such that $x + y = 0$. This states that every real number has an additive inverse.

Negating Nested Quantifiers

Statements involving nested quantifiers can be negated by successively applying the rules for negating statements involving a single quantifier.

EXAMPLE: Express the negation of the statement $\forall x \exists y(xy = 1)$ so that no negation precedes a quantifier.

Solution: By successively applying De Morgan's laws for quantifiers, we can move the negation in $\neg\forall x \exists y(xy = 1)$ inside all the quantifiers. We find that $\neg\forall x \exists y(xy = 1)$ is equivalent to $\exists x \neg\exists y(xy = 1)$, which is equivalent to $\exists x \forall y \neg(xy = 1)$. Because $\neg(xy = 1)$ can be expressed more simply as $xy \neq 1$, we conclude that our negated statement can be expressed as $\exists x \forall y(xy \neq 1)$.

Rules of Inference

Proofs in mathematics are valid arguments that establish the truth of mathematical statements.

Argument: An argument in propositional logic is a sequence of propositions.

Premise and Conclusion: All but the final proposition in the argument are called premises and the final proposition is called the conclusion.

Valid: An argument is valid if the truth of all its premises implies that the conclusion is true.

Argument Form: An argument form in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is valid no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

Rules of Inference: Templates for constructing valid arguments is called rules of inference. Rules of inference are our basic tools for establishing the truth of statements.

Fallacies: Incorrect reasoning which lead to invalid arguments is called fallacies.

Valid Arguments in Propositional Logic

- An argument is valid if the truth of all its premises implies that the conclusion is true.
- From the definition of a valid argument form we see that the argument form with premises $p_1, p_2, ..., p_n$ and conclusion q is valid, when $(p_1 \land p_2 \land \cdots \land p_n) \rightarrow q$ is a tautology.

EXAMPLE: Consider the following argument involving propositions (which, by definition, is a sequence of propositions):

"If you have a current password, then you can log onto the network."

"You have a current password."

Therefore,

"You can log onto the network."

Use p to represent "You have a current password" and q to represent "You can log onto the network." Then, the argument has the form

$$\begin{array}{rl} & p \rightarrow q \\ & p \\ \hline \therefore & q \end{array}$$

where ∴ is the symbol that denotes "therefore."

We know that when p and q are propositional variables, the statement $((p \rightarrow q) \wedge p) \rightarrow q$ is a tautology.

Rules of Inference for Propositional Logic

The rules of inference can be used as building blocks to construct more complicated valid argument forms.

Some important results are listed below.

Rule of Inference	Tautology	Name
$\begin{array}{l} p \\ p \to q \\ \hline \therefore q \end{array}$	$(p \land (p \to q)) \to q$	Modus ponens
$\begin{array}{l} \neg q \\ p \to q \\ \hline \therefore \neg p \end{array}$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$\begin{array}{l} p \lor q \\ \neg p \\ \hline \therefore q \end{array}$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
$\begin{array}{l} p \\ \hline \therefore p \lor q \end{array}$	$p \to (p \lor q)$	Addition
$\begin{array}{l} p \land q \\ \hline \therefore p \end{array}$	$(p \land q) \to p$	Simplification
$\begin{array}{l} p \\ q \\ \hline \therefore p \land q \end{array}$	$((p) \land (q)) \to (p \land q)$	Conjunction
$\begin{array}{l} p \lor q \\ \neg p \lor r \\ \hline \therefore q \lor r \end{array}$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

EXAMPLE 1: State which rule of inference is the basis of the following argument: “It is below freezing now. Therefore, it is either below freezing or raining now.”

Solution: Let p be the proposition “It is below freezing now” and q the proposition “It is raining now.” Then this argument is of the form

$$\begin{array}{l} \quad p \\ \hline \therefore p \lor q \end{array}$$

This is an argument that uses the addition rule.

EXAMPLE 2: State which rule of inference is the basis of the following argument: “It is below freezing and raining now. Therefore, it is below freezing now.”

Solution: Let p be the proposition “It is below freezing now,” and let q be the proposition “It is raining now.” This argument is of the form

$$\begin{array}{rl} & p \wedge q \\ \hline \therefore & p \end{array}$$

This argument uses the simplification rule.

EXAMPLE 3: State which rule of inference is used in the argument:

> If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow.

Solution: Let p be the proposition "It is raining today," let q be the proposition "We will not have a barbecue today," and let r be the proposition "We will have a barbecue tomorrow." Then this argument is of the form

$$\begin{array}{rl} & p \rightarrow q \\ & q \rightarrow r \\ \hline \therefore & p \rightarrow r \end{array}$$

Hence, this argument is a hypothetical syllogism.

Using Rules of Inference to Build Arguments

When there are many premises, several rules of inference are often needed to show that an argument is valid.

EXAMPLE 1: Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."

Solution: Let p be the proposition "It is sunny this afternoon," q the proposition "It is colder than yesterday," r the proposition "We will go swimming," s the proposition "We will take a canoe trip," and t the proposition "We will be home by sunset." Then the premises become $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$. The conclusion is simply t. We need to give a valid argument with premises $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$ and conclusion t. We construct an argument to show that our premises lead to the desired conclusion as follows.

Step	**Reason**
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7)

EXAMPLE 2: Show that the premises "If you send me an e-mail message, then I will finish writing the program," "If you do not send me an e-mail message, then I will go to sleep early," and "If I go to sleep early, then I will wake up feeling refreshed" lead to the conclusion "If I do not finish writing the program, then I will wake up feeling refreshed."

Solution: Let p be the proposition "You send me an e-mail message," q the proposition "I will finish writing the program," r the proposition "I will go to sleep early," and s the proposition "I will wake up feeling refreshed." Then the premises are $p \to q$, $\neg p \to r$, and $r \to s$. The desired conclusion is $\neg q \to s$. We need to give a valid argument with premises $p \to q$, $\neg p \to r$, and $r \to s$ and conclusion $\neg q \to s$.

This argument form shows that the premises lead to the desired conclusion

Step	**Reason**
1. $p \to q$	Premise
2. $\neg q \to \neg p$	Contrapositive of (1)
3. $\neg p \to r$	Premise
4. $\neg q \to r$	Hypothetical syllogism using (2) and (3)
5. $r \to s$	Premise
6. $\neg q \to s$	Hypothetical syllogism using (4) and (5)

Rules of Inference for Quantified Statements

Rules of inference for statements involving quantifiers are used extensively in mathematical arguments, often without being explicitly mentioned.

Universal instantiation

- Universal instantiation is the rule of inference used to conclude that P (c) is true, where c is a particular member of the domain, given the premise ∀xP (x).
- For example, "All women are wise" that "Lisa is wise," where Lisa is a member of the domain of all women.

Universal generalization

- Universal generalization is the rule of inference that states that ∀xP (x) is true, given the premise that P (c) is true for all elements c in the domain.

Existential instantiation

- Existential instantiation is the rule that allows us to conclude that there is an element c in the domain for which P (c) is true if we know that ∃xP (x) is true.

Existential generalization

- Existential generalization is the rule of inference that is used to conclude that ∃xP (x) is true when a particular element c with P (c) true is known.
- That is, if we know one element c in the domain for which P (c) is true, then we know that ∃xP (x) is true.

Table below shows the rules of inference in summarized form

Rule of Inference	*Name*
$\dfrac{\forall x P(x)}{\therefore P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$	Existential generalization

TABLE: Rules of Inference for Quantified Statements.

EXAMPLE 1: Show that the premises "Everyone in this discrete mathematics class has taken a course in computer science" and "Marla is a student in this class" imply the conclusion "Marla has taken a course in computer science."

Solution: Let D(x) denote "x is in this discrete mathematics class," and let C(x) denote "x has taken a course in computer science." Then the premises are ∀x(D(x) → C(x)) and D(Marla). The conclusion is C(Marla).

The following steps can be used to establish the conclusion from the premises.

Step	**Reason**
1. $\forall x(D(x) \to C(x))$	Premise
2. $D(\text{Marla}) \to C(\text{Marla})$	Universal instantiation from (1)
3. $D(\text{Marla})$	Premise
4. $C(\text{Marla})$	Modus ponens from (2) and (3)

EXAMPLE 2: Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

Solution: Let C(x) be "x is in this class," B(x) be "x has read the book," and P (x) be "x passed the first exam." The premises are ∃x(C(x) ∧ ¬B(x)) and ∀x(C(x) → P (x)). The conclusion is ∃x(P (x) ∧ ¬B(x)). These steps can be used to establish the conclusion from the premises.

Step	**Reason**
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \to P(x))$	Premise
5. $C(a) \to P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conjunction from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	Existential generalization from (8)

Resolution

- Computer programs have been developed to automate the task of reasoning and proving theorems.
- Many of these programs make use of a rule of inference known as resolution.
- This rule of inference is based on the tautology.

Fallacies

- Several common fallacies arise in incorrect arguments.
- These fallacies resemble rules of inference, but are based on contingencies rather than tautologies.

i. *fallacy of affirming the conclusion*

- The proposition $((p \rightarrow q) \land q) \rightarrow p$ is not a tautology, because it is false when p is false and q is true.
- However, there are many incorrect arguments that treat this as a tautology.
- In other words, they treat the argument with premises $p \rightarrow q$ and q and conclusion p as a valid argument form, which it is not.
- This type of incorrect reasoning is called the fallacy of affirming the conclusion.

EXAMPLE: Is the following argument valid?

If you do every problem in this book, then you will learn discrete mathematics. You learned discrete mathematics. Therefore, you did every problem in this book.

Solution: Let p be the proposition "You did every problem in this book." Let q be the proposition "You learned discrete mathematics." Then this argument is of the form: if $p \rightarrow q$ and q, then p. This is an example of an incorrect argument using the fallacy of affirming the conclusion. Indeed, it is possible for you to learn discrete mathematics in some way other than by doing every problem in this book. (You may learn discrete mathematics by reading, listening to lectures, doing some, but not all, the problems in this book, and so on.)

ii. *fallacy of denying the hypothesis*

- The proposition $((p \rightarrow q) \land \neg p) \rightarrow \neg q$ is not a tautology, because it is false when p is false and q is true.
- Many incorrect arguments use this incorrectly as a rule of inference.
- This type of incorrect reasoning is called the fallacy of denying the hypothesis.

EXAMPLE: Let p and q be as in above Example. If the conditional statement $p \rightarrow q$ is true, and $\neg p$ is true, is it correct to conclude that $\neg q$ is true? In other words, is it correct to assume that you did not learn discrete mathematics if you did not do every problem in the book, assuming that if you do every problem in this book, then you will learn discrete mathematics?

Solution: It is possible that you learned discrete mathematics even if you did not do every problem in this book. This incorrect argument is of the form $p \rightarrow q$ and $\neg p$ implies $\neg q$, which is an example of the fallacy of denying the hypothesis.

3.2 Introduction to Proofs

A proof is a valid argument that establishes the truth of a mathematical statement.

Basic Terminologies

Theorem:

A theorem is a statement that can be shown to be true. Theorems can also be referred to as *facts* or *results*. Less important theorems sometimes are called *propositions*.

Proof:

A proof is a valid argument that establishes the truth of a theorem.

Axioms (or postulates):

Statements which are statements we assume to be true

Lemma:

A less important theorem that is helpful in the proof of other results is called a lemma (plural lemmas or lemmata).

Corollary:

A corollary is a theorem that can be established directly from a theorem that has been proved.

Conjecture:

A conjecture is a statement that is being proposed to be a true statement, usually on the basis of some partial evidence, a heuristic argument, or the intuition of an expert.

Proof Methods

Direct Proofs

In a direct proof, we assume that p is true and use axioms, definitions, and previously proven theorems, together with rules of inference, to show that q must also be true.

EXAMPLE 1: Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

Solution:

- Let us assume that n is odd.
- By the definition of an odd integer, it follows that n = 2k + 1, where k is some integer.
- We want to show that n^2 is also odd.

$$n = 2k + 1$$

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

- By the definition of an odd integer, we can conclude that n^2 is an odd.

EXAMPLE 2: Give a direct proof that if m and n are both perfect squares, then nm is also a perfect square.

Solution:

- To produce a direct proof of this theorem, we assume that the hypothesis of this conditional statement is true, namely, we assume that m and n are both perfect squares.

- By the definition of a perfect square, it follows that there are integers s and t such that $m = s^2$ and $n = t^2$.

  $mn = s^2t^2 = (ss)(tt) = (st)(st) = (st)^2$

- By the definition of perfect square, it follows that mn is also a perfect square, because it is the square of st, which is an integer.

Indirect Proofs

Proofs of theorems of that do not start with the premises and end with the conclusion are called *indirect proofs*.

Proof by Contraposition

- An extremely useful type of *indirect proof* is known as proof by contraposition.
- Proofs by contraposition make use of the fact that the conditional statement $p \rightarrow q$ is equivalent to its contrapositive, $\neg q \rightarrow \neg p$.
- This means that the conditional statement $p \rightarrow q$ can be proved by showing that its contrapositive, $\neg q \rightarrow \neg p$, is true.

EXAMPLE 1: Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Solution:

- The first step in a proof by contraposition is to assume that the conclusion of the conditional statement "If $3n + 2$ is odd, then n is odd" is false; namely, assume that n is even.
- Then, by the definition of an even integer, $n = 2k$ for some integer k.
- Substituting 2k for n, we find that $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$.
- This tells us that $3n + 2$ is even (because it is a multiple of 2), and therefore not odd.
- This is the negation of the premise of the theorem.
- Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true.
- Therefore, "If $3n + 2$ is odd, then n is odd."

EXAMPLE 2: Prove that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Solution:

- The first step in a proof by contraposition is to assume that the conclusion of the conditional statement "If $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$" is false. That is, we assume that the statement $(a \leq \sqrt{n}) \vee (b \leq \sqrt{n})$ is false.
- Using De Morgan's law, implies that $a > \sqrt{n}$ and $b > \sqrt{n}$.
- We can multiply these inequalities together to obtain $ab > \sqrt{n} \cdot \sqrt{n} = n$ which contradicts the statement $n = ab$.

- Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true.
- Hence, we have proved that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

DEFINITION:

The real number r is rational if there exist integers p and q with $q \neq 0$ such that $r = p/q$. A real number that is not rational is called irrational.

Proofs by Contradiction

- To prove a statement p is true, we begin by assuming p false and show that this leads to a contradiction; something that always false.
- Many of the statements we prove have the form $p \to q$ which, when negated, has the form $p \to \neg q$. Often proof by contradiction has the form
    - *Proposition:*
        - $p \to q$
    - *Proof:*
        - Assume, for the sake of contradiction p is true but q is false.
        - Since we have a contradiction, it must be that q is true.

EXAMPLE 1: Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Solution:

- Let p be the proposition "$\sqrt{2}$ is irrational."
- To start a proof by contradiction, we suppose that $\neg p$ is true which says that $\sqrt{2}$ is rational.
- If $\sqrt{2}$ is rational, there exist integers a and b with $\sqrt{2} = a/b$, where $b \neq 0$ and a and b have no common factors (so that the fraction a/b is in lowest terms.)
- Because $\sqrt{2} = a/b$, when both sides of this equation are squared, it follows that

$$2 = a^2 / b^2$$

- Hence,

$$2b^2 = a^2$$

- By the definition of an even integer it follows that a^2 is even.
- We next use the fact that if a^2 is even, a must also be even.
- Furthermore, because a is even, by the definition of an even integer, $a = 2c$ for some integer c.
- Thus,

$$2b^2 = 4c^2$$

- Dividing both sides of this equation by 2 gives

$$b^2 = 2c^2$$

- By the definition of even, this means that b^2 is even.
- Again using the fact that if the square of an integer is even, then the integer itself must be even, we conclude that b must be even as well.
- We have now shown that the assumption of ¬p leads to the equation √2 = a/b, where a and b have no common factors, but both a and b are even, that is, 2 divides both a and b.
- Because our assumption of ¬p leads to the contradiction that 2 divides both a and b and 2 does not divide both a and b, ¬p must be false.
- That is, the statement p, "√2 is irrational," is true.
- We have proved that √2 is irrational.

EXAMPLE 2: Give a proof by contradiction of the theorem "If 3n + 2 is odd, then n is odd."

Solution:

- Let p be "3n + 2 is odd" and q be "n is odd."
- To construct a proof by contradiction, assume that both p and ¬q are true.
- That is, assume that 3n + 2 is odd and that n is not odd.
- Because n is not odd, we know that it is even.
- Because n is even, there is an integer k such that n = 2k.
- This implies that 3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1).
- Because 3n + 2 is 2t, where t = 3k + 1, 3n + 2 is even.
- Note that the statement "3n + 2 is even" is equivalent to the statement ¬p, because an integer is even if and only if it is not odd.
- Because both p and ¬p are true, we have a contradiction.
- This completes the proof by contradiction, proving that if 3n + 2 is odd, then n is odd.

Proof by Counter Example

To show that a statement of the form ∀xP (x) is false, we need only find a counterexample, that is, an example x for which P (x) is false.

EXAMPLE: Show that the statement "Every positive integer is the sum of the squares of two integers" is false.

Solution:

- To show that this statement is false, we look for a counterexample, which is a particular integer that is not the sum of the squares of two integers.
- It does not take long to find a counterexample, because 3 cannot be written as the sum of the squares of two integers.
- To show this is the case, note that the only perfect squares not exceeding 3 are $0^2 = 0$ and $1^2 = 1$.
- Furthermore, there is no way to get 3 as the sum of two terms each of which is 0 or 1.
- Consequently, we have shown that "Every positive integer is the sum of the squares of two integers" is false.

Vacuous Proofs

- We can quickly prove that a conditional statement $p \rightarrow q$ is true when we know that p is false, because $p \rightarrow q$ must be true when p is false.
- Consequently, if we can show that p is false, then we have a proof, called a vacuous proof, of the conditional statement $p \rightarrow q$.
- Vacuous proofs are often used to establish special cases of theorems that state that a conditional statement is true for all positive integers.

EXAMPLE: Show that the proposition P (0) is true, where P (n) is "If $n > 1$, then $n^2 > n$" and the domain consists of all integers.

Solution:

- Note that P (0) is "If $0 > 1$, then $0^2 > 0$." We can show P (0) using a vacuous proof. Indeed, the hypothesis $0 > 1$ is false.
- This tells us that P (0) is automatically true.

Trivial Proofs

- We can also quickly prove a conditional statement $p \rightarrow q$ if we know that the conclusion q is true.
- By showing that q is true, it follows that $p \rightarrow q$ must also be true.
- A proof of $p \rightarrow q$ that uses the fact that q is true is called a trivial proof.
- Trivial proofs are often important when special cases of theorems are proved.

EXAMPLE: Let P(n) be "If a and b are positive integers with $a \geq b$, then $a^n \geq b^n$," where the domain consists of all nonnegative integers. Show that P (0) is true.

Solution:

- The proposition P(0) is "If $a \geq b$, then $a^0 \geq b^0$." Because $a^0 = b^0 = 1$, the conclusion of the conditional statement "If $a \geq b$, then $a^0 \geq b^0$" is true.
- Hence, this conditional statement, which is P (0), is true.

Exhaustive and Proof by Cases

- To prove a conditional statement of the form

$$(p_1 \vee p_2 \vee \cdots \vee p_n) \rightarrow q$$

the tautology

$$[(p_1 \vee p_2 \vee \cdots \vee p_n) \rightarrow q] \leftrightarrow [(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \cdots \wedge (p_n \rightarrow q)]$$

can be used as a rule of inference.

- This shows that the original conditional statement with a hypothesis made up of a disjunction of the propositions $p_1, p_2, \ldots, p_n$ can be proved by proving each of the n conditional statements $p_i \rightarrow q$, $i = 1, 2, \ldots, n$, individually.

- Such an argument is called a proof by cases.
- Sometimes to prove that a conditional statement $p \rightarrow q$ is true, it is convenient to use a disjunction $p_1 \lor p_2 \lor \cdots \lor p_n$ instead of p as the hypothesis of the conditional statement, where p and $p_1 \lor p_2 \lor \cdots \lor p_n$ are equivalent.

EXHAUSTIVE PROOF:

- Some theorems can be proved by examining a relatively small number of examples.
- Such proofs are called exhaustive proofs, or proofs by exhaustion because these proofs proceed by exhausting all possibilities.
- An exhaustive proof is a special type of proof by cases where each case involves checking a single example.

EXAMPLE 1: Prove that $(n + 1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$.

Solution:

- We use a proof by exhaustion.
- We only need verify the inequality $(n + 1)^3 \geq 3^n$ when n = 1, 2, 3, and 4.
- For n = 1, we have $(n + 1)^3 = 2^3 = 8$ and $3^n = 3^1 = 3$;
- for n = 2, we have $(n + 1)^3 = 3^3 = 27$ and $3^n = 3^2 = 9$;
- for n = 3, we have $(n + 1)^3 = 4^3 = 64$ and $3^n = 3^3 = 27$; and
- for n = 4, we have $(n + 1)^3 = 5^3 = 125$ and $3^n = 3^4 = 81$.
- In each of these four cases, we see that $(n + 1)^3 \geq 3^n$.
- We have used the method of exhaustion to prove that $(n + 1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$.

PROOF BY CASES:

A proof by cases must cover all possible cases that arise in a theorem.

EXAMPLE: Prove that if n is an integer, then $n^2 \geq n$.

Solution:

- We can prove that $n^2 \geq n$ for every integer by considering three cases, when n = 0, when $n \geq 1$, and when $n \leq -1$.
- We split the proof into three cases because it is straightforward to prove the result by considering zero, positive integers, and negative integers separately.
    - Case (i): When n = 0, because $0^2 = 0$, we see that $0^2 \geq 0$. It follows that $n^2 \geq n$ is true in this case.
    - Case (ii): When $n \geq 1$, when we multiply both sides of the inequality $n \geq 1$ by the positive integer n, we obtain $n \cdot n \geq n \cdot 1$. This implies that $n^2 \geq n$ for $n \geq 1$.
    - Case (iii): In this case $n \leq -1$. However, $n^2 \geq 0$. It follows that $n^2 \geq n$. Because the inequality $n^2 \geq n$ holds in all three cases, we can conclude that if n is an integer, then $n^2 \geq n$.

Existence Proofs

- Many theorems are assertions that objects of a particular type exist.
- A theorem of this type is a proposition of the form ∃xP (x), where P is a predicate.
- A proof of a proposition of the form ∃xP (x) is called an existence proof.
- There are several ways to prove a theorem of this type.
- Sometimes an existence proof of ∃xP (x) can be given by finding an element a, called a witness, such that P (a) is true.
- This type of existence proof is called constructive.
- It is also possible to give an existence proof that is nonconstructive; that is, we do not find an element a such that P (a) is true, but rather prove that ∃xP (x) is true in some other way.
- One common method of giving a nonconstructive existence proof is to use proof by contradiction and show that the negation of the existential quantification implies a contradiction.

EXAMPLE: Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways.

Solution: After considerable computation (such as a computer search) we find that

1729 = 103 + 93 = 123 + 13

Because we have displayed a positive integer that can be written as the sum of cubes in two different ways, we are done.

Uniqueness Proofs

- Some theorems assert the existence of a unique element with a particular property.
- In other words, these theorems assert that there is exactly one element with this property.
- To prove a statement of this type we need to show that an element with this property exists and that no other element has this property.
- The two parts of a uniqueness proof are:
    - *Existence:* We show that an element x with the desired property exists.
    - *Uniqueness:* We show that if y ≠ x, then y does not have the desired property.
- Equivalently, we can show that if x and y both have the desired property, then x = y.

EXAMPLE: Show that if a and b are real numbers and $a \neq 0$, then there is a unique real number r such that $ar + b = 0$.

Solution:

- First, note that the real number $r = -b/a$ is a solution of $ar + b = 0$ because $a(-b/a) + b = -b + b = 0$.

- Consequently, a real number r exists for which $ar + b = 0$. This is the existence part of the proof.
- Second, suppose thats is a real number such that $as + b = 0$.
- Then $ar + b = as + b$, where $r = -b/a$.
- Subtracting b from both sides, we find that $ar = as$.
- Dividing both sides of this last equation by a, which is nonzero, we see that $r = s$.
- This means that if $s \neq r$, then $as + b \neq 0$.
- This establishes the uniqueness part of the proof.

Mistakes in Proof

There are many common errors made in constructing mathematical proofs. Among the most common errors are mistakes in arithmetic and basic algebra. Even professional mathematicians make such errors, especially when working with complicated formulae.

EXAMPLE: What is wrong with this famous supposed "proof" that 1 = 2?

"Proof:" We use these steps, where a and b are two equal positive integers.

Step	**Reason**
1. $a = b$	Given
2. $a^2 = ab$	Multiply both sides of (1) by a
3. $a^2 - b^2 = ab - b^2$	Subtract b^2 from both sides of (2)
4. $(a - b)(a + b) = b(a - b)$	Factor both sides of (3)
5. $a + b = b$	Divide both sides of (4) by $a - b$
6. $2b = b$	Replace a by b in (5) because $a = b$ and simplify
7. $2 = 1$	Divide both sides of (6) by b

Solution: Every step is valid except for one, step 5 where we divided both sides by a − b. The error is that a − b equals zero; division of both sides of an equation by the same quantity is valid as long as this quantity is not zero.

Assignment

1. State division and remainder algorithm. Suppose that the domain of the propositional function P(x) consists of the integer 0, 1, 2, 3 and 4. Write out each of the following propositions using disjunctions, conjunctions and negations.
   a. ∃x P(x)
   b. ∀x P(x)
   c. ∃x ¬P(x)
   d. ∀x ¬P(x)
   e. ¬∃x P(x)
   f. ¬∀x P(x)
2. Represent the argument "If it does not rain or if is not foggy then the sailing race will be held and the lifesaving demonstration will go on. If sailing race is held then trophy

will be awarded. The trophy was not awarded. Therefore it not rained" in propositional logic and prove the conclusion by using rules of inferences. {2+3}

3. Let A = "Aldo is Italian" and B = "Bob is English". Formalize the following sentences in proposition.
    a. Aldo isn't Italian.
    b. Aldo is Italian while Bob is English.
    c. If Aldo is Italian then Bob Bob is not English.
    d. Aldo is Italian or if Aldo isn't Italian then Bob is English.
    e. Either Aldo is Italian and Bob is English, or neither Aldo is Italian nor Bob is English.
4. Discuss common mistakes in proof briefly. Show that n is even if n^3+5 is odd by using indirect proof. {2+3}
5. Define preposition. Consider the argument "John, a student in this class knows how to write program in C. Everyone who knows how to write program in C can get a high paying job. Therefore, someone in this class can get high paying job". Now, explain which rules of inferences are set for each step.
6. Prove that the product xy is odd if and only if both x and y are odd integers.
7. All over smart people are stupid. Children of stupid people are naughty. John is a children of Jane. Jane is over smart. Represent these statements in FOPL and prove that John is naughty.
8. Prove that if n is positive integer, then n is odd if and only if 5n + 6 is odd.