Unit 1
Basic Discrete Structures

1.1 Sets

Sets

- A set is an unordered collection of objects, called elements or members of the set.
- A set is said to contain its elements.
- We write $a \in A$ to denote that a is an element of the set A.
- The notation $a \notin A$ denotes that a is not an element of the set A.

EXAMPLE 1: The set V of all vowels in the English alphabet can be written as V = {a, e, i, o, u}.

EXAMPLE 2: The set O of odd positive integers less than 10 can be expressed by O = {1, 3, 5, 7, 9}.

Set Builder Notation of Sets

Set can be describe by using **set builder notation**.

EXAMPLE: The set O of all odd positive integers less than 10 can be written as:

O = {x | x is an odd positive integer less than 10},

OR

$O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}$.

Venn Diagram

Sets can be represented graphically using Venn diagrams.

EXAMPLE 1: Draw a Venn diagram that represents V, the set of vowels in the English alphabet.

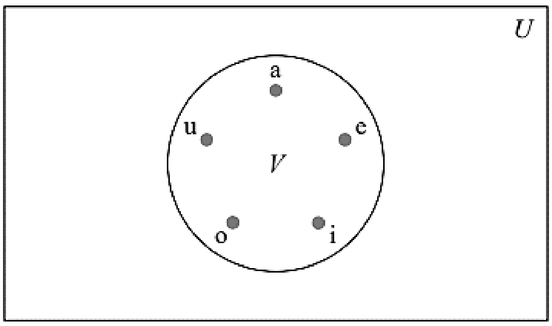


Fig: Venn Diagram for the Set of Vowels.

Subsets

- The set A is a subset of B if and only if every element of A is also an element of B.
- We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

- A set A is a subset of a set B but that $A \neq B$, we write $A \subset B$ and say that A is a *proper subset* of B.

The Size of a Set

- Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the cardinality of S.
- The cardinality of S is denoted by $|S|$.

EXAMPLE 1: Let A be the set of odd positive integers less than 10. Then $|A| = 5$.

Power Sets

- Given a set S, the power set of S is the set of all subsets of the set S.
- The power set of S is denoted by P(S).

EXAMPLE 1: What is the power set of the set {0, 1, 2}?

Solution:

The power set P({0, 1, 2}) is the set of all subsets of {0, 1, 2}.

Hence, $P(\{0, 1, 2\}) = \{\emptyset,\{0\},\{1\},\{2\},\{0, 1\},\{0, 2\},\{1, 2\},\{0, 1, 2\}\}$.

Cartesian Products

- Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

EXAMPLE 1: What is the Cartesian product of A = {1, 2} and B = {a, b, c}? Solution: The Cartesian product $A \times B$ is $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$.

- Note that the Cartesian products $A \times B$ and $B \times A$ are not equal
- The *Cartesian product* of the sets A1, A2,...,An, denoted by $A1 \times A2 \times\cdots\times An$, is the set of ordered n-tuples (a1, a2,...,an), where ai belongs to Ai for i = 1, 2,...,n. In other words,

$$A1 \times A2 \times\cdots\times An = \{(a_1, a_2,...,a_n) \mid a_i \in Ai \text{ for } i = 1, 2,...,n\}.$$

EXAMPLE 2: What is the Cartesian product $A \times B \times C$, where A = {0, 1}, B = {1, 2}, and C = {0, 1, 2}?

Solution: The Cartesian product $A \times B \times C$ consists of all ordered triples (a, b, c), where $a \in A$, $b \in B$, and $c \in C$. Hence,

$A \times B \times C = \{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1),(1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}$.

- The notation A^2 to denote A × A, the Cartesian product of the set A with itself. Similarly, A^3 = A × A × A, A^4 = A × A × A × A, and so on. More generally,

$$A^n = \{(a_1, a_2,...,a_n) \mid a_i \in A \text{ for } i = 1, 2,...,n\}$$

EXAMPLE 3: Suppose that A = {1, 2}. It follows that A2 = {(1, 1), (1, 2), (2, 1), (2, 2)} and A3 = {(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)}.

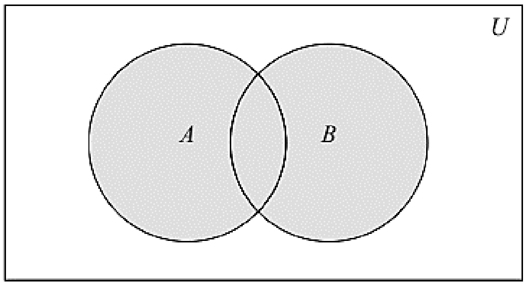
Set Operations

Union

- Let A and B be sets. The union of the sets A and B, denoted by A ∪ B, is the set that contains those elements that are either in A or in B, or in both.

$$A \cup B = \{x \mid x \in A \lor x \in B\}.$$

- The Venn diagram shown in figure below represents the union of two sets A and B.



$A \cup B$ is shaded.

Fig: Venn Diagram of the Union of A and B.

EXAMPLE 1: The union of the sets {1, 3, 5} and {1, 2, 3} is the set {1, 2, 3, 5}; that is,

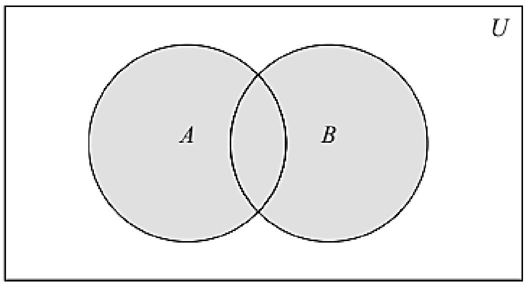
{1, 3, 5} ∪ {1, 2, 3}={1, 2, 3, 5}.

Intersection

- Let A and B be sets. The intersection of the sets A and B, denoted by A ∩ B, is the set containing those elements in both A and B.

$$A \cap B = \{x \mid x \in A \land x \in B\}.$$

- Two sets are called **disjoint** if their intersection is the empty set.
- The Venn diagram shown in figure below represents the intersection of two sets A and B.



$A \cup B$ is shaded.

Fig: Venn Diagram of the Intersection of A and B.

EXAMPLE 1: The intersection of the sets {1, 3, 5} and {1, 2, 3} is the set {1, 3}; that is,

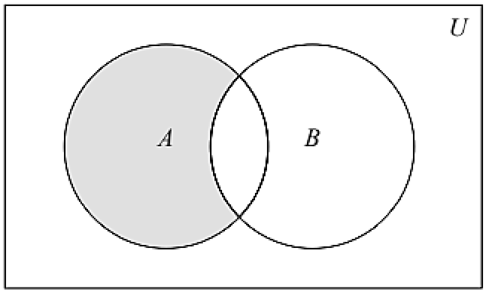
$$\{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}.$$

Difference

- Let A and B be sets. The difference of A and B, denoted by A − B, is the set containing those elements that are in A but not in B.
- The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \land x \notin B\}.$$

- The Venn diagram shown in Figure below represents the difference of the sets A and B.



$A - B$ is shaded.

Fig: Venn Diagram for the Difference of A and B.

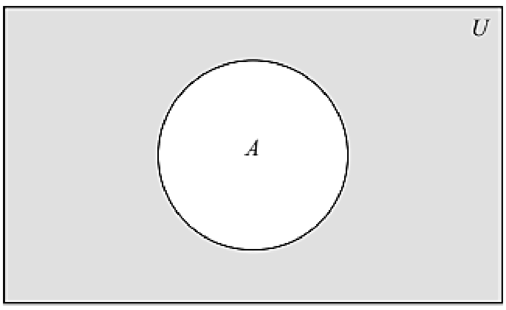
EXAMPLE 1: The difference of {1, 3, 5} and {1, 2, 3} is the set {5}; that is, {1, 3, 5} − {1, 2, 3}={5}. This is different from the difference of {1, 2, 3} and {1, 3, 5}, which is the set {2}.

Complement

- Let U be the universal set. The complement of the set A, denoted by $\bar{A}$, is the complement of A with respect to U.
- Therefore, the complement of the set A is U − A.

$$\bar{A} = \{x \in U \mid x \notin A\}.$$

- In figure below, the shaded area outside the circle representing A is the area representing $\bar{A}$



$\bar{A}$ is shaded.

Fig: Venn Diagram for the Complement of the Set A.

EXAMPLE 1: Let A = {a, e, i, o, u} (where the universal set is the set of letters of the English alphabet). Then

$$\bar{A} = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}.$$

Generalized Unions and Intersections

- The **union** of a collection of sets is the set that contains those elements that are members of at least one set in the collection.
- We use the notation $A1 \cup A2 \cup \cdots \cup An = \bigcup_{i=1}^{n} \mathrm{Ai}$ to denote the union of the sets A1, A2,...,An.
- The **intersection** of a collection of sets is the set that contains those elements that are members of all the sets in the collection.
- We use the notation $A1 \cap A2 \cap \cdots \cap An = \bigcap_{i=1}^{n} \mathrm{Ai}$ to denote the intersection of the sets A1, A2,...,An.

EXAMPLE 1: Let A = {0, 2, 4, 6, 8}, B = {0, 1, 2, 3, 4}, and C = {0, 3, 6, 9}. What are A ∪ B ∪ C and A ∩ B ∩ C?

Solution: The set A ∪ B ∪ C contains those elements in at least one of A, B, and C. Hence, A ∪ B ∪ C = {0, 1, 2, 3, 4, 6, 8, 9}. The set A ∩ B ∩ C contains those elements in all three of A, B, and C. Thus, A ∩ B ∩ C = {0}.

Inclusion-Exclusion Principle

- The number of elements in the union of the two sets A and B is the sum of the numbers of elements in the sets minus the number of elements in their intersection.
- That is,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Computer Representation of Sets

- There are various ways to represent sets using a computer. One method is to store the elements of the set in an unordered fashion. However, if this is done, the operations of computing the union, intersection, or difference of two sets would be time-consuming, because each of these operations would require a large amount of searching for elements.
- Another method of representing is storing elements using an arbitrary ordering of the elements of the universal set. This method of representing sets makes computing combinations of sets easy.
- Assume that the universal set U is finite (and of reasonable size so that the number of elements of U is not larger than the memory size of the computer being used). First, specify an arbitrary ordering of the elements of U, for instance $a_1, a_2, \ldots, a_n$. Represent a subset A of U with the bit string of length n, where the i^{th} bit in this string is 1 if a_i belongs to A and is 0 if a_i does not belong to A.
- To obtain the bit string for the union and intersection of two sets we perform bitwise Boolean operations on the bit strings representing the two sets.

EXAMPLE 1: Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$. What bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U?

Solution: The bit string that represents the set of odd integers in U, namely, {1, 3, 5, 7, 9}, has a one bit in the first, third, fifth, seventh, and ninth positions, and a zero elsewhere. It is

10 1010 1010.

Similarly, we represent the subset of all even integers in U, namely, {2, 4, 6, 8, 10}, by the string

01 0101 0101.

The set of all integers in U that do not exceed 5, namely, {1, 2, 3, 4, 5}, is represented by the string

11 1110 0000.

EXAMPLE 2: The bit strings for the sets {1, 2, 3, 4, 5} and {1, 3, 5, 7, 9} are 11 1110 0000 and 10 1010 1010, respectively. Use bit strings to find the union and intersection of these sets. Solution: The bit string for the union of these sets is

$$11\ 1110\ 0000 \lor 10\ 1010\ 1010 = 11\ 1110\ 1010,$$

which corresponds to the set {1, 2, 3, 4, 5, 7, 9}. The bit string for the intersection of these sets is

$$11\ 1110\ 0000 \land 10\ 1010\ 1010 = 10\ 1010\ 0000,$$

which corresponds to the set {1, 3, 5}.

1.2 Functions

Definition of Function

- Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A.
- We write f (a) = b if b is the unique element of B assigned by the function f to the element a of A.
- If f is a function from A to B, we write $f : A \rightarrow B$.
- Functions are sometimes also called mappings or transformations.
- If f is a function from A to B, we say that A is the *domain* of f and B is the *codomain* of f.
- If f (a) = b, we say that b is the *image* of a and a is a *preimage* of b.
- The *range, or image*, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f maps A to B.
- Figure below represents a function f from A to B.

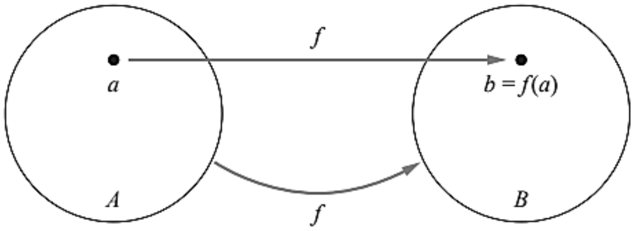


Fig: The Function f Maps A to B.

EXAMPLE 1: What are the domain, codomain, and range of the function that assigns grades to students described in the first paragraph of the introduction of this section?

Solution:

- Let G be the function that assigns a grade to a student in our discrete mathematics class.
- Note that G(Adams) = A, for instance.
- The domain of G is the set {Adams, Chou, Goodfriend, Rodriguez, Stevens}, and the codomain is the set {A, B, C, D, F}.
- The range of G is the set {A, B, C, F}, because each grade except D is assigned to some student.

Equal Functions

Two functions are equal when they have the same domain, have the same codomain, and map each element of their common domain to the same element in their common codomain.

Real and Integer Valued Functions

- A function is called real-valued if its codomain is the set of real numbers, and it is called integer-valued if its codomain is the set of integers.
- Two real-valued functions or two integer-valued functions with the same domain can be added, as well as multiplied.

Subset Rule of Image of the Function

- Let f be a function from A to B and let S be a subset of A. The image of S under the function f is the subset of B that consists of the images of the elements of S.
- We denote the image of S by f (S), so f (S) = {t | ∃s ∈S (t = f (s))}.
- We also use the shorthand {f (s) | s ∈ S} to denote this set.

EXAMPLE 1: Let A = {a, b, c, d, e} and B = {1, 2, 3, 4} with f (a) = 2, f (b) = 1, f (c) = 4, f (d) = 1, and f (e) = 1. The image of the subset S = {b, c, d} is the set f (S) = {1, 4}.

One-to-One and Onto Functions

- A function *f* is said to be *one-to-one,* or an *injunction,* if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of *f.*
- A function is said to be ***injective*** if it is one-to-one.
- Note that a function *f* is one-to-one if and only if $f(a) \neq f(b)$ whenever $a \neq b$.

EXAMPLE 1: Determine whether the function *f* from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one.

Solution: The function *f* is one-to-one because *f* takes on different values at the four elements of its domain. This is illustrated in figure below.

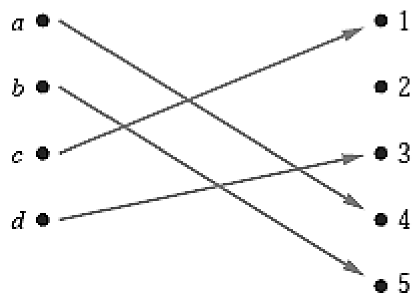


Fig: A One-to-One Function.

EXAMPLE 2: Determine whether the function f (x) = x^2 from the set of integers to the set of integers is one-to-one.

Solution: The function f (x) = x^2 is not one-to-one because, for instance, f (1) = f (−1) = 1, but 1 = −1.

EXAMPLE 3: Determine whether the function f (x) = x + 1 from the set of real numbers to itself is one-to-one.

Solution: The function f (x) = x + 1 is a one-to-one function. To demonstrate this, note that $x + 1 \neq y + 1$ when $x \neq y$.

Onto Functions

- For some functions the range and the codomain are equal. That is, every member of the codomain is the image of some element of the domain.
- Functions with this property are called **onto functions**.
- A function f from A to B is called onto, or a **surjection**, if and only if for every element $b \in B$ there is an element $a \in A$ with f (a) = b.
- A function f is called **surjective** if it is onto.

EXAMPLE 1: Let f be the function from {a, b, c, d} to {1, 2, 3} defined by f (a) = 3, f (b) = 2, f (c) = 1, and f (d) = 3. Is f an onto function?

Solution: Because all three elements of the codomain are images of elements in the domain, we see that f is onto. This is illustrated in Figure below. Note that if the codomain were {1, 2, 3, 4}, then f would not be onto.

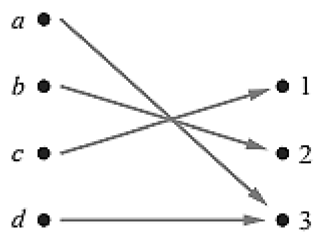


Fig: An Onto Function.

Bijection Function

- The function f is a **one-to-one correspondence**, or a **bijection**, if it is both one-to-one and onto. We also say that such a function is **bijective**.

EXAMPLE 1: Let f be the function from {a, b, c, d} to {1, 2, 3, 4} with f (a) = 4, f (b) = 2, f (c) = 1, and f (d) = 3. Is f a bijection?

Solution: The function f is one-to-one and onto. It is one-to-one because no two values in the domain are assigned the same function value. It is onto because all four elements of the codomain are images of elements in the domain. Hence, f is a bijection.

Increasing and Decreasing Functions

- A function f whose domain and codomain are subsets of the set of real numbers is called increasing if $f(x) \le f(y)$, and strictly increasing if $f(x) < f(y)$, whenever $x<y$ and x and y are in the domain of f.
- Similarly, f is called decreasing if $f(x) \ge f(y)$, and strictly decreasing if $f(x) > f(y)$, whenever $x < y$ and x and y are in the domain of f.
- A function that is either strictly increasing or strictly decreasing must be one-to-one.
- However, a function that is increasing, but not strictly increasing, or decreasing, but not strictly decreasing, is not one-to-one.

Inverse Functions

- Let f be a one-to-one correspondence from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that f (a) = b.
- The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when f (a) = b.

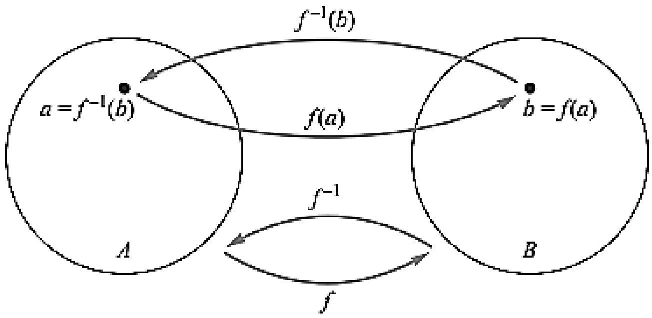


Fig: The Function f^{-1} Is the Inverse of Function f.

- A one-to-one correspondence is called **invertible** because we can define an inverse of this function.
- A function is **not invertible** if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

EXAMPLE 1: Let f be the function from {a, b, c} to {1, 2, 3} such that f (a) = 2, f (b) = 3, and f (c) = 1. Is f invertible, and if it is, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f −1 reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

EXAMPLE 2: Let f : **Z** → **Z** be such that f (x) = x + 1. Is f invertible, and if it is, what is its inverse?

Solution: The function f has an inverse because it is a one-to-one correspondence. To reverse the correspondence, suppose that y is the image of x, so that y = x + 1. Then x = y − 1. This means that y − 1 is the unique element of Z that is sent to y by f. Consequently, $f^{-1}(y) = y - 1$.

EXAMPLE 3: Let f be the function from R to R with f (x) = x^2. Is f invertible?

Solution: Because f (−2) = f (2) = 4, f is not one-to-one. If an inverse function were defined, it would have to assign two elements to 4. Hence, f is not invertible. (Note we can also show that f is not invertible because it is not onto.)

Compositions of Functions

- Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g, denoted for all a ∈ A by f ∘ g, is defined by

$$(f \circ g)(a) = f(g(a)).$$

- In other words, f ∘ g is the function that assigns to the element a of A the element assigned by f to g(a).
- That is, to find (f ∘ g)(a) we first apply the function g to a to obtain g(a) and then we apply the function f to the result g(a) to obtain (f ∘ g)(a) = f (g(a)).
- In Figure below the composition of functions is shown.

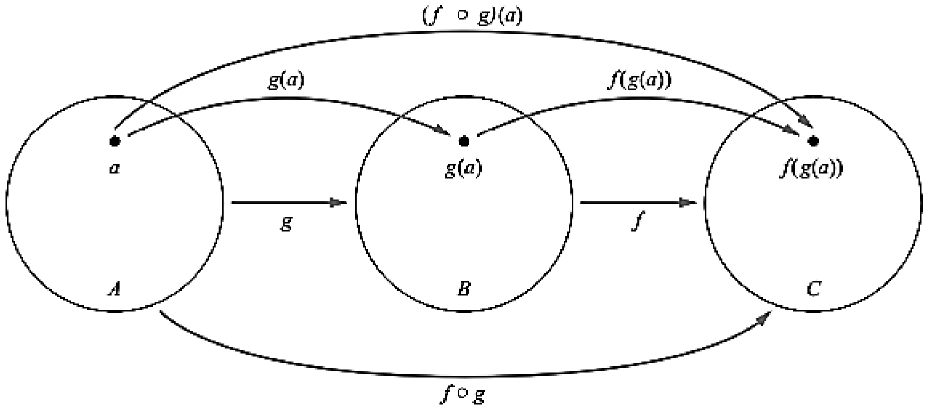


Fig: The Composition of the Functions f and g.

EXAMPLE 1: Let f and g be the functions from the set of integers to the set of integers defined by f (x) = 2x + 3 and g(x) = 3x + 2. What is the composition of f and g? What is the composition of g and f?

Solution: Both the compositions f ∘ g and g ∘ f are defined. Moreover,

$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$ and

$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$

The Graphs of Functions

Let f be a function from the set A to the set B. The graph of the function f is the set of ordered pairs {(a, b) | a ∈ A and f (a) = b}.

EXAMPLE 1: Display the graph of the function f (x) = x^2 from the set of integers to the set of integers.

Solution: The graph of f is the set of ordered pairs of the form (x, f (x)) = (x, x^2), where x is an integer.

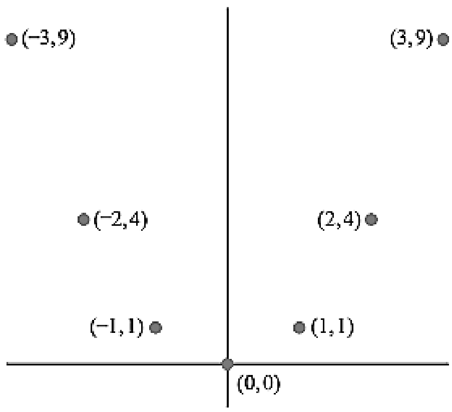


*Fig: The Graph of f (x) = x^2 from **Z** to **Z**.*

Floor and Ceiling Functions

- The floor function assigns to the real number x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$.
- The ceiling function assigns to the real number x the smallest integer that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$.

EXAMPLE 1: These are some values of the floor and ceiling functions:

a. $\lfloor 1/2 \rfloor = 0$ and $\lceil 1/2 \rceil = 1$
b. $\lfloor -1/2 \rfloor = -1$ and $\lceil -1/2 \rceil = 0$
c. $\lfloor 3.1 \rfloor = 3$ and $\lceil 3.1 \rceil = 4$
d. $\lfloor 7 \rfloor = 7$ and $\lceil 7 \rceil = 7$

Boolean Function

- Boolean algebra deals with binary variables and logic operation.
- A Boolean Function is described by an algebraic expression called Boolean expression which consists of binary variables, the constants 0 and 1, and the logic operation symbols.

EXAMPLE 1:

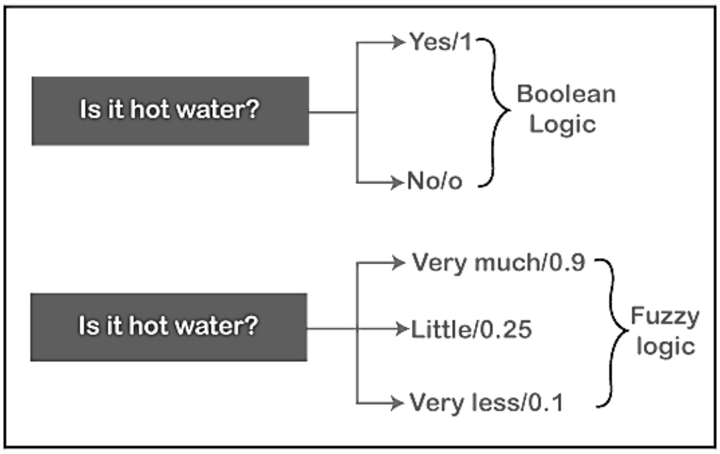
$$\underbrace{F(A, B, C, D)}_{\text{Boolean Function}} = \underbrace{A + \overline{BC} + ADC}_{\text{Boolean Expression}}$$

Exponential Function

A relation of the form $y = a^x$, with the independent variable x ranging over the entire real number line as the exponent of a positive number a.

Fuzzy Sets

- Fuzzy set is a set having degrees of membership between 1 and 0.
- Fuzzy sets are represented with tilde character(~).
- Partial membership exists when member of one fuzzy set can also be a part of other fuzzy sets in the same universe.



Membership Function

- For any set X, a membership function on X is any function from X to the real unit interval [0,1].
- The membership function which represents a fuzzy set A is usually denoted by μ_A.
- For $\mu_A(x)$ is called membership degree of x in the fuzzy set A.
- The membership degree $\mu_A(x)$ quantifies the grade of membership of the element x to the fuzzy set A.
- The value 0 means that x is not a member of the fuzzy set, the value 1 means the x is fully a member of the fuzzy set.
- The values between 0 and 1 characterize fuzzy members, which belongs to the fuzzy set only partially.

Fuzzy Sets Operations

1. *Union Operation*

- The membership function of the intersection of two fuzzy sets A and B with membership function μ_A and μ_B respectively is defined as the maximum of the two individuals' membership functions.
- This is called maximum criteria.

$$\mu_{A \cup B}(x) = max\ (\mu_A(x), \mu_B(x))$$

EXAMPLE:

- Let's suppose A is a set which contains following elements:

A = {(X1, 0.6), (X2, 0.2), (X3, 1), (X4, 0.4)}

- And, B is a set which contains following elements:

B = {(X1, 0.1), (X2, 0.8), (X3, 0), (X4, 0.9)}

- then,

AUB = {(X1, 0.6), (X2, 0.8), (X3, 1), (X4, 0.9)}

2. *Intersection Operation*

- The membership function of the intersection of two fuzzy sets A and B with membership function μ_A and μ_B respectively is defined as the minimum of the two individuals' membership functions.
- This is called minimum criteria.

$$\mu_{A \cap B}(x) = min\ (\mu_A(x), \mu_B(x))$$

EXAMPLE:

- Let's suppose A is a set which contains following elements:

A = {(X1, 0.3), (X2, 0.7), (X3, 0.5), (X4, 0.1)}

- And, B is a set which contains following elements:

B = {(X1, 0.8), (X2, 0.2), (X3, 0.4), (X4, 0.9)}

- then,

A∩B = {(X1, 0.3), (X2, 0.2), (X3, 0.4), (X4, 0.1)}

3. *Difference Operation*

- The membership function of the complement of a fuzzy set A with membership function μ_A is defined as the negation of the specified membership function.
- This is called the negation criterion.

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x)$$

EXAMPLE:

- Let's suppose A is a set which contains following elements:

A = {(X1, 0.3), (X2, 0.8), (X3, 0.5), (X4, 0.1)}

- then,

Ā= {(X1, 0.7), (X2, 0.2), (X3, 0.5), (X4, 0.9)}

1.3 Sequences and Summations

- Sequences are ordered lists of elements, used in discrete mathematics in many ways.
- For example, they can be used to represent solutions to certain counting problems.
- The terms of a sequence can be specified by providing a formula for each term of the sequence.

Sequences

- A *sequence* is a discrete structure used to represent an ordered list.
- For example, 1, 2, 3, 5, 8 is a sequence with five terms and 1, 3, 9, 27, 81,... , 3^n,... is an infinite sequence.

- A *sequence* is a function from a subset of the set of integers (usually either the set {0, 1, 2,...} or the set {1, 2, 3,...}) to a set S.
- We use the notation a_n to denote the image of the integer n. We call a_n a term of the sequence.
- We use the notation $\{a_n\}$ to describe the sequence. (Note that a_n represents an individual term of the sequence $\{a_n\}$.

EXAMPLE: Consider the sequence $\{a_n\}$,

where $a_n = 1 / n$.

The list of the terms of this sequence, beginning with a_1, namely,

$a_1, a_2, a_3, a_4,...,$

starts with

1, 1/2 , 1/3 , 1/4 ,....

Geometric Progression

- A *geometric progression* is a sequence of the form

$a, ar, ar^2,... , ar^n,...$

where the *initial term a* and the *common ratio r* are real numbers.

- A geometric progression is a discrete analogue of the exponential function $f(x) = ar^x$.

EXAMPLE: The sequences $\{b_n\}$ with $b_n = (-1)^n$, $\{c_n\}$ with $c_n = 2 \cdot 5^n$, and $\{d_n\}$ with $d_n = 6 \cdot (1/3)^n$ are geometric progressions with initial term and common ratio equal to 1 and −1; 2 and 5; and 6 and 1/3, respectively, if we start at n = 0. The list of terms $b_0, b_1, b_2, b_3, b_4,...$ begins with

1, −1, 1, −1, 1,...;

the list of terms $c_0, c_1, c_2, c_3, c_4,...$ begins with

2, 10, 50, 250, 1250,...;

and the list of terms $d_0, d_1, d_2, d_3, d_4,...$ begins with

$$6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots.$$

Arithmetic Progression

- An arithmetic progression is a sequence of the form

$a, a + d, a + 2d,... ,a + nd, . . .$

where the initial term a and the common difference d are *real numbers*.

- An arithmetic progression is a discrete analogue of the linear function $f(x) = dx + a$.

EXAMPLE: The sequences $\{s_n\}$ with $s_n = -1 + 4n$ and $\{t_n\}$ with $t_n = 7 - 3n$ are both arithmetic progressions with initial terms and common differences equal to −1 and 4, and 7 and −3, respectively, if we start at n = 0. The list of terms $s_0, s_1, s_2, s_3,...$ begins with −1, 3, 7, 11,..., and the list of terms $t_0, t_1, t_2, t_3,...$ begins with 7, 4, 1, −2,....

Strings

- Sequences of the form $a_1, a_2,...,a_n$ are often used in computer science. These finite sequences are also called **strings**. This string is also denoted by $a_1a_2 ...a_n$. (Recall that bit strings, which are finite sequences of bits.)
- The **length** of a string is the number of terms in this string.
- The **empty string**, denoted by λ, is the string that has no terms. The empty string has length zero.

EXAMPLE: The string abcd is a string of length four.

Summations

- The notation used to express the sum of the terms

  $a_m, a_{m+1},...,a_n$

  from the sequence $\{a_n\}$ is:

$$\sum_{j=m}^{n} a_j, \qquad \textstyle\sum_{j=m}^{n} a_j, \qquad \text{or} \qquad \textstyle\sum_{m\le j\le n} a_j$$

- Read as the sum from j = m to j = n of a_j.
- Here, the variable j is called the **index of summation.**
- Here, the index of summation runs through all integers starting with its lower limit m and ending with its upper limit n.
- A large uppercase Greek letter sigma, Σ, is used to denote summation.
- The usual laws for **arithmetic** apply to summations.
- For example, when a and b are real numbers, we have

$$\sum_{j=1}^{n} (ax_j + by_j) = a\sum_{j=1}^{n} x_j + b\sum_{j=1}^{n} y_j$$

  where $x_1, x_2,...,x_n$ and $y_1, y_2,...,y_n$ are real numbers.

EXAMPLE: Use summation notation to express the sum of the first 100 terms of the sequence $\{a_j\}$, where $a_j = 1/j$ for j = 1, 2, 3,....

Solution: The lower limit for the index of summation is 1, and the upper limit is 100. We write this sum as:

$$\sum_{j=1}^{100} \frac{1}{j}$$

EXAMPLE: What is the value of $\sum_{j=1}^{5} j^2$?

Solution: We have

$$\sum_{j=1}^{5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

= 1 + 4 + 9 + 16 + 25

= 55

Double summations

Double summations arise in many contexts (as in the analysis of nested loops in computer programs).

EXAMPLE: An example of a double summation is

$$\sum_{i=1}^{4} \sum_{j=1}^{3} ij$$

To evaluate the double sum, first expand the inner summation and then continue by computing the outer summation:

$$\sum_{i=1}^{4} \sum_{j=1}^{3} ij = \sum_{i=1}^{4} (i + 2i + 3i)$$

$$= \sum_{i=1}^{4} 6i$$

$$= 6 + 12 + 18 + 24 = 60$$

Summation and Functions

- We can also use summation notation to add all values of a function, or terms of an indexed set, where the index of summation runs over all values in a set.
- That is, we write

$$\sum_{s \in S} f(s)$$

to represent the sum of the values f(s), for all members s of S.

EXAMPLE: What is the value of $\sum_{S \in \{0,2,4\}} S$?

Solution: Because $\sum_{S\in\{0,2,4\}} S$ represents the sum of the values of s for all the members of the set {0, 2, 4}, it follows that

$$\sum_{S\in\{0,2,4\}} S = 0 + 2 + 4 = 6$$

Exercise

1. Consider a set U = {1,2,3,4,5,6,7,8,9,10}. What will be the computer representation for set containing the numbers which are multiple of 3 not exceeding 6?
2. Prove that $\overline{A \cap B} = \bar{A} \cup \bar{B}$ by using set builder notation.
3. How sets are represented by using bit string? Why it is preferred over unordered representation of sets?
4. How can you relate domain and co-domain of functions with functions in programming language? Discuss composite and inverse of function with suitable examples.
5. Describe injective, surjective and bijective functions with suitable examples.