Unit 5
Counting and Discrete Probability

5.1 Counting

Basic Counting Principles

THE PRODUCT RULE:

- Suppose that a procedure can be broken down into a sequence of two tasks.
- If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are n_1n_2 ways to do the procedure.

EXAMPLE: A new company with just two employees, Sharma and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Solution:

- The procedure of assigning offices to these two employees consists of assigning an office to Sharma, which can be done in 12 ways, then assigning an office to Patel different from the office assigned to Sharma, which can be done in 11 ways.
- By the product rule, there are $12 \cdot 11 = 132$ ways to assign offices to these two employees.

THE SUM RULE:

If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

EXAMPLE: Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?

Solution:

- There are 37 ways to choose a member of the mathematics faculty and there are 83 ways to choose a student who is a mathematics major.
- Choosing a member of the mathematics faculty is never the same as choosing a student who is a mathematics major because no one is both a faculty member and a student.
- By the sum rule it follows that there are $37 + 83 = 120$ possible ways to pick this representative.

The Subtraction Rule (Inclusion–Exclusion for Two Sets)

- If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

- The subtraction rule is also known as the principle of *inclusion–exclusion*, especially when it is used to count the number of elements in the union of two sets.

EXAMPLE: How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution:

- We can construct a bit string of length eight that either starts with a 1 bit or ends with the two bits 00, by constructing a bit string of length eight beginning with a 1 bit or by constructing a bit string of length eight that ends with the two bits 00.
- String of length eight that begins with a 1 in $2^7 = 128$ ways.
- String of length eight that ends with a 1 in $2^6 = 64$ ways.
- String of length eight that begins with a 1 and ends with 00 in $2^5 = 32$ ways.
- Consequently, the number of bit strings of length eight that begin with a 1 or end with a 00, which equals the number of ways to construct a bit string of length eight that begins with a 1 or that ends with 00, equals $128 + 64 - 32 = 160$.

The Pigeonhole Principle

If k is a positive integer and k + 1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Proof:

- We prove the pigeonhole principle using a proof by contraposition.
- Suppose that none of the k boxes contains more than one object.
- Then the total number of objects would be at most k.
- This is a contradiction, because there are at least k + 1 objects.

EXAMPLE: Among any group of 366 people, there must be at least two with the same birthday, because there are only 365 possible birthdays.

COROLLARY:

A function f from a set with k + 1 or more elements to a set with k elements is not one-to-one.

Proof:

- Suppose that for each element y in the codomain of f we have a box that contains all elements x of the domain of f such that f (x) = y.
- Because the domain contains k + 1 or more elements and the codomain contains only k elements, the pigeonhole principle tells us that one of these boxes contains two or more elements x of the domain.
- This means that f cannot be one-to-one.

The Generalized Pigeonhole Principle

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Proof:

- We will use a proof by contraposition.
- Suppose that none of the boxes contains more than $\lceil N/k \rceil - 1$ objects.
- Then, the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

where the inequality $\lceil N/k \rceil < (N/k) + 1$ has been used.

- This is a contradiction because there are a total of N objects.

EXAMPLE: Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

Permutations and Combinations

Permutations with Repetition

- A *permutation* of a set of distinct objects is an ordered arrangement of these objects.
- An ordered arrangement of r elements of a set is called an *r-permutation*.

a. Permutations without Repetition

$$P(n, r) = \frac{n!}{(n-r)!}$$

EXAMPLE 1: In how many ways can we select three students from a group of five students to stand in line for a picture? In how many ways can we arrange all five of these students in a line for a picture?

Solution:

Here,

Case I:

$$n = 5$$

$$r = 3$$

$$P(n, r) = \frac{n!}{(n-r)!}$$

$$P(5,3) = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 60$$

Case II:

$$n = 5$$

$$r = 3$$

$$P(n,r) = \frac{n!}{(n-r)!}$$

$$P(5,3) = \frac{5!}{(5-5)!} = \frac{5!}{0!} = 120$$

EXAMPLE 2: How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution:

$$n = 100$$

$$r = 5$$

$$P(n,r) = \frac{n!}{(n-r)!}$$

$$P(100,5) = \frac{100!}{(100-3)!} = 100.99.98 = 970200$$

b. ***Permutations with Repetition***

- Counting permutations when repetition of elements is allowed can easily be done using the product rule.
- The number of r-permutations of a set of n objects with repetition allowed is:

$$C(n,r) = n^r$$

EXAMPLE: How many strings of length 5 can be formed from the uppercase letters of the English alphabet?

Solution:

- By the product rule, because there are 26 uppercase English letters, and because each letter can be used repeatedly.
- Here,

$$n = 26$$

$$r = 5$$

$$C(n,r) = n^r$$

$C(26,5) = 26^5 = 11881376$

Combinations

- An r-combination of elements of a set is an unordered selection of r elements from the set.
- Thus, an r-combination is simply a subset of the set with r elements.
- The number of r-combinations of a set with n distinct elements is denoted by C(n, r).
- Note that C(n, r) is also denoted by $\binom{n}{r}$and is called a *binomial coefficient*.

a. *Combination without Repetition*

$$C(n,r) = \frac{n!}{r!\,(n-r)!}$$

EXAMPLE 1: How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?

Solution:

- Because the order in which the five cards are dealt from a deck of 52 cards does not matters.
- So, we will calculate combination

Case I:

$$n = 52$$
$$r = 5$$

$$C(n,r) = \frac{n!}{r!\,(n-r)!}$$

$$C(52,5) = \frac{52!}{5!\,(52-5)!} = \frac{52.51.50.49.48}{5.4.3.2.1} = \ 2598960$$

Case II:

$$n = 52$$
$$r = 47$$

$$C(n,r) = \frac{n!}{r!\,(n-r)!}$$

$$C(52,47) = \frac{52!}{47!\,(52-47)!} = \frac{52.51.50.49.48}{5.4.3.2.1} = \ 2598960$$

b. Combination with Repetition

- There are $C(n + r - 1, r) = C(n + r - 1, n - 1)$ r-combinations from a set with n elements when repetition of elements is allowed.

EXAMPLE: Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen? Assume that only the type of cookie, and not the individual cookies or the order in which they are chosen, matters.

Solution:

- The number of ways to choose six cookies is the number of 6-combinations of a set with four elements.
- We know that,

$$C(4 + 6 - 1, 6) = C(9, 6)$$

$$C(9, 6) = C(9, 3) = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84$$

- So, there are 84 different ways to choose the six cookies.

Binomial Coefficients

- The number of r-combinations from a set with n elements is often denoted by $\binom{n}{r}$.
- This number is also called a binomial coefficient because these numbers occur as coefficients in the expansion of powers of binomial expressions such as $(a + b)^n$.

The Binomial Theorem

Let x and y be variables, and let n be a nonnegative integer. Then

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

EXAMPLE: What is the expansion of $(x + y)^4$?

Solution: From the binomial theorem it follows that

$$\begin{aligned}(x + y)^4 &= \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j \\ &= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4 \\ &= x^4 + 4x^3 y + 6x^2 y^2 + 4x y^3 + y^4.\end{aligned}$$

Pascal's Identity and Triangle

- Let n and k be positive integers with n ≥ k. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

- Pascal's identity is the basis for a geometric arrangement of the binomial coefficients in a triangle.
- The n^{th} row in the triangle consists of the binomial coefficients

$$\binom{n}{k}, k = 0,1,\dots,n$$

- This triangle is known as Pascal's triangle.
- Pascal's identity shows that when two adjacent binomial coefficients in this triangle are added, the binomial coefficient in the next row between these two coefficients is produced.

EXAMPLE:

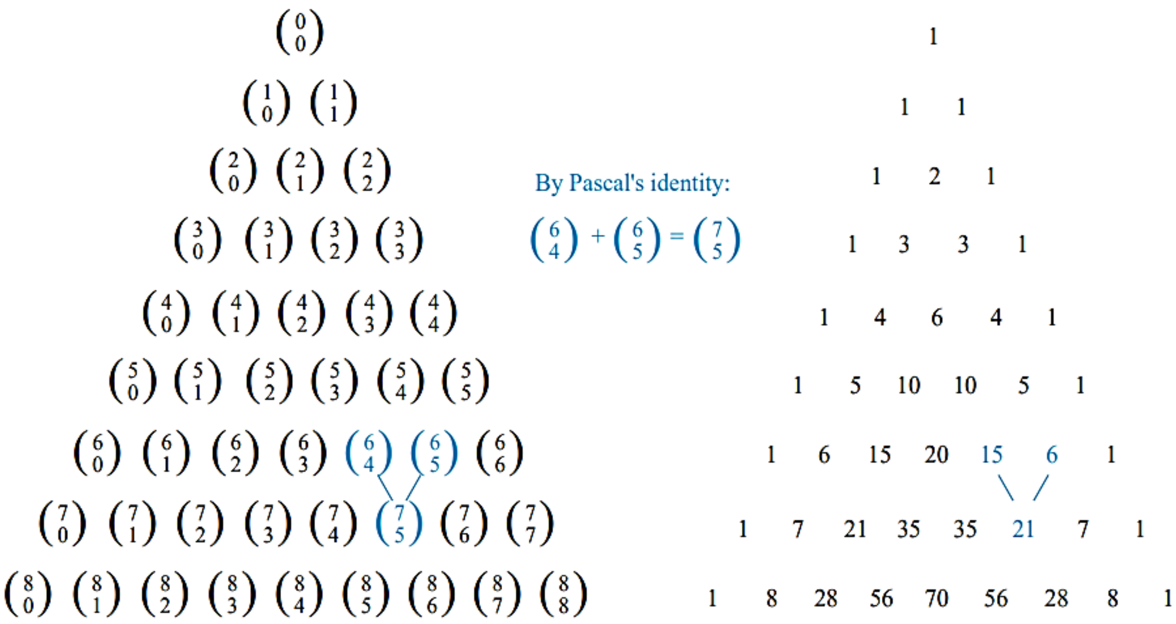

Fig: Pascal's Triangle.

5.2 Discrete Probability

Some Terminologies

Experiment:

An experiment is a procedure that yields one of a given set of possible outcomes.

Sample Space:
The sample space of the experiment is the set of possible outcomes.

Event:
An event is a subset of the sample space.

Finite Probability

If S is a finite nonempty sample space of equally likely outcomes, and E is an event, that is, a subset of S, then the probability of E is:

$$P(E) = \frac{|E|}{|S|}$$

EXAMPLE 1: An urn contains four blue balls and five red balls. What is the probability that a ball chosen at random from the urn is blue?

Solution:

- To calculate the probability, note that there are nine possible outcomes, and four of these possible outcomes produce a blue ball.
- Hence, the probability that a blue ball is chosen is 4/9.

EXAMPLE 2: What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 7?

Solution:

- There are a total of 36 equally likely possible outcomes when two dice are rolled.
- There are six successful outcomes, namely, (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), and (6, 1), where the values of the first and second dice are represented by an ordered pair.
- Hence, the probability that a seven comes up when two fair dice are rolled is 6/36 = 1/6.

Probabilities of Complements

Let E be an event in a sample space S. The probability of the event $\bar{E} = S - E$, the complementary event of E, is given by

$$P(\bar{E}) = 1 - P(E)$$

EXAMPLE: A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?

Solution:

- Let E be the event that at least one of the 10 bits is 0.
- Then $\bar{E}$ is the event that all the bits are 1s.
- Because the sample space S is the set of all bit strings of length 10, it follows that

$$p(E) = 1 - p(\overline{E}) = 1 - \frac{|\overline{E}|}{|S|} = 1 - \frac{1}{2^{10}}$$
$$= 1 - \frac{1}{1024} = \frac{1023}{1024}.$$

- Hence, the probability that the bit string will contain at least one 0 bit is 1023/1024.

Unions of Events

Let E1 and E2 be events in the sample space S. Then

$$p(E1 \cup E2) = p(E1) + p(E2) - p(E1 \cap E2)$$

EXAMPLE: What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

Solution:

- Let E1 be the event that the integer selected at random is divisible by 2, and let E2 be the event that it is divisible by 5.
- Then E1 ∪ E2 is the event that it is divisible by either 2 or 5.
- Also, E1 ∩ E2 is the event that it is divisible by both 2 and 5, or equivalently, that it is divisible by 10.
- Because |E1| = 50, |E2| = 20, and |E1 ∩ E2| = 10, it follows that

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$
$$= \frac{50}{100} + \frac{20}{100} - \frac{10}{100} = \frac{3}{5}.$$

Conditional Probability

Let E and F be events with $p(F) > 0$. The conditional probability of E given F, denoted by $p(E/F)$, is defined as

$$p(E/F) = \frac{p(E \cap F)}{p(F)}$$

EXAMPLE: A bit string of length four is generated at random so that each of the 16 bit strings of length four is equally likely. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0? (We assume that 0 bits and 1 bits are equally likely.)

Solution:

- Let E be the event that a bit string of length four contains at least two consecutive 0s, and let F be the event that the first bit of a bit string of length four is a 0.

- The probability that a bit string of length four has at least two consecutive 0s, given that its first bit is a 0, equals

$$p(E/F) = \frac{p(E \cap F)}{p(F)}$$

- Because E ∩ F = {0000, 0001, 0010, 0011, 0100}, we see that p(E ∩ F) = 5/16.
- Because there are eight bit strings of length four that start with a 0, we have

  p(F) = 8/16 = 1/2.
- Consequently,

$$p(E/F) = \frac{5/16}{1/2} = \frac{5}{8}$$

Random Variables

- A random variable is a function from the sample space of an experiment to the set of real numbers.
- That is, a random variable assigns a real number to each possible outcome.

EXAMPLE: Suppose that a coin is flipped three times. Let X(t) be the random variable that equals the number of heads that appear when t is the outcome. Then X(t) takes on the following values:

X(HHH) = 3

X(HHT) = X(HTH) = X(THH) = 2

X(TTH) = X(THT) = X(HTT) = 1

X(TTT) = 0

The Distribution of a Random Variable

- The distribution of a random variable X on a sample space S is the set of pairs (r, p(X = r)) for all r ∈ X(S), where p(X = r) is the probability that X takes the value r.

EXAMPLE:

- Each of the eight possible outcomes when a fair coin is flipped three times has probability 1/8.
- So, the distribution of the random variable X(t), where X(t) be the random variable that equals the number of heads that appear when t is the outcome, is determined by the probabilities P (X = 3) = 1/8, P (X = 2) = 3/8, P (X = 1) = 3/8, and P (X = 0) = 1/8.
- Consequently, the distribution of X(t) is the set of pairs (3, 1/8), (2, 3/8), (1, 3/8), and (0, 1/8).

Probabilistic Primility Testing

Format’s Little Theorem

- If p is a prime number and a is a positive integer not divisible by p then $a^{p-1} \equiv 1 (\text{mod } p)$.
- The Format’s little theorem holds only for when p is prime.

EXAMPLE: Show that 7 is a prime number by using Format’s little theorem.

Solution:

- Let’s take the value of a = 2, then according to the Format’s little theorem,

$$2^{7-1} \equiv 1 (\text{mod } 7)$$
$$2^{6} \equiv 1 (\text{mod } 7)$$

- The above congruency is holds, therefore 7 is a prime number.

Concept and Examples of Randomized Algorithms

- An algorithm that uses random numbers to decide what to do next anywhere in its logic is called a Randomized Algorithm.
- For example, in Randomized Quick Sort, we use a random number to pick the next pivot (or we randomly shuffle the array).

5.3 Advanced Counting

Recurrence Relations

- A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1,...,a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.
- A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.
- A recurrence relation is said to *recursively define* a sequence.
- The *initial conditions* for a recursively defined sequence specify the terms that precede the first term where the recurrence relation takes effect.

EXAMPLE 1: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for n = 1, 2, 3,..., and suppose that $a_0 = 2$. What are a_1, a_2, and a_3?

Solution:

- We see from the recurrence relation that $a_1 = a_0 + 3 = 2 + 3 = 5$.
- It then follows that $a_2 = 5 + 3 = 8$ and $a_3 = 8 + 3 = 11$

EXAMPLE 2: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for n=2, 3, 4,..., and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

Solution:

- We see from the recurrence relation that $a_2 = a_1 - a_0 = 5 - 3 = 2$ and $a_3 = a_2 - a_1 = 2 - 5 = -3$.

- We can find a_4, a_5, and each successive term in a similar way.

EXAMPLE 3: Fibonacci sequence

- The Fibonacci sequence, $f_0, f_1, f_2, \ldots$, is defined by the initial conditions $f_0 = 0$, $f_1 = 1$, and the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

for $n = 2, 3, 4, \ldots.$

- Find the Fibonacci numbers f_2, f_3, f_4, f_5, and f_6.

Solution:

- The recurrence relation for the Fibonacci sequence tells us that we find successive terms by adding the previous two terms.
- Because the initial conditions tell us that $f_0 = 0$ and $f_1 = 1$, using the recurrence relation in the definition we find that

$$f_2 = f_1 + f_0 = 1 + 0 = 1,$$

$$f_3 = f_2 + f_1 = 1 + 1 = 2,$$

$$f_4 = f_3 + f_2 = 2 + 1 = 3,$$

$$f_5 = f_4 + f_3 = 3 + 2 = 5,$$

$$f_6 = f_5 + f_4 = 5 + 3 = 8.$$

Closed Formula

- We say that we have solved the recurrence relation together with the initial conditions when we find an explicit formula, called a *closed formula*, for the terms of the sequence.

Solution of Recurrence Relation

- Many methods have been developed for solving recurrence relations.

EXAMPLE 1: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots$, and suppose that $a_1 = 2$. Solve the recurrence relation.

Solution:

- Starting with the initial condition $a_1 = 2$, and working upward until we reach a_n to deduce a closed formula for the sequence.
- We see that

$$a_2 = 2 + 3$$

$$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$$

$$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3 \ldots$$

$$a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1).$$

- We can also successively apply the recurrence relation, starting with the term a_n and working downward until we reach the initial condition $a_1 = 2$ to deduce this same formula.
- The steps are

$$\begin{aligned} a_n &= a_{n-1} + 3 \\ &= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\ &= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3 \\ &\;\;... \\ &= a_2 + 3(n - 2) = (a_1 + 3) + 3(n - 2) = 2 + 3(n - 1). \end{aligned}$$

- At each iteration of the recurrence relation, we obtain the next term in the sequence by adding 3 to the previous term.
- We obtain the nth term after $n - 1$ iterations of the recurrence relation.
- Hence, we have added $3(n - 1)$ to the initial term $a_1 = 2$ to obtain a_n.
- This gives us the closed formula $a_n = 2 + 3(n - 1)$.
- Note that this sequence is an arithmetic progression.

Exercise

1. Solve the recurrence relation
   $a_n = 2a_{n-1}$, $n \geq 1$ and $a_0 = 3$
2. Solve the recurrence relation $a_n = a_{n-1} + 4$, subject to the initial condition $a_1 = 3$.

General Form of Linear Homogeneous Recurrence Relation

- The homogeneous recurrence relation of degree k with constant coefficient has the general form

  $a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k}$ ………………..(i)

- If $a_n = r^n$ is a solution of equation (i), then, it must satisfy equation (i)
- i.e
  $r^n = c_1r^{n-1} + c_2r^{n-2} + \dots + c_kr^{n-k}$ ………………….(ii)
- Dividing by r^{n-k} on both side, we get
  $r^k = c_1r^{k-1} + c_2r^{k-2} + \dots + c_k$
  $r^k - c_1r^{k-1} + c_2r^{k-2} + \dots + c_k = 0$ ………………….(iii)
- This equation is known as characteristics equation of given recurrence relation and it provides characteristics root of recurrence relation which are used to give on explicit formula for all the solution of recurrence relation.

Solving Linear Homogeneous Recurrence Relation with Constant Coefficient

- A linear homogeneous recurrence relation of degree k with constant coefficient is a recurrence relation of the form

  $$a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k}$$

  where,

  $c_1, c_2, \dots, c_k$ are real numbers and $c_k \neq 0$.

- In solving the recurrence relation of the type above, the approach is look for the solution of the form $a_n = r^n$, where r is a constant.
- $a_n = r^n$ is the solution of a recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k}$ if and only if $r^n = c_1r^{n-1} + c_2r^{n-2} + \dots + c_kr^{n-k}$
- Where we divide both side by r^{n-k} and transpose the right hand side, we get
  $r^k - c_1r^{k-1} - c_2r^{k-2} - \dots - c_k = 0$
- Here we say that $a_n = r^n$ is a solution if and only if r is the solution if the equation

  $r^k - c_1r^{k-1} - c_2r^{k-2} - \dots - c_k = 0$

- And solution are called characteristics root.

Theorem 1: (without proof)

Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if $a_n = \alpha_1r_1^n + \alpha_2r_2^n$ for n = 0, 1, 2, … , where α_1 and α_2 are constants.

EXAMPLE: Solve the recurrence relation $a_n = 5a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$ and $a_1 = 0$.

Solution:

- The given recurrence relation is
  $a_n = 5a_{n-1} + 6a_{n-2}$ ……………………….(i)
- The characteristics equation is:
  $r^2 - 5r + 6 = 0$
  $r^2 - 3r - 2r + 6 = 0$
  $r(r - 3) - 2(r - 3) = 0$
  $(r - 3)(r - 2) = 0$
- Therefor r = 2, 3 i.e. $r_1 = 2$ and $r_2 = 3$
- Since two characteristics roots are distinct, we use the theorem1, to write the solution.
- The general form of solution is:
  $a_n = \propto_1r_1^n + \propto_2r_2^n$
  $a_n = \propto_12^n + \propto_23^n$ ……………………….(ii)
- From initial conditions $a_0 = 1$,
  $a_0 = \propto_12^0 + \propto_23^0$
  $1 = \propto_12^0 + \propto_23^0$
  $1 = \propto_1 + \propto_2$

$\propto_1 = 1 - \propto_2$ ……………………………(iii)

- Again $a_1 = 0$,

  $a_1 = \propto_1 2^1 + \propto_2 3^1$

  $0 = 2\propto_1 + 3\propto_2$ ……………………(iv)

- Now,

  $0 = 2(1 - \propto_2) + 3\propto_2$

  $0 = 2 - 2\propto_2 + 3\propto_2$

  $0 = 2 + \propto_2$

  $\therefore \propto_2 = -2$ and $\propto_1 = 3$

- Therefore, the solution of given recurrence relation is:

  $a_n = \propto_1 r_1^n + \propto_2 r_2^n$

  $a_n = 3.2^n + (-2).3^n$

  $a_n = 3.2^n - 2.3^n$

Exercise

1. Solve the recurrence relation $a_n = 6a_{n-1} - 8a_{n-2}$ for $n \geq 2$, $a_0 = 4$ and $a_1 = 10$.
2. What is the solution of recurrence relation $a_n = a_{n-2} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$.
3. Solve the recurrence relation $a_n = a_{n-1} + 6a_{n-2}$ for $n > 2$, $a_0 = 3$ and $a_1 = 6$.
4. Find the solution of recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n > 2$, $f_0 = 0$ and $f_1 = 1$.
5. What is the solution of recurrence relation $a_n = a_{n-1} - 2a_{n-2}$ with initial conditions, $a_0 = 2$ and $a_1 = 7$.

<u>Theorem 2 (Without Proof)</u>

Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_2 r - c_2 = 0$ has only one root r_0. Then the sequence $\{a_n\}$ is a solution of recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \propto_1 r_0^n + \propto_2 n r_0^n$ from n = 0, 1, 2… where $\propto_1$ and $\propto_2$ are constants.

EXAMPLE: Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ for $n \geq 2$, $a_0 = 1$ and $a_1 = 6$.

Solution:

- Characteristics equation of given relation is

  $r^2 - 6r + 9 = 0$

  $(r - 3)^2 = 0$

  $\therefore r = 3, 3$

- Therefore, it has only one root, r = 3
- Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if $a_n = \propto_1 3^n + \propto_2 3^n$, for some constant $\propto_1$ and $\propto_2$.
- From the initial conditions, we have,

  $a_0 = 1 = \propto_1$ and

  $a_1 = 6 = \propto_1 .3 + \propto_2 .3$

- Then we get, $\propto_1 = 1$ and $\propto_2 = 1$

- Hence, the solution of the sequence $\{a_n\}$ is:
  $a_n = 1.3^n + 1.n.3^n$
  $a_n = 3^n (1 + n)$

Exercise

1. Solve the recurrence relation $a_n = 2a_{n-1} – a_{n-2}$ for $n \geq 2$, $a_0 = 4$ and $a_1 = 1$.
2. Solve the recurrence relation $a_n = 2a_{n-1} – a_{n-2}$ for $n \geq 2$, $a_0 = 3$ and $a_1 = 6$.
3. Solve the recurrence relation $a_n = –6a_{n-1} – 9a_{n-2}$ for $n \geq 2$, $a_0 = 5$ and $a_1 = –1$.

Theorem 3 (Without Proof)

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that $r^k – c_1r^{k+1} – \ldots – c_k = 0$ has k distinct roots $r_1, r_2, \ldots, r_k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$ if and only if $a_n = \propto_1 r_1^n + \propto_2 r_2^n + \ldots + \propto_k r_k^n$ for n = 0, 1, 2… where $\propto_1, \propto_2, \ldots, \propto_k$ are constants.

EXAMPLE: Find the solution to the recurrence relation $a_n = 6a_{n–1} − 11a_{n–2} + 6a_{n–3}$ with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.

Solving Linear Non-homogeneous Recurrence Relation with Constant Coefficients

The recurrence relation of the form $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + F(n)$, where $c_1, c_2, \ldots, c_k$ are real numbers and F(n) is a function depending upon n. The recurrence relation preceding F(n) is called associated homogeneous recurrence relation.

For example, $a_n = 7a_{n-1} + 3a_{n-2} + 6n$ is a linear non-homogeneous recurrence relation with the constant coefficients.

Theorem (Without Proof)

If $\{a_n^{(p)}\}$ is a particular solution of the non-homogeneous linear recurrence relation with constant coefficient $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + F(n)$, then every solution of the form $\{a_n^{(p)} + a_n^{(n)}\}$, where $a_n^{(n)}$ is a solution of the associated homogeneous recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$.

EXAMPLE: Find all the solution of the recurrence relation $a_n = 4a_{n-1} + n^2$. Also find the solution of the recurrence relation with initial condition $a_1 = 1$.

Solution:

- We have associated linear homogeneous recurrence relation as $a_n = 4a_{n-1}$. The root is 4, so solutions are $a_n(n) = \propto 4^n$, where $\propto$ is a constant.
- Since $F(n) = n^2$ is a polynomial of degree 2, a trial solution is a quadratic function n, say, $p_n = an^2 + bn + c$, where a, b, and c are constants.

- To determine whether there are any solution of this form, suppose that $p_n = an^2 + bn + c$ is such solution.
- Then the equation $a_n = 4a_{n-1} + n^2$ becomes

  $an^2 + bn + c = 4(a(n-1)^2) + b(n-1) + c) + n^2$
  $= 4an^2 + 8an + 4a + 4bn – 4b – 4c + n^2$
  $= (4a + 1)n^2 + (-8a + 4b)n + (4a – 4b + 4c)$
- Here $an^2 + bn + c$ is the solution if and only if

  $4a + 1 = a$ i.e. $a = - 1/3$
  $-8a + 4b = b$ i.e. $b = -8/9$
  $4a – 4b + 4c = c$ i.e. $c = -28/27$
- So, $a_n(p) = -1/3 (n^2 + (8/3)n + 28/9) + \propto 4^n$ where $\propto$ is a constant.
- For solution with $a_1 = 1$, we have

  $a_1 = 1 = -1/3 (1 + (8/3) + 28/9) + \propto.4$
  i.e. $\propto = 22/27$

- Then, the solution is $a_n = -1/3(1 + (8/3) + 28/9) + (22/27)4^n$

Exercise

1. Find all the solution of recurrence relation $a_n = 2a_{n-1} + 3^n$ and a solution with initial condition $a_1 = 5$.

Recurrences Applications

One of the application area of recurrence relations is analysis of divide and conquer algorithms.

Divide and Conquer Algorithms

- Divide and Conquer algorithms divide a problem of larger size to the problem of smaller size so continuously such that the problem of the smallest size that has trivial solution is obtained.
- If f(n) represents the number of operations required to solve the problem of size n, then follows the recurrence relation $f(n) = af(n/b) + g(n)$, called divide and conquer recurrence relation.
- In the relation above the problem of size n is partitioned into a parts of the problem of the size n/b and g(n) is the operations required to conquer the solutions.

EXAMPLE: Binary Search

- The binary search algorithm reduces the search for an element in a search sequence of size n to the binary search for this element in a search sequence of size n/2, when n is even.
- The problem size n has been reduced to one problem of size n/2.
- So to comparisons are needed to implement this reduction.

- Hence if f(n) is the number of comparisons required to search for an element in a search sequence of size n, then

$$f(n) = f(n/2) + 2 \quad \text{when n is even.}$$

Assignment

1. A group of 8 scientist is composed of 5 chemist and 3 biologist. In how many ways can a committee of 5 be formed that has 3 chemist and 2 biologist? [TU 2079]
2. Show that if there are 30 students in a class, then at least two have same names that begin with the same letter. Explain the pascal's triangle. [TU 2078]
3. How many 3 digits numbers can be formed from the digits 1,2,3,4 and 5 assuming that:
   a. Repetitions of digits are allowed
   b. Repetitions of digits are not allowed [TU 2076]
4. State pigeonhole principle. Solve the recurrence relation $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ with initial conditions $a_0=1$,$a_1 = 3$, $a_2=7$. (10) [TU 2076]
5. What does primality testing means? Describe how Fermat's Little Theorem tests for a prime number with suitable example. [TU 2075]
6. Solve the recurrence relation $a_n = 5a_{n-1} – 6a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 2$. (5) [TU 2075]
7. List any two applications of conditional probability. You have 9 families you would like to invite to a wedding. Unfortunately, you can only invite 6 families. How many different sets of invitations could you write? [TU 2075]
8. Define linear homogeneous recurrence relation. Solve the recurrence relation $an=a_{n/2}+n+1$, with $a_1=1$. Also discuss about probabilistic primality testing with example. (2+4+4) [TU Model Question]
9. State and prove generalized pigeonhole principle? How many cards should be selected from a deck of 52 cards to guarantee at least three cards of same suit? (2.5+ 2.5) [TU Model Question]