Mathematical Models - Exercises

Data Modelling and Analysis

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Contents

Population Modelling: Fish Harvesting

You are the manager of OdysSea Inc., a fish farm that specialises in harvesting Albacore fish.

Albacore can be divided into three categories, according to their age in years:

- Young: fish in the range of (0,1) years
- Adult: fish in the range of (1,2) years
- Mature: fish in the range >2 years

Your goal is to chart the fish stocks as time elapses, making sure that you suitably harvest a proportion of each category without over- or under-harvesting, as that could lead to the destruction of stocks. As a manager, you want a steady state of Albacore so that the business is under control.

Suppose that 80% of young Albacore survive into adulthood, and that 75% of adult Albacore mature. Also suppose that 40% of mature Albacore survive without dying out from one year to the next. As you farm only has Albacore and no other species in its ecosystem, depletion is taken only as natural causes (age, disease, etc).

Using this information:

1. Create a mathematical model to monitor Albacore stocks in time.
2. Solve your mathematical model if your starting population is 100 young fish, 50 adult fish, and 20 mature fish. Consider the birth rates for adult Albacore is 2 and for mature Albacore is 1. You want to study each group in the next 10 years.

Solution

This problem is a clear example of a population growth problem. Other examples include the modelling of sustainable resources, global warming, genetically modified foods, etc. Many mathematical modelling authors have written extensively on the formulation of simple population models and also on interactive population behaviour in the animal kingdom such as the “prey-predator" model we have seen in Lab 2 with Moose and Wolves.

This problem in particular is referred to as an “age-stratified" problem, since populations are categorised into age groups for growth investigation.

Let’s remember the steps for model construction:

1. Identify question or problem
2. System analysis
3. Formulate mathematical model
4. Solve mathematical model
5. Interpret solution

1. Identify Question or Problem:

Our general system can be defined as: S: {*OdysSea Inc’s Albacore fish farm*}.

In this case, the general Q could be:{*How will the poulation of each age group change in time?*}

2. System analysis

In this particular exercise the situation is simplified to only one type of fish, no predators, and no consideration of migration and/or reproduction patterns within the year. With all of this in mind, let’s define our reduced system, which has variables, parameters and constants that will be used in the creation of the model:

- Y_n: Young: fish in the range of (0,1)
- A_n: Adult: fish in the range of (1,2)
- M_n: Mature: fish in the range >2
- α: birth rate for adults ($\alpha = 2$)
- β: birth rate for mature ($\beta = 1$)
- y_s: survival rate of young into adulthood (80%)
- a_s: survival rate of adult into maturity (75%)
- m_s: survival rate of mature (40% per year)
- n: time (in years)

3. Formulate mathematical model

As part of our model, we will have to consider three groups Y, A, and M

Before we start trying to come up with mathematical statements, it might be a good idea to write down, just with words, how each Albacore group will change in time:

- Young at a time (n+1) = adult birth rate X adult at a time n + mature birth rate * mature at a time n
- Adult at a time (n+1) = survival rate of young into adulthood * young at a time n
- Mature at a time (n+1) = survival rate of adult into mature * adult at a time n + survival rate of mature * mature at a time n

Using this information, we can now create our model for the three groups using time series:

$$Y_{n+1} = \alpha * A_n + \beta * M_n \tag{1}$$

$$A_{n+1} = y_s * Y_n \tag{2}$$

$$M_{n+1} = a_s * A_n + m_s * M_n \tag{3}$$

As you can see, all three equations are a system of equations. Also, they have certain similarities (they all depend on the past of the groups). You can rewrite them as a matrices as follows:

$$\begin{pmatrix} Y \\ A \\ M \end{pmatrix}_{n+1} = R \begin{pmatrix} Y \\ A \\ M \end{pmatrix}_{n} \tag{4}$$

Where R is:

$$\begin{pmatrix} 0 & \alpha & \beta \\ y_s & 0 & 0 \\ 0 & a_s & m_s \end{pmatrix} \tag{5}$$

This R matrix is called the *Leslie* matrix in the literature.

4. Solve mathematical model

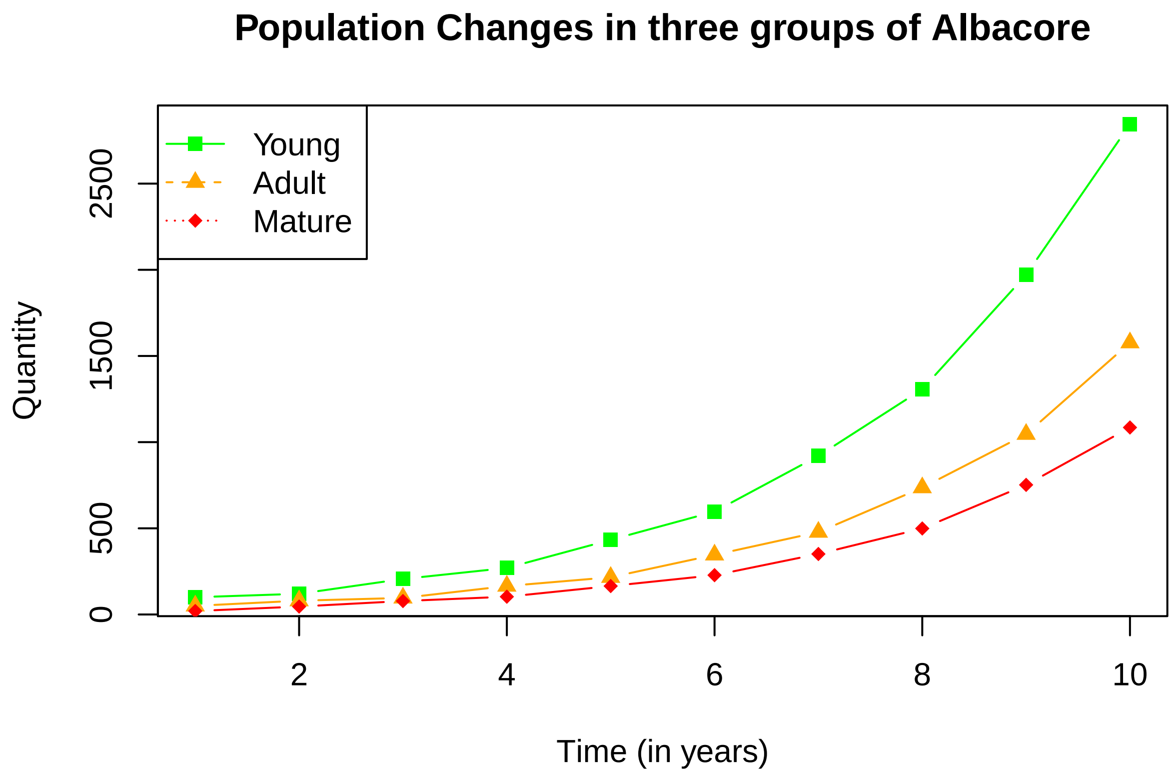
You may solve this model manually. However, because we need to calculate the groups population in 10 years, you will have to solve 30 equations to get the information that you need.

We can use R to implement the model as a function, and then quickly solve it by calling it with the corresponding initialisation parameters.

```r
odysSea_farming = function(Y, A, M, y_s, a_s,m_s, alpha, beta,n){

  albacore = data.frame(m=1, Y = Y, A = A, M = M)

  for (i in 2:n){

    Y_n = alpha * albacore[i-1,]$A + beta * albacore[i-1,]$M
    A_n = y_s * albacore[i-1,]$Y
    M_n = a_s * albacore[i-1,]$A + m_s * albacore[i-1,]$M

    total = data.frame(m = i, Y = Y_n, A=A_n, M= M_n)

    albacore = rbind(albacore, round(total))

  }

# Can be done in more sophisticated ways with ggplot2 - but we won't cover that
# until Lab 6
plot(albacore$m, albacore$Y, pch=15, type = "b", col = "green",
     xlab = "Time (in years)", ylab = "Quantity",
     main = "Population Changes in three groups of Albacore")
points(albacore$A, pch = 17, type = "b",col ="orange")
points(albacore$M, pch =18,type = "b", col="red")
legend("topleft",legend=c("Young","Adult","Mature"), col=c("green","orange","red"),
                                   pch=c(15,17,18),lty=c(1,2,3), ncol=1)
return(albacore)
}
```

To solve the model, let's call the function:

```r
Y = 100
A = 50
M = 20
y_s = 0.8
a_s = 0.75
m_s = 0.4
n = 10
alpha = 2
beta= 1

odysSea_farming(Y, A, M, y_s, a_s,m_s, alpha, beta, n)
```



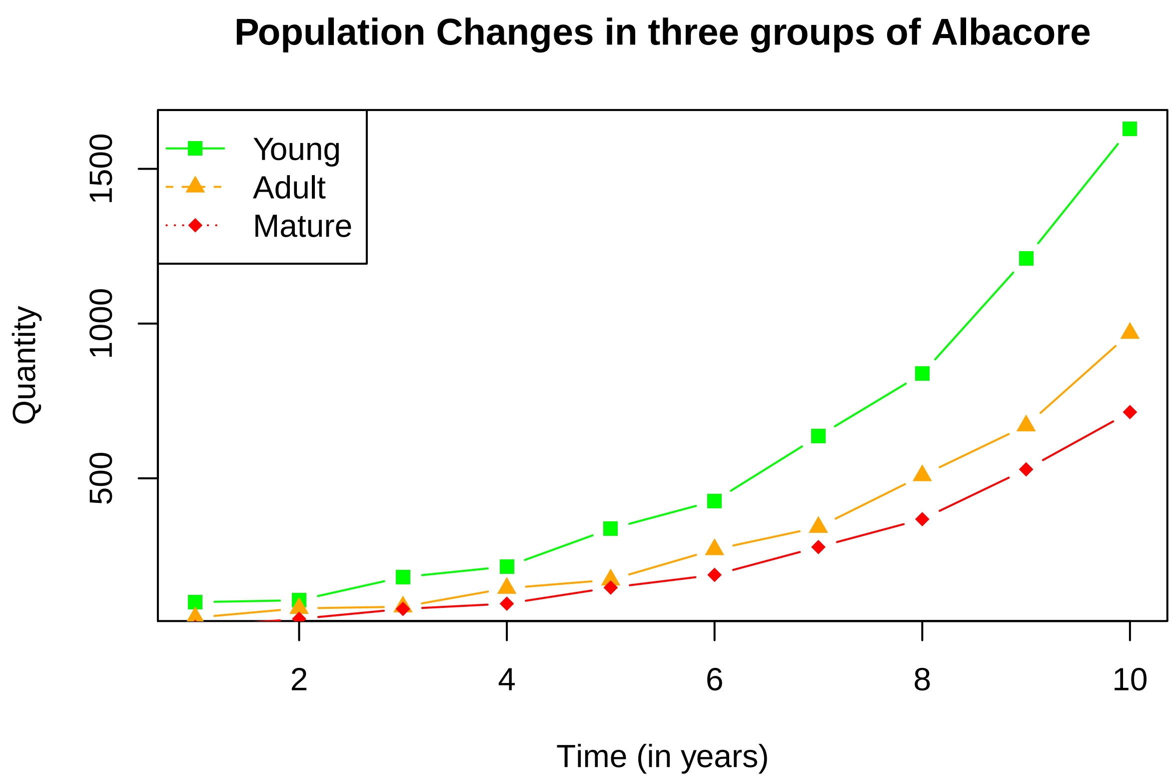

```
##      m    Y    A    M
## 1    1  100   50   20
## 2    2  120   80   46
## 3    3  206   96   78
## 4    4  270  165  103
## 5    5  433  216  165
## 6    6  597  346  228
## 7    7  920  478  351
## 8    8 1307  736  499
## 9    9 1971 1046  752
## 10  10 2844 1577 1085
```

5. Interpret solution

We can see that at the end of the 10th year, all groups will have grown considerable, with over 2k young Albacore, over 1.5k adult albacore and over £1k mature albacore. Depending on the circumstances, this can be considered a dangerous case, close to having an overpopulated farm.

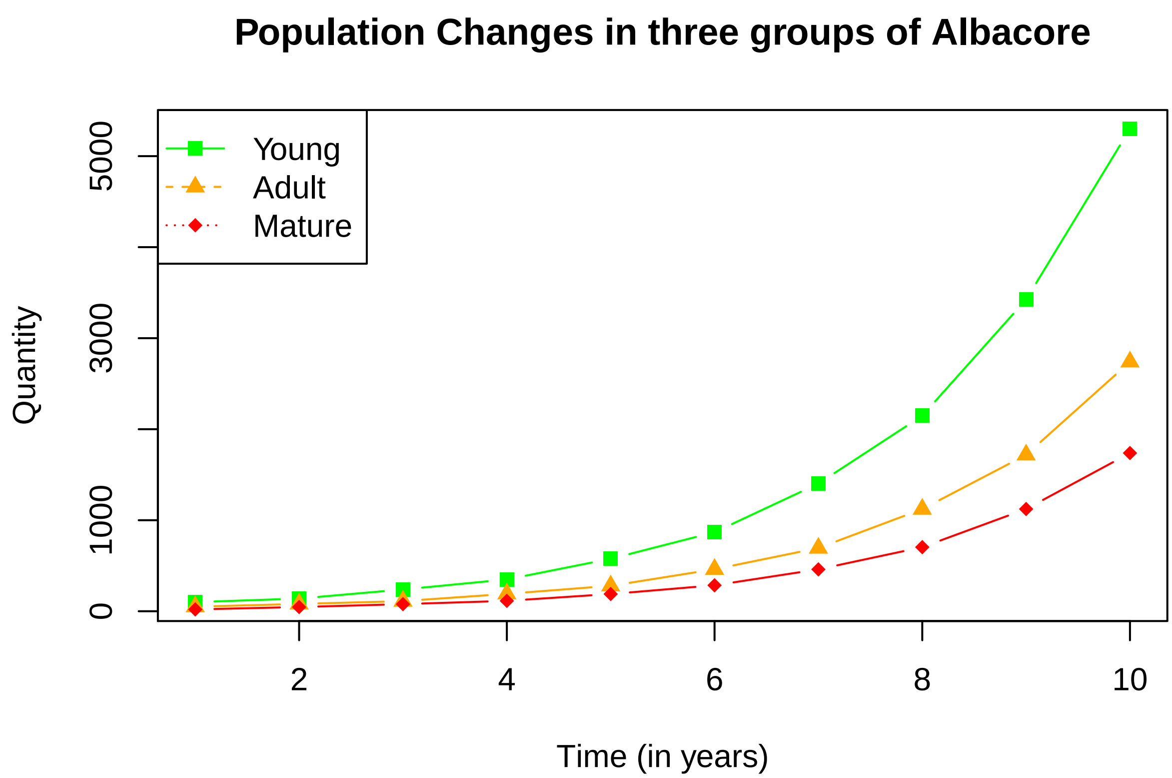
Let's carry out a quick simulation, with smaller and larger values of the birth rates:

```
alpha = 1.8
beta = 0.8

odysSea_farming(Y, A, M, y_s, a_s,m_s, alpha, beta, n)
```




```
##      m    Y   A   M
## 1    1  100  50  20
## 2    2  106  80  46
## 3    3  181  85  78
## 4    4  215 145  95
## 5    5  337 172 147
## 6    6  427 270 188
## 7    7  636 342 278
## 8    8  838 509 368
## 9    9 1211 670 529
## 10  10 1629 969 714
```

```
alpha = 2.3
beta= 1.2

odysSea_farming(Y, A, M, y_s, a_s,m_s, alpha, beta, 10)
```




```
##     m    Y    A    M
## 1   1  100   50   20
## 2   2  139   80   46
## 3   3  239  111   78
## 4   4  349  191  114
## 5   5  576  279  189
## 6   6  868  461  285
## 7   7 1402  694  460
## 8   8 2148 1122  704
## 9   9 3425 1718 1123
## 10 10 5299 2740 1738
```

As we can see, smaller changes in these rates entail large changes in the numbers of each group. These changes are particularly striking in Young and Mature Albacores.

With these model conditions in mind, let's (for the sake of curiosity) investigate a case that could be of interest to you as the manager: Are there any birth rates that would guarantee a constant state in the farm (i.e. all groups remain the same)?

1. Constant state

This case is slightly different from the *Equilibrium state*, which we covered in Lab 2. In that case, our goal was to calculate the initial populations that would make the model constant from n=0. In this case, we are looking for parameters α and β that guarantee that, eventually, an equilibrium will be reached.

The procedure, however, is similar. For the model to reach a constant state, it will have to stop depending on time. Therefore, we need to calculate α and β such as:

$$Y = \alpha * A + \beta * M \tag{6}$$

$$A = 0.8 * Y \tag{7}$$

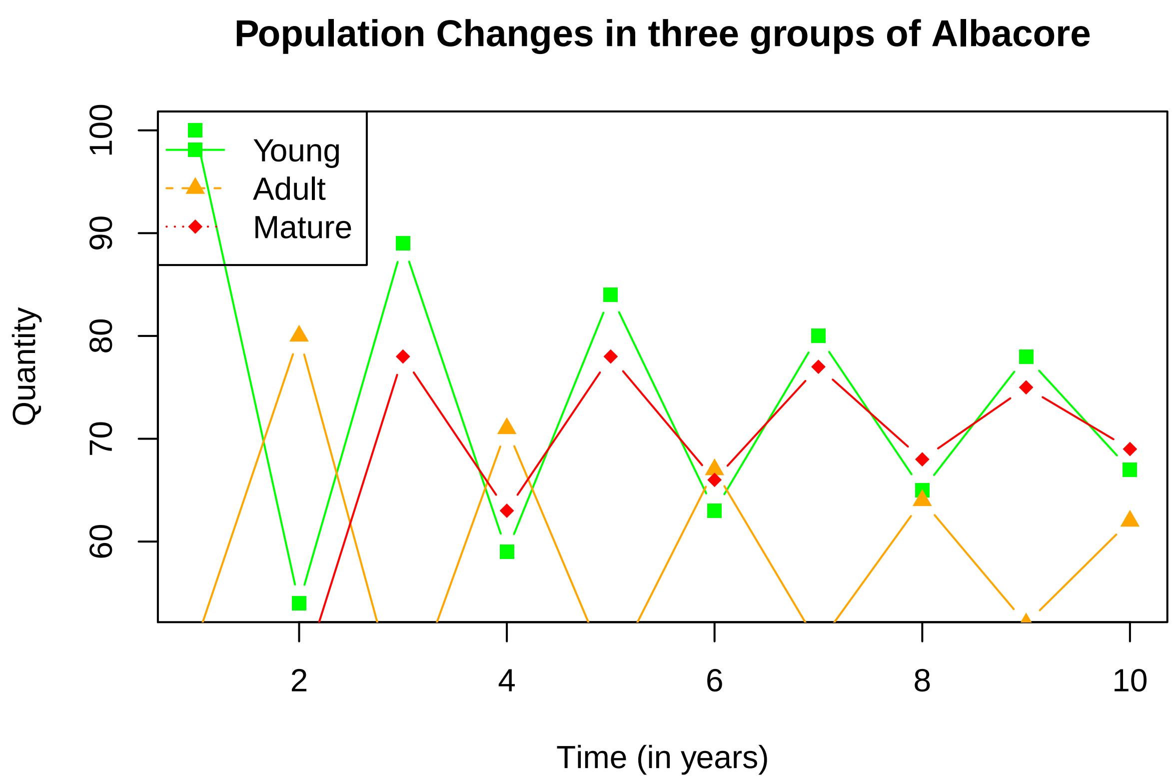
$$M = 0.75 * A_n + 0.4 * M \tag{8}$$

If you solve that set of linear equations by substitution, you'll reach two potential results:

- Y = 0, A = 0, M = 0, which can be rejected
- 0.8 * $\alpha + \beta = 1$

Let's see if that is true!

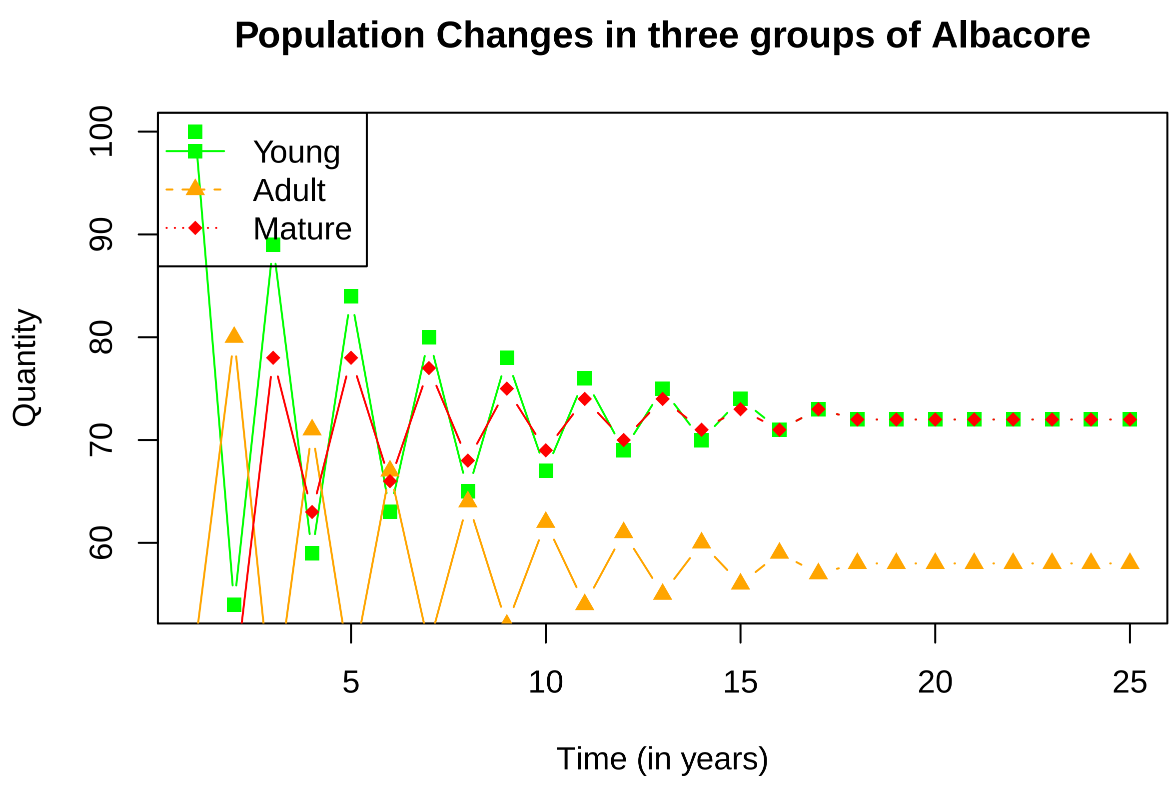
```
alpha = 1
beta = 1 - 0.8*alpha
odysSea_farming(Y, A, M, y_s, a_s,m_s, alpha, beta, 10)
```



```
##      m   Y  A  M
## 1    1 100 50 20
## 2    2  54 80 46
## 3    3  89 43 78
## 4    4  59 71 63
## 5    5  84 47 78
## 6    6  63 67 66
## 7    7  80 50 77
## 8    8  65 64 68
## 9    9  78 52 75
## 10 10  67 62 69
```

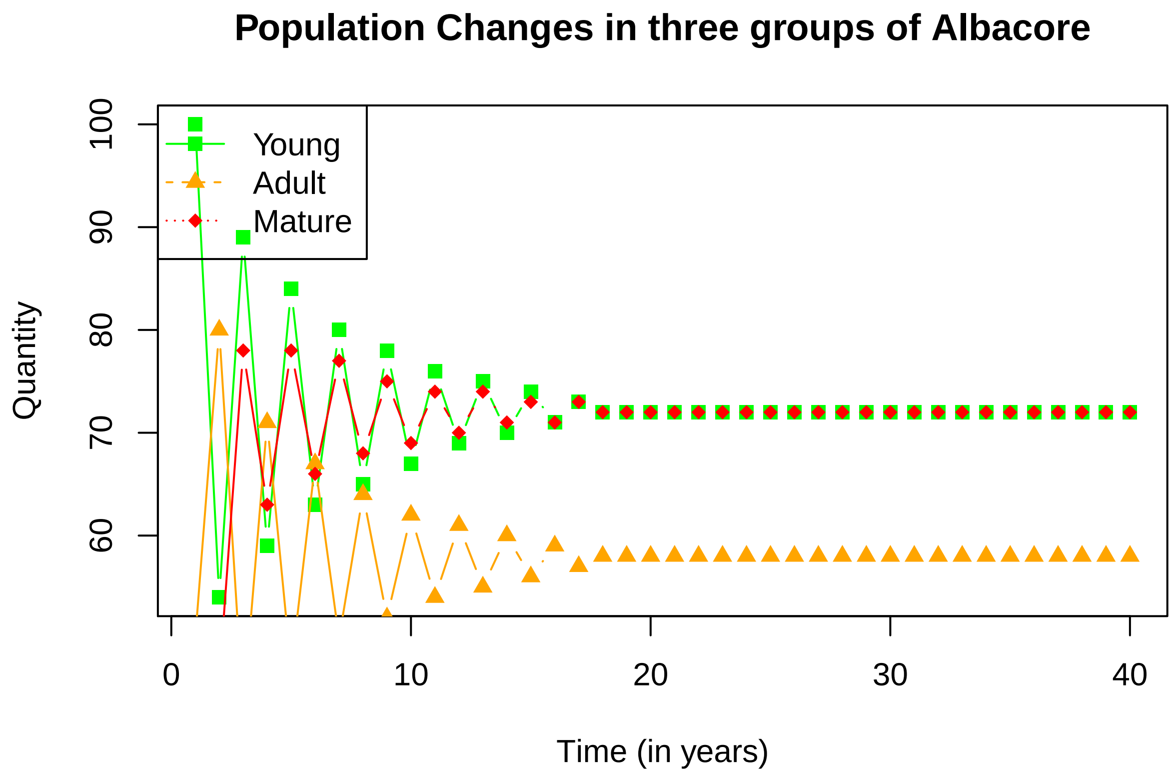
Here, it is difficult to see. Let's see what happens when we increase the time period:

```
alpha = 1
beta = 1 - 0.8*alpha
res = odysSea_farming(Y, A, M, y_s, a_s,m_s, alpha, beta, 25)
```



Let's increase it even more!

```
alpha = 1
beta = 1 - 0.8*alpha
res = odysSea_farming(Y, A, M, y_s, a_s,m_s, alpha, beta, 40)
```



As you can see, with an α and a β constructed in such a way, we can guarantee that the farm will reach a constant state at some point.

This further shows that these values of α and β will reach a constant state, but also shows that in that state, Y = M.

Discrete Event Models: Flu Spread in Jubilee Campus[1]

A flu is spreading across Jubilee Campus. The University's Health Centre is interested in knowing and experimenting with a model for this flu, to see if it could become an epidemic[2]. They have hired you to create and use this model.

The Health Centre is interested in modelling three different populations:

- Susceptible subjects: those who haven't been infected.
- Infected subjects: those who are currently experiencing the flu and can transmit it
- Removed subjects: those who were infected by the flu, but are no longer sick with it and cannot transmit it any more.

You are asked to incorporate the following assumptions into your model:

1. No one enters or leaves the community
2. There is no contact with the outside.
3. Each person is either Susceptible (S), Infected (I), or Removed (R).
4. Initially, each person is either S or I.
5. Once someone gets the flu this year, they cannot get the flu again.
6. The average length of the flu is 5/3 weeks (1 and 2/3 weeks).
7. You need to monitor weekly changes.
8. The disease spreads in 0.14% of cases with contact

Answer the following questions:

1. According to the information given below, create a mathematical model ({S (Sr), Q, M}) to study the weekly spread of the disease.
2. If the total population of the campus is 1000 people and the number of infected subjects on Week 0 was 5, obtain the changes in each population for the first 25 weeks. Plot all of them together to visualise how they develop during that time. (Hint: You may create an R script to do this. It will be faster than calculating 25x3 population values!)

Solution

Remember that the steps to build a model can be summarised as:

1. Identify question or problem
2. System analysis
3. Formulate mathematical model
4. Solve mathematical model
5. Interpret solution

Let's begin creating our model following those steps.

[1] I am aware of the irony of teaching remotely about how to model flu-spread in the middle of a flu-adjacent pandemic, but alas, mathematical modelling deals with these problems by nature! ¯\\(•_•)/¯

[2] *sighs*

1. Identify question or problem

The system in this case can be defined as: S:{*The Jubilee Campus population, the flu and its characeteristics*}

In this case, the question has been given to us by the Health Centre: Q:{*How can we model the spread of the flu in campus?*}

2. System analysis

With this in mind, let's define our reduced system, S_r, with out parameters and constants:

- S: Susceptible population
- I: Infected population
- R: Removed population
- n: month in which we are looking at the populations
- a: infection rate (0.0014)
- b: removal rate

3. Formulate mathematical model

Let's first create a general model that we can then "customize" with the characteristics of this particular flue, such as infection and removal rate.

Similarly to what we did before, instead of jumping directly to mathematical functions, let's first start modelling the populations with natural language:

- S in week n = those susceptible from the week before - those infected in week n
- I in week n = those who didn't get removed from the week before + those infected in week n
- R in week n = those removed so far + those removed this week

At this stage, you might have realised that we have almost all the information that we need, except for two things:

1. Those infected in a given week
2. The rate of removal

Let's begin to find each of them:

1. Those infected in a given week: to find this, we need to model the "infection" procedure in itself. In other words: when does an infection happens? Informally, we can say that an infection at a time n happens with a probability a when a susceptible person comes in contact with an infected person. However, something to consider is that there is not 1 infected person at a time n, but I(n), so we have to make sure our model covers all possible interactions between all susceptible and all infected people. In other words:
$$a * S(n) * I(n)$$

2. The rate of removal: The rate of removal is ratio from the infected population at a time (n-1) that gets better at a time n. In order to obtain this, you need to know how long the flu lasts. The removal rate will be approximated as the inverse of this duration. So, we know from our specifications that the flu lasts 5/3 weeks. Therefore, 5/3 or 60% of the people infected in week (n) get better at week (n+1) (the remaining 40% continue being sick).

Now, let's bring mathematical notation into the mix:

$$S_{n+1} = S_n - a * S_n * I_n \tag{9}$$

$$I_{n+1} = (1 - b) * I_n + a * S_n * I_n \tag{10}$$

$$R_{n+1} = R_n + b * I_n \tag{11}$$

M is then: M={S(n), I(n), R(n)}

These models are known in the literature as SIR (Susceptible-Infected-Removed) models.

4. Solve mathematical model

To solve the model, let's first write a function in R:

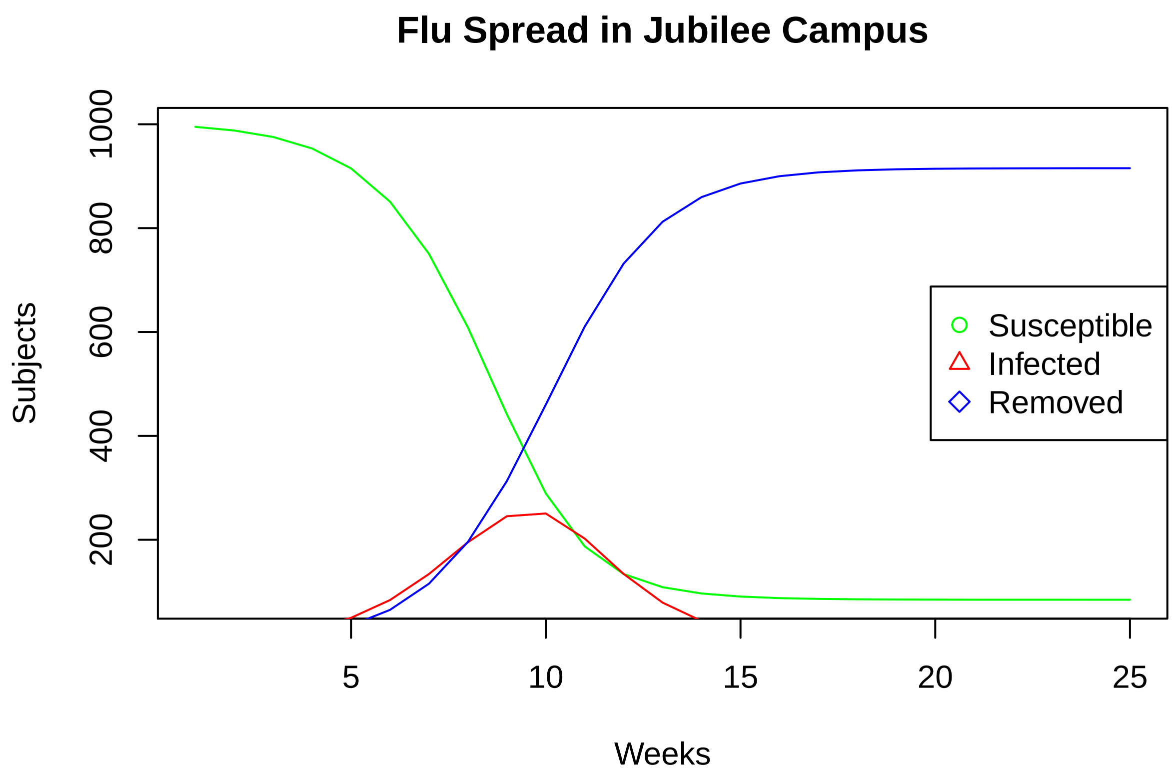
```
# SOLUTION: {S,Q,M}

  # S = Jubilee Campus population and flu

  # Sr =
  # S: Susceptible population
  # I: Infected population
  # R: Removed Population

  # Q = How is the flu going to be spread accross campus?

  # M:
  # As we have seen in class, you want to model these 3 populations:
  # S(n) = number of susceptible subjects after a period n
  # I(n) = number of infected subjects after a period n
  # R(n) = number if removed subjects after a period n

  # M was found to be:
    # S(n+1) = S(n) - 0.001407*S(n)*I(n)
    # I(n+1) = I(n) - 0.6* I(n) + 0.001407*S(n)*I(n)
    # R(n+1) = R(n) + 0.6* I(n)

diseaseModel = function (susceptible, infected, iterations)
{
  sir = matrix(NA, iterations, 3);
```

```r
# sir is a table of Nx3, where N is the no. of weeks (iterations) we are looking into.
# The first column has the susceptible subjects (S), the second columns has the
# infected subjects (I) and the third column has the retired subjects (R)
#
# NOTE ABOUT IMPLEMENTATION: This implementation uses a matrix, a much more elegant
# model would use a data frame. Try to implement it with a dataframe!
# Hint: look up the function tail.

sir[1,1] = susceptible;
sir[1,2] = infected;
sir[1,3] = 0;
a = 0.001407;

for (n in 2:iterations){
  # n represents the week we are looking into

  # look at the indeces! Instead of starting at 1 and having to add 1 in the
  # model (i.e S(n+1), I(n+1), R(n+1)), we are going to start at 2 and use
  # n-1. Both solutions are valid, of course.

  sir[n,1] = sir[n-1,1] - a*sir[n-1,1]*sir[n-1,2];
  # a*s[n-1,1]*i[n-1,2] -> is the number of people that get sick each week,
  # which is why we subtract it from the health people (S) and...

  sir[n,2] = sir[n-1,2] - 0.6 * sir[n-1,2] +
    a*sir[n-1,1]*sir[n-1,2];  #... and we add it to the infected people

  sir[n,3] = sir[n-1,3] + 0.6 * sir[n-1,2];

}

# Simple plot with all three populations:
plot(sir[,1],col="green", pch=1,xlab="Weeks",
     ylab="Subjects",type="l", main="Flu Spread in Jubilee Campus");
lines(sir[,2],col="red", type="l", pch=2);
lines(sir[,3],col="blue", type="l", pch=5);
legend("right", legend = c("Susceptible", "Infected","Removed"),
       pch=c(1,2,5),col=c("green","red","blue"))

# NOTE ABOUT PLOTTING MODELS:
# If your plot does not show all lines, it might be because
# the range of the Y axis is not large enough.
# To fix this, you can use the ylim parameter in plot with
# the range you want. An example of this:
# a = c(1, 10, 3, 10)
# b = c(100, 1000, 1000)
# plot(a)
# lines(b) # b is not shown!
# To fix this:
# plot(a, ylim=c(min(a,b), max(a,b)))
# lines(b)

sir = as.table(sir)
```

```
  colnames(sir) = c("Susceptible", "Infected","Removed");
  rownames(sir) = seq(1:iterations)
  return(sir)
}
```

Now, let's call the model with the relevant data to solve it:

```
sir=diseaseModel(995,5,25)
```




```
sir
```

```
##    Susceptible      Infected      Removed
## 1  995.00000000   5.00000000   0.00000000
## 2  988.00017500   8.99982500   3.00000000
## 3  975.48937205  16.11073295   8.39989500
## 4  953.37717284  28.55649239  18.06633477
## 5  915.07144591  49.72832389  35.20023020
## 6  851.04595419  83.91682128  65.03722453
## 7  750.56213496 134.05054774 115.38731730
## 8  608.99927068 195.18308338 195.81764594
## 9  441.75430860 245.31819544 312.92749597
## 10 289.27719827 250.60438850 460.11841323
## 11 187.27794979 202.24100388 610.48104633
## 12 133.98743003 134.18692131 731.82564865
```

```
## 13 108.69046948  78.97172907 812.33780144
## 14  96.61352113  43.66563998 859.72083889
## 15  90.67782257  23.40195456 885.92022288
## 16  87.69211470  12.34648969 899.96139561
## 17  86.16877017   6.46194041 907.36928942
## 18  85.38532801   3.36821833 911.24645367
## 19  84.98067983   1.75193550 913.26738466
## 20  84.77120473   0.91024930 914.31854596
## 21  84.66263649   0.47266796 914.86469555
## 22  84.60633213   0.24537155 915.14829633
## 23  84.57712282   0.12735792 915.29551925
## 24  84.56196723   0.06609876 915.37193401
## 25  84.55410289   0.03430385 915.41159327
```

5. Interpret solution

As can be seen in the graph, after a spike in the Infected population between weeks 5 and 15, all of the population will have moved from Susceptible to Removed.