P477 Project Report
Anisotropic Hydrodynamics

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1 Introduction

The existence of the Quark Gluon Plasma (QGP) has been firmly established by the heavy ion colliders. It is characterised by large pressure anisotropies in early times and along the dilute edges of the fireball. Earlier, QGP was described using ideal isotropic hydrodynamics which predicted that the QGP thermalizes at sub-fm/c scale. Dissipative corrections accounted for the large anisotropies present. But, theoretical studies firmly established that sub-fm/c isotropisation is not possible. This led to the study of large momentum space anisotropies in the QGP, which motivates us to study anisotropic Hydrodynamics (aHydro).

In this short review, we closely follow [1], and study a simple anisotropic flow. We start with basics of hydrodynamics and obtain the hydrodynamic equations for our simple anisotropic flow. We then compare our results with viscous hydrodynamics and check that our anisotropic hydrodynamics reduces to the correct ideal isotropoic limit and free streaming limit.

2 Prelude to hydrodynamics from kinetic theory

We start the analysis with the Boltzmann equation,

$$p^\mu \partial_\mu f(x,p) = -C[f] \tag{1}$$

where p^μ and x^μ are the particle momentum and position respectively and $C[f]$ is the collision kernel which includes both elastic and inelastic collisions. The equations of hydrodynamics are obtained by taking various moments of the Boltzmann equation (BE). Some notations are in order.

$$\int dP \equiv N_{dof} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{p^0} \tag{2}$$

where N_{dof} is the number of degrees of freedom a.k.a degeneracy. We define the following integrals,

$$\mathcal{C}_r^{\mu_1\mu_2\cdots\mu_n} \equiv -\int dP (p.u)^r p^{\mu_1} p^{\mu_2} \cdots p^{\mu_n} \mathcal{C}[f] \tag{3}$$

we call this integral the n^{th} moment of $C[f]$ for a given r. We need only the $r=0$ set of moments for our study. The 0^{th} moment of the BE is,

$$\partial_\mu \left(\int dP p^\mu f \right) = -\int dP\, C[f] \tag{4}$$

The quantity in the bracket is nothing but the particle number four current.

$$J^{\mu} \equiv \int dP\, p^{\mu} f \tag{5}$$

hence,

$$\partial_{\mu} J^{\mu} = \mathcal{C}_0 \tag{6}$$

Subsequently, the 1^{st} moment gives the equation for the stress tensor,

$$T^{\mu\nu} \equiv \int dP\, p^{\mu} p^{\nu} f \tag{7}$$

hence,

$$\partial_{\mu} T^{\mu\nu} = \mathcal{C}_0^{\nu} \tag{8}$$

In any microscopic model, the first moment of the collision kernel vanishes because of the four momentum conserving delta function. But, we resort to effective collision kernel, mainly the collision kernel assuming the *relaxation time approximation*. In such cases, the vanishing of C_0^{ν} leads to constrains. In the similar spirit as above, one can define the higher moments as follows,

$$I^{\mu\nu_1\nu_2\cdots\nu_n} \equiv \int dP\, p^{\mu} p^{\nu_1} p^{\nu_2} \cdots p^{\nu_n} f \tag{9}$$

and hence,

$$\partial_{\mu} I^{\mu\nu_1\nu_2\cdots\nu_n} = \mathcal{C}_0^{\nu_1\nu_2\cdots\nu_n} \tag{10}$$

To express the most general particle current, stress tensor and higher moments, we choose the orthonormal tetrads $X_i\,\{i = 0, 1, 2, 3\}$ and expand the tensors in that basis. In the local rest frame,

$$X^{\mu}_{i,LRF} = \delta^{\mu}_{i} \quad i = 0, 1, 2, 3 \tag{11}$$

Usually the X_0 is taken as the fluid four velocity u and the other vectors are denoted as $X_1 = X$, $X_2 = Y$ and $X_3 = Z$. The general form of the current and stress tensor are as follows, we only use the fact that the tensors of rank more than two are fully symmetric.

$$\begin{aligned} J^{\mu} &= n u^{\mu} + n^{i} X^{\mu}_{i} \\ T^{\mu\nu} &= t_{00} g^{\mu\nu} + \sum_{i=1}^{3} t_{ii} X^{\mu}_{i} X^{\nu}_{i} + \sum_{i,j\neq 0, i>j}^{3} t_{ij} X^{\mu}_{i} X^{\nu}_{j} \end{aligned} \tag{12}$$

where in this co-ordinate system, $g^{\mu\nu} = u^{\mu}u^{\nu} - \sum_{i=1}^{3} X^{\mu}_{i} X^{\nu}_{i}$.

3 Introduction to 0+1d flow

For pedagogy, we consider the system with the following symmetries,

- Longitudinal boost invariance
- Transverse homogeneity
- Conformal invariance (Massless case [1])

[1] A simple way to see this implication is to consider scale invariance, a subset of conformal invariance, which naively implies that there must not be any length scale in the problem. If we consider massive case, there is an inherent length scale, the Compton wavelength and hence field theories with mass term lacks scale invariance and thereby conformal invariance

- Reflection symmetric along spatial directions

What does 0+1d mean?: The homogeneity in the transverse direction, helps us get rid of two independent variables: x and y. We transform to the Milne co-ordinates (τ, ζ, x, y), where $\tau = \sqrt{t^2 - z^2}$ is the proper time and $\zeta = arctanh(\frac{z}{t})$ is the spacetime rapidity. For boost invariant flows, the spacetime rapidity is equal to the usual rapidity, $(\rho = arctanh(\beta))$, where β is the velocity of the boosted frame, $\beta = \zeta$. Hence there is only one variable on which the equations shall be dependent.

The current and the stress tensor are specialised to obey the above symmetries. The spatial components of the current vanishes because of the spatial reflection symmetry. Considering the z axis as the longitudinal axis,the stress tensor in the LRF takes the form,

$$T^{\mu\nu} = (\epsilon + P_T)u^\mu u^\nu - P_T g^{\mu\nu} + (P_L - P_T)Z^\mu Z^\nu \tag{13}$$

where ϵ is the energy density, P_T and P_L are the transverse and longitudinal pressures. One can derive the above form by imposing the symmetries on the general form of $T^{\mu\nu}$ as given in equation 12. Similarly for the 3rd moment in the LRF using the $0 + 1$d symmetries,

$$I_{\mu\nu\sigma} = I_u u_{(\mu} u_\nu u_{\sigma)} + I_x u_{(\mu} X_\nu X_{\sigma)} + I_y u_{(\mu} Y_\nu Y_{\sigma)} + I_z u_{(\mu} Z_\nu Z_{\sigma)} \text{[2]} \tag{14}$$

We will be requiring the lab frame tetrads, which are obtained by boosting with the spacetime rapidity ζ,

$$\begin{aligned} u^\mu &= (cosh\zeta, 0, 0, sinh\zeta)\text{[3]} \\ X^\mu &= (0, 1, 0, 0) \\ Y^\mu &= (0, 0, 1, 0) \\ Z^\mu &= (sinh\zeta, 0, 0, cosh\zeta) \end{aligned} \tag{15}$$

4 Conformal 0+1d aHydro

Just like isotropic hydrodynamics, where we considered an one-particle distribution function and proceeded to evaluate weighted integrals, aHydro proceeds by postulating a general distribution function that may become highly anisotropic along one direction in momentum space. The following is the ansatz considered by Romatschke and Strickland [2].

$$f(x,p) = f_{\rm eq}\left(\frac{\sqrt{\mathbf{p}^2 + \xi(x)p_z^2}}{\Lambda(x)}, \frac{\mu(x)}{\Lambda(x)}\right) \tag{17}$$

Here $f_{eq}(\hat{E}) = \frac{1}{[exp(\hat{E})+a]}$ is an equilibrium distribution function with $a = 0, 1,$ and -1 corresponding to classical, Fermi-Dirac, and Bose-Einstein statistics, respectively, ξ(x) is the local anisotropy parameter, Λ(x) is the local scale parameter which reduces to the temperature in the isotropic limit, ξ(x) $\to$ 0, and μ(x) is the local chemical potential.

[2] Note that $A_{(\mu}B_\nu c_{\sigma)} = A_\mu B_\nu c_\sigma + B_\mu A_\nu c_\sigma + B_\mu C_\nu A_\sigma$

[3] Notice that this is nothing but the Bjorken flow discussed in class,

$$\frac{u_z}{u_t} = \frac{z}{t} \tag{16}$$

which is the boost invariance we are discussing in this note.

We see that in the Local Rest Frame (LRF) f is intrinsically momentum-space anisotropic along the beam axis direction after introducing the $\xi(x)p_z^2$ term. As we go ahead, the form of f and further properties of the parameters would be derived. We will be working in the zero chemical potential case ($\mu = 0$).

Visualising the ansatz: In our understanding, the way to visualise the effect of introducing ξ is to first think of the non-relativistic limit. In that limit the energy dispersion relation is quadratic in p_x, p_y and p_z. The effect of ξ in front of the p_z^2 is then to change the width of the distribution around the mean momentum (in the case of Boltzmann statistics, the standard deviation σ of the Gaussian is scaled by $\xi^{-1/2}$). Similar effects will be seen for any of the statistics and for relativistic dispersion relation also. If ξ is -1, the distribution is independent of p_z term making it homogeneous in that direction. $\xi = 0 \implies$ isotropic distribution in momentum space, while larger positive values mean higher anisotropies.

Why the limit?: $\xi \in (-1, \infty)$? The limit ensures that the coefficient of p_z is non-negative which is necessary for the square root in the energy-momentum dispersion relation to be well defined. This simple constraint, as we will see, leads to many important results. For example, this is linked to the positivity of the longitudinal and transverse pressures, it also puts bounds on the shear stress tensor component which is used to get viscous-Hydro from aHydro in the small anisotropy limit. Even the positivity of the entropy production can be tracked back to this.

Why introduce Λ?: With large scale momentum space anisotropies, it is reasonable to expect that thermodynamic variables might not be well defined. This stops us from writing T (for temperature) as required in the f_{eq}. However, as we shall see, the Landau matching condition which corresponds to construction of an artificial equilibrium, helps us by providing a relation between ξ and T.

Let us look at expressions of the number density (n), energy density (ϵ) and pressures using the new distribution function:

$$n(\xi, \Lambda) \equiv \int \frac{d^3p}{(2\pi)^3} f_{\text{eq}} \left(\sqrt{\mathbf{p}^2 + \xi(x)p_z^2}/\Lambda(x) \right)$$

(After a change in parameter $p_z'^2 = (1+\xi)p_z^2$)

$$n(\xi, \Lambda) = \frac{1}{\sqrt{1+\xi}} \int \frac{d^3p}{(2\pi)^3} f_{\text{eq}}(|\mathbf{p}|/\Lambda(x))$$

$$n(\xi, \Lambda) = \frac{1}{\sqrt{1+\xi}} n_{\text{eq}}(\Lambda) \tag{18}$$

We arrive from the first line to the second after re-parameterising and then relabelling $p_z'^2$ as p_z^2 (notice that the integral does not change when you do so).

We now calculate the energy density:

$$\epsilon = \int dPE^2 f_{\text{eq}} \left(\sqrt{\mathbf{p}^2 + \xi(x)p_z^2}/\Lambda(x) \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \sqrt{p_x^2 + p_y^2 + p_z^2} f_{\text{eq}} \left(\sqrt{\mathbf{p}^2 + \xi(x)p_z^2}/\Lambda(x) \right)$$

(After introducing parameter change similar to what was done for n, but in spherical coordinates and we also get another term under a square root.)

$$= \frac{1}{\sqrt{1+\xi}} \int \frac{d^3p}{(2\pi)^3} |\mathbf{p}| \sqrt{\sin^2\theta + \frac{\cos^2\theta}{1+\xi}} f_{\text{eq}}(|\mathbf{p}|/\Lambda(x))$$

[The integral over θ does not involve f_{eq}, so we take it out and multiply and divide by another $\int d\theta$ factor (this way integral over d^3p remains, and we get another 1/2 factor)]

$$\epsilon = \left(\frac{1}{2\sqrt{1+\xi}}\int d(\cos\theta)\sqrt{\sin^2\theta + \frac{\cos^2\theta}{1+\xi}}\right)\int \frac{d^3p}{(2\pi)^3}|\mathbf{p}|f_{\text{eq}}(|\mathbf{p}|/\Lambda(x)) \tag{19}$$

We write the equation 19 in a simpler form: $\epsilon = \mathcal{R}(\xi)\epsilon_{\text{eq}}(\Lambda)$, where $\mathcal{R}(\xi) = \frac{1}{2}\left[\frac{1}{1+\xi} + \frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right]$

Following a similar procedure, we get

$$\begin{aligned} P_T &= \mathcal{R}_T(\xi)P_{\text{eq}}(\Lambda) \\ P_L &= \mathcal{R}_L(\xi)P_{\text{eq}}(\Lambda) \end{aligned} \tag{20}$$

where $\mathcal{R}_T(\xi) = \frac{3}{2\xi}\left[\frac{1+(\xi^2-1)\mathcal{R}(\xi)}{\xi+1}\right]$ and $\mathcal{R}_L(\xi) = \frac{3}{\xi}\left[\frac{(\xi+1)\mathcal{R}(\xi)-1}{\xi+1}\right]$.

Let us evaluate the three-rank tensor further using the aHydro distribution function, which will be of help to us later down the line:

$$\begin{aligned} I_u &= \mathcal{S}_u(\xi)I_{\text{eq}}(\Lambda), \\ I_x = I_y &= \mathcal{S}_T(\xi)I_{\text{eq}}(\Lambda), \\ I_z &= \mathcal{S}_L(\xi)I_{\text{eq}}(\Lambda) \end{aligned} \tag{21}$$

where $\mathcal{S}_u(\xi) = \frac{3+2\xi}{(1+\xi)^{3/2}}$, $\mathcal{S}_T(\xi) = \frac{1}{\sqrt{1+\xi}}$ and $\mathcal{S}_L(\xi) = \frac{1}{(1+\xi)^{3/2}}$.

Performing these integrations are very similar to those in equation 19, but instead of a complicated integral like $\mathcal{R}(\xi)$, we have a simpler integrand that can be calculated to obtain the above equations.

4.1 Equations of motion

The objective of this section is to obtain aHydro equations of motion for the zeroth, first and second moments of the the zeroth order of $\mathcal{C}[f]$. Starting with equations 6 and 12, we get:

$$\mathcal{D}_\mu n + n\theta_\mu = \mathcal{C}_0 = \partial_\tau n + \frac{n}{\tau} \tag{22}$$

To move ahead we will assume that the collisional kernal takes up the RTA form (Relaxation Time Approximation):

$$\mathcal{C}[f] = \frac{p.u}{\tau_{eq}}[f - f_{eq}(T)] \tag{23}$$

where $\tau_{eq} = \frac{5\bar{\eta}}{T}$ is the relaxation time [1], and $\bar{\eta} = \frac{\eta}{s}$ is the ratio of shear viscosity to entropy density.

As explained in [3], it is necessary to implement the Landau condition $(\epsilon(\xi,\Lambda) = \epsilon_{eq}(T))$ to determine the isotropic equilibrium energy density $\epsilon_{eq}(T)$ from an anistropic single-particle distribution function.

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3}|\mathbf{p}|\frac{1}{\exp\left(\sqrt{\mathbf{p}^2+\xi p_z^2}/\Lambda\right)+a} = \int \frac{d^3\mathbf{p}}{(2\pi)^3}|\mathbf{p}|\frac{1}{\exp(|\mathbf{p}|/T)+a}$$

(We exactly get the form of equation, we obtained while deriving equation 19)

$$\mathcal{R}(\xi)\int \frac{d^3\mathbf{p}}{(2\pi)^3}|\mathbf{p}|\frac{1}{\exp(|\mathbf{p}|/\Lambda)+a} = \int \frac{d^3\mathbf{p}}{(2\pi)^3}|\mathbf{p}|\frac{1}{\exp(|\mathbf{p}|/T)+a}$$

(We change to spherical coordinates and on the left hand side we reparameterise $|\mathbf{p}| = \text{y}\Lambda$ and on the right hand side, we use $|\mathbf{p}| = \text{y'T}$. And since Λ and T does not depend on $|\mathbf{p}|$, we can take it outside the integral.)

$$\mathcal{R}(\xi)\Lambda^4 \int \frac{d^3\mathbf{y}}{(2\pi)^3}|\mathbf{y}|\frac{1}{\exp\left(|\mathbf{y}|\right)+a} = T^4 \int \frac{d^3\mathbf{y}'}{(2\pi)^3}|\mathbf{y}'|\frac{1}{\exp(|\mathbf{y}'|)+a}$$

$$T = \mathcal{R}^{1/4}(\xi)\Lambda \tag{24}$$

Evaluating $\mathcal{C}_0$ using equations 3, 18 and 23, one obtains:

$$\mathcal{C}_0 = \int \frac{p.u}{\tau_{eq}}[f - f_{eq}(T)]dP = \frac{n_{\text{eq}}}{\tau_{\text{eq}}}\left(\frac{1}{\sqrt{1+\xi}} - \mathcal{R}^{3/4}(\xi)\right) \tag{25}$$

(The first term above comes trivially using equation 18, but the calculation for the second term involving f_{eq} (writing $T = \mathcal{R}^{1/4}(\xi)(\xi)\Lambda$ and reparameterising p=p'$\mathcal{R}^{1/4}$, in spherical co-ordinates) is as shown below.)

$$-\int \frac{d^3\mathbf{p}}{(2\pi)^3 p^0}\frac{p.u}{\tau_{eq}}\frac{1}{\exp(|\mathbf{p}|/T)+a} = -\frac{\mathcal{R}^{3/4}(\xi)}{\tau_{eq}}\int \frac{d^3\mathbf{p}'|p'^2|}{(2\pi)^3}\frac{1}{\exp(|\mathbf{p}'|/\Lambda)+a} = -R^{3/4}(\xi)\frac{n_{eq}}{\tau_{eq}}$$

Again using equation 18, we can expand the left-hand-side of equation 22 in terms of derivatives of ξ and Λ, this gives us:

$$\frac{1}{1+\xi}\partial_\tau\xi - \frac{6}{\Lambda}\partial_\tau\Lambda - \frac{2}{\tau} = \frac{-2}{\tau_{\text{eq}}}\left(1 - R^{3/4}(\xi)\sqrt{1+\xi}\right) \tag{26}$$

NOTE: The above equation is as written in [1], but we get an additional minus sign on the full term in the RHS. Fortunately we do not use the zeroth moment (soon we will know why) to obtain further results, hence we have no other contradictions w.r.t [1].

In equation 26, the first and third term on the LHS, and the RHS are trivially obtained, but the second term on the LHS ($\frac{1}{\sqrt{1+\xi}}\partial_\tau n_{eq}(\Lambda) = \frac{6}{\Lambda}\partial_\tau\Lambda$) needs some manipulation:

$$\frac{1}{\sqrt{1+\xi}}\partial_\tau n_{eq}(\Lambda) = \frac{\partial_\tau}{\sqrt{1+\xi}}\int \frac{d^3p}{(2\pi^3)}|p|^2\frac{1}{e^{|p|/\Lambda}+a}$$

(Using y = p/Λ)

$$\frac{\partial_\tau}{\sqrt{1+\xi}}\int \Lambda^3\frac{d^3y}{(2\pi^3)}|y|^2\frac{e^{-y}}{1+ae^y}$$

(We see that apart from Λ, no term in the integral is dependent on τ, so the derivative acts only on Λ. Λ can also be taken out of the integral as it does not depend on y. We also see that the $1/(1+ae^{-y})$ can be Taylor expanded for small y, and then can be expressed as an infinite GP sum, and then the integral can be reduced as a gamma function using a substitution y'=(n+1)x, and the integral will finally reduce to something like $\sum_{n=0}^{\infty}\frac{a^2\Gamma(3)}{(n+1)^3}$ but those are just steps in extra effort to obtain a value for an integral, and are irrelevant.)

$$\frac{\partial_\tau(\Lambda^3)}{\sqrt{1+\xi}}\int \frac{d^3x}{(2\pi^3)}|x|^2\frac{e^{-x}}{1+ae^x} = \frac{3\Lambda^2\partial_\tau\Lambda}{\sqrt{1+\xi}}\int \frac{d^3x}{(2\pi^3)}|x|^2\frac{e^{-x}}{1+ae^x}$$
$$= \frac{3\partial_\tau\Lambda}{\Lambda\sqrt{1+\xi}}\int \Lambda^3\frac{d^3x}{(2\pi^3)}|x|^2\frac{e^{-x}}{1+ae^x} = \frac{3n_{eq}\partial_\tau\Lambda}{\Lambda\sqrt{1+\xi}}$$

Performing a similar manipulation on the first moment equation of $\mathcal{C}_0$, we get a simplified form of energy conservation in $0+1$d, i.e.

$$\frac{\partial \epsilon(\tau)}{\partial \tau} = -\frac{\epsilon(\tau) + P_L(\tau)}{\tau} \tag{27}$$

(Use the $T^{\mu\nu}$ as given in 13, and construct its derivative along u^μ (that is, $u_\mu \partial_\nu T^{\mu\nu} = 0$).)

we get:

$$\frac{\mathcal{R}'(\xi)}{\mathcal{R}(\xi)} \partial_\tau \xi + \frac{4}{\Lambda} \partial_\tau \Lambda = \frac{1}{\tau} \left[\frac{1}{\xi(1+\xi)\mathcal{R}(\xi)} - \frac{1}{\xi} - 1 \right] \tag{28}$$

And turning to second moment of $\mathcal{C}_0$, [1] obtains the zz and xx (=yy)projections of equation 10 respectively, as:

$$(\log \mathcal{S}_L)' \, \partial_\tau \xi + 5 \partial_\tau \log \Lambda + \frac{3}{\tau} = \frac{1}{\tau_{\text{eq}}} \left[\frac{\mathcal{R}^{5/4}}{\mathcal{S}_L} - 1 \right]$$
$$(\log \mathcal{S}_T)' \, \partial_\tau \xi + 5 \partial_\tau \log \Lambda + \frac{1}{\tau} = \frac{1}{\tau_{\text{eq}}} \left[\frac{\mathcal{R}^{5/4}}{\mathcal{S}_T} - 1 \right]$$

The equation of motion for the second moment (equation 29) is thus obtained from the above equations, after solving further:

$$\frac{1}{1+\xi} \partial_\tau \xi - \frac{2}{\tau} + \frac{\mathcal{R}^{5/4}(\xi)}{\tau_{\text{eq}}} \xi \sqrt{1+\xi} = 0 \tag{29}$$

We now have fully obtained the equations of motion for the zeroth, first and second moment of $\mathcal{C}_0$, and this can be continued till whenever we want, but since we have two unknowns, ξ and Λ, we need only two at a time to decouple and solve them.

Apparently, higher moments are sensitive to high-momentum behavior of the distribution, so the authors naturally consider the lowest possible moments. Thus there are two viable possibilities, 0^{th}+1^{st} or 1^{st}+2^{nd} moments. But the authors point out that 0^{th}+1^{st} moment equations do not reproduce correct near-equilibrium equations in the small anisotropic limit, and that 1^{st}+2^{nd} does reproduce the same. But with hindsight, zeroth, first and second moments are needed in the non-conformal case (to solve for chemical potential).

5 Consistency checks

Now that we have equation 29, in principle, we know the evolution of the anisotropy parameter, and hence the distribution function (as given by the ansatz 17). All the hydrodynamic variables can then be derived. But how do we know that the predicted dynamics are correct? In this section we take a closer look at the equation 29, analyse it in various limits, and perform the standard check of entropy production given by the second law of thermodynamics.

5.1 Relation to second-order viscous hydrodynamics in the small anisotropy limit

In order to make relation to viscous hydrodynamics, one usually adds dissipative terms to $T^{\mu\nu}$ decomposed along the various tensor degrees of freedom. However, we are dealing with the special case of $0+1d$ system and hence we only need the scalar term representing dissipation, π. The reflection symmetry, allows us to get away by adding it to the diagonal terms only. The trace of the new $T^{\mu\nu}$ should still give zero and hence with this constraint, we see that the new relations must be: $P_L = P_{eq}(T) - \pi$ and $P_T = P_{eq}(T) + \frac{\pi}{2}$. Using the relation, equation 27 can be written as:

$$\tau \partial_\tau log(\epsilon) = -\frac{4}{3} + \frac{\pi}{\epsilon} \tag{30}$$

Define

$$\overline{\pi} = \frac{\pi}{\epsilon} = \frac{1}{3}\left[1 - \frac{\mathcal{R}_L(\xi}{\mathcal{R}(\xi)}\right] \tag{31}$$

It is worthwhile to note here that $\xi \in (-1, \infty) \implies \overline{\pi} \in (-2/3, 1/3)$. Taking derivative of the first equality in equation 31 gives:

$$\frac{\partial_\tau \pi}{\epsilon} = \overline{\pi}^{'} \partial_\tau \xi + \overline{\pi} \partial_\tau log(\epsilon) \tag{32}$$

Where $\overline{\pi}^{'} = d\overline{\pi}/d\xi$ Using the definition of $\overline{\pi}$ and 29 a dynamical equation for π is obtained as:

$$\frac{\partial_\tau \pi}{\epsilon} + \frac{\pi}{\epsilon\tau}\left(\frac{4}{3} - \frac{\pi}{\epsilon}\right) = \left[\frac{2(1+\xi)}{\tau} - \frac{\mathcal{H}(\xi)}{\tau_{eq}}\right]\overline{\pi}^{'}(\xi) \tag{33}$$

with $\mathcal{H}(\xi) = \xi(1+\xi)^{3/2}\mathcal{R}^{5/4}(\xi)$. Note that the above equation has only π as the unknown as the ξ's can be expressed as $\xi(\pi)$ by inverting equation 31, and the energy density can be eliminated by the relation $\epsilon = \frac{15}{4}\frac{\eta}{\tau_{eq}}$.

Expanding all the non-linear terms in equation 31 in a Taylor series about $\xi = 0$, one could get $\overline{\pi} \equiv \overline{\pi}(\xi)$, $\overline{\pi}^{'} \equiv \overline{\pi}^{'}(\xi)$, and $\mathcal{H} \equiv \mathcal{H}(\xi)$ as a series is ξ; inverting which one obtains:

$$\xi = \frac{45}{8}\overline{\pi}\left[1 + \frac{195}{56}\overline{\pi} + \mathcal{O}(\overline{\pi})\right] \tag{34}$$

Plugging it back in the obtained series for $\overline{\pi}$, $\overline{\pi}^{'}$ and $\mathcal{H}$, and putting those in equation 33 gives:

$$\partial_\tau \pi - \frac{4\eta}{3\tau_{eq}\tau} + \frac{38\pi}{21\tau} = -\frac{\pi}{\tau_{eq}} \tag{35}$$

The above equation is in agreement with vHydro equations that were obtained by [4] with the RTA approximation. Hence the $0 + 1d$ aHydro reduces to vHydro in the small anisotropy limit.

5.2 Free streaming and ideal limits

The small anisotropy limit means that $\overline{\eta} \longrightarrow 0$ (or $T \longrightarrow \infty$) and hence $\tau_{eq} \longrightarrow 0$ owing to the relation $\tau_{eq} = 5\overline{\eta}/T$. In this limit, the equation 35 gives the ideal hydrodynamics.

However, there is one more limiting case which it correctly reproduces, and that is the free streaming of particles which corresponds to the opposite limit: $\overline{\eta} \longrightarrow \infty$ (or equivalently, $\tau_{eq} \longrightarrow \infty$). Subject to this limit, one obtains from equation 29:

$$\frac{1}{1+\xi}\partial_\tau \xi = \frac{2}{\tau} \tag{36}$$

Same limit, and the above equation reduces equation 28 to: $\partial_\tau \Lambda = 0$. The above two have solutions: $\xi \sim \tau^2$ and $\lambda = \lambda_0$.

5.3 Thermodynamic check

Similar to the derivations of equations 18, 19 and 20, one can get the entropy as a function of the anisotropy parameter ξ and hard momentum scale Λ.

$$\mathcal{S} = \frac{S_{iso}(\Lambda)}{\sqrt{1+\xi}} \tag{37}$$

Now, the second law of thermodynamics demands that the defined entropy have a positive divergence, that is $\partial_\mu \mathcal{S}^\mu \geq 0$. In the $0+1d$ case this reduces to $\partial_\tau(\tau\mathcal{S}) \geq 0$ which on using equation 37 and equation 22 appears as:

$$\frac{\partial_\tau(\tau\mathcal{S})}{\mathcal{S}} = \frac{2\tau}{\tau_{eq}}\left[\mathcal{R}^{3/4}(\xi)\sqrt{1+\xi}-1\right] \tag{38}$$

For all $\xi \in (-1,\infty)$, $\left[\mathcal{R}^{3/4}(\xi)\sqrt{1+\xi}\right] \geq 1$ and hence the right-hand side is positive. This confirms that the formalism developed here, respects the second law of thermodynamics.

6 Numerical solution to aHydro equation

We solve equation 29 numerically and evaluate the pressure anisotropy (P_T/P_L) till $\tau = 100 fm/c$. The behaviour of the direct solution of equation 29, that is $\xi(\tau)$ is also plotted to show that ξ does eventually converge to zero as expected. The convergence of ξ to zero naturally corresponds to convergence of pressure anisotropy to 1 (that is, the isotropic limit). The time taken for convergence depends on $\overline{\eta}$ among other variables as can be seen in figure 1. The plots were generated in python using mid-point method of solving differential equations, with the initial conditions: $T_0 = 0.3GeV$, $\xi_0 \approx 0$, $\tau = 2.5fm/c$. It is worthwhile to note that the pressure anisotropy converges to 1, the isotropic limit, at time scales much larger than $1fm/c$.

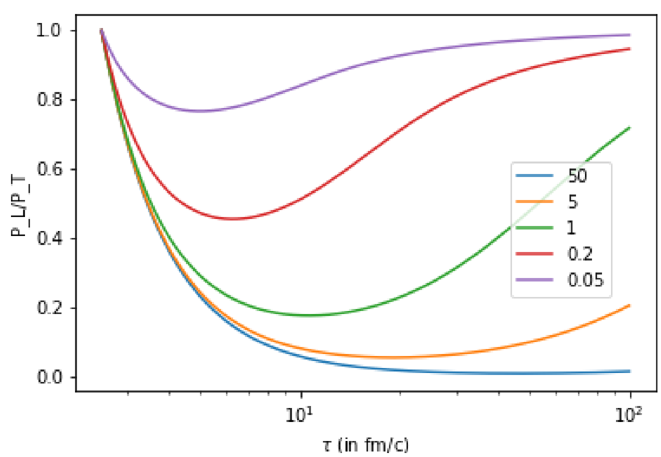

Figure 1: The 0+1d pressure anisotropy P_L/P_T predicted by the aHydro equation 29. The legend contains the $\overline{\eta}$ values used for each curve. Note that the free streaming limit would be a steep vertical line with negative slope, while the ideal hydro limit $\overline{\eta} \longrightarrow 0$ will be a horizontal line at $P_L/P_T = 1$.

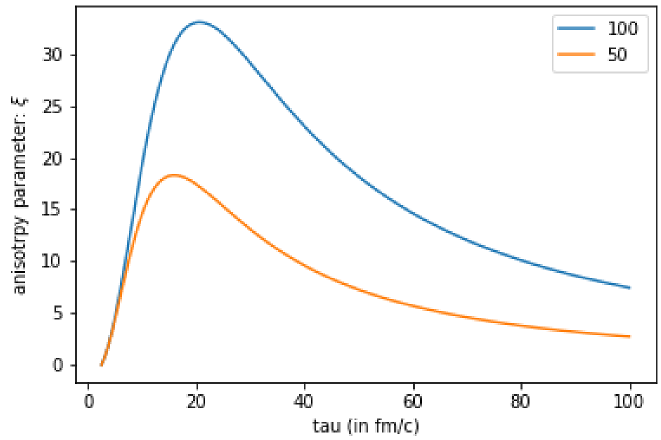

Figure 2: Evolution of ξ between $\tau = 2.5fm/c$ and $\tau = 100fm/c$ with the mentioned initial conditions.

7 Summary

We have considered the simplest case of anisotropic flow and exhibited that it reduces to the ideal and free streaming limits. Looking ahead, one relaxes the symmetry assumptions of $0 + 1$d case. The analysis is the same as above, but with increased complexity.

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