Introduction to Anisotropic Hydrodynamics: The case of longitudinal expansion

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What do we know?

- Heavy ion experiments indicate collective behaviour, existence of QGP, well described by non-dissipative hydrodynamics and also has significant momentum anisotropy, in the early times.
- Studies of this behaviour using the isotropic non-dissipative hydrodynamics (kindergarten hydrodynamics) predicts sub-fm/c isotropisation.
- Inclusion of dissipation in isotropic hydrodynamics introduces momentum anisotropies, still isotropisation scale is unaltered.

What is the problem?

Theoretical studies firmly establish the following,

- Sub-fm/c isotropisation is unlikely.
- Large momentum space anisotropy in the early times and dilute edges of the fireball

Possible resolution

We should then alter our microscopic picture and include anisotropy in hydrodynamics, the one particle distribution function f.

Overview

- Motivation
- Prelude to Hydrodynamics from Kinetic Theory
- Conformal 0+1d aHydro
- Equations of Motion
- Consistency Checks
- Numerical solution to aHydro equations

Moments of Boltzmann equation

Start from the Boltzmann equation,

Boltzmann equation

$$p^\mu \partial_\mu f(x,p) = -C[f] \tag{1}$$

By taking various moments of the Boltzmann equation, we obtain the hydrodynamic equations. Consider the following definitions,

$$\int dP \equiv N_{dof} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{p^0} \tag{2}$$

$$C_r^{\mu_1\mu_2\cdots\mu_n} \equiv -\int dP (p.u)^r p^{\mu_1} p^{\mu_2} \cdots p^{\mu_n} C[f] \tag{3}$$

0^{th} moment

The 0^{th} moment of the BE is,

$$\partial_\mu \left(\int dP p^\mu f \right) = - \int dP\, C[f] = C_0 \tag{4}$$

The particle four current is,

$$J^\mu \equiv \int dP\, p^\mu f \tag{5}$$

hence,

$$\partial_\mu J^\mu = C_0 \tag{6}$$

This is the equation dictating the particle current. C_0 is obtained once we plug in a $C[f]$ in Equation 4.

1^{st} moment

1^{st} moment gives the equation for the stress tensor,

$$T^{\mu\nu} \equiv \int dP \, p^\mu p^\nu f \tag{7}$$

hence,

$$\partial_\mu T^{\mu\nu} = C_0^\nu \tag{8}$$

In any microscopic model, the 1st moment of $C[f]$ vanishes because of the four momentum conserving delta function.
For effective collision kernel (e.g. under RTA), the vanishing of C_0^ν leads to additional constrains on the system.
Higher moments can be obtained like so.

Higher Moment

In the similar spirit as above, one can define the higher moments as follows,

$$I^{\mu\nu_1\nu_2\cdots\nu_n} \equiv \int dP\, p^{\mu} p^{\nu_1} p^{\nu_2} \cdots p^{\nu_n} f \tag{9}$$

and hence,

$$\partial_{\mu} I^{\mu\nu_1\nu_2\cdots\nu_n} = \mathcal{C}_0^{\nu_1\nu_2\cdots\nu_n} \tag{10}$$

Orthonormal Tetrads

Choose the orthonormal tetrads $X_i\,\{i = 0, 1, 2, 3\}$ and expand the tensors in that basis. In the local rest frame,

$$X^{\mu}_{i,LRF} = \delta^{\mu}_{i} \quad i = 0, 1, 2, 3 \tag{11}$$

Usually the X_0 is taken as the fluid four velocity u and the other vectors are denoted as $X_1 = X$, $X_2 = Y$ and $X_3 = Z$. The current and stress tensor are written in this basis.

$$\begin{aligned} J^{\mu} &= nu^{\mu} + n^{i} X^{\mu}_{i} \\ T^{\mu\nu} &= t_{00} g^{\mu\nu} + \sum_{i=1}^{3} t_{ii} X^{\mu}_{i} X^{\nu}_{i} + \sum_{i,j\neq 0, i>j}^{3} t_{ij} X^{\mu}_{i} X^{\nu}_{j} \end{aligned} \tag{12}$$

where in this co-ordinate system, $g^{\mu\nu} = u^{\mu} u^{\nu} - \sum_{i=1}^{3} X^{\mu}_{i} X^{\nu}_{i}$.

Symmetries of 0+1d flow

We consider the system with the following symmetries,

- Longitudinal boost invariance
- Transverse homogeneity
- Conformal invariance (Massless case)
- Reflection symmetric along spatial directions

n^i's vanishes because of the spatial reflection symmetry.
Considering the z axis as the longitudinal axis, the stress tensor takes the form,

$$T^{\mu\nu} = (\epsilon + P_T)u^\mu u^\nu - P_T g^{\mu\nu} + (P_L - P_T)Z^\mu Z^\nu \tag{13}$$

where ϵ is the energy density, P_T and P_L are the transverse and longitudinal pressures. Similarly for the 3rd moment using the $0 + 1$d symmetries,

$$I_{\mu\nu\sigma} = I_u u_{(\mu} u_\nu u_{\sigma)} + I_x u_{(\mu} X_\nu X_{\sigma)} + I_y u_{(\mu} Y_\nu Y_{\sigma)} + I_z u_{(\mu} Z_\nu Z_{\sigma)} \tag{14}$$

Lab frame

For the $0+1$d case the lab frame tetrads are obtained by boosting with the spacetime rapidity $\zeta = arctanh(z/t)$, since the system is boost invariant

$$\begin{aligned} u^\mu &= (cosh\zeta, 0, 0, sinh\zeta) \\ X^\mu &= (0, 1, 0, 0) \\ Y^\mu &= (0, 0, 1, 0) \\ Z^\mu &= (sinh\zeta, 0, 0, cosh\zeta) \end{aligned} \tag{15}$$

Conformal 0+1d aHydro

$$f(x,p) = f_{\mathrm{eq}}\left(\frac{\sqrt{\mathbf{p}^2+\xi(x)p_z^2}}{\Lambda(x)}, \frac{\mu(x)}{\Lambda(x)}\right) \tag{16}$$

Here $f_{eq}(\hat{E}) = \frac{1}{[exp(\hat{E})+a]}$ is an equilibrium distribution function with a = 0, 1, and -1 corresponding to classical, Fermi-Dirac, and Bose-Einstein statistics, respectively, ξ(x) is the local anisotropy parameter, Λ(x) is the local scale parameter which reduces to the temperature in the isotropic limit, ξ(x) $\rightarrow$ 0, and μ(x) is the local chemical potential.

Number Density

$$n(\xi,\Lambda) \equiv \int \frac{d^3p}{(2\pi)^3} f_{\rm eq}\left(\sqrt{\mathbf{p}^2+\xi(x)p_z^2}/\Lambda(x)\right)$$

(After a change in parameter $p_z'^2 = (1+\xi)p_z^2$)

$$n(\xi,\Lambda) = \frac{1}{\sqrt{1+\xi}} \int \frac{d^3p}{(2\pi)^3} f_{\rm eq}(|\mathbf{p}|/\Lambda(x))$$

$$n(\xi,\Lambda) = \frac{1}{\sqrt{1+\xi}} n_{\rm eq}(\Lambda) \tag{17}$$

Energy Density

$$
\begin{aligned}
\epsilon &= \int dP E^2 f_{\mathrm{eq}}\left(\sqrt{\mathbf{p}^2+\xi(x)p_z^2}/\Lambda(x)\right)\\
&= \int \frac{d^3p}{(2\pi)^3}\sqrt{p_x^2+p_y^2+p_z^2}\, f_{\mathrm{eq}}\left(\sqrt{\mathbf{p}^2+\xi(x)p_z^2}/\Lambda(x)\right)\\
&= \frac{1}{\sqrt{1+\xi}}\int \frac{d^3p}{(2\pi)^3}|\mathbf{p}|\sqrt{\sin^2\theta+\frac{\cos^2\theta}{1+\xi}}\, f_{\mathrm{eq}}(|\mathbf{p}|/\Lambda(x))
\end{aligned}
$$

$$
\epsilon = \left(\frac{1}{2\sqrt{1+\xi}}\int d(\cos\theta)\sqrt{\sin^2\theta+\frac{\cos^2\theta}{1+\xi}}\right)\int \frac{d^3p}{(2\pi)^3}|\mathbf{p}| f_{\mathrm{eq}}(|\mathbf{p}|/\Lambda(x)) \tag{18}
$$

We write the Equation 18 in a simpler form: $\epsilon = \mathcal{R}(\xi)\epsilon_{\mathrm{eq}}(\Lambda)$, where $\mathcal{R}(\xi) = \frac{1}{2}\left[\frac{1}{1+\xi}+\frac{\arctan\sqrt{\xi}}{\sqrt{\xi}}\right]$

Transverse and Longitudinal Pressures

$$
\begin{aligned}
P_T &= \mathcal{R}_T(\xi) P_{\mathrm{eq}}(\Lambda) \\
P_L &= \mathcal{R}_L(\xi) P_{\mathrm{eq}}(\Lambda)
\end{aligned}
\tag{19}
$$

where $\mathcal{R}_T(\xi) = \frac{3}{2\xi}\left[\frac{1+\left(\xi^2-1\right)\mathcal{R}(\xi)}{\xi+1}\right]$ and $\mathcal{R}_L(\xi) = \frac{3}{\xi}\left[\frac{(\xi+1)\mathcal{R}(\xi)-1}{\xi+1}\right]$.

The 3-rank Tensor I_{eq}

$$
\begin{aligned}
I_{eq} &= \frac{1}{3}\int dP E^3 f_{eq} \\
I_u &= \mathcal{S}_u(\xi) I_{\rm eq}(\Lambda), \\
I_x = I_y &= \mathcal{S}_T(\xi) I_{\rm eq}(\Lambda), \\
I_z &= \mathcal{S}_L(\xi) I_{\rm eq}(\Lambda)
\end{aligned} \tag{20}
$$

where $\mathcal{S}_u(\xi) = \frac{3+2\xi}{(1+\xi)^{3/2}}$, $\mathcal{S}_T(\xi) = \frac{1}{\sqrt{1+\xi}}$ and $\mathcal{S}_L(\xi) = \frac{1}{(1+\xi)^{3/2}}$.

Equations of Motion

This section would involve deriving the equations of motion for the zeroth, first and second order of $\mathcal{C}_0$. Before we begin, there are some general quantities we should look at, that would be constantly used. Starting from what was done in class:

$$\partial_\tau n + \frac{n}{\tau} = \mathcal{C}_0 \tag{21}$$

$$\text{RTA: } \mathcal{C}[f] = \frac{p.u}{\tau_{eq}}[f - f_{eq}(T)] \tag{22}$$

As explained in [1], we implement the Landau condition $(\epsilon(\xi, \Lambda) = \epsilon_{eq}(T))$ which gives:

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3}|\mathbf{p}|\frac{1}{\exp\left(\sqrt{\mathbf{p}^2 + \xi p_z^2}/\Lambda\right) + a} = \int \frac{d^3\mathbf{p}}{(2\pi)^3}|\mathbf{p}|\frac{1}{\exp(|\mathbf{p}|/T) + a}$$

(We exactly get the form of equation, we obtained while deriving the energy density term.)

$$\mathcal{R}(\xi) \int \frac{d^3\mathbf{p}}{(2\pi)^3} |\mathbf{p}| \frac{1}{\exp\left(|\mathbf{p}|/\Lambda\right) + a} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} |\mathbf{p}| \frac{1}{\exp(|\mathbf{p}|/T) + a}$$

$$\mathcal{R}(\xi)\Lambda^4 \int \frac{d^3\mathbf{y}}{(2\pi)^3} |\mathbf{y}| \frac{1}{\exp\left(|\mathbf{y}|\right) + a} = T^4 \int \frac{d^3\mathbf{y}'}{(2\pi)^3} |\mathbf{y}'| \frac{1}{\exp(|\mathbf{y}'|) + a}$$

$$T = \mathcal{R}^{1/4}(\xi)\Lambda \tag{23}$$

EOM for Zeroth Moment

$$\mathcal{C}_0 = \int \frac{p.u}{\tau_{eq}}[f - f_{eq}(T)]dP = \frac{n_{\rm eq}}{\tau_{\rm eq}}\left(\frac{1}{\sqrt{1+\xi}} - R^{3/4}(\xi)\right) \qquad (24)$$

(The first term above comes trivially using $n(\xi,\Lambda) = \frac{1}{\sqrt{1+\xi}} n_{\rm eq}(\Lambda)$. But its the second term that involves some manipulation.)

We need to evaluate:

$$-\int \frac{d^3\mathbf{p}}{(2\pi)^3 p^0} \frac{p.u}{\tau_{eq}} \frac{1}{\exp(|\mathbf{p}|/T) + a}$$

EOM for Zeroth Moment

The calculation for the second term involving f_{eq} (writing $T = \mathcal{R}^{1/4}(\xi)(\xi)\Lambda$ and reparameterising p=p'$\mathcal{R}^{1/4}$, in spherical co-ordinates) is as shown below:

$$= -\frac{\mathcal{R}^{3/4}(\xi)}{\tau_{eq}} \int \frac{d^3\mathbf{p}'|p'^2|}{(2\pi)^3} \frac{1}{\exp(|\mathbf{p}'|/\Lambda) + a}$$

$$= -\mathcal{R}^{3/4}(\xi)\frac{n_{eq}}{\tau_{eq}} \tag{25}$$

Plugging $\mathcal{C}_0$ in Equation $\partial_\tau n + \frac{n}{\tau} = \mathcal{C}_0$ would gives us:

$$\frac{1}{1+\xi}\partial_\tau \xi - \frac{6}{\Lambda}\partial_\tau \Lambda - \frac{2}{\tau} = \frac{2}{\tau_{\mathrm{eq}}}\left(1 - R^{3/4}(\xi)\sqrt{1+\xi}\right) \tag{26}$$

EOM for First Moment

Performing a similar manipulation on the first moment equation of $\mathcal{C}_0$, starting from the simplified form of energy conservation in $0+1$d, i.e.

$$\frac{\partial \epsilon(\tau)}{\partial \tau} = -\frac{\epsilon(\tau) + P_L(\tau)}{\tau} \tag{27}$$

we get:

$$\frac{\mathcal{R}'(\xi)}{\mathcal{R}(\xi)} \partial_\tau \xi + \frac{4}{\Lambda} \partial_\tau \Lambda = \frac{1}{\tau} \left[\frac{1}{\xi(1+\xi)\mathcal{R}(\xi)} - \frac{1}{\xi} - 1 \right] \tag{28}$$

EOM for Second Moment

And turning to second moment of $\mathcal{C}_0$, [2] obtains the zz and xx (=yy) projections of Equation 10 respectively, as:

$$(\log \mathcal{S}_L)' \, \partial_\tau \xi + 5\partial_\tau \log \Lambda + \frac{3}{\tau} = \frac{1}{\tau_{\mathrm{eq}}} \left[\frac{\mathcal{R}^{5/4}}{\mathcal{S}_L} - 1 \right]$$

$$(\log \mathcal{S}_T)' \, \partial_\tau \xi + 5\partial_\tau \log \Lambda + \frac{1}{\tau} = \frac{1}{\tau_{\mathrm{eq}}} \left[\frac{\mathcal{R}^{5/4}}{\mathcal{S}_T} - 1 \right]$$

The equation of motion for the second moment (Equation 29) is thus obtained from the above equations, after solving further.

$$\frac{1}{1+\xi} \partial_\tau \xi - \frac{2}{\tau} + \frac{\mathcal{R}^{5/4}(\xi)}{\tau_{\mathrm{eq}}} \xi \sqrt{1+\xi} = 0 \qquad (29)$$

What Now!?

We now have fully obtained the equations of motion for the zeroth, first and second moment of $\mathcal{C}_0$, and this can be continued till whenever we want, but since we have two unknowns, ξ and Λ, we need only two at a time to decouple and solve them.

- According to [2], higher moments are sensitive to high-momentum behavior of the distribution, so they naturally considered the lowest possible moments.
- Thus two viable possibilities: $0^{th}+1^{st}$ or $1^{st}+2^{nd}$ moments to decouple and solve them.
- But the $0^{th}+1^{st}$ moment equations do not reproduce correct near-equilibrium equations in the small anisotropic limit, and that $1^{st}+2^{nd}$ does reproduce the same. But with hindsight, zeroth, first and second moments are needed to in the non-conformal case (to solve for chemical potential).

Does aHydro $<=>$ Second order vHydro?

In order to make relation to viscous hydrodynamics, one adds dissipative terms to $T^{\mu\nu}$ decomposed along the various tensor degrees of freedom. We are dealing with the special case of $0+1d$ system and hence we only need the scalar term representing dissipation, π. tr($T^{\mu\nu}$) should still give zero and hence, we see that:

$$P_L = P_{eq}(T) - \pi \text{ and } P_T = P_{eq}(T) + \frac{\pi}{2}$$

Using this and $\frac{\partial\epsilon(\tau)}{\partial\tau} = -\frac{\epsilon(\tau)+P_L(\tau)}{\tau}$, we get:

$$\tau\partial_\tau log(\epsilon) = -\frac{4}{3} + \frac{\pi}{\epsilon} \tag{30}$$

We define:

$$\overline{\pi} = \frac{\pi}{\epsilon} = \frac{1}{3}\left[1 - \frac{\mathcal{R}_L(\xi}{\mathcal{R}(\xi)}\right] \tag{31}$$

Taking derivative of the first equality in the last equation gives:

$$\frac{\partial_\tau \pi}{\epsilon} = \overline{\pi}' \partial_\tau \xi + \overline{\pi} \partial_\tau \mathit{log}(\epsilon) \tag{32}$$

where $\overline{\pi}' = d\overline{\pi}/d\xi$.
Using the above equation and the second order EOM we obtain:

$$\frac{\partial_\tau \pi}{\epsilon} + \frac{\pi}{\epsilon\tau}\left(\frac{4}{3} - \frac{\pi}{\epsilon}\right) = \left[\frac{2(1+\xi)}{\tau} - \frac{\mathcal{H}(\xi)}{\tau_{eq}}\right]\overline{\pi}'(\xi) \tag{33}$$

where $\mathcal{H}(\xi) = \xi(1+\xi)^{3/2}\mathcal{R}^{5/4}(\xi)$. Note that the above equation has only π as the unknown as the ξ's can be expressed as $\xi(\pi)$ by inverting Equation 31, and the energy density can be eliminated by the relation $\epsilon = \frac{15}{4}\frac{\eta}{\tau_{eq}}$.

Expanding all the non-linear terms in Equation 31 in a Taylor series about $\xi = 0$, one could get $\overline{\pi} \equiv \overline{\pi}(\xi)$, $\overline{\pi}' \equiv \overline{\pi}'(\xi)$, and $\mathcal{H} \equiv \mathcal{H}(\xi)$ as a series is ξ; inverting which one obtains:

$$\xi = \frac{45}{8}\overline{\pi}\left[1 + \frac{195}{56}\overline{\pi} + \mathcal{O}(\overline{\pi}\right] \tag{34}$$

Plugging it back in the obtained series for $\overline{\pi}$, $\overline{\pi}'$ and $\mathcal{H}$, and putting those in Equation 33 gives:

$$\partial_\tau \pi - \frac{4\eta}{3\tau_{eq}\tau} + \frac{38\pi}{21\tau} = -\frac{\pi}{\tau_{eq}} \tag{35}$$

The above equation is in agreement with vHydro equations that were obtained by **??** with the RTA approximation. Hence the $0 + 1d$ aHydro reduces to vHydro in the small anisotropy limit.

Evolution of ξ

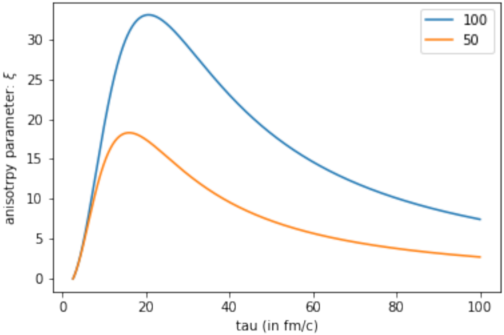

Figure: Evolution of ξ between $\tau = 2.5fm/c$ and $\tau = 100fm/c$ with the initial conditions: $T_0 = 0.3GeV$, $\xi_0 = 0$, $\tau_0 = 2.5fm/c$.

Pressure anisotropy for various $\overline{\eta}$

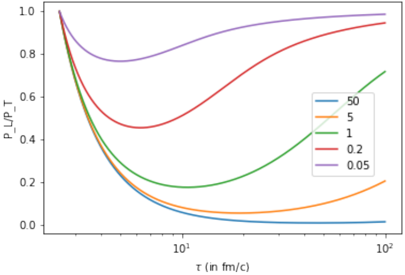

Figure: The 0+1d pressure anisotropy PL /P_T predicted by the aHydro equation 29. The legend contains the $\overline{\eta}$ values used for each curve. Note that the free streaming limit would be a steep vertical line with negative slope, while the ideal hydro limit $\overline{\eta} \longrightarrow 0$ will be a horizontal line at $P_L/P_T = 1$.

References

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