Algorithm Analysis

Lecture 2

https://livecodestream.dev/post/complete-guide-to-understanding-time-and-space-complexity-of-algorithms/

Recap Lecture 1

Data structures has wide applications

- Design an **efficient** search engine, as good as Google's
- Build an **efficient** security system based on face recognition
- Understand the human genome and trace your ancestry

Data structures are a key for designing efficient algorithms

What is an efficient algorithm ?
How do we know that algorithm is efficient ?

Recap Lecture 1

We need to analyze algorithm in term of time (efficiency)

- Independent of Hardware, OS, and compiler

Introduced the concept of T(n) …**logical time unit**

- Consider each operation (comparison, assignment) as a program step
- To keep stuff simple we consider each operation cost is 1 unit of time

A sequence of operations:

`count = count + 1;`	1 unit of time
`sum = sum + count;`	1 unit of time

➔ Total Cost = 2

Don't count comments and declarations

Algorithm Analysis

Estimate the performance of an algorithm through
The **number of operations** required to process an **input of certain size**

Algorithm	Cost (time complexity)
`i=0;`	1
`sum = 0;`	1
`while(i<N ){`	N+1
`sum ++;`	N
`i++`	N
`}`	

The condition always executes one more time than the loop itself

T(N) = 1 + 1 + N+1 + N + N

T(N) = 3N+3

https://www.ft.com/content/d24fce5e-58fc-11e6-9f70-badea1b336d4

LECTURE 2

Complexity *Analysis*

For student

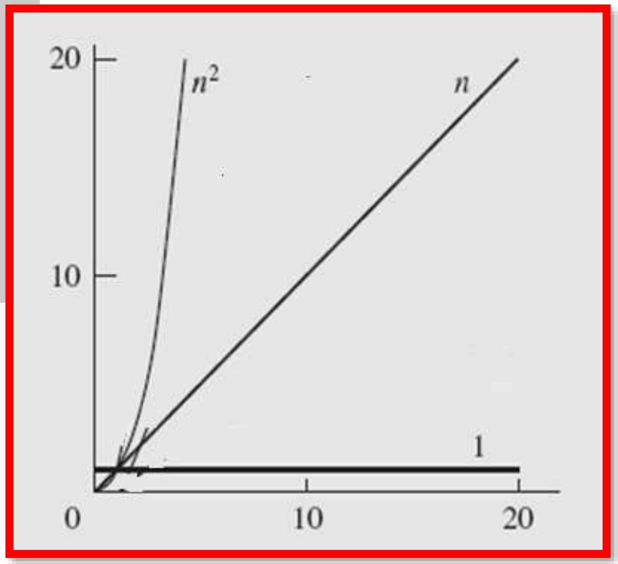
Chapter 2 of Alan Weiss's book

Chapter 2 of Adam Drozdek's book

Analysis of Loop

Algorithm	Cost (time complexity)
`i=0;`	1
`sum = 0;`	1
`while(i<N ){`	N+1
`sum ++;`	N
`i++`	N
`}`	

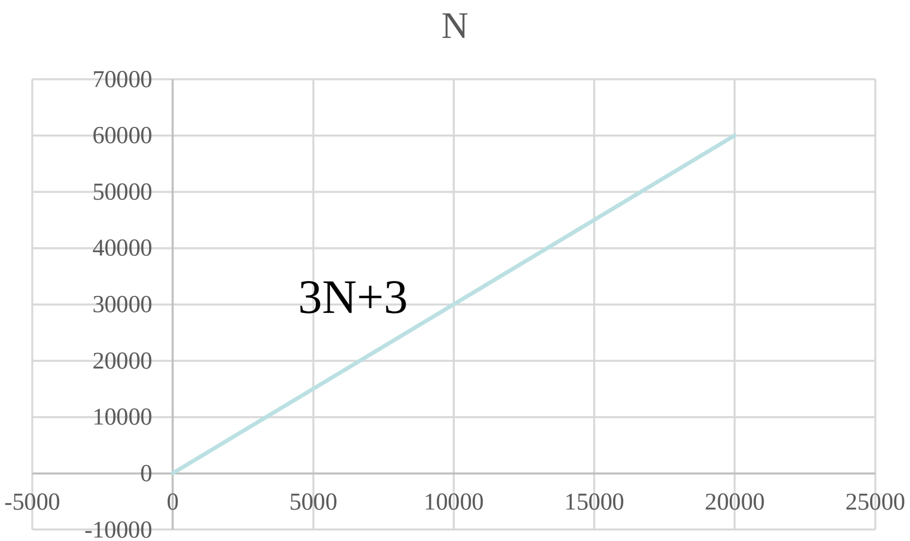
```
T(N) = 1 + 1 + N+1 + N + N
T(N) = 3N+3
```



TWO important things to note

- What is the effect of constant in *T(N)* = 3N + 3 **?**
- How T(n) varies for different N (input) ?
    - Check for N =5, 10, 20, 50, 100, 500, 1000, 10000, 100000

Analysis of Loop



N	T(N)= 3N+3
10	33
20	63
30	93
40	123
50	153
100	303
150	453
200	603
5000	15003
10000	30003
15000	45003
20000	60003

NESTED Loop

Example: Nested Loop

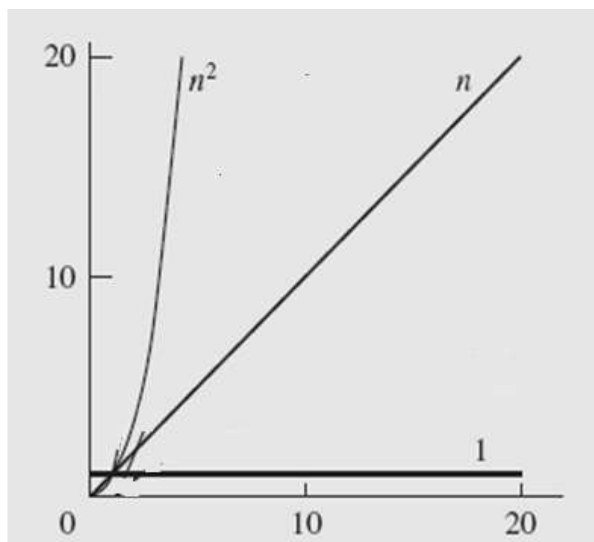
Code	Cost	Times
`i=1;`	1	1
`sum = 0;`	1	1
`while (i <= n) {`	1	n+1
`j=1;`	1	n
`while (j <= n) {`	1	n*(n+1)
`sum += i;`	1	n*n
`j++;`	1	n*n
`}`		
`i++;`	1	n
`}`		

T(N) = 1 + 1 + (n+1) + n + n*(n+1)+n*n+ n*n+ n = $\mathbf{3n^2 + 3n + 3}$

➔ The time required for this algorithm is **proportional to n^2**

Analysis of Nested Loop

- **2 important things to Note**
  - What is the effect of constant and low order term in in $T(N) = 3N^2 + 3N+3$ *for large N ?*
  - How T(n) varies for different N (input) ?
    - Check for N =5, 10, 20,50,100, 500, 1000, 10000 ,100000



Example: Effect of low order terms

- If **T(n) = 7n+100**
- What is T(n) for different values of n???

n	T(n)	Comment
1	107	Contributing factor is 100

Example: Effect of low order terms

- If **T(n) = 7n+100**
- What is T(n) for different values of n???

n	T(n)	Comment
1	107	Contributing factor is 100
5	135	Contributing factor is 7n and 100
10	170	Contributing factor is 7n and 100

Example: Effect of low order terms

- If **T(n) = 7n+100**
- What is T(n) for different values of n???

n	T(n)	Comment
1	107	Contributing factor is 100
5	135	Contributing factor is 7n and 100
10	170	Contributing factor is 7n and 100
100	800	Contribution of 100 is small

Example: Effect of low order terms

- If **$T(n) = 7n+100$**
- What is T(n) for different values of n???

n	T(n)	Comment
1	107	Contributing factor is 100
5	135	Contributing factor is 7n and 100
10	170	Contributing factor is 7n and 100
100	800	Contribution of 100 is small
1000	7100	Contributing factor is 7n

Example: Effect of low order terms

- If **T(n) = 7n+100**
- What is T(n) for different values of n???

n	T(n)	Comment
1	107	Contributing factor is 100
5	135	Contributing factor is 7n and 100
10	170	Contributing factor is 7n and 100
100	800	Contribution of 100 is small
1000	7100	Contributing factor is 7n
10000	70100	Contributing factor is 7n

Example: Effect of low order terms

- If **T(n) = 7n+100**
- What is T(n) for different values of n???

n	T(n)	Comment
1	107	Contributing factor is 100
5	135	Contributing factor is 7n and 100
10	170	Contributing factor is 7n and 100
100	800	Contribution of 100 is small
1000	7100	Contributing factor is 7n
10000	70100	Contributing factor is 7n
10^6	7000100	What is the contributing factor????

***DEDUCTION**: When approximating T(n) we can IGNORE the 100 term for very large value of n and say that T(n) can be approximated by 7(n)*

$T(N)=cN$ and $T(N)=N$

- **Note** we are estimating Algorithm complexity with reference to size of the input
- $T_1(N)=N$ and $T_2(N)=30N$ will have same effect with increase in input size
- Effect of input size on above T(N)

N	$T_1(N)=N$	$T_2(N)=30N$
10	$t_1=10$	$t_2=300$

T(N)=cN and T(N)=N

- **Note** we are estimating Algorithm complexity with reference to size of the input
- $T_1(N)=N$ and $T_2(N)=30N$ will have same effect with increase in input size
- Effect of input size on above T(N)

N	$T_1(N)=N$	$T_2(N)=30N$
10	$t_1=10$	$t_2=300$
20	$2t_1$	$2t_2$

$T(N)=cN$ and $T(N)=N$

- **Note** we are estimating Algorithm complexity with reference to size of the input
- $T_1(N)=N$ and $T_2(N)=30N$ will have same effect with increase in input size
- Effect of input size on above T(N)

N	$T_1(N)=N$	$T_2(N)=30N$
10	$t_1=10$	$t_2=300$
20	$2t_1$	$2t_2$
100	$10t_1$	$10t_2$
1000	$100t_1$	$100t_2$

NO difference in T(N) = N and T(N) = cN in terms of the growth of time w.r.t input size

Example 2

$$T(n) = n^2 + 100n + \log_{10}n + 1000$$

n	T(n)	n^2		100n		$\log_{10}n$		1000	
		Val	%	Val	%	Val	%	Val	%
1	1101	1	0.1%	100	9.1%	0	0%	1000	90.8%

Example 2

$$T(n) = n^2 + 100n + \log_{10}n + 1000$$

n	T(n)	n^2		100n		$\log_{10}n$		1000	
		Val	%	Val	%	Val	%	Val	%
1	1101	1	0.1%	100	9.1%	0	0%	1000	90.8%
10	2101	100	5.8%	1000	47.6%	1	0.05%	1000	47.6%

Example 2

$$T(n) = n^2 + 100n + \log_{10}n + 1000$$

n	T(n)	n^2		100n		$\log_{10}n$		1000	
		Val	%	Val	%	Val	%	Val	%
1	1101	1	0.1%	100	9.1%	0	0%	1000	90.8%
10	2101	100	5.8%	1000	47.6%	1	0.05%	1000	47.6%
100	21002	10000	47.6%	10000	47.6%	2	0.99%	1000	4.76%

Example 2

$$T(n) = n^2 + 100n + \log_{10}n + 1000$$

n	T(n)	n^2		100n		$\log_{10}n$		1000	
		Val	%	Val	%	Val	%	Val	%
1	1101	1	0.1%	100	9.1%	0	0%	1000	90.8%
10	2101	100	5.8%	1000	47.6%	1	0.05%	1000	47.6%
100	21002	10000	47.6%	10000	47.6%	2	0.99%	1000	4.76%
10^5	10,010,001,005	10^{10}	99.9%	10^7	.099%	5	0.0%	1000	0.00%

When approximating T(n) we can ***IGNORE the last 3 terms*** *and say that T(n) can be approximated by* n^2

Rate of Growth

- Consider the example of buying *Gold and Metal jewelry*
-

+

Cost: cost_of_gold + cost_of_metal

Cost ~ cost_of_gold (approximation)

Rate of Growth

- The low order terms in a function are relatively insignificant for **large** *n*

$$n^4 + 100n^2 + 10n + 50 \quad \sim \quad n^4$$

i.e., we say that $n^4 + 100n^2 + 10n + 50$ and n^4 have the same **rate of growth**

Thus,
we throw away leading constants &
we also throw away low order terms.

Partial SUM

	Cost

<pre>int sum(int n)
{
    int partialSum;
    1 partialSum = 0;
    2 **for(int i = 1; i <= n;
    ++i)**
 	
    3 **partialSum += i * i * i;**
 	
    4 return partialSum;
}</pre>	

T(N)= 6N+4

If we had to perform all this work everytime we needed to analyze a program, the task would quickly **become infeasible**.

Partial SUM

	Cost
`int sum( int n )`	0
`{`	
`int partialSum;`	
`1 partialSum = 0;`	1
`2 for( int i = 1; i <= n; ++i )`	i=1 cost is 1, i<=N cost is N+1 ++i cost is N
`3 partialSum += i * i * i;`	
`4 return partialSum;`	
`}`	

T(N)= 6N+4

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Partial SUM

	Cost
`int sum( int n )`	0
`{`	
`int partialSum;`	
`1 partialSum = 0;`	1
`2 for( int i = 1; i <= n; ++i )`	i=1 cost is 1, i<=N cost is N+1 ++i cost is N
`3 partialSum += i * i * i;`	3 ops 1 assignment, 2 multiplication, 1 add Total 4 cost for line 3
`4 return partialSum;`	0
`}`	

T(N)= 6N+4

If we had to perform all this work everytime we needed to analyze a program, the task would quickly **become infeasible**.

Problems with T(n)

T(n) is difficult to calculate

T(n) is also not very meaningful as step size is not exactly defined

T(n) is usually very complicated so we need an **approximation of T(n)….close to T(n).**

Approximation of T(n) is called ASYMPTOTIC COMPLEXITY

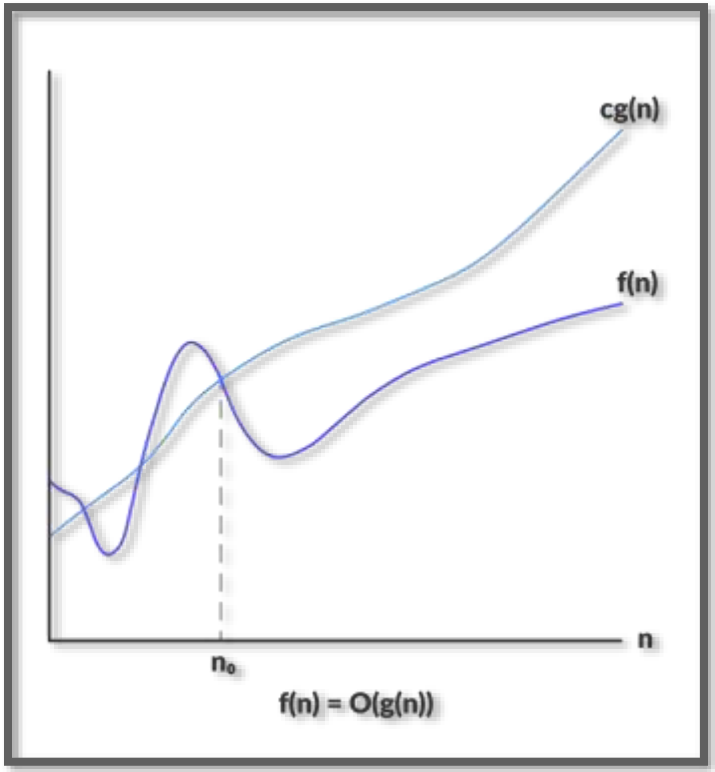
Asymptotic complexity studies the efficiency of an algorithm as the input size becomes large

Big-Oh or Big-O

f(n) is O(g(n)) *if there exist positive numbers* $\boldsymbol{c}$ **&** $\boldsymbol{n_0}$ *such that* $\boldsymbol{f(n) <= cg(n)}$ *for all* $n >= n_0$

g(n) is called the upper bound on f(n) OR

f(n) grows at the most as large as g(n)



Big-Oh or Big-O

f(n) is O(g(n)) if there exist positive numbers c & n_0 such that f(n)<=cg(n) for all n>= n_0

g(n) is called the upper bound on f(n) OR

f(n) grows at the most as large as g(n)

Example:

$T(n) = n^2 + 3n + 4$

$n^2 + 3n + 4 <= 2n^2$ for all $n_0 > 10$

What is f(n) and what is g(n)?

What is c & n

so we can say that T(n) is $O(n^2)$ OR

T(n) is in the order of n^2.

T(n) is bounded above by a + real multiple of n^2

How to choose c and N

$$f(n) = 2n^2 + 3n + 1 = O(n^2)$$

What is c & N

- **The definition of big-Oh states only that there must exist certain *c* and *N*, but it does not give any hint of how to calculate these constants.**
- **Second, it does not put any restrictions on these values and gives little guidance in situations when there are many candidates.**
- **In fact, there are usually infinitely many pairs of *c*s and *N*s that can be given for the same pair of functions *f* and *g*.**

How to choose c and N

We obtain the value of c & n by solving the inequality

$$2n^2 + 3n + 1 \le cn^2$$

or equivalently

$$2 + \frac{3}{n} + \frac{1}{n^2} \le c$$

Put value of n=1,2,3, in this equation to get value of c for that n

Because it is one inequality with two unknowns, different pairs of constants c and n for the same function g(n^2) can be determined.

For a fixed g, an infinite number of pairs of c's and n's can be identified.

How to choose c and N

$$f(n) = 2n^2 + 3n + 1 = O(n^2) \tag{2.2}$$

where $g(n) = n^2$, candidate values for c and N are shown in Figure 2.2.

Here N is same as n_0

Different values of c and N for function $f(n) = 2n^2 + 3n + 1 = O(n^2)$ calculated according to the definition of big-O.

c	≥ 6	$\geq 3\frac{3}{4}$	$\geq 3\frac{1}{9}$	$\geq 2\frac{13}{16}$	$\geq 2\frac{16}{25}$	...	$\rightarrow$	2
N	1	2	3	4	5	...	$\rightarrow$	∞

More Examples

- Show that $3n+3$ is $O(n)$.
  - Show $\exists c, n_0$: $3n+3 \le cn$, $\forall n>n_0$.
    - $3+3/n <= c$ ….
    - $n_0=1$ $c>=6$
    - $n_0=2$ $c>=4.5$ so on

- Show that $2n+2$ is $O(n)$.
  - $n_0=1$, $c>=4$
  - $c=3$, $n_0=2$.

More Examples

- Show that $2n^2 + 2n+2$ is $O(n^2)$.
  - Hold if we let $c=6$, $n_0=1$.
  - Hold if we let $c=5$, $n_0=2$
  - Hold if we let $c=4$ $n_0=2$

- Show that $3n^2 + 3n+3$ is $O(n^2)$.
  - Hold if we let $c=9$, $n_0=1$.
  - Hold if we let $c=7$, $n_0=3$

How to choose c and N

$$f(n) = 2n^2 + 3n + 1 = O(n^2) \quad (2.2)$$

where $g(n) = n^2$, candidate values for c and N are shown in Figure 2.2.

Different values of c and N for function $f(n) = 2n^2 + 3n + 1 = O(n^2)$ calculated according to the definition of big-O.

Here N is same as n_0

c	≥ 6	$\geq 3\frac{3}{4}$	$\geq 3\frac{1}{9}$	$\geq 2\frac{13}{16}$	$\geq 2\frac{16}{25}$	...	$\rightarrow$	2
N	1	2	3	4	5	...	$\rightarrow$	∞

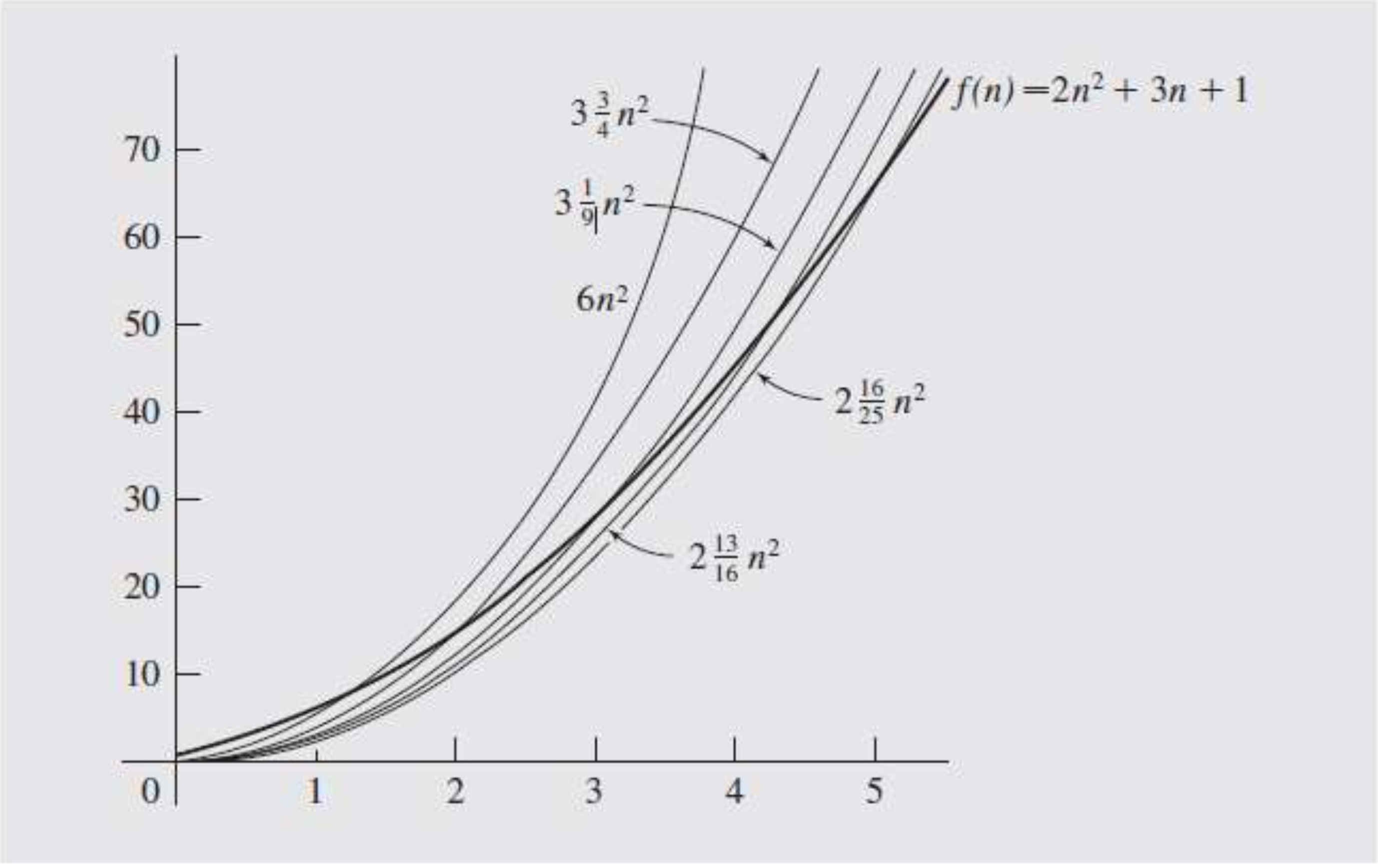
The crux of the matter is that the value of c depends on which N is chosen, and vice versa.

To choose the best c and N, it should be determined for which N a certain term in f becomes the largest and stays the largest.

In Equation 2.2, the only candidates for the largest term are $2n^2$ and $3n$;
these terms can be compared using the inequality $2n^2 > 3n$ that holds for $n > 1.5$.
Thus, $N = 2$ and $c \geq 3$}

How to choose c and N

- The point is that *f* and *g* grow at the same rate.
- g is almost always greater than or equal to f if it is multiplied by a constant c.



Big-O example, graphically

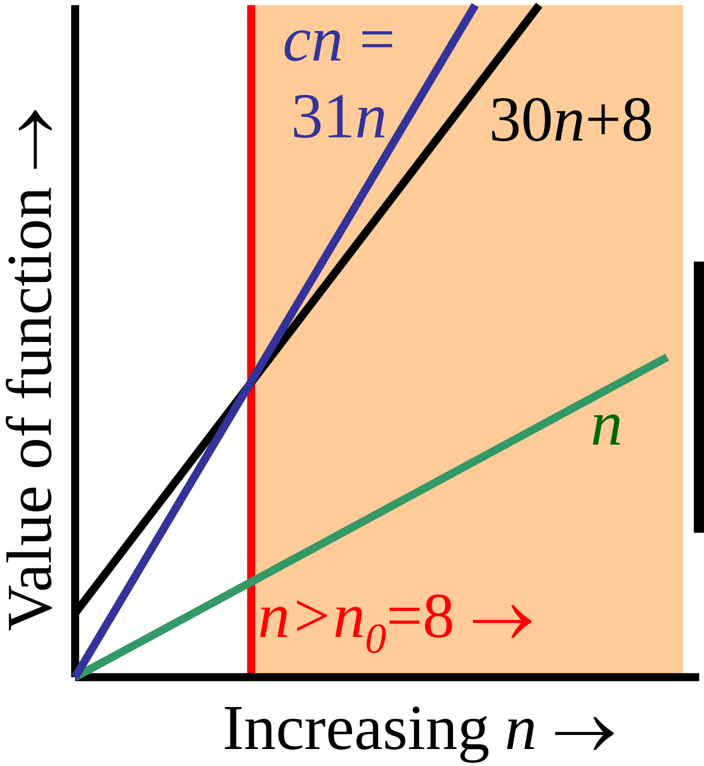
Show that $30n+8$ is $O(n)$.

Show $\exists c, n_0$: $30n+8 \le cn$, $\forall n > n_0$.

Let $c=31$, $n_0=8$. Assume $n>n_0=8$. Then

$cn = 31n = 30n + n > 30n+8$, so $30n+8 < cn$.

- Note $30n+8$ isn't less than n *anywhere* ($n>0$).
- It isn't even less than $31n$ *everywhere*.
- But it *is* less than $31n$ everywhere to the right of $n=8$.



$30n+8 \in O(n)$

Example

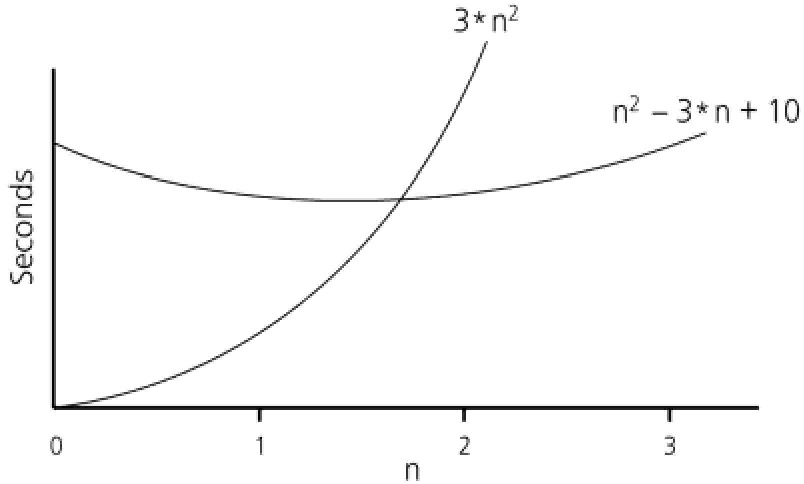
- For $n^2 – 3n+10$ Find constants c and n_0 exist such that

$$cn^2 > n^2 – 3n+10 \quad \text{for all } n \geq n_0 .$$

c is 3 and n_0 is 2

$$3n^2 > n^2 – 3n+10 \quad \text{for all } n \geq 2 .$$

Thus, the algorithm requires no more than kn^2 time units for $n \geq n_0$, So it is **$O(n^2)$**



Big Oh Example

- There is no unique set of values for n_0 and c in proving the asymptotic bounds
- Prove that $100n + 5 = O(n^2)$
  - $100n + 5 \le 100n + n = 101n \le 101n^2$

    for all $n \ge 5$

    $n_0 = 5$ and $c = 101$ is a solution
  - $100n + 5 \le 100n + 5n = 105n \le 105n^2$

    for all $n \ge 1$

    $n_0 = 1$ and $c = 105$ is also a solution

Must find **SOME** constants c and n_0 that satisfy the asymptotic notation relation

ALGORITHM ANALYSIS OF DIFFERENT CODES

Nested For Loops

- **For Loop** *can be confusing as three statements are embedded in one …As T(N) is rough estimate, so some analysis use 1 for entire for loop …1 cost for each line or step*

```
sum = 0;                              O(1)
for(i=0; i<N; i++)                    O(N)
    for(j=0; j<N; j++)                O(N^2)
        sum += arr[i][j];             O(N^2)
                              ------------
```

$$T(N) \approx O(1) + O(N) + O(N^2) + O(N^2) = O(N^2)$$

The total number of times a statement executes =
*outer loop times * inner loop times*

Nested For Loops

Rough estimate

`for(i=0; i<N; i++)`	$O(N)$
`    arr[i][i] =0;`	$O(N)$
`  sum = 0;`	$O(1)$
`  for(i=0; i<N; i++)`	$O(N)$
`    for(j=0; j<N; j++)`	$O(N^2)$
`      sum += arr[i][j];`	$O(N)^2$

$$O(N)+O(N^2) = O(N^2)$$