Graph Theory

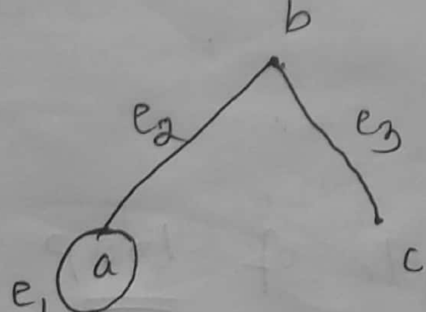
Graph theory, a fascinating branch of mathematics, has numerous applications to such diverse areas as comp. sci. engineering and many more.

A **graph** 'G' consists of a non-empty finite set 'V' of vertices (or points or nodes) and a set 'E' of un-ordered pairs of elements in V called edges.

The graph G is the ordered pair (V, E).

For example let $V = \{a, b, c\}$ and
$E = \{(a,b), (b,c), (a,a)\}$

Fig(1):



let $(a,a) = e_1$ $(a,b) = e_2$ &

$(b,c) = e_3$

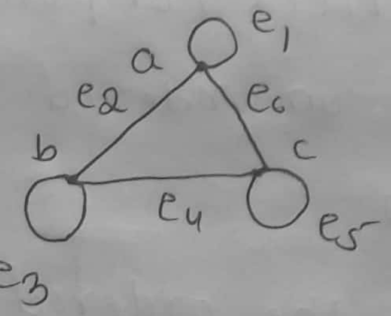
An edge emanating from and terminating at the same vertex is called a LOOP.

Parallel Edges have the same vertices

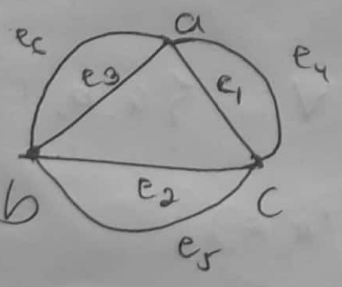
A graph is said to be simple graph if it contains no loops or parallel edges

For example

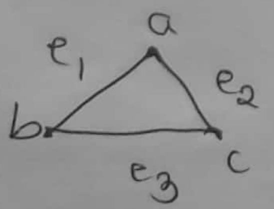
G_1 :



G_2 :



G_3 :



Here G_1 is example of loop

G_2 is example of parallel edges or multiple edges

G_3 is example of simple graph

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Adjacency & Incidence.

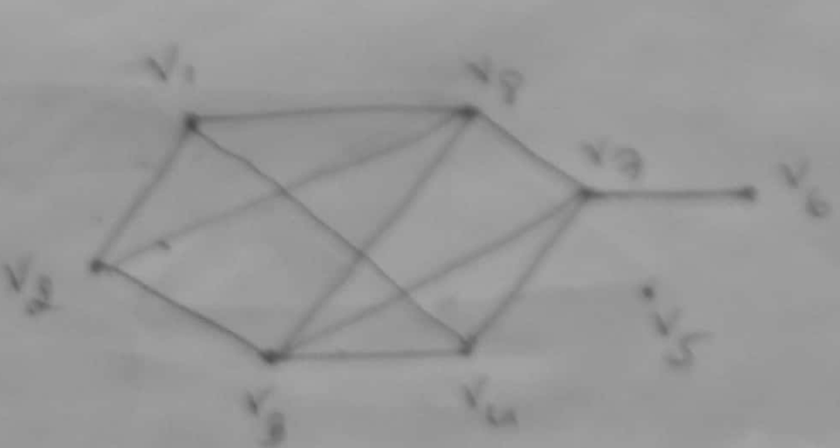
Two vertices v & w in a graph are adjacent if an edge runs between them.

An edge 'e' is incident with two end vertices.

Degree of a vertex.

The degree of a vertex 'v' in a graph is the number of edges meeting at v. It is denoted by deg(v).

For example



Vertex	degree
v_1	3
v_2	3
v_3	4
v_4	3
v_5	0
v_6	1
v_7	4
v_8	4

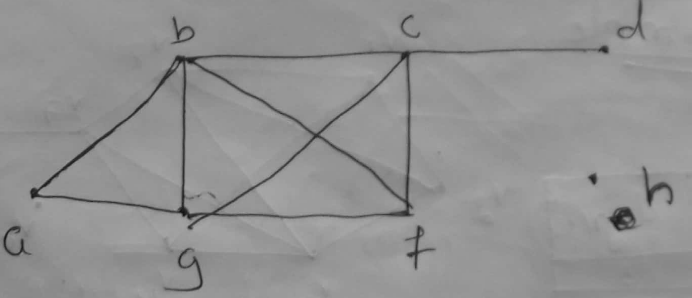
Note: A vertex of degree **zero** is called **isolated vertex**. A vertex of degree **1 (one)** is called **pendant vertex**.

In the above graph $deg(v_5) = 0$ and $deg(v_6) = 1$.

$\therefore$ v_5 is called isolated vertex

v_6 is called pendant vertex

Example: Find the degree of each vertex in the following graph and find the sum of degree of each vertex.



~~deg~~ Number of vertex $= |V| = 7$

Number of Edges $= |E| = e = 9$

~~deg(a) = 2, deg(b) = deg(g) = deg(f) =~~

$deg(a) = 2$, $deg(b) = 4$, $deg(c) = 4$,

$deg(d) = 1$, $deg(h) = 0$, $deg(f) = 3$

$deg(g) = 4$

$\therefore$ sum of degree of each vertex $= 2 + 4 + 4$
$+ 1 + 0 + 3 + 4 = 18$

In other words it is equal to $2 \times e$

$= 2 \times$ number of edges

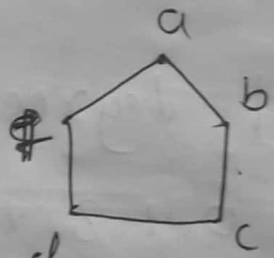
$= 2 \times 9 = 18.$

using the above relation we can define ~~the bel~~ the important Handshaking property (or Handshaking Theorem)

Handshaking Theorem:

Let $G = (V, E)$ be a undirected graph e be the number of edges

then
$$\boxed{\sum_{i=1}^{n} deg(v_i) = 2e}$$

For example let G:  be an undirected graph. Given that

number of edges $= e = 5$

$$\therefore \sum_{i=1}^{5} deg(v_i) = 2 \times 5 = 10$$

Example: Is a graph with four vertices a, b, c & d with $deg(a) = 4$, $deg(b) = 5$, $deg(d) = 5$ and $deg(c) = 2$ possible?

Sol^n: sum of degrees $= 4 + 5 + 2 + 5 = 16$

since sum is even there might be a graph with 8 edges.

Note: From the Handshaking property it is clear that the sum of degree of each vertex is always even.

Example: How many edges are there in a graph with 10 vertices each of degree is six?

Sol^n: Given number of vertices $= 10$

degree of each vertex $= 6$

$\therefore \quad \sum \deg(v_i) = 6 \times 10 = 60$

using handshaking property

$$\sum \deg(v_i) = 2e$$

$$\Rightarrow \quad 2e = 60 \quad \Rightarrow \quad e = \frac{60}{2} = 30$$

$\therefore$ number of edges $= 30$

Note: An undirected graph has an even number of odd degree

Ex: In a hydrocarbon molecule, a hydrogen atom is bonded to exactly one carbon atom, and a carbon atom bonds to four atoms (carbon or hydrogen). can a hydrocarbon molecule with three carbon atoms and five hydrogen atoms exists?

Sol: Given that a hydrocarbon molecule with 3 carbon atoms & 5 hydrogen-atoms.

A hydrogen atom is bonded exactly one carbon atom.

i.e degree of each vertex (hydrogen) is = 1

A carbon atom is bonded with four atoms

∴ degree of each vertex (carbon) = 4

$$\therefore \Sigma \deg(v_i) = 3\times4+5\times1$$
$$= 12+5 = 17$$

since sum is not even, such a graph does not exist. i.e hydrocarbon molecule not exist

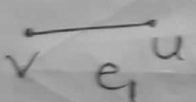
Some special simple graphs:

complete graph: A simple graph with an edge between every pair of vertices is called complete graph

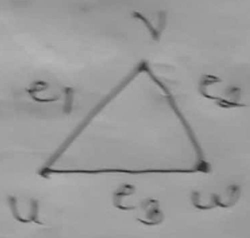
A complete graph with 'n' vertices is called denoted by 'K_n'

For example

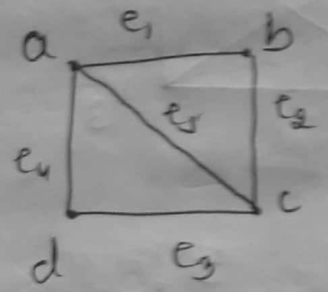
G_1:



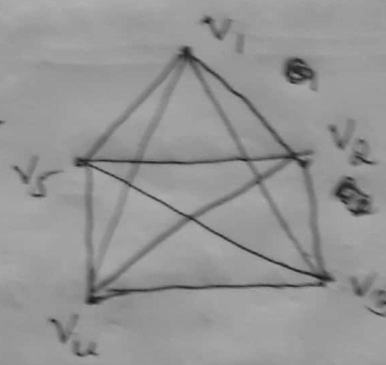
G_2:



G_3:



G_4:



Here $\boxed{G_1}$ is complete graph $\boxed{K_2}$

$\boxed{G_2}$ is complete graph with 3 vertices $\boxed{K_3}$

$\boxed{G_4}$ is complete graph with 5 vertices & is given by $\boxed{K_5}$

$\boxed{G_3}$ is not complete graph because there is no edge bet^n (b, d)

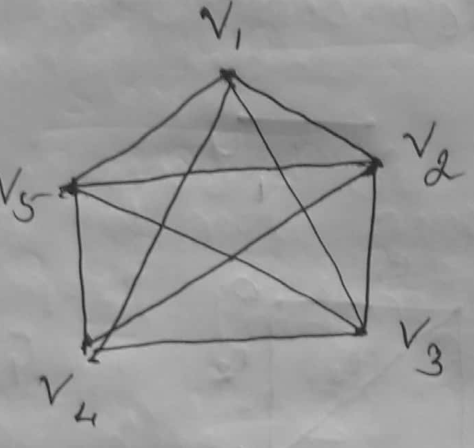
Note:

① The complete graph K_n has $C(n,2)$ or $^nC_2$ edges

② The degree of each vertex in complete graph K_n is $(n-1)$

For example, consider a complete graph with 5 vertices

K_5:



number of edges $= {^5C_2} = \frac{5!}{3!\,2!}$

$= \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1}$

$= 10$

degree of each vertex $= n-1$

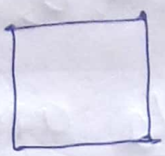
$= 5-1$

$= 4$

Cycles :

The cycle graph C_n of length $n (\geq 3)$ consists of n vertices $v_1, v_2, \cdots v_n$ and edges $\{v_i, v_{i+1}\}$, where $1 \leq i \leq n$ and $v_{n+1} = v_1$.

For example

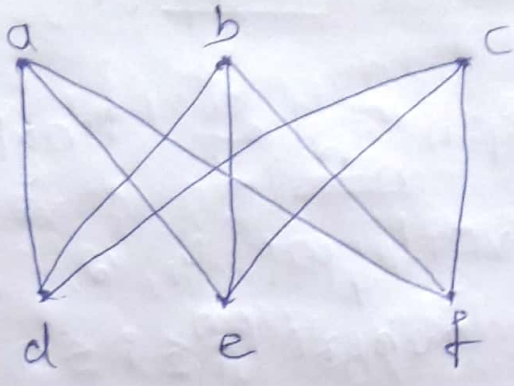
C_3 :  C_4 :  C_5 : 

Bipartite Graph :

If the vertex set 'V' of a simple graph $G = (V, E)$ can be ~~partition~~ devided into two disjoint subsets 'V_1' & 'V_2', so every edge in graph 'G' is incident with a vertex in 'V_1' and a vertex in 'V_2'. This is called Bipartite graph.

For example

G_1 :



Here vertex set $V = \{a, b, c, d, e, f\}$

Let $V_1 = \{a, b, c\}$ and $V_2 = \{d, e, f\}$

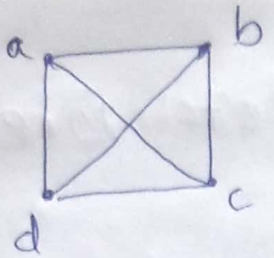
where V_1 & V_2 are disjoint subsets of V.

i.e. $V_1 \cap V_2$ is empty.

$\therefore$ every ~~e~~ edge in 'G_1' is incident with a vertex in V_1 and a vertex in V_2.

we can note that there is no edge between vertices of V_1 itself. and also ~~th~~ ~~there does not~~

there is no edge between vertices of V_2 itself.

G_2 :  is not Bipartite graph.

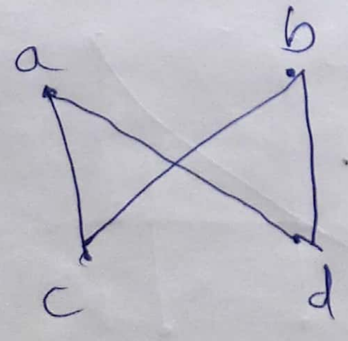
Complete Bipartite Graph :

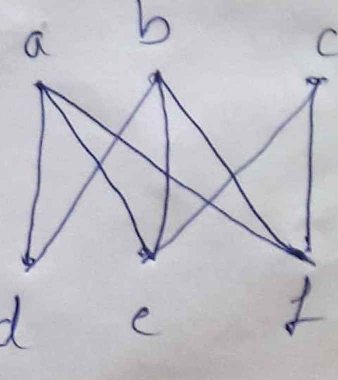
Let G be a bipartite graph with vertex set V & edge set E. Let V be partitioned into two disjoint subsets (devided) . Let the number of vertices in set V_1 & V_2. Let $V_1 = |V_1| = m$.

Let the number of vertices in set $V_2 = |V_2| = n$.

The graph G is said to be complete Bipartite graph if $\exists$ edge betn every vertex in V_1 & V_2.

For example

G_1 : 

G_2 : 

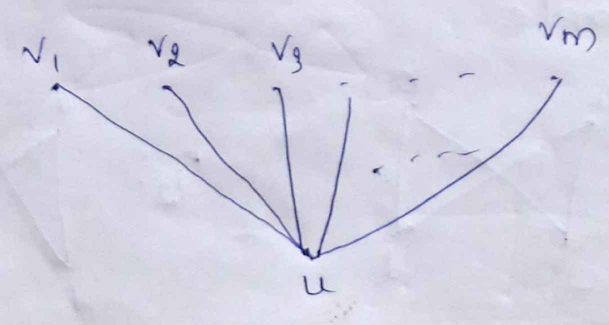
Here G_1 is called Complete Bipartite graph.

Here G_2 is not complete Bipartite graph because there is no edge between the vertices c & d.

But G_2 is Bipartite graph.

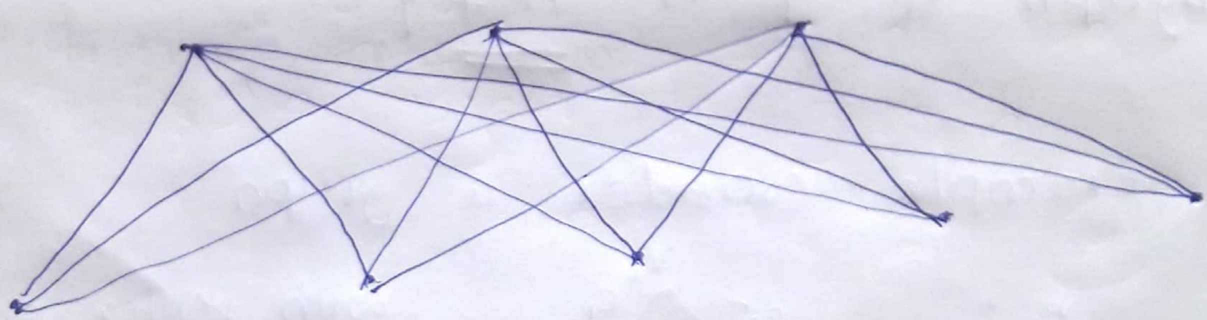
Note: The complete Bipartite graph is denoted by $K_{m,n}$

$K_{1,m}$ is called star graph and it is given by

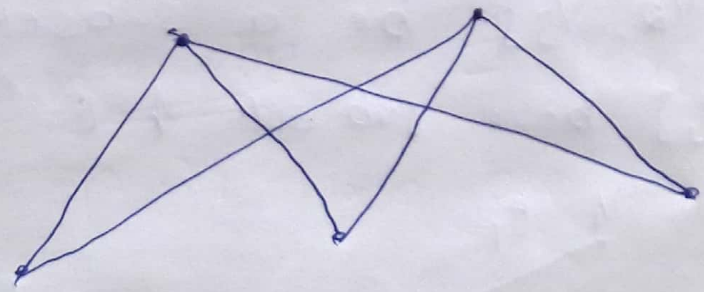


Example: Draw complete Bipartite graph $K_{3,5}$ & $K_{2,3}$

Sol^n: The complete graph $K_{3,5}$
↓
Bipartite



The complete Bipartite graph $K_{2,3}$

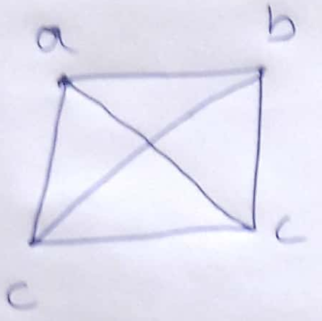


Note:

Regular graph: A graph is said to be Regular graph if the degree of each vertex is same. ~~It is repr~~

For example

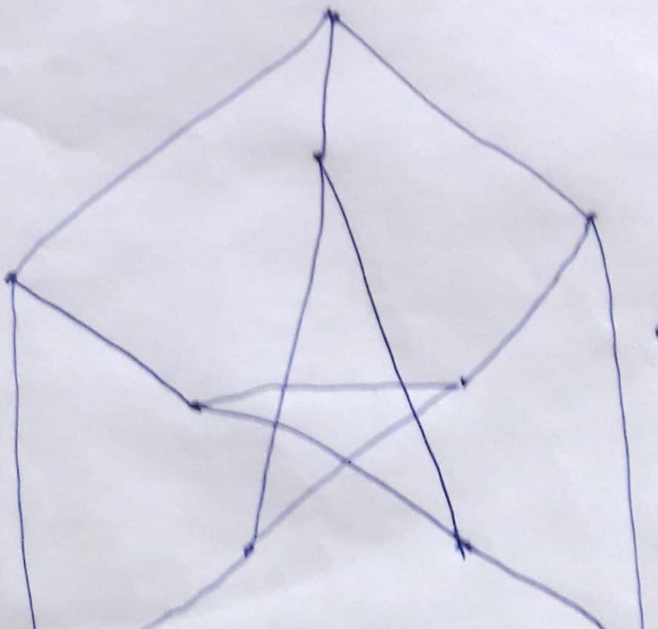
G_1:



G_1 is called 3-Regular graph.

degree of each vertex = 3

G_2:



G_2 is called 2-Regular graph

G_2 is called Petersen graph

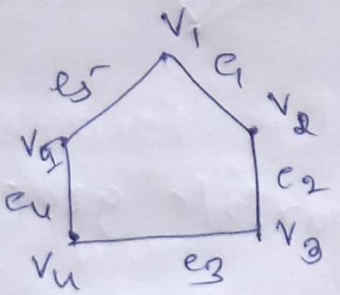
Subgraph:

A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ & $F \subseteq E$.

A subgraph $\underline{H}$ of a graph $\underline{G}$, is a proper subgraph of G if $\underline{H \neq G}$.

For example consider a graph

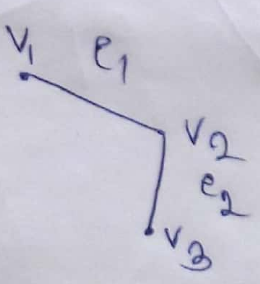
G :



with vertex set
$V = \{v_1, v_2, v_3, v_4, v_5\}$
edge set
$E = \{e_1, e_2, e_3, e_4, e_5\}$

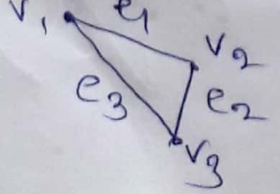
Let $W = \{v_1, v_2, v_3\}$ be a subset of $\underline{V}$
and $F = \{e_1, e_2\}$ be a sub set of E.

Then H :



is called sub graph of G.

But the below graph is not sub graph



because the edge $\underline{e_3}$ does not exist in $\underline{G}$.

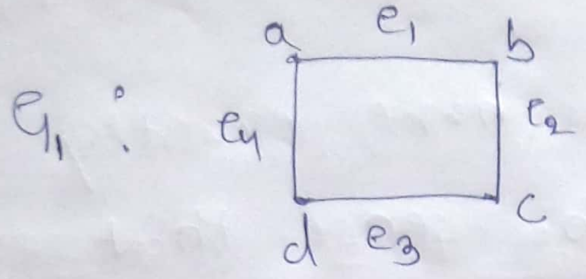
Graph Isomorphism:

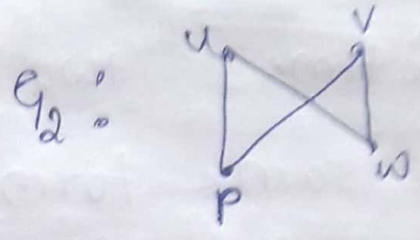
sometimes, two graphs have exactly the same form in the sense that there is a one-to-one relation between their ~~relation~~ vertex sets that preserves edges. In such case, we say that the two graphs are isomorphic.

Isomorphism of Graphs:

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a one-to-one and onto function f from V_1 to V_2 with property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. such a function f is called isomorphism

For example consider two graphs





Here the ~~stor~~ structure of graphs G_1 and G_2 are different. Now we develop a relation f as follows

$V_1 = \{a, b, c, d\}$, $V_2 = \{u, v, w, p\}$

Let $u = f(a)$, $w = f(b)$ $p = f(d)$

and $v = f(c)$

Here a and b are adjacent in G_1 corresponding $f(a) = u$ & $f(b) = w$ are adjacent in G_2. a and d are adjacent in G_1 corresponding $f(a) = u$ and $f(d) = p$ are adjacent in G_2.

In G_1, b and c are adjacent in ~~in~~ G_1 corresponding $f(b) = w$ and $f(c) = v$ are adjacent in G_2

In G_1, c and d are adjacent, corresponding $f(c) = v$ and $p = f(d)$ are adjacent in G_2.

From above all adjacency preserves.

Now we observe the degree of each vertex.

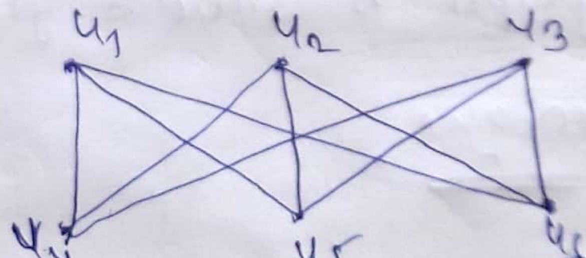
in G_1, $\deg(a) = 2$, $\deg(b) = 2$, $\deg(c) = 3$
$\deg(d) = 2$

In G_2, $\deg(u) = 2$, $\deg(w) = 2$, $\deg(v) = 2$
$\deg(p) = 2$

$\therefore$ graph G_1 & G_2 are Isomorphism to each other.

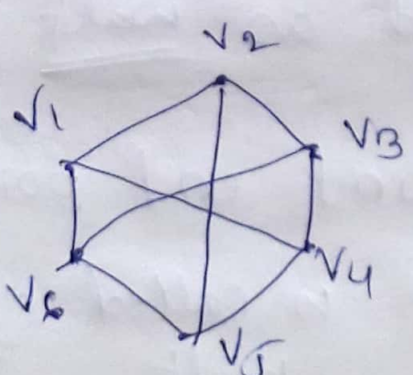
HW Show that the following two graph's are Isomorphic

G_1 :



where $G_1 = K_{3,3}$

G_2 :



where G_2 = 3-Regular graph

Paths, cycles and circuites

Let v_0 and v_n be two vertices in a graph. A path of length "n" from v_0 to v_n is a sequence of vertices v_i and edges e_i of the form $v_0 - e_1 - v_1 - e_2 - \cdots - e_n - v_n$ where each edge e_i is incident with the vertices v_{i-1} and v_i, $1 \le i \le n$.

The vertices v_0 and v_n are end points of the paths.

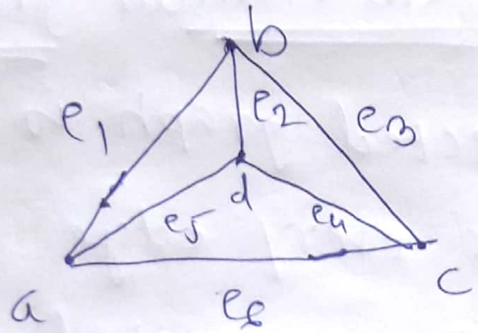
Note: ① If the graph is simple graph then the path is unique.

② Path is also called as walk

③ If starting point and end point are same, then walk is called closed walk.

(4) The walks or path is called Trail if walks that has no repeated edge.

For example consider the below graph



The sequence $a - e_1 - b - e_3 - c - e_4 - d$ is a path of length $= 3$.

consider a sequence $a - e_6 - c - e_3 - b - e_1 - a$ is called closed walks or closed path or circuite or cycle.

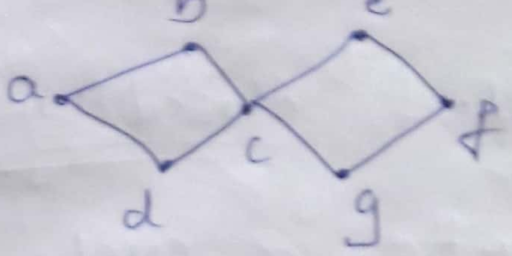
Note: A closed path with no repeated edges is called circuit.

Theorem: There is a simple path between any two distinct vertices in a connected graph.

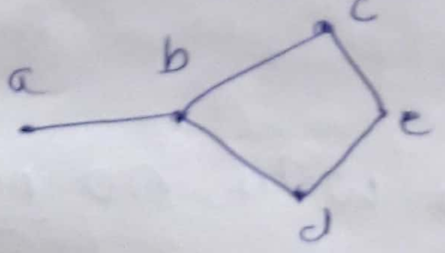
Eulerian and Hamiltonian graphs:

A path in a connected graph is an Eulerian path if it contains every edge exactly once. A circuit in a connected graph is an Eulerian circuit if it contains every edge of the graph. A connected graph with an Eulerian circuit is an Eulerian graph.

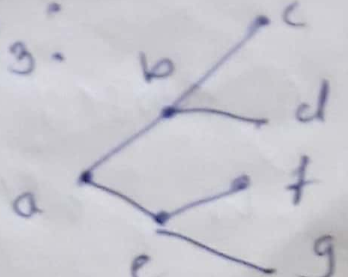
For example G_1:



G_2:



G_3:



Here G_1 is an Eulerian since it

- contains an Eulerian circuit,

is a-b-c-e-f-g-c-d-a.

In fact ~~it~~ G_1 contains several Eulerian

- circuits.

In G_2, it has only Eulerian path

namely a-b-c-e-d-b. It has several

Eulerian path.

But G_2 does not include Eulerian circuits

G_3 has no Eulerian paths or circuits.

Theorem: A connected graph G is Eulerian if and only if every vertex of G has even degree

Proof: suppose G is Eulerian

$\Rightarrow$ G contains Eulerian circuit of the form $v_0 - e_1 - v_1 - e_2 - \cdots - v_{n-1} - e_n - v_0$.

Since the edges e_1 and e_n are incident with v_0.

$\therefore$ degree of v_0 is atleast two. Because there may be more than two edges incident with v_0.

$\therefore$ Every time the circuit passes through the vertex, the degree of vertex increased by two.

$\therefore$ The degree of every vertex including v_0, is an even integer.

Conversely, suppose every vertex of G has even degree.

Let v_0 be an arbitrary vertex in G.

Let $c_1 = v_0 - v_1 - v_2 - \cdots - v_{n-1} - v_0$ be a circuit.

This is possible since every vertex has even degree.

If C_1 is Eulerian $\Rightarrow$ G is also Eulerian.

Suppose C_1 is not Eulerian.

consider the subgraph $\underline{H}$ obtained by deleting all the edges in C_1 and vertices not incident with remaining edges. Note that all vertices of H have even degree. Because G is connected. construct a circuit C_2 for sub graph H.

Now combining C_1 & C_2 to form a larger circuit. If it is Eulerian then G is also Eulerian

If this is not Eulerian continue the above process.

Hamiltonian graph:

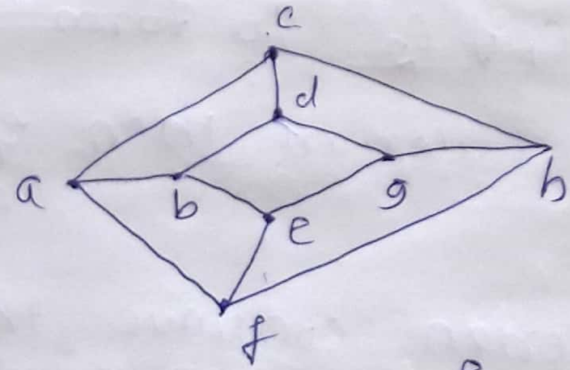
A simple path in a connected graph is Hamiltonian if it contains every vertex.

A cycle in a connected graph that contains every vertex is a Hamiltonian.

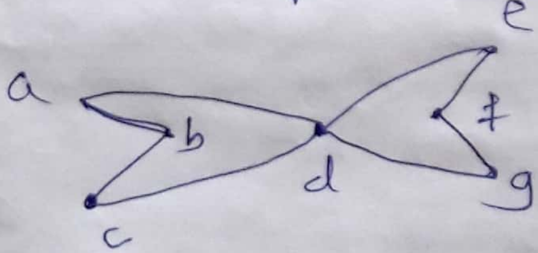
A connected graph that contains a Hamiltonian cycle is a Hamiltonian graph.

For example consider the following graphs

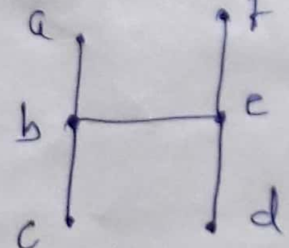
G_1:



G_2:



G_3:



In the above graphs

Graph G_1 contains a Hamiltonian cycle for example a - c - d - g - h - f - e - b - a
$\therefore$ G_1 is Hamiltonian.

Similar path's we can observe in G_1.

In graph G_2 suppose we start at a, b or c. Then we have to pass through d to visit vertices e, f & g. so to return to begining vertex, we have to pass through d again.
$\therefore$ G_2 does not contain Hamiltonian path
$\therefore$ G_2 is not Hamiltonian.

Similarly G_3.

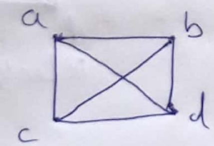
Planar Graphs :

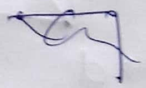
A graph is planar if it can be drawn in the plane, so its edges meet only at the vertices. Such a drawing is a planar-representation of the graph.

A graph is called planar if it can be drawn in the plane without any edge ~~repetation~~ crossing. Such a drawing is called a planar representation of the graph.

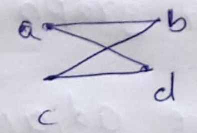
For example the graph's

G_1 :



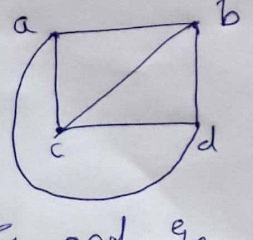


G_2 :

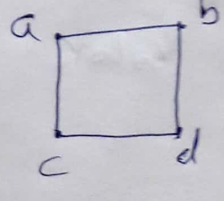


The above graphs G_1 & G_2 are drawn as

G_1 :



G_2 :



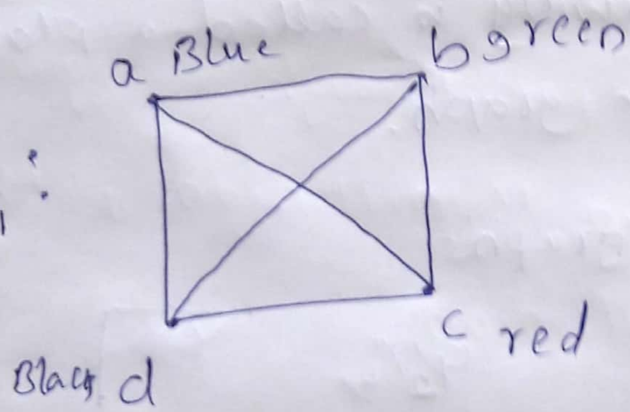
$\therefore$ G_1 and G_2 are planar.

Graph coloring:

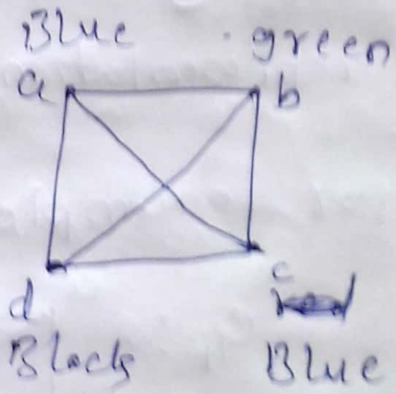
A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

For example

G_1:



G_2:



Here coloring is wrong for graph G_2

~~Bte~~ Because a and c are adjacent

∴ adjacent vertices ~~have~~ does not have same color

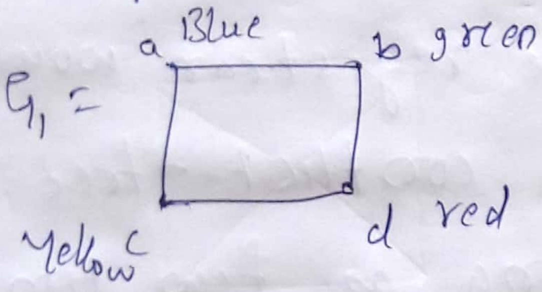
chromatic number :

The chromatic number of a graph is the least number of colors needed for a coloring of this graph.

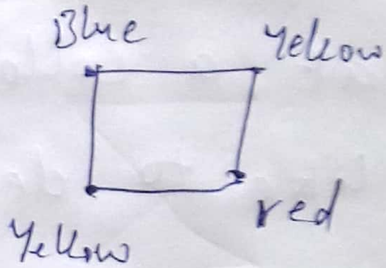
The chromatic number of a graph G is denoted by $\chi(G)$.

For example

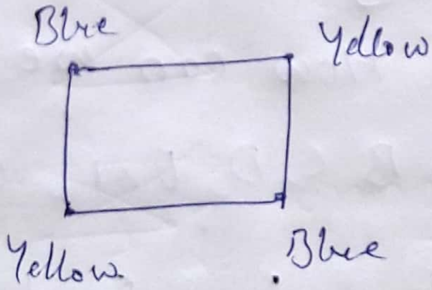
G_1 =



G_2 :



G_3 :



$\therefore$ For the above graph the chromatic number is 2

$\therefore \chi(G) = 2$

Note: what is chromatic number of K_n?

Since K_n is complete graph and No two vertices can be assigned the same color

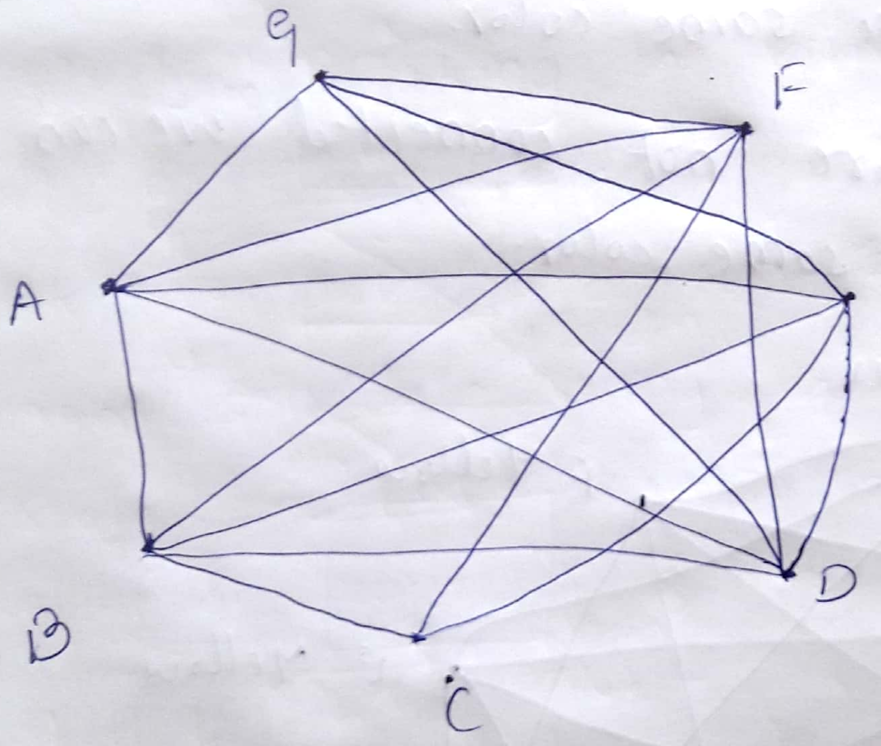
$\therefore \quad \chi(G) = \chi(K_n) = n$

Example: In the below table, the student's taking the various courses at a college. The registrar would like to develop a conflict-free final exam schedule using as few time slots as possible. How can we help her?

consider the courses A, B, C, D, E, F, G

A	B	C	D	E	F	G	→ courses
B_1	C_1	C_2	B_1	B_1	A_1	A_1	↓ students
B_2	E_1	E_1	F_1	G_1	F_1	B_1	
C_1	N_1	G_1	H_1	C_3	G_1	C_3	
F_1	P_1	N_1	H_2	F_2	N_2	C_4	
H_1	R_1	N_2	N_1	N_1	R_1	H_2	
W_1							

Sol^n: First we construct a graph model for the problem. We represent each course by a vertex. Two vertices are adjacent if the corresponding courses are incompatible, that is, they have a common student.



Apply the coloring to it so that no two vertices have same color.

Since the vertices ~~have~~ A, B, D & E are connected to each other

∴ A, B, D & E have different colors

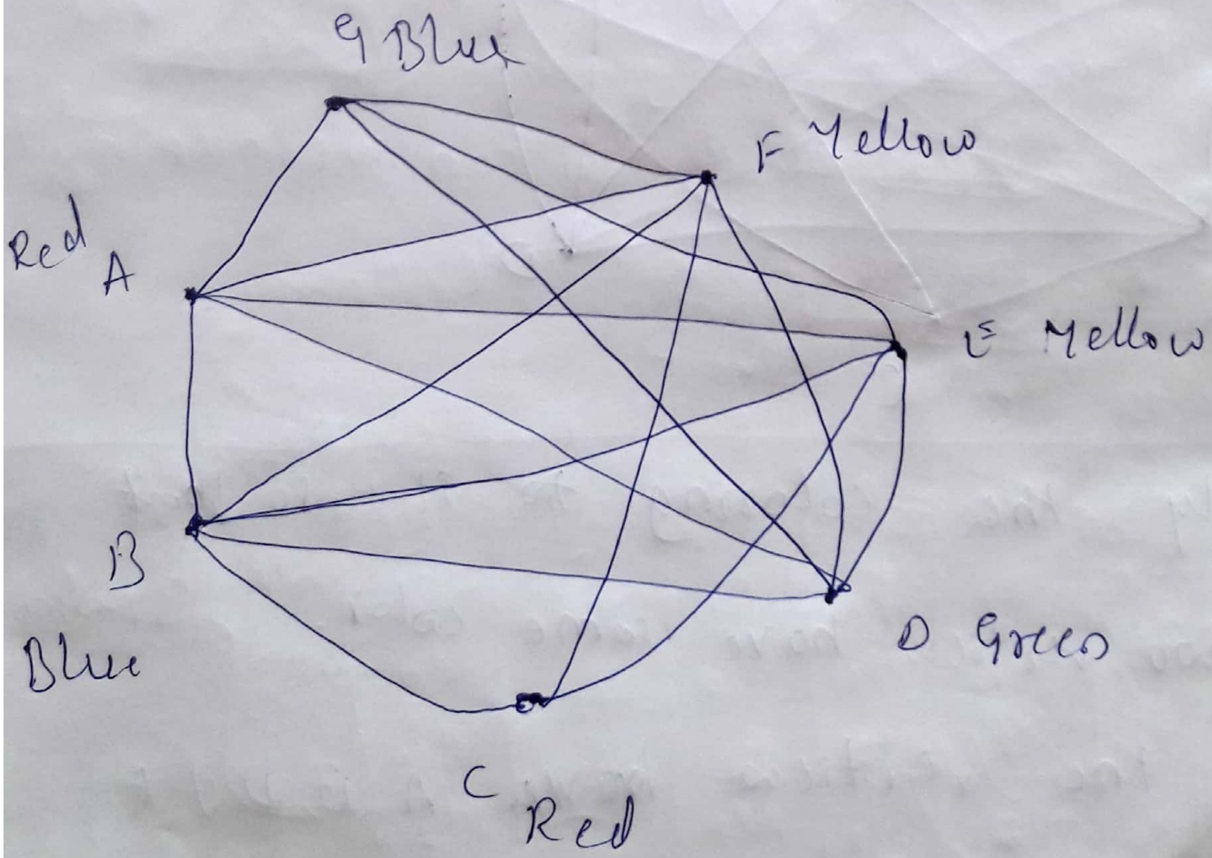
Now we assign colors for remaining vertices as follows

since E & F are not adjacent we can assign same color to both vertices.

since B & G are not connected

∴ B & G have same color.

since A & C are not connected we can assign the same color.



$\therefore$ The chromatic number of the above graph is $\chi(G) = 4$.

We can give the final exam schedule schedule as follows

Day ~~Blocks~~	1	2	3	4
course	A, C	B, G	D	E, F

$\longrightarrow$ $\longrightarrow$ $\longrightarrow$ $\longrightarrow$

HW The mathematics department has six committees, each meeting once a-month. How many different meeting ~~tim~~ -times must be used to ensure that no member is scheduled to attend two meetings at the same time if the committees are

$C_1 = \{$Arlinghaus, Brand, Zaslavsky$\}$

$C_2 = \{$ Brand, Lee, Rosen $\}$

$C_3 = \{$ Arlinghaus, Rosen, Zaslavsky $\}$

$C_4 = \{$ Lee, Rosen, Zaslavsky $\}$

$C_5 = \{$ Arlinghaus, Brand $\}$

$C_6 = \{$ Brand, Rosen, Zaslavsky $\}$