COL 351 : Analysis & Design of Algorithms

Lecture 15

Greedy Algorithms II :

Job Scheduling and Huffman Coding

Aug 28, 2024 | Rohit Vaish

Greedy Algorithms

Do what looks best right now and hope everything works out at the end

JOB SCHEDULING

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input : a set of n jobs with positive lengths $l_1, l_2, \ldots, l_n$ and positive weights $w_1, w_2, \ldots, w_n$

output : a job schedule (or sequence) that minimizes the sum of weighted completion times $\sum_{j=1}^{n} w_j \cdot C_j$.

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Brute force : $O(n!)$ ☹

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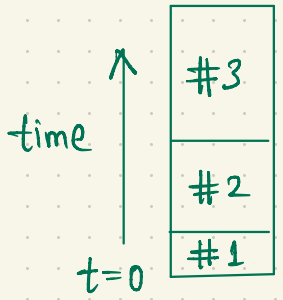
E.g., $l_1 = 1$, $l_2 = 2$, $l_3 = 3$



$C_1 = 1$ $C_2 = 3$ $C_3 = 6$

OBJECTIVE FUNCTION

goal : minimize the weighted sum of completion times

$$\min \sum_{j=1}^{n} w_j \cdot C_j$$

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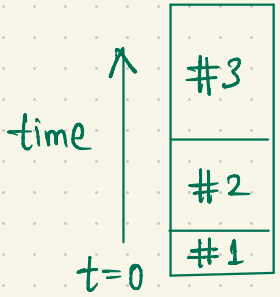
weights $w_1 = 3$, $w_2 = 2$, $w_3 = 1$



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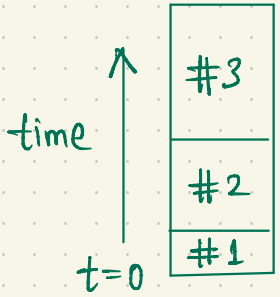

$$\sum_{j=1}^{n} w_j \cdot C_j = \underset{w_1}{3} \times \overset{C_1}{1} + \underset{w_2}{2} \times \overset{C_2}{3} + \underset{w_3}{1} \times \overset{C_3}{6}$$

$$= 15$$

SPECIAL CASES

goal : min $\sum_{j=1}^{n} w_j \cdot C_j$

(I) All lengths equal : schedule heavier job earlier

(II) All weights equal : schedule shorter job earlier

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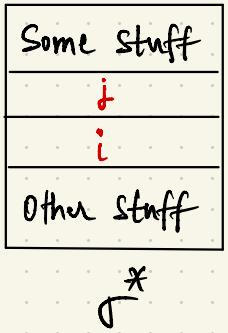

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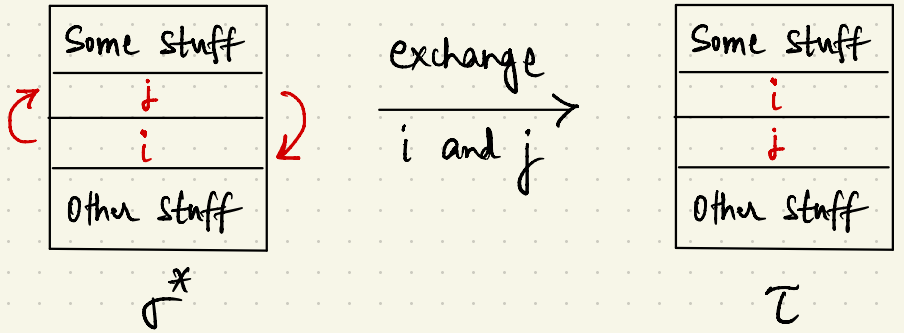

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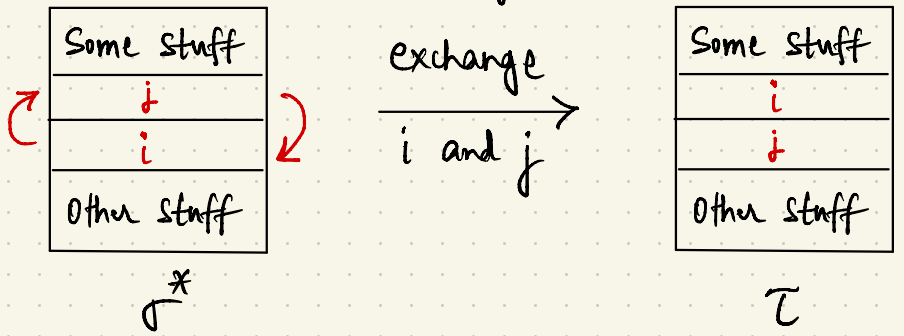

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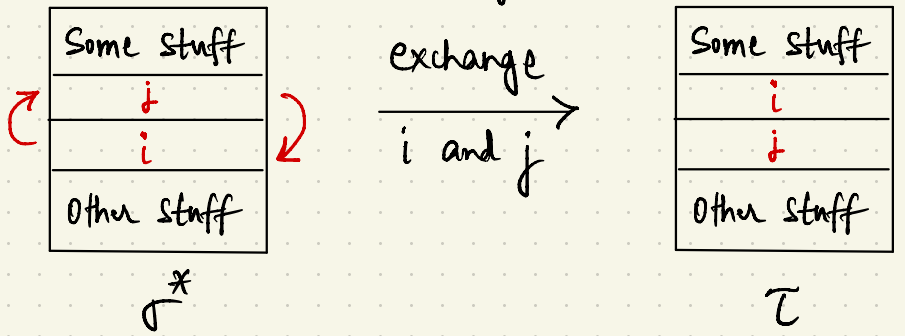

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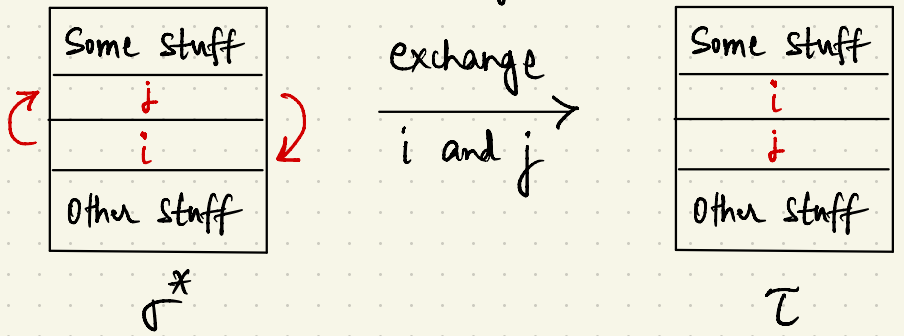

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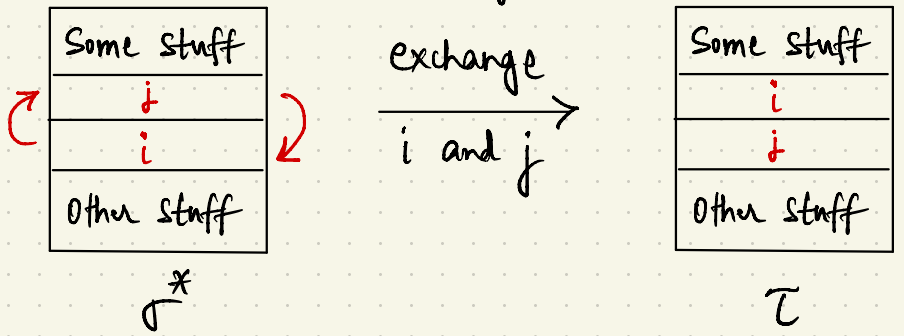

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τ is strictly better than σ^*

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Will show that σ is at least as good as τ

$\Rightarrow$ greedy schedule is optimal.

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↳ consecutive jobs i, j in τ such that $i > j$.

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Observe,

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$\Rightarrow$ σ must be optimal

HUFFMAN CODING

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B	01	B	01	How to interpret 001?
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How to compute an optimal prefix-free code ?

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brute force: at least $n!$

one symbol with one bit,
another symbol with two bits,
another symbol with three bits, ...

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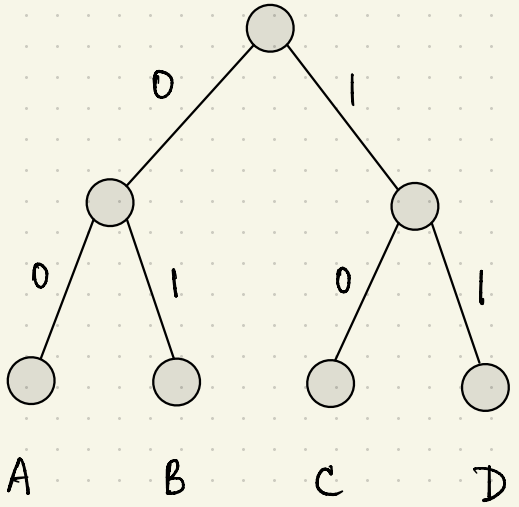

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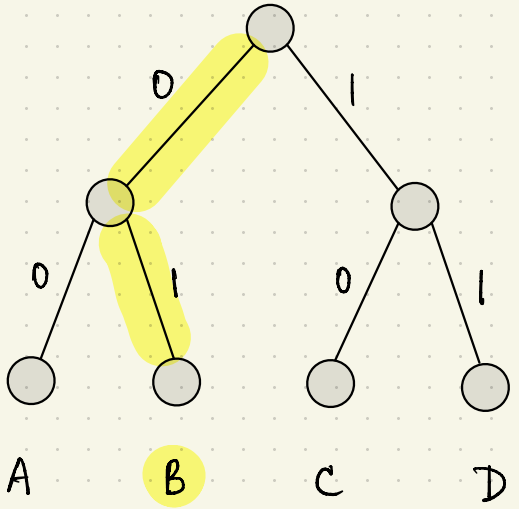

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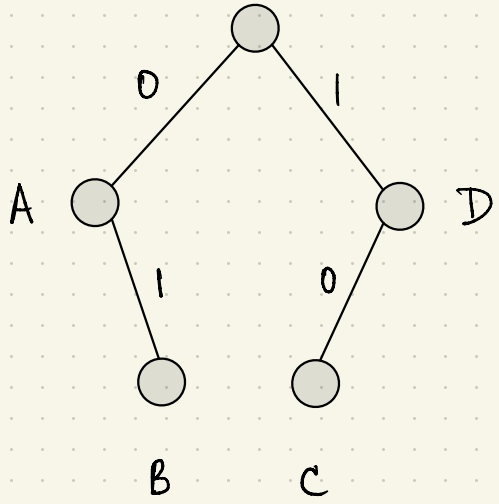

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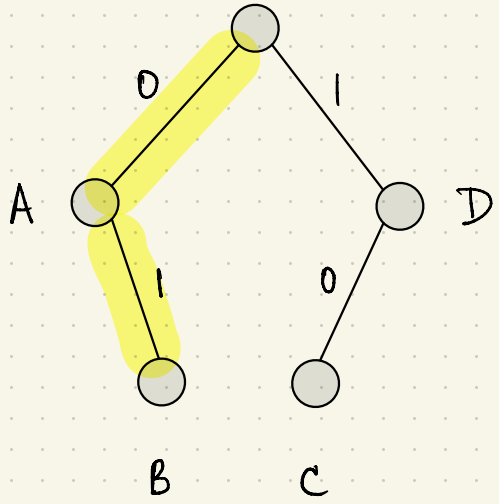

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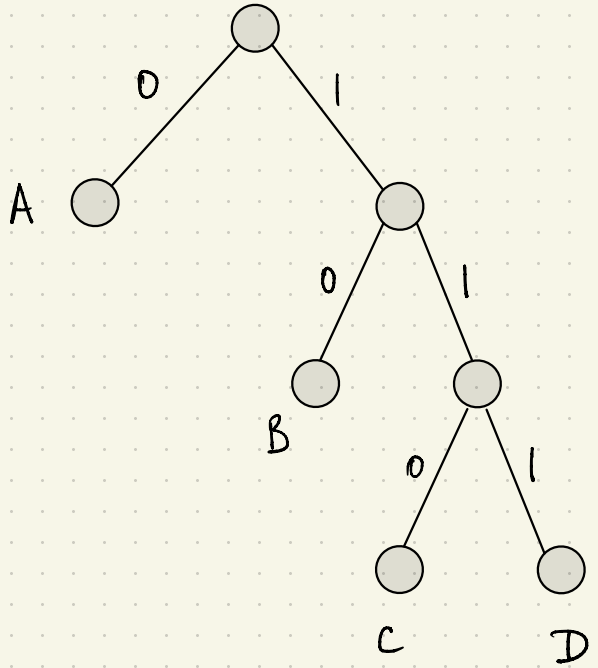
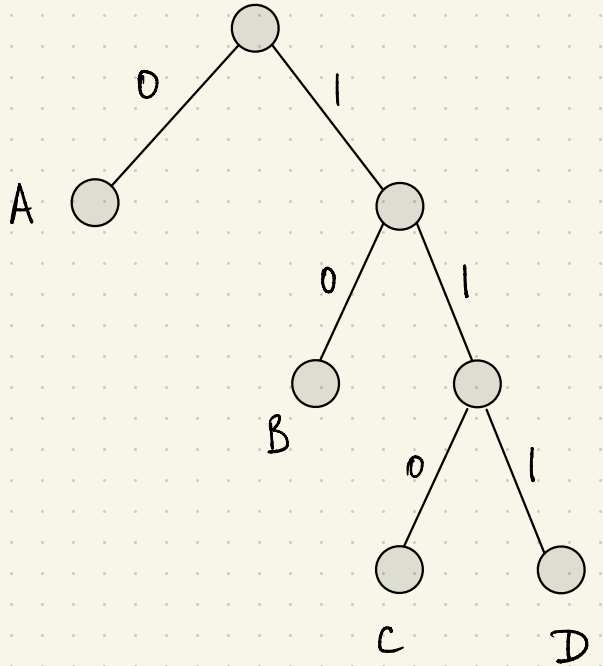


CODES AS TREES

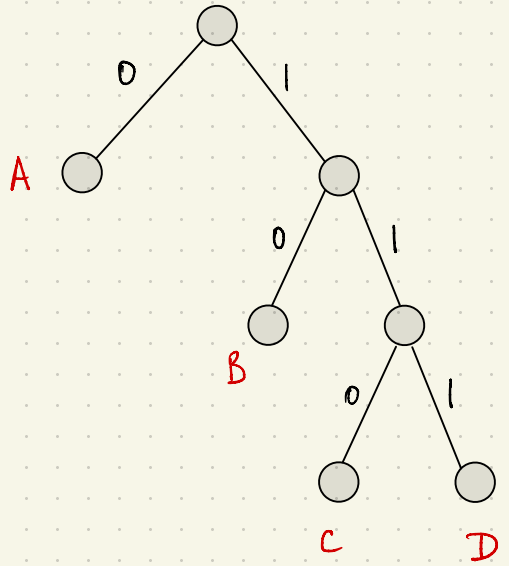
$\Sigma = \{A, B, C, D\}$

Symbol	variable length
A	0
B	10
C	110
D	111



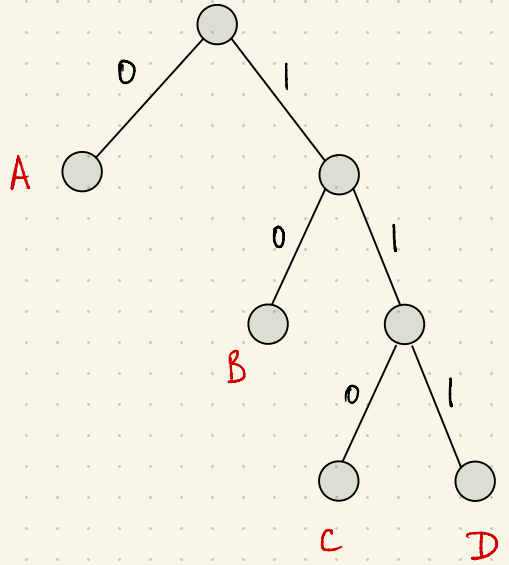
prefix-free

PREFIX-FREE CODES AS TREES



PREFIX-FREE CODES AS TREES

* Left child "0", right child "1"



PREFIX-FREE CODES AS TREES

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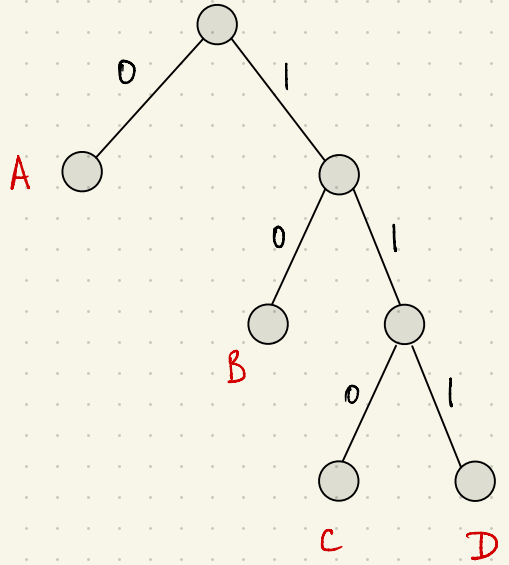
* for each $i \in \Sigma$, exactly one node labeled "i"



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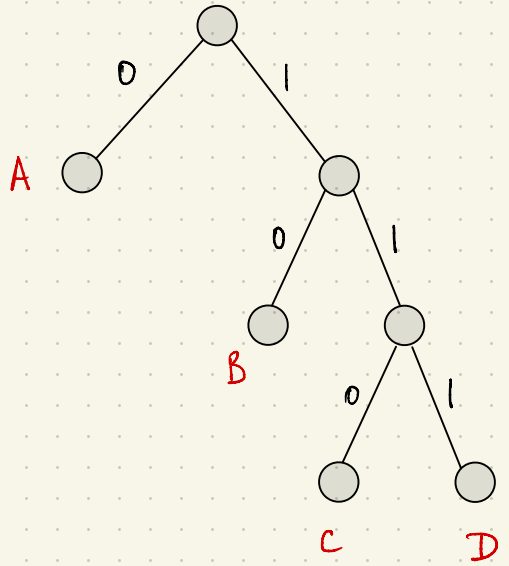

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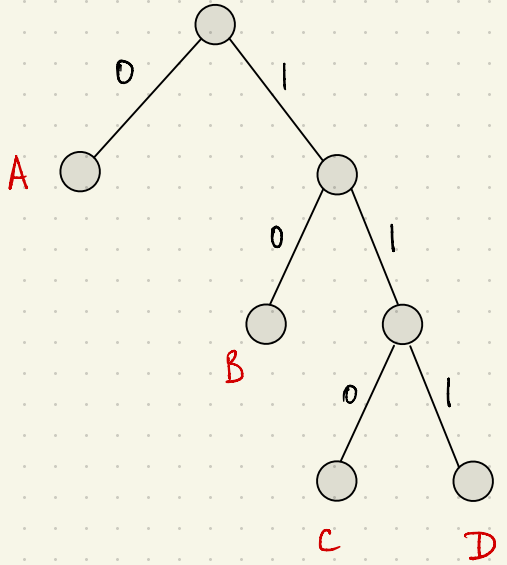

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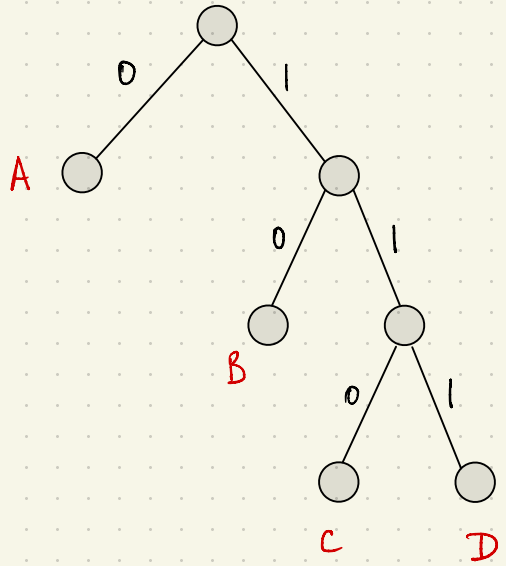

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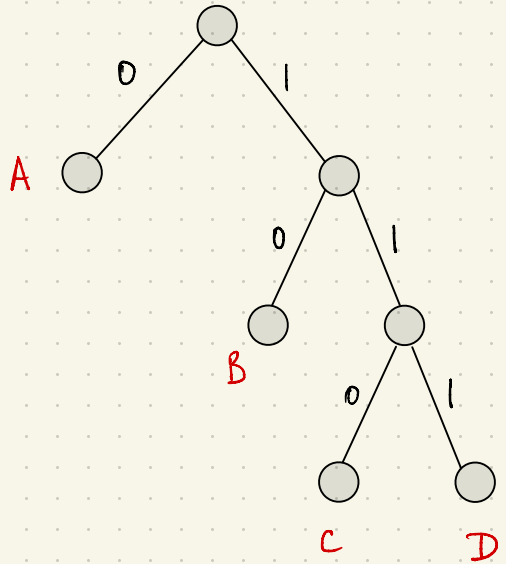

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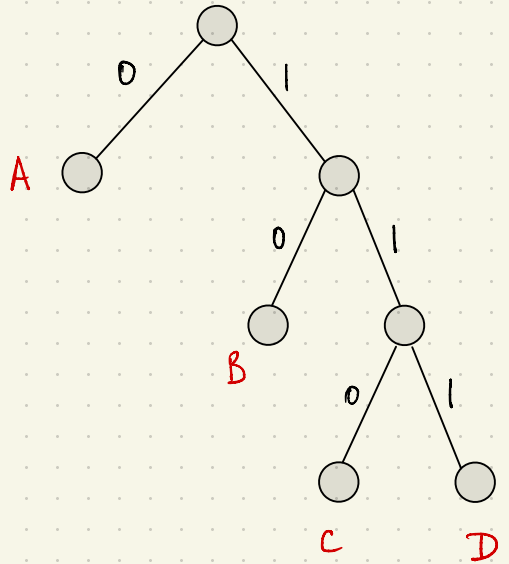

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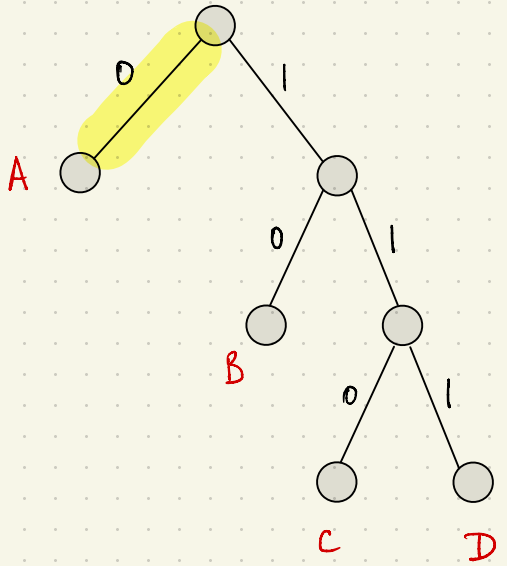

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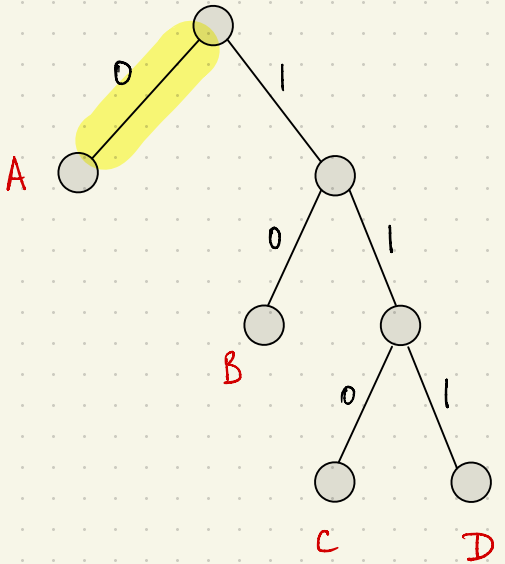
  (0 = A)



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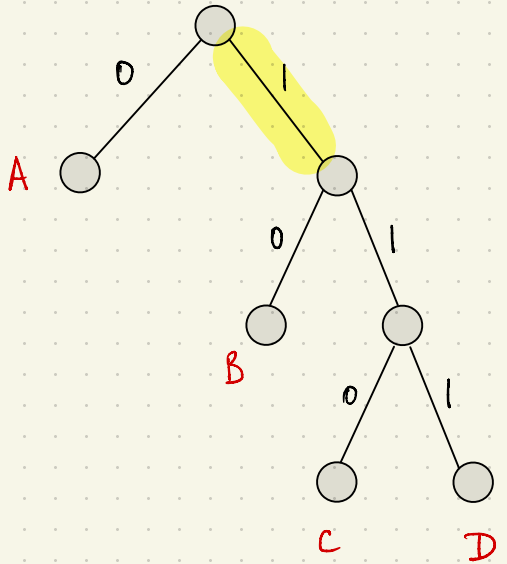
A



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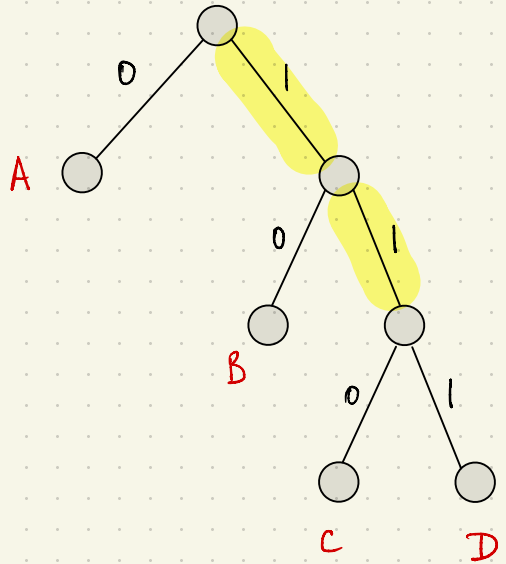

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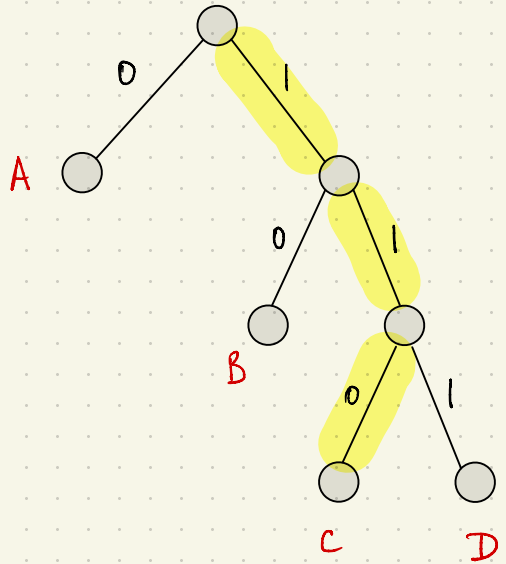

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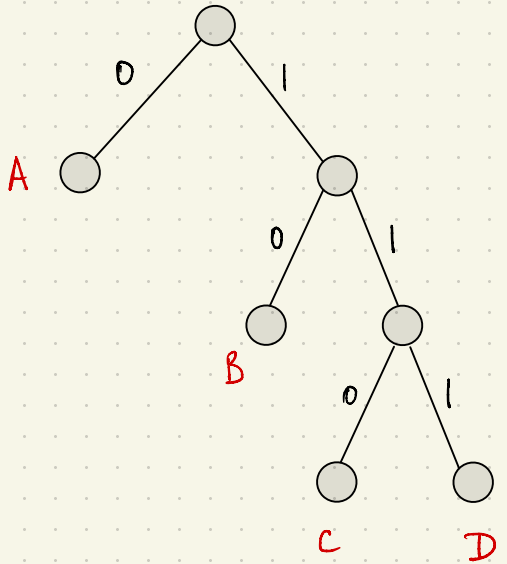
A C



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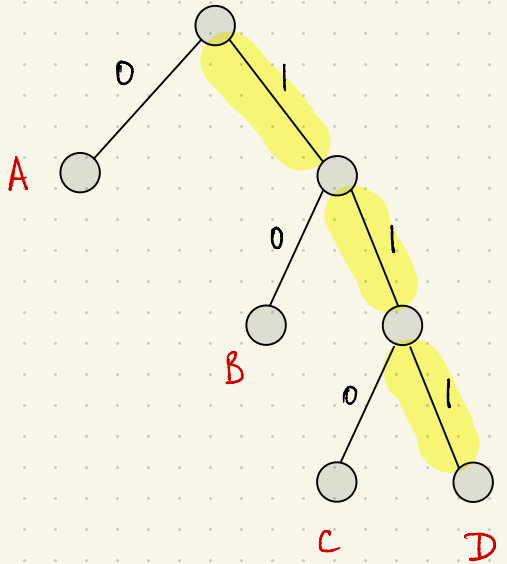

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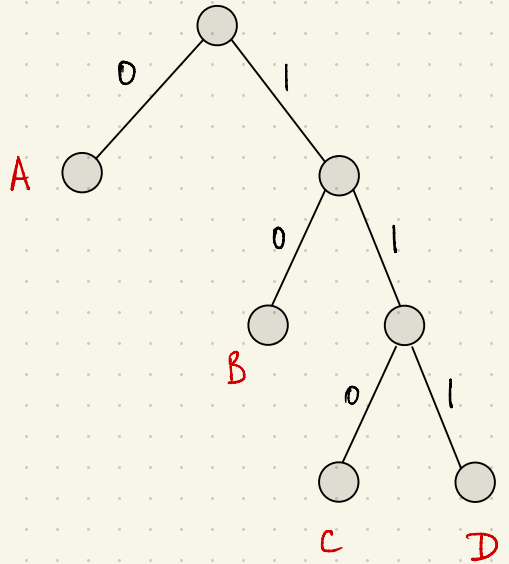
A C D



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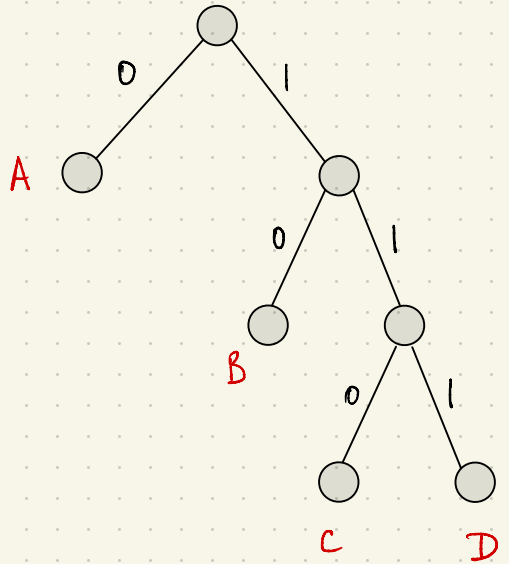
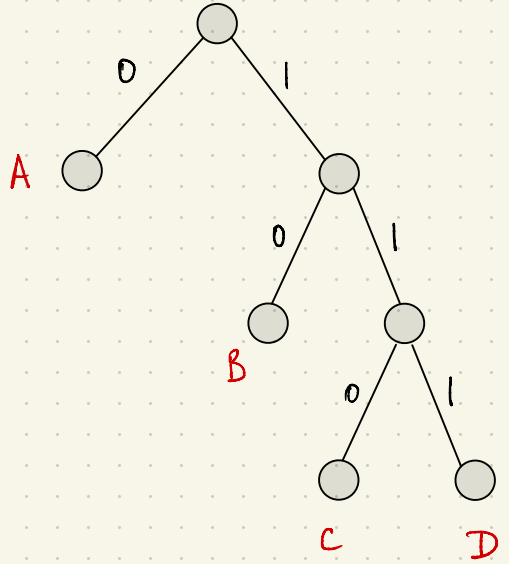
* Decoding is easy! e.g., 0 1 1 0 1 1 1 (A C D)
* Encoding length of $i \in \Sigma$ = depth of i in the tree



PREFIX-FREE CODES AS TREES

Σ tree: a binary tree with leaves labeled in one-to-one correspondence with symbols of Σ.



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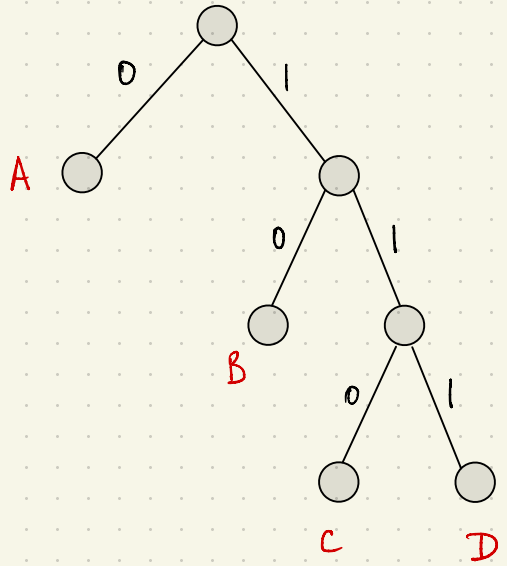

Note : For any Σ, some Σ-tree always exists

e.g., $\{0, 10, 110, 1110, \ldots\}$

Optimal Prefix-free codes

input : a non negative frequency p_i for each symbol i of alphabet Σ of size $|\Sigma| = n \geq 2$

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Optimal Prefix-free codes

input: a nonnegative frequency p_i for each symbol i of alphabet Σ of size $|\Sigma| = n \geq 2$

output: A Σ-tree with minimum possible average leaf depth

$$L(T) := \sum_{i \in \Sigma} p_i \cdot \text{depth of } i \text{ in } T$$

Optimal Prefix-free codes

symbol	frequency	fixed length	variable length
A	60%	00	0
B	25%	01	10
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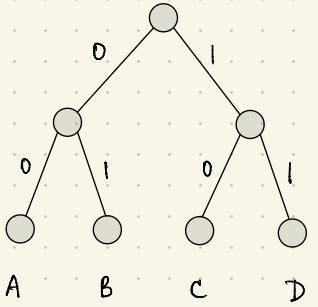

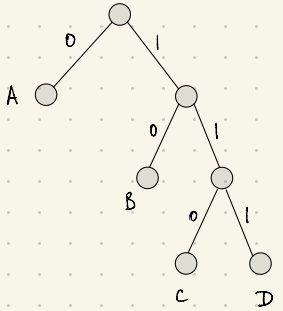

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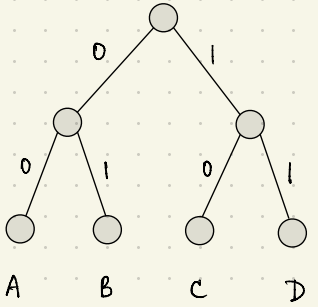

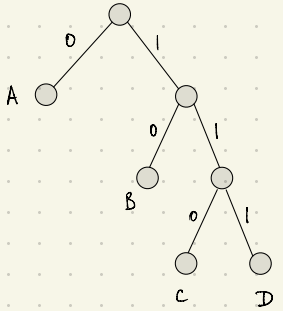

Average leaf depth : 2 1.55

$= 0.6 \times 1 + 0.25 \times 2 + 0.15 \times 3$

HUFFMAN'S GREEDY ALGORITHM

BUILDING A TREE

What's a principled approach for building a Σ-tree?

BUILDING A TREE

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Top down approach [aka Fano-Shannon encoding]

BUILDING A TREE

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BUILDING A TREE

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- partition Σ into Σ_1, Σ_2 each with $\approx 50\%$ of total frequency

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What's a principled approach for building a Σ-tree?

Top down approach [aka Fano-Shannon encoding]

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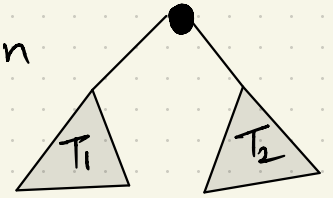
- recursively compute T_1 for Σ_1, T_2 for Σ_2, return



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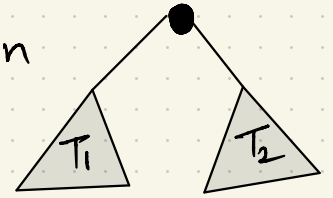

Suboptimal ☹ [Tutorial sheet 6]

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Bottom up approach [Huffman encoding]

BUILDING A TREE

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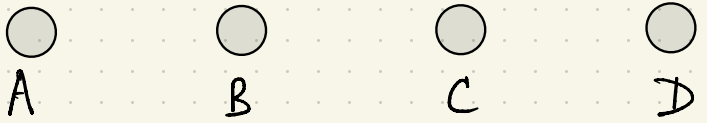
Bottom up approach [Huffman encoding]

- greedy via

  Successive mergers



BUILDING A TREE

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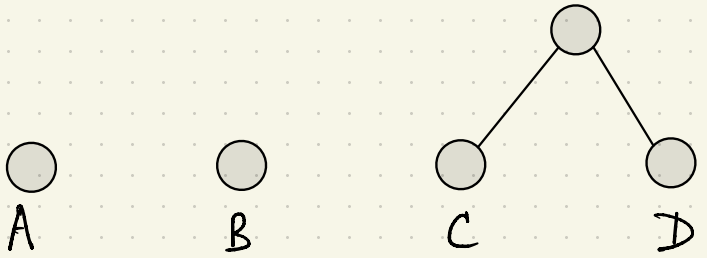

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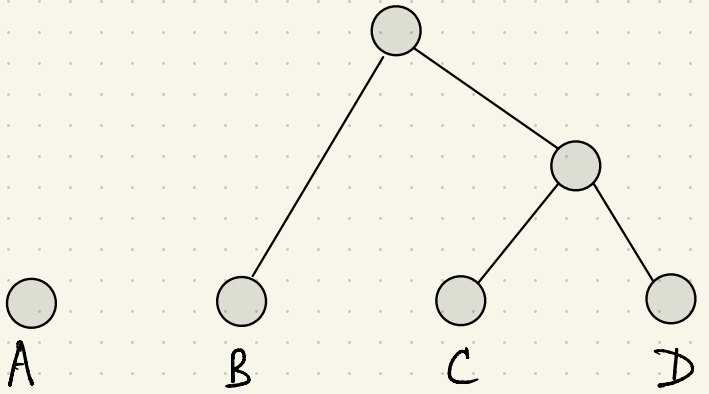

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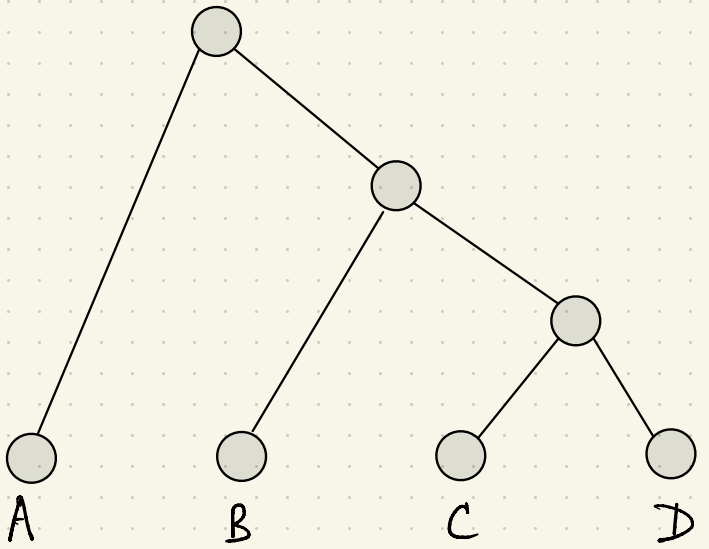

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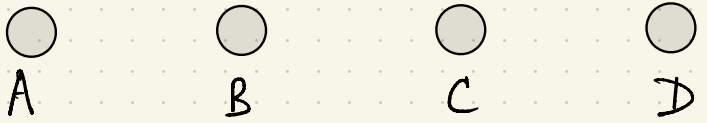

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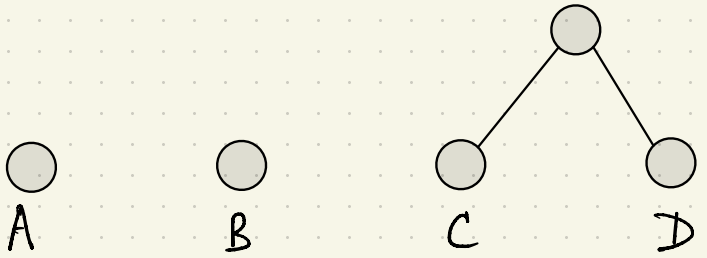

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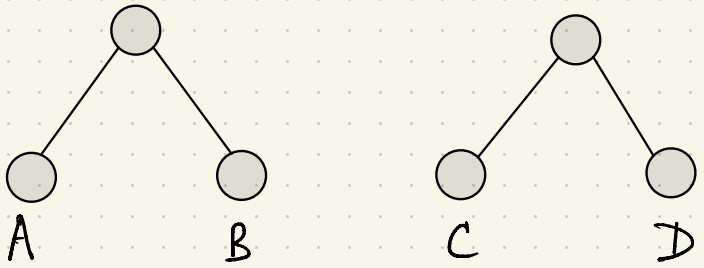

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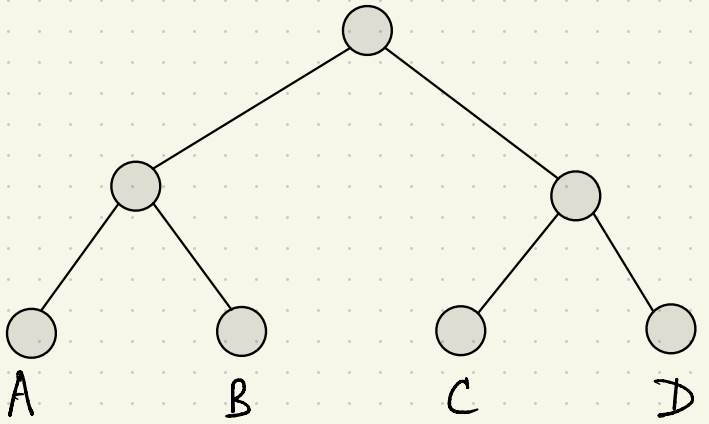
- greedy via

Successive mergers



BUILDING A TREE

What's a principled approach for building a Σ-tree?

Bottom up approach [Huffman encoding]

- after each merge

👍

👎

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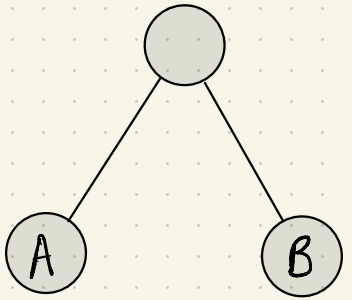

Not much to do

BUILDING A TREE

Which pair of trees is "safe" to merge?

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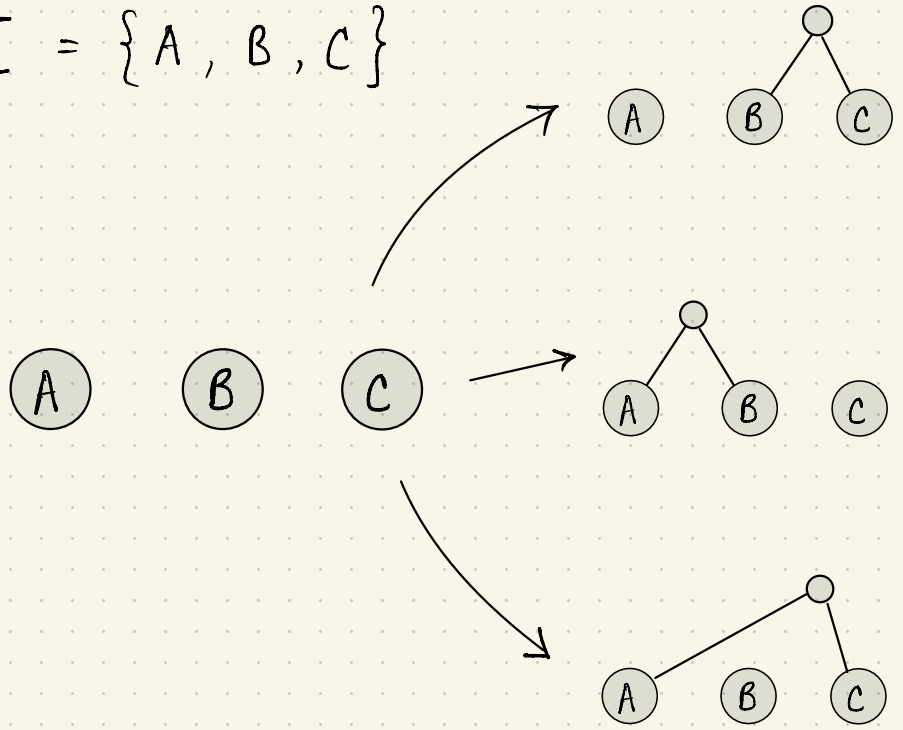

BUILDING A TREE

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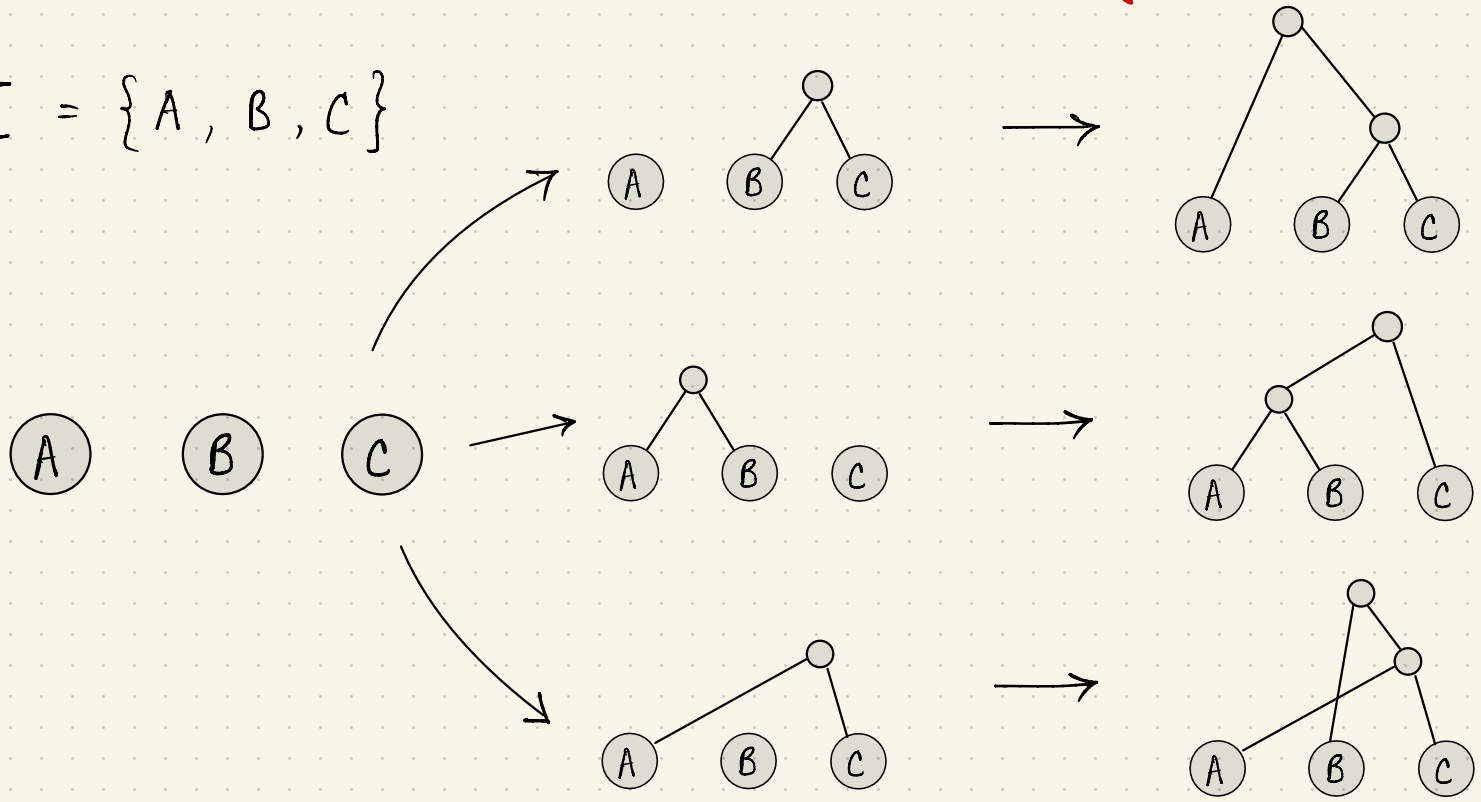
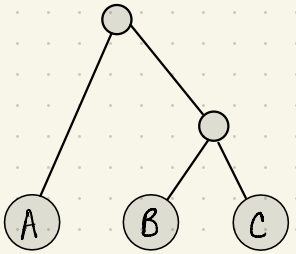


BUILDING A TREE

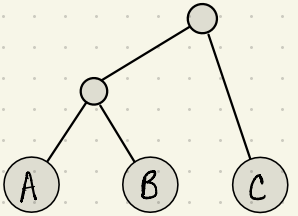
Which pair of trees is "safe" to merge?

$\Sigma = \{A, B, C\}$

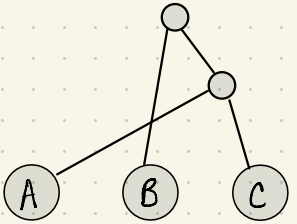
$p_a + 2p_b + 2p_c$



$2p_a + 2p_b + p_c$



$2p_a + p_b + 2p_c$

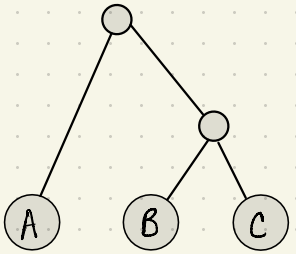


BUILDING A TREE

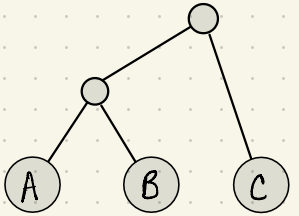
Which pair of trees is "safe" to merge?

$\Sigma = \{A, B, C\}$

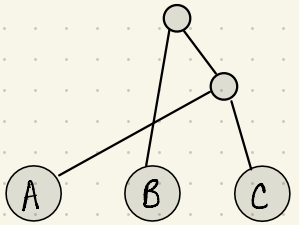
$(p_a + p_b + p_c) + (p_b + p_c)$



$(p_a + p_b + p_c) + (p_a + p_b)$



$(p_a + p_b + p_c) + (p_a + p_c)$



BUILDING A TREE

Which pair of trees is "safe" to merge?

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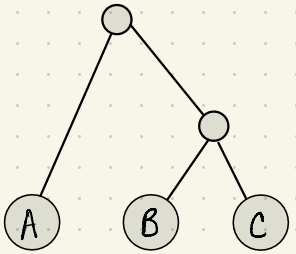
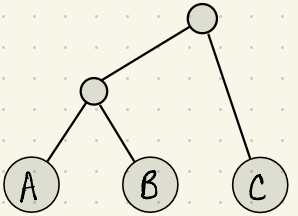
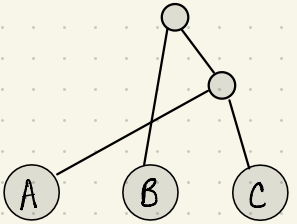


$(p_a + p_b + p_c) + (p_a + p_b)$



$(p_a + p_b + p_c) + (p_a + p_c)$



BUILDING A TREE

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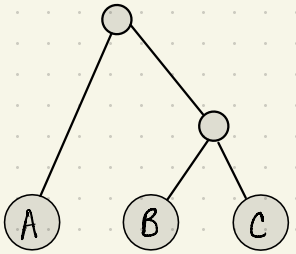
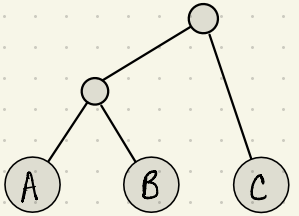
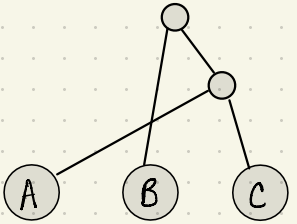


The first merge should involve the two least-frequent symbols

$(p_a + p_b + p_c) + (p_a + p_b)$



$(p_a + p_b + p_c) + (p_a + p_c)$



BUILDING A TREE

Which pair of trees is "safe" to merge?

$\Sigma = \{A, B, C, D\}$

How to merge?