COL 351 : Analysis & Design of Algorithms

Lecture 18

Greedy Algorithms II : Huffman's algorithm

Sept 04, 2024 | Rohit Vaish

Optimal Prefix-free codes

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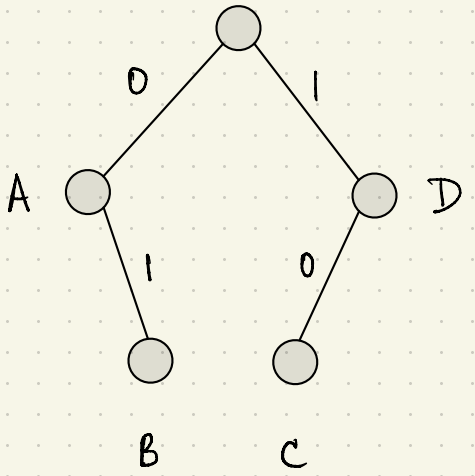
BINARY CODES ⟷ LABELED BINARY TREES

$\Sigma = \{A, B, C, D\}$

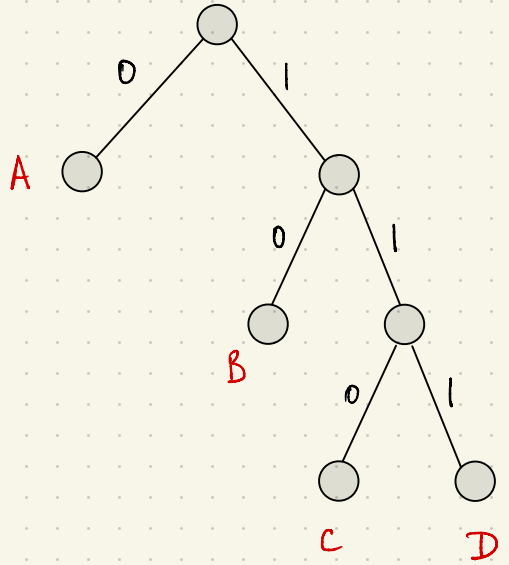
Symbol	variable length
A	0
B	01
C	10
D	1

⟷



PREFIX-FREE CODES AS TREES

* prefix-free code $\iff$ only leaves are labeled



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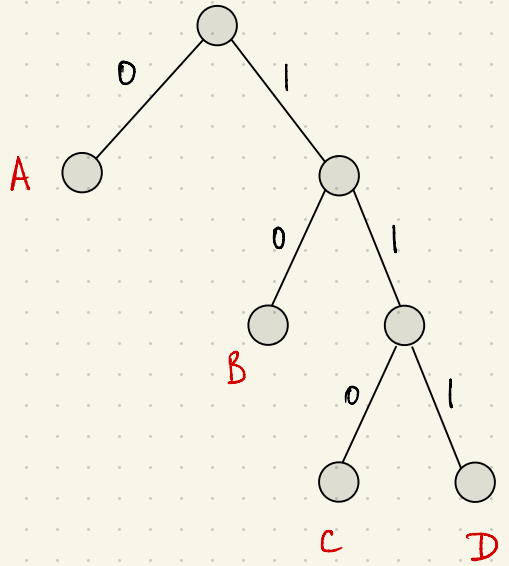

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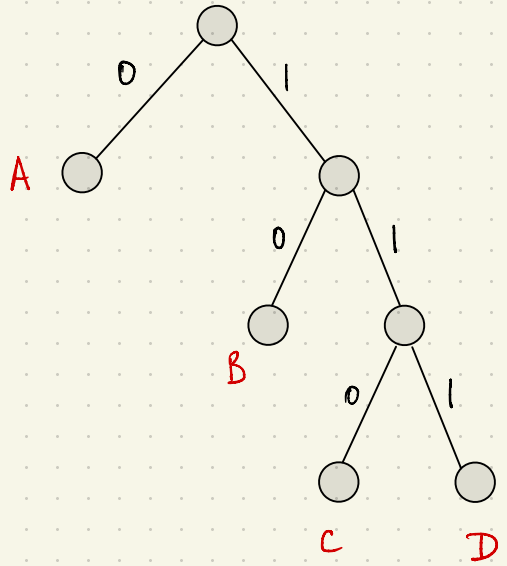
* Σ tree: a binary tree with leaves labeled in one-to-one correspondence with symbols of Σ.



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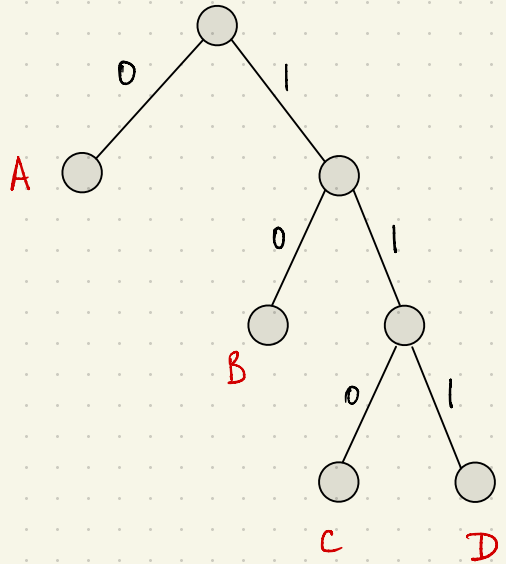
* For any Σ, some Σ-tree always exists
  e.g., $\{0, 10, 110, 1110, \ldots\}$



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Optimal Prefix-free codes

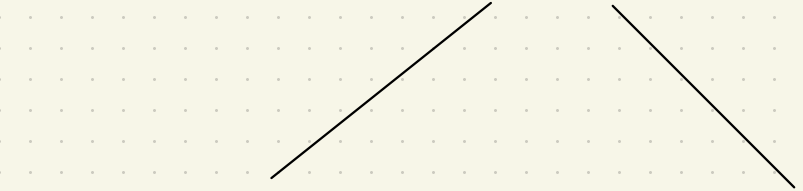
input: a nonnegative frequency p_i for each symbol i of alphabet Σ of size $|\Sigma| = n \geq 2$

output: A Σ-tree T with minimum possible average leaf depth

$$L(T) := \sum_{i \in \Sigma} p_i \cdot \text{depth of leaf labeled } i \text{ in } T$$

BUILDING A TREE

What's a principled approach for building a Σ-tree?



Top down

(divide and conquer)

Bottom up

(greedy)

TOP DOWN APPROACH

[Shannon - Fano encoding]

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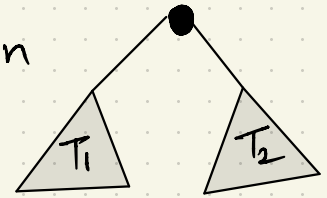
* partition Σ into Σ_1, Σ_2 each with $\approx 50\%$ of total frequency
* recursively compute T_1 for Σ_1, T_2 for Σ_2, return



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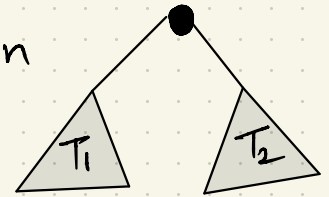

* Suboptimal ☹ [Tutorial sheet 6]

HUFFMAN'S GREEDY ALGORITHM

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leveraging partial knowledge of optimal solution

Recall : prefix-free code $\longleftrightarrow$ labeled binary tree
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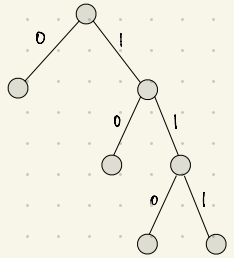

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  - if we knew this
- a labeling of T^*

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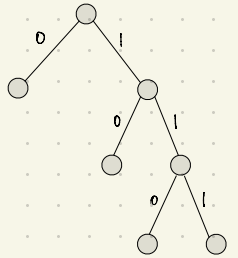
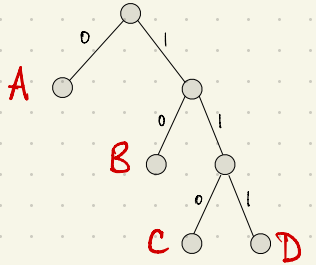


Want

- an optimal Σ tree T^* — if we knew this
- a labeling of T^* — Can we determine this?



LABELING AN OPTIMAL SOLUTION

Lemma: Let u and v be leaves of T^* such that $\text{depth}(u) < \text{depth}(v)$. Further, suppose in the labeling of T^* corresponding to an optimal prefix code, u is labeled $a \in \Sigma$ and v is labeled $b \in \Sigma$. Then, $p_a \geq p_b$.

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Proof: Exercise

Hint: Apply exchange argument to $L(T^*) := \sum_{i \in \Sigma} p_i \cdot$ depth of leaf labeled i in T^*.

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Labeling algorithm:

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- and so on ...

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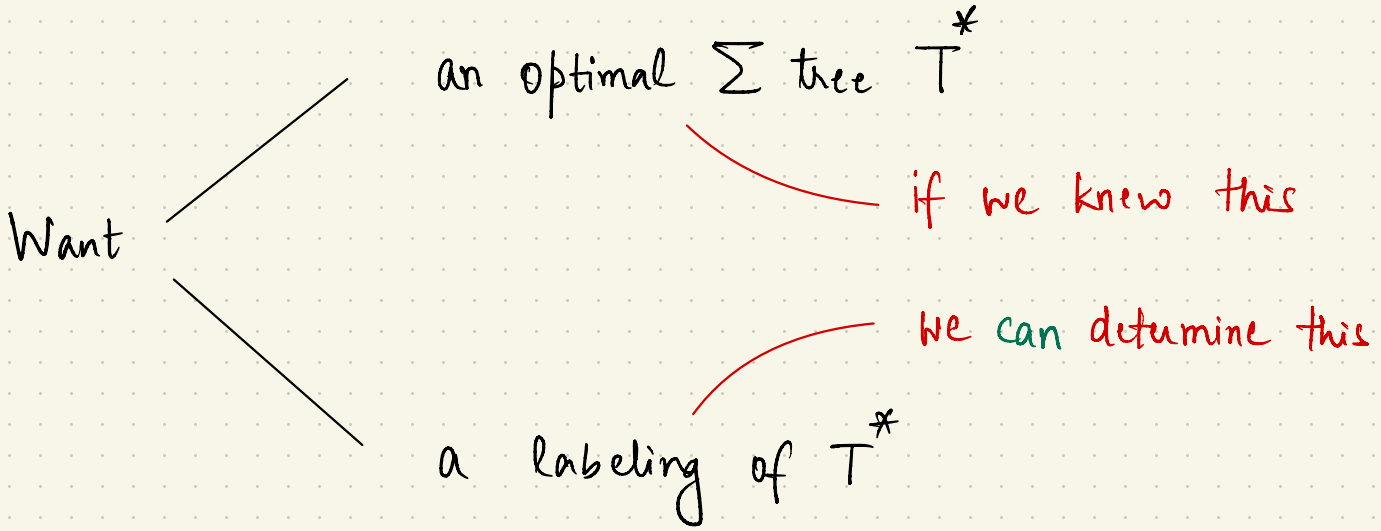

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Want
- an optimal Σ tree T^*
  - how to determine this?
  - brute force : exponentially-many binary trees ☹
- a labeling of T^*

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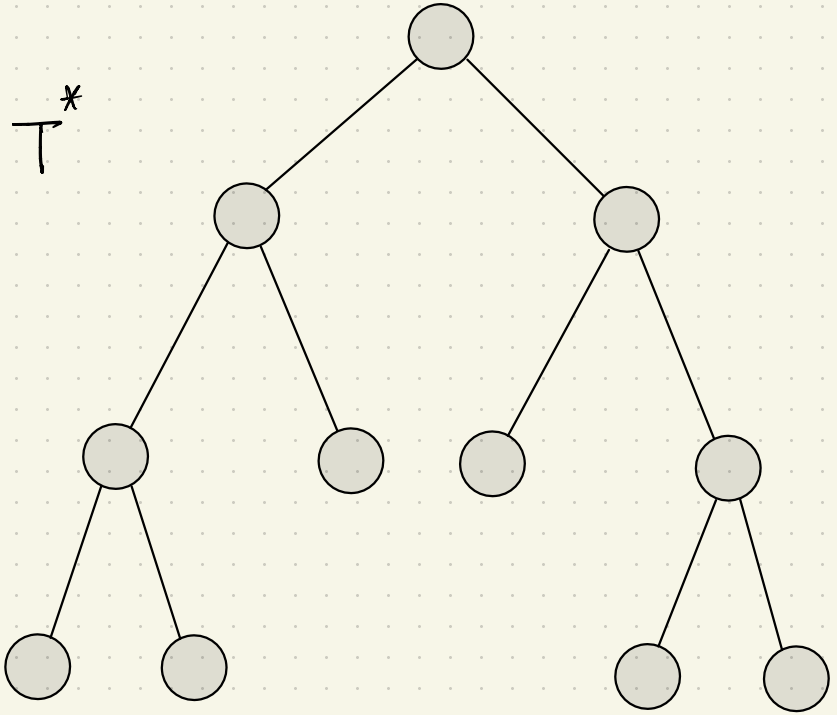
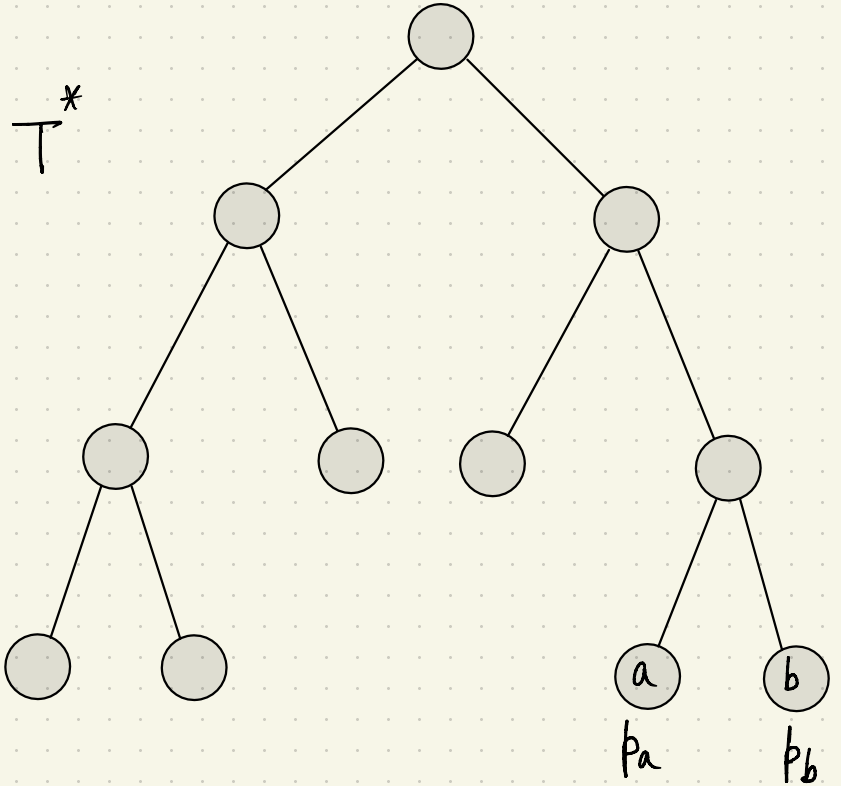
Check: If u is a leaf at maximum depth, then u's sibling is also a leaf at maximum depth.

Corollary: There is an optimal prefix code for input (Σ, p) with corresponding Σ-tree T^* in which the two lowest-frequency symbols are assigned to sibling leaves at maximum depth.

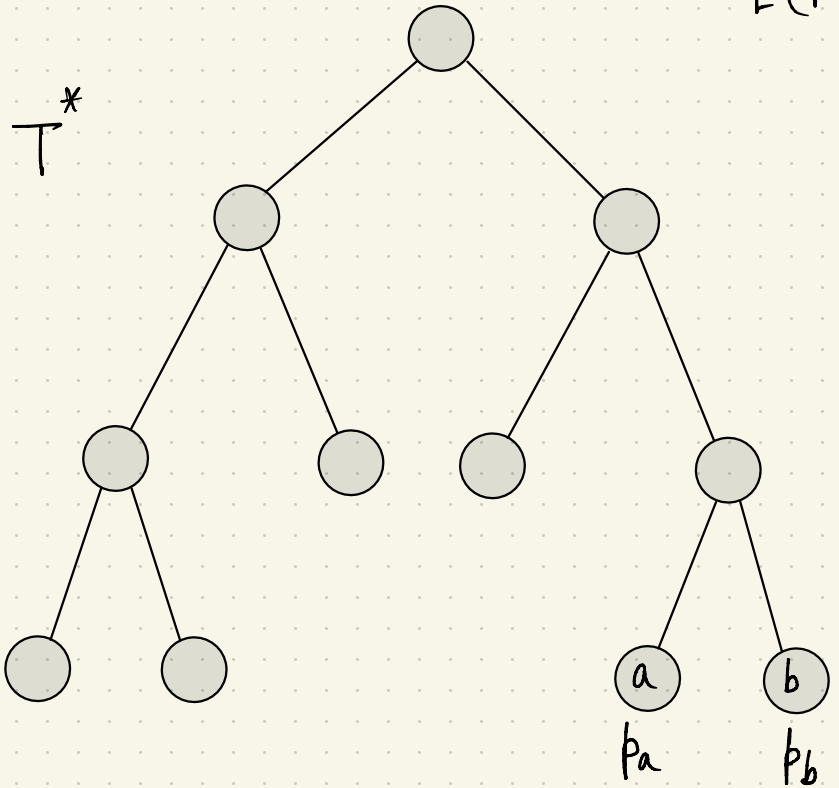

T^*



Two lowest-frequency symbols

("safely" locked in")

$$L(T^*) = \sum_{i \in \Sigma} p_i \cdot \text{depth of leaf labeled } i$$



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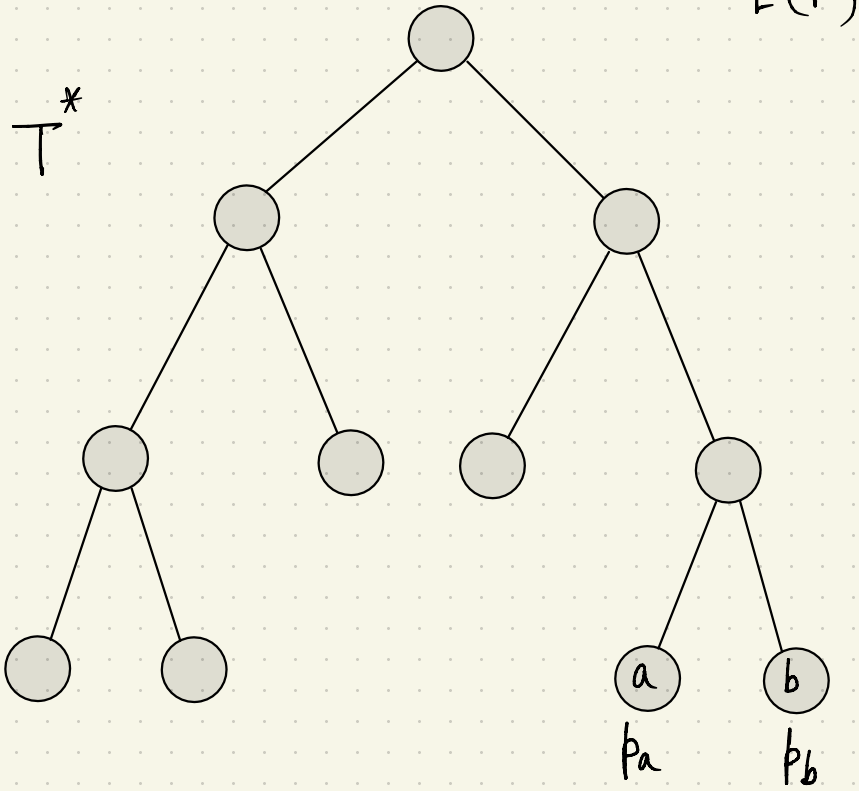

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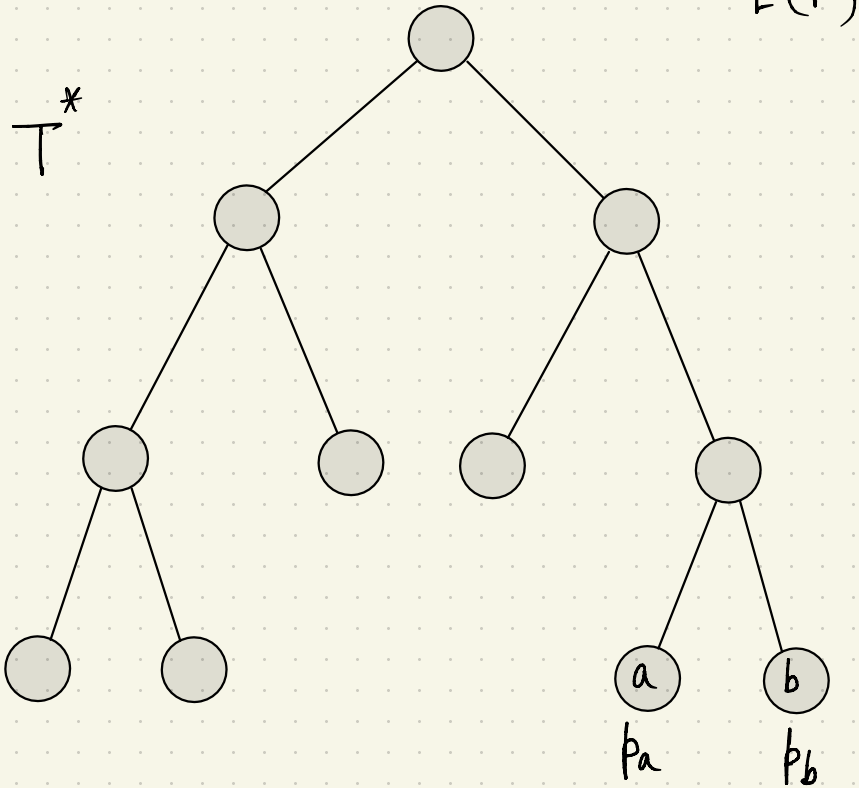

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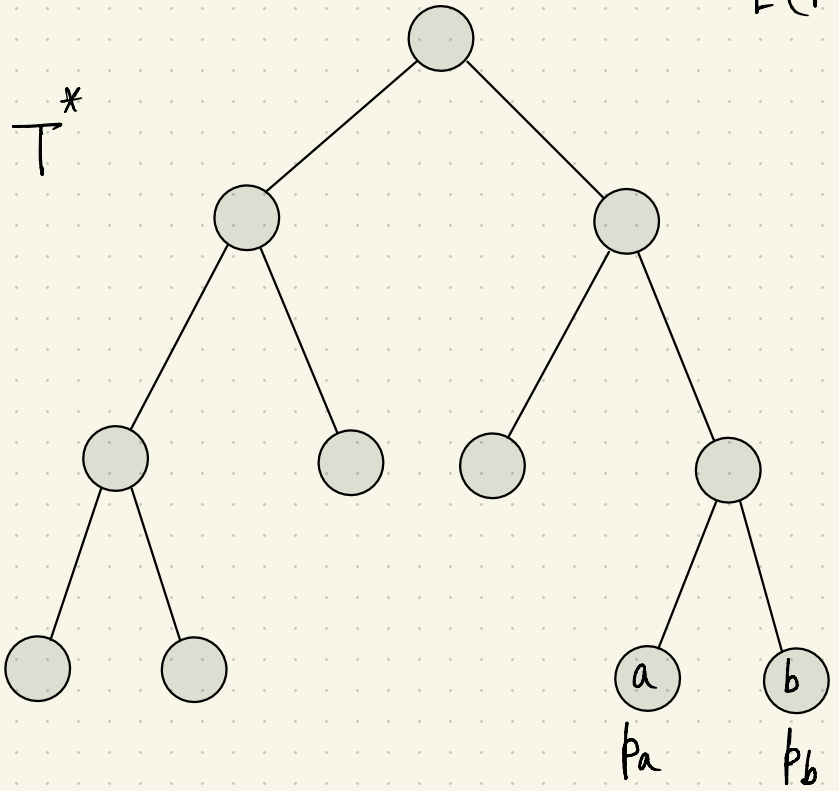

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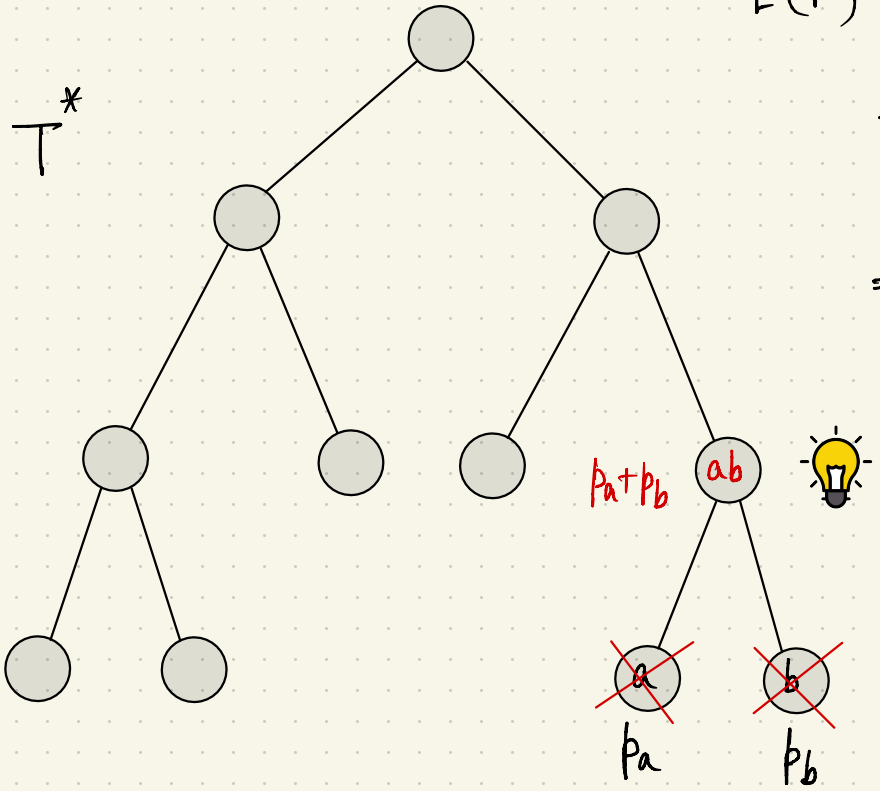

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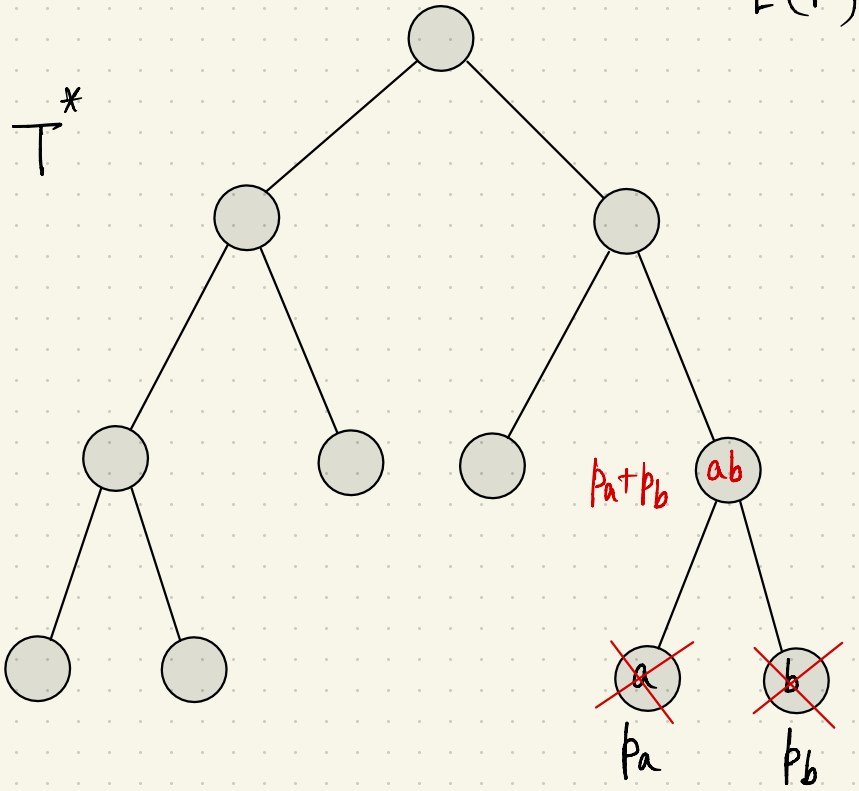

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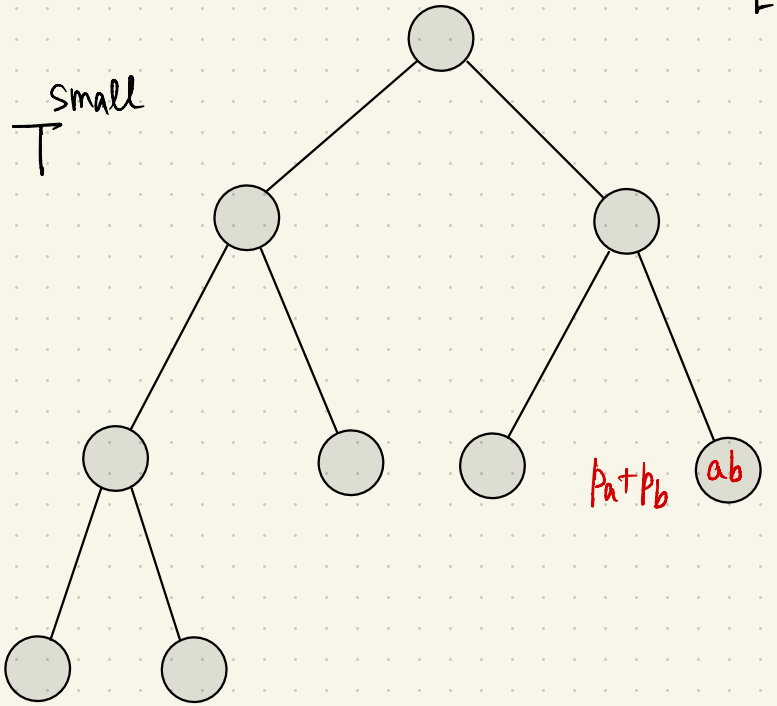

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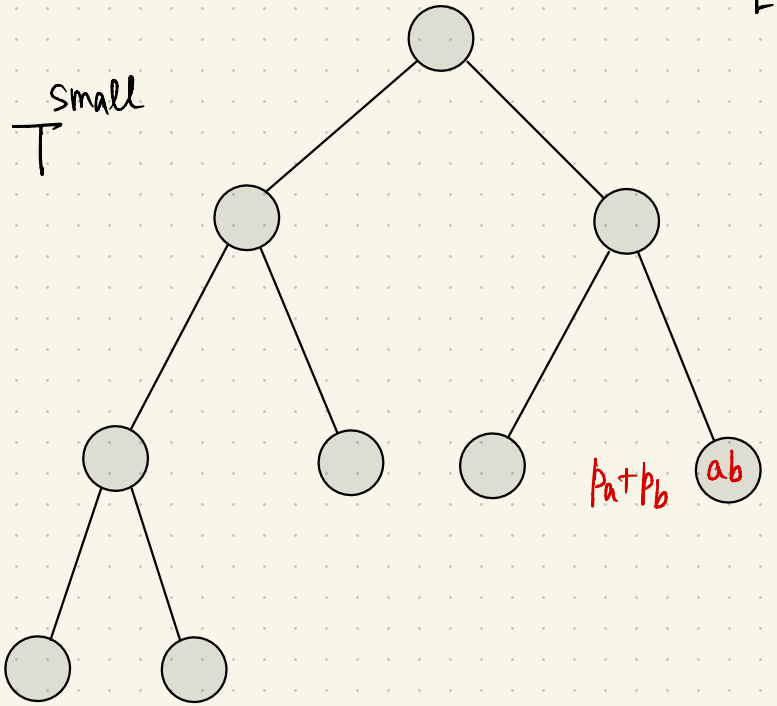

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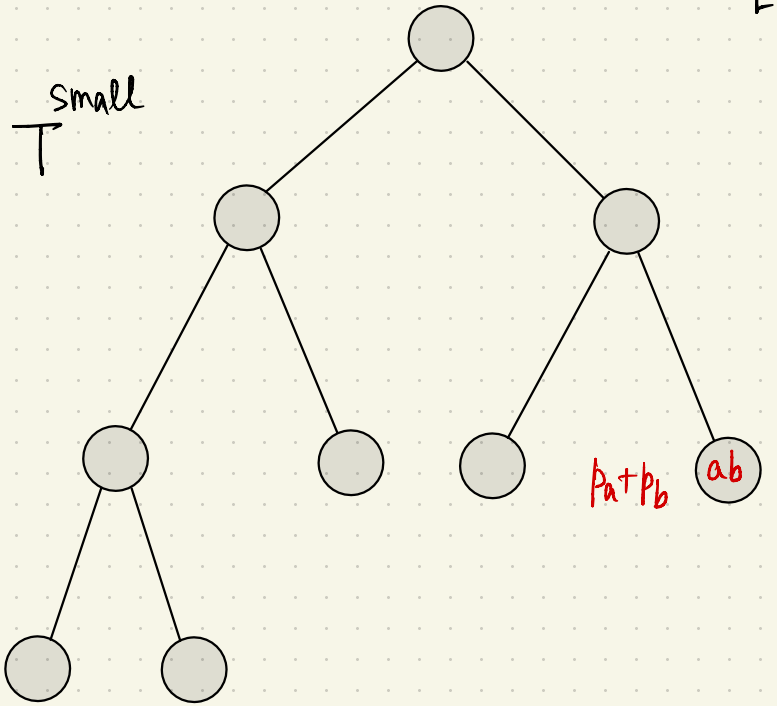
fixed



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fixed

recurse!

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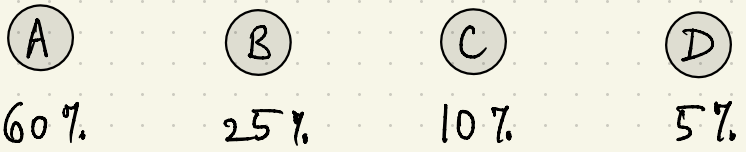

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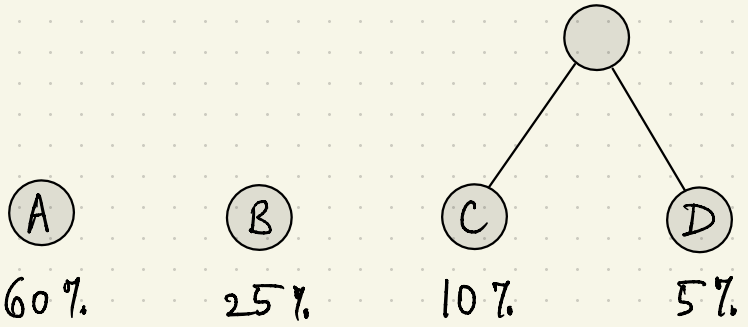

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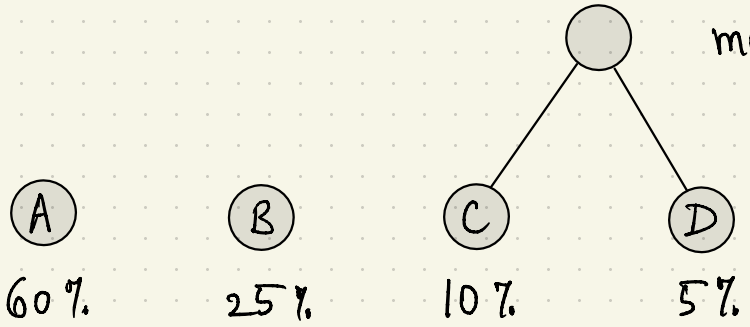

meta-symbol "CD" with frequency = 10+5 = 15%

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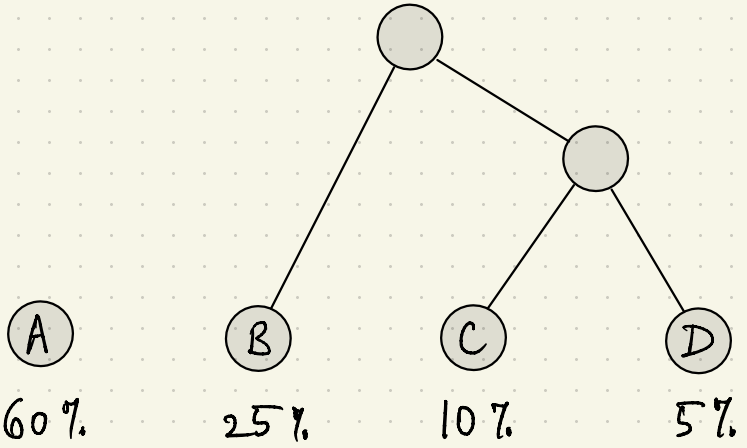

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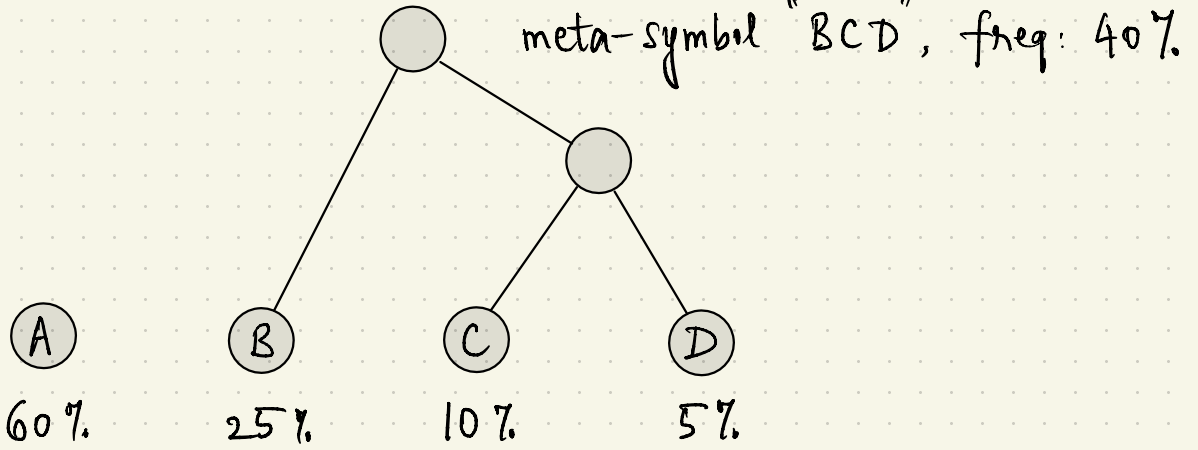

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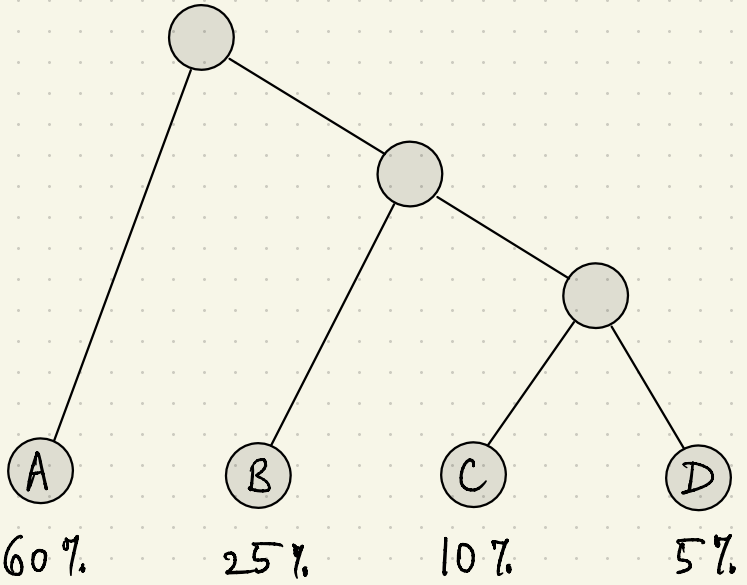

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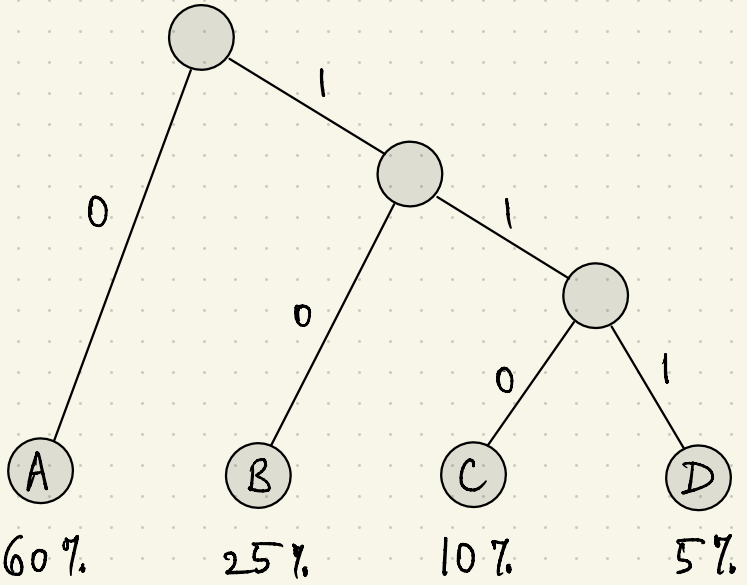

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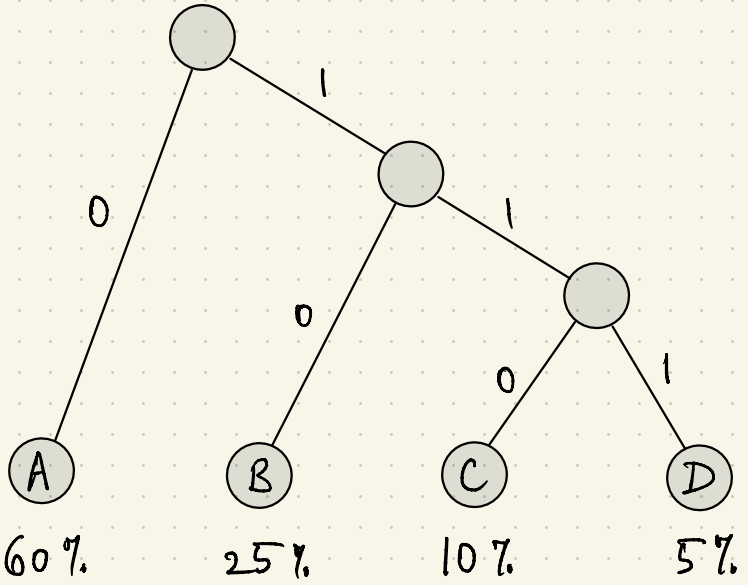


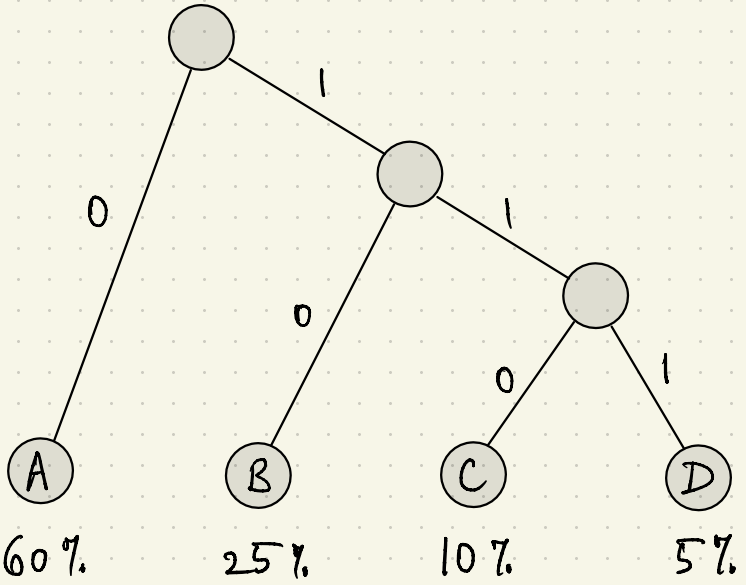
A : 0

B : 01

C : 110

D : 111

HUFFMAN'S ALGORITHM



Huffman code

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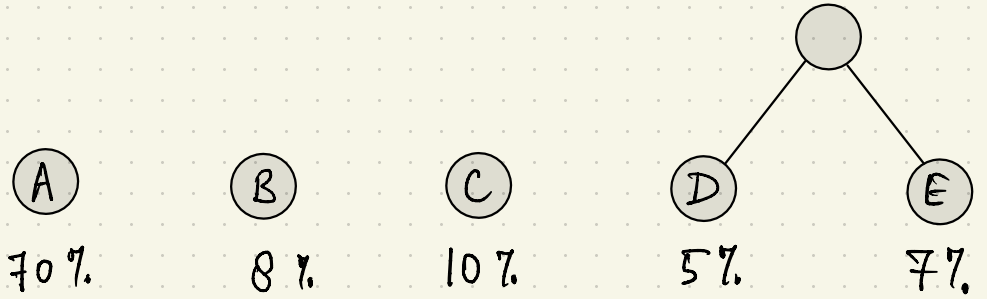

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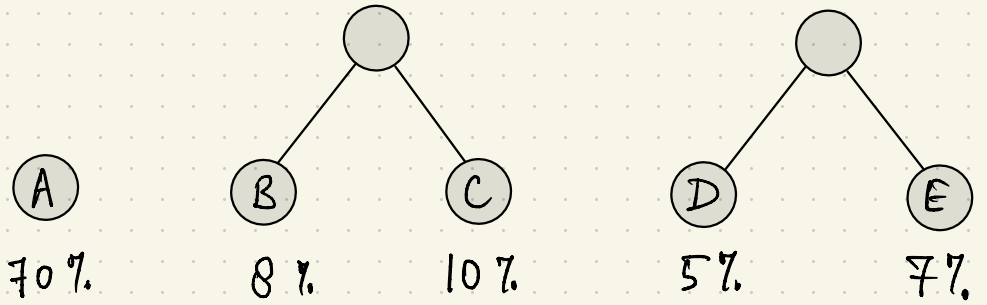

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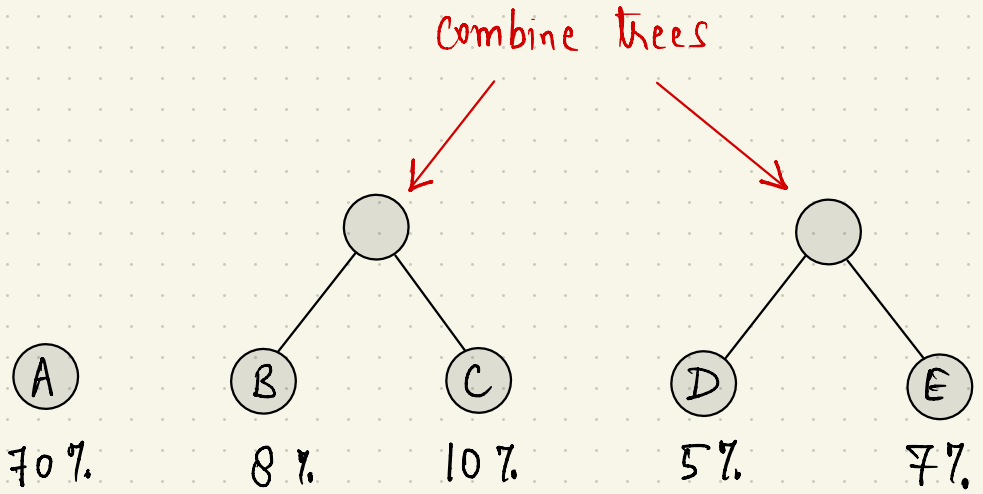

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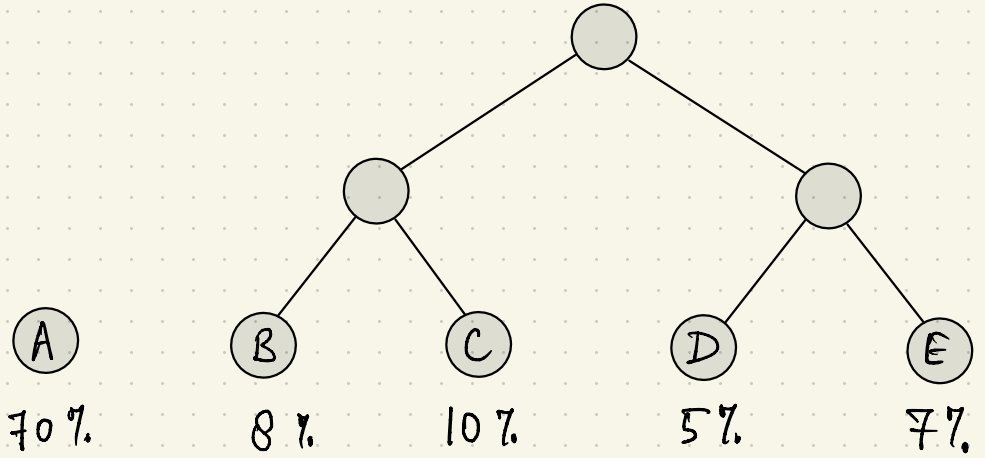

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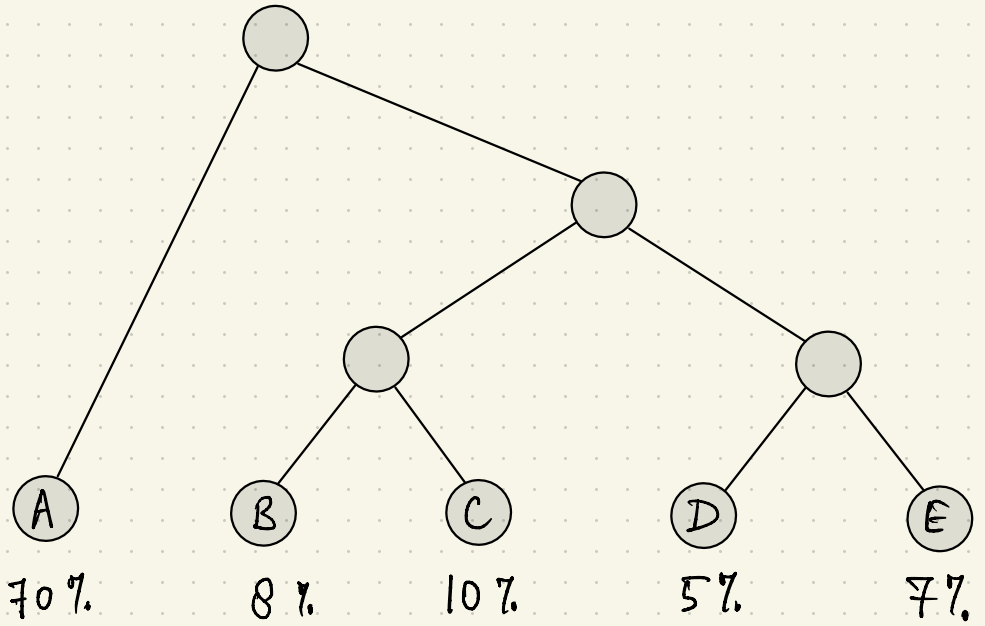

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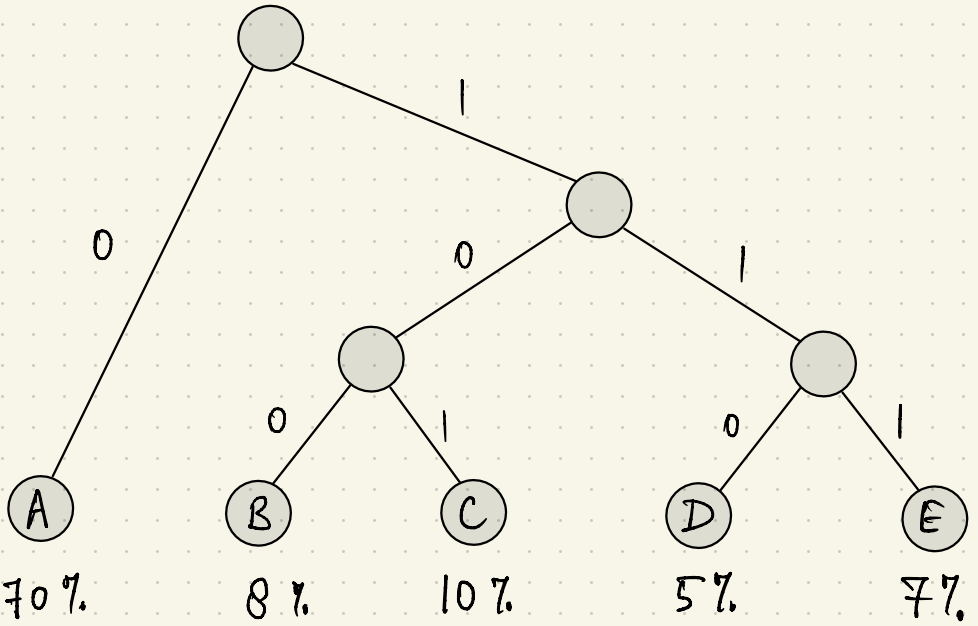
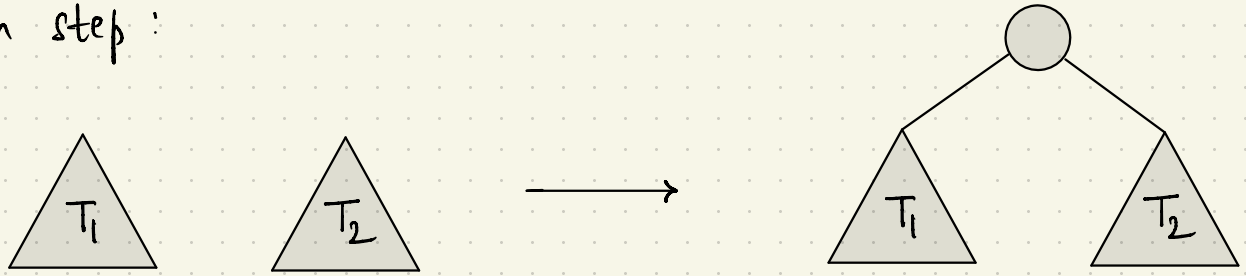


A: 0
B: 100
C: 101
D: 110
E: 111

AN ITERATIVE VIEW OF HUFFMAN'S ALGORITHM

At each step:



after each merge

👍 # trees in the forest decreases by 1

👎 average depth of leaves increases

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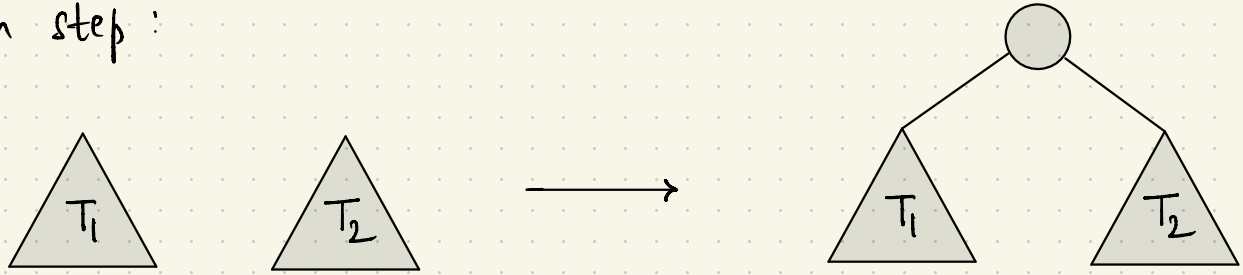

$$\text{increase in average leaf-depth} = \sum_{a \in T_1} p_a + \sum_{b \in T_2} p_b$$

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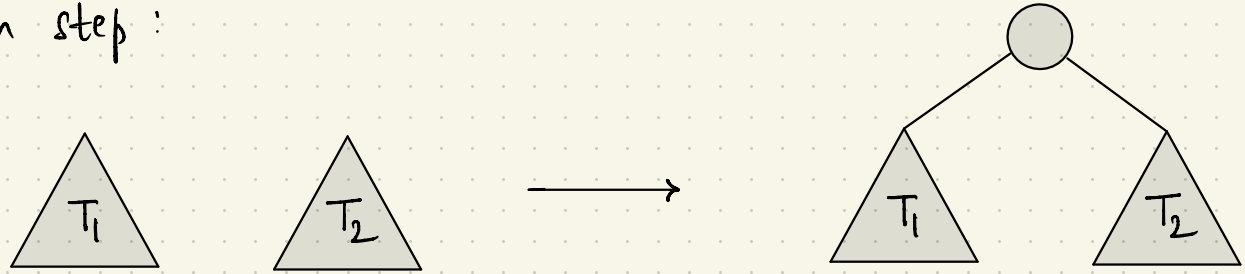

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Huffman's greedy criterion

Merge the pair of trees that causes minimum possible increase in average leaf depth.

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input: alphabet Σ, frequencies $\{p_a\}_{a \in \Sigma}$

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return unique tree in F.

Running time of Huffman's algorithm

RUNNING TIME OF HUFFMAN'S ALGORITHM

if $|\Sigma| = 2$

  encode one letter using 0 and other using 1

else

  $x, y \leftarrow$ two lowest-frequency letters

  Create new alphabet Σ' by merging x and y into z with $p_z := p_x + p_y$.

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Correctness of Huffman's algorithm

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Theorem: For every alphabet Σ and nonnegative frequencies $\{p_a\}_{a \in \Sigma}$, Huffman's algorithm returns a prefix-free code with minimum possible average encoding length.

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Base case: $n = 2$

Huffman uses one bit per symbol — optimal!

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Let T' be algorithm's output for $\Sigma' \Rightarrow T'$ is optimal!

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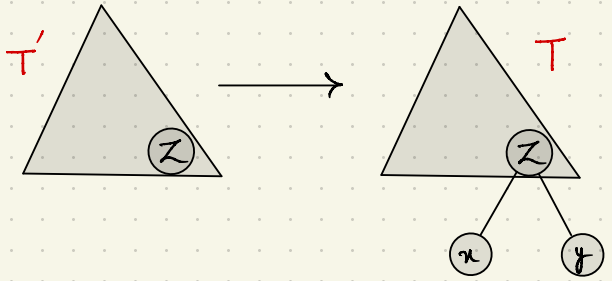

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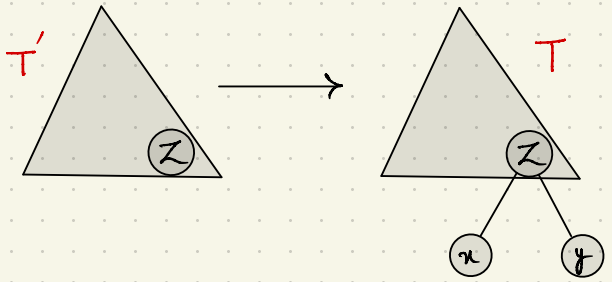
$$L(T') = L(T) - (p_x + p_y) \quad \text{(exercise)}$$



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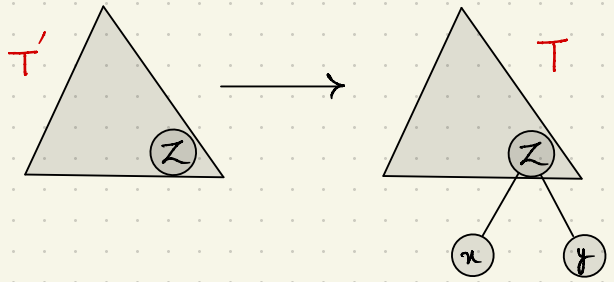

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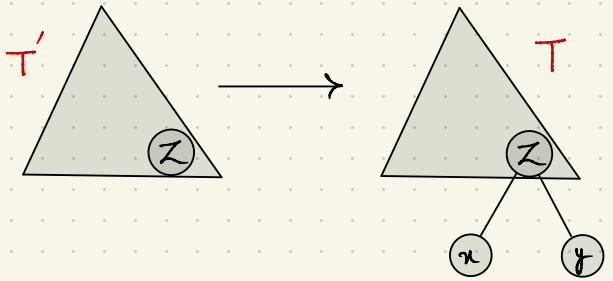

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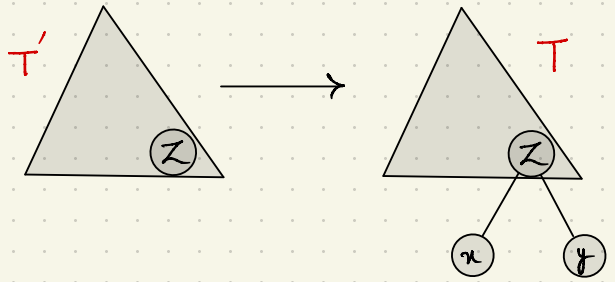

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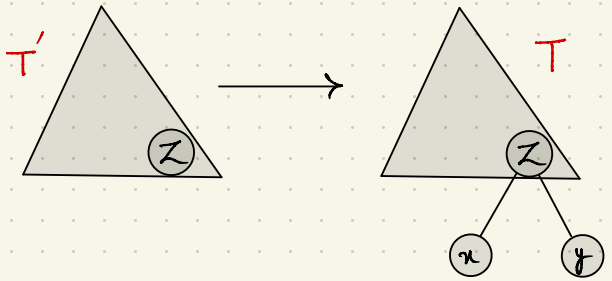

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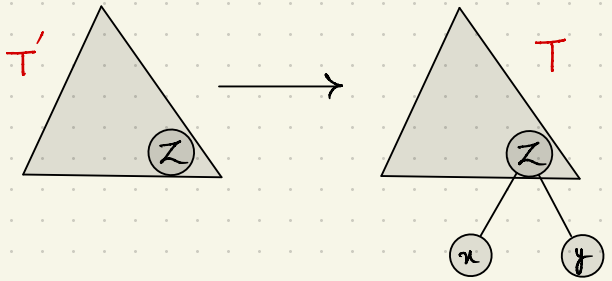

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SUMMARY

Goal : Prefix-free code with minimum average encoding length

Key ideas : (1) Reimagining codes as trees

(2) Two lowest-frequency symbols are "locked in" as deepest leaves.

(3) "Merging" into meta-symbol facilitates recursion

Minimum Spanning Trees

SPANNING TREES

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A spanning tree of a connected graph

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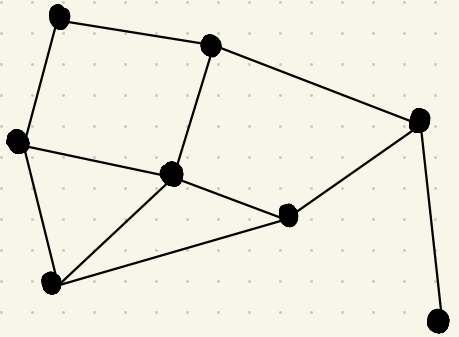
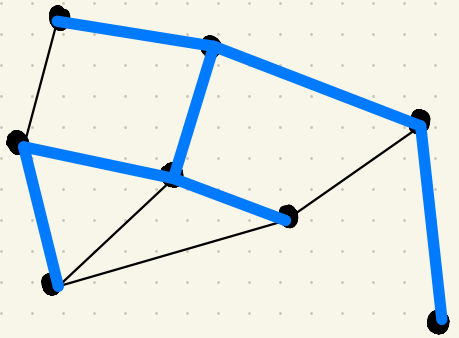


SPANNING TREES

A spanning tree of a connected graph is a subgraph that is itself a tree and whose vertex set is the same as that of the original graph.



Theorem: Every connected graph has a spanning tree.

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Proof: By contradiction.

Suppose, for contradiction, that some connected graph G has no spanning tree.

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↑ T is well-defined because of well ordering principle.

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Suppose, for contradiction, that some connected graph G has no spanning tree.

Let T be a connected subgraph of G with the same set of vertices and the smallest number of edges possible.

Then, by contradiction assumption, T is NOT a tree.

Theorem: Every connected graph has a spanning tree.

Proof: Therefore, T must have a cycle.

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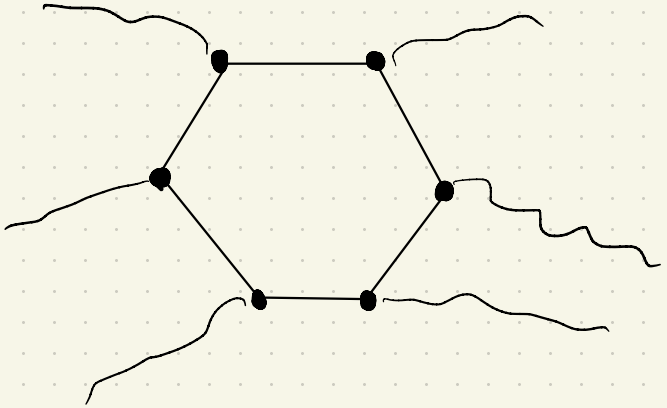

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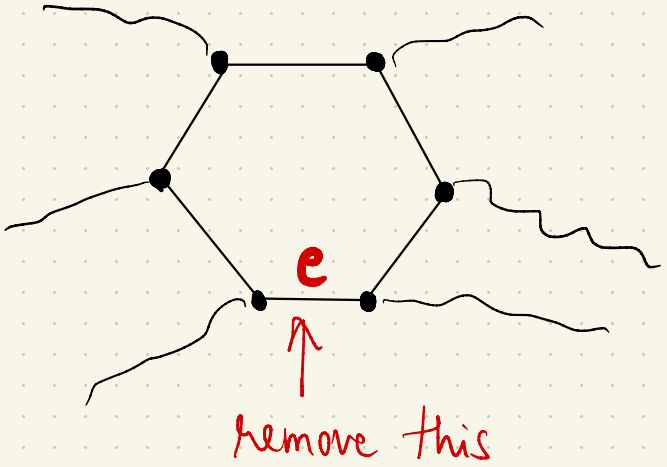

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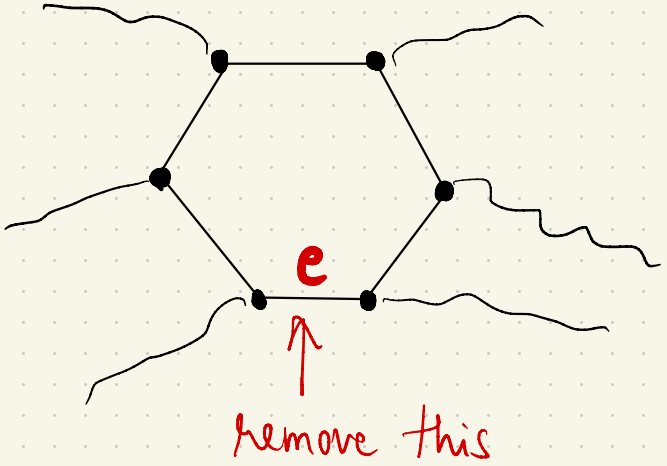

Case I:

does not contain e.

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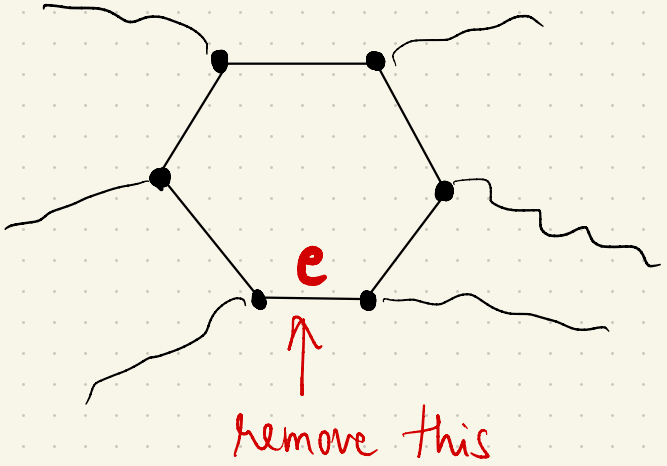
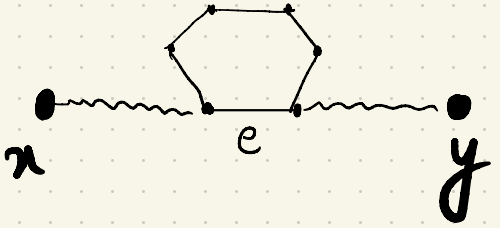


Case I:

does not contain e.

Case II:



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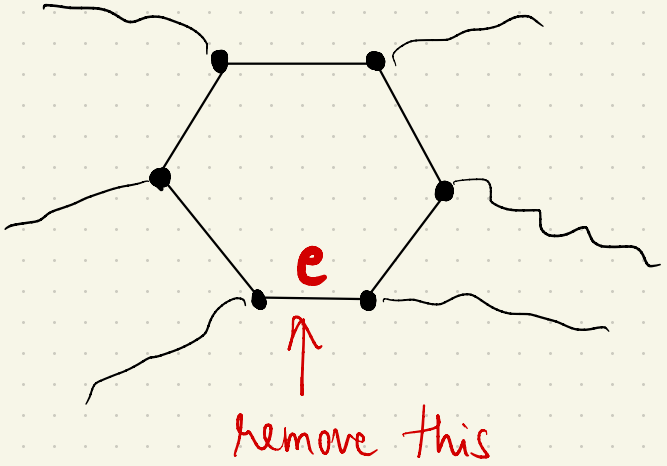

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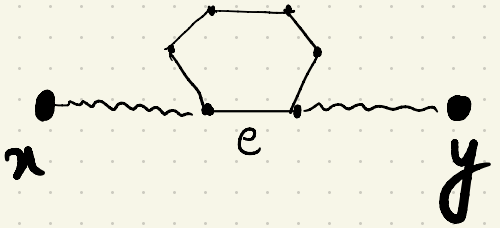
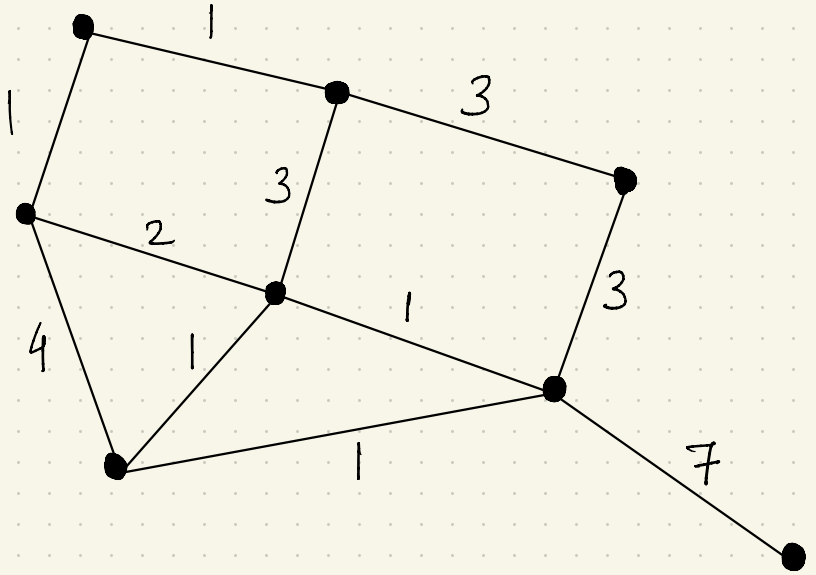
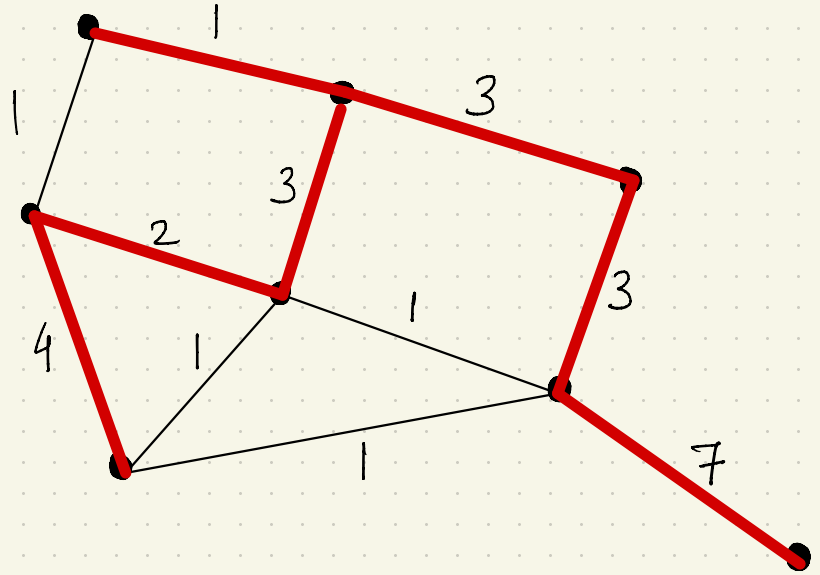


All vertices in T are still connected after removing e.

Minimum weight spanning tree

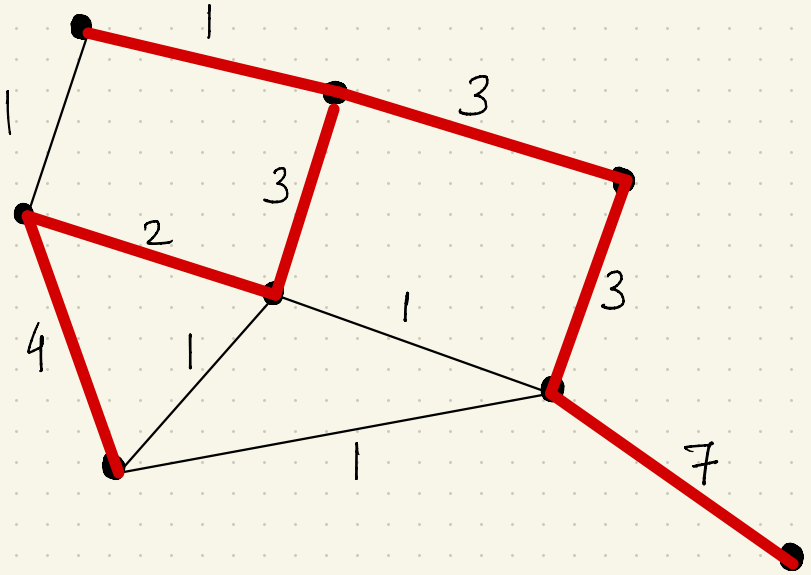


Minimum weight spanning tree



Weight = 1 + 3 + 3 + 3 + 2 + 4 + 7 = 23

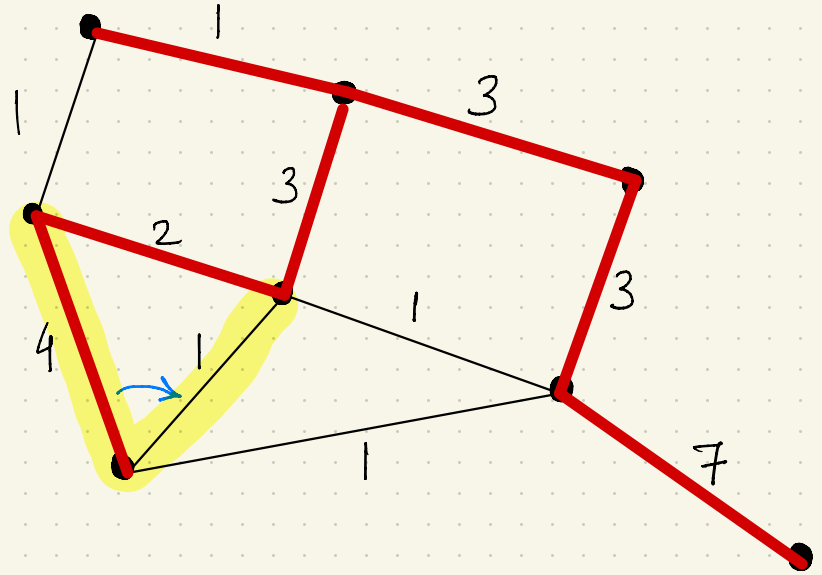
Minimum weight spanning tree



Weight = $1+3+3+3+2+4+7 = 23$

Can we do better?

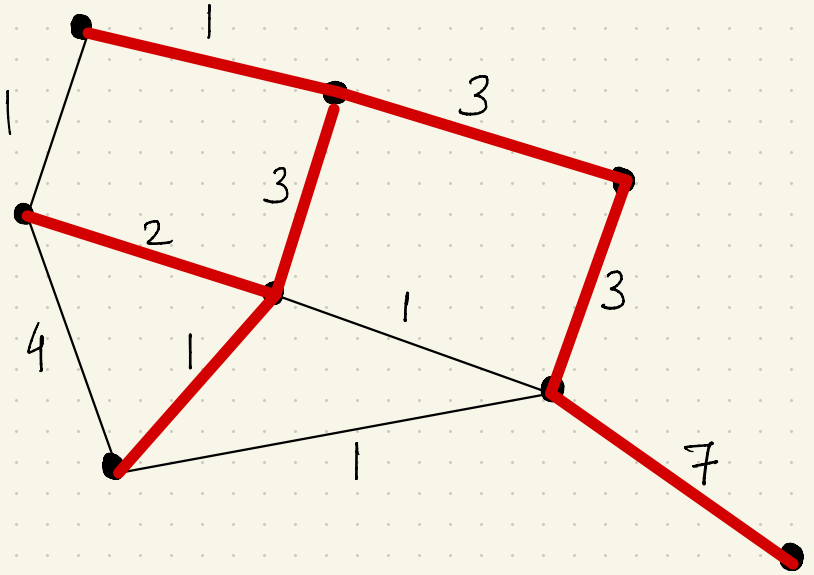
Minimum weight spanning tree



Weight = $1+3+3+3+2+4+7 = 23$

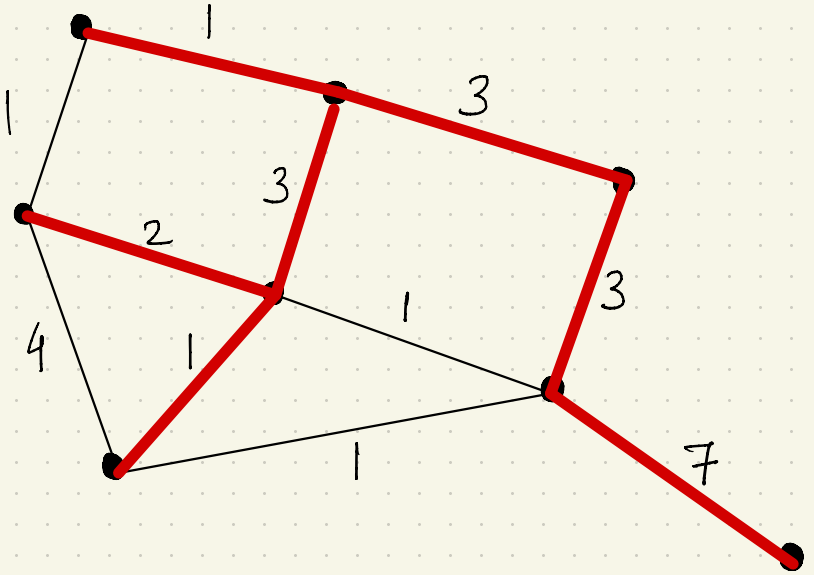
Can we do better?

Minimum weight spanning tree



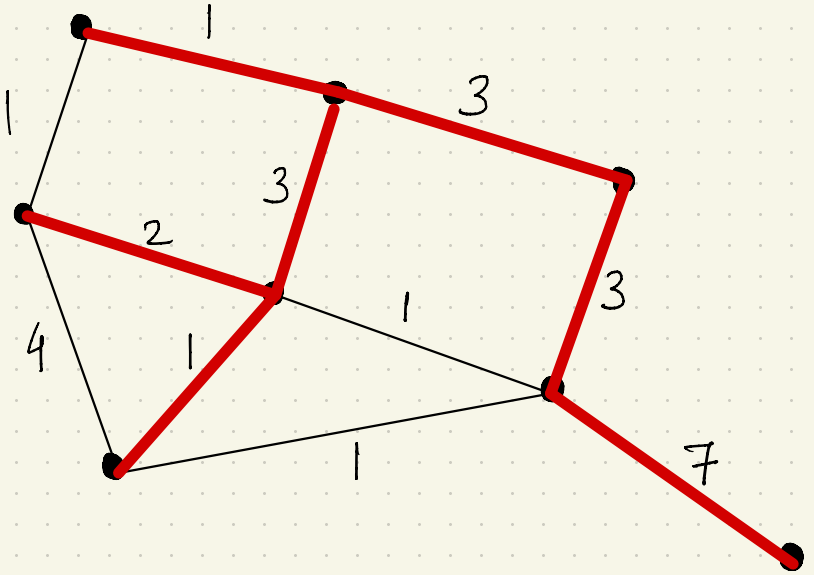
Weight = $1+3+3+3+2+\cancel{4}\ 1+7 = \cancel{23}\ 20$

Minimum weight spanning tree



Weight = 1 + 3 + 3 + 3 + 2 + 1 + 7 = 20

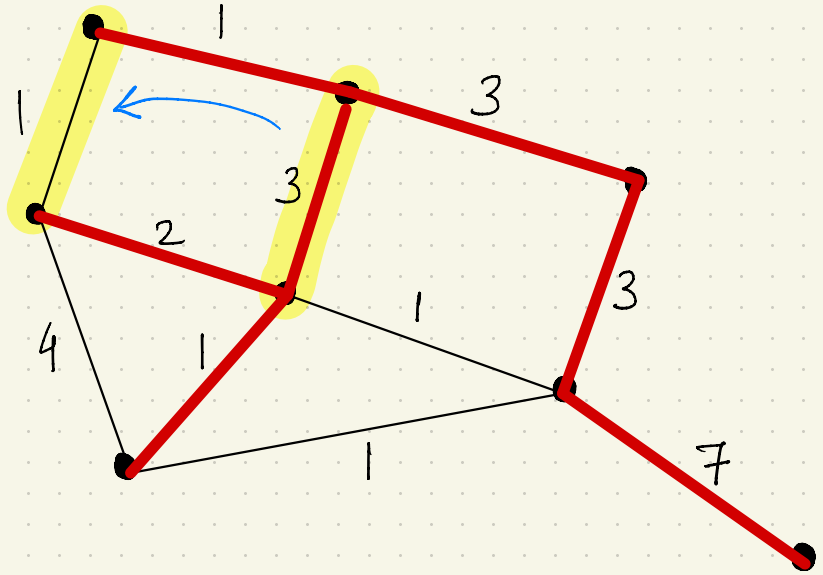
Minimum weight spanning tree



Weight = $1+3+3+3+2+1+7=20$

Can we do better?

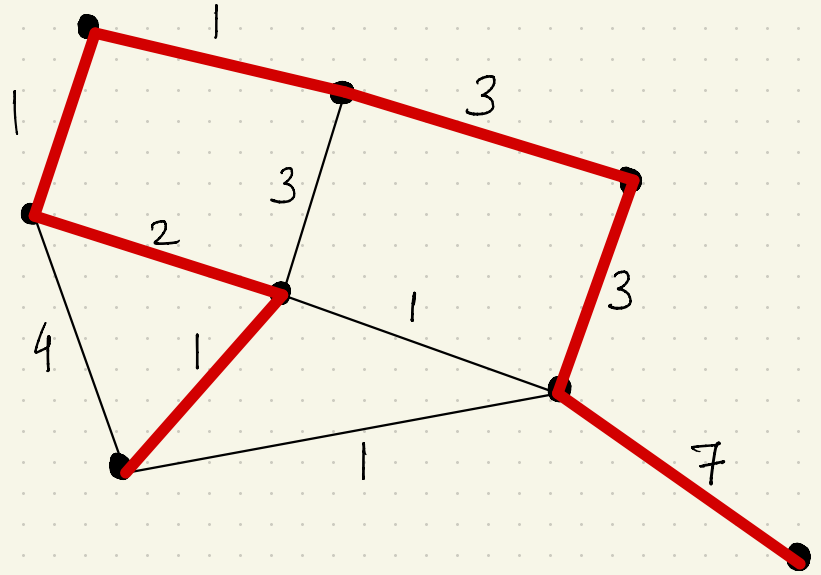
Minimum weight spanning tree



Weight = $1+3+3+3+2+1+7 = 20$

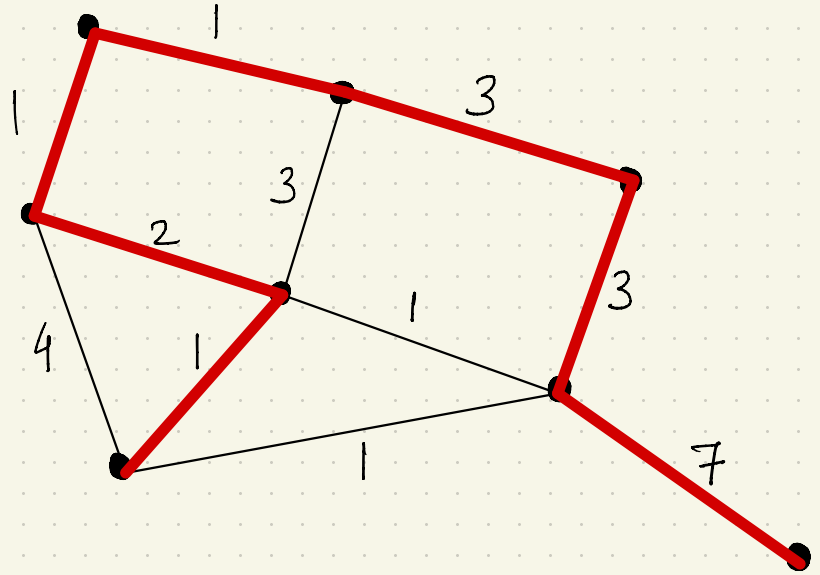
Can we do better?

Minimum weight spanning tree



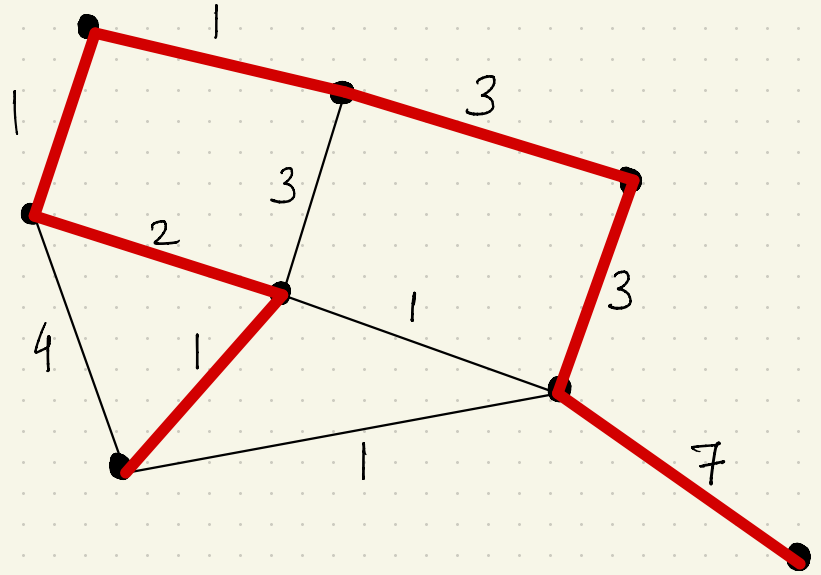
Weight = $1 + \cancel{2}\ 1 + 3 + 3 + 2 + 1 + 7 = \cancel{20}\ 18$

Minimum weight spanning tree



Weight = 1 + 1 + 3 + 3 + 2 + 1 + 7 = 18

Minimum weight spanning tree



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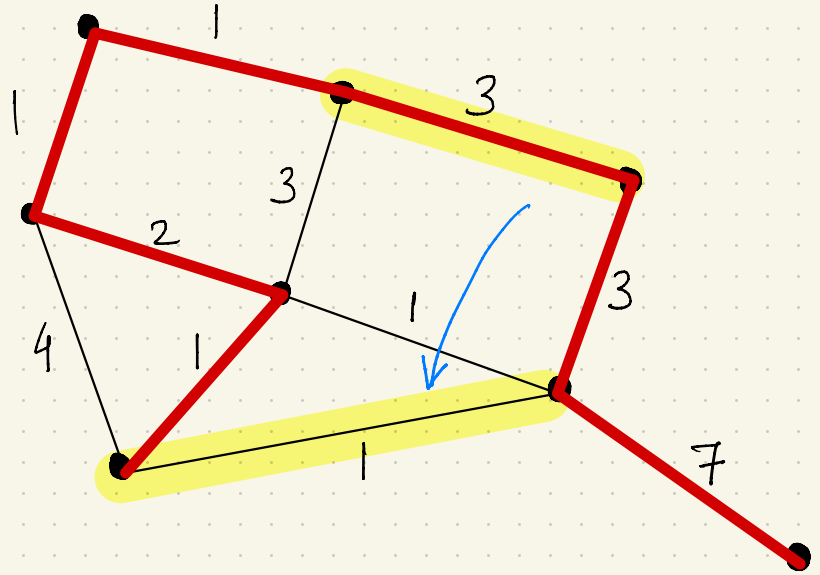
Can we do better?

Minimum weight spanning tree



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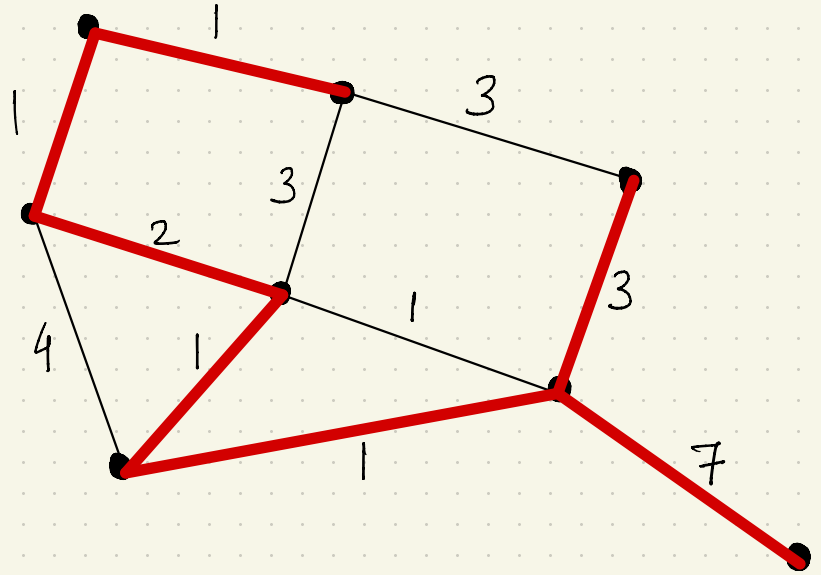
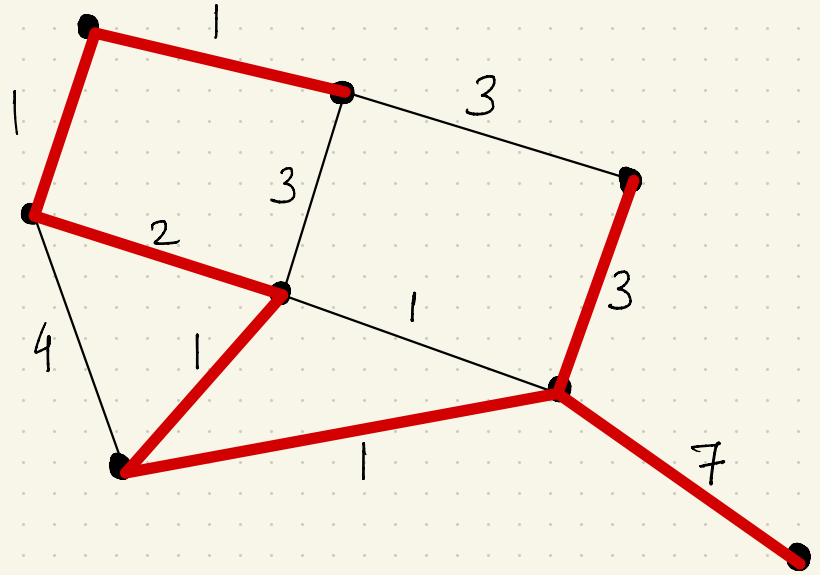
Can we do better?

Minimum weight spanning tree



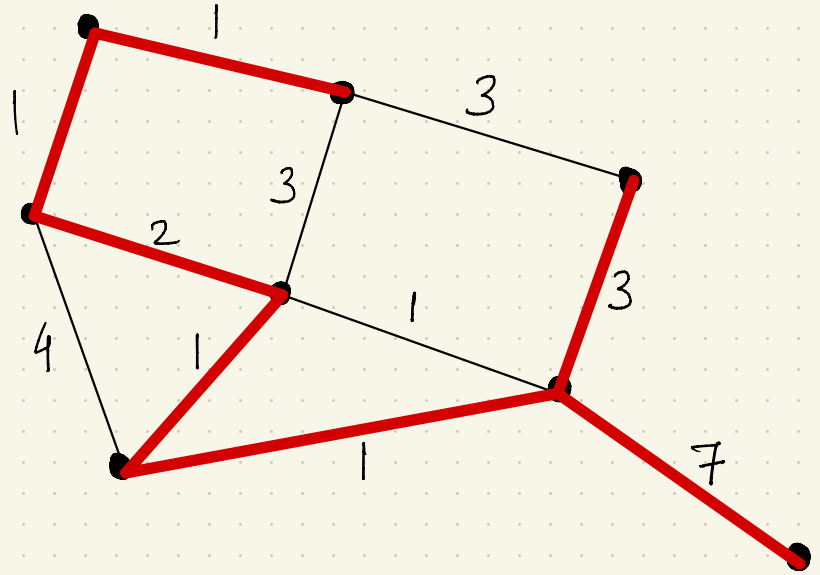
Weight = 1 + 1 + ~~2~~ 1 + 3 + 2 + 1 + 7 = ~~18~~ 16

Minimum weight spanning tree



Weight = 1 + 1 + 1 + 3 + 2 + 1 + 7 = 16

Minimum weight spanning tree



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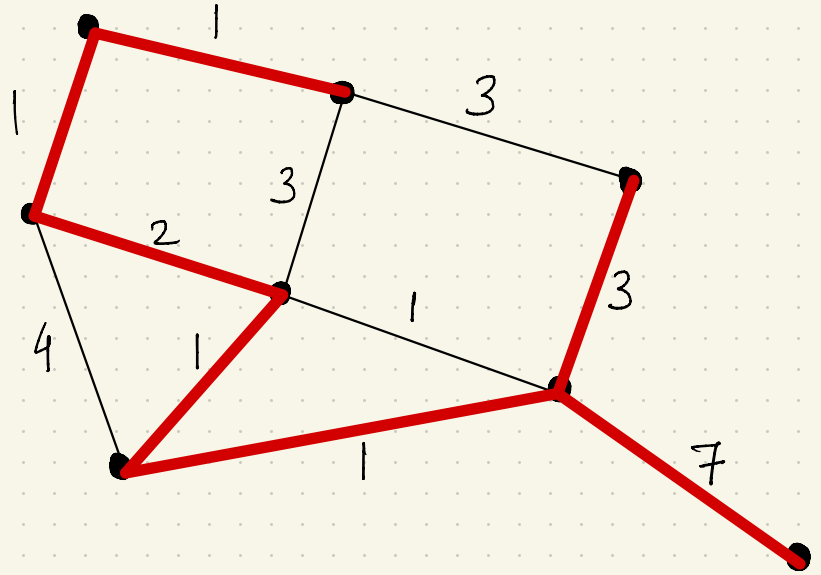
Can we do better?

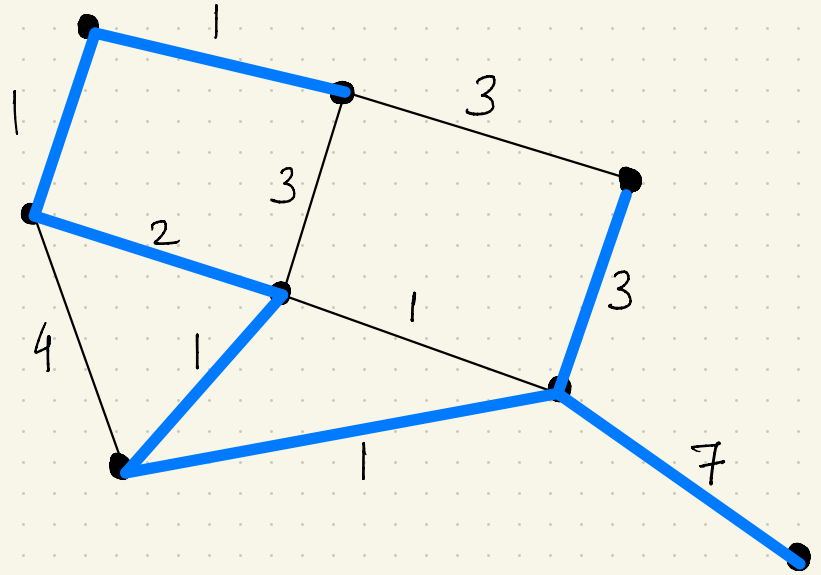
Minimum weight spanning tree



Weight = 1 + 1 + 1 + 3 + 2 + 1 + 7 = 16

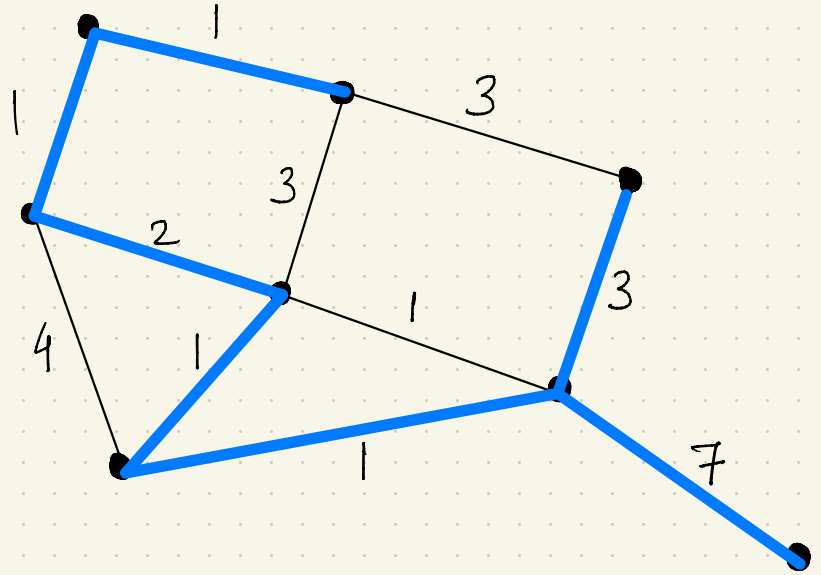
Can we do better? No!

Minimum weight spanning tree



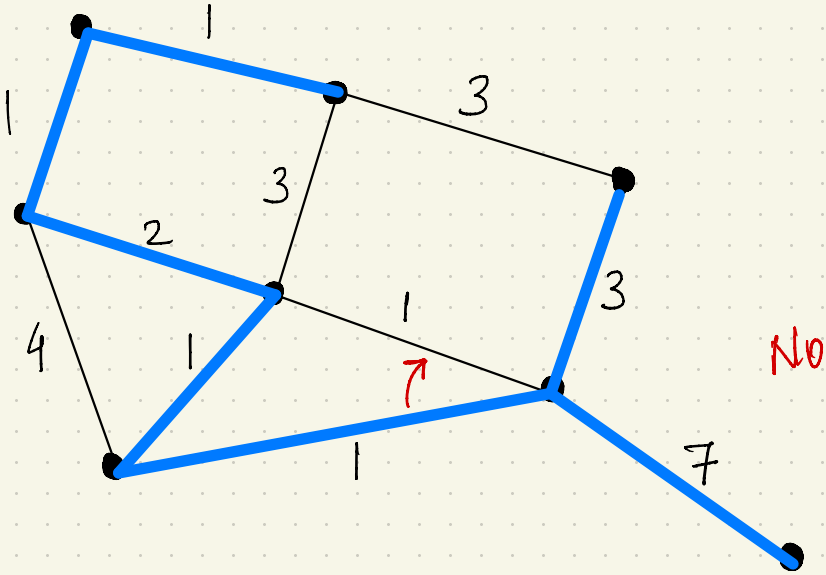
Spanning tree of minimum weight

Minimum weight spanning tree



Weight = 1 + 1 + 1 + 1 + 2 + 3 + 7 = 16

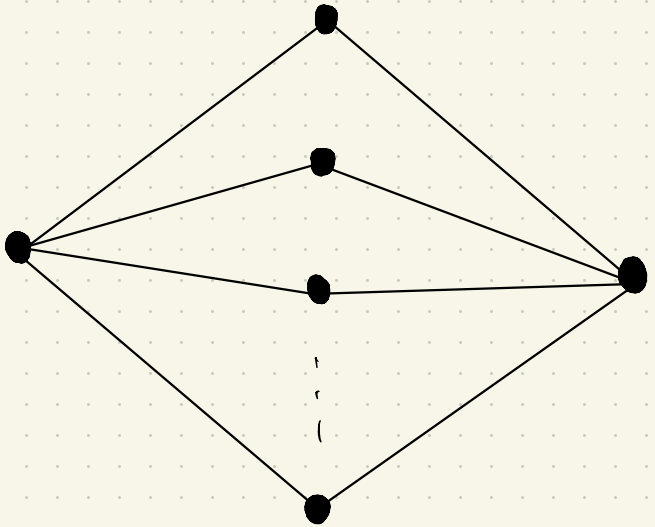
Minimum weight spanning tree



NOT UNIQUE!

Weight = 1 + 1 + 1 + 1 + 2 + 3 + 7 = 16

EXPONENTIALLY MANY SPANNING TREES



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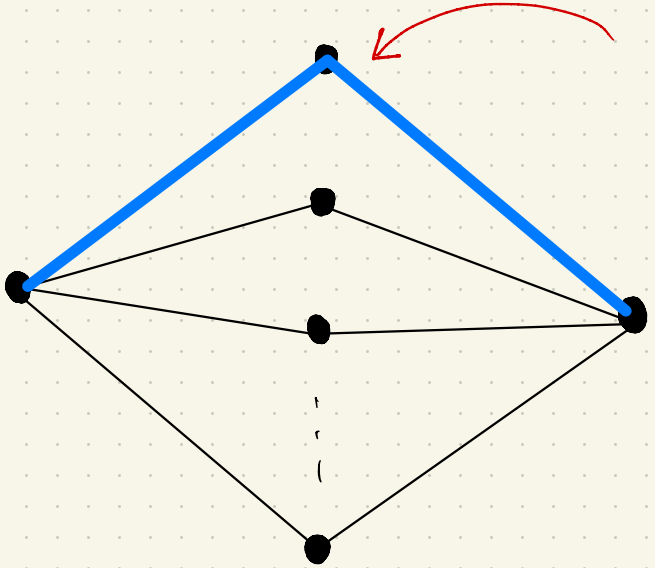
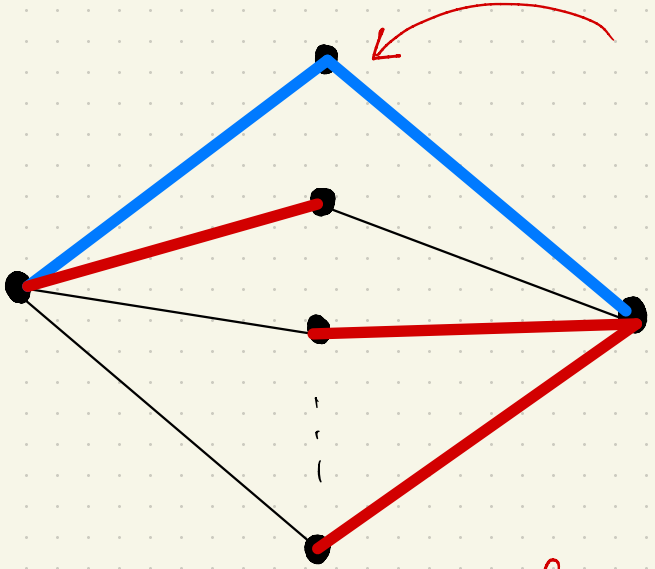


Some middle node must have both edges in spanning tree

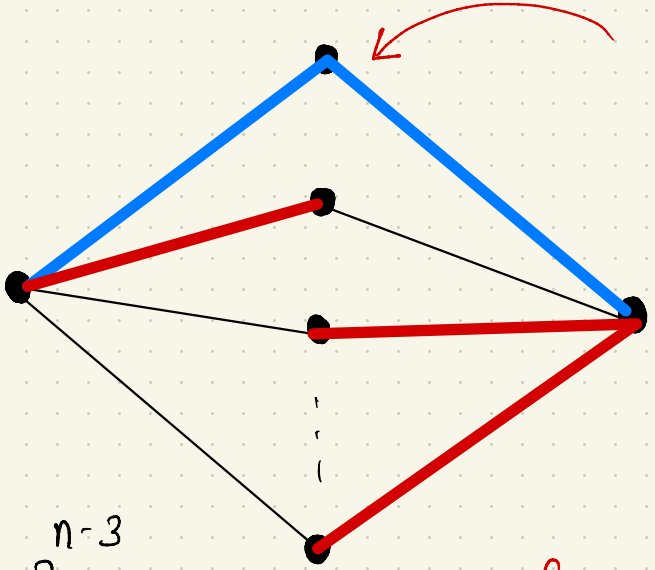
EXPONENTIALLY MANY SPANNING TREES



Some middle node must have both edges in spanning tree

for every other node, pick either left or right edge

EXPONENTIALLY MANY SPANNING TREES



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$\#$ spanning trees $\geq 2^{n-3}$

for every other node, pick either left or right edge

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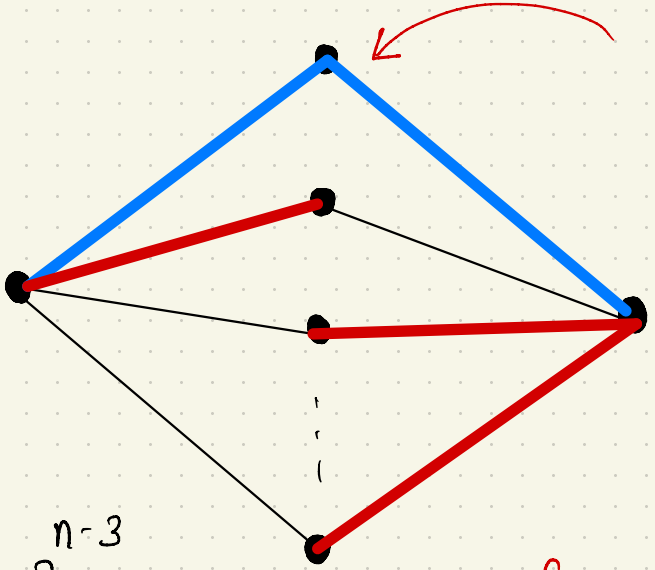

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Cannot hope to find an MST via brute force enumeration