COL 351 : Analysis & Design of Algorithms

Lecture 10

Graph Algorithms III :

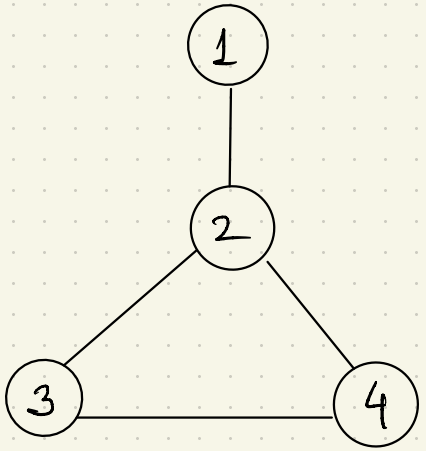
BFS, DFS, and Applications

Aug 14, 2024 | Rohit Vaish

REPRESENTING GRAPHS

Adjacency matrix

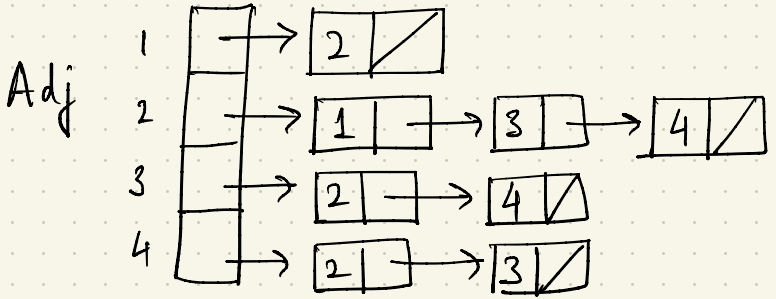
$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

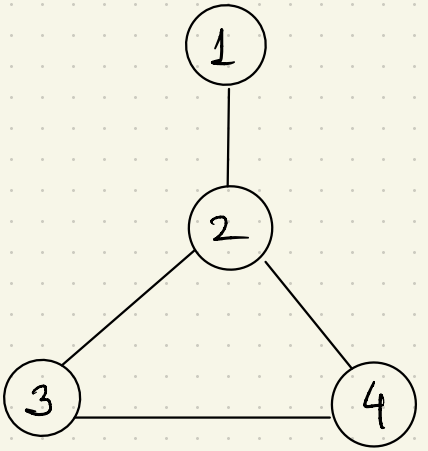


$\Theta(n^2)$ space

REPRESENTING GRAPHS

Adjacency list



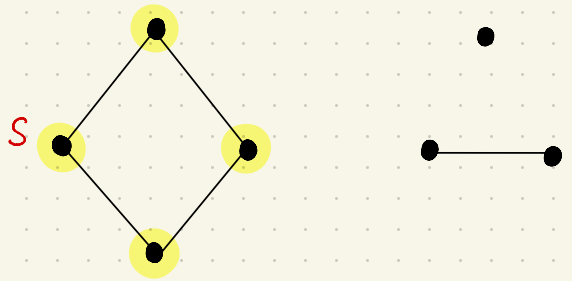


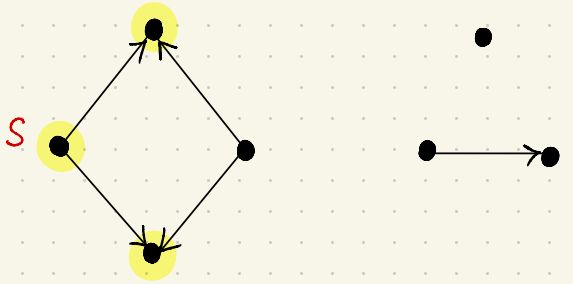
$\Theta(m+n)$ space

Generic Graph Search

input: an undirected or directed graph $G=(V,E)$, starting vertex s

goal: identify the vertices of V reachable from s





GENERIC GRAPH SEARCH

input: an undirected or directed graph $G=(V,E)$, starting vertex s

goal: identify the vertices of V reachable from s efficiently

$O(|V|+|E|)$ time

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Generic Algorithm

initially s is explored, other vertices unexplored

while some edge $(u,v) \in E$ with u explored
and v unexplored

  mark v explored

Generic Graph Search

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    mark v explored

Claim: v is marked explored $\iff$ G has a path from s to v

Generic Algorithm

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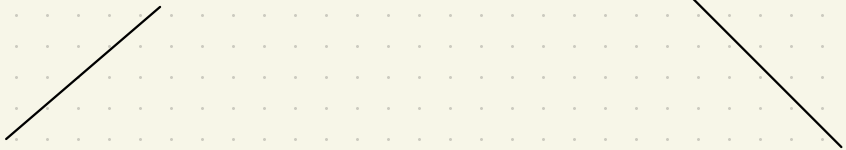


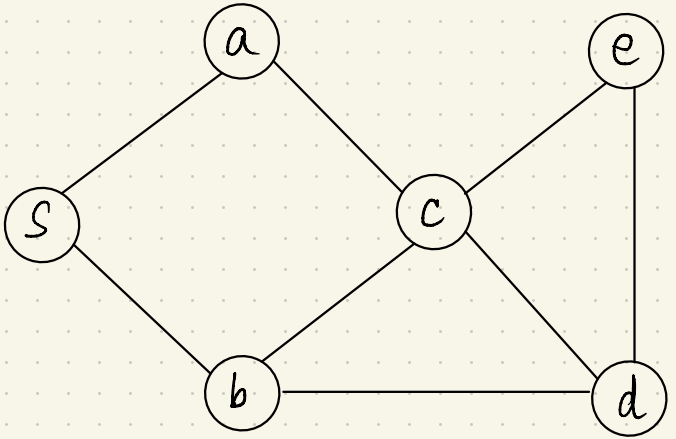
Breadth-first search (BFS)

explore layerwise

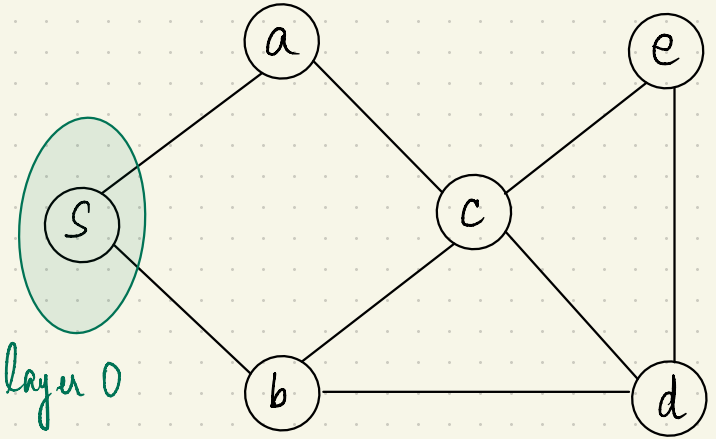
Depth-first search (DFS)

explore like a maze

BREADTH - FIRST SEARCH



BREADTH - FIRST SEARCH



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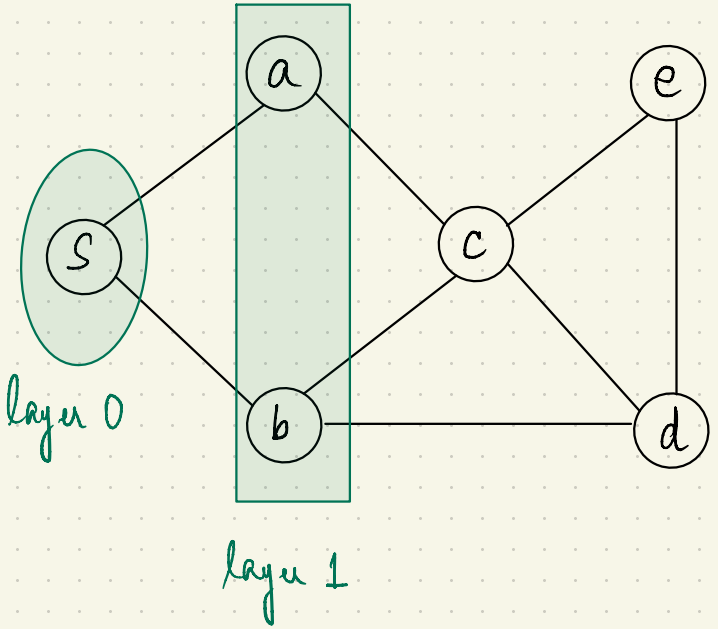

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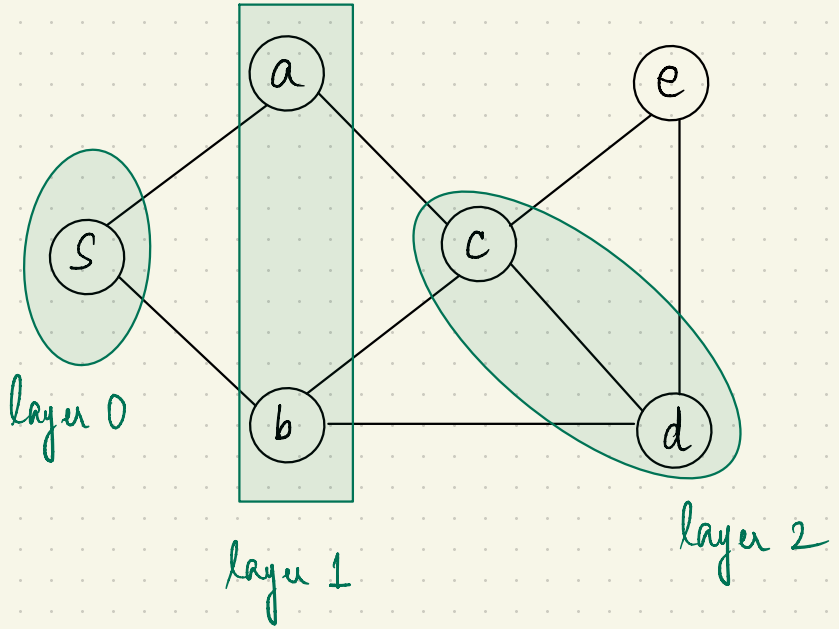
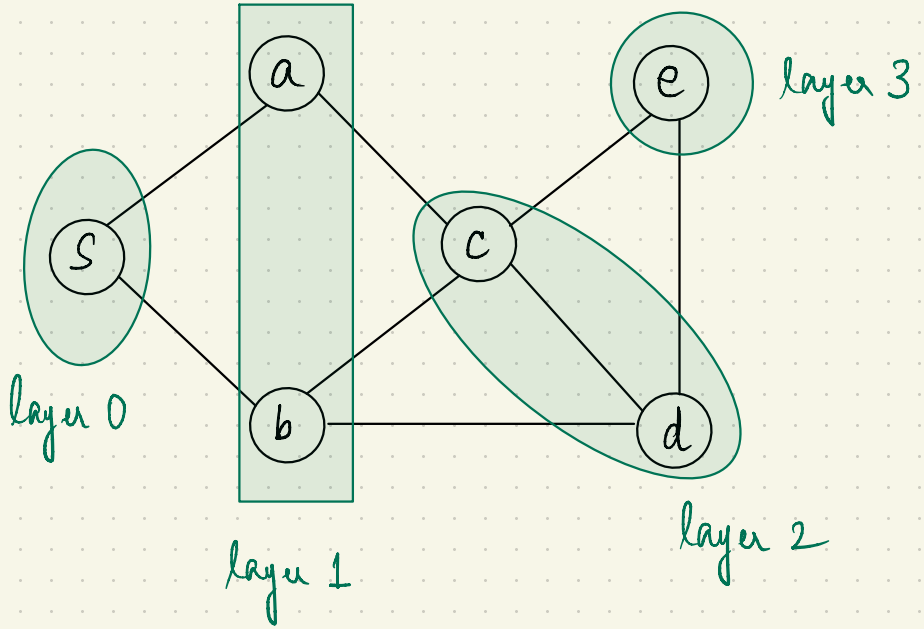


BREADTH - FIRST SEARCH



BREADTH-FIRST SEARCH

mark s as explored, all other vertices unexplored

Q := a queue data structure (FIFO), initialized with s

while $Q \neq \emptyset$

    remove the first node of Q, say v

    for each $(v, w) \in E$

        if w is unexplored

            mark w as explored

            add w to the end of Q

BREADTH-FIRST SEARCH

Claim 1: At the end of BFS

v explored $\iff$ there is a path from s to v

Claim 2: Running time of while-loop of BFS is $O(n_s + m_s)$

where n_s = # vertices reachable from s

m_s = # edges " " "

BREADTH-FIRST SEARCH

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```
while Q ≠ ∅
    remove the first node of Q, say v
    for each (v, w) ∈ E
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```

Applications of BFS

Shortest paths

Connected Components

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SHORTEST PATHS

input: an undirected / directed graph $G=(V,E)$, a starting vertex s

output: $dist(s,v)$ for every vertex $v \in V$

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number of edges

$$dist(s,v) = \begin{cases} \text{length of shortest path from } s \text{ to } v \text{ if one exists,} \\ \infty \quad \text{otherwise} \end{cases}$$

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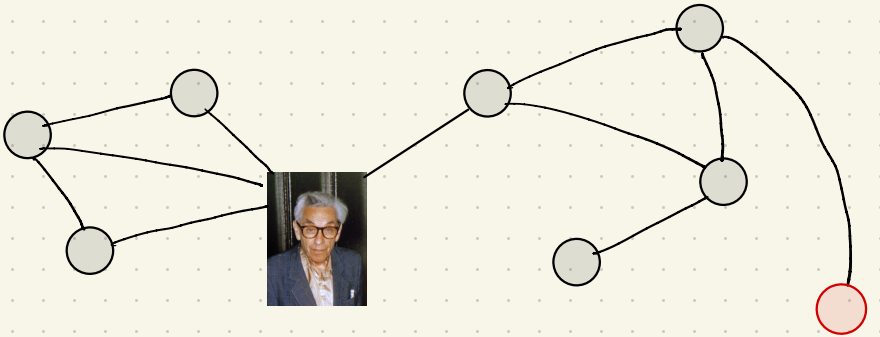

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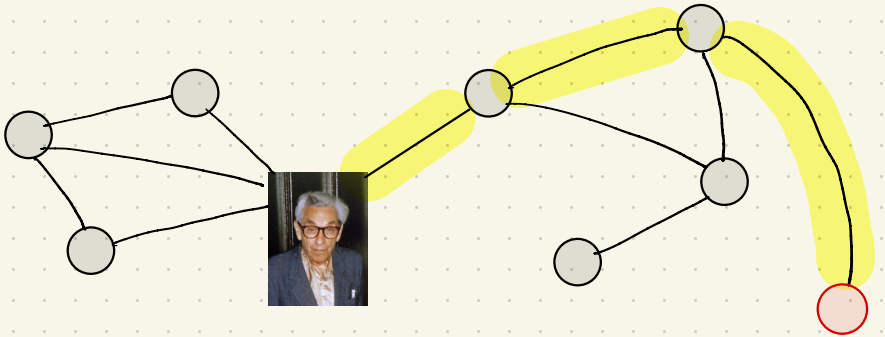

SHORTEST PATHS

BFS pseudo code

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$dist(s) := 0$, $dist(v) = \infty$ for all $v \neq s$.

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while $Q \neq \emptyset$

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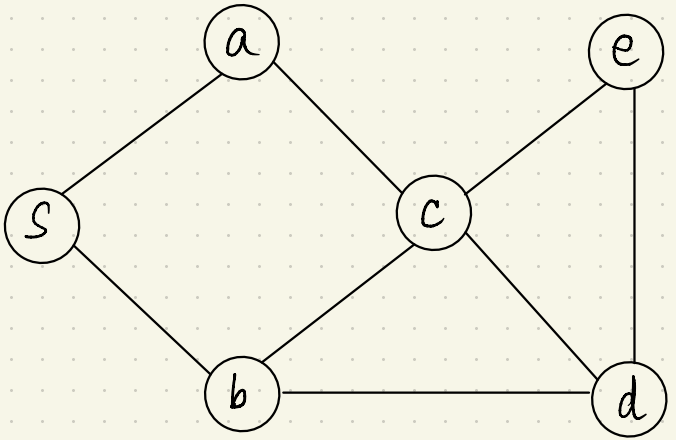
        if w is unexplored

            mark w as explored

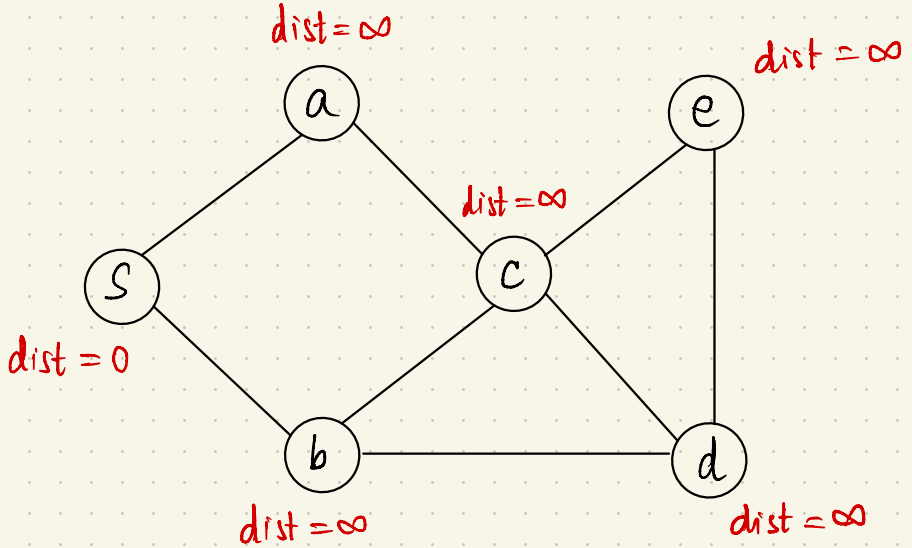
            $dist(w) := dist(v) + 1$

            add w to the end of Q

Shortest Paths



SHORTEST PATHS



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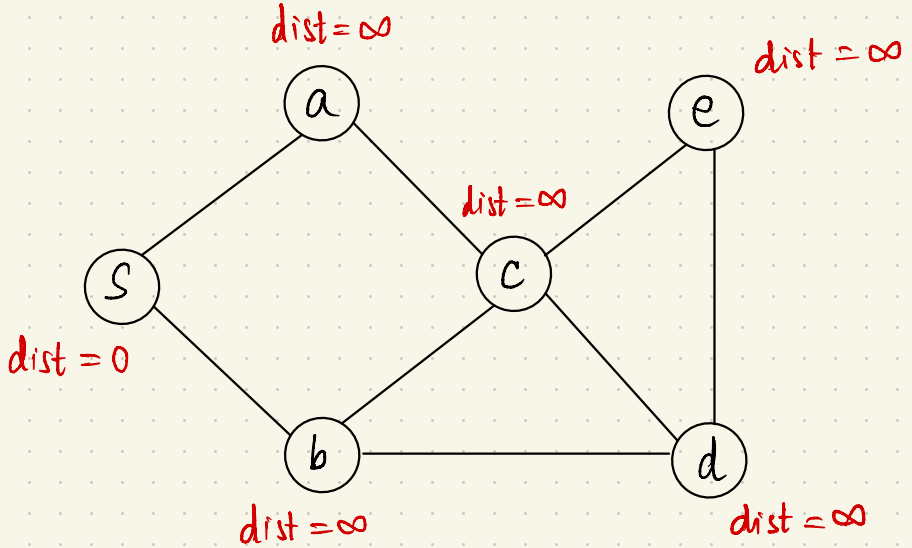
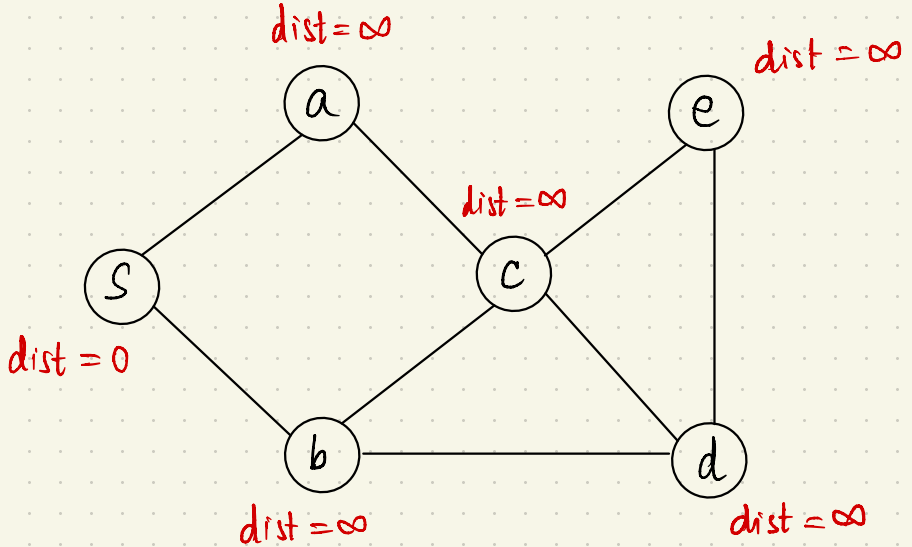


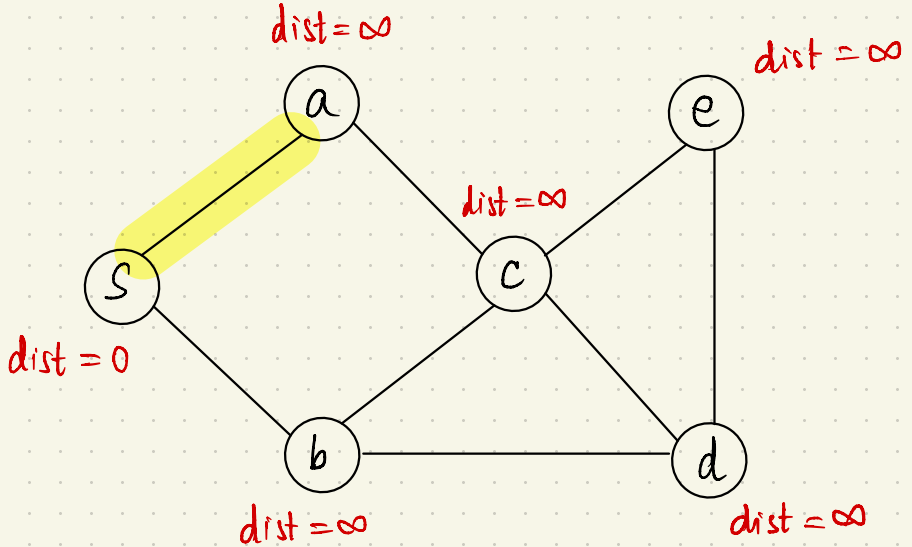
$Q = S$

SHORTEST PATHS



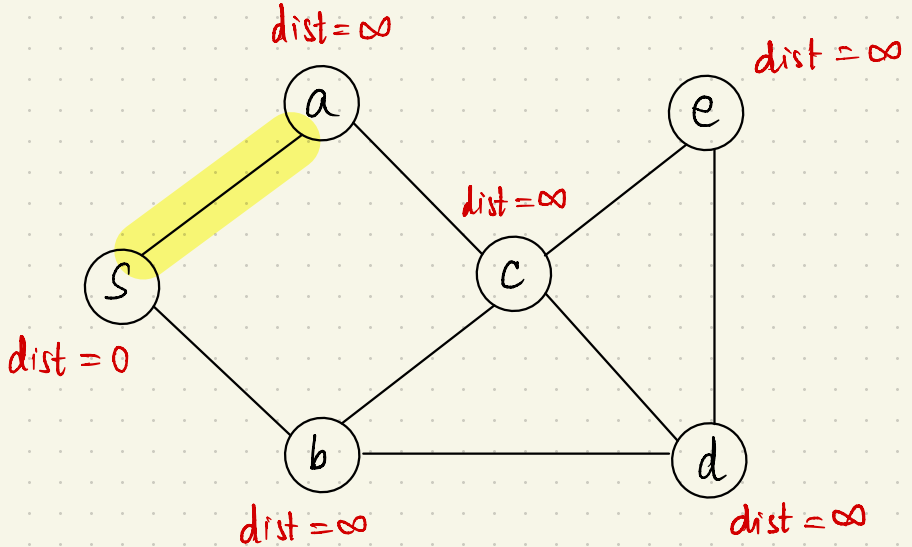
$Q = \{S\}$

SHORTEST PATHS



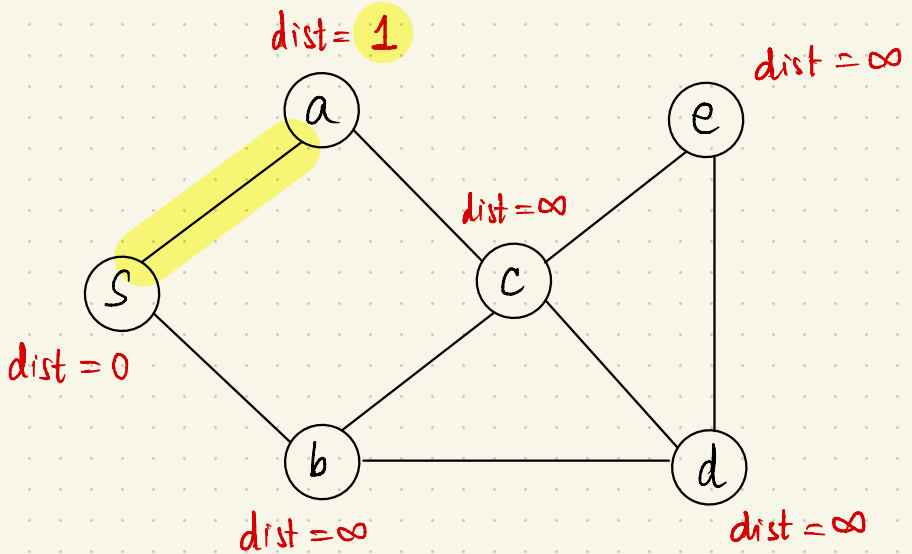
$Q = \cancel{\{\}}$

SHORTEST PATHS



$Q = \cancel{s}\ a$

SHORTEST PATHS



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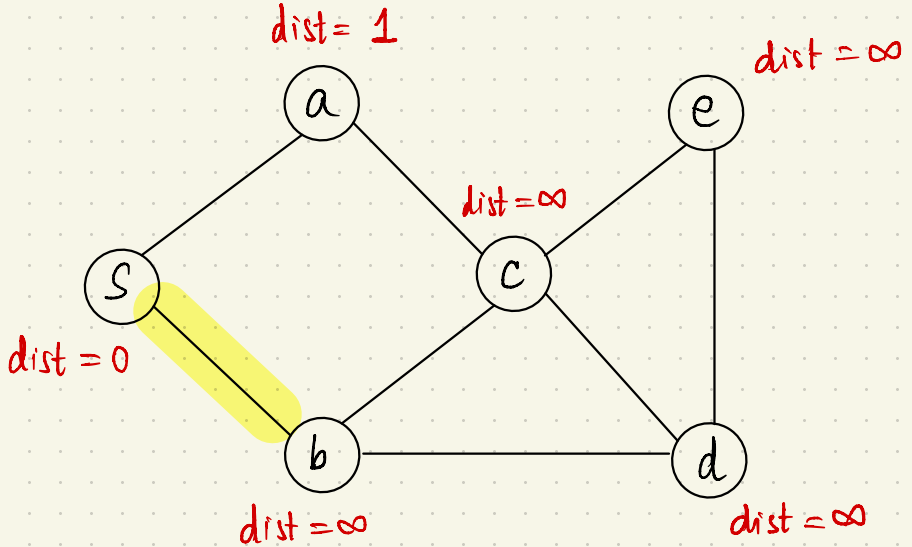

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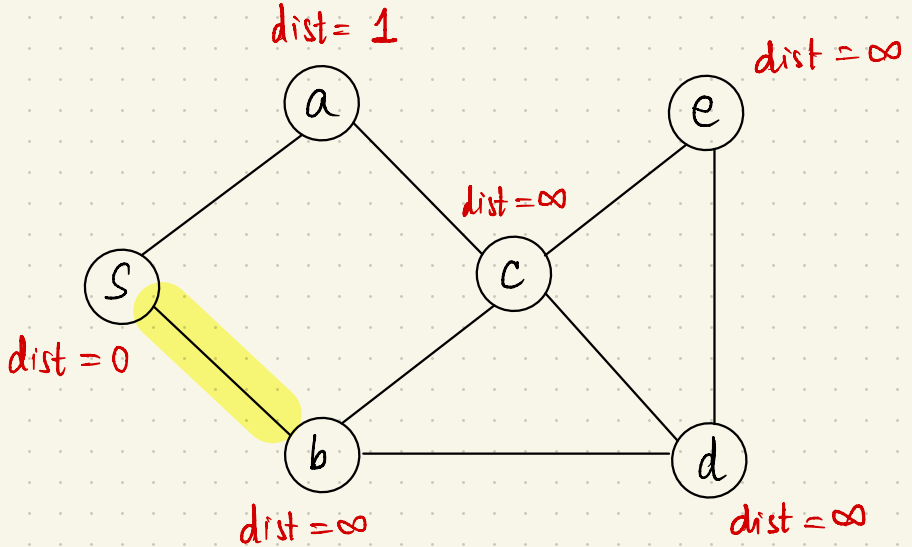

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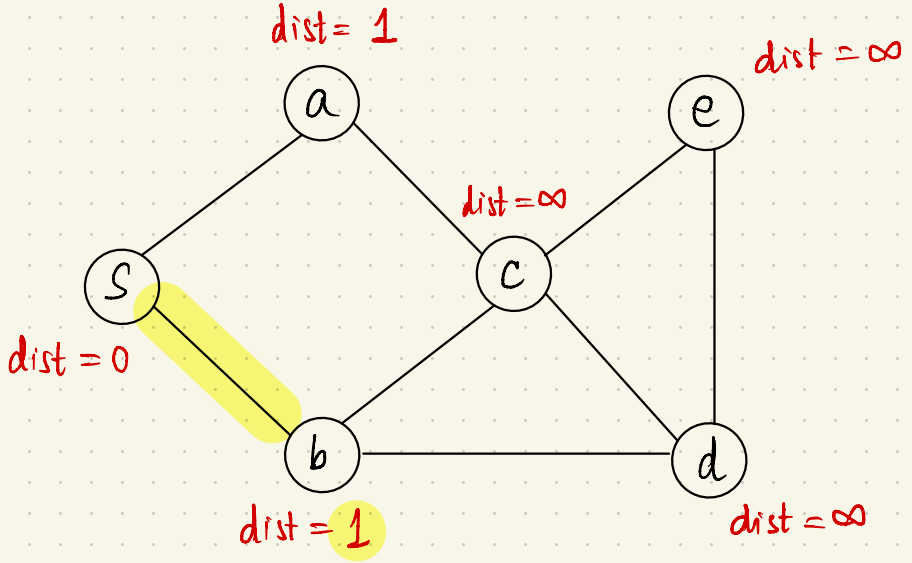

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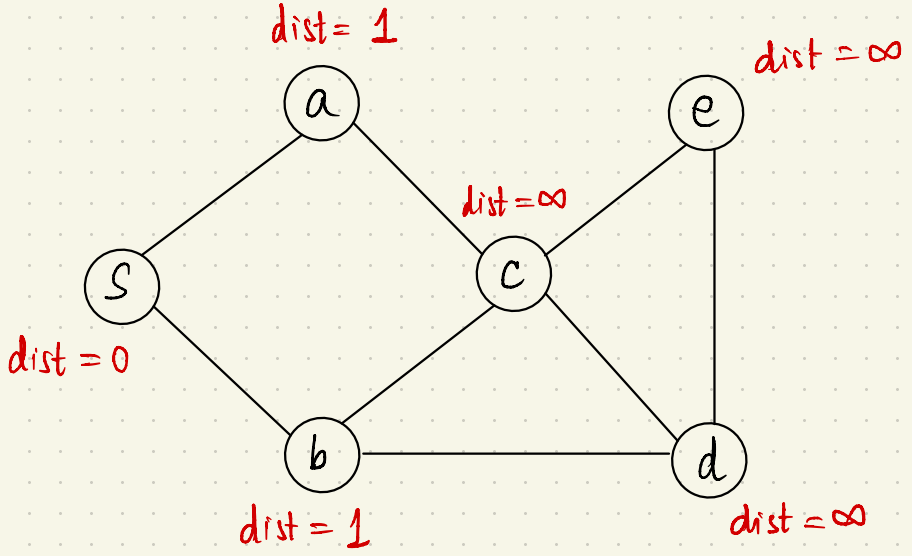
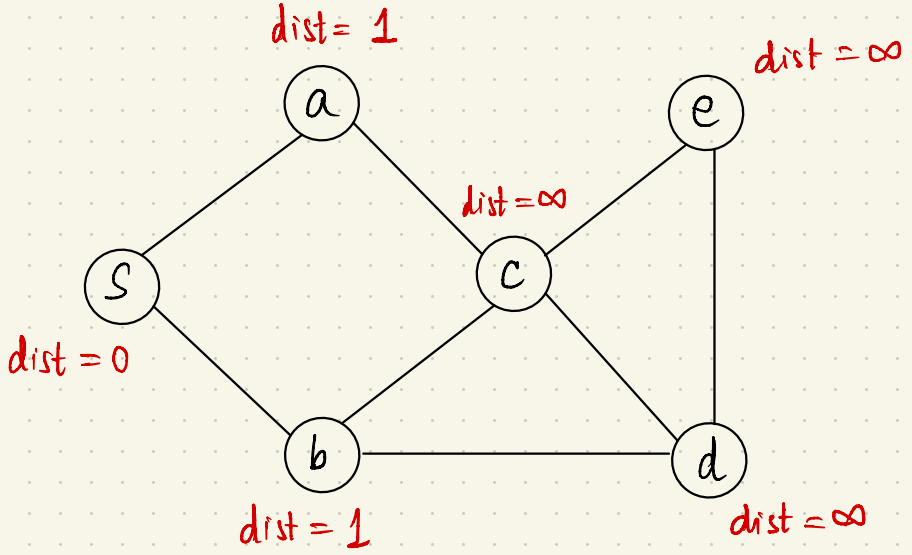


$Q = \cancel{s}\ a\ b$

fully explored

SHORTEST PATHS



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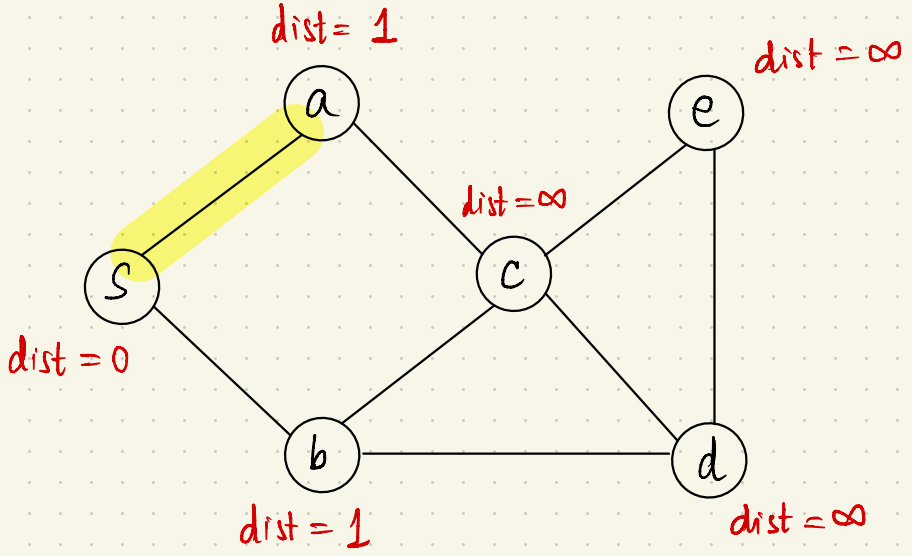

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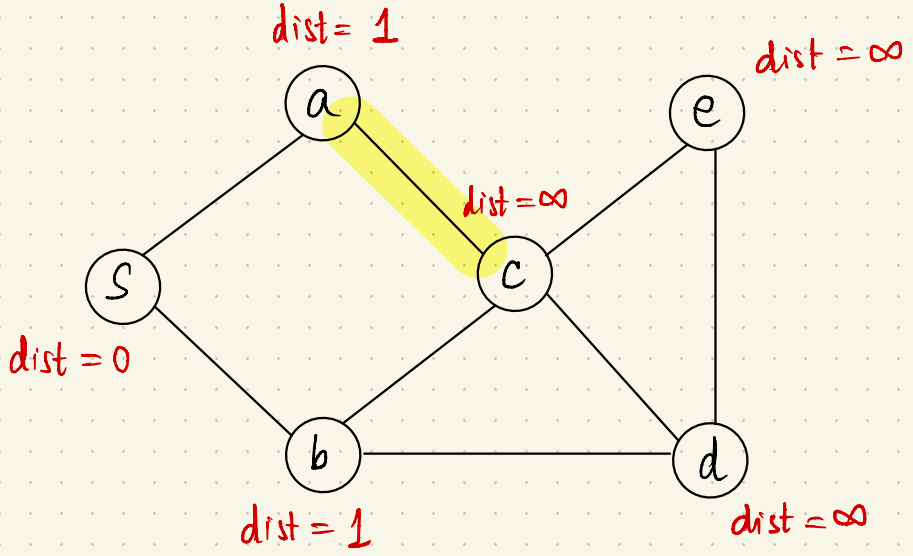

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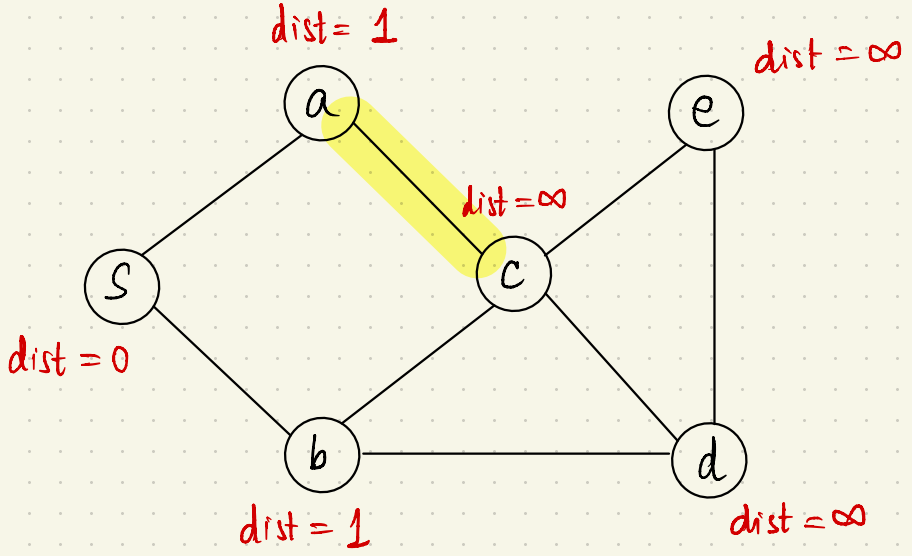

$Q = \not{s}\ \not{a}\ b\ c$

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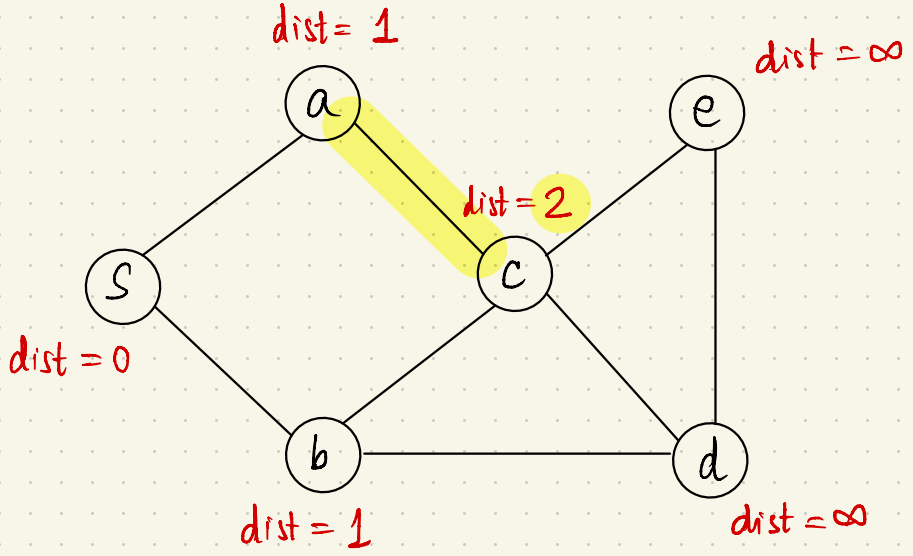

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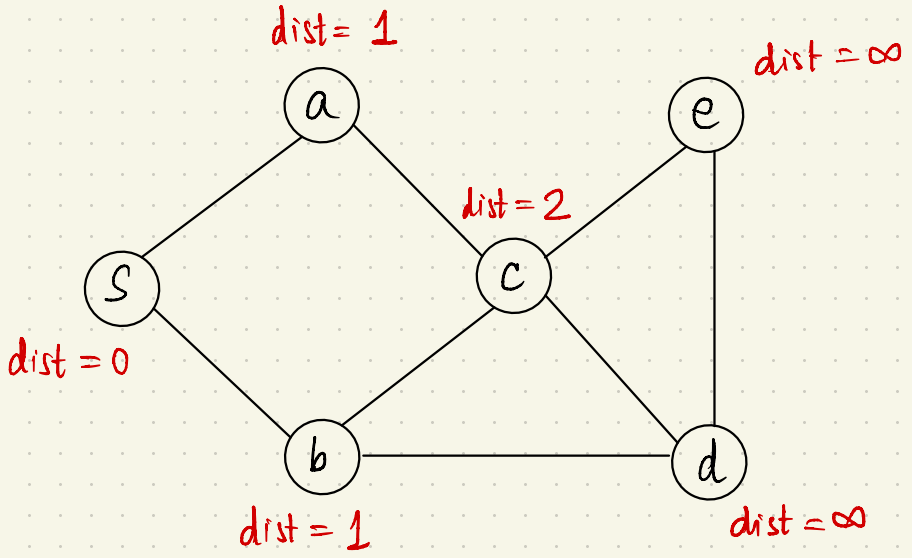
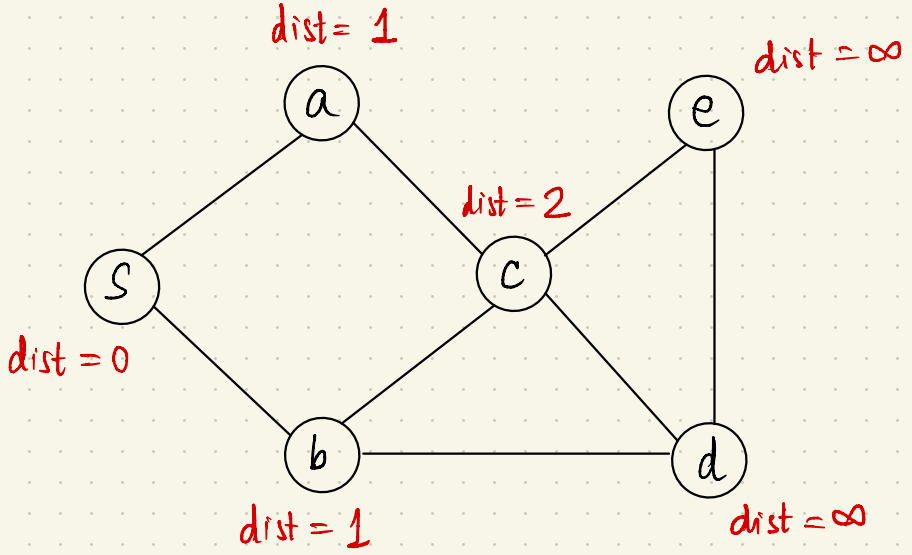


$Q = \cancel{S}\ \cancel{a}\ b\ c$

fully explored

SHORTEST PATHS



Q = ~~s~~ ~~a~~ ~~b~~ c

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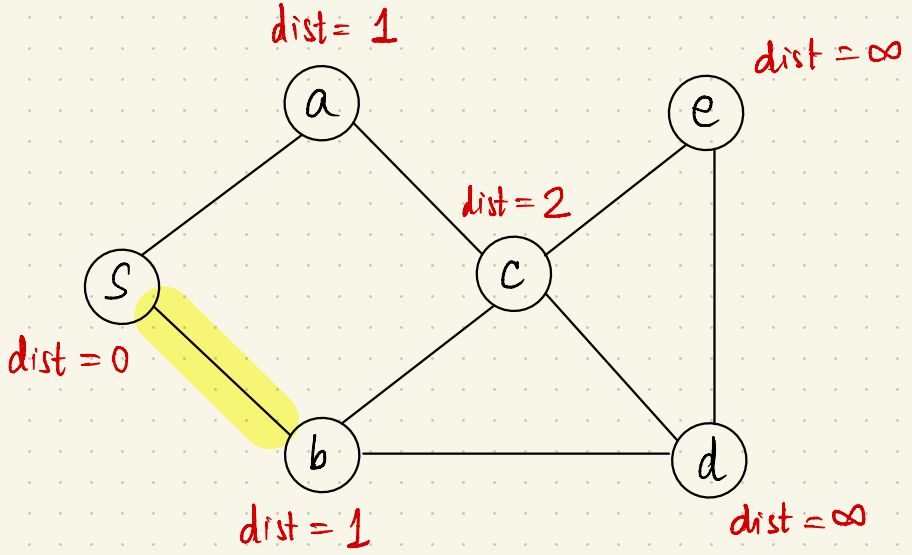

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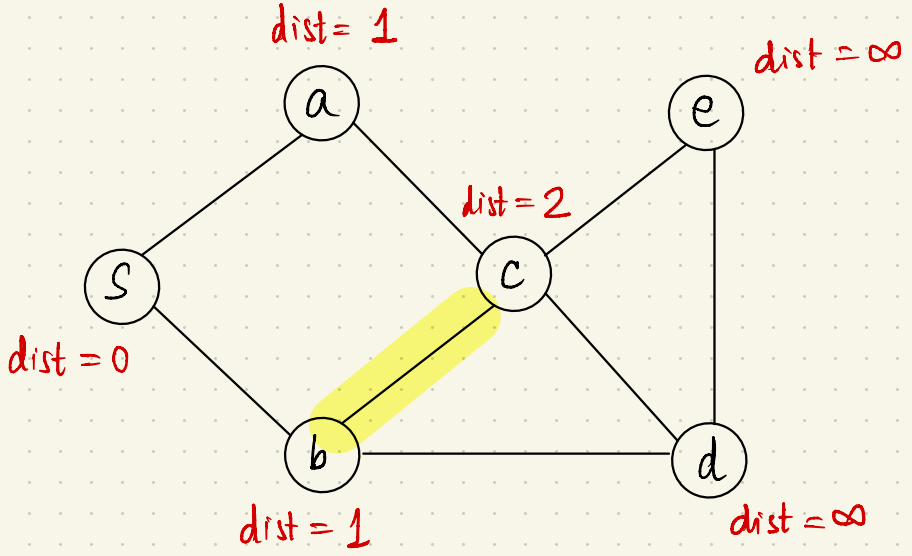

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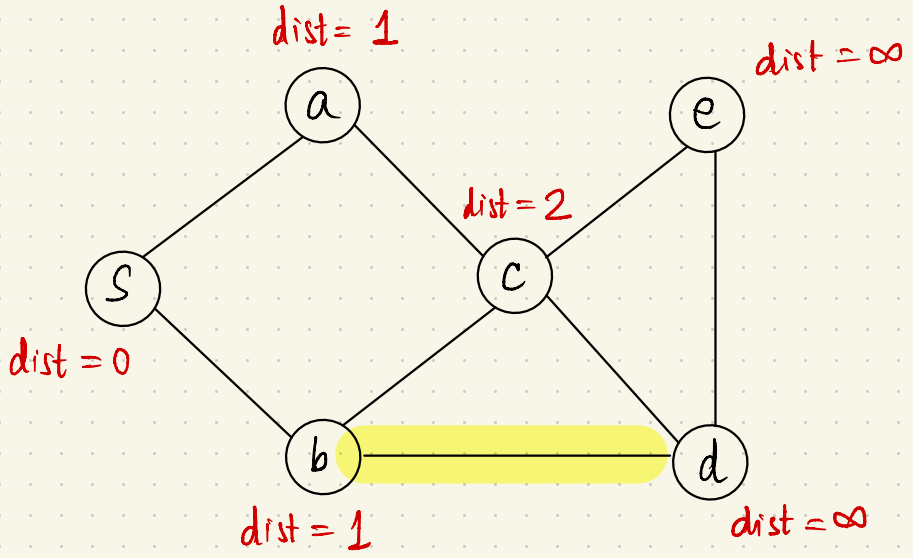

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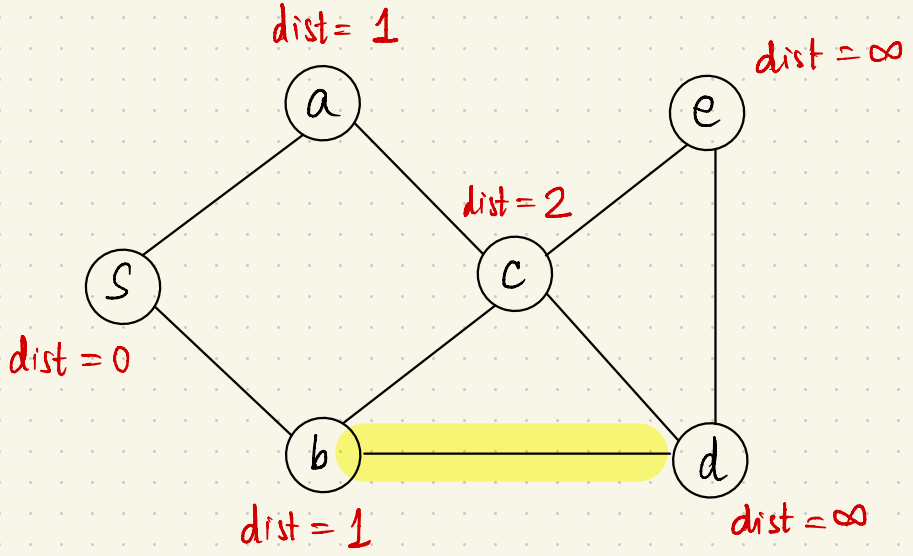
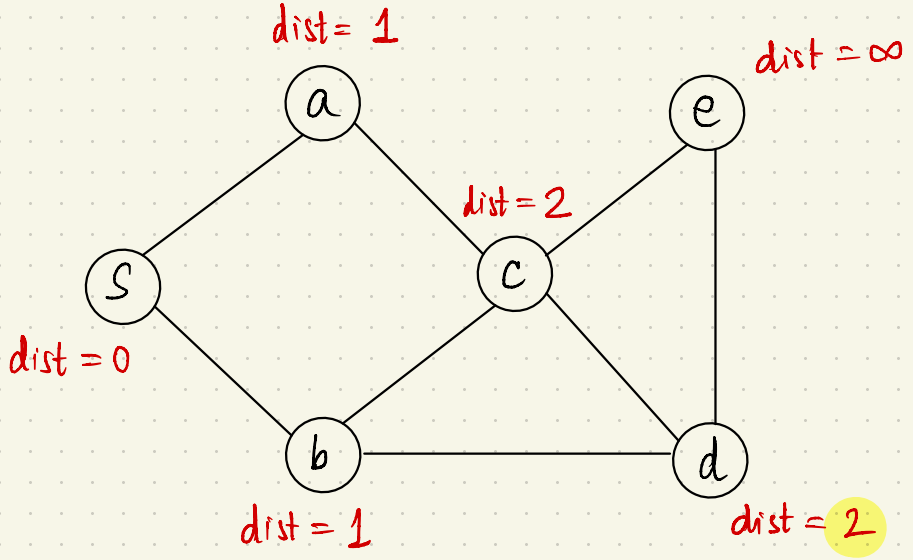


$Q = \not{S}\ \not{a}\ \not{b}\ c\ d$

SHORTEST PATHS



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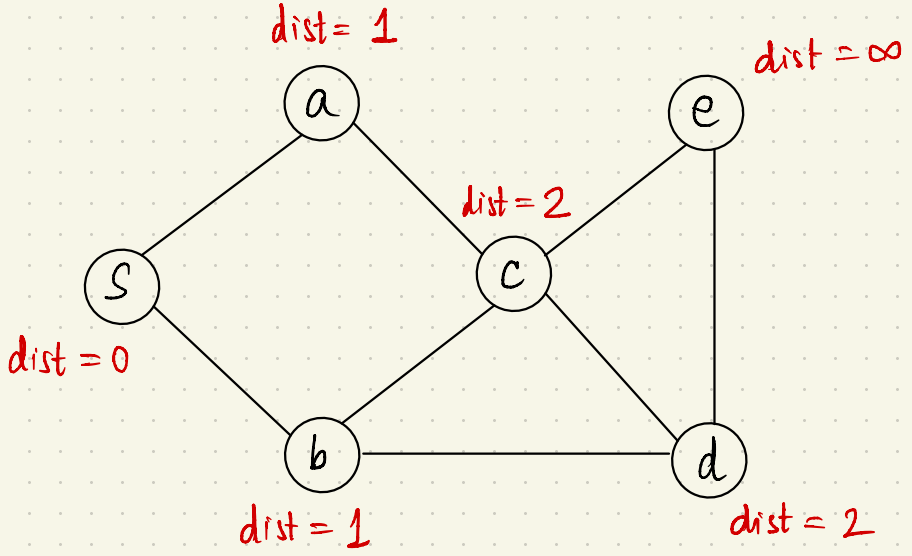

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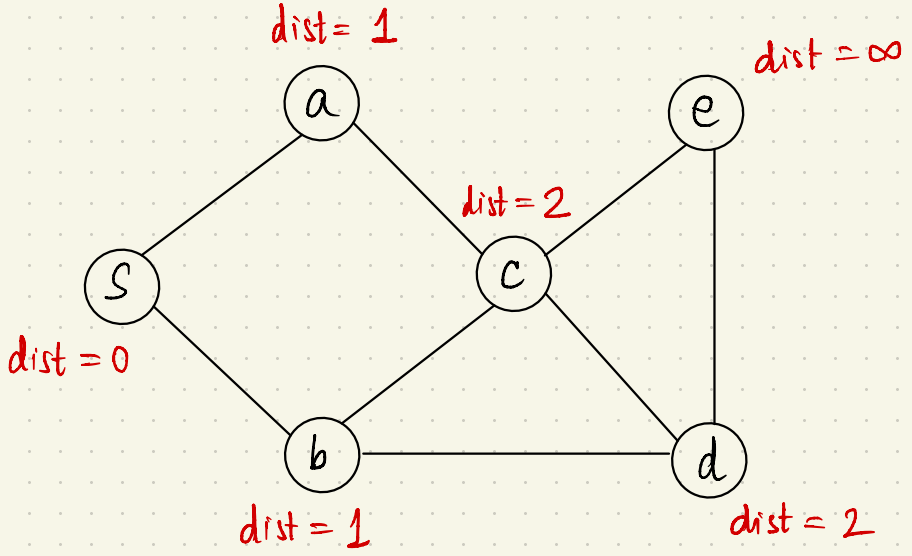
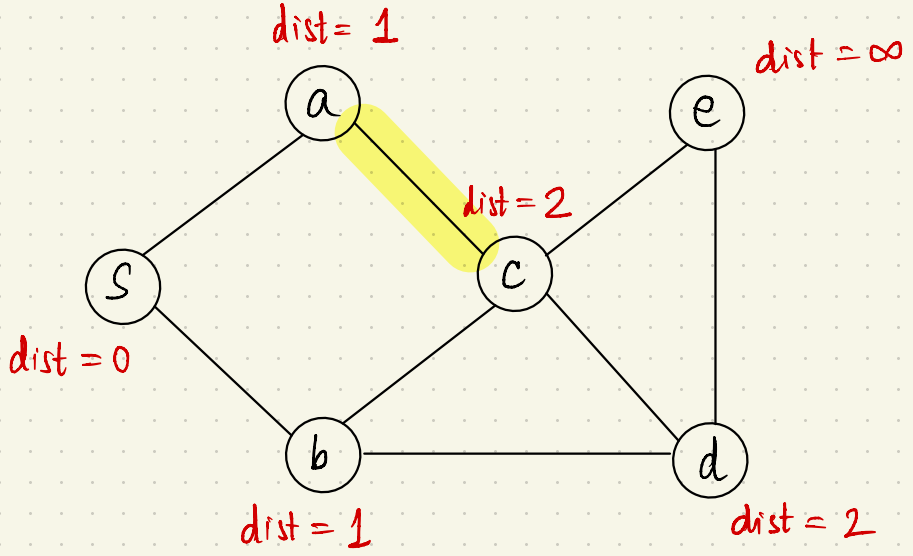
fully explored

SHORTEST PATHS



$Q = \not{s}\ \not{a}\ \not{b}\ \not{c}\ d$

SHORTEST PATHS



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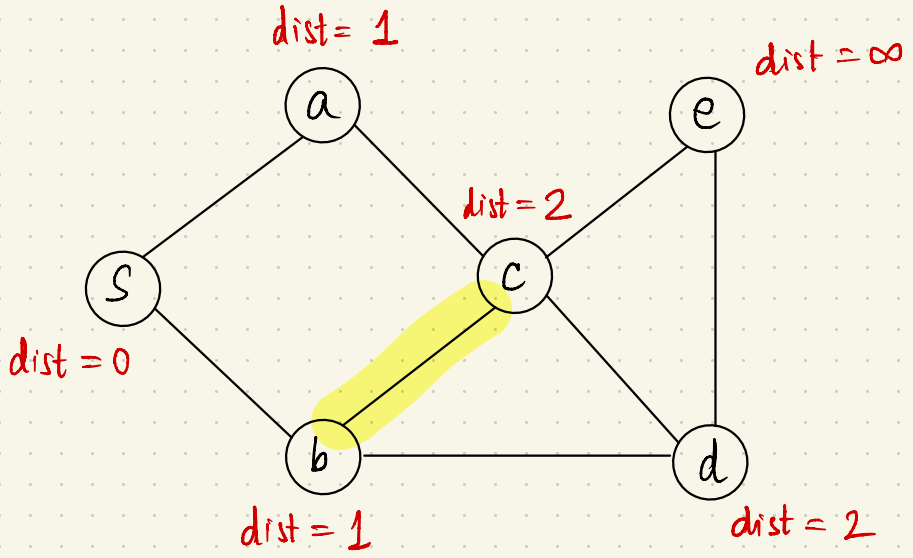

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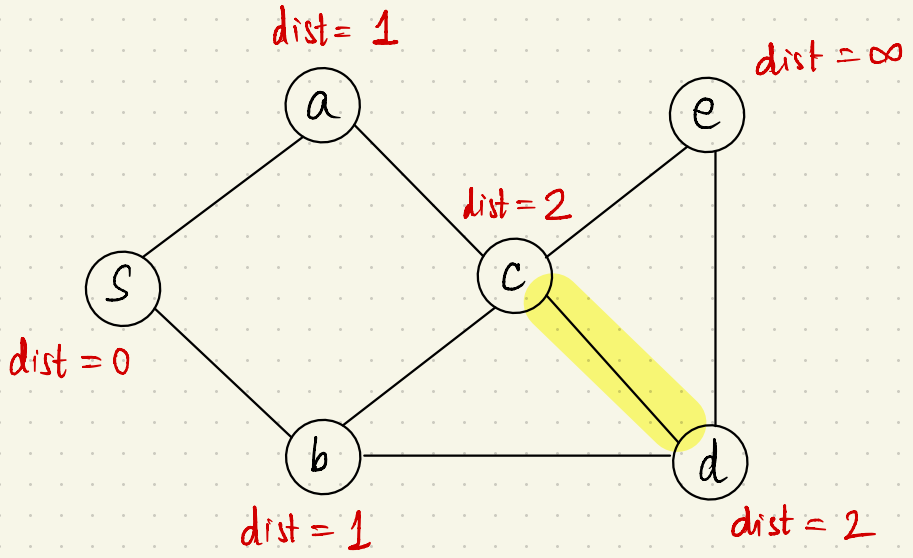

$Q = \cancel{s}\,\cancel{a}\,\cancel{b}\,\cancel{c}\,d$

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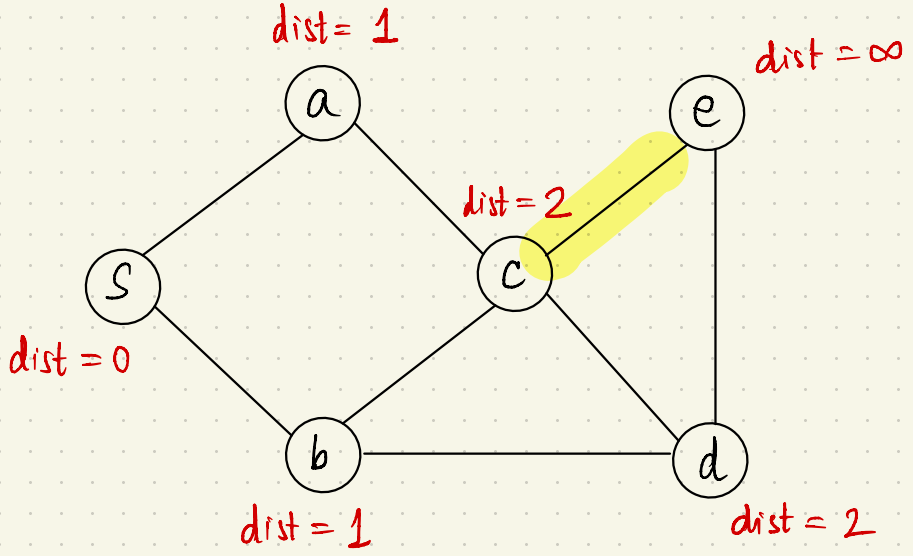

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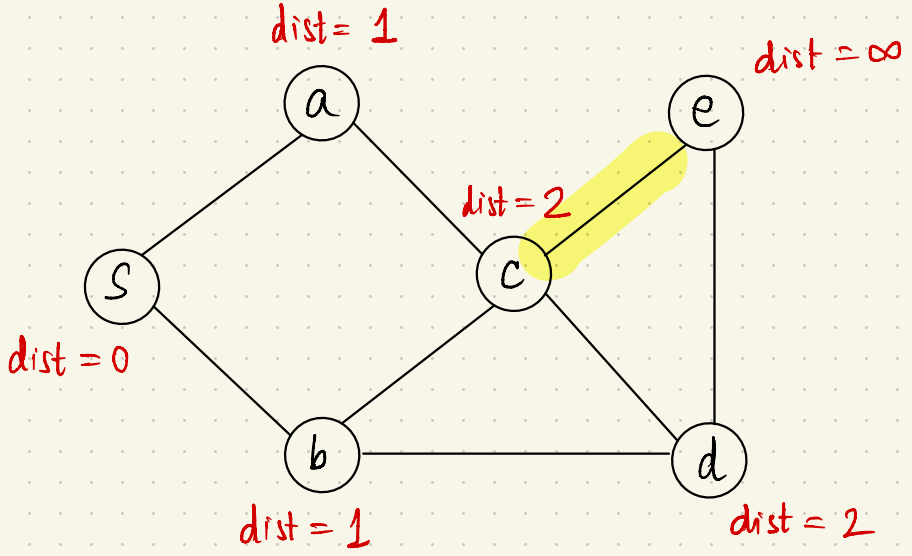

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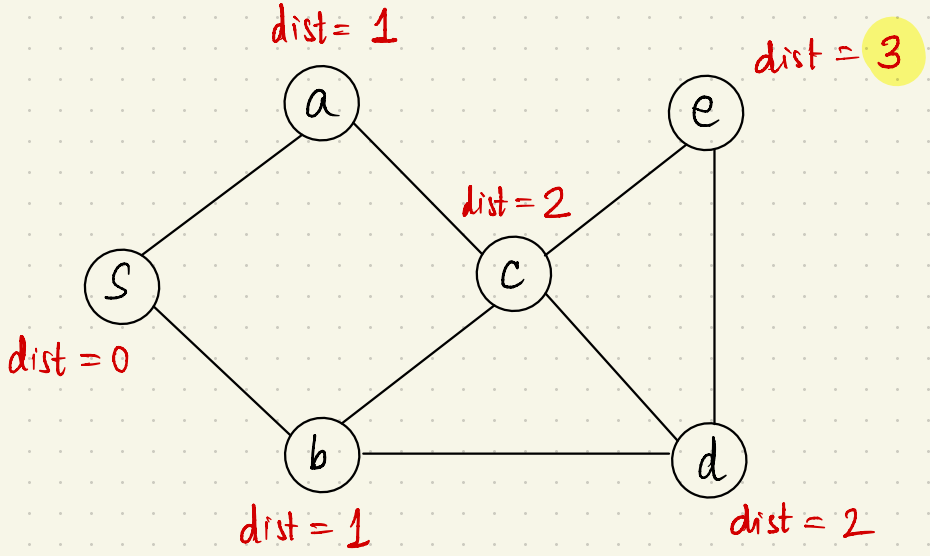

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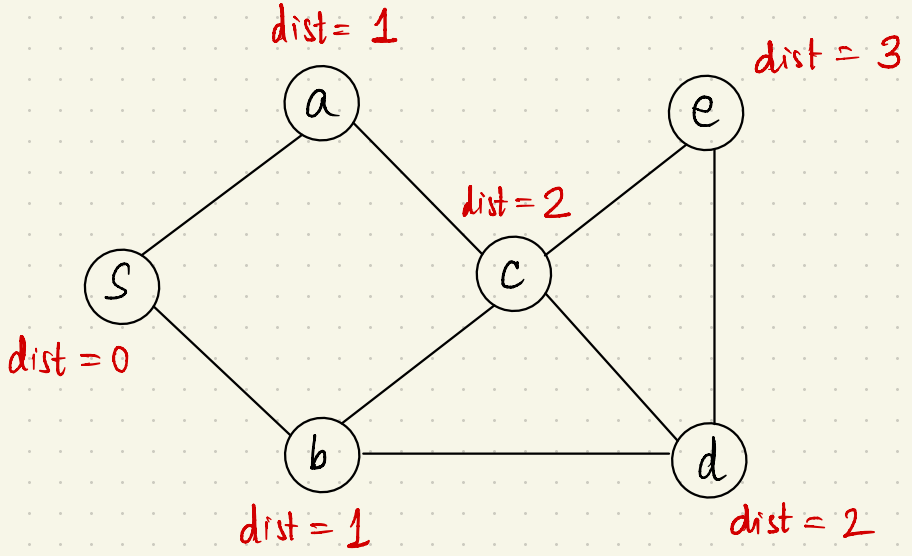

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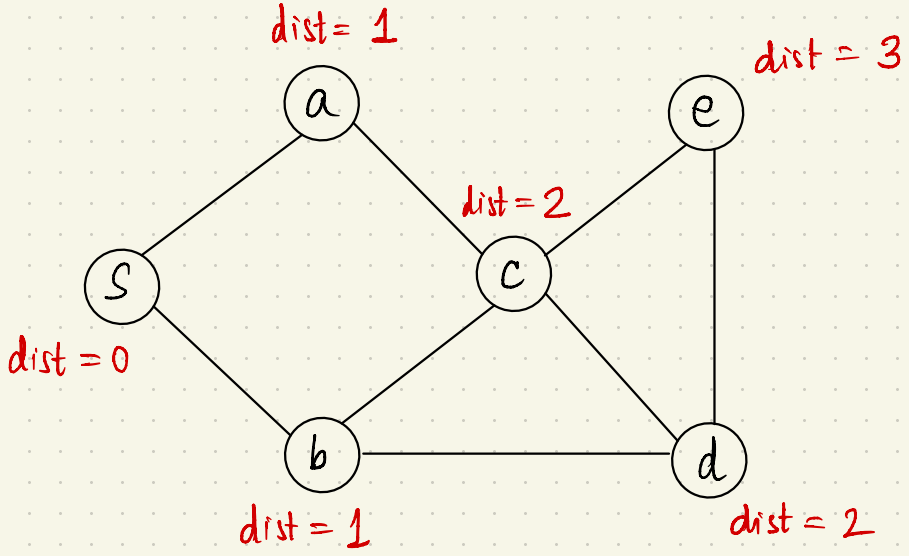
fully explored

SHORTEST PATHS



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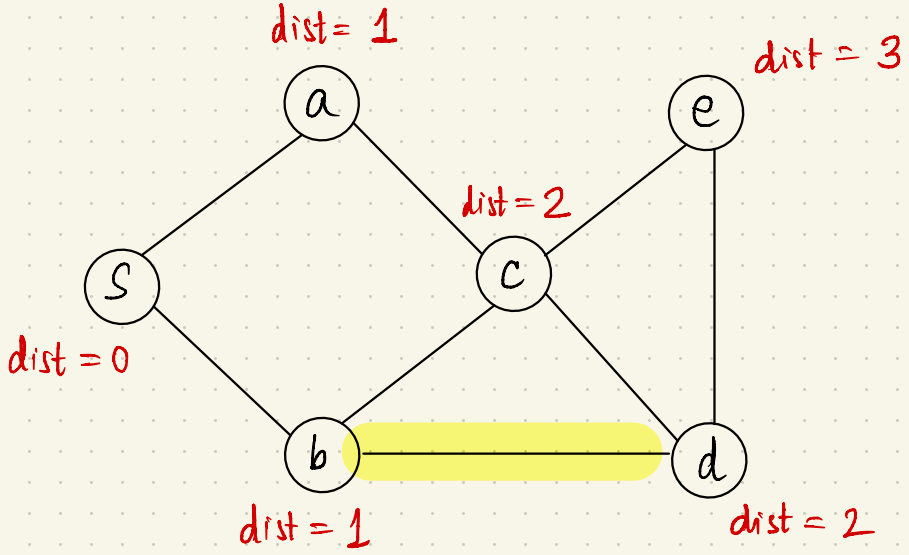

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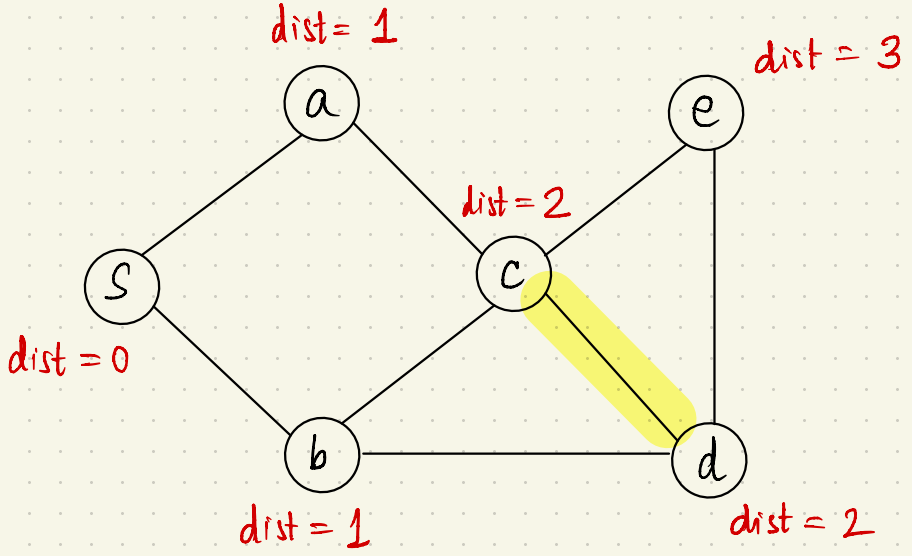

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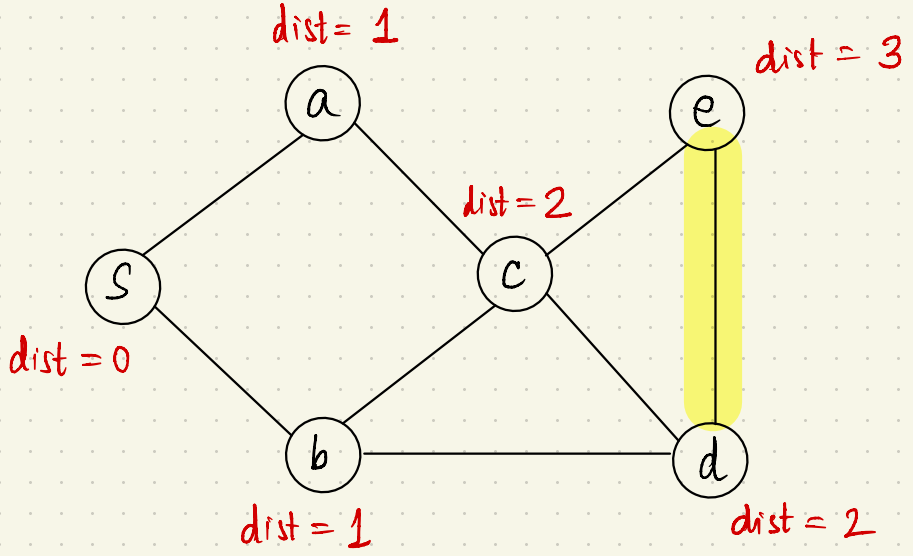

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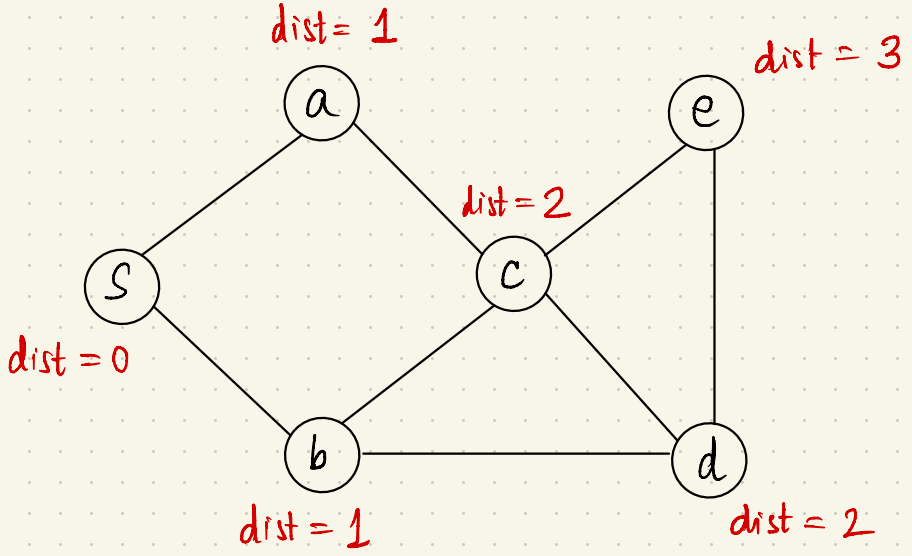

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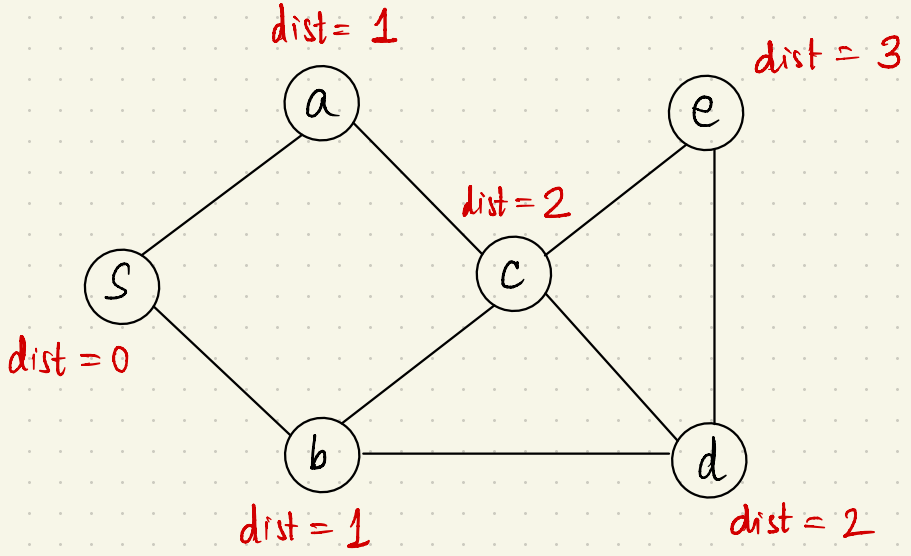

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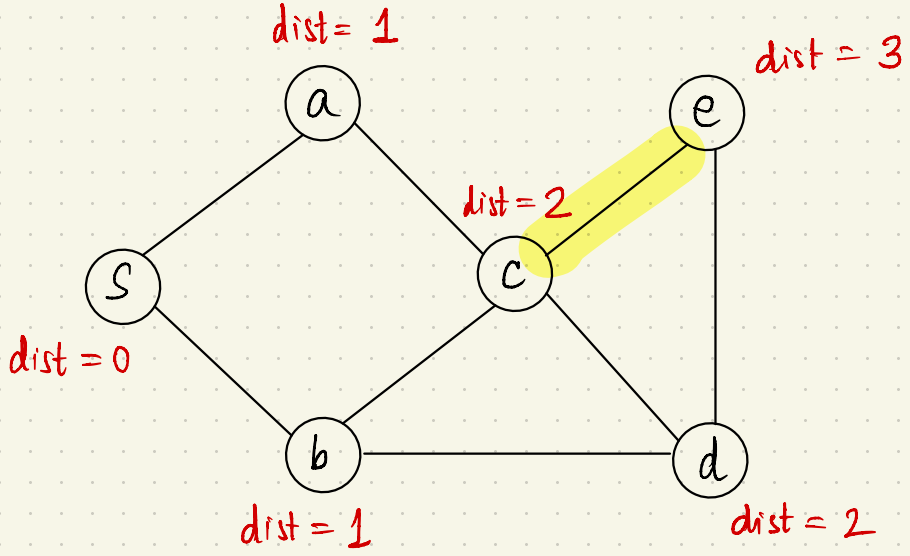

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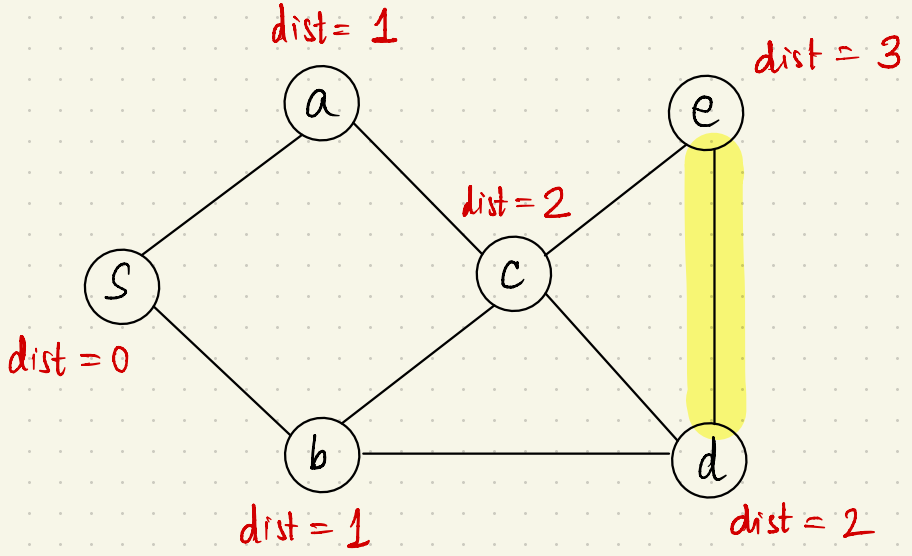

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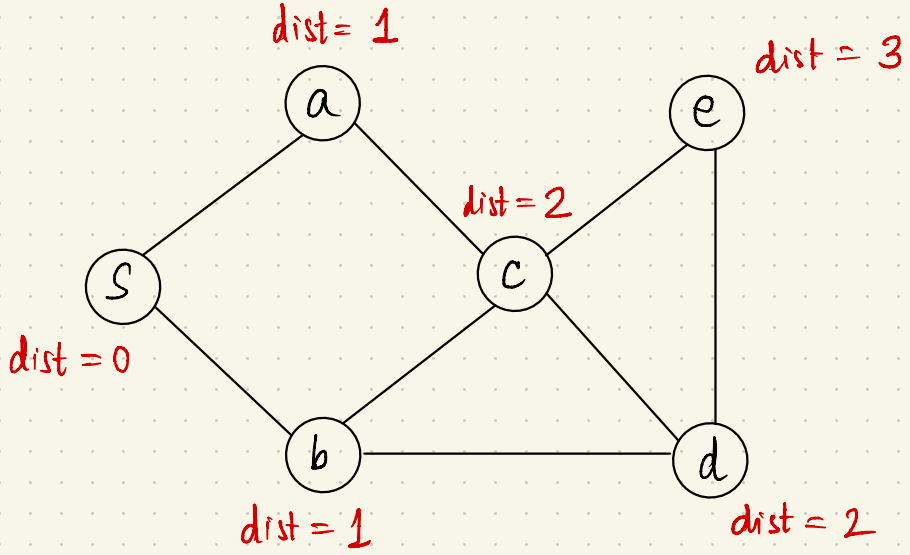
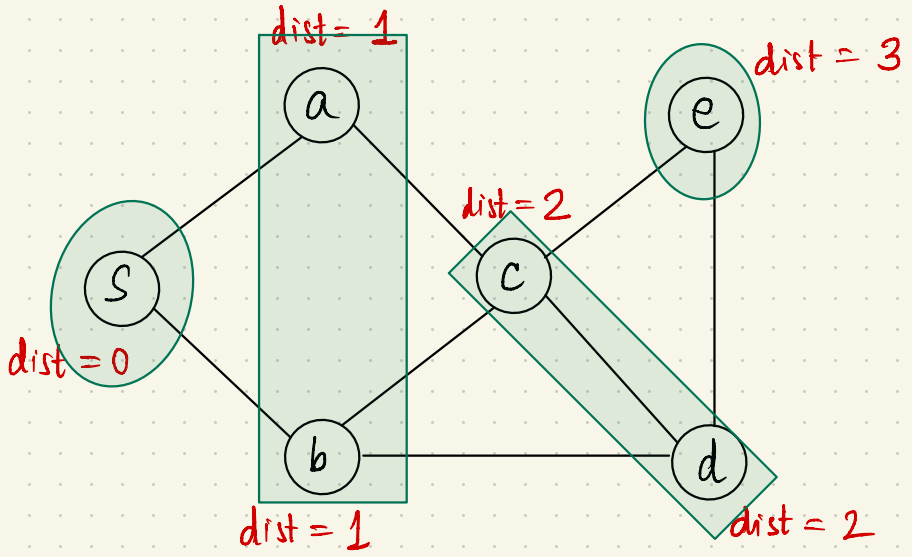


Q = ~~S a b c d e~~ done!

SHORTEST PATHS



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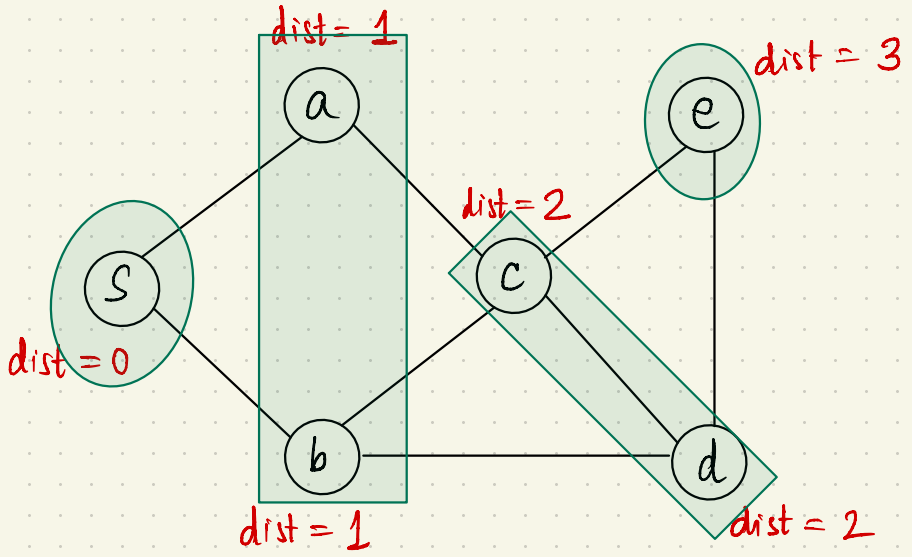

Claim: at termination, $dist(v) = i \iff v$ is in i^{th} layer

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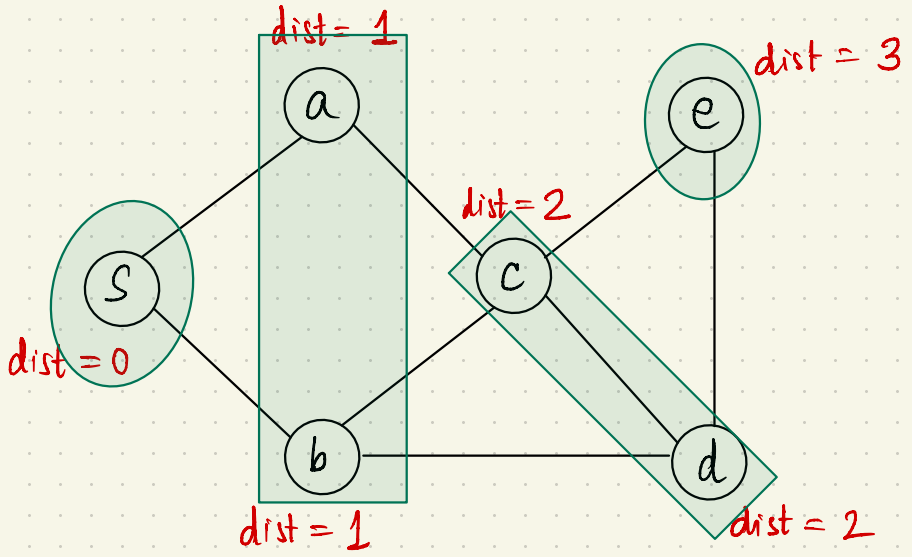

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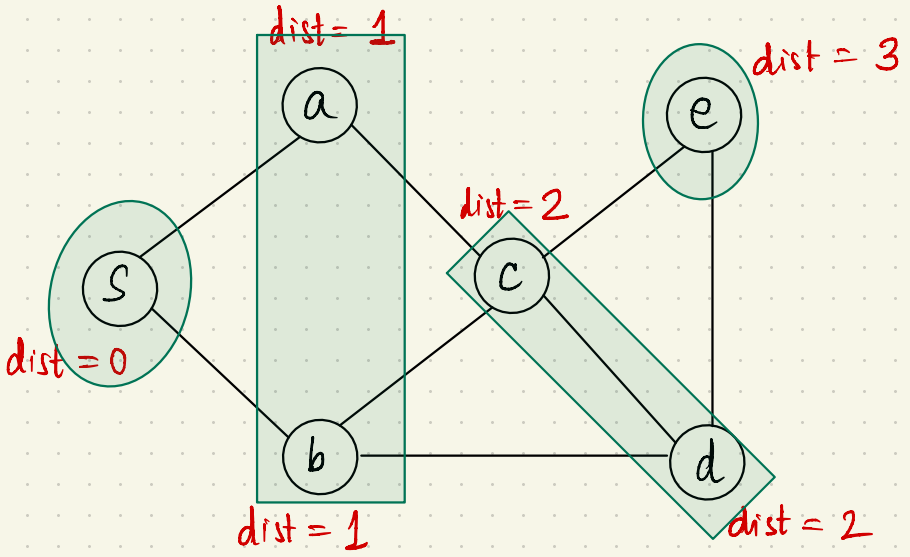
shortest path $s \to v$ has i edges

SHORTEST PATHS



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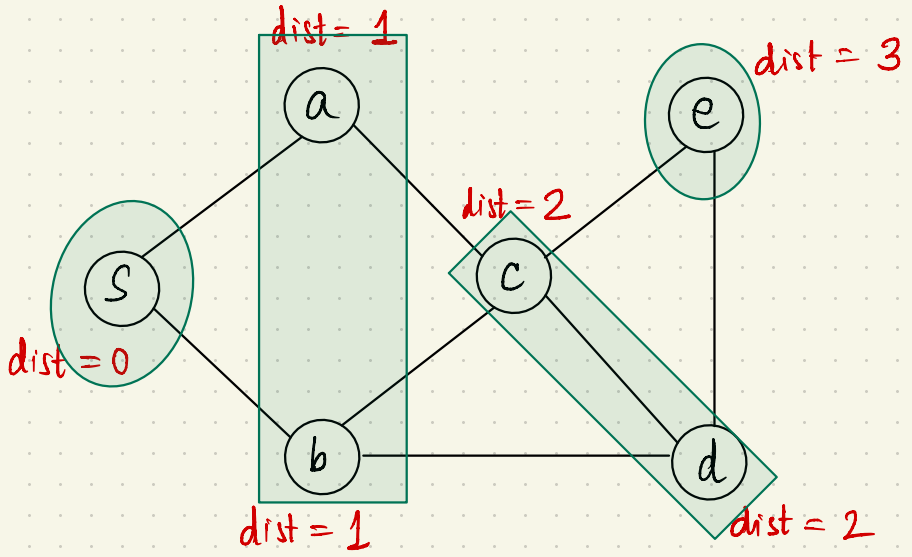
Proof: Exercise (by induction)

SHORTEST PATHS



Claim: The augmented BFS has running time $O(m+n)$.

APPLICATIONS OF BFS

Shortest paths

Connected Components

Applications of BFS

Shortest paths

Connected Components

CONNECTED COMPONENTS

Undirected graph $G = (V, E)$

CONNECTED COMPONENTS

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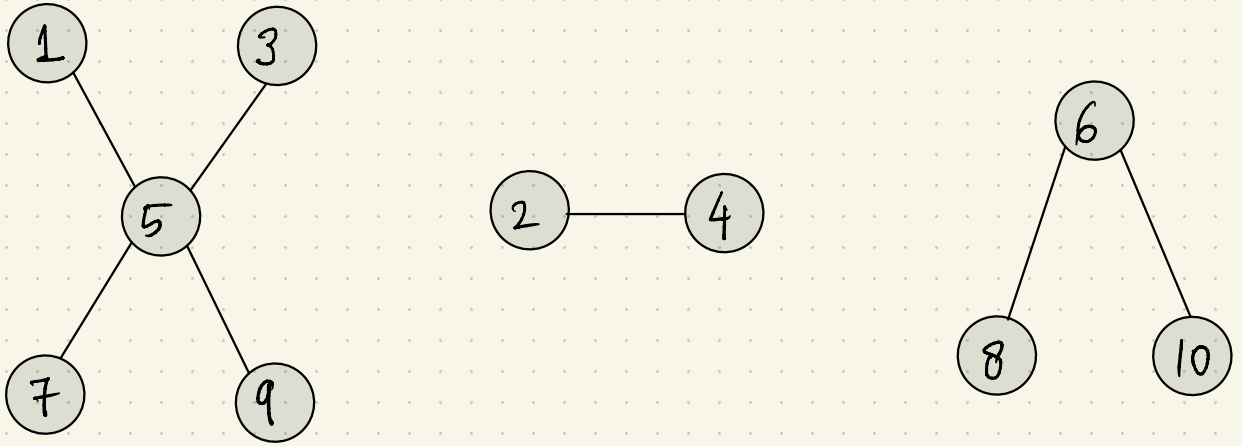
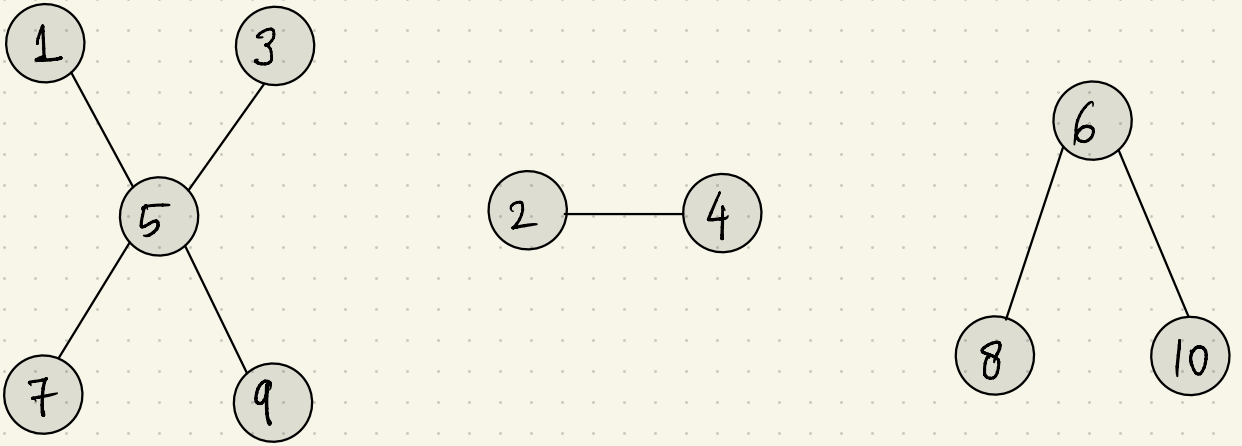


CONNECTED COMPONENTS

Undirected graph $G = (V, E)$



Connected components : maximal subsets of reachable vertices

(or equivalence class)

CONNECTED COMPONENTS

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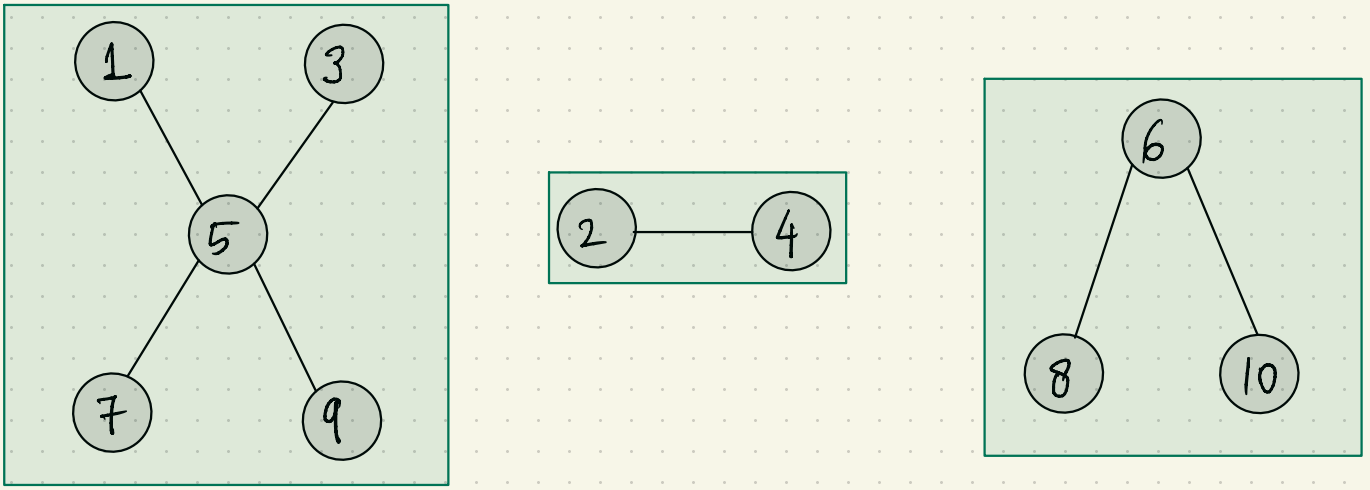

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input: an undirected graph $G = (V, E)$

output: identify the connected components of G

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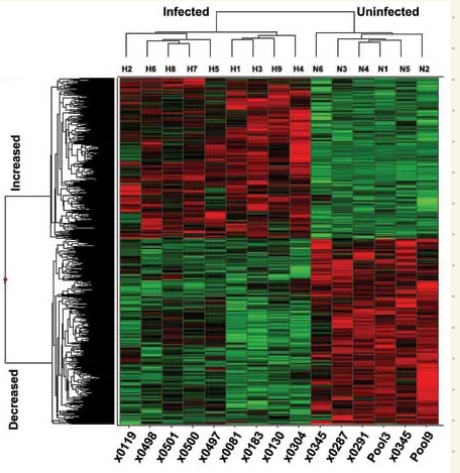
Detecting network failures



Genome clustering

Connected components via BFS

CONNECTED COMPONENTS via BFS

mark all vertices $1, 2, \ldots, n$ as unexplored

initialize CC count := 0

CONNECTED COMPONENTS via BFS

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CONNECTED COMPONENTS via BFS

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mark all vertices 1, 2, ..., n as unexplored
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for i = 1 to n                    // try all vertices
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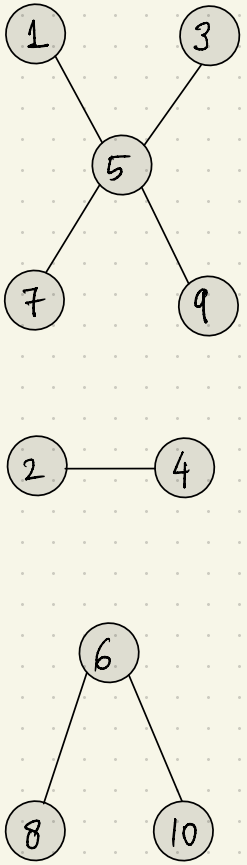

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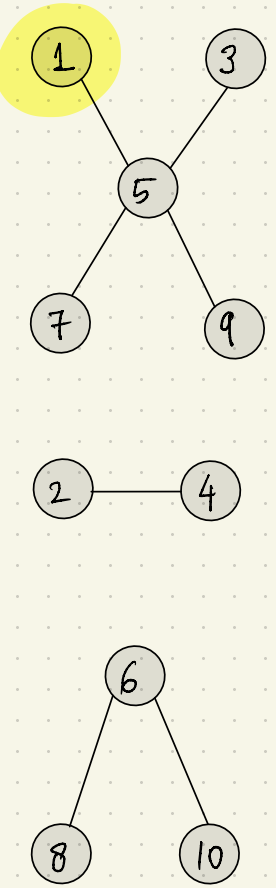

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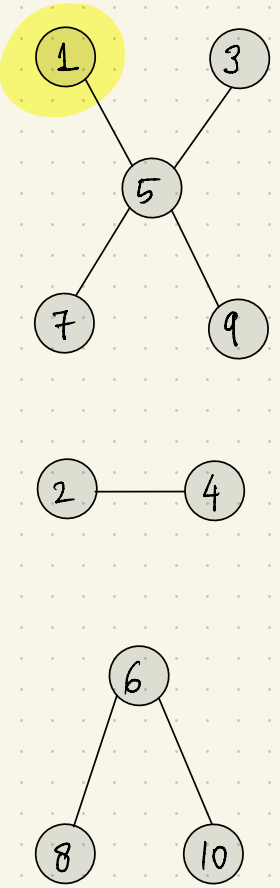

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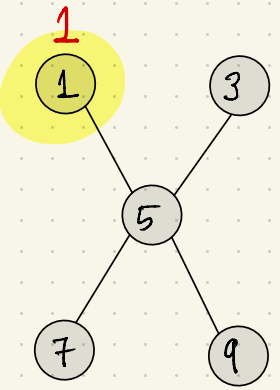

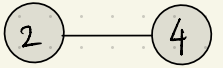

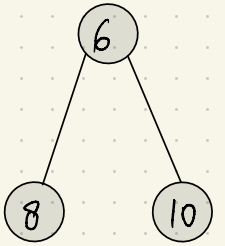

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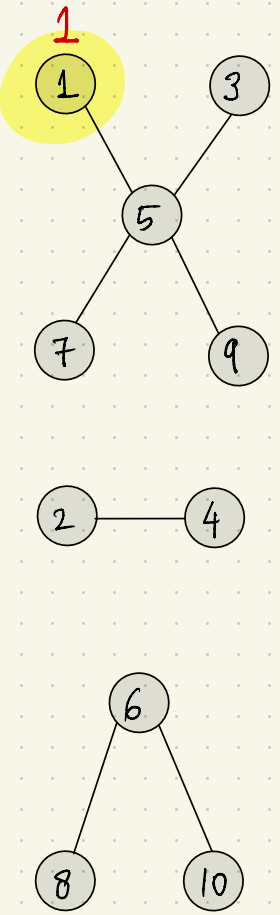

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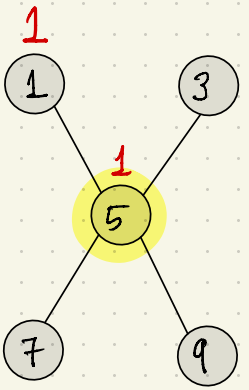

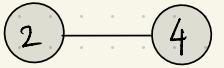

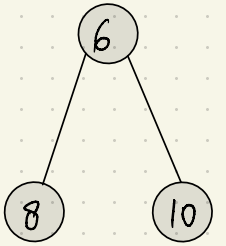

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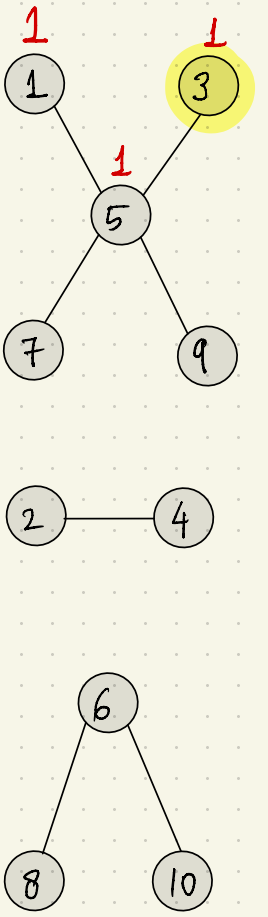

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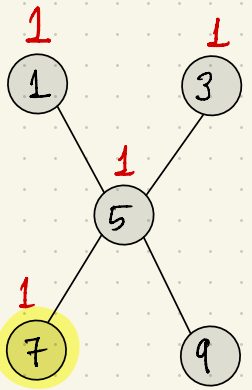

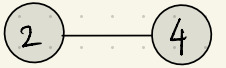

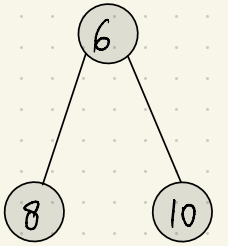

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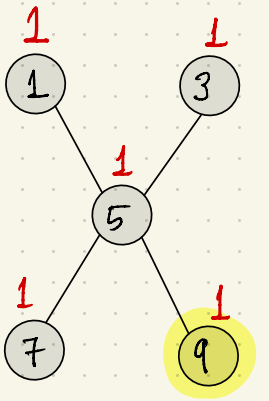

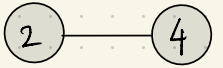

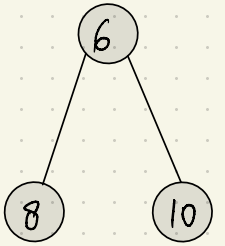

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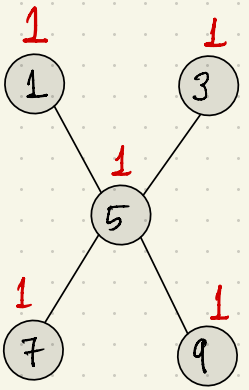

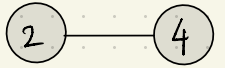

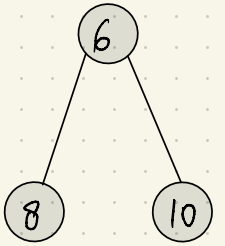

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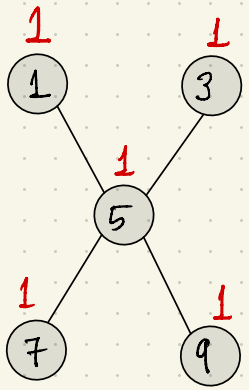

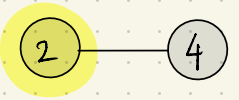

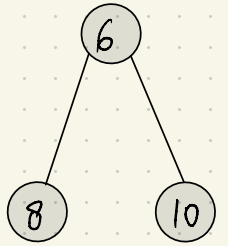

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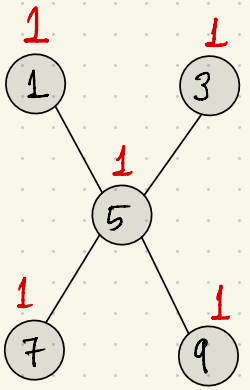

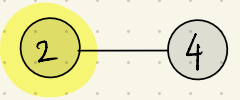

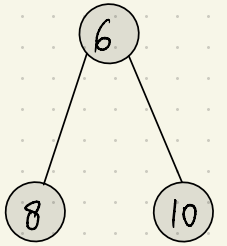

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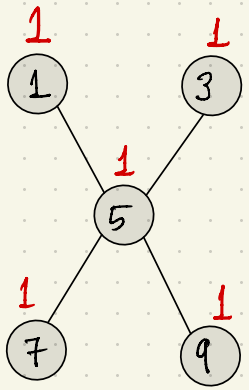

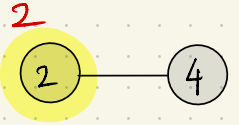

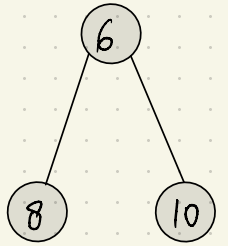

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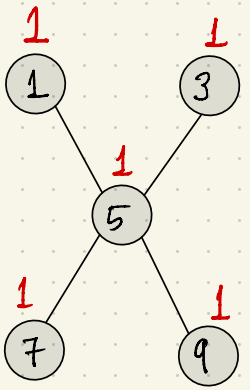

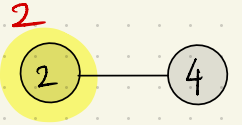

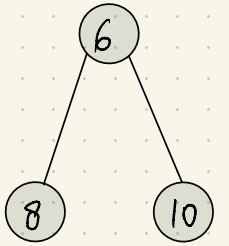

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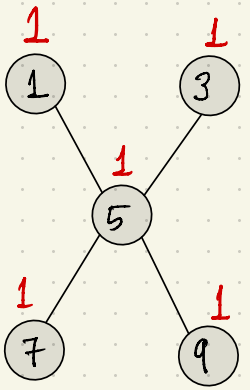

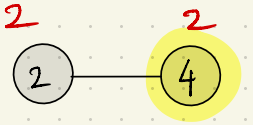

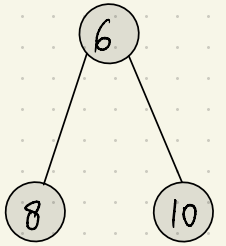

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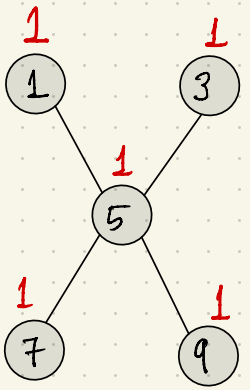

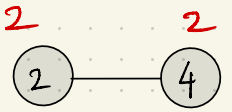

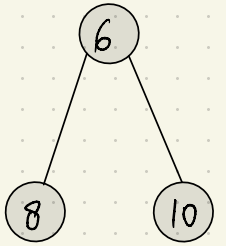

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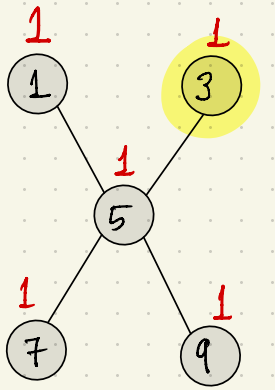

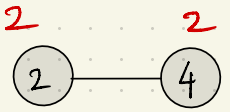

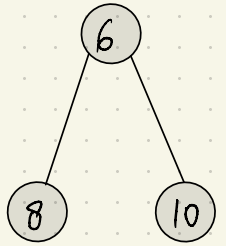

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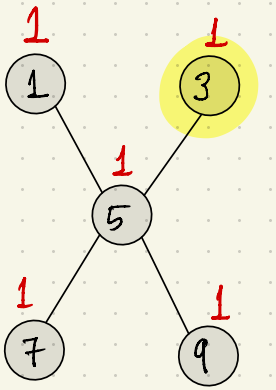

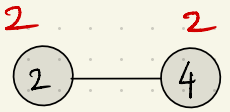

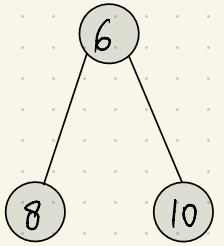

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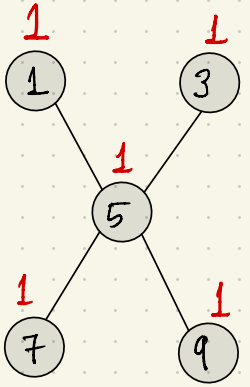

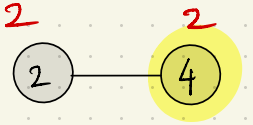

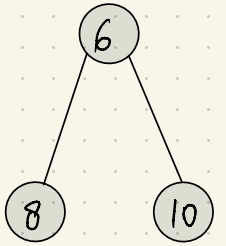

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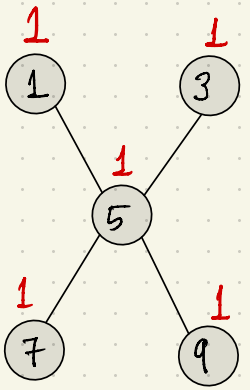

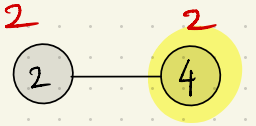

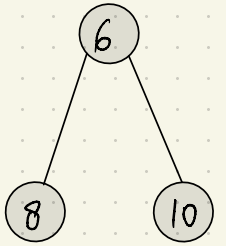

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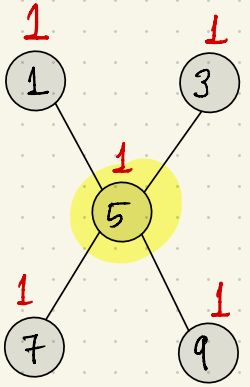

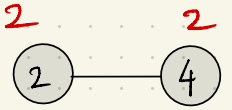

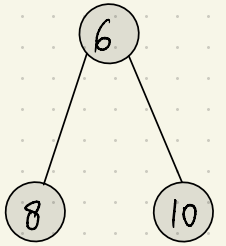

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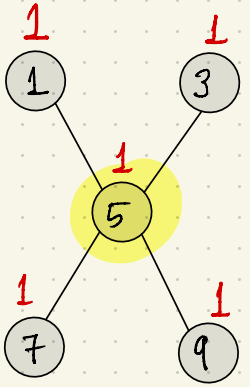

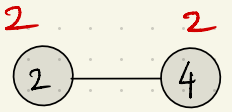

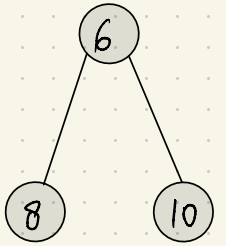

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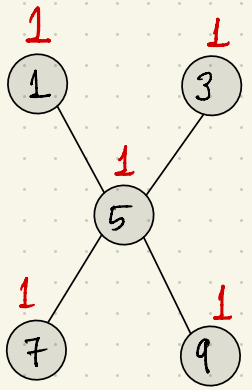

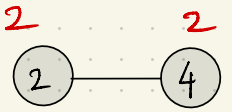

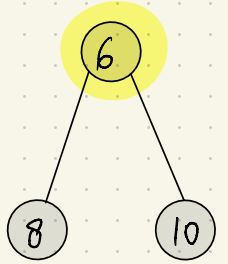

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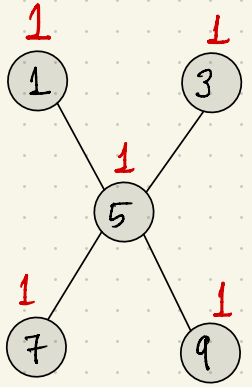

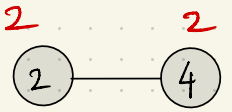

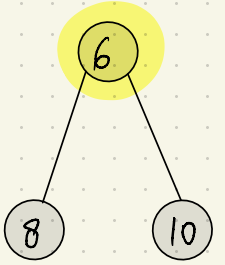

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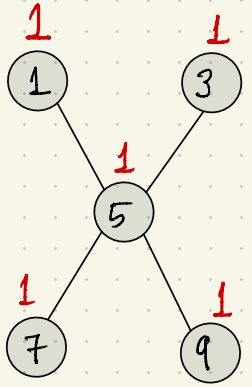
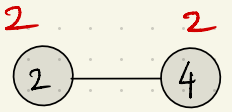
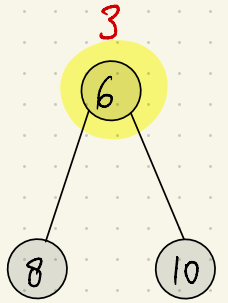






CONNECTED COMPONENTS via BFS

mark all vertices 1, 2, ..., n as unexplored

initialize CC count := 0

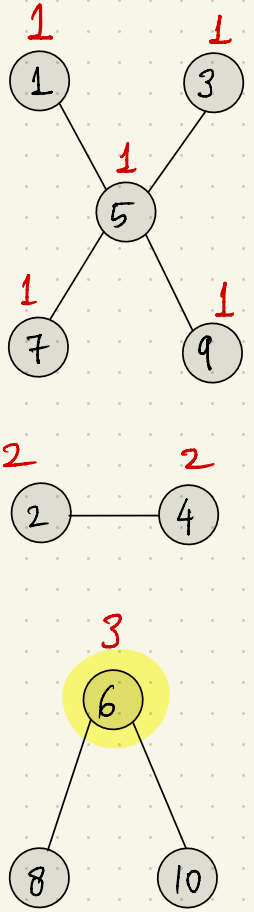
for i = 1 to n // try all vertices

  if i is unexplored

    increment CC count by 1

    BFS (G, i)

      while updating CC number of a vertex extracted from queue



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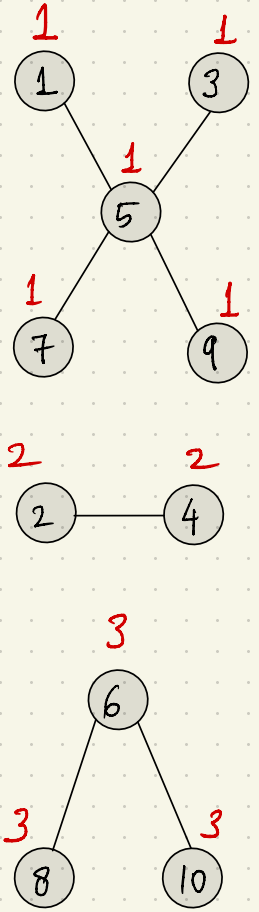
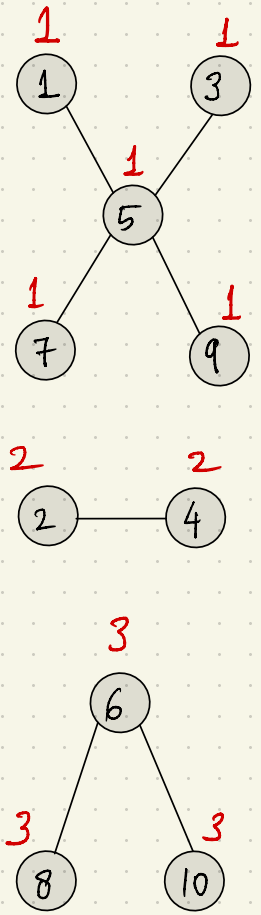


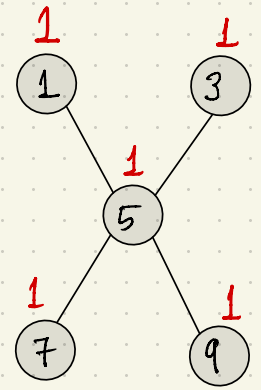
CONNECTED COMPONENTS via BFS

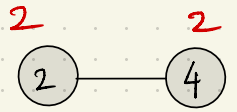


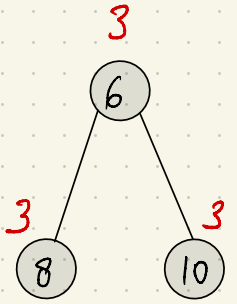
CONNECTED COMPONENTS via BFS

Claim: (a) At termination,

$cc[u] = cc[v] \iff u$ and v in same connected component







CONNECTED COMPONENTS via BFS

Claim: (a) At termination,

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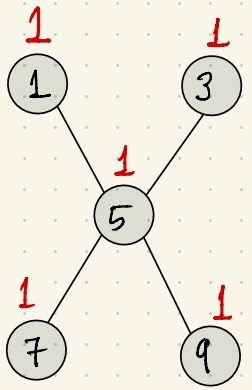
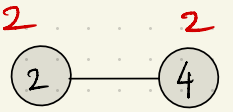
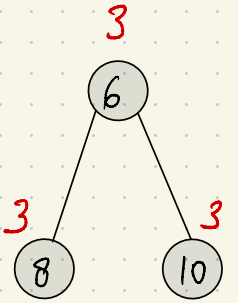
Idea: correctness of BFS + explored vertex is never marked unexplored







CONNECTED COMPONENTS via BFS

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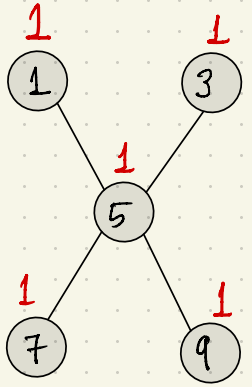
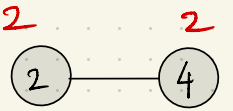
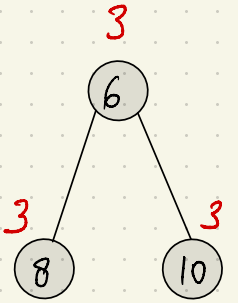
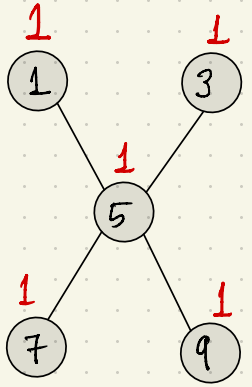
(b) Algo runs in $O(m+n)$ time







CONNECTED COMPONENTS via BFS



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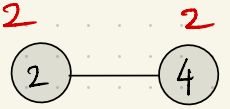
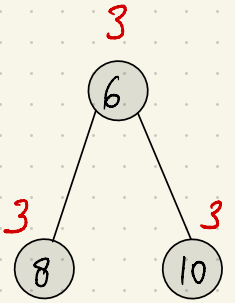
explored vertex is never marked unexplored

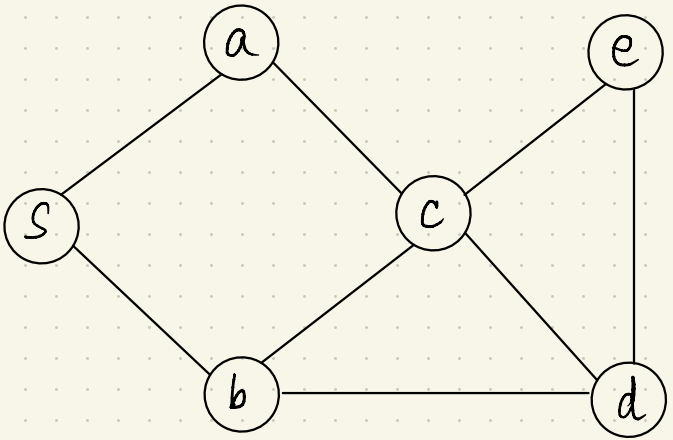


(b) Algo runs in $O(m+n)$ time

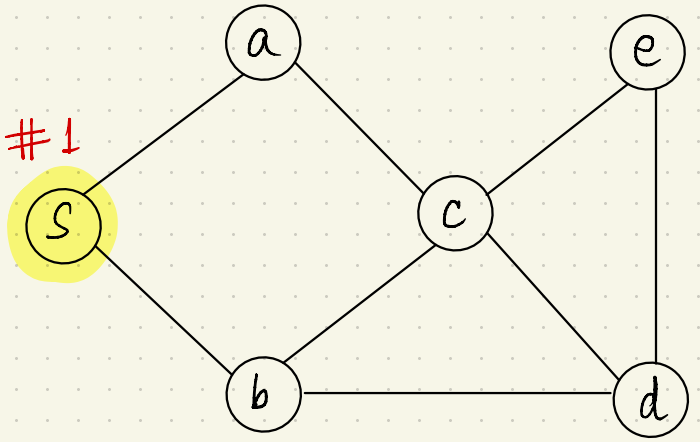
Idea: Time per component = $O(m_s + n_s)$.



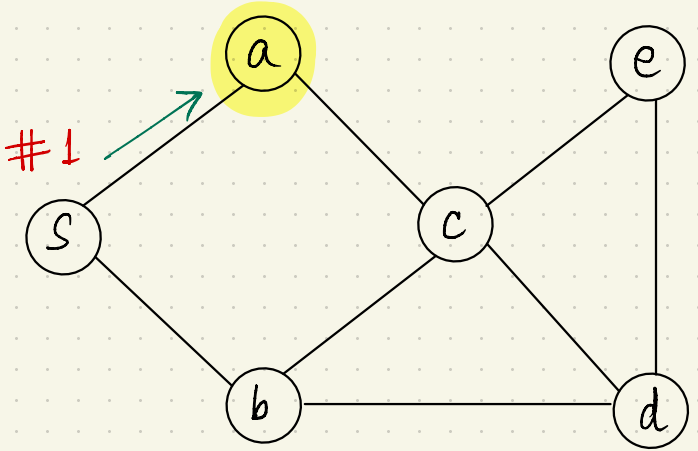
DEPTH-FIRST SEARCH



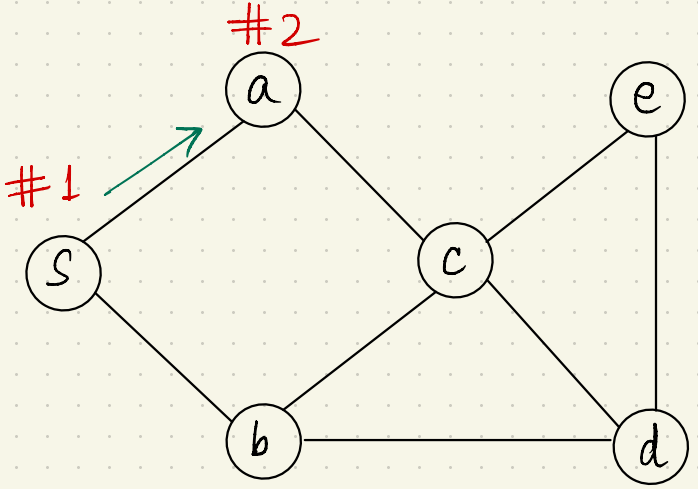
DEPTH-FIRST SEARCH



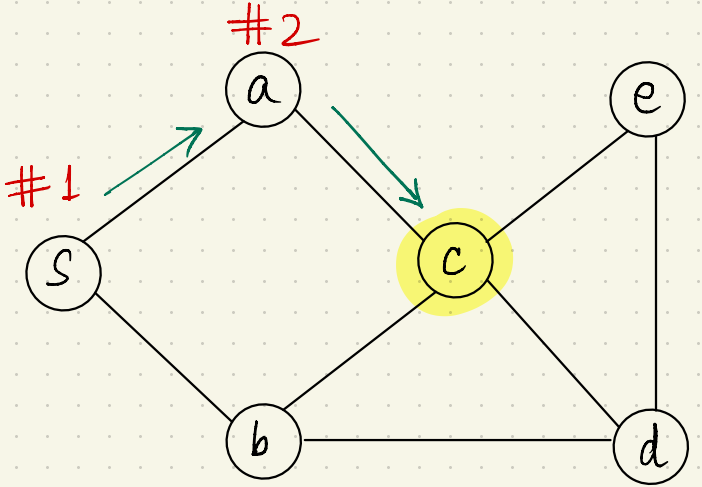
DEPTH-FIRST SEARCH



DEPTH-FIRST SEARCH

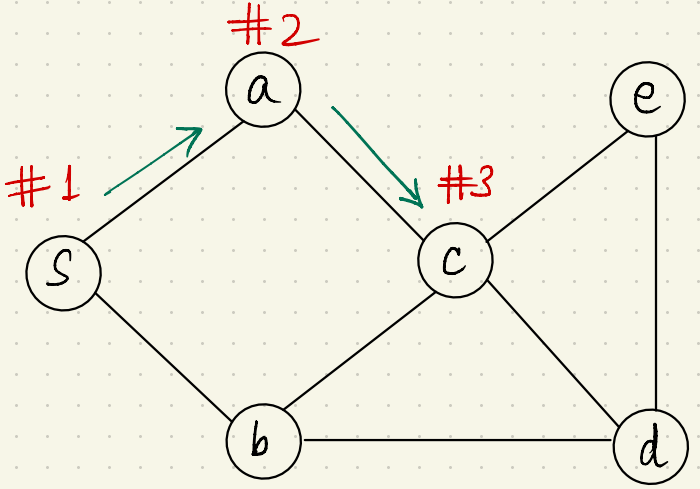


DEPTH-FIRST SEARCH



Differs from BFS !

DEPTH-FIRST SEARCH



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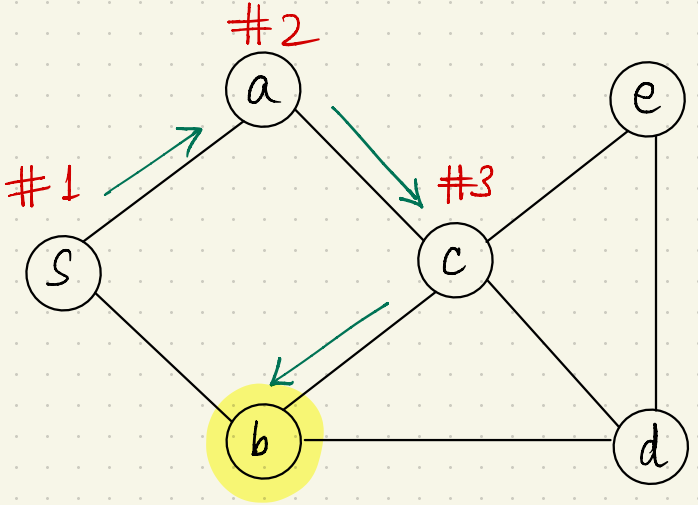

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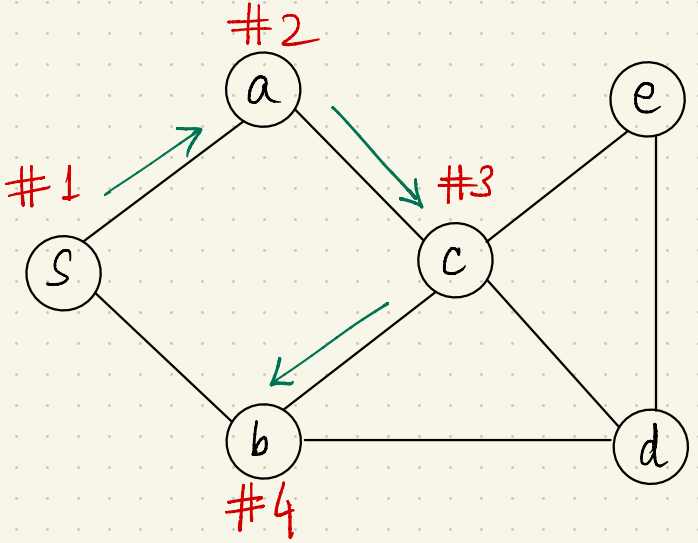
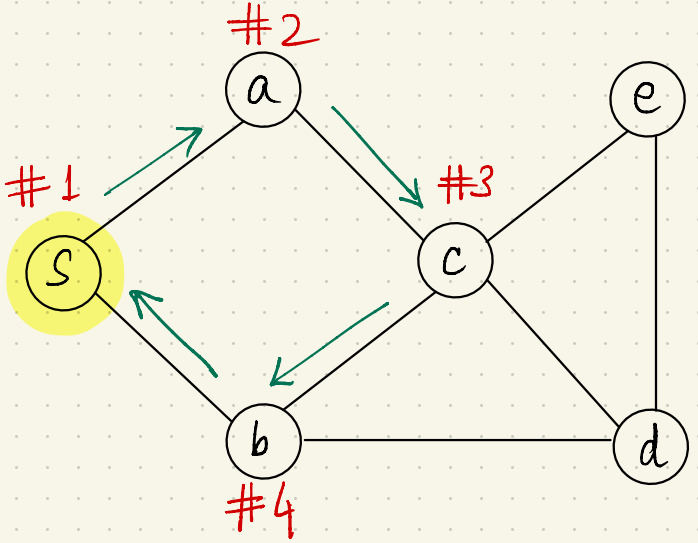
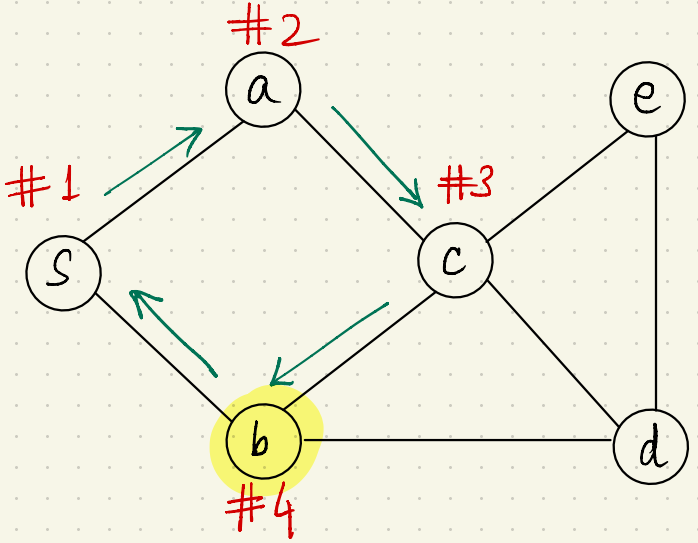


DEPTH-FIRST SEARCH



s is already explored ⟹ backtrack!

DEPTH-FIRST SEARCH



s is already explored $\Rightarrow$ backtrack!

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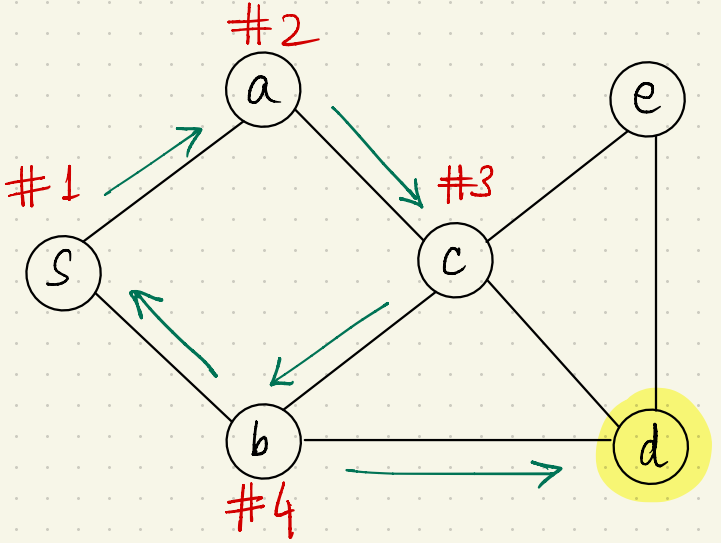
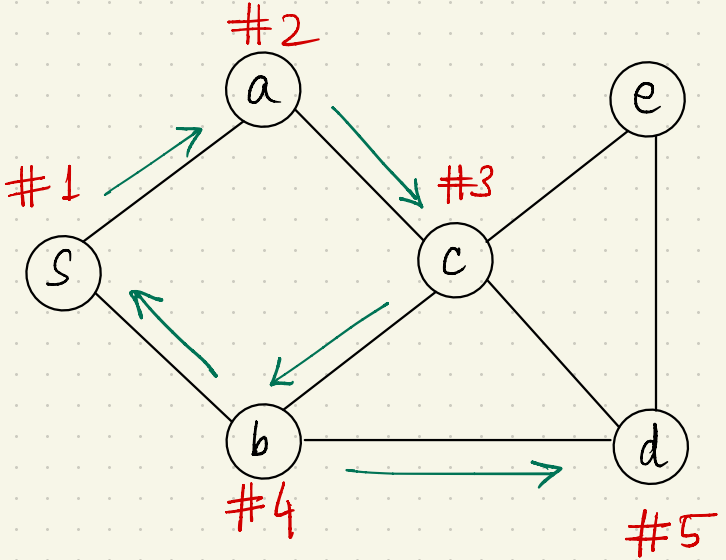
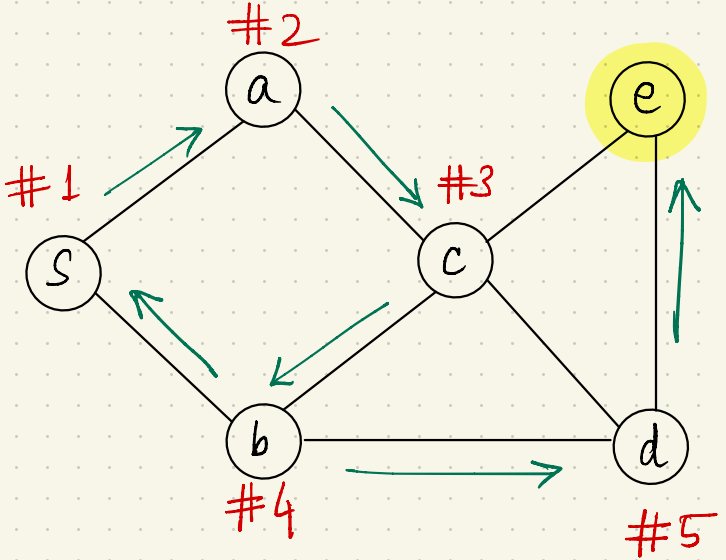


DEPTH-FIRST SEARCH



DEPTH-FIRST SEARCH



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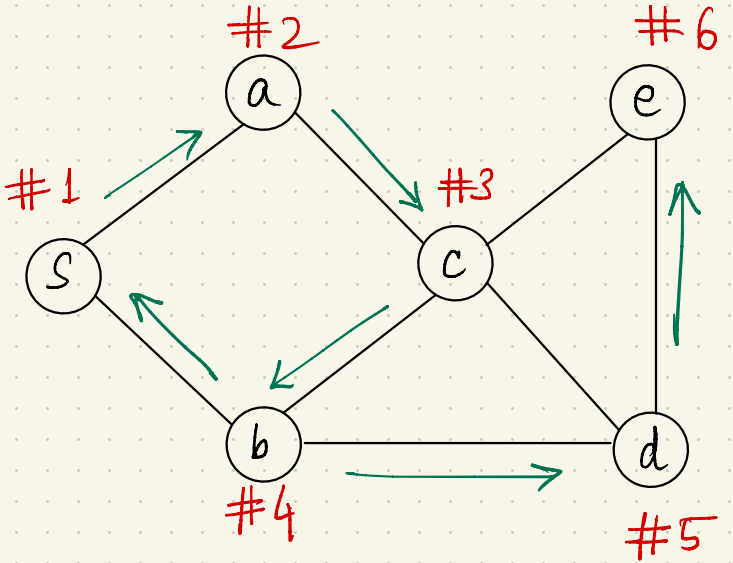
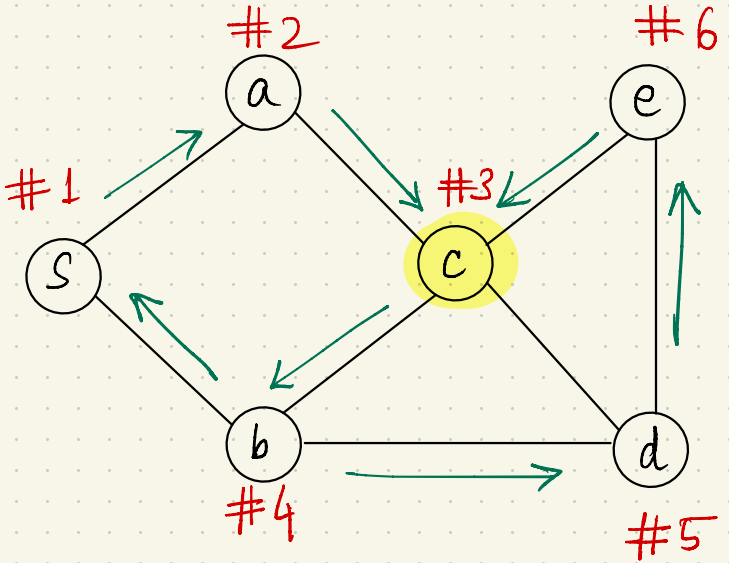
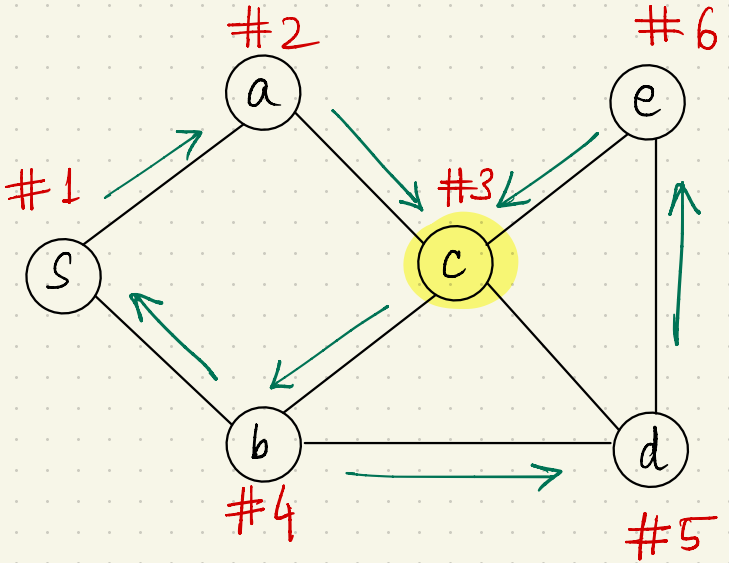


DEPTH-FIRST SEARCH



DEPTH-FIRST SEARCH



c is already explored $\Rightarrow$ backtrack!

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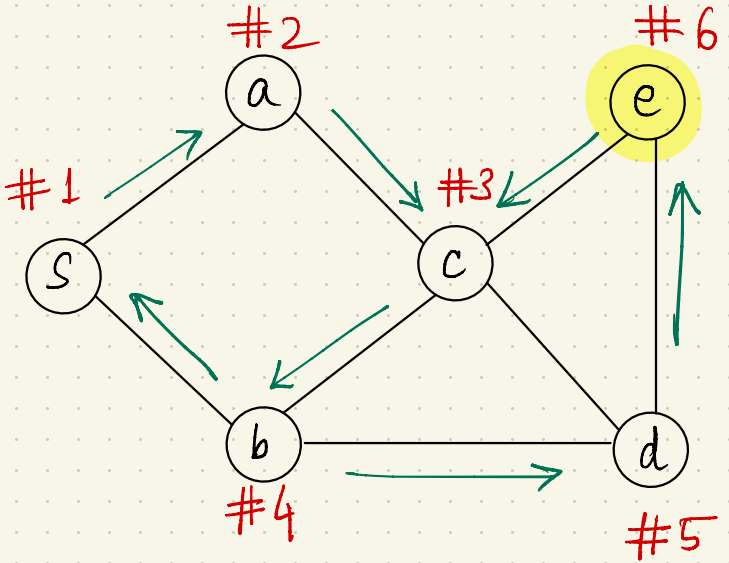

c is already explored ⟹ backtrack!

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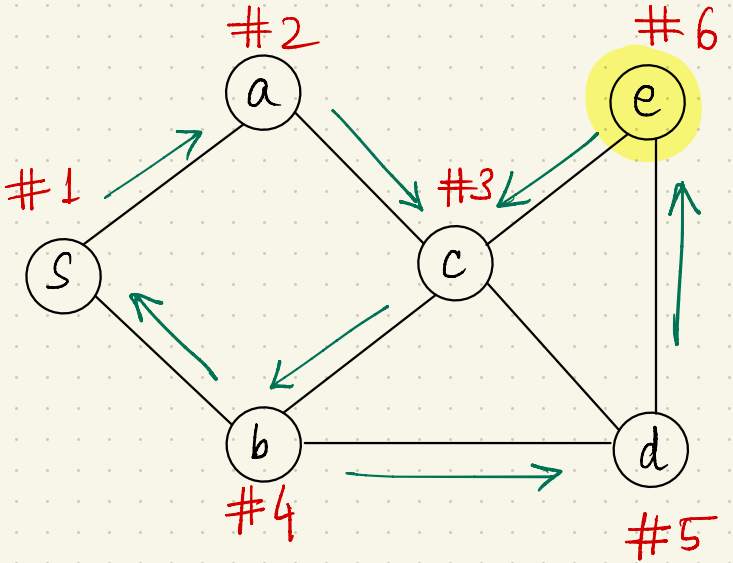
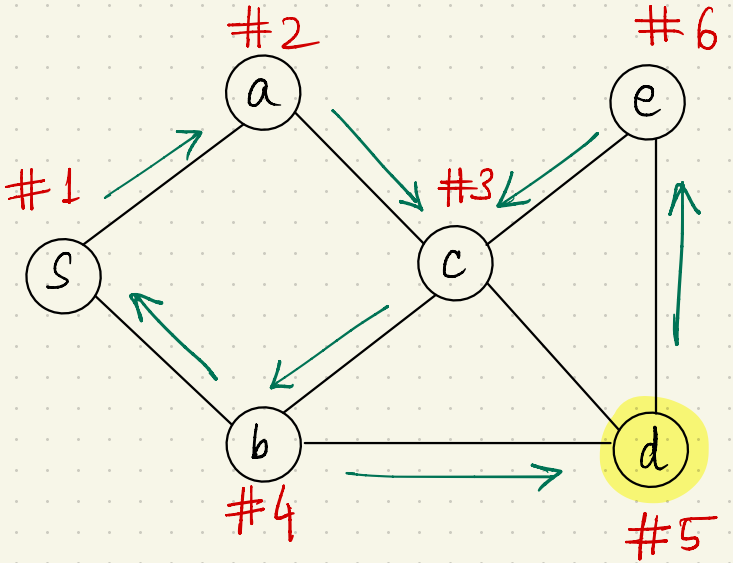


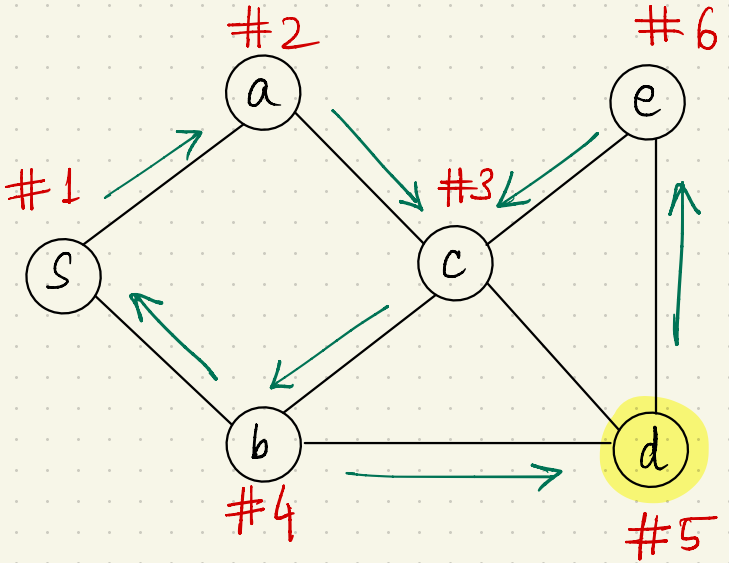
No unexplored neighbors of e ⇒ backtrack!

DEPTH-FIRST SEARCH

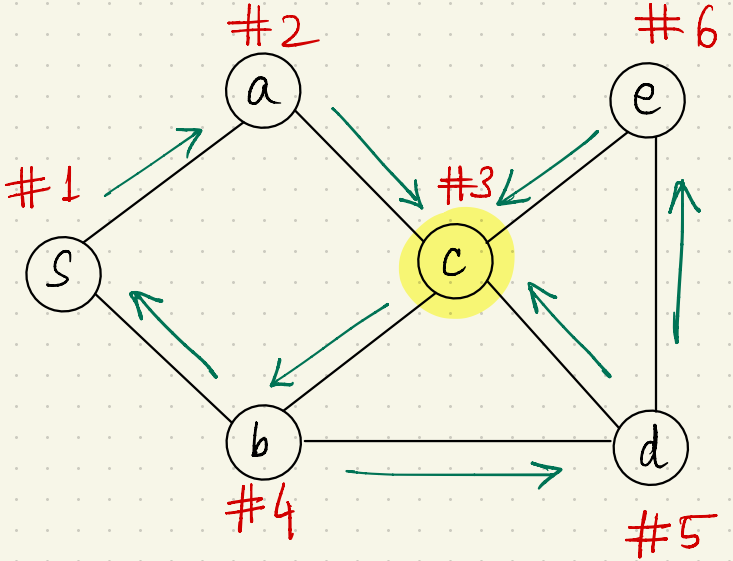


No unexplored neighbors of e ⇒ backtrack!

DEPTH-FIRST SEARCH



DEPTH-FIRST SEARCH



c is already explored ⟹ backtrack!

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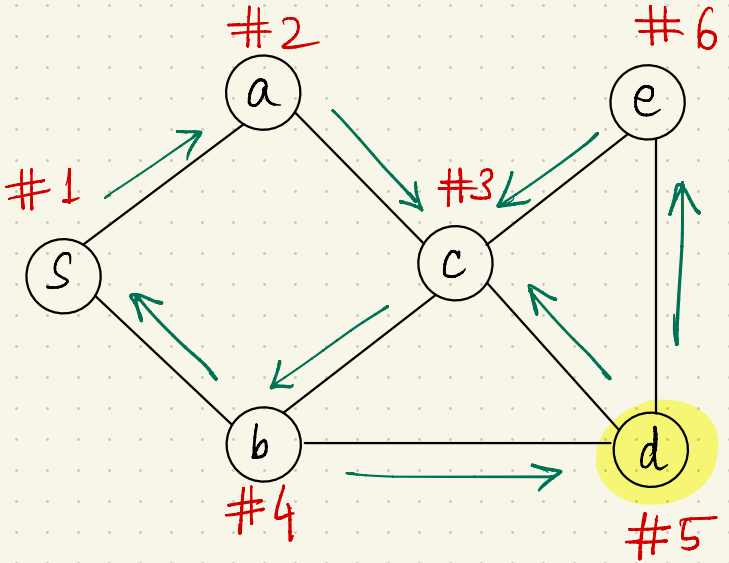
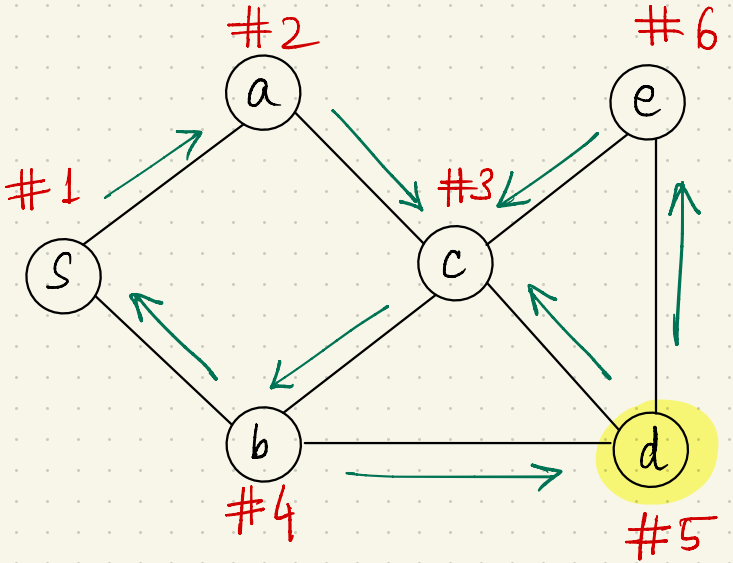


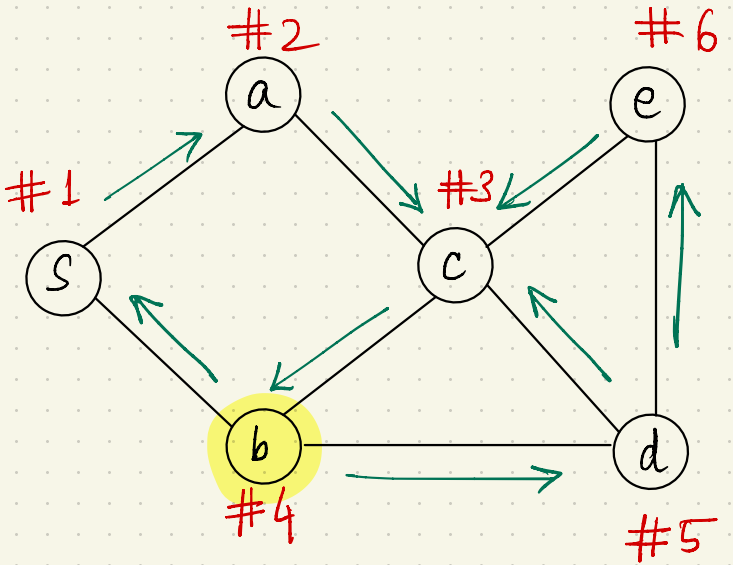
c is already explored ⟹ backtrack!

DEPTH-FIRST SEARCH



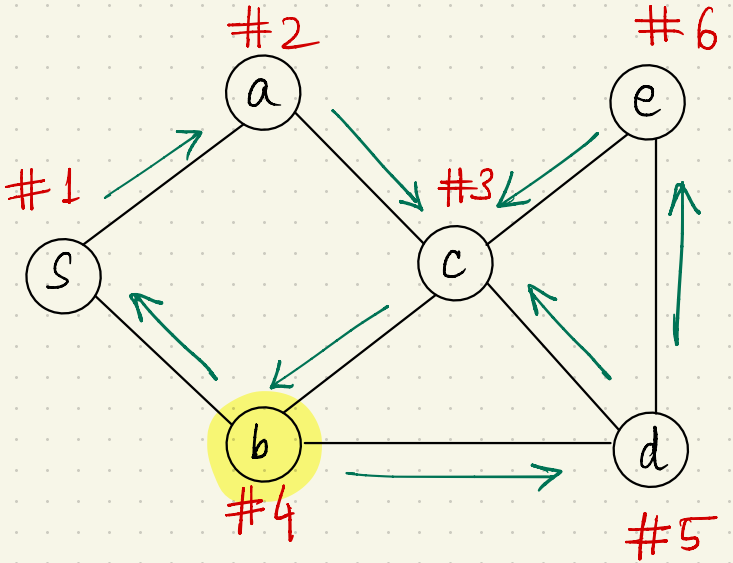
No unexplored neighbors of d ⇒ backtrack!

DEPTH-FIRST SEARCH



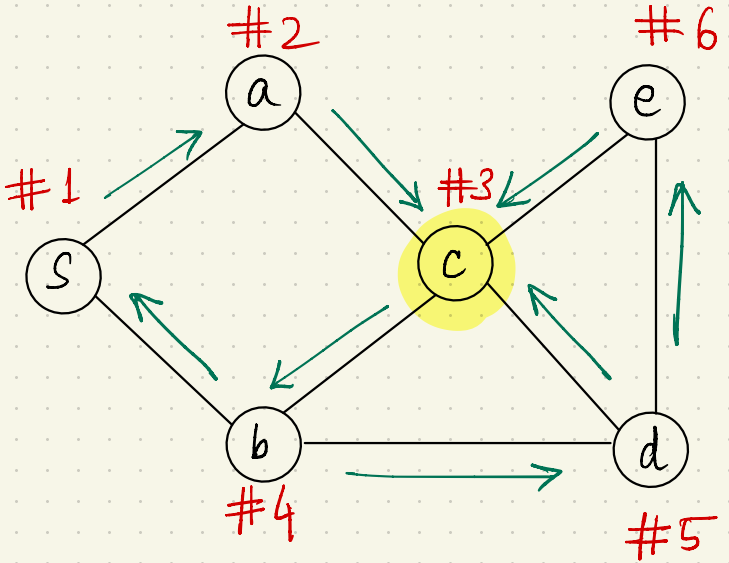
No unexplored neighbors of d ⇒ backtrack!

DEPTH-FIRST SEARCH



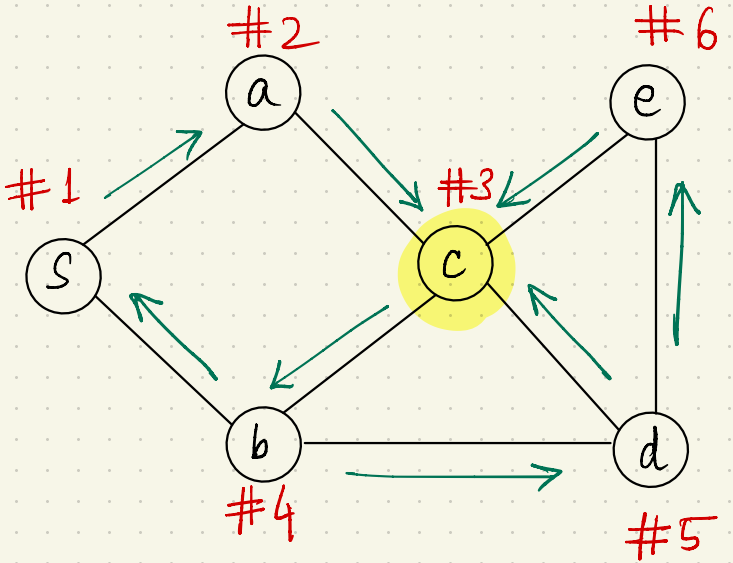
No unexplored neighbors of b ⇒ backtrack!

DEPTH-FIRST SEARCH



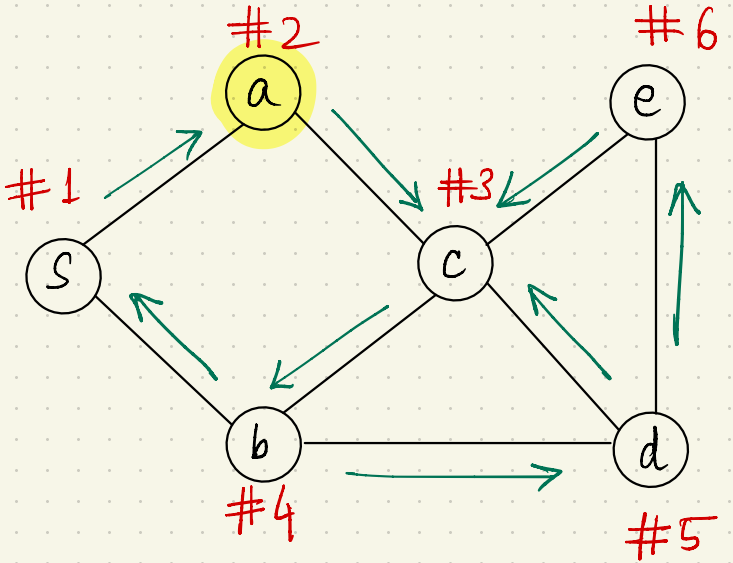
No unexplored neighbors of b ⇒ backtrack!

DEPTH-FIRST SEARCH



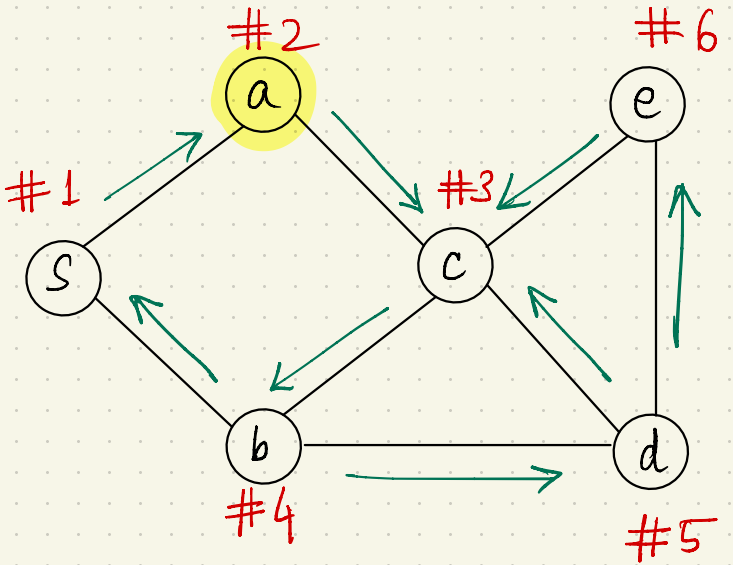
No unexplored neighbors of c ⇒ backtrack!

DEPTH-FIRST SEARCH



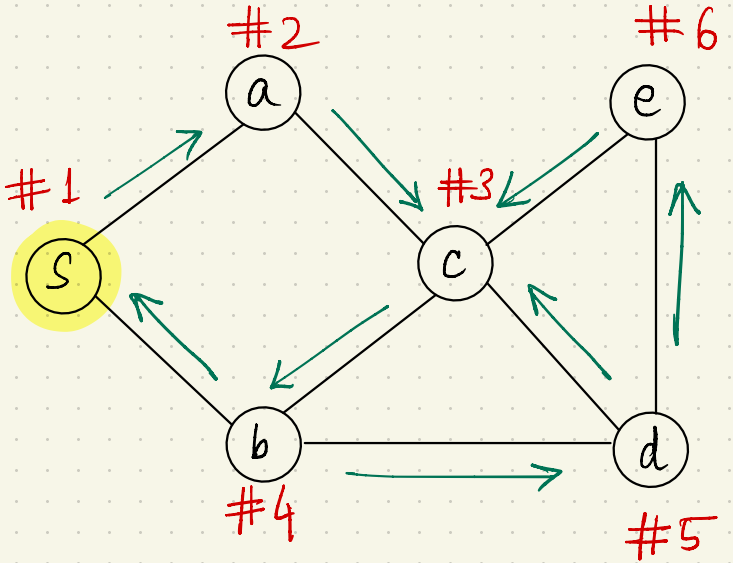
No unexplored neighbors of c ⇒ backtrack!

DEPTH-FIRST SEARCH



No unexplored neighbors of a ⇒ backtrack!

DEPTH-FIRST SEARCH



Back home!

DEPTH-FIRST SEARCH

Iterative version

DEPTH-FIRST SEARCH

Iterative version

mark all vertices as unexplored

S := a stack data structure (LIFO), initialized with s

while $S \neq \emptyset$

- remove the top node of S, say v ("pop")
- if v is unexplored
  - mark v as explored
  - for each (v, w) in adj. list of v
    - add w to the front of S ("push")

DEPTH-FIRST SEARCH

Iterative version

mark all vertices as unexplored

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Recursive version

DEPTH-FIRST SEARCH

Recursive version

```
DFS (G, s)          // all vertices unexplored before the call

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Corollary: DFS can compute connected components in linear time.

APPLICATIONS OF DFS

Topological ordering

Strongly Connected Components

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TOPOLOGICAL ORDERING

Directed graph $G = (V, E)$

TOPOLOGICAL ORDERING

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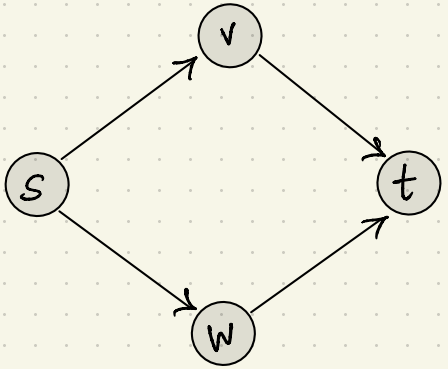
A labeling f of G's vertices such that:

* unique $f(v) \in \{1, 2, \ldots, n\}$ for every $v \in V$

* for every $(v, w) \in E$ $\quad f(v) < f(w)$.

TOPOLOGICAL ORDERING



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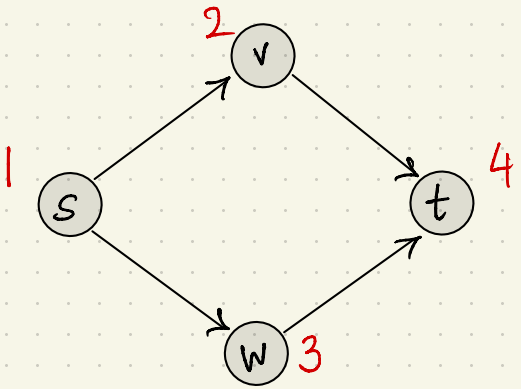

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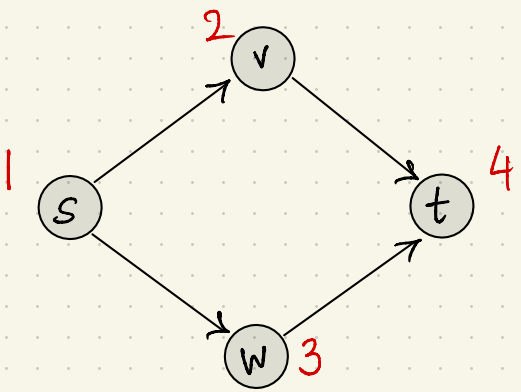

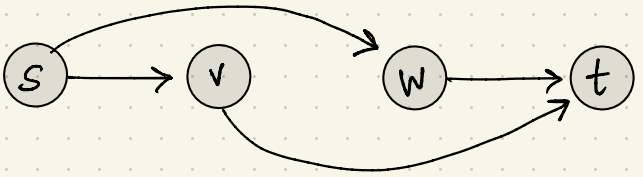

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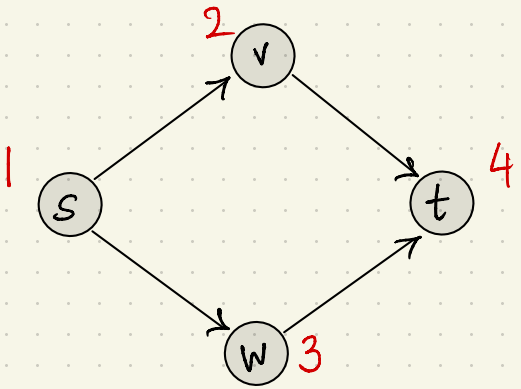

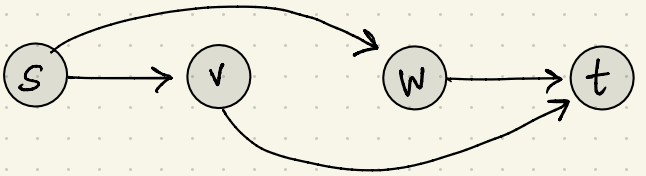

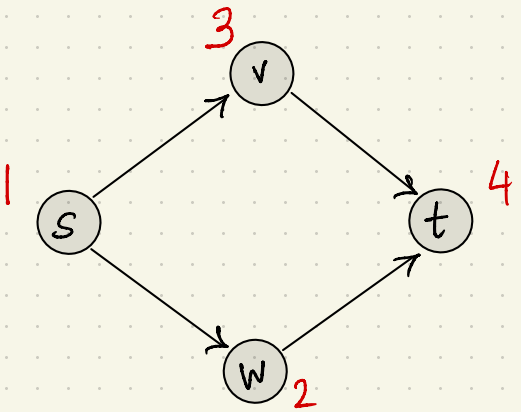

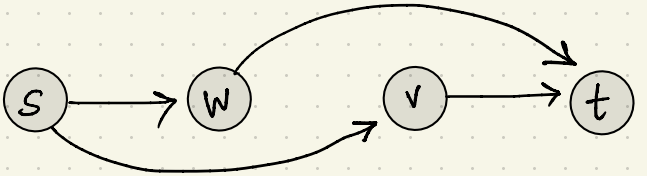

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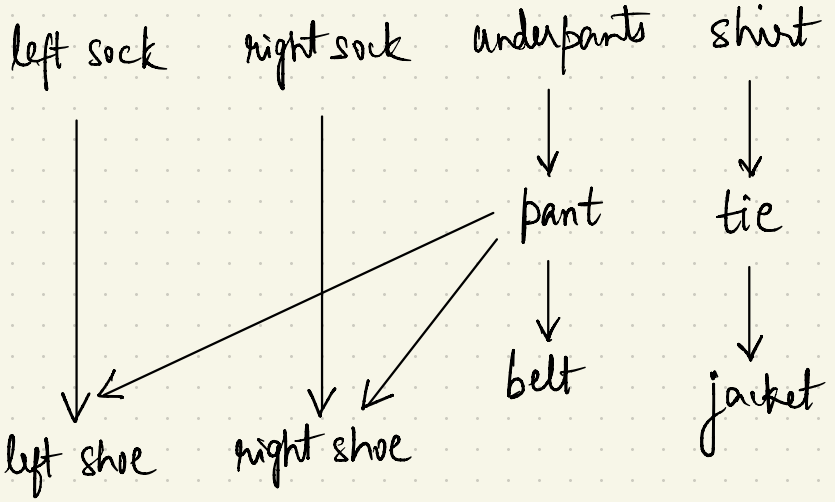

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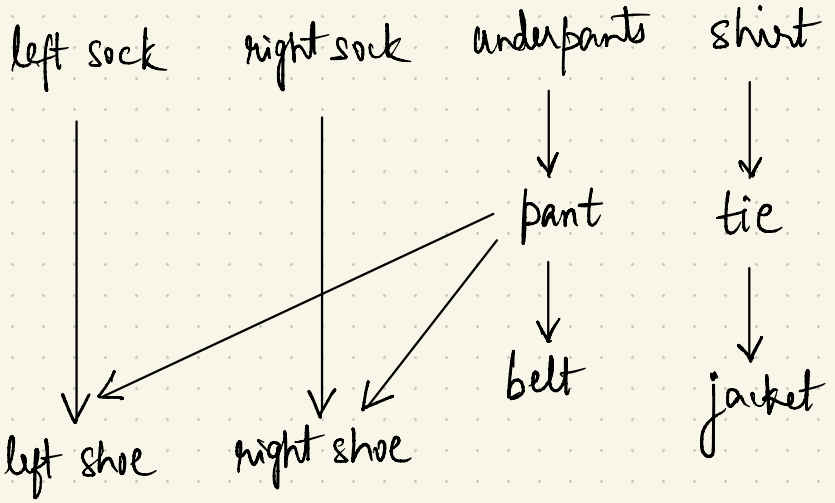
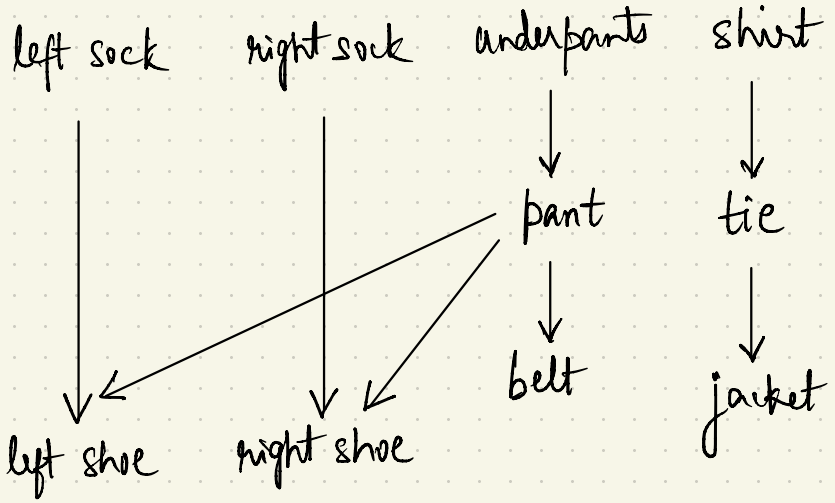


underpants
shirt
pant
belt
tie
jacket
left sock
right sock
left shoe
right shoe

TOPOLOGICAL ORDERING



underpants	left sock
shirt	shirt
pant	tie
belt	underpants
tie	right sock
jacket	pants
left sock	right shoe
right sock	belt
left shoe	jacket
right shoe	left shoe

TOPOLOGICAL ORDERING

When does a directed graph have a topological ordering?

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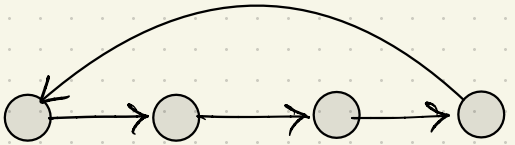
Necessary : No cycle



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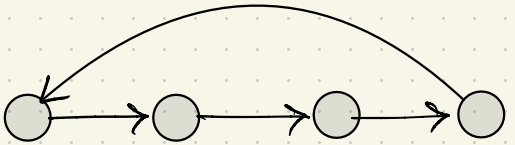

Also sufficient!

TOPOLOGICAL ORDERING

Theorem : Every directed acyclic graph has at least one topological ordering.

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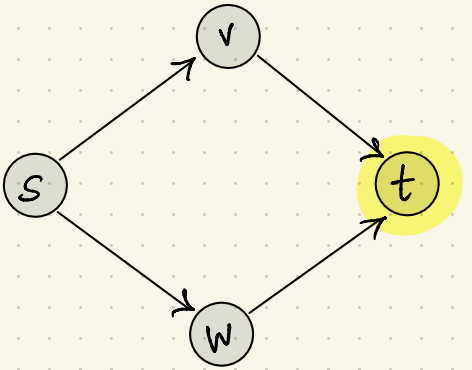

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Proof of Lemma: Suppose not. Then, every vertex has some outgoing edge

"Following your nose" will give a cycle. Contradiction!

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Lemma: Every directed acyclic graph has at least one sink.

a vertex with no outgoing edges

Lemma: Every directed acyclic graph has at least one source.

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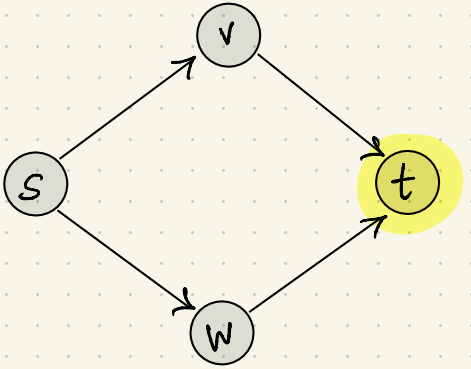

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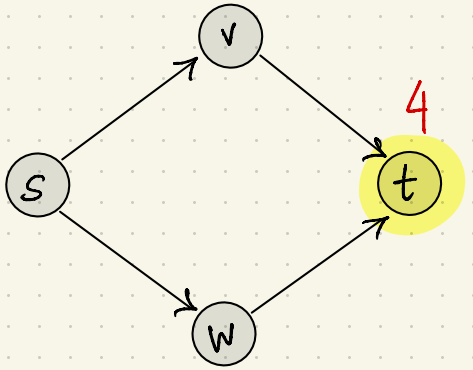

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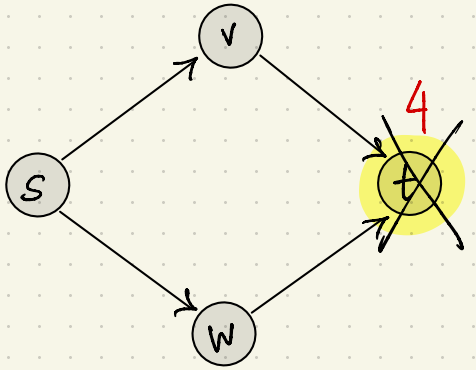

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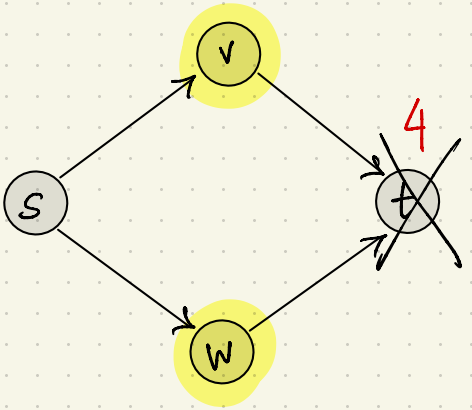

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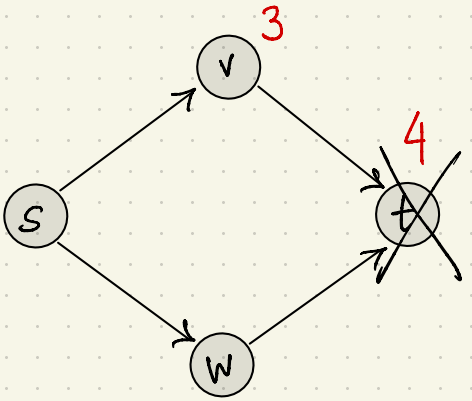

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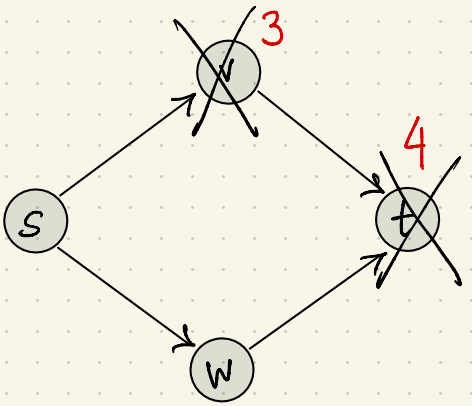

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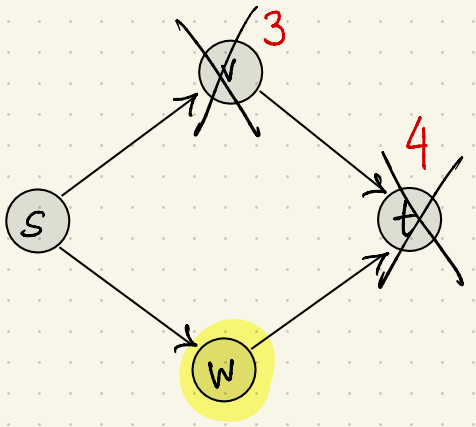

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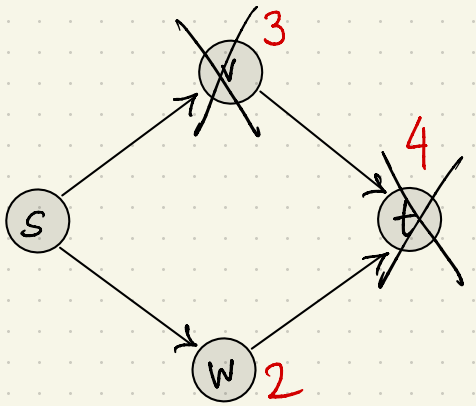

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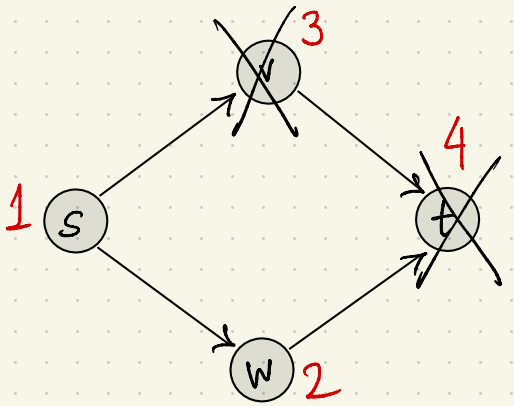

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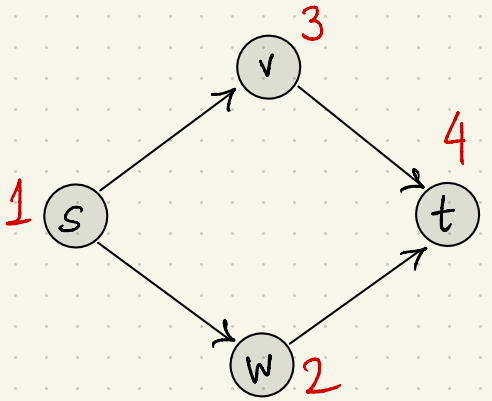

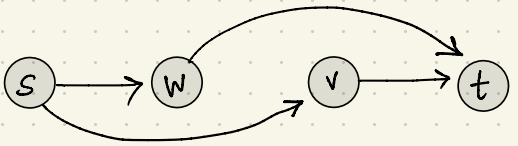

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Algorithm

1. Find a sink vertex v.
2. Assign to it the largest available label and recurse on $G \setminus \{v\}$.

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Correctness

If $f[v] = i$, no edges to vertices with $f[v] < i$.

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Running time

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Can we do better for sparse graphs?

TOPOLOGICAL ORDERING via DFS

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DFS pseudocode

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DFS (G, s)        // all vertices unexplored before the call

    mark s as explored

    for each (s, v) in adj. list of s
        if v is unexplored
            DFS (G, v)
```

TOPOLOGICAL ORDERING via DFS

```
DFS (G, s)   // all vertices initially unexplored
mark s as explored
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TOPOLOGICAL ORDERING via DFS

DFS-Loop (G)

DFS (G, s) // all vertices initially unexplored

mark s as explored

for each (s, v) in adj. list of s

    if v is unexplored

        DFS (G, v)

TOPOLOGICAL ORDERING via DFS

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DFS-Loop(G)
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current_label := |V| // labeling f
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DFS(G, s)  // all vertices initially unexplored
mark s as explored
for each (s, v) in adj. list of s
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        DFS(G, v)
Set f[s] = current_label
decrease current_label by 1
```
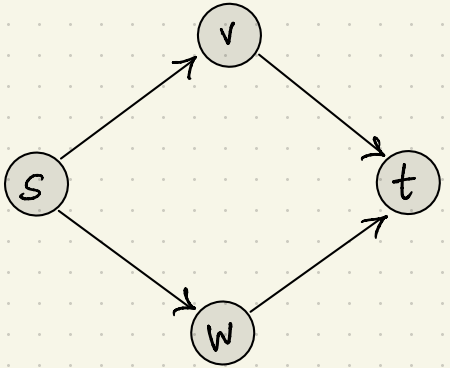

TOPOLOGICAL ORDERING via DFS



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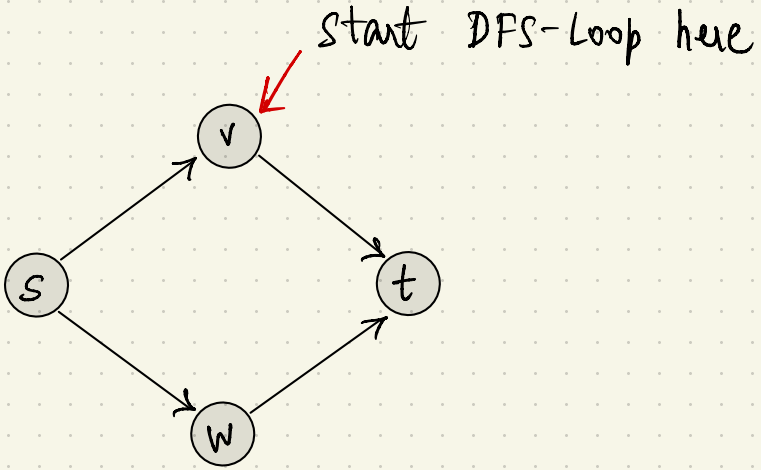

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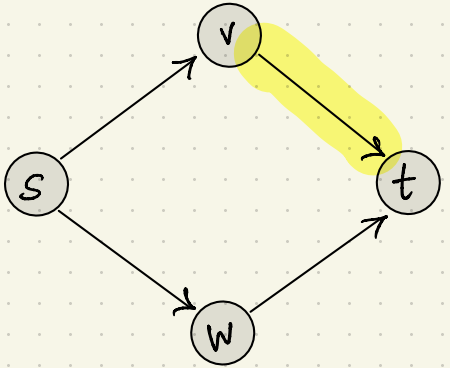

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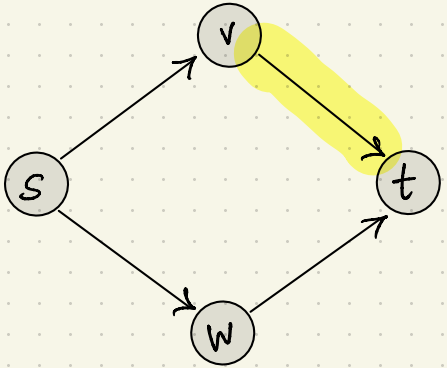
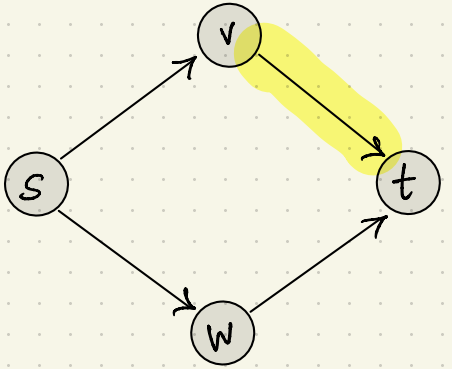


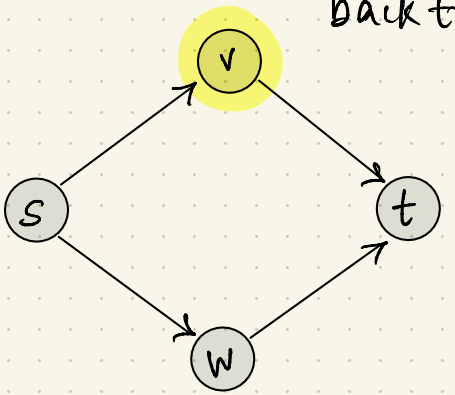
nowhere to go from t

TOPOLOGICAL ORDERING via DFS



$f[t] = 4$

TOPOLOGICAL ORDERING via DFS

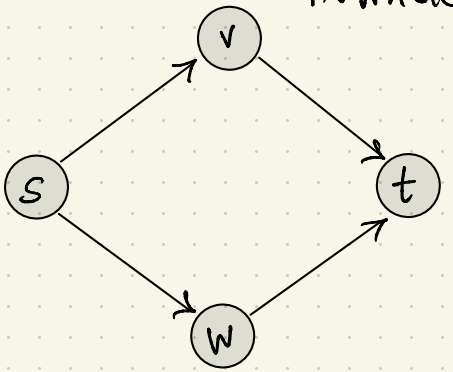


backtrack to v

$f[t] = 4$

TOPOLOGICAL ORDERING via DFS

nowhere else to go from v



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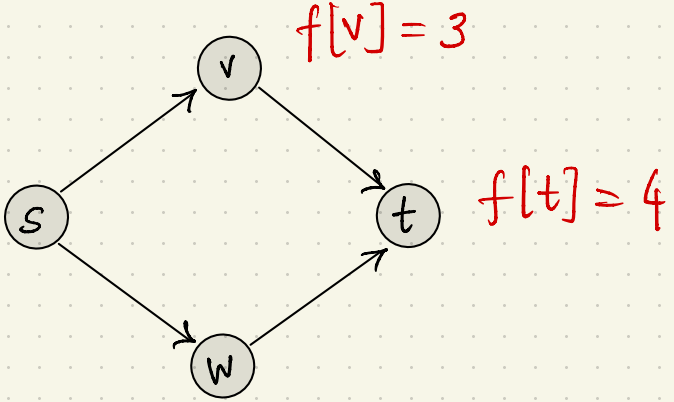
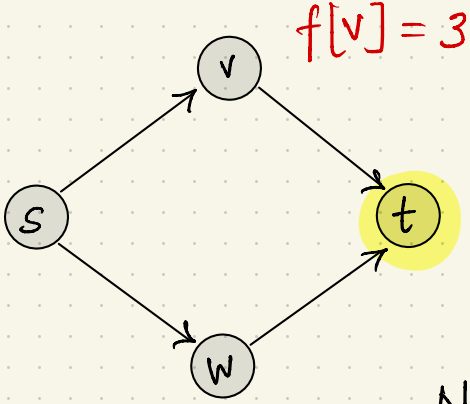


TOPOLOGICAL ORDERING via DFS



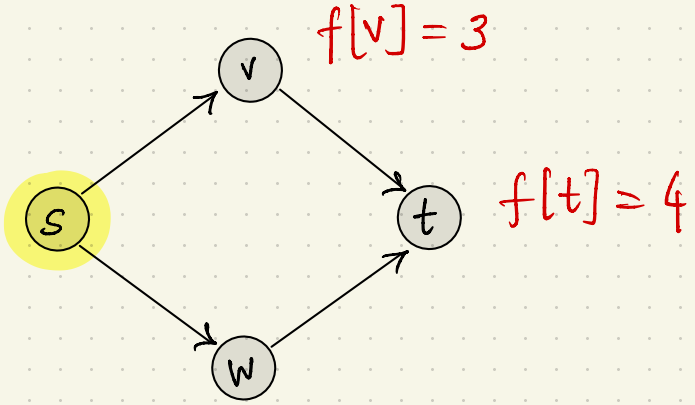
Next, DFS-loop considers t

but it is already explored

so skip it.

TOPOLOGICAL ORDERING via DFS

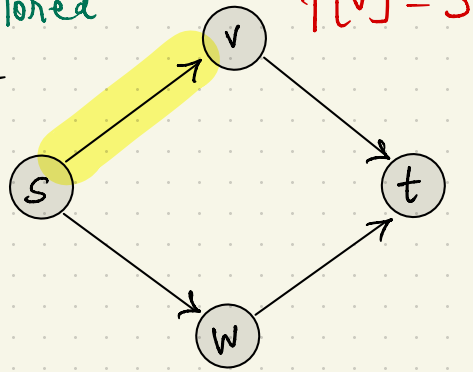
Next, DFS-Loop considers s.



TOPOLOGICAL ORDERING via DFS

v is explored

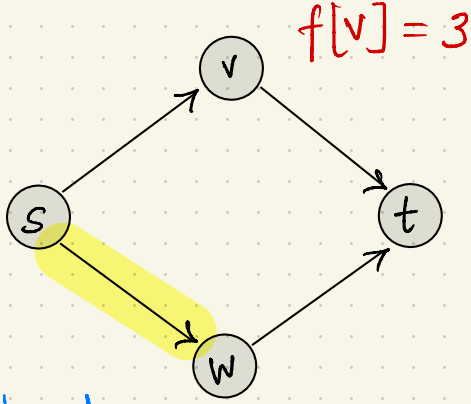
so skip it



f[v] = 3

f[t] = 4

TOPOLOGICAL ORDERING via DFS

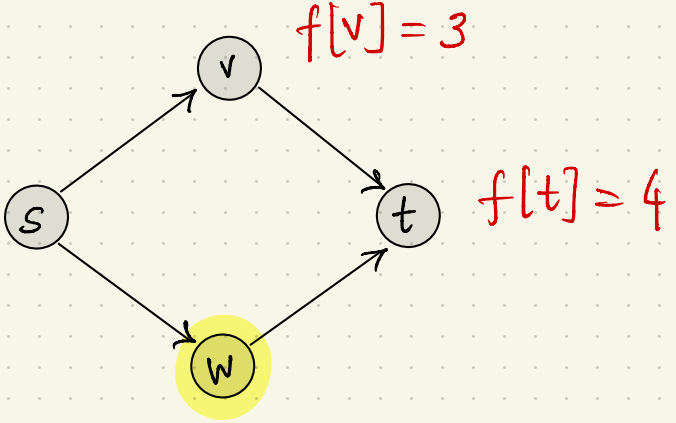


$f[v] = 3$

$f[t] = 4$

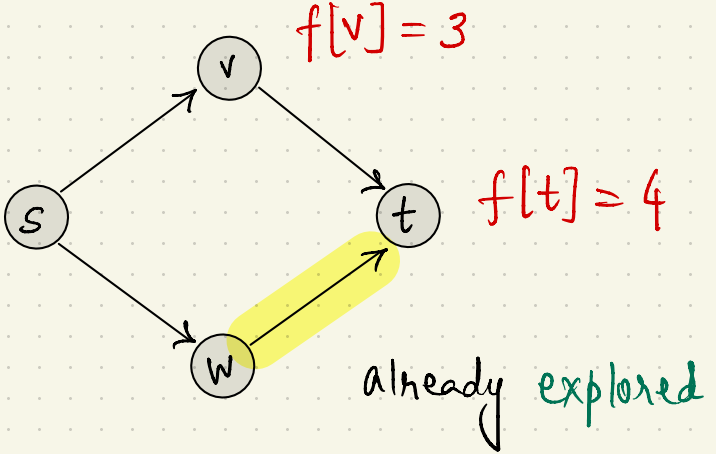
w is unexplored

TOPOLOGICAL ORDERING via DFS

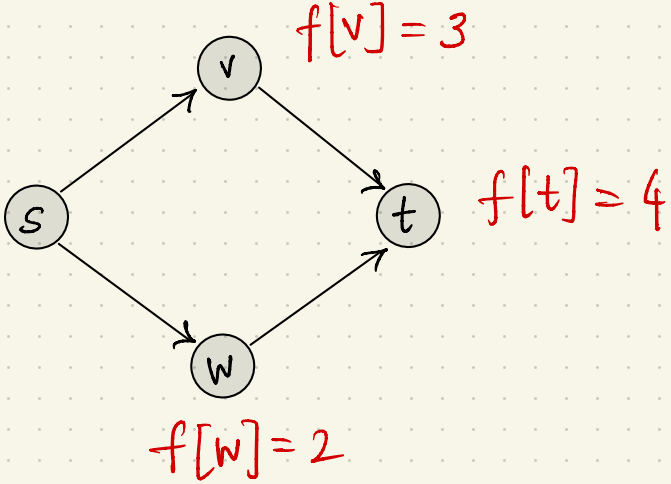


Call DFS on w

TOPOLOGICAL ORDERING via DFS

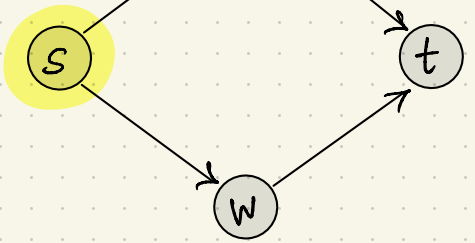


TOPOLOGICAL ORDERING via DFS



TOPOLOGICAL ORDERING via DFS

backtrack to s

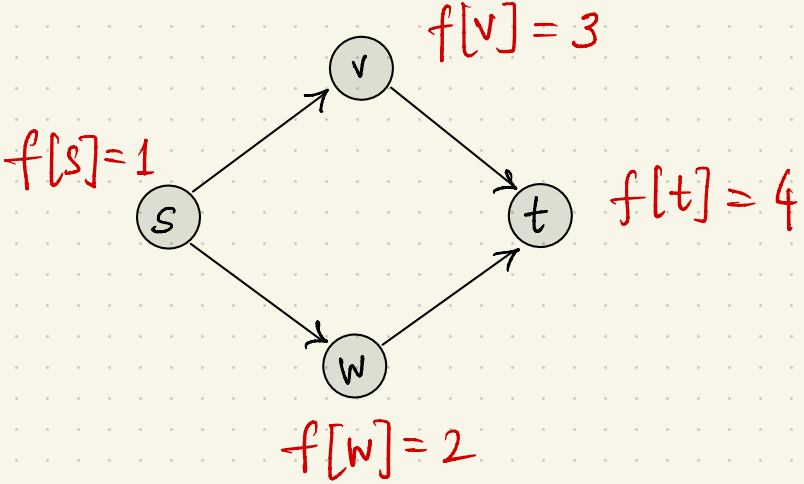


f[v] = 3

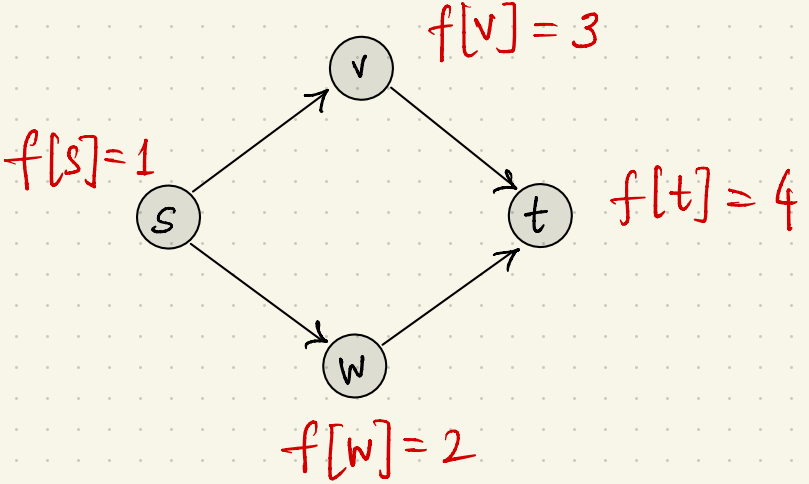
f[t] = 4

f[w] = 2

TOPOLOGICAL ORDERING via DFS



TOPOLOGICAL ORDERING via DFS



DFS - Loop runs out of vertices and stops!

TOPOLOGICAL ORDERING via DFS

Claim 1: DFS-Loop algorithm runs in $O(m+n)$ time.

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