COL351: Analysis and Design of Algorithms **Fall 2024**

Tutorial Sheet 5

Announced on: Aug 22 (Thurs)

Problems marked with ($\star$) will not be asked in the tutorial quiz.

1. You are given a directed graph $G = (V, E)$ in the adjacency list format. You need to return the reverse graph G^{rev} where all edges have been flipped. The graph G^{rev} should also be represented using adjacency lists. Design a linear time algorithm for this problem.

2. Design a linear-time algorithm to check if a given directed graph is strongly connected.

3. Consider a directed graph $G = (V, E)$ with nonnegative edge lengths and a starting vertex s. Define the *bottleneck* of a path to be the maximum length of one of its edges (as opposed to the sum of the lengths of its edges). Show how to modify Dijkstra's algorithm to compute, for each vertex $v \in V$, the smallest bottleneck of any $s - v$ path.

4. Consider a directed, weighted graph G where all edge weights are positive. You have one *voucher* that lets you traverse the edge of your choice for free (by changing its weight to zero). Design an $\mathcal{O}((|E| + |V|) \cdot \log |V|)$ time algorithm to find a minimum weight path between two vertices s and t using the voucher.

5. You are given a directed graph $G = (V, E)$ with a source vertex s and target vertex t. Each edge $e \in E$ has an associated probability $p_e \in [0, 1]$ of failure, and each edge fails independently of all others. Design an algorithm to find the most reliable path from s to t, that is, the one whose probability of failure is the least. Note that a path fails if any of the edges in the path fails. You may assume any operation with real numbers is $\mathcal{O}(1)$ time.

6. Can we solve the single-source *longest* path problem by changing minimum to maximum in Dijkstra's algorithm? If so, prove the correctness of your algorithm. If not, provide a counterexample.

7. Consider a directed graph in which the only negative edges are those that leave s; all other edges are positive. Can Dijkstra's algorithm, starting at s, fail on such a graph? Prove your answer.

8. ($\star$) Alice and Bob are taking a road trip from City X to City Y. They decide to switch the driving duties at each rest stop they visit; however, because Alice has a better sense of direction than Bob, she should be driving both when they depart and when they arrive (to navigate the city streets).

   Given a weighted undirected graph $G = (V, E, w)$ with positive edge weights, where the vertices represent rest stops and the edges represent routes between rest stops, devise a polynomial-time algorithm to find a route (if possible) of minimum distance between City X and City Y such that Alice and Bob alternate edges and Alice drives the first and last edge.

9. ($\star$) A *tournament* is a directed graph $G = (V, E)$ such that for all $u, v \in V$, exactly one of (u, v) or (v, u) is in E.

   a) Show that every tournament has a *Hamiltonian path*, i.e., a path that visits every vertex exactly once.

   b) Design a polynomial-time algorithm to find this path.