COL 351 : Analysis & Design of Algorithms

Lecture 17

Greedy Algorithms III : Huffman Coding (Contd.)

Sept 03, 2024 | Rohit Vaish

BINARY CODES

Alphabet Σ : finite nonempty set of symbols

e.g., $\Sigma = \{A, B, \ldots, Z\}$, $\Sigma = \{\bullet, *\}$, etc.

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e.g., A = 00000

B = 00001

C = 00010

⋮

fixed length binary codes

VARIABLE LENGTH CODES

When some symbols of the alphabet occur more frequently than others

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When some symbols of the alphabet occur more frequently than others

e.g.,

	fixed length		variable length
A	00	A	0
B	01	B	01
C	10	C	10
D	11	D	1

VARIABLE LENGTH CODES

When some symbols of the alphabet occur more frequently than others

e.g.,

	fixed length		variable length	
A	00	A	0	Ambiguous ☹
B	01	B	01	How to interpret 001?
C	10	C	10	AB ? AAD ?
D	11	D	1	

PREFIX-FREE CODES

For each pair of distinct symbols $i, j \in \Sigma$,

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e.g., 10 110 0 110

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e.g., 1 0 1 1 0 0 1 1 0

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symbol	frequency	fixed length	variable length (prefix-free)
A	60%	00	0
B	25%	01	10
C	10%	10	110
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On average:		2 bits/symbol	1.55 bits/symbol

more efficient

$= 0.6 \times 1 + 0.25 \times 2 + 0.15 \times 3$

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How to compute an optimal prefix-free code ?

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brute force : ?

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brute force: at least $n!$

n symbols encoded as

0, 10, 110, 1110, ...

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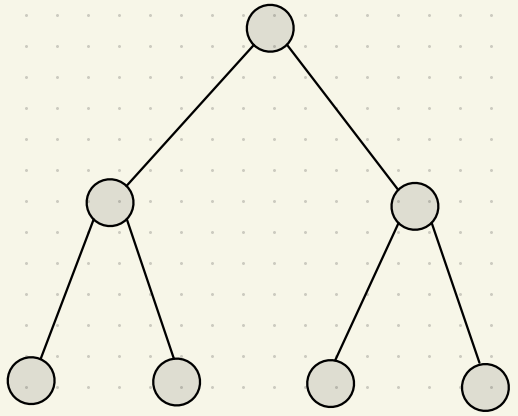

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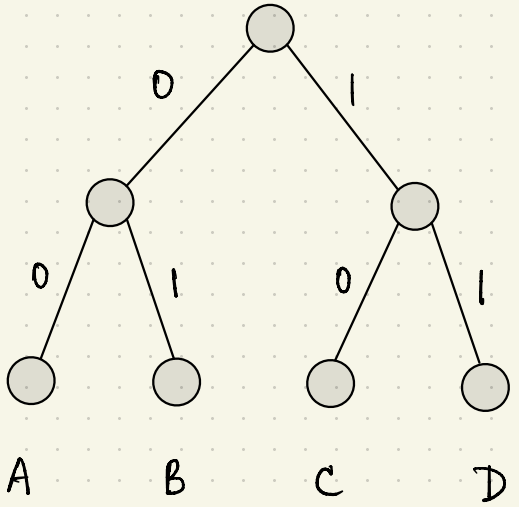

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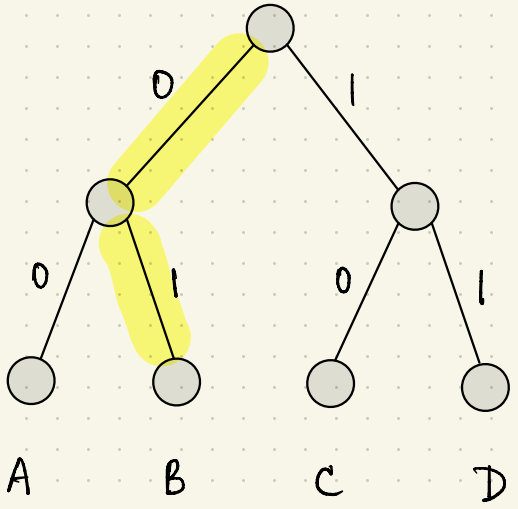

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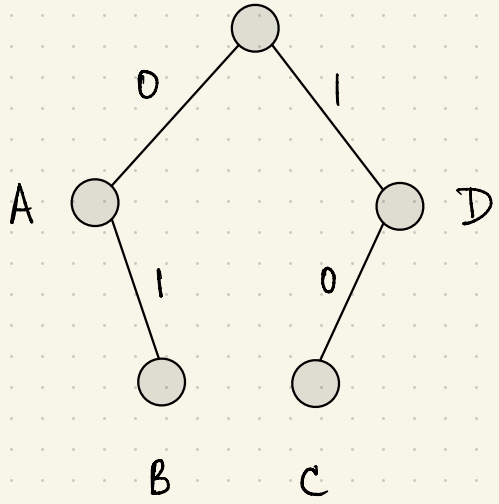

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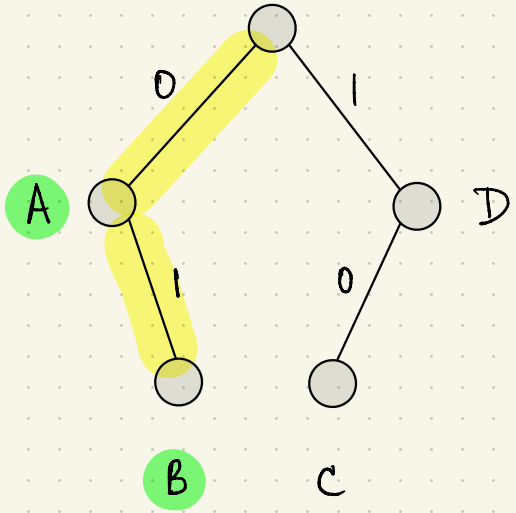

not prefix-free

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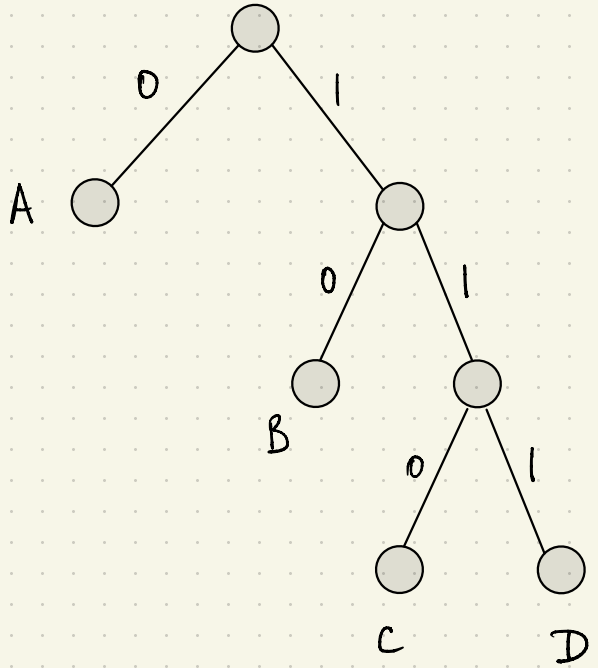

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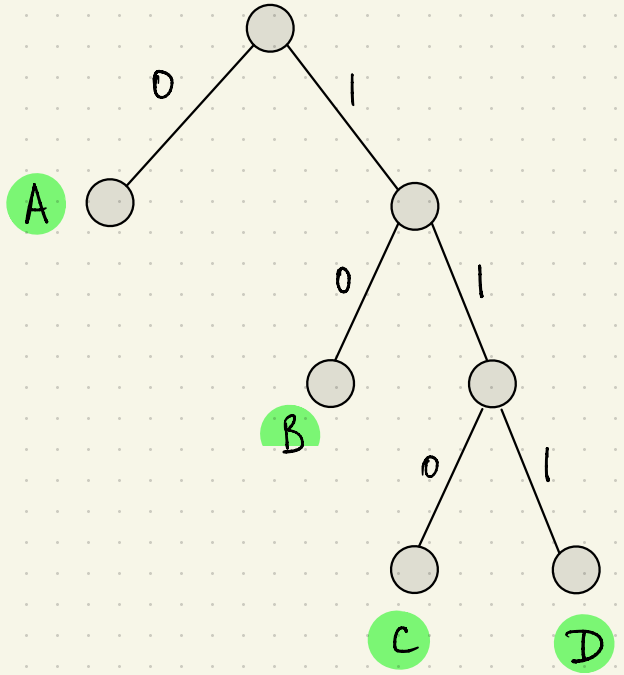

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BINARY CODES ⟷ LABELED BINARY TREES

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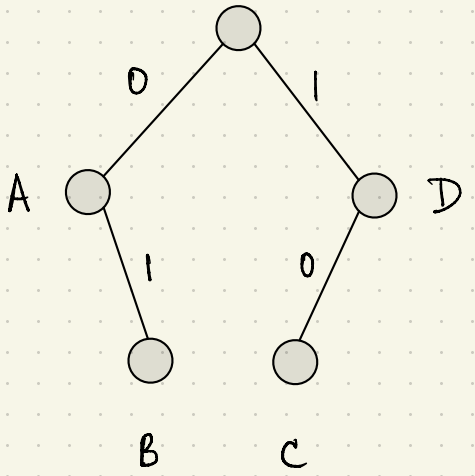
⟷



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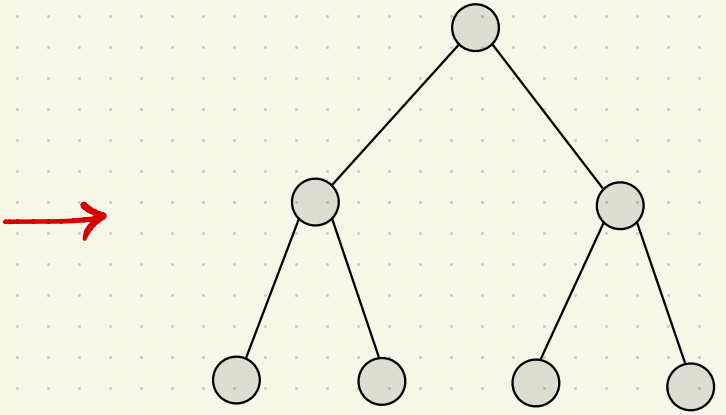

Depth = max # bits used for any symbol

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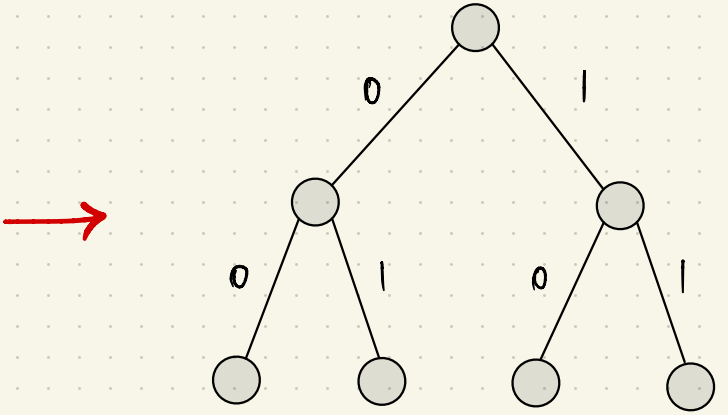

Left child : 0

Right child : 1

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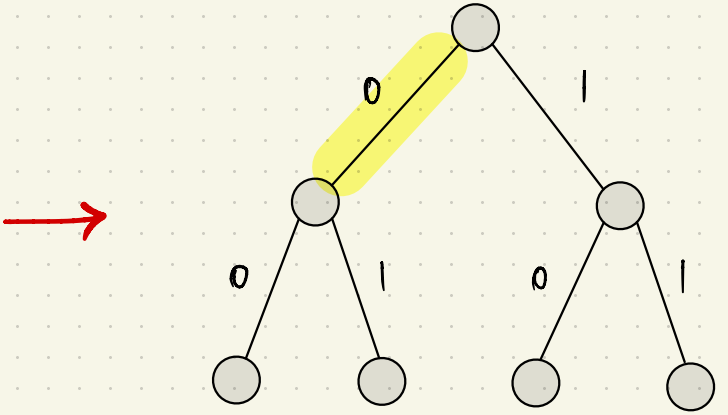

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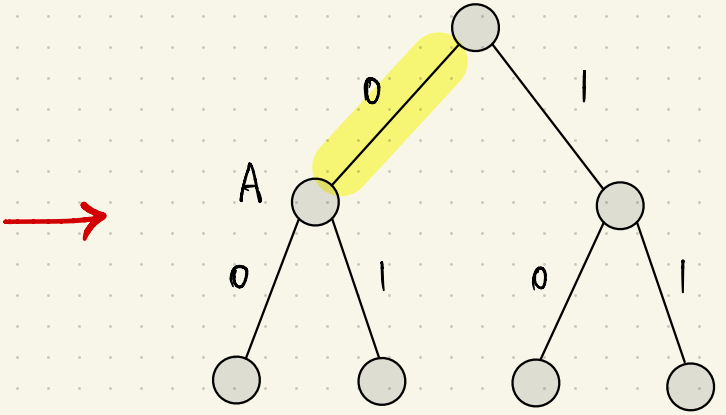

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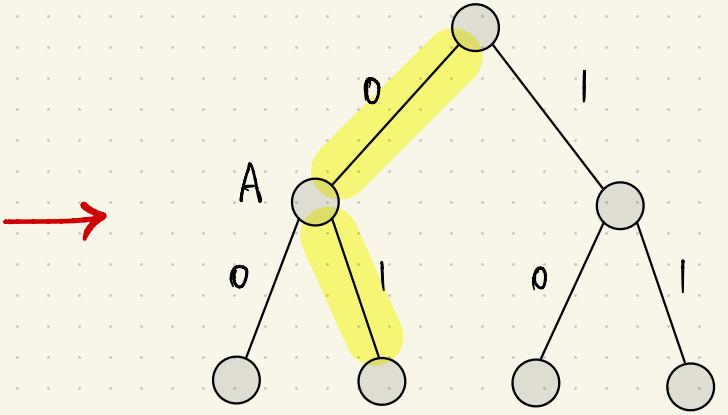

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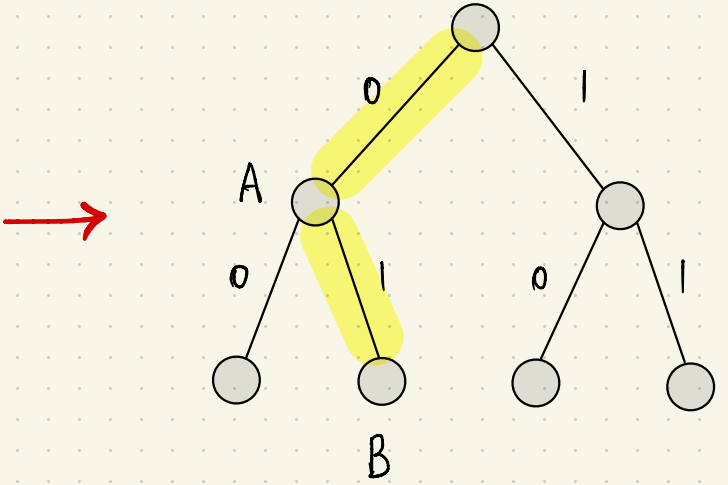

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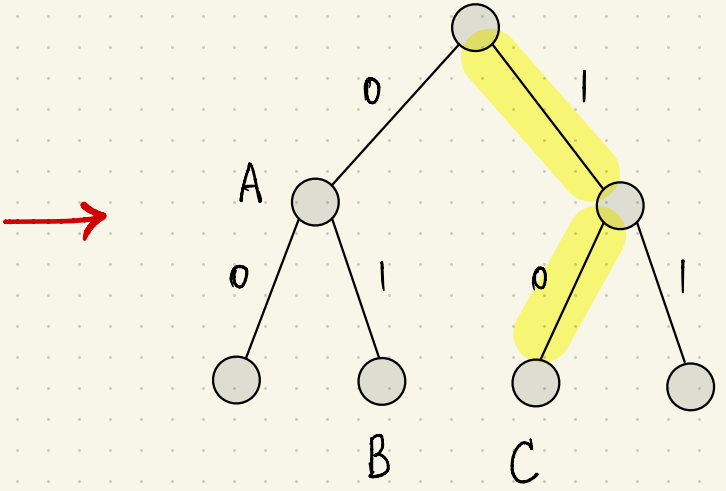

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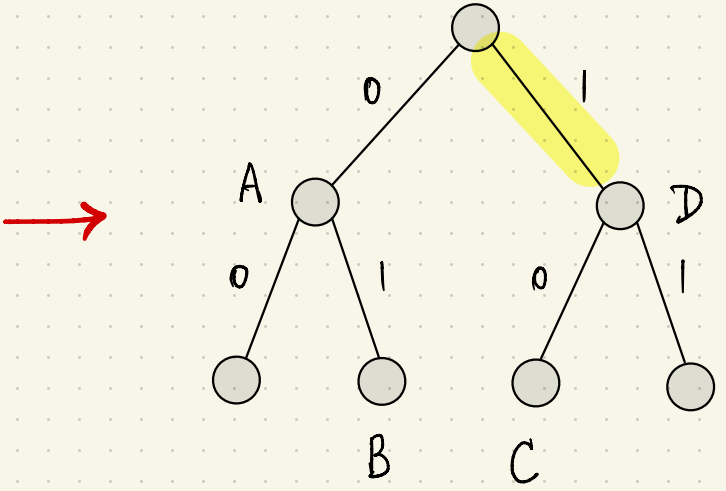

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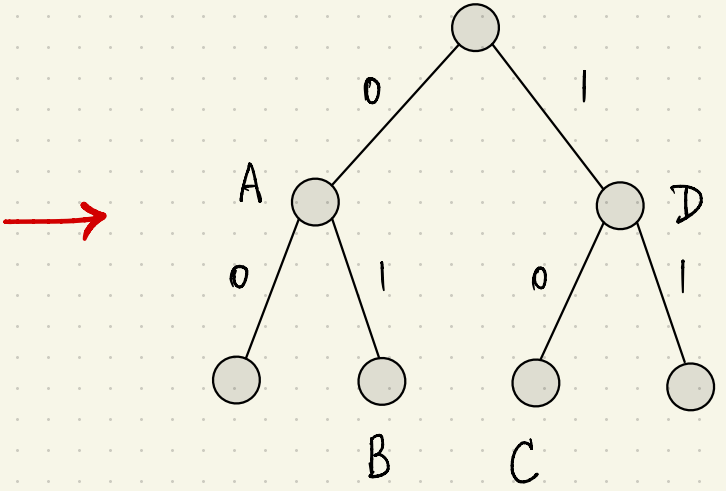

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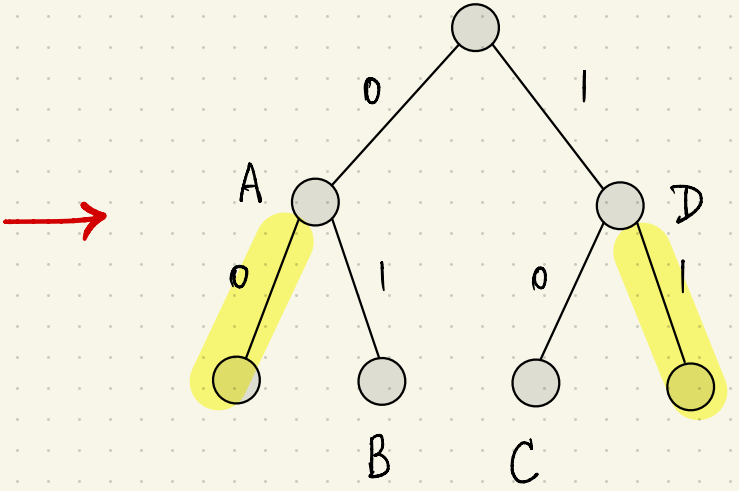

Repeatedly prune unlabeled leaves

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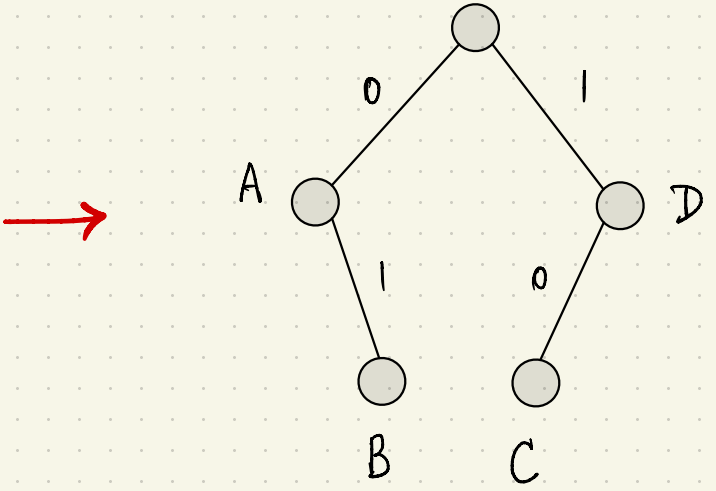

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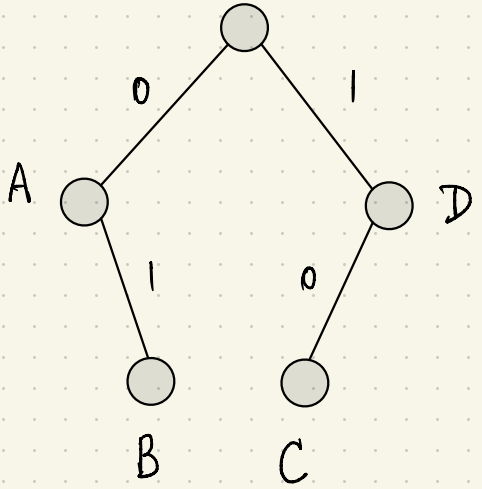

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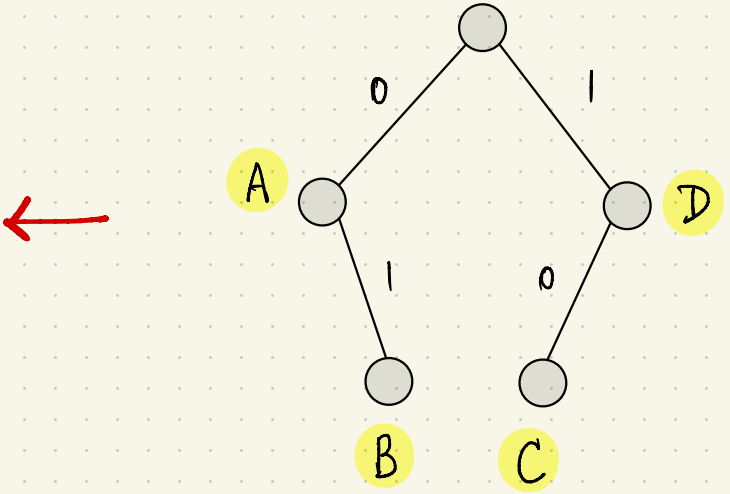

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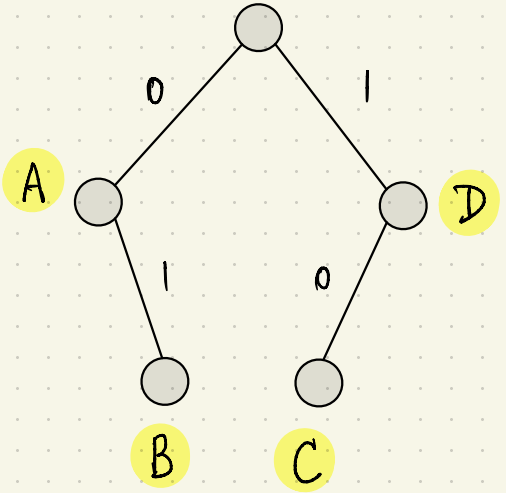

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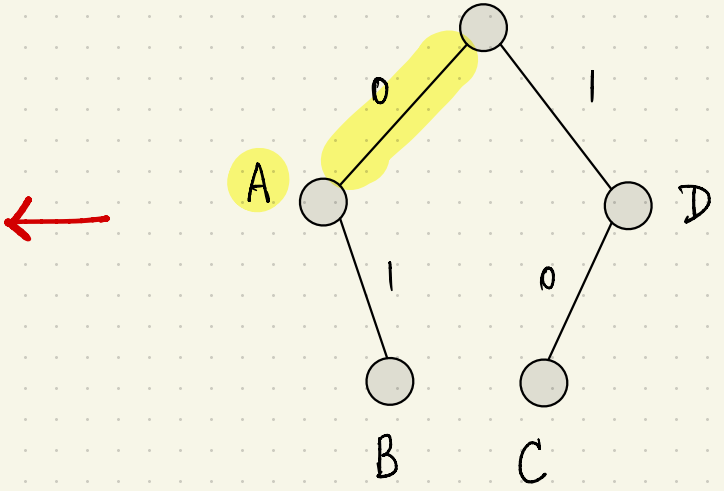

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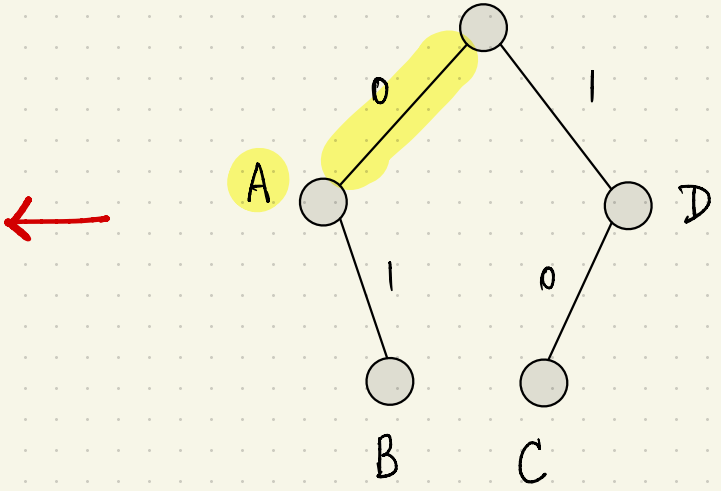

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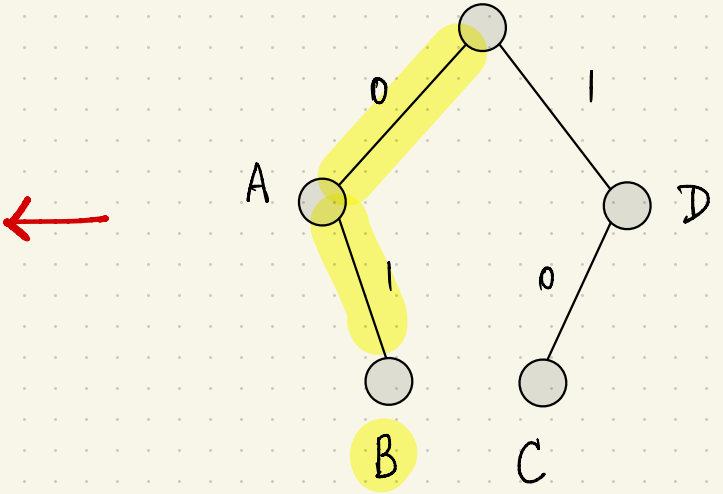

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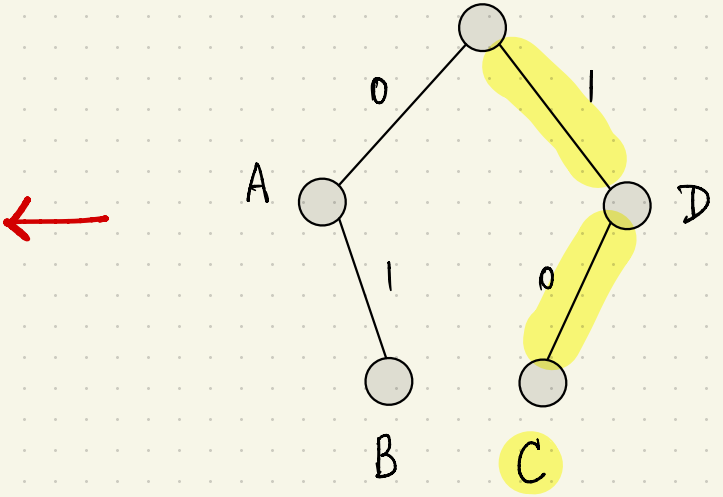
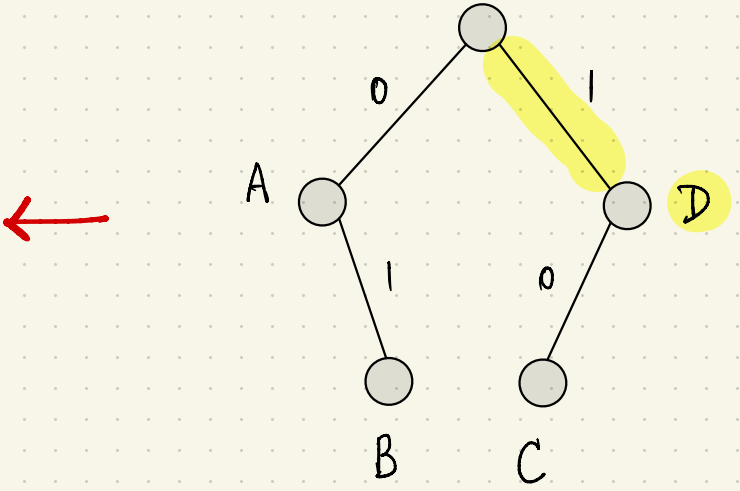
C	10
D	



BINARY CODES ⟷ LABELED BINARY TREES

Symbol	variable length
A	0
B	01
C	10
D	1



BINARY CODES ⟷ LABELED BINARY TREES

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A	0
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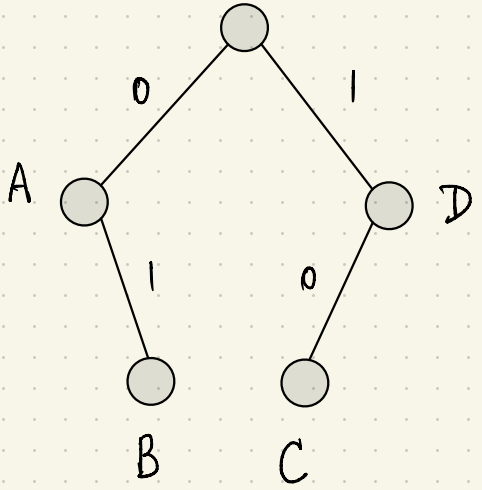
⟷



BINARY CODES ⟷ LABELED BINARY TREES

* Left child "0", right child "1"
* for each $i \in \Sigma$, exactly one node labeled "i"
* encoding of $i \in \Sigma$: bits along the path from root to node "i".

PREFIX-FREE CODES AS TREES

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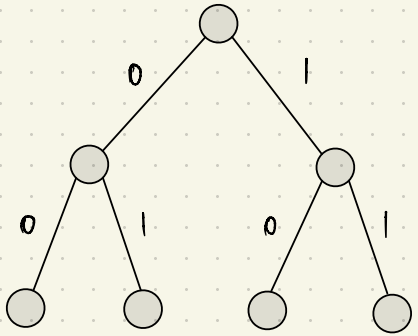
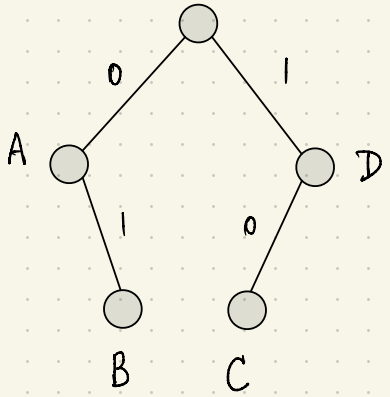
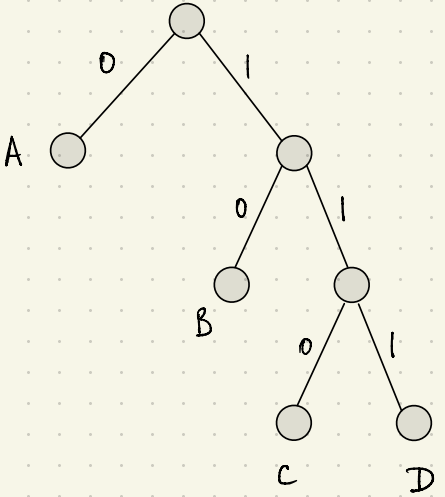


prefix-free



not prefix-free



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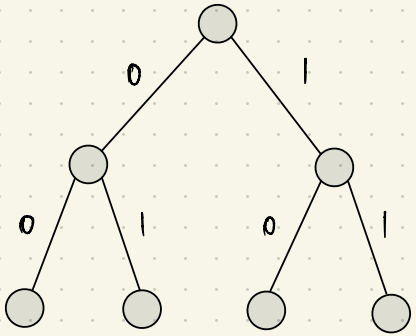

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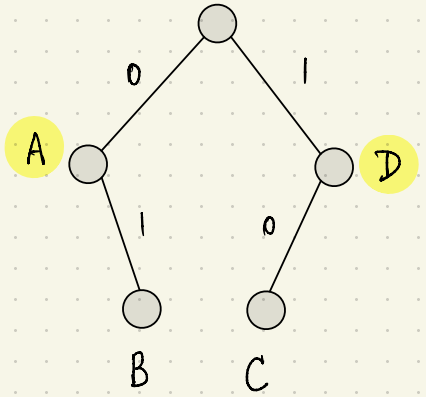

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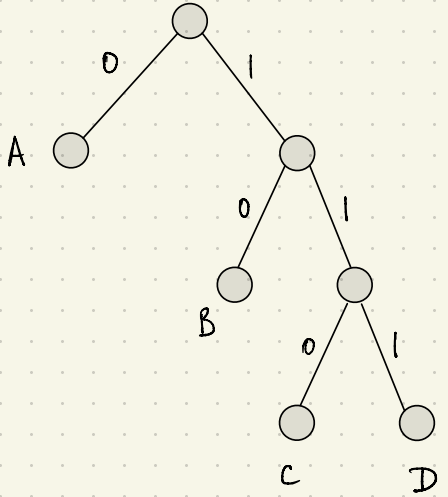

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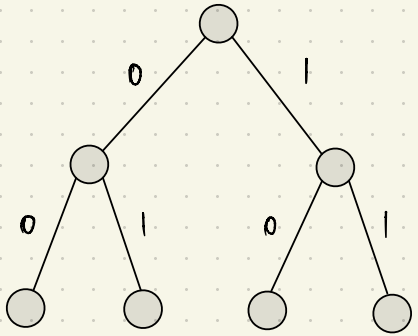

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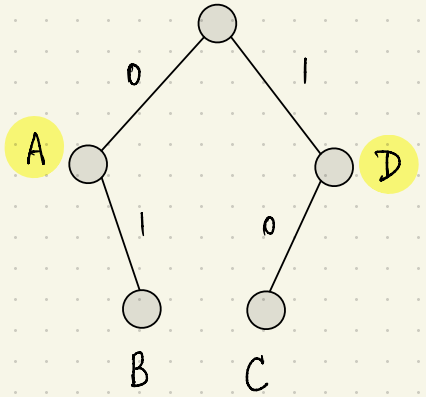

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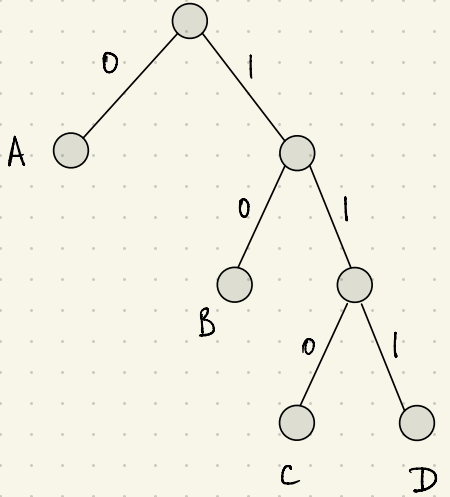

prefix-free

symbol i prefix of symbol j $\iff$ node i ancestor of node j

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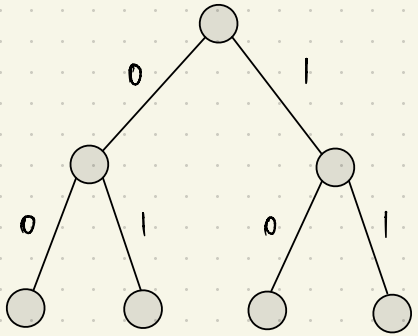

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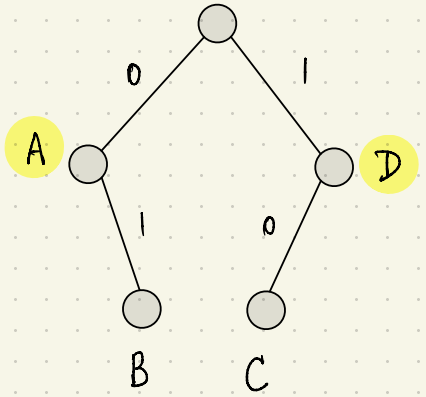

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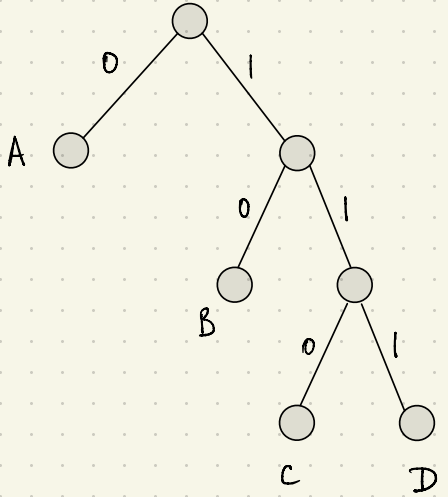


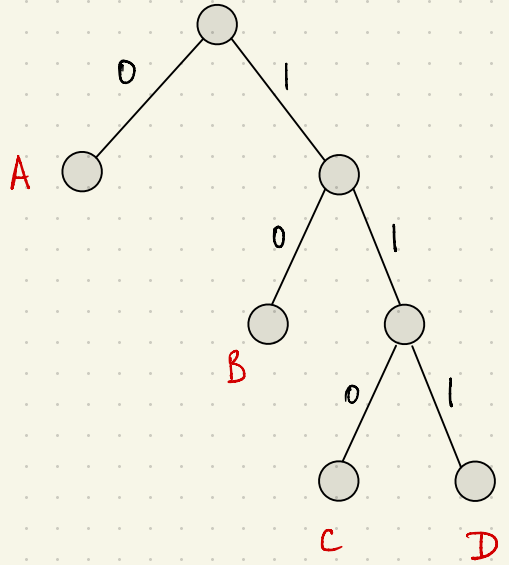
prefix-free

symbol i prefix of symbol j $\iff$ node i ancestor of node j

prefix-free code $\iff$ only leaves are labeled

PREFIX-FREE CODES AS TREES

* Decoding is easy! e.g., 0 1 1 0 1 1 1



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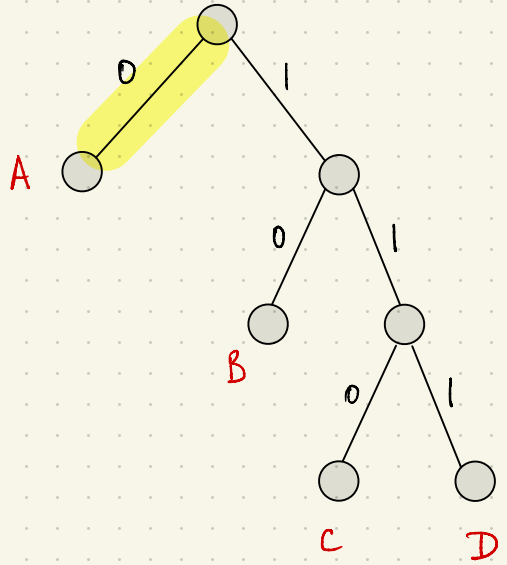

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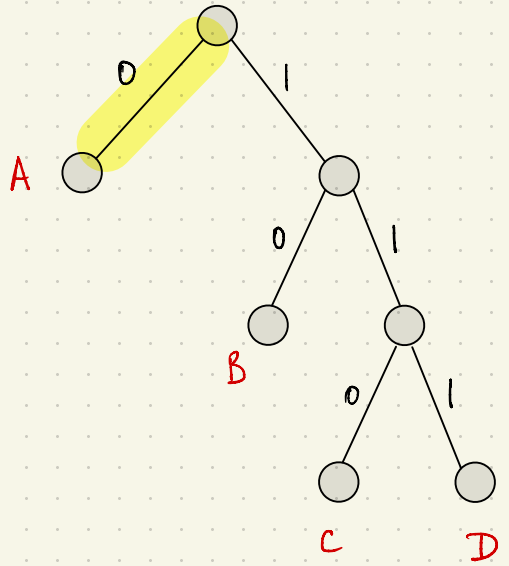
A



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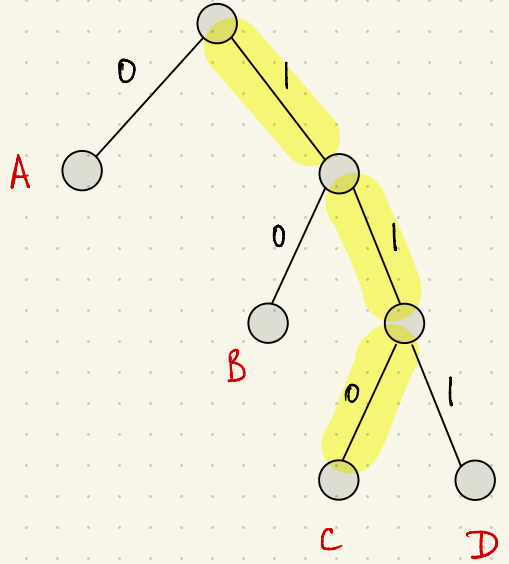

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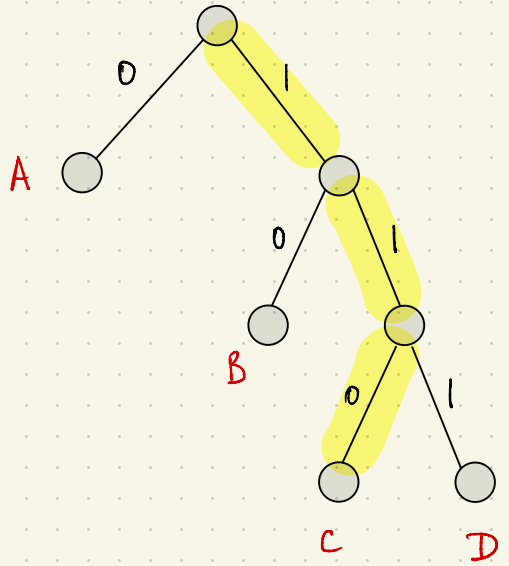
A C



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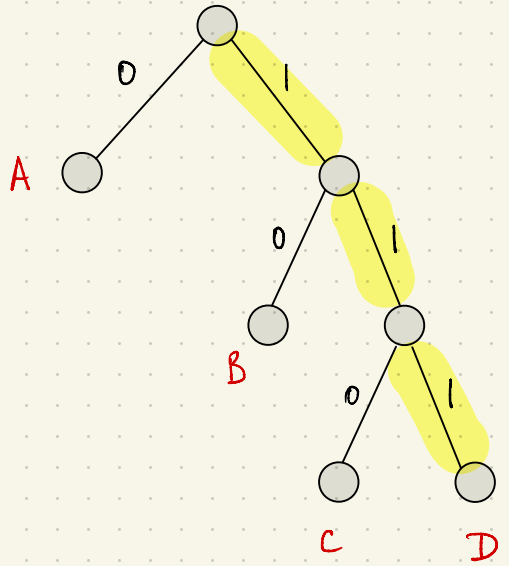
A C D



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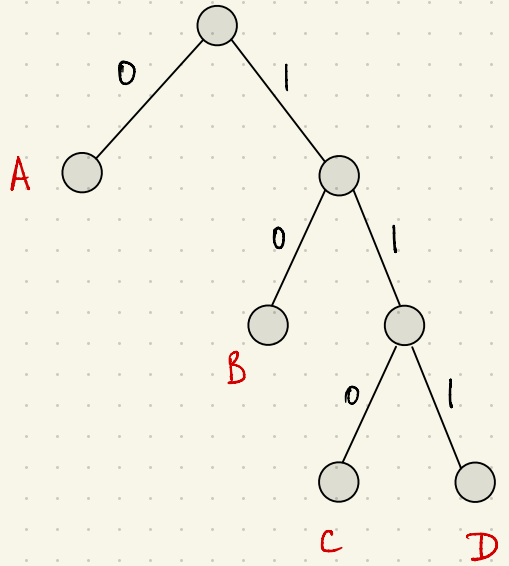

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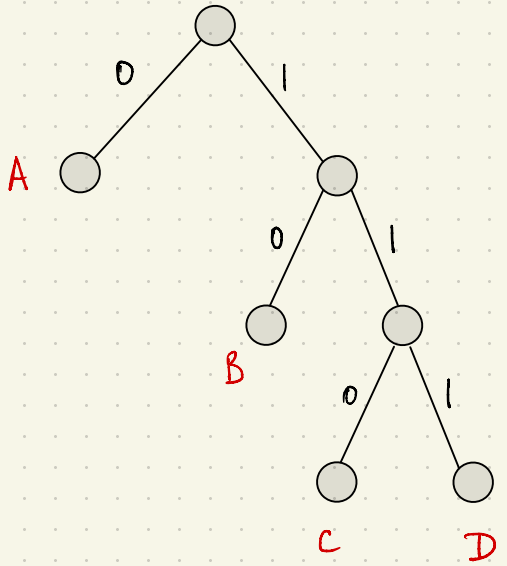
* Encoding length of symbol $i \in \Sigma$
  = depth of leaf labeled i in the tree



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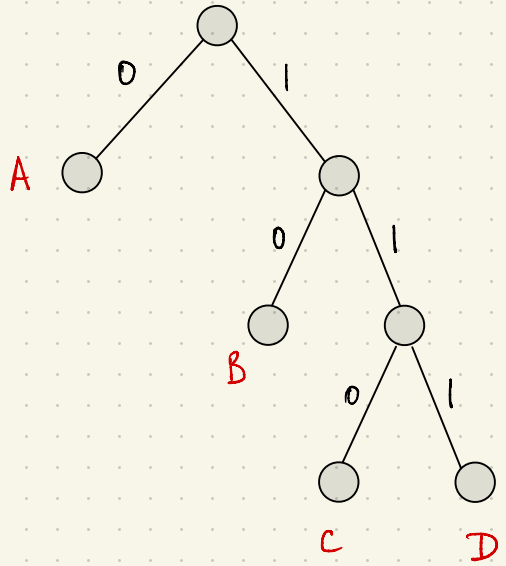

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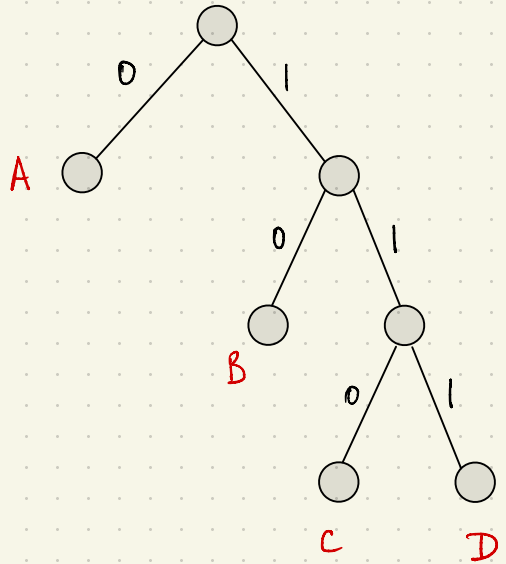
* For any Σ, some Σ-tree always exists
  e.g., $\{0, 10, 110, 1110, \ldots\}$



Optimal Prefix-free codes

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input: a nonnegative frequency p_i for each symbol i of alphabet Σ of size $|\Sigma| = n \geq 2$

output: A Σ-tree T with minimum possible average leaf depth

$$L(T) := \sum_{i \in \Sigma} p_i \cdot \text{depth of leaf labeled } i \text{ in } T$$

Optimal Prefix-free codes

symbol	frequency	fixed length	variable length
A	60%	00	0
B	25%	01	10
C	10%	10	110
D	5%	11	111

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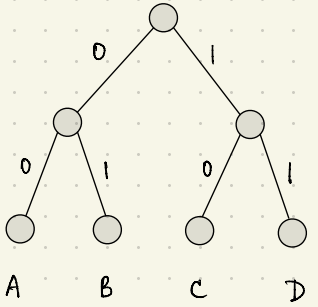

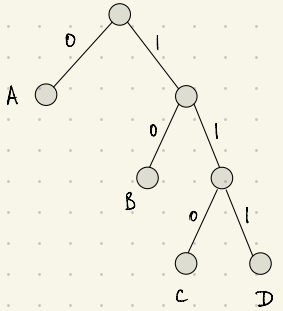

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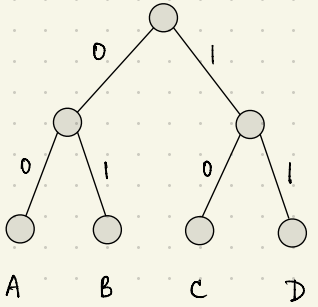

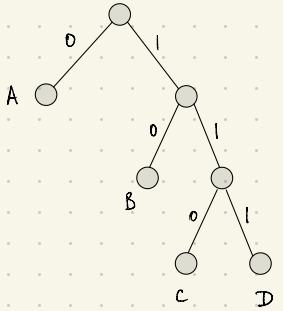

Average leaf depth : 2 1.55

$= 0.6 \times 1 + 0.25 \times 2 + 0.15 \times 3$

BUILDING A TREE

What's a principled approach for building a Σ-tree?

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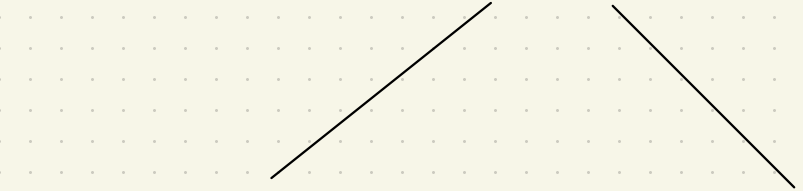

Top down

(divide and conquer)

Bottom up

(greedy)

TOP DOWN APPROACH

[Shannon - Fano encoding]

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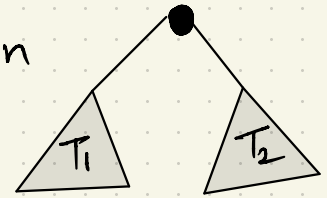

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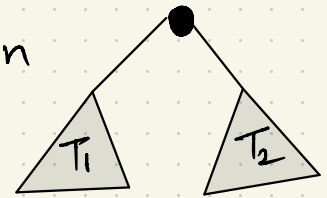

e.g., $\Sigma = \{ a \quad b \quad c \quad d \quad e \}$

9% 41% 35% 5% 10%

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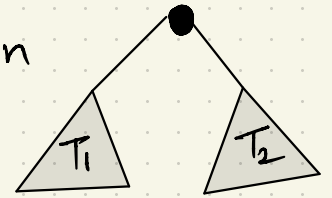

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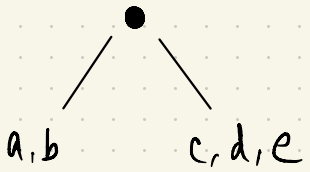

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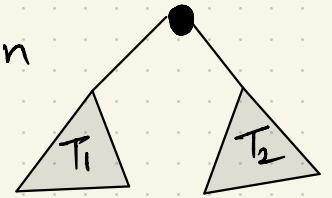

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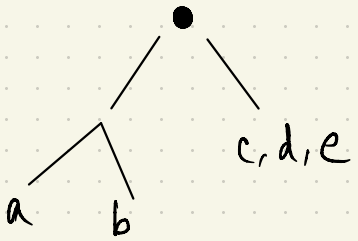

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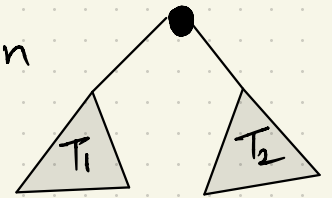

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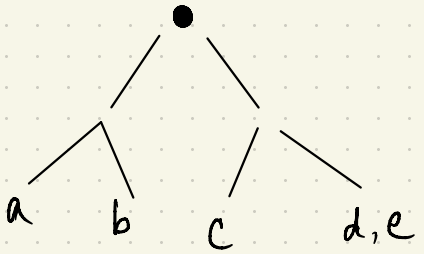

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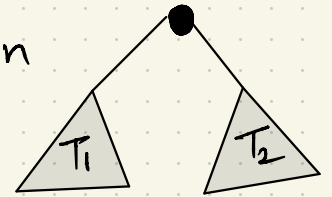

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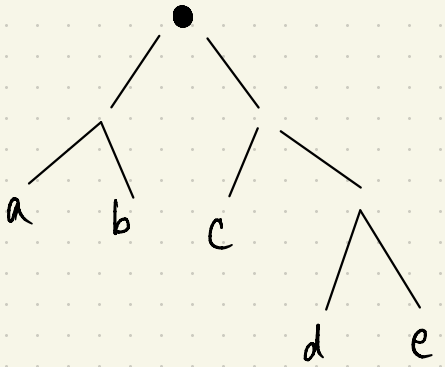

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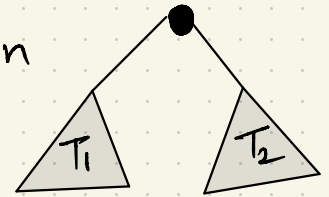
* partition Σ into Σ_1, Σ_2 each with $\approx 50\%$ of total frequency
* recursively compute T_1 for Σ_1, T_2 for Σ_2, return



* Suboptimal ☹ [Tutorial sheet 6]