Mathematics Analysis and Approaches SL

Exploring Barycentric Coordinates to solve difficult Olympiad Problems

Course: MCV4U7-1

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Contents

1 Introduction

1.1 Rationale

Why would anyone want to solve such a problem? As someone with great exposure to math-contest intensive environments, I am passionate about taking on complex and rigorous problems. I am invested in the field, and I channel my dedication in pursuing prestigious contests such as the Olympiads. Elevating my mathematical skill has my top priority for the past 2 years, having been an active goal long before enter the IB programme.

During my studies in IB, I was exposed to basic forms of coordinate systems, such as Polar coordinates or Cartesian coordinates. While tackling contest problems, I was surprised to discover that most could not be trivially solved with these coordinates, and necessitated the introduction of coordinates of a more sophisticated construction. In addition, a second lemma for why I chose this topic is to fight the stigma behind mathematical Olympiads as a whole: The resources for these higher level contests are not "self-learning" friendly or not made for those who are a bit slower than the geniuses that use them in the Olympiads. But this displaces a majority who are still highly interested and willing to pursue the rough journey of math competitions. As one born with mediocre intuition and problem-solving skills, I still entertain great dreams; I hope this paper can be more than an IA - to serve as a guide to the eager but novice reader, providing an opportunity to learn the beautiful but computation-intensive topic of Barycentric Coordinates.

1.2 Aim

The purpose of my IA is to build up concepts necessary to explore Barycentric Coordinates with rigor, and adapt them to various math contest problems to demonstrate their advantages over other coordinate systems and their efficiency in contest environments.

2 Preliminary Mathematics and Notation

Before diving deeper into Barycentric Coordinates a few prerequisites are required. In this section, I reference various ideas in geometry and linear algebra to help support the construction of Barycentric Coordinates later in this text. Additionally, in the following sections, I will build up notation that will be frequently referenced in later parts.

2.1 Reference Triangles

During the discussion of Barycentric Coordinates, sometimes the construction of a reference triangle is useful. In this section, I will discuss various aspects of the reference triangle. Let this reference triangle be denoted by $\triangle ABC$, with a representing side BC, b representing the side AC, c representing the side AB, and shortening $\angle BAC$ to $\angle A$, $\angle CBA$ to $\angle B$, and $\angle ACB$ to $\angle C$.

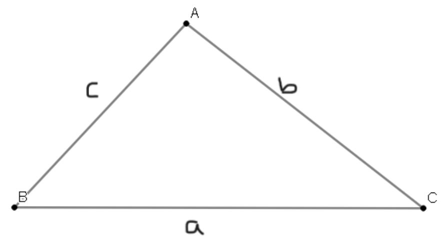

Figure 1: Standard Reference Triangle

I define non-degenerate triangles as a triangle with non-zero area. In contrast, a degenerate triangle is simply a triangle with zero area. In this limiting case, a **degenerate** triangle is just a straight line.
Additionally, in this paper $[P_1P_2...P_n]$, will be synonymous with the **signed**-area of the polygon $P_1P_2...P_n$. What this means that the orientation of the points determine if the value of the area is positive or negative. To be specific, the area of $[ABC]$ is positive if the vertices A, B, C are oriented in counterclockwise order as seen below.

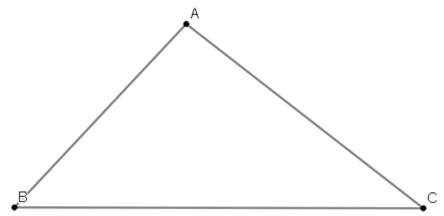

Figure 2: $[ABC]$ is positive. $[ACB]$ is negative,etc..

In contrast, its negative if it is clockwise (Like $[ACB]$ in Figure 2). This will be extremely important to include the scope of **negative** coordinates to our Barycentric Coordinate System. In addition, it is also important to brush upon some terminology regarding symmetrical results, which is significant in of itself because it helps to include some simplification in the overall bashfulness of this coordinate system throughout the paper. Namely, when one says **without the loss of generality**, it means that the proposition given, does not simply apply for the current specific case being considered, but it is also valid for all general cases. If these terminologies or notations are not used, then appropriate context will be used to subsidize these.

2.2 Vectors

The bulk of Barycentric Coordinates lies heavily on linear algebra, and besides the knowledge learned from the curriculum, there are a few things you need to be familiar with. In a heavily watered-down sense, Barycentric Coordinates are useful for its ability to relate seemingly unrelated points to the sides of the reference triangle. Therefore, connecting vectors to lengths is an incentive to pursue in this internal assessment.
Moving on, extra knowledge that you need to be aware, is that when I reference a vector $\vec{A}$, with only one endpoint, this usually means the tail of the vector is at the origin of the Cartesian plane.

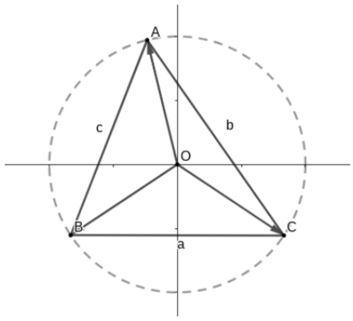

Figure 3: Origin at the Circumcenter of the Triangle

Now, like previously mentioned, Barycentric Coordinates are often associated with triangles, and usually the location of the origin is set at the circumcenter. The reasoning for this is that the circumcenter is is equidistant from all vertices of the triangle, like seen in Figure 3. So, defining R as the circumradius:

$$|\vec{A}| = |\vec{B}| = |\vec{C}| = R$$

From this with the dot product and without the loss of generality, I get the corollary, as the angle between the angle itself is 0:

$$\begin{aligned}\vec{A} \cdot \vec{A} &= |\vec{A}||\vec{A}|\cos(0) \\ &= |\vec{A}|^2(1) \\ &= R^2\end{aligned}$$

$$\boxed{\therefore \vec{A} \cdot \vec{A} = R^2}$$

Additionally, what about $\vec{A} \cdot \vec{B}$, well I need two things: the magnitudes of the vectors, which is simply R and the angle between them, $\angle AOB = 2C$ (Since $\angle AOB = 2 \times \angle ACB = 2C$). Thus, without the loss of generality:

$$\begin{aligned}\vec{A} \cdot \vec{B} &= |\vec{A}||\vec{B}|\cos 2C \\ &= R^2(1 - 2\sin^2 C)\end{aligned}$$

Now using the extended Law of Sines:

$$\begin{aligned}\frac{c}{\sin C} &= 2R \\ c &= 2R(\sin C) \\ \sin C &= \frac{c}{2R}\end{aligned}$$

Thus substituting this value back in to the former equation:

$$\vec{A}\cdot\vec{B} = R^2\left(1-2\left(\frac{c}{2R}\right)^2\right)$$
$$= R^2(1-\frac{2c^2}{4R^2})$$
$$= R^2 - \frac{1}{2}c^2$$

Similarly, $\vec{B}\cdot\vec{C} = R^2 - \frac{1}{2}a^2$, and $\vec{C}\cdot\vec{A} = R^2 - \frac{1}{2}b^2$.

These properties are useful for later constructing the distance formula while also being demonstrating the nice tool of translation for problem-solving!

2.3 Determinants

The notion of a determinant is imperative to Barycentric Coordinates. For any matrix, A, the determinant of A, called $detA$, or $|A|$, is a special **scalar** value associated with the peculiar matrix.

For a 2×2 matrix called A:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

its the determinant yields,

$$|A| = ad - bc$$

For a 3×3 matrix called B:

$$B = \begin{bmatrix} x & y & z \\ a & b & c \\ d & e & f \end{bmatrix}$$

the determinant yields,

$$|B| = x\begin{vmatrix} b & c \\ e & f \end{vmatrix} + y\begin{vmatrix} a & c \\ d & f \end{vmatrix} + z\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

Which can be simplified even more although although messily:

$$|B| = x(bf-ec) + y(dc-af) + z(ae-bd)$$

. Knowing how to calculate the determinant is great, but at this point, you might be asking: ***But what is a determinant?***

Well...it can mean many things, but the only meaning of the determinant that is necessary in this IA is that it can be represented by, for a 3×3 matrix: the volume of the parallelpiped, a distorted rectangular-prism in which all the faces are parallelograms, formed by the point vectors described from the matrix. Thus the parallelpiped formed by the matrix:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

is represented by the diagram sourced from Geogebra:

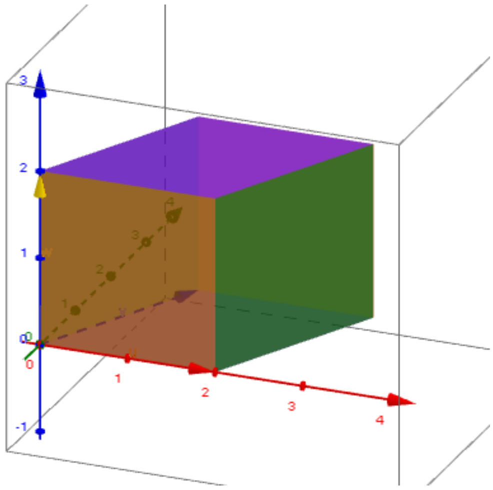

Figure 4: Visual Representation of a parallelpiped from Geogebra

Therefore, the volume of this parallelpiped is represented by:

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 2 \times (2 \times 2 - 0 \times 0) + 0 \times (0 \times 2 - 0 \times 1) + 0 \times (0 \times 0 - 2 \times 1)$$
$$= 8$$

Note that although the determinant can be represented by the volume of a parallelpiped, it is not **actually** the volume. This can be seen by noting that determinants have no units, but explaining why and what the true precise definition of a determinant will be out of the scope of the paper. The only thing you need to know is that **the determinant can be represented by the volume of the parallelpiped formed by the point vectors shown through the matrix.**

3 Barycentric Coordinates

Intuitively, the Barycentric Coordinate System is a type of coordinate system in which a point is determined in reference to a shape, which is usually a triangle – kind of like how points in a Cartesian coordinate system are determined with respect to the axes.

In a more technical and precise sense: without the loss of generality, I fix a non-degenerate triangle $\triangle ABC$ to be used as the reference triangle for our Barycentric Coordinates.
Now then, there are many ways to define points in Barycentric Coordinates; Lets see the first one:

Definition 1. *Each Point P in Barycentric Coordinates are represented by 3 values such that, $P = (x, y, z)$ where $x, y, z \in \mathbb{R}$ and:*

$$\vec{P} = x\vec{A} + y\vec{B} + z\vec{C} \text{ and } x + y + z = 1.$$

From this, one can reason to themselves of this rigor **vector** definition.
If it helps, it can also be considered as assigning weights to the point vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, to describe the vector $\vec{P}$.
But for the sake of understanding and simplicity, it might be more convenient to interpret Barycentric Coordinates by its **area** definition. What this means, is that for the point P, the coordinates can be determined by the areas of the triangles formed by the sides and the point. So, without the loss of generality, of a point P.

Definition 2. *For each point P, $[PBC] = x[ABC]$ and $[PCA] = y[ABC]$, and $[PAB] = z[ABC]$, where $x, y, z \in \mathbb{R}$ so:*

$$P = (x, y, z) = (\frac{[PBC]}{[ABC]}, \frac{[PCA]}{[ABC]}, \frac{[PAB]}{[ABC]}) = \frac{1}{[ABC]}([PBC], [PCA], [PAB])$$

$$\text{and } x + y + z = 1$$

So basically assigning 3 individual weights to the total area of the reference triangle.
To remain focused on the goal of this Math IA and to be wary of the length, it will be assumed without proof that these two definitions are synonymous with each other.
Lean on either in order to help understand the concepts to be presented in the paper. They will be both be useful.
Namely, I can get the Barycentric Coordinates of the vertices of the reference triangle:

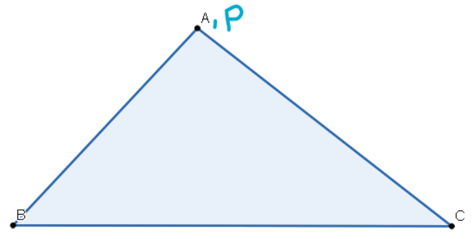

Figure 5: Point P is at the same position as the vertex A

It can be seen that as point P and A share a point $\triangle BPC \cong \triangle ABC$, and $\triangle APB$ and $\triangle APC$ are degenerate triangles so:

$$P = (\frac{[PBC]}{[ABC]}, \frac{[PCA]}{[ABC]}, \frac{[PAB]}{[ABC]})$$
$$= (\frac{[ABC]}{[ABC]}, \frac{0}{[ABC]}, \frac{0}{[ABC]})$$
$$= (1, 0, 0)$$

Through similar means it can be found that $B = (0, 1, 0)$ and $C = (0, 0, 1)$.
The simple-form of the special points of the triangle highlights one the strength of Barycentric Coordinates.

3.1 Area

The **heart** of Barycentric Coordinates, or more formally, the most fundamental concept most of the applicable tools in Barycentric Coordinates are built on is the area formula of a triangle. I will showcase the latter, prove it, then show the consequences of this theorem.

Theorem 1. *Let points P, Q, R, denote points with Barycentric Coordinates: $P = (x, y, z)$, $Q = (a, b, c)$, $R = (d, e, f)$, then the signed area of $\triangle PQR$ is found through the determinant*

$$\frac{[PQR]}{[ABC]} = \begin{vmatrix} x & y & z \\ a & b & c \\ d & e & f \end{vmatrix}$$

Proof. Let point J be a point not in the plane of $\triangle ABC$. And now, if I forego Barycentric Coordinates for one second, I now have a 3-dimesional coordinate system with 3 axes, such that $J = (0, 0, 0)$, $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$, where these axes are not necessarily perpendicular with each other. One can now intuitively argue that the coordinates of any point in the plane of $\triangle ABC$, whose coordinates sum up to 1, correspond to the same point in Barycentric Coordinates that has $\triangle ABC$ as reference.

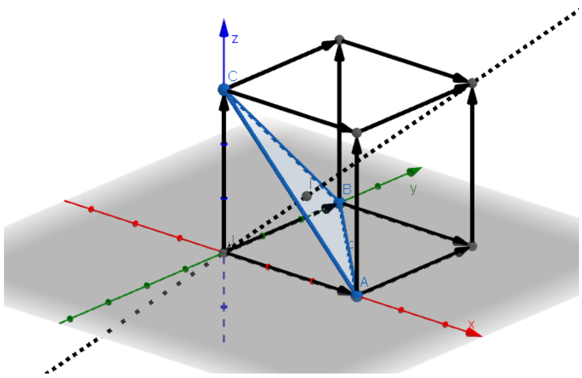

Figure 6: The plane of $\triangle ABC$ is highlighted in blue.

Now, define 3 points $P = (x, y, z)$, $Q = (a, b, c)$, $R = (d, e, f)$, that lie on the $\triangle ABC$ plane, to form the general triangle. Thus let PV_{PQR} denote the volume of the parallelpiped formed by the point vectors of the points P, Q, R

from the origin J. And similarly for PV_{ABC}:

$$\frac{PV_{PQR}}{PV_{ABC}} = \frac{\begin{vmatrix} x & y & z \\ a & b & c \\ d & e & f \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}} = \begin{vmatrix} x & y & z \\ a & b & c \\ d & e & f \end{vmatrix}$$

Now all I have is the volume of the parallelpiped , PV_{PQR}. How could I relate that to the area of the triangles $\triangle PQR$ and $\triangle ABC$ found in the area formula?

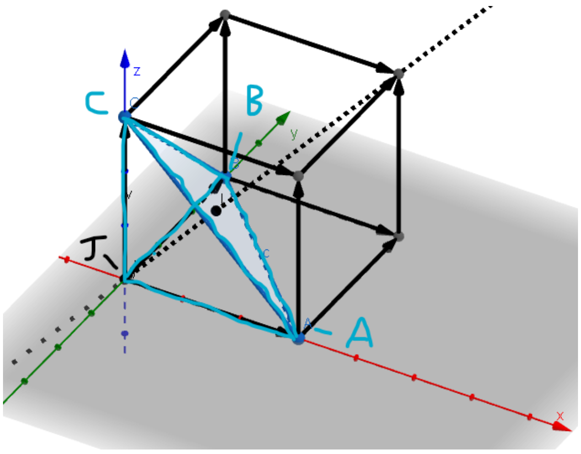

Figure 7: JABC is highlighted in light blue.

Well look at $JABC$ in Figure 7. It has the same height as the parallelpiped, its base area is half of the parallelpiped, it is a triangular pyramid, and most importantly, one of its sides has the area, $[ABC]$. Therefore if B and h represents the area of the base and height of the PV_{ABC} respectively I have:

$$PV_{ABC} = Bh$$

So:

$$\begin{aligned} V_{JABC} &= (\frac{1}{3})(\frac{B}{2})(h) \\ &= \frac{PV_{ABC}}{6} \\ PV_{ABC} &= 6 \times V_{JABC} \end{aligned}$$

Now, letting H represent the height from origin J to the plane ABC,

$$6V_{JABC} = 6\left(\frac{1}{3}[ABC]H\right)$$
$$6V_{JABC} = 2[ABC]H = PV_{ABC}$$

Notice that since $\triangle PQR$ is on plane $\triangle ABC$, the triangular pyramid $JPQR$ has the same height of H. So, just like with $\triangle ABC$, let B_2 and h_2 represent the area of the base and height of the parallelpiped formed by the vector points P, Q, R:

$$V_{JPQR} = \frac{1}{3}(\frac{B_2}{2})(h_2)$$
$$PV_{PQR} = 6V_{JPQR}$$

So:

$$6V_{JPQR} = 6\left(\frac{1}{3}[PQR][H]\right)$$
$$6V_{JPQR} = 2[PQR]H = PV_{PQR}$$

Finally, I have:

$$\frac{PV_{PQR}}{PV_{ABC}} = \frac{2[PQR]H}{2[ABC]H}$$
$$= \frac{[PQR]}{[ABC]}$$

To complete this proof:

$$\frac{[PQR]}{[ABC]} = \frac{PV_{PQR}}{PV_{ABC}} = \begin{vmatrix} x & y & z \\ a & b & c \\ d & e & f \end{vmatrix}$$

□

For every successful coordinate system in both the math contest setting and the field, there must exist a few "tools". Specifically, effective representation of common mathematical objects like a line can lead to new information. So as a direct consequence of the area formula, I can represent a line:

Theorem 2. *The equation of a line in Barycentric Coordinates has the general appearance:*

$$0 = ux + uy + uz$$

where u,y,$z \in \mathbb{R}$.

Proof. More specially, note that a line is essentially a degenerate triangle; This means the area is 0. So for 3 arbitrary points $P = (x, y, z)$, $Q = (a, b, c)$, $R = (d, e, f)$, I have:

$$0 = \begin{vmatrix} x & y & z \\ a & b & c \\ d & e & f \end{vmatrix}$$
$$= x(b \times f - c \times e) + y(a \times f - c \times d) + z(a \times e - b \times d)$$
$$= ux + vy + wz$$

where the constants, $u, v, w \in \mathbb{R}$ make the equation clean by replacing the other constants. ☐

From this I can create some corollaries that will be used in the example problem later on:

Corollary 1. *The equation of the line that passes through A is*
$y = kz$

where $k \in \mathbb{R}$.

Proof. The proof is as simple as putting in the coordinates of A into the line equation.

$$0 = u(1) + v(0) + w(0)$$
$$u = 0$$

So the equation is:

$$0 = vy + wz$$
$$-vy = wz$$
$$y = \frac{-w}{v} z$$

Let k replace the constant, $\frac{-w}{v}$.

$$y = kz$$

☐

I can similarly find that the equations of the lines that passes through B and C as: $x = k_1 z$ and $x = k_2 y$ respectively. $k_i \in \mathbb{R}$ where $i = 1, 2$, replacing the messy fraction with the constants.

3.2 Displacement Vectors

Since the background information section, the only vectors that was introduced were point-vectors whose tail was at the origin. Besides this type of vector, one must also be familiar with the vector between two points.

Definition 3. *A displacement vector of two arbitrary points, $P = (x, y, z)$ and $Q = (a, b, c)$, $\vec{PQ}$ yields the Barycentric Coordinates:*

$$(x - a, y - b, z - c)$$

.

Leading to the direct consequence:

Corollary 2. *The sum of the coordinates of a displacement vector is 0.*

Proof. Simply add the coordinates and rearrange the variables.

$$\begin{aligned} x - a + y - b + z - c &= x + y + z - a - b - c \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

□

3.3 Distance Formula

Lengths are useful in Barycentric Coordinates therefore wouldn't it be convenient if there was an explicit formula? The only method I currently know to get the "difference" between two points, is the displacement vector. Therefore, if I define two general points in their vector definition, would it not be convenient if I could relate these vectors to length values like the side lengths of the reference triangle? Well, I already have the necessary tools for this exact occasion. But first I must translate O, the circumcenter to the origin of the Cartesian plane to use them:

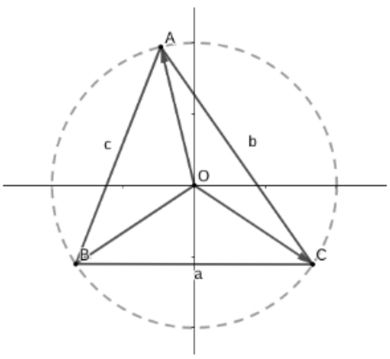

Figure 8: As points are in reference with the triangle, translating does change distance.

Therefore, consider two points P and Q. To get the distance between these general points, it is obvious to find the *displacement* vector between them. Therefore, define the displacement vector PQ as (x, y, z) such that $x + y + z = 0$ and $\vec{PQ} = x\vec{A} + y\vec{B} + z\vec{C}$. But now what? I have the displacement vector now; What concept is there that allows me to explicitly connect the magnitude (distance) of the vector, to the vector itself??

The DOT product!

Thus, as the angle between the same angle is 0°:

$$(\vec{PQ})^2 = |\vec{PQ}||\vec{PQ}|\cos(0)$$
$$(\vec{PQ})^2 = |\vec{PQ}|^2$$

Therefore to find the magnitude/length of the displacement vector $|\vec{PQ}|$, all I need to do is to relate $\vec{PQ}$ to length values. Thus, using the relations: $|\vec{A}| = |\vec{B}| = |\vec{C}| = R$, $\vec{A}\cdot\vec{B} = R^2 - \frac{1}{2}c^2$, $\vec{B}\cdot\vec{C} = R^2 - \frac{1}{2}a^2$, and $\vec{C}\cdot\vec{A} = R^2 - \frac{1}{2}b^2$, that was previously established in the preliminary mathematics section:

$$\begin{aligned}
(\vec{PQ})^2 &= (x\vec{A} + y\vec{B} + z\vec{C}) \cdot (x\vec{A} + y\vec{B} + z\vec{C}) \\
&= x^2\vec{A}\cdot\vec{A} + xy\vec{A}\cdot\vec{B} + xz\vec{A}\cdot\vec{C} + yx\vec{B}\cdot\vec{A} + y^2\vec{B}\cdot\vec{B} + yz\vec{B}\cdot\vec{C} + xz\vec{C}\cdot\vec{A} + zy\vec{C}\cdot\vec{B} + z^2\vec{C}\cdot\vec{C} \\
&= x^2(R^2) + y^2(R^2) + z^2(R^2) + 2xy(R^2 - \frac{1}{2}c^2) + 2xz(R^2 - \frac{1}{2}b^2) + 2yz(R^2 - \frac{1}{2}a^2) \\
&= R^2(x^2 + y^2 + z^2) + 2xyR^2 - xyc^2 + 2xzR^2 - xzb^2 + 2yzR^2 - yza^2 \\
&= R^2(x^2 + y^2 + z^2 + 2xy + 2xz + 2yz) - (xyc^2 + xzb^2 + yza^2) \\
&= (x + y + z)^2 - (xyc^2 + xzb^2 + yza^2)
\end{aligned}$$

Now since $\vec{PQ}$ is a displacement vector, its coordinates yield: $(x + y + z = 0)$ So:

$$(\vec{PQ})^2 = |\vec{PQ}|^2 = -yza^2 - xzb^2 - xyc^2$$

Knowing the distance formula is great, because I now have access to a very useful consequence of the distance formula.

3.4 Circle Formula

Theorem 3. *The standard formula for a circle is:*
-$a^2yz - b^2zx - c^2xy + (ux + vy + wz)(x + y + z) = 0$. With $u, v, w \in \mathbb{R}$.

Proof. Assume a circle has the center with coordinates (j, k, l), with R radius. So now, just like how a circle is defined in a Cartesian plane, I will use the distance formula:

$$-a^2(y - k)(z - l) - b^2(x - j)(z - l) - c^2(x - j)(y - k) = R^2$$

I'm expanding this out to hopefully find a cleaner form, ironically:

$$\begin{aligned}R^2 &= -a^2(yz - yl - kz + kl) - b^2(zx - zj - lx + lj) - c^2(xy - xk - jy + jk)\\ &= -a^2yz + a^2yl + a^2kz - a^2kl - b^2zx + b^2zj + b^2lx - b^2lj - c^2xy + c^2xk + c^2jy - c^2jk\end{aligned}$$

Organizing yields:

$$R^2 + a^2kl + b^2lj + c^2jk = -a^2yz - b^2xz - c^2xy + z(a^2k + b^2j) + y(a^2l + c^2j) + x(b^2l + c^2k)$$

As I have many constants, replacing them with one variable will greatly simplify the equation so, letting C_1, C_2, C_3, C represent the constants in the equation I have:

$$C = -a^2yz - b^2xz - c^2xy + C_1x + C_2y + C_3z$$

Can I simplify any more? Well I can use the fact of $x + y + z = 1$ for something very clever:

$$\begin{aligned}C(1) &= -a^2yz - b^2xz - c^2xy + C_1x + C_2y + C_3z\\ Cx + Cy + Cz &= -a^2yz - b^2xz - c^2xy + C_1x + C_2y + C_3z\\ 0 &= -a^2yz - b^2xz - c^2xy + C_1x - Cx + C_2y - Cy + C_3z - Cz\\ 0 &= -a^2yz - b^2xz - c^2xy + x(C_1 - C) + y(C_2 - C) + z(C_3 - C)\end{aligned}$$

This now looks synonymous with the equation of the circle I was looking for. The only thing now is to define the constants $u = C_1 - C, v = C_2 - C, w = C_3 - C$ for:

$$\begin{aligned}0 &= -a^2yz - b^2xz - c^2xy + ux + vy + wz\\ &= -a^2yz - b^2xz - c^2xy + (ux + vy + wz)(x + y + z)\end{aligned}$$

Which concludes the proof. □

The first thing to note is the messiness of this theorem, which is true, making circles something to avoid in Barycentric Coordinates. Note though, that if x, y, z are 0, I can remove many of the terms in the equation. As a consequence, and will be used in the first example problem, the equation for the circumcircle, the circle that passes through the vertices of a general $\triangle ABC$ yields:

Corollary 3. *The circumcircle of the reference triangle has equation*
$a^2yz + b^2zx + c^2xy = 0$

Proof. Since this circle passes through the vertices of the reference triangle I can find:

For $(1, 0, 0)$:

$$
\begin{aligned}
0 &= -a^2(0)(0) - b^2(1)(0) - c^2(1)(0) + (u(1) + v(0) + w(0))(1 + 0 + 0) \\
u &= 0
\end{aligned}
$$

For $(0, 1, 0)$:

$$
\begin{aligned}
0 &= -a^2(1)(0) - b^2(0)(0) - c^2(0)(1) + (u(0) + v(1) + w(0))(0 + 1 + 0) \\
v &= 0
\end{aligned}
$$

For $(0, 0, 1)$:

$$
\begin{aligned}
0 &= -a^2(0)(1) - b^2(0)(1) - c^2(0)(0) + (u(0) + v(0) + w(1))(0 + 0 + 1) \\
w &= 0
\end{aligned}
$$

Therefore using the constants I have found:

$$
\begin{aligned}
0 &= -a^2yz - b^2xz - c^2xy + ((0)x + (0)y + (0)z)(x + y + z) \\
0 &= -a^2yz - b^2xz - c^2xy \\
0 &= a^2yz + b^2xz + c^2xy
\end{aligned}
$$

□

3.5 Example Problem

To fulfill my goal of solving Olympiad-level problems, lets start with an entry-level International Math Olympiad Problem!

Points P and Q lie on side BC of acute-angled $\triangle ABC$ so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ, respectively, such that P is the midpoint of AM, and Q is the midpoint of AN. Prove that lines BM and CN intersect on the circumcircle of $\triangle ABC$. **(IMO 2014/4)** [9]

First lets see what is happening:

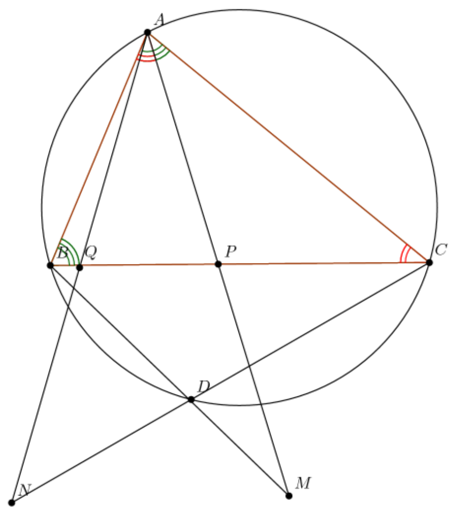

Figure 9: 4th Question of the 2014 IMO

Three things.
First, let me denote point D, as the intersection between $\overline{BM}$ and $\overline{CN}$.
Second, luckily there is only 1 notable triangle that can be used as the reference triangle, $\triangle ABC$. So $A = (1, 0, 0)$, $B = (0, 1, 0)$, and $C = (0, 0, 1)$.
Third, since this question involves lines, side-lengths, and reveals information about angles, I had the intuitive feeling this problem needed similar triangles or congruent triangles.
And I quickly saw that because: $\angle ABP = \angle ABC$ and $\angle BAP = \angle ACB$:

$$\triangle ABP \sim \triangle ABC$$

. Additionally since $\angle QAC = \angle ABC$ and $\angle ACQ = \angle ACB$, I conclusively have:

$$\triangle ABC \sim \triangle ABP \sim \triangle AQC$$

At this point, I now have the additional information:

$$\frac{BC}{AB} = \frac{AB}{BP} = \frac{AC}{AQ}$$

Which is extremely useful considering Barycentric Coordinates almost exclusively works with lengths and areas. Now, reviewing the end-goal of finding D to see if its on the circumcircle of $\triangle ABC$, I noticed that I need to find the equations of lines $\overline{BM}$ and $\overline{CN}$, but to do that I need to find M and N. But to find those points I realized I can find the easier-to-find P and Q first — so:

Defining P in its area definition, the triangles relevant are seen below:

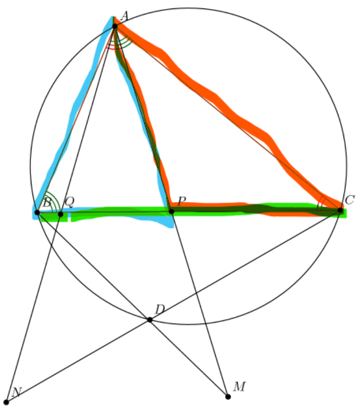

Figure 10: Green: $\triangle BPC$, Orange: $\triangle APC$, Blue:$\triangle BPA$

$$\begin{aligned} P &= (\frac{[BPC]}{[ABC]}, \frac{[APC]}{[ABC]}, \frac{[ABP]}{[ABC]}) \\ &= (0, \frac{[APC]}{[ABC]}, \frac{[ABP]}{[ABC]}) \end{aligned}$$

Now the question becomes: How can I get the ratios of these triangle areas from the information I have now? Well I have ratios of side-lengths from similar triangles, so that means I have the ratios of their areas as well! So through the similarity $\triangle APB \sim \triangle ABC$:

$$\begin{aligned} \frac{[ABP]}{[ABC]} &= (\frac{AB}{BC})^2 \\ &= \frac{c^2}{a^2} \end{aligned}$$

Therefore I can get the last ratio $\frac{[APC]}{[ABC]}$, from

$$\begin{aligned} 0 + \frac{[APC]}{[ABC]} + \frac{c^2}{a^2} &= 1 \\ &= 1 - \frac{c^2}{a^2} \\ &= \frac{a^2 - c^2}{a^2} \end{aligned}$$

So $P = (0, \frac{a^2-c^2}{a^2}, \frac{c^2}{a^2})$.

Solving for $Q = (\frac{[BQC]}{[ABC]}, \frac{[AQC]}{[ABC]}, \frac{[BQA]}{[ABC]})$ is mostly identical. I will use the similarity between $\triangle AQC \sim \triangle ABC$:

$$\begin{aligned} \frac{[AQC]}{[ABC]} &= (\frac{AC}{BC})^2 \\ &= \frac{b^2}{a^2} \end{aligned}$$

Therefore I can get the last ratio $\frac{[BQA]}{[ABC]}$, from

$$\begin{aligned} 0 + \frac{b^2}{a^2} + \frac{[BQA]}{[ABC]} &= 1 \\ &= 1 - \frac{b^2}{a^2} \\ &= \frac{a^2 - b^2}{a^2} \end{aligned}$$

So: $Q = (0, \frac{b^2}{a^2}, \frac{a^2-b^2}{a^2})$.

Finally, I can find the points M and N using the midpoint property of the problem. For M, the ratios to be found are:

$$M = (\frac{[BMC]}{[ABC]}, \frac{[AMC]}{[ABC]}, \frac{[BMA]}{[ABC]})$$

Now, lets look at each ratio separately:

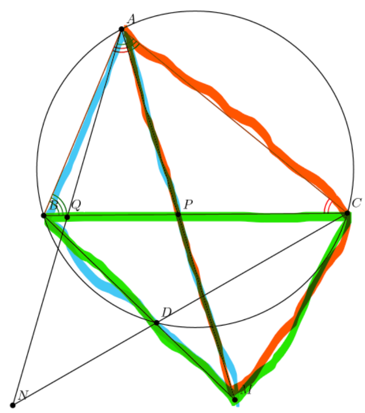

Figure 11: Green: $\triangle BMC$, Orange: $\triangle AMC$, Blue: $\triangle BMA$

For $\frac{[BMC]}{[ABC]}$, because $PM = AP$, the point M is the reflection of A through point P. Since they are opposites, it can be reasoned that $\triangle BMC$ is the literal inverse of $\triangle ABC$, therefore:

$$\begin{aligned} \frac{[BMC]}{[ABC]} &= \frac{-[ABC]}{[ABC]} \\ &= -1 \end{aligned}$$

For $\frac{[AMC]}{[ABC]}$, notice how $PM = AP \iff AM = 2AP$? So because $\triangle AMC$ same height as $\triangle APC$ from point C to line AM, this means the area is doubled!

$$\begin{aligned} \frac{[AMC]}{[ABC]} &= \frac{2 \times [APC]}{[ABC]} \\ &= \frac{2a^2 - 2c^2}{a^2} \end{aligned}$$

For the final ratio $\frac{[BMA]}{[ABC]}$, due to the midpoint P on $\overline{AM}$, it can be noted that:

$$\begin{aligned}\frac{[BMA]}{[ABC]} &= \frac{2 \times [BPA]}{[ABC]} \\ &= \frac{2c^2}{a^2}\end{aligned}$$

Therefore:

$$M = (-1, \tfrac{2a^2-2c^2}{a^2}, \tfrac{2c^2}{a^2})$$

.

Symmetrically, N can be solven for through the same process as M for:

$$(-1, \tfrac{2b^2}{a^2}, \tfrac{2a^2-2b^2}{a^2})$$

To be efficient as the process is literally identical, the proof will be left as an exercise for the reader.

I can now find the equations of lines $\overline{BM}$ and $\overline{CN}$.

For $\overline{BM}$, as it passes through B and substituting in M I have:

$$\begin{aligned}x &= kz \\ -1 &= k(\frac{2c^2}{a^2}) \\ k &= \frac{-a^2}{2c^2}\end{aligned}$$

So:

$$x = \tfrac{-a^2}{2c^2} z$$

For $\overline{CN}$ as it passes through C and N I have:

$$\begin{aligned}x &= ky \\ -1 &= k(\frac{2b^2}{a^2}) \\ k &= \frac{-a^2}{2b^2}\end{aligned}$$

So:

$$x = \tfrac{-a^2}{2b^2} y$$

. This implies I can solve for D: Let D have the coordinates (x, y, z). So, along with $x + y + z = 1$, and the equations of the line I have:

$$\begin{aligned}
\frac{-a^2}{2b^2}y &= \frac{-a^2}{2c^2}z \\
\frac{-a^2}{2b^2}(1-x-z) &= \frac{-a^2}{2c^2}z \\
\frac{-a^2}{2b^2}\left(1-(\frac{-a^2}{2c^2}z)-z\right) &= \frac{-a^2}{2c^2}z \\
\frac{-a^2}{2b^2}(\frac{2c^2}{2c^2}+\frac{a^2z}{2c^2}-\frac{2c^2z}{2c^2}) &= \frac{-a^2}{2c^2}z \\
\frac{-a^2}{2b^2}(2c^2+a^2z-2c^2z) &= -a^2z \\
\frac{-2c^2-a^2z+2c^2z}{2b^2} &= -z \\
0 &= \frac{-2c^2-a^2z+2c^2z+2b^2z}{2b^2} \\
&= \frac{-2c^2}{2b^2}+\frac{z(-a^2+2c^2+2b^2)}{2b^2} \\
z &= \frac{c^2}{b^2}\times\frac{2b^2}{(-a^2+2c^2+2b^2)} \\
&= \frac{2c^2}{2c^2+2b^2-a^2}
\end{aligned}$$

Therefore x yields:

$$\begin{aligned}
x &= \frac{-a^2}{2c^2}z \\
&= \frac{-a^2(2c^2)}{2c^2(2c^2+2b^2-a^2)} \\
&= \frac{-a^2}{2c^2+2b^2-a^2}
\end{aligned}$$

Now finally I can get y:

$$\begin{aligned}
x+y+z &= 1 \\
y &= 1-x-z \\
&= \frac{2c^2+2b^2-a^2-(-a^2+2c^2)}{2c^2+2b^2-a^2} \\
&= \frac{2b^2}{2c^2+2b^2-a^2}
\end{aligned}$$

Therefore D, the intersection between $\overline{BM}$ and $\overline{CN}$, is at

$$D = (\frac{-a^2}{2c^2+2b^2-a^2}, \frac{2b^2}{2c^2+2b^2-a^2}, \frac{2c^2}{2c^2+2b^2-a^2})$$

Now, I can use circumcircle equation found as a corollary of the general equation of a circle to see if this point lies on the circumcircle of $\triangle ABC$

$$\begin{aligned} a^2yz + b^2zx + c^2xy &= 0 \\ \frac{1}{(2c^2+2b^2-a^2)^2}[a^2(2b^2)(2c^2) + b^2(2c^2)(-a^2) + c^2(-a^2)(2b^2)] &= 0 \\ 4a^2b^2c^2 - 2a^2b^2c^2 - 2a^2b^2c^2 &= 0 \\ 0 &= 0 \end{aligned}$$

Since it lies on circumcircle, I am done.

4 Conclusion

In essence, I built the concept of Barycentric Coordinates from the ground up, to find that Barycentric Coordinates played a huge role in providing a new way to look at problems. It was exciting to utilize these Barycentric Coordinates to explore new approaches for math contest questions. Additionally, throughout my journey, I also realized the beauty of linear algebra, creating additional interest in math. I am curious if there exists more devices in formal linear algebra that could benefit me during in contest environments.

4.1 Strengths

In essence the advantages from my experience I saw in Barycentric Coordinates can be summed up in 3 points: First, I noticed it works well with Ratios of lengths due to its inherent structure. Second, there are simple equations for many mathematical objects like lines and circles; Especially if it passes through a vertex of the reference triangle. In addition, this also allows Barycentric Coordinates to find intersection points easily. Third, its symmetrical; Oddly, through the ugliness of this specific type of Mathematics, there was many symmetrical results which meant I did not need to think as much as finding the points/equations were very similar.

There are many more exciting applications and advantages of Barycentric Coordinates but due to the page limit, there simply is no room to show them.

4.2 Weaknesses

Symmetrically, there are 2 notable weaknesses. First, Barycentric Coordinates are incredibly bashful. Namely, the equation of a circle can get messy if it does not pass through the vertices of the reference triangle. Even arbitrary points can be messy as well, which I saw in the example problem. This means that the longer the problem, the more messy the problem becomes and thus leads to more area for error. Even the most simple problems can become long. In this lense, I guess I understand why it is unpopular in the olympiad-setting. But personally I like the idea of an approach that guarantees an answer to a lot of geometry problems, regardless of the time and work it takes. Second, those that are observant might have noticed I have completely neglected the approaches for angles. Even though there is an approach for specific angles of the reference triangle (omitted from the paper), generally most angles are hard to deal with due to the limited tools in the toolbox. That is why I noticed getting length conditions from the angles are the only certain way of making use of angles effectively. For instance, in the example problem, I got length ratios from the angles through similar triangles.

4.3 Comments on learning

My experience trying to interpret the content in the few papers on Barycentric Coordinates, was to say the least, frustrating. I had to work backwards, infer, prove individually, google, and do everything in order to actually understand what was going on. Reading 1 page, sometimes even just getting past 1 line can take hours. On one side, its great, this minimal-form of learning is efficient at helping individuals figure out concepts by themselves, leading to better comprehension. Which I do agree and noticed as I tackled Olympiad-level questions.
But on the other side, it deters people from Olympiad-Mathematics. It creates a barrier between what might be called high-school mathematics and Olympiad mathematics. Just like the barrier between university mathematics and high-school mathematics. Why must this barrier exist? Why must individuals think and treat university-math and Olympiad math as harder than the math they are currently learning so they can justify poor explanations? It makes no sense to me, and throughout working on this IA, seeing this minimalism just made me more motivated to write.

4.4 Why Barycentric?

Throughout my Math IA, I was stuck between choosing 4 other coordinate systems to go with: Cartesian Coordinates, The Complex Number System, the Vector System, and Barycentric Coordinates. But why did I end up choosing Barycentric Coordinates? One main reason: The applicability: Overall from what I noticed in Barycentric Coordinates, compared with other approaches to geometry problems: Cartesian coordinates, vectors, or Complex Numbers, it allows for much computation freedom. For instance, in Cartesian Coordinates, the special points of a triangle are incredibly ugly and the equations of a circle are extremely difficult to deal with. In addition, dealing with equations of lines for the Vector and Complex Number systems are simply too complicated to be used. In addition, Barycentric Coordinates are apparently something not well-known throughout the contest space, which allows for even the highest-level math problems to be solved relatively easily!
But at the end of the day, there is no "best" coordinate system. Each coordinate system is created with an unique feature that mathematicians can take advantage of to be more well-equipped to tackle problems. In this peculiar case, I saw Barycentric Coordinates as the most applicable out of the four for math contest questions.

4.5 What's Next

The limits of the Math Internal Assessment is enormous; The page limit is a huge barrier, especially for tedious and long mathematics like Barycentric Coordinates. Multiple insightful concepts, like Evan's Favourite Factoring Trick, a way to approach perpendiculars, and the simple forms of triangle centers were all neglected due to this page limit. I hope to anyone reading explore Barycentric Coordinates themselves because a mere 20 pages is simply not enough to fully appreciate the beautiful subject.

In terms of learning though, the most important and universal thing that I learned was the techniques used to derive Barycentric Coordinates. In peculiar, substituting messy results that appeared while proving the circle formula or the line formula with just one variable made the bashful technique just a bit easier; Its a potentially an useful heuristic to approach future messy problems! I love the ingenuity that came from Barycentric Coordinates, and through this IA, helped solidify my belief that improving your mathematical intuition through mathematical proofs is one of the best ways to become better problem-solvers.

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