Theory_and_Examples

November 4, 2017

1 Mousai: An Open-Source General Purpose Harmonic Balance Solver

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Joseph C. Slater, October 23, 2017

```
In [1]: %matplotlib inline
        %load_ext autoreload
        %autoreload 2
        import scipy as sp
        import numpy as np
        import matplotlib.pyplot as plt
        import matplotlib
        import mousai as ms
        from scipy import pi, sin
        matplotlib.rcParams['figure.figsize'] = (11, 5)
```

1.1 Overview

A wide array of contemporary problems can be represented by nonlinear ordinary differential equations with solutions that can be represented by Fourier Series: * **Limit cycle oscillation of wings/blades** * Flapping motion of birds/insects/ornithopters * Flagellum (threadlike cellular structures that enable bacteria etc. to swim) * Shaft rotation, especially including rubbing or non-linear bearing contacts * **Engines** * Radio/sonar/radar electronics * Wireless power transmission * Power converters * Boat/ship motions and interactions

- **Cardio systems** (heart/arteries/veins)
- Ultrasonic systems transversing nonlinear media
- Responses of composite materials or materials with cracks
- Near buckling behavior of vibrating columns
- Nonlinearities in power systems
- **Energy harvesting systems**
- **Wind turbines**
- Radio Frequency Integrated Circuits
- **Any system with nonlinear coatings/friction damping, air damping, etc.**

These can all be observed in a quick literature search on 'Harmonic Balance'.

1.2 Why (did I) write Mousai?

- The ability to code harmonic balance seems to be publishable by itself
  - It's not research- it's just application of a known family of technique
- A limited number of people have this knowledge and skill
  - Most cannot access this technique
  - "Research effort" is spent coding the technique, not doing research

1.2.1 Why write Mousai? (continued)

- Matlab command eig unleashed power to the masses
- Very few papers are published on eigensolutions- they have to be better than `eig`
- `eig` only provides simple access to high-end eigensolvers written in `C` and `Fortran`
- Undergraduates with no practical understanding of the algorithms easily solve problems that were intractable a few decades ago.
- *Access* and *ease of use* of such techniques enable *greater science* and *greater research*.
- The real world is nonlinear, but **linear analysis dominates because the tools are easier to use**.
- With `Mousai`, an undergraduate can solve a nonlinear harmonic response problem easier then a PhD can today.

1.3 Theory:

1.3.1 Linear Solution

- Most dynamics systems can be modeled as a first order differential equation

$$\ddot{\mathbf{z}}(t) = \mathbf{f}(\mathbf{z}(t), \mathbf{u}(t)) \tag{1}$$

  - Use finite differences
  - Use Galerkin methods (Finite Elements)
  - Of course discrete objects

- This is the common *State-Space* form: -solutions exceedingly well known if it is linear

- Finding the oscillatory response, after dissipation of the transient response, requires **long** time marching.

  - Without damping, this may not even been feasible.
  - With damping, tens, hundreds, or thousands of cycles, therefore thousands of times steps at minimum.

For a linear system in the frequency domain this is

$$j\omega\mathbf{Z}(\omega) = \mathbf{f}(\mathbf{Z}(\omega), \mathbf{U}(\omega)) \tag{2}$$

$$j\omega\mathbf{Z}(\omega) = A\mathbf{Z}(\omega) + B\mathbf{U}(\omega) \tag{3}$$

where

$$A = \frac{\partial \mathbf{f}(\mathbf{Z}(\omega), \mathbf{U}(\omega))}{\partial \mathbf{Z}(\omega)}, \qquad B = \frac{\partial \mathbf{f}(\mathbf{Z}(\omega), \mathbf{U}(\omega))}{\partial \mathbf{U}(\omega)} \tag{4}$$

are constant matrices.
The solution is:

$$\mathbf{Z}(\omega) = (Ij\omega - A)^{-1} B\mathbf{U}(\omega) \tag{5}$$

Where the magnitudes and phases of the elements of $\mathbf{Z}$ provide the amplitudes and phases of the harmonic response of each state at the frequency ω.

1.3.2 Nonlinear solution

- For a nonlinear system in the frequency domain we assume a Fourier series solution

$$\mathbf{z}(t) = \lim_{N\to\infty} \sum_{n=-N}^{N} \mathbf{Z}_n e^{jn\omega t} \tag{6}$$

- $N = 1$ for a single harmonic. $n = 0$ is the constant term.
- This can be substituted into the governing equation to find $\dot{\mathbf{z}}(t)$:

$$\dot{\mathbf{z}}(t) = \mathbf{f}(\mathbf{z}(t), \mathbf{u}(t)) \tag{7}$$

- This is actually a function call to a Finite Element Package, CFD, Matlab function, - whatever your solver uses to get derivatives
- We can also find $\dot{\mathbf{z}}(t)$ from the derivative of the Fourier Series:

$$\dot{\mathbf{z}}(t) = \lim_{N\to\infty} \sum_{n=-N}^{N} jn\omega \mathbf{Z}_n e^{jn\omega t} \tag{8}$$

- The difference between these methods is zero when $\mathbf{Z}_n$ are correct.

$$\mathbf{0} \approx \sum_{n=-N}^{N} jn\omega \mathbf{Z}_n e^{jn\omega t} - \mathbf{f}\left(\sum_{n=-N}^{N} \mathbf{Z}_n e^{jn\omega t}, \mathbf{u}(t)\right) \tag{9}$$

- These operations are wrapped inside a function that returns this error
- This function is used by a Newton-Krylov nonlinear algebraic solver.
- Calls any solver in the SciPy family of solvers with the ability to easily pass through parameters to the solver *and* to the external derivative evaluator.

1.4 Examples:

1.4.1 Duffing Oscillator

$$\ddot{x} + 0.1\dot{x} + x + 0.1x^3 = \sin(\omega t)$$

In [3]:
```
# Define our function (Python)
def duff_osc_ss(x, params):
    omega = params['omega']
    t = params['cur_time']
    xd = np.array([[x[1]],
                   [-x[0] - 0.1 * x[0]**3 - 0.1 * x[1] + 1 * sin(omega * t)]])
    return xd
```

In [4]:
```
# Arguments are name of derivative function, number of states, driving frequency,
# form of the equation, and number of harmonics

t, x, e, amps, phases = ms.hb_so(duff_osc_ss, num_variables=2, omega=.1,
                                 eqform='first_order', num_harmonics=5)
print('Displacement amplitude is ', amps[0])
print('Velocity amplitude is ', amps[1])
```

```
Displacement amplitude is  0.946956354668
Velocity amplitude is  0.0946956354558
```

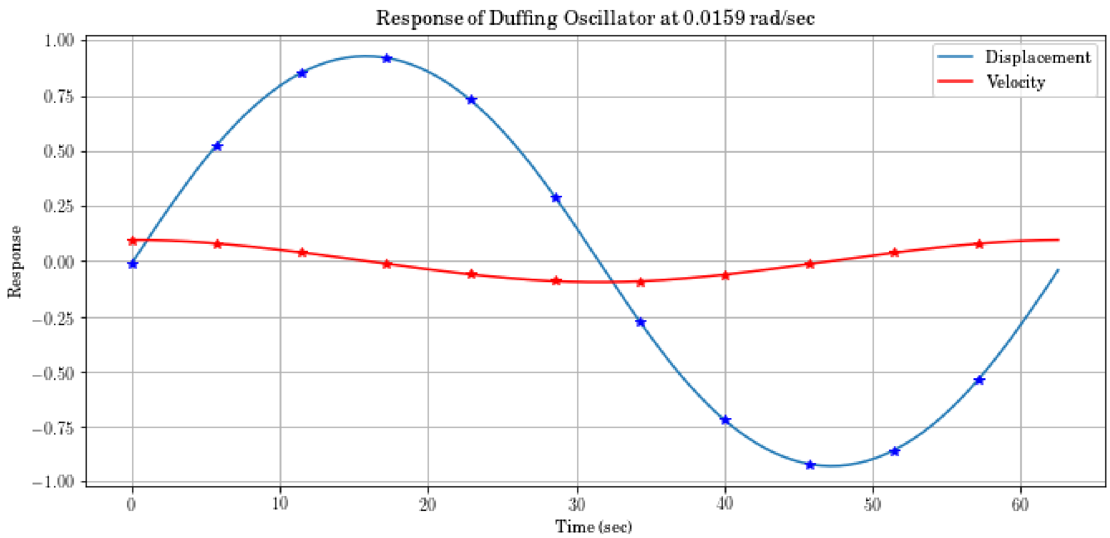
Mousai can easily recreate the near-continuous response

```
time, xc = ms.time_history(t, x)
```

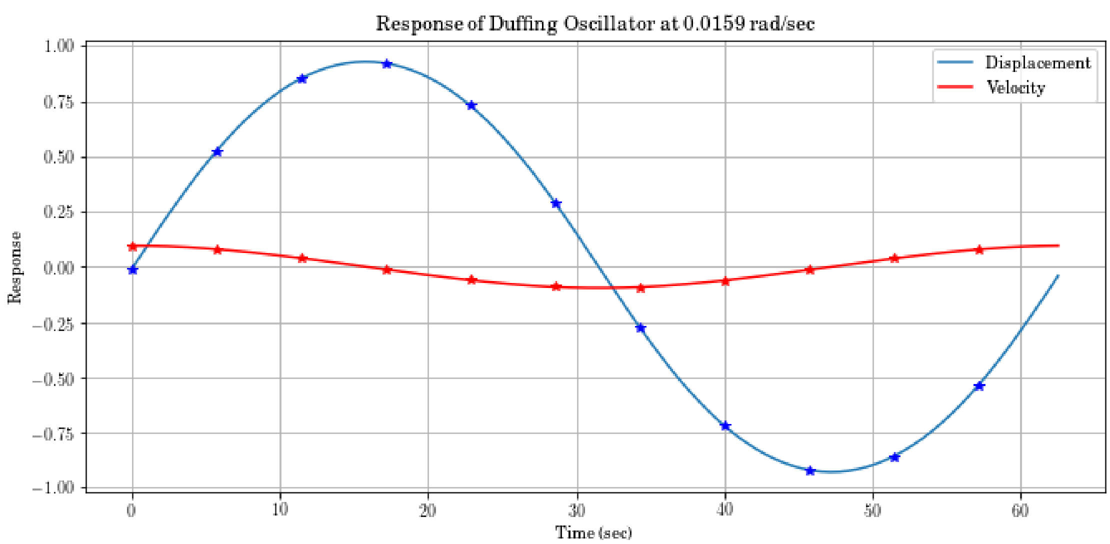
In [17]:
```
def pltcont():
    time, xc = ms.time_history(t, x)
    disp_plot, _ = plt.plot(time, xc.T[:, 0], t,
                            x.T[:, 0], '*b', label='Displacement')
    vel_plot, _ = plt.plot(time, xc.T[:, 1], 'r',
                           t, x.T[:, 1], '*r', label='Velocity')
    plt.legend(handles=[disp_plot, vel_plot])
    plt.xlabel('Time (sec)')
    plt.title('Response of Duffing Oscillator at 0.0159 rad/sec')
    plt.ylabel('Response')
    plt.legend
    plt.grid(True)
```

In [18]:
```
fig=plt.figure()
ax=fig.add_subplot(111)
time, xc = ms.time_history(t, x)
disp_plot, _ = ax.plot(time, xc.T[:, 0], t,
                       x.T[:, 0], '*b', label='Displacement')
vel_plot, _ = ax.plot(time, xc.T[:, 1], 'r',
                      t, x.T[:, 1], '*r', label='Velocity')
```

```
        ax.legend(handles=[disp_plot, vel_plot])
        ax.set_xlabel('Time (sec)')
        ax.set_title('Response of Duffing Oscillator at 0.0159 rad/sec')
        ax.set_ylabel('Response')
        ax.legend
        ax.grid(True)
```

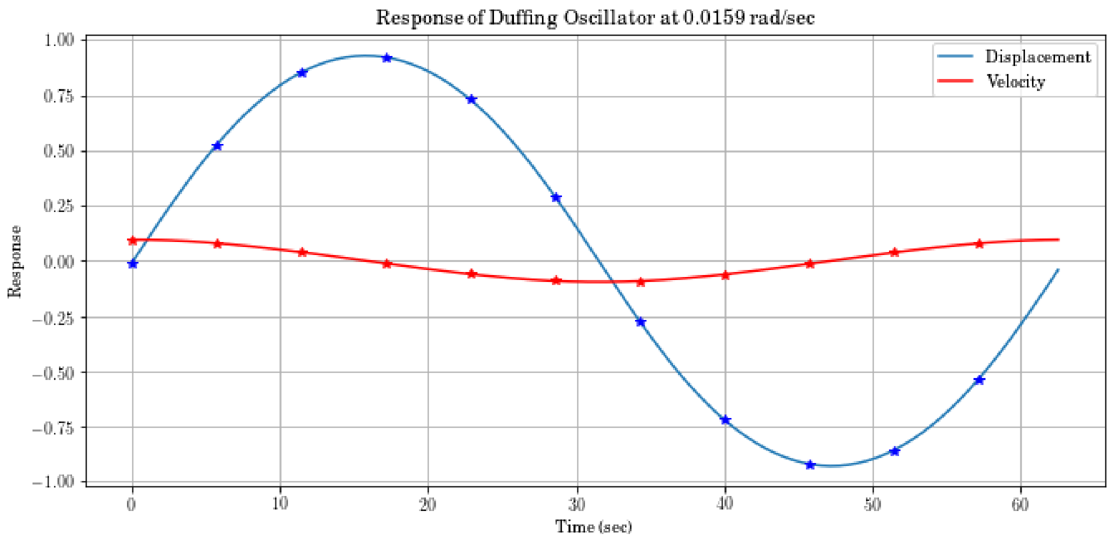



```
In [19]: pltcont()# abbreviated plotting function
```

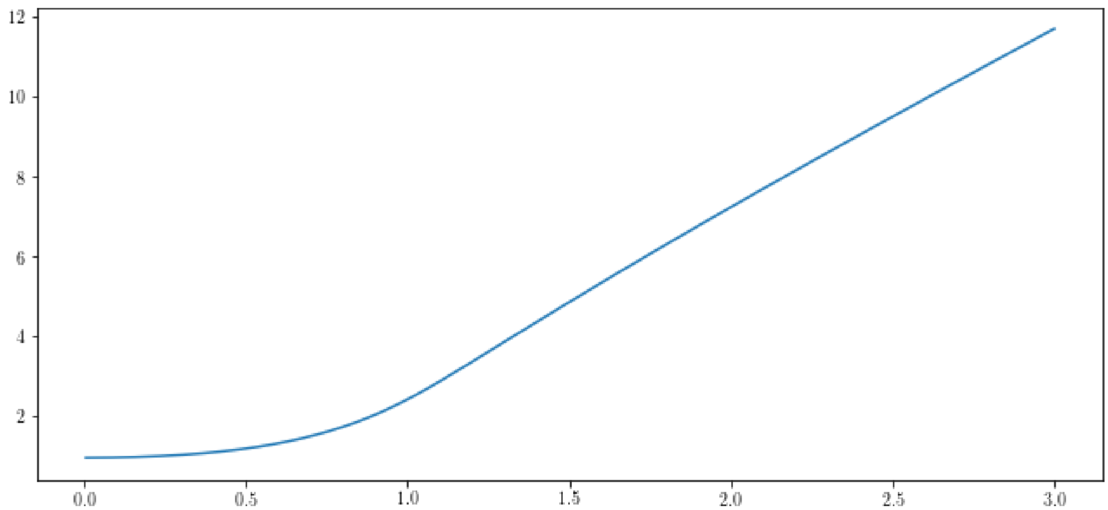



```
In [20]: time, xc = ms.time_history(t, x)
         disp_plot, _ = plt.plot(time, xc.T[:, 0], t,
```

```
                             x.T[:, 0], '*b', label='Displacement')
vel_plot, _ = plt.plot(time, xc.T[:, 1], 'r',
                            t, x.T[:, 1], '*r', label='Velocity')
plt.legend(handles=[disp_plot, vel_plot])
plt.xlabel('Time (sec)')
plt.title('Response of Duffing Oscillator at 0.0159 rad/sec')
plt.ylabel('Response')
plt.legend
plt.grid(True)
```



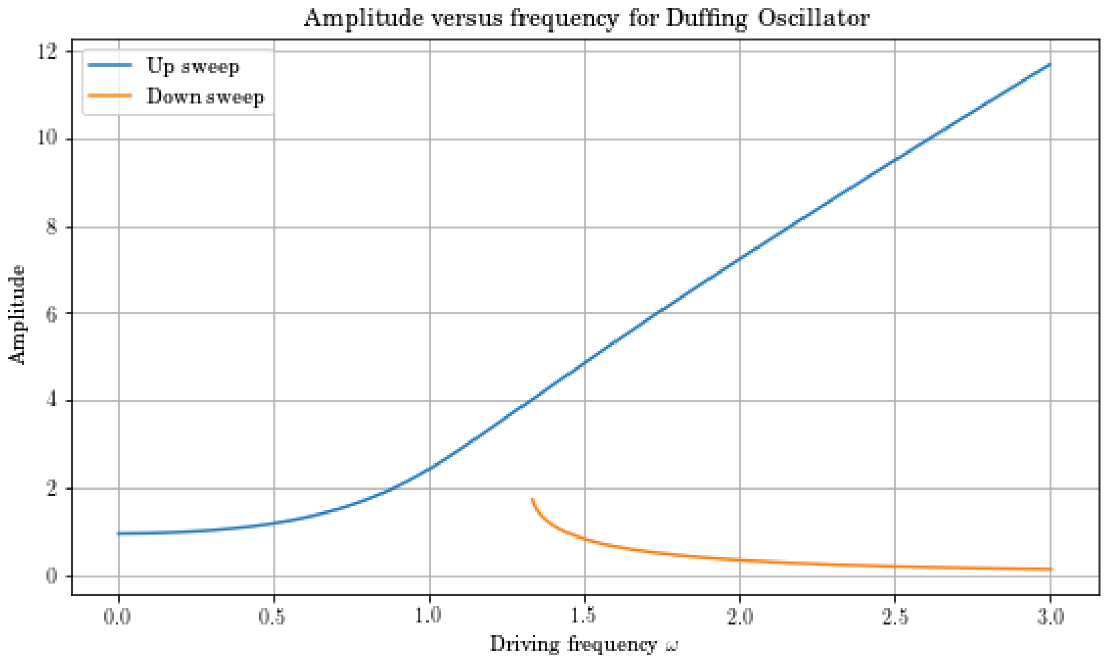

```
In [21]: omega = np.arange(0, 3, 1 / 200) + 1 / 200
         amp = sp.zeros_like(omega)
         amp[:] = np.nan
         t, x, e, amps, phases = ms.hb_so(duff_osc_ss, num_variables=2,
                                  omega=1 / 200, eqform='first_order', num_harmonics=1)
         for i, freq in enumerate(omega):
             # Here we try to obtain solutions, but if they don't work,
             # we ignore them by inserting `np.nan` values.
             x = x - sp.average(x)
             try:
                 t, x, e, amps, phases =
                 ms.hb_so(duff_osc_ss, x0=x,
                          omega=freq, eqform='first_order', num_harmonics=1)
                 amp[i] = amps[0]
             except:
                 amp[i] = np.nan
             if np.isnan(amp[i]):
                 break
         plt.plot(omega, amp)
```



Let's sweep through driving frequencies to find a frequency response function

```
In [7]: omegal = np.arange(3, .03, -1 / 200) + 1 / 200
        ampl = sp.zeros_like(omegal)
        ampl[:] = np.nan
        t, x, e, amps, phases = ms.hb_so(duff_osc_ss, num_variables=2,
                                         omega=3, eqform='first_order', num_harmonics=1)
        for i, freq in enumerate(omegal):
            # Here we try to obtain solutions, but if they don't work,
            # we ignore them by inserting `np.nan` values.
            x = x - np.average(x)
            try:
                t, x, e, amps, phases =
                ms.hb_so(duff_osc_ss, x0=x,
                         omega=freq, eqform='first_order', num_harmonics=1)
                ampl[i] = amps[0]
            except:
                ampl[i] = np.nan
            if np.isnan(ampl[i]):
                break
```

```
In [8]: plt.plot(omega,amp, label='Up sweep')
        plt.plot(omegal,ampl, label='Down sweep')
        plt.legend()
        plt.title('Amplitude versus frequency for Duffing Oscillator')
        plt.xlabel('Driving frequency $\\omega$')
        plt.ylabel('Amplitude')
        plt.grid()
```



1.4.2 Two degree of freedom system

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \alpha x_1^3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ A\sin(\omega t) \end{bmatrix}$$

```
In [9]: def two_dof_demo(x, params):
            omega = params['omega']
            t = params['cur_time']
            force_amplitude = params['force_amplitude']
            alpha = params['alpha']
            # The following could call an external code to obtain the state derivatives
            xd = np.array([[x[1]],
                           [-2 * x[0] - alpha * x[0]**3 + x[2]],
                           [x[3]],
                           [-2 * x[2] + x[0]]] + force_amplitude * np.sin(omega * t))
            return xd
```

Let's find a response.

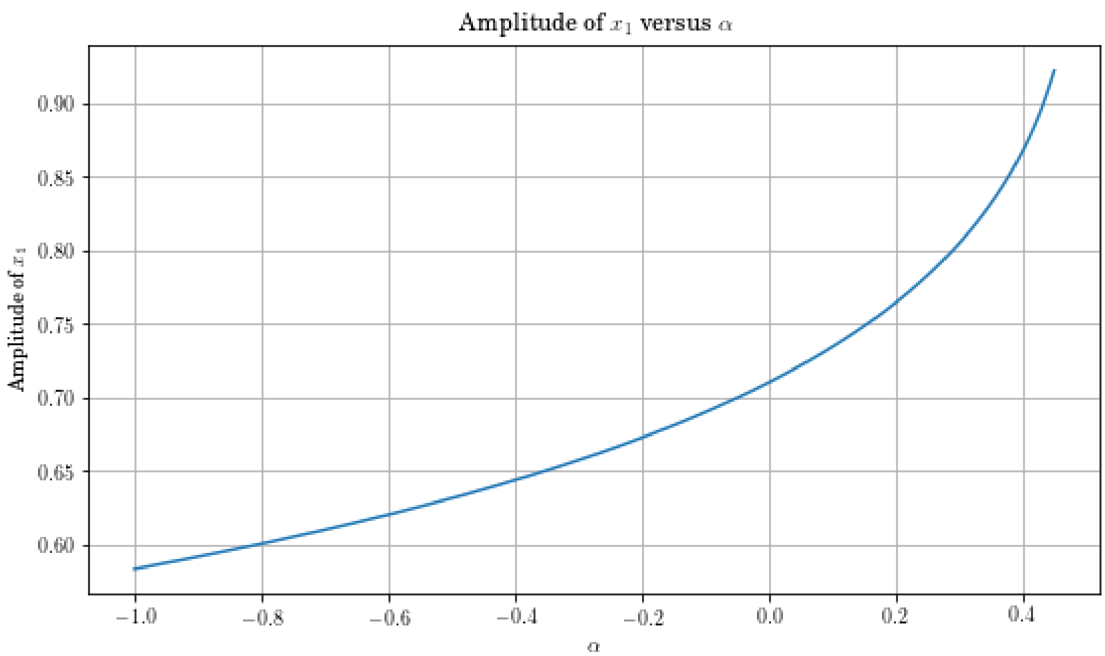
```
In [10]: parameters = {'force_amplitude': 0.2}
         parameters['alpha'] = 0.4
         t, x, e, amps, phases = ms.hb_so(two_dof_demo, num_variables=4,
                                          omega=1.2, eqform='first_order', params=parameters)
         amps

Out[10]: array([ 0.86696762,  0.89484597,  0.99030411,  1.04097851])
```

Or a parametric study of response amplitude versus nonlinearity.

```
In [11]: alpha = np.linspace(-1, .45, 2000)
         amp = np.zeros_like(alpha)
         for i, alphai in enumerate(alpha):
             parameters['alpha'] = alphai
             t, x, e, amps, phases = ms.hb_so(two_dof_demo, num_variables=4, omega=1.2,
                                              eqform='first_order', params=parameters)
             amp[i] = amps[0]
```

```
In [12]: plt.plot(alpha,amp)
         plt.title('Amplitude of $x_1$ versus $\\alpha$')
         plt.ylabel('Amplitude of $x_1$')
         plt.xlabel('$\\alpha$')
         plt.grid()
```



1.4.3 Two degree of freedom system with Coulomb Damping

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\ddot{x}_1\\\ddot{x}_2\end{bmatrix}\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}\mu|\dot{x}|_1\\0\end{bmatrix}=\begin{bmatrix}0\\A\sin(\omega t)\end{bmatrix}$$

```
In [13]: def two_dof_coulomb(x, params):
             omega = params['omega']
             t = params['cur_time']
             force_amplitude = params['force_amplitude']
             mu = params['mu']
             # The following could call an external code to obtain the state derivatives
```

```
        xd = np.array([[x[1]],
                       [-2 * x[0] - mu * np.abs(x[1]) + x[2]],
                       [x[3]],
                       [-2 * x[2] + x[0]]] + force_amplitude * np.sin(omega * t))
        return xd
```

In [14]:
```
parameters = {'force_amplitude': 0.2}
parameters['mu'] = 0.1
t, x, e, amps, phases = ms.hb_so(two_dof_coulomb, num_variables=4,
                                 omega=1.2, eqform='first_order', params=parameters)
amps
```

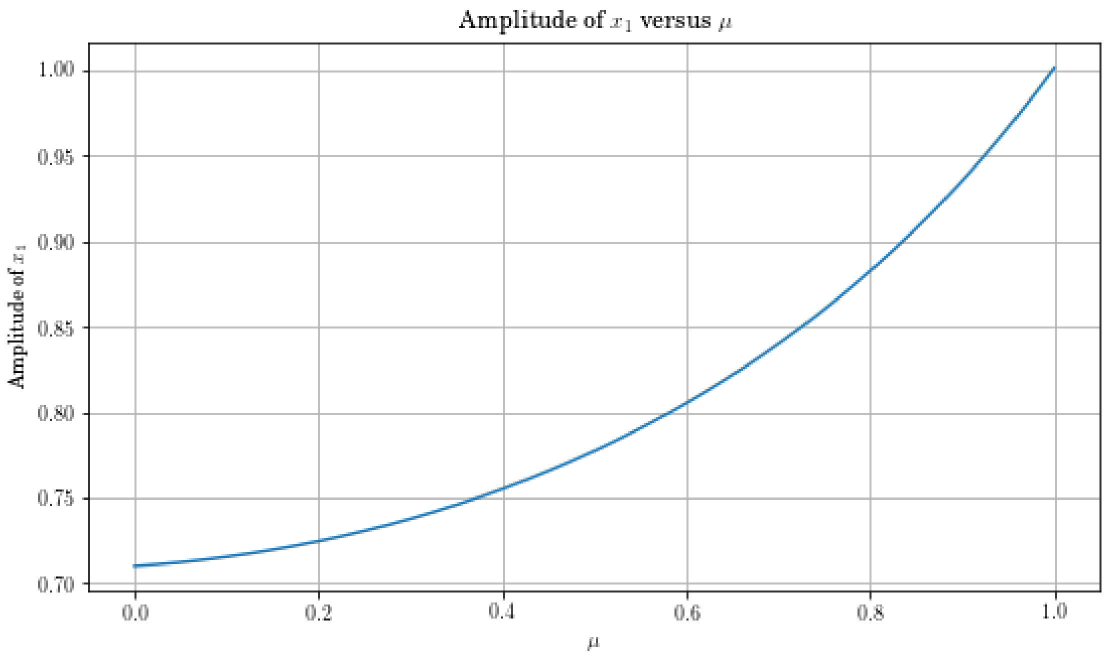
Out[14]:
```
array([ 0.68916938,  0.68228249,  0.67299987,  0.66065012])
```

In [31]:
```
mu = np.linspace(0, 1.0, 200)
amp = np.zeros_like(mu)
for i, mui in enumerate(mu):
    parameters['mu'] = mui
    t, x, e, amps, phases = ms.hb_so(two_dof_coulomb, num_variables=4, omega=1.2,
                                     eqform='first_order', num_harmonics=3, params=para
    amp[i] = amps[0]
```

Too much Coulomb friction can increase the response.

- Did you know that?
- This damping shifted resonance.

In [32]:
```
plt.plot(mu,amp)
plt.title('Amplitude of $x_1$ versus $\\mu$')
plt.ylabel('Amplitude of $x_1$')
plt.xlabel('$\\mu$')
plt.grid()
```



1.4.4 But can I solve an equation in one line? Yes!!!

Damped Duffing oscillator in one command.

```
In [25]: out = ms.hb_so(lambda x, v,
                        params: np.array([[-x - .1 * x**3 - .1 * v + 1 *
                                           sin(params['omega'] * params['cur_time'])]]),
                        num_variables=1, omega=.7, num_harmonics=1)
         out[3][0]
```

```
Out[25]: 1.4779630014433971
```

OK - that's a bit obtuse. I wouldn't do that normally, but Mousai can.

1.5 How to get this?

- Install Scientific Python from SciPy.org
  - AFRL: See your tech support to get the Enthought distribution installed
- See the Mousai documents for installation instructions
  - `pip install mousai`
  - AFRL: Talk to me- install is easy if I send you the files.
- See Mousai on GitHub (https://github.com/josephcslater/mousai)

1.6 Conclusions

- Nonlinear frequency solutions are within reach of undergraduates
- Installation is trivial
- Already in use (GitHub logs indicate dozens of users)
- Custom special case and proprietary solvers such as BDamper can be replaced for free
- Research potential is about to be unleashed

1.7 Future

- Add time-averaging method
  - currently requires high number of harmonics for non-smooth systems
- Add masking of known harmonics (average is often fixed and known)
- Automated sweep control
- Branch following
- Condense the one-line method
- Evaluate on large scale problems
  - Currently attempting to hook to ANSYS
- Parallelize
- Leverage CUDA