# Projective Geometery

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# Chapter 1

# Conics

## 1.1 Dandelin Spheres

Germinal Pierre Dendelin, a 19th century French-Belgian Professor, discovered this beautiful proof to demonstrate that any plane that cuts through a right circular cone produces a quadratic curve.

**Theorem 1.** *When a plane intersects a right circular cone, the curve produced will either be an ellipse, a parabola or a hyperbola.*

*Proof.* Place a sphere tangent to the intersecting plane $\pi$ and the cone such that it touches the plane at $F$, and the cone in a circle $C$ with centre $O$ that lies on a horizontal plane $\epsilon$ [1].
Take an aribtrary point $P$ on the curve $Q$, and extend the line $VP$ from the vertex $V$ of the cone to meet $C$ at point $L$. Let $D$ be the point on the intersection on the planes $\pi$ and $\epsilon$ such that $PD$ is perpendicular to the line of intersection.(If the planes do not intersect, $Q$ will be a circle)

Drop a perpendicular $PM$ on $OL$ such that $\triangle PML$ and $\triangle PMD$ are both right angled. Denote $\angle PLM$ as $\alpha$, and $\angle PDM$ as $\beta$.
From the triangles $\triangle PML$ and $\triangle PMD$

$$\begin{aligned} \sin\alpha &= \frac{PM}{PD} \\ \text{and} \quad \sin\beta &= \frac{PM}{PL} \\ \text{i.e.} \quad \frac{PL}{PD} &= \frac{\sin\alpha}{\sin\beta} \end{aligned}$$

[1] Assuming that there exists atleast one such sphere

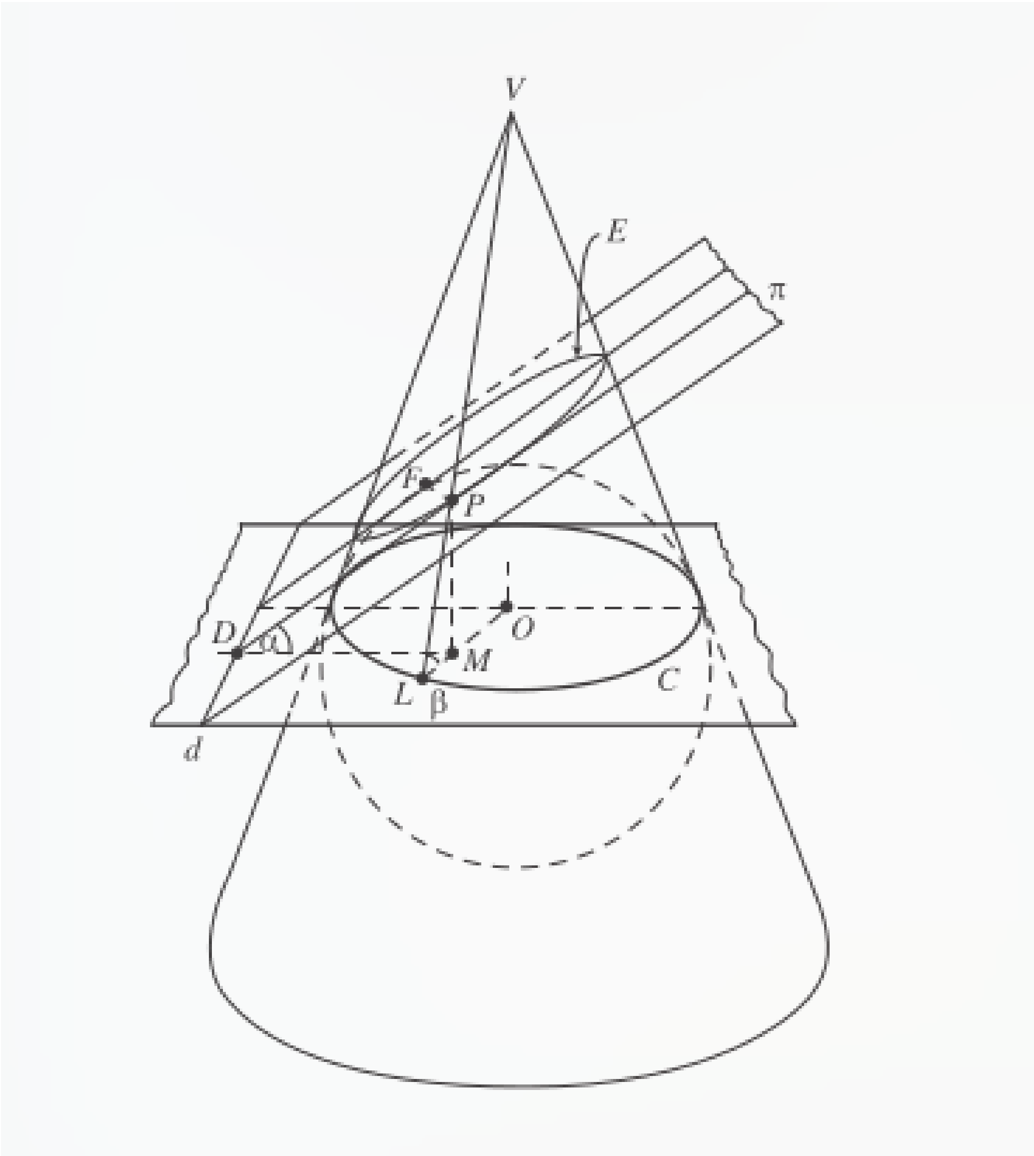


Figure 1.1: When $0 < \alpha < \beta < \frac{\pi}{2}$

Since $PL$ and $PF$ are both tangents from $P$ to the sphere, $PF = PL$. Therfore,

$$\frac{PF}{PD} = \frac{\sin\alpha}{\sin\beta}$$

i.e. $PF = e \cdot PD$, where $e = \sin\alpha / \sin\beta$

It follows from the focus - directrix definition that $Q$ will be an ellipse if $\alpha < \beta$, a parabola if $\alpha = \beta$, or a hyperbola if $\alpha > \beta$. $\square$

## 1.2 Group Laws on Conics