PROJECTIVE GEOMETRY

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CHAPTER 1
Conics

The majority of this chapter is based on the ideas presented in Shirali's article on conic groups [Shi09]. We'll also develop this further with the chapter on conics over characteristic two fields.

1.1 Group Laws on Conics

Consider a non-degenerate conic section $\mathcal{C}$ and a point $O \in \mathcal{C}$. For any points $P, Q \in \mathcal{C}$, let ℓ' be the line passing through O such that $\ell' \parallel \ell$ where ℓ is the line joining P and Q. If ℓ' intersects $\mathcal{C}$ at a point other than O, call that point R. Otherwise, take $R = O$. Define a binary operation $\oplus_O \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ as $P \oplus_O Q := R$.

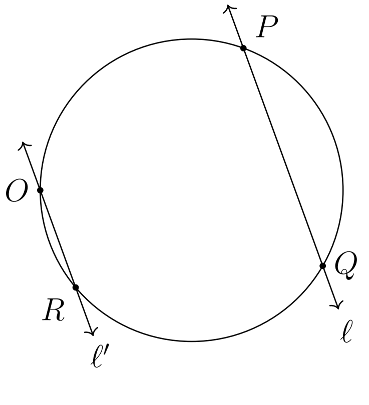


Figure 1.1 : $P \oplus_O Q$ when $\mathcal{C}$ is a circle.

We'll first find formulae to calculate $P \oplus_O Q$ and then proceed to prove that $\mathcal{C}$ is a group with $\oplus_O$.

A Note on Standard Forms

Throughout this section, we'll only use standard forms of non-degenerate conics i.e. circle, rectangular hyperbola and parabola with equations $x^2+y^2 = 1$, $xy = 1$ and $y = x^2$ respectively. The other non-degenerate conics in their respective classes are affine-congruent (we'll introduce this notion in Chapter 3) to these standard forms [BEG12, § 2.5 Thm. 4].

Circle

If $\mathcal{C} = \mathcal{S}$ with equation $x^2 + y^2 = 1$, any point $P \in \mathcal{S}$ has coordinates $(\cos t, \sin t)$ where $t \in [0, 2\pi)$ is the angle P forms with the positive x-axis in the counter-clockwise direction.

Let $O, P, Q, R \in \mathcal{P}$ be points with parameters t_0, t_1, t_2 and t_3 respectively such that $P \oplus_O Q = R$. By definition of $P \oplus_O Q$, we have $PQ \parallel OR$. Note that if $P = Q$, then slope at P is

$$y'|_{x=t_1} = \left(-\frac{x}{y}\right)_{t=t_1} = \left(-\frac{\cos t}{\sin t}\right)_{t=t_1} = -\cot t_1 = -\cot\left(\frac{t_1+t_2}{2}\right)$$

and if $P \neq Q$, then $t_1 \neq t_2$ and slope of PQ is

$$\frac{\sin t_2 - \sin t_1}{\cos t_2 - \cos t_1} = -\frac{\sin\left(\frac{t_2-t_1}{2}\right)\cos\left(\frac{t_2+t_1}{2}\right)}{\sin\left(\frac{t_2-t_1}{2}\right)\sin\left(\frac{t_2+t_1}{2}\right)} = -\cot\left(\frac{t_2+t_1}{2}\right)$$

Also note that $\sin\left(\frac{t_2-t_1}{2}\right)$ can be cancelled as it's only zero when $t_2 = t_1 + 2n\pi$ which means $P = Q$. So, we don't need to consider the points being same as a separate case. Equating slopes of PQ and OR, we get,

$$\begin{aligned} -\cot\left(\frac{t_2+t_1}{2}\right) &= -\cot\left(\frac{t_3+t_0}{2}\right) \\ \implies \frac{t_2+t_1}{2} &= n\pi + \frac{t_3+t_0}{2} \\ \implies t_3 &= t_2 + t_1 - t_0 - 2n\pi \end{aligned}$$

As shifts of $2n\pi$ don't affect t_3, we can ignore that term on the RHS. Thus for any $P, Q \in \mathcal{S}$ with parameters t_1 and t_2 respectively for circle $\mathcal{S}$, $P \oplus_O Q = R$ has parameter $t_3 = t_1 + t_2 - t_0$ where t_0 is the parameter for point O. Note that we always add or subtract multiples of 2π to make sure $t_3 \in [0, 2\pi)$.

It is easy to see that $\oplus_O$ satisfies closure for $\mathcal{S}$. We'll verfiy each of the group axioms now.

1. **Identity:** For any $P \in \mathcal{S}$ with parameter t, $P \oplus_O O$ will have parameter

   $$t' = t + t_0 - t_0 = t$$

   Thus O acts as the identity element for $\oplus_O$.

2. **Inverse:** The point $Q \in \mathcal{S}$ with parameter $2t_0 - t$ gives the parameter of $P \oplus_O Q$ to be

   $$t' = t + 2t_0 - t - t_0 = t_0$$

   Hence, Q is the inverse of P.

3. **Associativity:** For any $P, Q, R \in \mathcal{S}$ with parameters t_1, t_2 and t_3 respectively, $P \oplus_O (Q \oplus_O R)$ has parameter

$$t_1 + (t_2 + t_3 - t_0) - t_0 = t_1 + t_2 + t_3 - 2t_0$$

   On the other hand, $(P \oplus_O Q) \oplus_O R$ has parameter

$$(t_1 + t_2 - t_0) + t_3 - t_0 = t_1 + t_2 + t_3 - 2t_0$$

   Thus $\oplus_O$ is associative.

This shows that $\mathcal{S}$ is a group with $\oplus_O$.

Theorem 1. $\langle \mathcal{S}, \oplus_O \rangle \cong \langle S^1, \cdot \rangle$ *where* $S^1 = \{e^{i\theta} \in \mathbb{C} \colon \theta \in [0, 2\pi)\}$.

Proof. Consider $\varphi \colon \mathcal{S} \to S^1$ given by $\varphi((\cos\theta, \sin\theta)) = e^{i(\theta - \theta_0)}$. For any points $P, Q \in \mathcal{S}$ parametrized by θ_1 and θ_2 respectively, $P \oplus_O Q$ has parameter $\theta_1 + \theta_2 - \theta_0$. So,

$$\varphi(P \oplus_O Q) = e^{i(\theta_1 + \theta_2 - 2\theta_0)} = e^{i(\theta_1 - \theta_0)} e^{i(\theta_2 - \theta_0)} = \varphi(P)\varphi(Q)$$

Thus φ is a homomorphism.

If $\varphi(P) = \varphi(Q)$ for some $P, Q \in \mathcal{S}$ parametrized by θ_1 and θ_2 respectively, then

$$e^{i(\theta_1 - \theta_0)} = e^{i(\theta_2 - \theta_0)} \implies e^{i\theta_1} e^{-i\theta_0} = e^{i\theta_2} e^{-i\theta_0} \implies e^{i\theta_1} = e^{i\theta_2} \implies \theta_1 = 2n\pi + \theta_2$$

i.e. $P = Q$. Thus φ is injective.

For any $e^{i\theta} \in S^1$, we have the point $P = (\cos(\theta + \theta_0), \sin(\theta + \theta_0)) \in \mathcal{S}$ such that

$$\varphi(P) = e^{i(\theta + \theta_0 - \theta_0)} = e^{i\theta}$$

Thus φ is surjective. This shows that φ is a bijective homomorphism i.e. an isomorphism from $\langle \mathcal{S}, \oplus_O \rangle$ to $\langle S^1, \cdot \rangle$. ■

Parabola

If $\mathcal{C} = \mathcal{P}$ is the parabola with equation $y = x^2$, any point on it can be parametrized as (t, t^2) where $t \in \mathbb{R}$.

Let $O, P, Q, R \in \mathcal{P}$ be points with parameters t_0, t_1, t_2 and t_3 respectively such that $P \oplus_O Q = R$. By definition of $P \oplus_O Q$, we have $PQ \parallel OR$. Note that if $P = Q$, then slope at P is

$$y'|_{x=t_1} = (2x)_{x=t_1} = 2t_1 = t_1 + t_2$$

and if $P \neq Q$, then $t_1 \neq t_2$ and slope of PQ is

$$\frac{t_2^2 - t_1^2}{t_2 - t_1} = t_1 + t_2$$

So, we don't need to consider the points being same as a separate case. Equating slopes of PQ and OR, we get,

$$t_1 + t_2 = t_0 + t_3 \implies t_3 = t_1 + t_2 - t_0$$

Thus, for any points $P, Q \in \mathcal{P}$ with parameters t_1 and t_2 respectively for a parabola $\mathcal{P}$, $P \oplus_O Q = R$ has parameter $t_3 = t_1 + t_2 - t_0$ where t_0 is the parameter for point O.

It is easy to see that $\oplus_O$ satisfies closure for $\mathcal{P}$. We'll verfiy each of the group axioms now.

1. **Identity:** For any $P \in \mathcal{P}$ with parameter t, $P \oplus_O O$ will have parameter

$$t' = t + t_0 - t_0 = t$$

   Thus O acts as the identity element for $\oplus_O$.

2. **Inverse:** The point $Q \in \mathcal{P}$ with parameter $2t_0 - t$ gives the parameter of $P \oplus_O Q$ to be

$$t' = t + 2t_0 - t - t_0 = t_0$$

   Hence, Q is the inverse of P.

3. **Associativity:** For any $P, Q, R \in \mathcal{P}$ with parameters t_1, t_2 and t_3 respectively, $P \oplus_O (Q \oplus_O R)$ has parameter

$$t_1 + (t_2 + t_3 - t_0) - t_0 = t_1 + t_2 + t_3 - 2t_0$$

   On the other hand, $(P \oplus_O Q) \oplus_O R$ has parameter

$$(t_1 + t_2 - t_0) + t_3 - t_0 = t_1 + t_2 + t_3 - 2t_0$$

   Thus $\oplus_O$ is associative.

This shows that $\mathcal{P}$ is a group with $\oplus_O$.

Theorem 2. $\langle \mathcal{P}, \oplus_O \rangle \cong \langle \mathbb{R}, + \rangle$.

Proof. Consider $\varphi \colon \mathcal{P} \to \mathbb{R}$ given by $\varphi((t, t^2)) = t - t_0$. For any points $P, Q \in \mathcal{P}$ parametrized by t_1 and t_2 respectively, $P \oplus_O Q$ has parameter $t_1 + t_2 - t_0$. So,

$$\varphi(P \oplus_O Q) = t_1 + t_2 - 2t_0 = (t_1 - t_0) + (t_2 - t_0) = \varphi(P) + \varphi(Q)$$

Thus φ is a homomorphism.

If $\varphi(P) = \varphi(Q)$ for some $P, Q \in \mathcal{P}$ parametrized by t_1 and t_2 respectively, then

$$t_1 - t_0 = t_2 - t_0 \implies t_1 = t_2$$

i.e. $P = Q$. Thus φ is injective.

For any $t \in \mathbb{R}$, we have the point $P = (t + t_0, (t + t_0)^2) \in \mathcal{P}$ such that

$$\varphi(P) = t + t_0 - t_0 = t$$

Thus φ is surjective. This shows that φ is a bijective homomorphism i.e. an isomorphism from $\langle \mathcal{P}, \oplus_O \rangle$ to $\langle \mathbb{R}, + \rangle$. ■

Hyperbola

If $\mathcal{C} = \mathcal{H}$ is the rectangular hyperbola with equation $xy = 1$, any point on it can be parametrized as (t, t^{-1}) where $t \in \mathbb{R}^{\times}$.

Let $O, P, Q, R \in \mathcal{H}$ be points with parameters t_0, t_1, t_2 and t_3 respectively such that $P \oplus_O Q = R$. By definition of $P \oplus_O Q$, we have $PQ \parallel OR$. Note that if $P = Q$, then slope at P is

$$y'|_{x=t_1} = \left(-\frac{1}{x^2}\right)_{x=t_1} = -\frac{1}{t_1^2} = -\frac{1}{t_1 t_2}$$

and if $P \neq Q$, then $t_1 \neq t_2$ and slope of PQ is

$$\frac{t_2^{-1} - t_1^{-1}}{t_2 - t_1} = \frac{t_1 - t_2}{t_1 t_2 (t_2 - t_1)} = -\frac{1}{t_1 t_2}$$

So, we don't need to consider points being same as a separate case. Equating slopes of PQ and OR, we get,

$$-\frac{1}{t_1 t_2} = -\frac{1}{t_0 t_3} \implies t_3 = \frac{t_1 t_2}{t_0}$$

Thus, for any points $P, Q \in \mathcal{H}$ with parameters t_1 and t_2 respectively for a rectangular hyperbola $\mathcal{H}$, $P \oplus_O Q = R$ has parameter $t_3 = t_1 t_2 t_0^{-1}$ where t_0 is the parameter corresponding to point O.

It is easy to see that $\oplus_O$ satisfies closure for $\mathcal{H}$. We'll verfiy each of the group axioms now.

1. **Identity:** For any $P \in \mathcal{H}$ with parameter t, $P \oplus_O O$ will have parameter

   $$t' = t t_0 t_0^{-1} = t$$

   Thus O acts as the identity element for $\oplus_O$.

2. **Inverse:** The point $Q \in \mathcal{H}$ with parameter $t_0^2 t^{-1}$ gives the parameter of $P \oplus_O Q$ to be

   $$t' = t(t_0^2 t^{-1}) t_0^{-1} = t_0$$

   Hence, Q is the inverse of P.

3. **Associativity:** For any $P, Q, R \in \mathcal{H}$ with parameters t_1, t_2 and t_3 respectively, $P \oplus_O (Q \oplus_O R)$ has parameter

   $$t_1(t_2 t_3 t_0^{-1}) t_0^{-1} = t_1 t_2 t_3 t_0^{-2}$$

   On the other hand, $(P \oplus_O Q) \oplus_O R$ has parameter

   $$(t_1 t_2 t_0^{-1}) t_3 t_0^{-1} = t_1 t_2 t_3 t_0^{-2}$$

   Thus $\oplus_O$ is associative.

This shows that $\mathcal{H}$ is a group with $\oplus_O$.

Theorem 3. $\langle \mathcal{H}, \oplus_O \rangle \cong \langle \mathbb{R}^\times, \cdot \rangle$.

Proof. Consider $\varphi \colon \mathcal{H} \to \mathbb{R}^\times$ given by $\varphi((t, t^{-1})) = tt_0^{-1}$. For any points $P, Q \in \mathcal{H}$ parametrized by t_1 and t_2 respectively, $P \oplus_O Q$ has parameter $t_1 t_2 t_0^{-1}$. So,

$$\varphi(P \oplus_O Q) = t_1 t_2 t_0^{-2} = (t_1 t_0^{-1})(t_2 t_0^{-1}) = \varphi(P)\varphi(Q)$$

Thus φ is a homomorphism.

If $\varphi(P) = \varphi(Q)$ for some $P, Q \in \mathcal{H}$ parametrized by t_1 and t_2 respectively, then

$$t_1 t_0^{-1} = t_2 t_0^{-1} \implies t_1 = t_2$$

i.e. $P = Q$. Thus φ is injective.

For any $t \in \mathbb{R}$, we have the point $P = (tt_0, (tt_0)^{-1}) \in \mathcal{H}$ such that

$$\varphi(P) = tt_0 t_0^{-1} = t$$

Thus φ is surjective. This shows that φ is a bijective homomorphism i.e. an isomorphism from $\langle \mathcal{H}, \oplus_O \rangle$ to $\langle \mathbb{R}^\times, \cdot \rangle$. ■

1.2 Generalizing to any field

Note: *Throughout this section, we'll limit ourselves to fields whose characteristic is not 2 as fields with characteristic 2 require a more careful treatment. We'll have a look at these in Chapter 2.*

In the previous section, we've considered our conic as the set of points $(x, y) \in \mathbb{R}^2$ that make $f(x, y) = 0$ where $f \in \mathbb{R}[x, y]$ is square-free and has degree 2. We could very well have considered a similar set for any field $\mathbb{F}$ and we'll now show how a similar operation gives rise to a group structure.

We'll consider $\mathbb{F}^2$ as a vector space for the rest of this section. Consider a set

$$\mathcal{C} = \{(x, y) \in \mathbb{F}^2 : f(x, y) = 0\}$$

where $f \in \mathbb{F}[x, y]$ is square-free and has degree 2. Fix an $\vec{O} = (x_0, y_0) \in \mathcal{C}$. For any $\vec{A}, \vec{B} \in \mathcal{C}$ where $\vec{A} = (a_1, a_2)$ and $\vec{B} = (b_1, b_2)$.

Let

$$\vec{c} = \begin{cases} \vec{B} - \vec{A} & \text{if } \vec{A} \neq \vec{B} \\ \left(\frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x}\right)_{(x,y)=\vec{A}} & \text{otherwise} \end{cases}$$

$$\ell = \{\vec{x} \in \mathbb{F}^2 \colon \vec{x} = \vec{O} + \lambda \vec{c} \quad \forall \lambda \in \mathbb{F}\}$$

Note that the partial derivative above is a formal derivative since we considered f to be a polynomial in x and y. We aren't really considering any limits here. Clearly, $\vec{O} \in \mathcal{C} \cap \ell$. Now, $|\mathcal{C} \cap \ell|$ can either be 1 or 2 (from the Bézout bound). Define

$$\vec{A} \oplus_O \vec{B} := \begin{cases} \vec{C} & \text{if } \mathcal{C} \cap \ell = \{\vec{O}, \vec{C}\} \\ \vec{O} & \text{if } \mathcal{C} \cap \ell = \{\vec{O}\} \end{cases}$$

Hyperbola and Parabola

For $\mathcal{C} = \mathcal{P}$ and $\mathcal{C} = \mathcal{H}$, we get $f(x, y)$ to be $y - x^2$ and $xy - 1$ respectively. In both cases, the parametrization we used for $\mathbb{R}^2$ case works for $\mathbb{F}^2$ as well. Further, even our formula for the operation extends nicely to $\mathbb{F}^2$ as the derivation didn't really use any properties special to the vector space $\mathbb{R}^2$. So, we have $\langle \mathcal{P}, \oplus_O \rangle \cong \langle \mathbb{F}, + \rangle$ and $\langle \mathcal{H}, \oplus_O \rangle \cong \langle \mathbb{F}^\times, \cdot \rangle$.

Circle

For $\mathcal{C} = \mathcal{S}$, we get $f(x, y) = x^2 + y^2 - 1$. This curve has radial symmetry, so we can always apply a rotation to it such that $\vec{O} = (1, 0)$. Our goal is to find λ such that $\vec{O} + \lambda \vec{c} \in \mathcal{S}$. Suppose $\vec{c} = (z, w)$. Any point on $\mathcal{S}$ must satisfy $x^2 + y^2 = 1$. Thus

$$\begin{aligned}
&(1 + \lambda z)^2 + (0 + \lambda w)^2 = 1 \\
\implies &1 + \lambda^2(z^2 + w^2) + 2\lambda z = 1 \\
\implies &\lambda^2(z^2 + w^2) + 2\lambda z = 0 \\
\implies &\lambda((z^2 + w^2)\lambda + 2z) = 0 \\
\implies &\lambda = 0 \text{ or } \lambda = -\frac{2z}{z^2 + w^2}
\end{aligned}$$

Since $P \neq Q$, $(z, w) = (b_1 - a_1, b_2 - a_2)$. If $z^2 + w^2 = 0$, then

$$\begin{aligned}
&b^2 + a^2 + a^2 + b^2 - 2a_1b_1 - 2a_2b_2 = 0 \\
\implies &a_1b_1 = 1 - a_2b_2 \\
\implies &a_1^2b_1^2 = 1 + a_2^2b_2^2 - 2a_2b_2 \\
\implies &a_1^2b_1^2 = 1 + (1 - a_1^2)(1 - b_1^2) - 2a_2b_2 \\
\implies &2a_2b_2 = 1 - a_1^2 + 1 - b_1^2 \\
\implies &a_2^2 + b_2^2 - 2a_2b_2 = 0 \\
\implies &(a_2 - b_2)^2 = 0 \\
\implies &a_2 = b_2
\end{aligned}$$

It is now easy to see that $a_1^2 = b_1^2$ or $a_1 = \pm b_1$. If $a_1 = b_1$, then $P = Q$ which is a contradiction. If $a_1 = -b_1$, then $(z, w) = (2b_1, 0)$ but this means $4b_1^2 = 0$ or $b_1 = a_1 = 0$ or $P = Q$ which is again a contradiction. Hence, we can safely assume $z^2 + w^2 \neq 0$ when $P \neq Q$. The first solution just corresponds to $\vec{O}$, hence we take the second one. So, $\vec{A} \oplus_O \vec{B} = (1 + \lambda z, \lambda w)$.

If $\vec{A} \neq \vec{B}$, then $\vec{c} = (z, w) = (b_1 - a_1, b_2 - a_2)$. This means the first coordinate is

$$\begin{aligned}
1 + \lambda z &= \frac{z^2 + w^2 - 2z^2}{z^2 + w^2} \\
&= \frac{1 - b_1^2 - a_1^2 - a_2b_2 + a_1b_1}{1 - a_1b_1 - a_2b_2} \\
&= \frac{(1 - b_1^2 - a_1^2 - a_2b_2 + a_1b_1)(a_1b_1 - a_2b_2)}{(1 - a_1b_1 - a_2b_2)(a_1b_1 - a_2b_2)} \\
&= \frac{(1 - b_1^2 - a_1^2 - a_2b_2 + a_1b_1)(a_1b_1 - a_2b_2)}{1 - b_1^2 - a_1^2 - a_2b_2 + a_1b_1} \\
&= a_1b_1 - a_2b_2
\end{aligned}$$

and the second coordinate is

$$\begin{aligned}\lambda w &= \frac{-2zw}{z^2+w^2}\\ &= \frac{-(b_1b_2+a_1a_2-a_1b_2-a_2b_1)}{1-a_1b_1-a_2b_2}\\ &= \frac{-(b_1b_2+a_1a_2-a_1b_2-a_2b_1)(a_1b_2+a_2b_1)}{(1-a_1b_1-a_2b_2)(a_1b_2+a_2b_1)}\\ &= \frac{-(b_1b_2+a_1a_2-a_1b_2-a_2b_1)(a_1b_2+a_2b_1)}{a_1b_2+a_2b_1-b_1b_2-a_1a_2}\\ &= a_1b_2+a_2b_1\end{aligned}$$

If $\vec{A}=\vec{B}$, then $\vec{c}=(z,w)=(2a_2,-2a_1)$. So,

$$\begin{aligned}1+\lambda z &= 1+\frac{-4a_2(2a_2)}{4a_2^2+4a_1^2} = 1-2a_2^2 = a_1^2-a_2^2\\ \text{and } \lambda w &= \frac{-4a_2(-2a_1)}{4a_2^2+4a_1^2} = 2a_1a_2\end{aligned}$$

Hence, $\vec{A}\oplus_O\vec{B}=(a_1b_1-a_2b_2, a_1b_2+a_2b_1)$ for any points $\vec{A},\vec{B}\in\mathcal{S}$.

Theorem 4. *If S is defined over $\mathbb{F}^2$, $\langle\mathcal{S},\oplus_O\rangle\cong\langle\mathrm{SO}_2(\mathbb{F}),\cdot\rangle$.*

Proof. Consider $\varphi\colon\mathcal{S}\to\mathrm{SO}_2(\mathbb{F})$ given by

$$\varphi((a_1,a_2))=\begin{bmatrix}a_1 & -a_2\\ a_2 & a_1\end{bmatrix}$$

It is easy to see that $\det\varphi((a_1,a_2))=a_1^2+a_2^2=1$. Further, the columns are orthogonal to each other as $-a_1a_2+a_2a_1=0$.

For any $(a_1,a_2),(b_1,b_2)\in\mathcal{S}$,

$$\begin{aligned}\varphi((a_1,a_2))\varphi((b_1,b_2)) &= \begin{bmatrix}a_1 & -a_2\\ a_2 & a_1\end{bmatrix}\begin{bmatrix}b_1 & -b_2\\ b_2 & b_1\end{bmatrix}\\ &= \begin{bmatrix}a_1b_1-a_2b_2 & -a_1b_2-a_2b_1\\ a_1b_2+a_2b_1 & a_1b_1-a_2b_2\end{bmatrix}\\ &= \varphi((a_1b_1-a_2b_2, a_1b_2+a_2b_1))\\ &= \varphi((a_1,a_2)\oplus_O(b_1,b_2))\end{aligned}$$

Thus φ is a homomorphism.

For any $(a_1,a_2),(b_1,b_2)\in\mathcal{S}$,

$$\varphi((a_1,a_2))=\varphi((b_1,b_2)) \implies \begin{bmatrix}a_1 & -a_2\\ a_2 & a_1\end{bmatrix}=\begin{bmatrix}b_1 & -b_2\\ b_2 & b_1\end{bmatrix} \implies (a_1,a_2)=(b_1,b_2)$$

Thus φ is injective.

Consider any $M \in \mathrm{SO}_2(\mathbb{F})$, where

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then, by definition of $\mathrm{SO}_2(\mathbb{F})$, $ad - bc = 1$ and $MM^{\mathrm{T}} = I$. The second condition gives

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \implies a^2 + b^2 &= 1 \\ c^2 + d^2 &= 1 \\ ac + bd &= 0 \end{aligned}$$

Using these, we get $a = d$ and $b = -c$. Consider a point $(a, b) \in \mathbb{F}^2$. Since $a^2 + b^2 = 1$, $(a, b) \in \mathcal{S}$. Further, $\varphi((a, b)) = M$. Thus φ is surjective. This shows that φ is a bijective homomorphism i.e. an isomorphism from $\langle \mathcal{S}, \oplus_O \rangle$ to $\langle \mathrm{SO}_2(\mathbb{F}), \cdot \rangle$. ■

Theorem 5. *If $x^2 + 1 = 0$ has a solution in $\mathbb{F}$, then $\langle \mathrm{SO}_2(\mathbb{F}), \cdot \rangle \cong \langle \mathbb{F}^\times, \cdot \rangle$.*

Proof. Let $i \in \mathbb{F}$ be a solution to $x^2 + 1 = 0$. From the previous proof, we have, for any $M(a, b) \in \mathrm{SO}_2(\mathbb{F})$,

$$M(a, b) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

where $a, b \in \mathbb{F}$. The characteristic polynomial of $M(a, b)$ is $(a-\lambda)^2+b^2$ or $\lambda^2-2a\lambda+a^2+b^2$. Thus the eigenvalues are $a \pm ib$. The corresponding eigenvectors will be $(1, \mp i)$. We can then write M as a diagonal matrix,

$$M'(a, b) = \begin{bmatrix} a + ib & 0 \\ 0 & a - ib \end{bmatrix}$$

For any $z \in \mathbb{F}^\times$, $\exists\, a, b \in \mathbb{F}$ such that $z = a + ib$. In particular, $b = -i(z - a)$. Further, $a^2 + b^2 = 1$ gives $z^2 - 2az + 1 = 0$ i.e. $a = (z^{-1} + z)/2$ and $b = i(z^{-1} - z)/2$. Consider the map $\varphi : \mathbb{F}^\times \to \mathrm{SO}_2(\mathbb{F})$ given by

$$\varphi(z) = M\left(\frac{z^{-1} + z}{2}, \frac{i(z^{-1} - z)}{2}\right)$$

For any $z_1, z_2 \in \mathbb{F}^\times$,

$$\begin{aligned} &\varphi(z_1) = \varphi(z_2) \\ \implies &z_1 z_2^2 - (z_1^2 + 1)z_2 + z_1 = 0 \text{ and } z_2^{-1} - z_2 = z_1^{-1} - z_1 \\ \implies &z_2 = z_1, z_1^{-1} \text{ and } z_2^{-1} - z_2 = z_1^{-1} - z_1 \\ \implies &z_2 = z_1 \end{aligned}$$

So, φ is injective. Further, for any $M(a,b) \in \mathrm{SO}_2(\mathbb{F})$, $a+ib \neq 0$ (otherwise, $a^2+b^2=0$). Hence, $\varphi(a+ib) = M(a,b)$ and φ is surjective.

For any $z_1, z_2 \in \mathbb{F}^\times$,

$$\begin{aligned}
\varphi(z_1)\varphi(z_2) &= M\left(\frac{z_1^{-1}+z_1}{2}, \frac{i(z_1^{-1}-z_1)}{2}\right) M\left(\frac{z_2^{-1}+z_2}{2}, \frac{i(z_2^{-1}-z_2)}{2}\right) \\
&= \begin{bmatrix} \dfrac{(z_1z_2)^{-1}+z_1z_2}{2} & \dfrac{i((z_1z_2)^{-1}-z_1z_2)}{2} \\ \dfrac{-i((z_1z_2)^{-1}-z_1z_2)}{2} & \dfrac{(z_1z_2)^{-1}+z_1z_2}{2} \end{bmatrix} \\
&= M\left(\frac{(z_1z_2)^{-1}+z_1z_2}{2}, \frac{i((z_1z_2)^{-1}-z_1z_2)}{2}\right) \\
&= \varphi(z_1z_2)
\end{aligned}$$

Thus φ is bijective homomorphism i.e. an isomorphism from $\langle \mathrm{SO}_2(\mathbb{F}), \cdot\rangle$ to $\langle \mathbb{F}^\times, \cdot\rangle$. ■

The above theorem can better be understood by noting that applying $(x,y) \mapsto (x, iy)$ to the equation $x^2+y^2=1$ results in $x^2-y^2=1$ which is an equation of a hyperbola. Hence, the group $\langle \mathbb{F}^\times, \cdot\rangle$ corresponding to hyperbola is actually isomorphic to the group $\langle \mathrm{SO}_2(\mathbb{F}), \cdot\rangle$ corresponding to the circle if $x^2+1=0$ has a solution in $\mathbb{F}$.

1.3 Finding Pythagorean Triplets

Consider the set $\mathcal{C} = \{(x,y) \in \mathbb{Q}^2 \colon x^2+y^2=1\}$ and $P_0 = (1,0) \in \mathcal{C}$. For any $t, b \in \mathbb{Q}$, let $\ell_{t,b} = \{(x,y) \in \mathbb{Q} \colon y = tx+b\}$ such that $P_0 \in \ell_{t,b} \,\forall\, t, b \in \mathbb{Q}$. This means $0 = t+b$ or $b=-t$. Define $\ell_t \colon = \ell_{t,-t}$. We'll now find the intersection of ℓ_t and $\mathcal{C}$. From ℓ_t, we have $y = tx-t = t(x-1)$. Putting this in $x^2+y^2=1$,

$$x^2 + t^2(x^2+1-2x) = 1 \implies (1+t^2)x^2 - 2t^2x + (t^2-1) = 0$$

Applying the quadratic formula, we get

$$x = \frac{t^2 \pm \sqrt{t^4-(t^2+1)(t^2-1)}}{t^2+1} = \frac{t^2 \pm 1}{1+t^2}$$

Thus $x=1$ or $x = (t^2-1)/(t^2+1)$. $x=1$ corresponds to $y=0$ i.e. the point P_0. For $x = (t^2-1)/(t^2+1)$,

$$y = t\left(\frac{t^2-1}{t^2+1} - 1\right) = \frac{-2t}{t^2+1}$$

Call this point P_t. As $P_t \in \mathcal{C}$,

$$\left(\frac{t^2-1}{t^2+1}\right)^2 + \left(\frac{-2t}{t^2+1}\right)^2 = 1 \implies (t^2-1)^2 + (2t)^2 = (t^2+1)^2$$

If $t \in \mathbb{Z}$, then (t^2-1), $2t$ and (t^2+1) will all be in $\mathbb{Z}$. Hence, $(t^2-1, 2t, t^2+1)$ is a valid Pythagorean triple for all $t \in \mathbb{Z}$.

Note that this does **NOT** generate all Pythagorean triples. E.g. the triple $(5, 12, 13)$ will never be generated by this method as neither 5 nor 12 is one less than a perfect square.

We can adopt a similar strategy to generate rational or integer solutions to equations of the form $ax^2 + by^2 = cz^2$ where $a, b, c \in \mathbb{Q}$.

CHAPTER 2
Conics in Characteristic 2 Fields

Given a conic $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, we've classified it by writing it in matrix form as

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} A & \frac{1}{2}B \\ \frac{1}{2}B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} D & E \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + F = 0$$

and diagonalizing the symmetric matrix to obtain an orthonormal basis within which, the conic only has no xy term. However, this method no longer works if we're working in $\mathbb{F}^2$ such that $\mathrm{ch}(\mathbb{F}) = 2$ as $1 + 1 = 2 = 0$ and hence, we can't divide by 2.

In this chapter, we'll classify the conics in finite fields with characteristic 2 and investigate the conic groups that arise from them. We'll state the following theorem which will be used heavily throughout the chapter:

Theorem 6. *If $\mathbb{F}$ is a finite field with $\mathrm{ch}(\mathbb{F}) = 2$, then $\forall\, a \in \mathbb{F}\ \exists\, b \in \mathbb{F}$ such that $b^2 = a$. We'll write $b = \sqrt{a}$.*

Proof. Suppose $|\mathbb{F}| = q$. Since $\mathrm{ch}(\mathbb{F}) = 2$, q is even. So, $|\mathbb{F}^\times| = q - 1$ is odd. As $\mathbb{F}^\times$ is cyclic for any finite field $\mathbb{F}$ [DF04, Prop. 9.18], we have $\mathbb{F}^\times = \langle g \rangle$ for some $g \in \mathbb{F}^\times$.

Take any $a \in \mathbb{F}^\times$. Then, $a = g^k$ for some $k \in \mathbb{Z}/(q-1)\mathbb{Z}$. Now, 2 has a multiplicative inverse in $\mathbb{Z}/(q-1)\mathbb{Z}$ since $q - 1$ is odd. Hence, $\exists\, b \in \mathbb{F}^\times$ such that $b = g^{2^{-1}k}$. It is easy to see that $b^2 = a$. ■

2.1 Classification

Consider a non-degenerate, non-singular conic in a finite characteristic 2 field $\mathbb{F}$ given by

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where not all of A, B and C are zero.

Case 1: $A \neq 0, B \neq 0$ and $C \neq 0$

Apply the affine transformation $(x, y) \mapsto \left(\frac{x}{\sqrt{A}}, \frac{y}{\sqrt{C}}\right)$ and take $H = \frac{B}{\sqrt{A}\sqrt{C}}$ to get the following form

$$x^2 + Hxy + y^2 + \frac{D}{\sqrt{A}}x + \frac{E}{\sqrt{C}}y + F = 0$$

Further applying the affine transformation $(x, y) \mapsto \left(x + \frac{E}{H\sqrt{C}}, y + \frac{D}{H\sqrt{A}}\right)$ gives

$$x^2 + Hxy + y^2 + K = 0$$

where $K = F + \frac{D^2}{H^2A} + \frac{E^2}{H^2C} + \frac{DE}{H\sqrt{A}\sqrt{C}}$. Now, applying the affine transformation $(x, y) \mapsto \left(\frac{\sqrt{K}}{H}x, \frac{\sqrt{K}}{H}y\right)$ gives

$$\frac{K}{H^2}x^2 + \frac{K}{H^2}xy + \frac{K}{H^2}y^2 + K = 0$$

We can multiply by $\frac{H^2}{K}$ since $K = 0$ will make the conic singular. Thus, taking $L = H^2$ we have

$$\boxed{x^2 + xy + y^2 + L = 0}$$

where L can be any non-zero element of $\mathbb{F}$ due to Theorem 6.

Case 2: $A = 0, B \neq 0$ and $C \neq 0$

Apply the affine transformation $(x, y) \mapsto (x, x + y)$ to get

$$(B + C)x^2 + Bxy + Cy^2 + (D + E)x + Ey + F = 0$$

If $B \neq C$, we can proceed as Case 1. Otherwise, we have

$$Bxy + By^2 + Dx + Ey + F = 0$$

Applying the affine transformation $(x, y) \mapsto \left(\frac{x}{\sqrt{B}}, \frac{y}{\sqrt{B}}\right)$ and take $H = \frac{B}{\sqrt{C}}$ to get the following form

$$xy + y^2 + \frac{D}{\sqrt{B}}x + \frac{E}{\sqrt{B}}y + F = 0$$

Further applying the affine transformation $(x, y) \mapsto \left(x + \frac{E}{\sqrt{B}}, y + \frac{D}{\sqrt{B}}\right)$ gives

$$xy + y^2 + K = 0$$

where $K = F + \frac{D^2}{B} + \frac{DE}{B}$. Now, applying the affine transformation $(x, y) \mapsto (x + y, y)$ gives

$$xy + K = 0$$

We can now proceed as Case 3.

Case 3: $A = 0, B \neq 0$ and $C = 0$

Applying the affine transformation $(x, y) \mapsto \left(x + \frac{E}{B}, y + \frac{D}{B}\right)$ gives

$$Bxy + H = 0$$

where $H = F + \frac{DE}{B^2}$. Dividing by B and taking $K = \frac{H}{B}$, we get

$$xy + K = 0$$

Finally, applying the affine transformation $(x, y) \mapsto (\sqrt{K}x, \sqrt{K}y)$ and dividing by K ($K = 0$ implies conic is singular) results in

$$\boxed{xy + 1 = 0}$$

Case 4: $A \neq 0, B = 0$ and $C \neq 0$

Apply the affine transformation $(x, y) \mapsto \left(\frac{x}{\sqrt{A}}, \frac{y}{\sqrt{C}}\right)$ to get the following form

$$x^2 + y^2 + \frac{D}{\sqrt{A}}x + \frac{E}{\sqrt{C}}y + F = 0$$

Further applying the affine transformation $(x, y) \mapsto (x, x + y)$ gives

$$y^2 + Hx + Ky + F = 0$$

where $H = \left(\frac{D}{\sqrt{A}} + \frac{E}{\sqrt{C}}\right)$ and $K = \frac{E}{\sqrt{C}}$. We can proceed as Case 5 from here.

Case 5: $A = 0, B = 0$ and $C \neq 0$

Applying the affine transformation $(x, y) \mapsto \left(x, \frac{y}{\sqrt{C}}\right)$ to get the following form

$$y^2 + Dx + \frac{E}{\sqrt{C}}y + F = 0$$

Note that since $D = 0$ gives a quadratic equation in y, this case corresponds to a degenerate conic. Hence, we can assume $D \neq 0$. So, applying the affine transformation $(x, y) \mapsto \left(\frac{x}{D} + \frac{Ey}{D\sqrt{C}} + \frac{F}{D}, y\right)$ gives

$$\boxed{y^2 + x = 0}$$

Thus, upto affine congruence there are 3 classes of non-degenerate, non-singular conics in finite fields of characteristic two:

I. $y^2 + x = 0$

II. $xy + 1 = 0$

III. $x^2 + xy + y^2 + L = 0 \quad \forall\, L \in \mathbb{F}^\times$

From here on, we'll refer to these as Type I, Type II and Type III conics respectively.

Note that Type I and Type II have equations similar to parabola and hyperbola. Further, since $x^2 + 1 = 0$ has a solution in any field of characteristic two, Theorem 5 gives us that ellipses and hyperbolae will be affine congruent. Hence, all the non-degenerate conics we're used to in $\mathbb{R}$ are contained in the Type I and Type II cases. Type III, however, is a new class that appears only in the case of characteristic two fields.

2.2 Conic Groups

For Type I and Type II conics, we can achieve a similar parametrization as done for any field with characteristic not two in Chapter 1. This gives us the groups corresponding to Type I and Type II conics to be isomorphic to $\langle \mathbb{F}, + \rangle$ and $\langle \mathbb{F}^\times, \cdot \rangle$.

For Type III conics, we have to consider a quadratic field extension $\mathbb{F}(\alpha)$ as a two dimensional vector space over $\mathbb{F}$ with an ordered basis $\{1, \alpha\}$. Note that such an α is guaranteed to exist as the finite field of order $|\mathbb{F}|^2$ has a subfield isomorphic to $\mathbb{F}$ since $|\mathbb{F}| \mid |\mathbb{F}|^2$ [DF04, §14.3]. Suppose α is the root of the equation $x^2 + bx + c$ and hence $\alpha^2 = b\alpha + c$.

Note that for some fixed $a = a_1 + a_2\alpha \in \mathbb{F}(\alpha)$, multiplying any $x = x_1 + x_2\alpha \in \mathbb{F}(\alpha)$ by a is an $\mathbb{F}$-linear map since

$$\begin{aligned} ax &= (a_1 + a_2\alpha)(x_1 + x_2\alpha) \\ &= a_1x_1 + (a_1x_2 + a_2x_1)\alpha + a_2x_2\alpha^2 \\ &= (a_1 + a_2\alpha)x_1 + (a_2c + a_1\alpha + a_2b\alpha)x_2 \\ &= \begin{bmatrix} a_1 & a_2c \\ a_2 & a_1 + a_2b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

We'll consider a map $N \colon \mathbb{F}(\alpha) \to \mathbb{F}$ which sends $z = x + y\alpha \in \mathbb{F}(\alpha)$ to the determinant of the above matrix for multiplying by a i.e.

$$N(z) = N(x + y\alpha) = \begin{vmatrix} x & cy \\ y & x + by \end{vmatrix} = x(x + by) + cy^2 = x^2 + cy^2 + bxy$$

This map is known as the field norm on $\mathbb{F}(\alpha)$. It is easy to see that the equation $N(z) = k$ for some $k \in \mathbb{F}^\times$ corresponds to a conic in $\mathbb{F}^2$ that is Type III (since $A = 1$, $B = b$ and $C = c^2$). Further, N can be thought of as a group homomorphism from $\mathbb{F}(\alpha)^\times$ to $\mathbb{F}^\times$ since $N(1) = 1$ and

$$\begin{aligned} N((x + y\alpha)(z + w\alpha)) &= N(xz + yw\alpha^2 + (xw + yz)\alpha) \\ &= N(xz + cyw + (xw + yz + byw)\alpha) \\ &= x^2(z^2 + cw^2 + bzw) + cy^2(z^2 + cw^2 + bzw) \\ &\quad + bxy(z^2 + cw^2 + bzw) \\ &= (x^2 + cy^2 + bxy)(z^2 + cw^2 + bzw) \\ &= N(x + y\alpha)N(z + w\alpha) \end{aligned}$$

For any $z \in \mathbb{F}(\alpha)$ that satisfies $N(z) = k$, every element of the coset $(\ker N)z$ also satisfies $N(z) = k$. In fact, these are the only solutions as cosets are either equal or disjoint. Hence, the order of the conic group of Type III is $|\ker N|$.

This is where the investigation ends. There are two main directions to complete this theory – first is classifying the structure of the group of Type III conics – and second is looking at the case of infinite fields with characteristic two.

CHAPTER 3
Affine Geometry

Affine spaces extend the concept of vector spaces and linear transformations with the familiar notion of translation. The treatment of affine geometry given here is majorly inspired from Berger's textbook [Ber87]. The more application oriented parts such as properties of affine transformations are taken from Brannan's textbook on geometry [BEG12].

3.1 Affine space

Definition 1. Given a vector space $\vec{X}$ over $\mathbb{F}$, its set of points X and an operation $+\colon X \times \vec{X} \to X$ such that $\forall\, \vec{v}, \vec{w} \in \vec{X}$ and $\forall\, p \in \vec{X}$,

1. $p + \vec{0} = p$
2. $p + (\vec{v} + \vec{w}) = (p + \vec{v}) + \vec{w}$
3. $\theta_p\colon \vec{X} \to X$ given by $\theta_p(\vec{v}) = p + \vec{v}$ is a bijection.

Then X is called an affine space with underlying vector space $\vec{X}$.

Due to the third point above, we have the following definition:

Definition 2. Given an affine space X, for any $a, b \in X$,

$$b - a := \theta_a^{-1}(b)$$

3.2 Affine frames and coordinates

Definition 3. An $(n+1)$-tuple $(p_0, \vec{v}_1, \ldots, \vec{v}_n)$ where $p_0 \in X$ and $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis of $\vec{X}$ is called an affine frame.

Given $p \in X$ and an affine frame $(p_0, \vec{v}_1, \ldots, \vec{v}_n)$ of X, if $p - p_0 = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$, then p is said to have coordinates $(c_1, \ldots, c_n)$ in that frame.

3.3 Affine transformation

Definition 4. Given an affine space X, a funtion $f\colon X \to X$ is said to be an affine transformation if $\exists\, \vec{f} \in \mathrm{End}(\vec{X})\colon \vec{f}(b - a) = f(b) - f(a)\ \forall\, a, b \in X$.

Notation. We denote the set of affine transformations over X as $\mathrm{A}(X)$ and the set of invertible affine transformations over X as $\mathrm{GA}(X)$.

Theorem 7. *Given $f \in \mathrm{A}(X)$, $\vec{f}$ is unique. Further, given some $p_0 \in X$, $\exists!\, b \in A$ such that $f(p) = b + \vec{f}(p - p_0)\ \forall\, p \in X$.*

Proof. Suppose $\vec{f_1}, \vec{f_2} \in \mathrm{End}(\vec{X})$ such that for any $a, b \in X$

$$\vec{f_1}(b-a) = f(b) - f(a)$$
$$\vec{f_2}(b-a) = f(b) - f(a)$$

Assume $\exists\, \vec{v} \in \vec{X}\colon \vec{f_1}(\vec{v}) \neq \vec{f_2}(\vec{v})$. For some $a \in X$, we have $\theta_a(\vec{v}) \in X$ such that $\theta_a(\vec{v}) - a = \theta_a^{-1}(\theta_a(\vec{v})) = \vec{v}$. This means

$$\vec{f_1}(\vec{v}) = \vec{f_1}(\theta_a(\vec{v}) - a) = f(\theta_a(\vec{v})) - f(a) = \vec{f_2}(\theta_a(\vec{v}) - a) = \vec{f_2}(\vec{v})$$

This is a contradiction. Hence, our assumption that such a $\vec{v}$ exists must be wrong and so, $\vec{f_1} = \vec{f_2}$.

Fixing some $p_0 \in X$, we have $\vec{f}(p - p_0) = f(p) - f(p_0)\ \forall\, p \in X$. So,

$$f(p) = f(p_0) + \vec{f}(p - p_0)\ \forall\, p \in X$$

Hence, $b = f(p_0)$. For some $b_1, b_2 \in X$ and $b_1 \neq b_2$, assume

$$f(p) = b_1 + \vec{f}(p - p_0)\ \forall\, p \in X$$

$$f(p) = b_2 + \vec{f}(p - p_0)\ \forall\, p \in X$$

Note that

$$b_1 = b_1 + (\vec{f}(p-p_0) - \vec{f}(p-p_0)) = (b_1 + \vec{f}(p-p_0)) - \vec{f}(p-p_0) = f(p) - \vec{f}(p-p_0)$$

$$b_2 = b_2 + (\vec{f}(p-p_0) - \vec{f}(p-p_0)) = (b_2 + \vec{f}(p-p_0)) - \vec{f}(p-p_0) = f(p) - \vec{f}(p-p_0)$$

Hence, $b_1 = b_2$. ■

3.4 Properties of Affine Transformations

Definition 5. Given $a, b \in X$, we define the line passing through a and b as

$$\ell_{ab} := \{a + t(b-a)\colon t \in \mathbb{F}\}$$

Definition 6. Two lines ℓ_{ab} and ℓ_{pq} are said to be parallel if $b - a = k(p - q)$ for some $k \in \mathbb{F}$. We write this as $\ell_{ab} \parallel \ell_{pq}$.

Theorem 8. *Consider $f \in \mathrm{GA}(X)$ and ℓ_{ab} for some $a, b \in X$. Then,*

$$\exists\, p, q \in X\colon f(\ell_{ab}) = \ell_{pq}$$

Proof. Fixing $p_0 = a$ in Theorem 7, we have $p \in X$ such that

$$f(a + t(b - a)) = p + \vec{f}(t(b - a)) = p + t\vec{f}(b - a)\ \forall\, t \in \mathbb{F}$$

Since $\vec{v} \mapsto p + \vec{v}$ is a bijection, we have $q \in X$ such that $q - p = \vec{f}(b - a)$. Thus

$$f(a + t(b - a)) = p + t(q - p)\ \forall\, t \in \mathbb{F}$$

i.e. $f(\ell_{ab}) = \ell_{pq}$. ■

The above theorem can be interpreted as the following statement:

$\boxed{\textit{Affine transformations take straight lines to straight lines.}}$

Theorem 9. *For any $f \in \mathrm{GA}(X)$,*

$$\ell_{ab} \parallel \ell_{pq} \implies f(\ell_{ab}) \parallel f(\ell_{pq})$$

Proof. Since $\ell_{ab} \parallel \ell_{pq}$, we have $b - a = k(q - p)$ for some $k \in \mathbb{F}$. Using Theorem 7, we can write

$$\begin{aligned} f(\ell_{ab}) &= \{f(a + t(b - a)) \colon t \in \mathbb{F}\} \\ &= \{c + \vec{f}(a + t(b - a) - p_0) \colon t \in \mathbb{F}\} \\ &= \{c + \vec{f}((a - p_0) + t(b - a)) \colon t \in \mathbb{F}\} \\ &= \{c + \vec{f}(a - p_0) + t\vec{f}(b - a) \colon t \in \mathbb{F}\} \end{aligned}$$

Similarily, $f(\ell_{pq}) = \{c + \vec{f}(p - p_0) + t\vec{f}(q - p) \colon t \in \mathbb{F}\}$. Now,

$$b - a = k(q - p) \implies \vec{f}(b - a) = k\vec{f}(q - p)$$

By definition, this means that $f(\ell_{ab}) \parallel f(\ell_{pq})$.

■

The above theorem can be interpreted as the following statement:

$\boxed{\textit{Affine transformations take parallel lines to parallel lines.}}$

If the underlying vector space $\vec{X}$ of an affine space X has a norm $\|\cdot\|$ defined on it, we have the following theorem:

Theorem 10. *Given $f \in \mathrm{GA}(X)$, a line ℓ_{ac} and any $b \in \ell_{ac}$ such that $b \neq a$ and $b \neq c$, we have*

$$\frac{\|b - a\|}{\|c - b\|} = \frac{\|f(b) - f(a)\|}{\|f(c) - f(b)\|}$$

Proof. Since $b \in \ell_{ac}$, let $b = a + t_0(c - a)$. Now,

$$\frac{\|b - a\|}{\|c - b\|} = \frac{|t_0| \, \|c - a\|}{|1 - t_0| \, \|c - a\|} = \left| \frac{t_0}{1 - t_0} \right|$$

Using Theorem 7 with $p_0 = a$, we have $f(x) = p + \vec{f}(x - a)$ for some $p \in X$. So,

$$\begin{aligned} f(a) &= p + \vec{f}(a - a) = p \\ f(b) &= p + \vec{f}(a + t_0(c - a) - a) = p + t_0\vec{f}(c - a) \\ f(c) &= p + \vec{f}(c - a) \end{aligned}$$

Hence,

$$\frac{\|f(b) - f(a)\|}{\|f(c) - f(b)\|} = \frac{|t_0| \left\|\vec{f}(c - a)\right\|}{|1 - t_0| \left\|\vec{f}(c - a)\right\|} = \left| \frac{t_0}{1 - t_0} \right|$$

■

The above theorem can be interpreted as the following statement:

Affine transformations preserve the ratio of distances of 3 collinear points.

3.5 Fundamental theorem of Affine Geometry

Theorem 11. *If $A_0, A_1, \ldots, A_n, B_0, B_1, \ldots, B_n \in X$ such that $\{A_1 - A_0, \ldots, A_n - A_0\}$ and $\{B_1 - B_0, \ldots, B_n - B_0\}$ are linearly independent where $n = \dim \vec{X}$, then*

$$\exists!\, f \in \mathrm{GA}(X)\colon\ f(A_i) = B_i \ \forall\, i \in \{0, 1, \ldots, n\}$$

Proof. Let $\vec{v_i} = A_i - A_0$ and $\vec{w_i} = B_i - A_0$ $\forall\, i \in \{1, 2, \ldots, n\}$. Clearly, both $\beta_1 = \{\vec{v}_1, \ldots, \vec{v}_n\}$ and $\beta_2 = \{\vec{w}_1, \ldots, \vec{w}_n\}$ form a basis for $\vec{X}$. In fact, there are unique linear transformations $\vec{f_1}, \vec{f_2} \in \mathrm{GL}(\vec{X})$ such that $\vec{f_1}(\vec{v_i}) = \vec{e_i}$ and $\vec{f_2}(\vec{w_i}) = \vec{e_i}$ $\forall\, i \in \{1, 2, \ldots, n\}$ where $\{\vec{e}_1, \ldots, \vec{e}_n\}$ is the standard basis of $\vec{X}$.

Consider the affine transformations $f_1, f_2 \in \mathrm{GA}(X)$ given by

$$\begin{aligned} f_1(p) &= O + \vec{f_1}(p - A_0) \ \forall\, p \in X \\ f_2(p) &= O + \vec{f_1}(p - B_0) \ \forall\, p \in X \end{aligned}$$

Now, $f_1(A_0) = f_2(B_0) = O$ and $f_1(A_i) = f_2(B_i) = O + \vec{e_i}$ $\forall\, i \in \{1, 2, \ldots, n\}$. Since f_1 and f_2 are invertible, it is easy to see that $f = f_2^{-1} f_1$ satisfies $f(A_i) = B_i$ $\forall\, i \in \{0, 1, \ldots, n\}$.

Next, we need to prove that f is unique. Suppose there are two affine transformations $g_1, g_2 \in \mathrm{GA}(X)$ that satisfy $g_1(A_i) = g_2(A_i) = B_i$ $\forall\, i \in \{0, 1, \ldots, n\}$ but $\exists\, q_0 \in X$ such that $g_1(q_0) \neq g_2(q_0)$.

From Theorem 7, picking $p_0 = A_0$, $\exists!\, b_1, b_2 \in X$ such that $\forall\, q \in X$

$$g_1(p) = b_1 + \vec{g}_1(p - A_0)$$
$$g_2(p) = b_2 + \vec{g}_2(p - A_0)$$

Since $g_1(A_0) = g_2(A_0) = B_0$, we have $b_1 = b_2$. Further using

$$g_1(A_i) = g_2(A_i) = B_i \ \forall\, i \in \{1, 2, \ldots, n\}$$

we get the relations

$$\vec{g}_1(\vec{v}_i) = \vec{g}_2(\vec{v}_i) \ \forall\, i \in \{1, 2, \ldots, n\}$$

But note that β_1 is a basis of $\vec{X}$. Thus for any $\vec{a} \in \vec{X}$, we have scalars $c_1, \ldots, c_n$ such that $\vec{a} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$. Hence,

$$\vec{g}_1(\vec{a}) = c_1\vec{g}_1(\vec{v}_1) + \cdots + c_n\vec{g}_1(\vec{v}_2) = c_1\vec{g}_2(\vec{v}_1) + \cdots + c_n\vec{g}_2(\vec{v}_2) = \vec{g}_2(\vec{a}) \ \forall\, \vec{a} \in \vec{X}$$

So, $b_1 = b_2$ and $\vec{g}_1 = \vec{g}_2$. But this contradicts that $\exists\, q_0 \in X \colon g_1(q_0) \neq g_2(q_0)$. Hence, $g_1 = g_2$. ■

Intuitively, this theorem says that there exists an affine transformation in $\mathrm{GA}(X)$ which takes an n-simplex in an affine space X with $\dim \vec{X} = n$ to another n-simplex in X. Note that an n-simplex is a generalization of the concept of triangles and tetrahedra in 2D and 3D respectively. In particular, a triangle is a 2-simplex and a tetrahedron is a 3-simplex. So, if we consider the affine space $\mathbb{R}^2$, this theorem says that there is an affine transformation that takes any triangle to any other triangle. We can also state it as

$$\boxed{\textit{All triangles in } \mathbb{R}^2 \textit{ are affine-congruent.}}$$

In general, we say two figures are affine-congruent if there is an invertible affine transformation taking one to the other.

CHAPTER 4

Projective Geometry

The treatment of projective geometry given throughout most of this chapter except the last sections is taken from Audin's textbook [Aud02].

4.1 Projective Spaces

Definition 7. Let $\vec{E}$ be a finite dimensional vector space. The *projective space* $\mathrm{P}(\vec{E})$ deduced from $\vec{E}$ is the set of all 1 dimensional linear subspaces of $\vec{E}$.

Remark. The dimension of $\mathrm{P}(\vec{E})$ is $\dim \vec{E} - 1$. If $\vec{E}$ consists only of the point 0, it does not contain any lines, and $\mathrm{P}(\vec{E})$ is empty. Thus it shall be implicitly assumed that $\dim \vec{E} \geq 1$. If $\dim \vec{E} = 1$, $\vec{E}$ itself is a line, and thus the set of lines contains a unique element, $\mathrm{P}(\vec{E})$ is a point.

4.1.1 Projective Subspace

A subset V of $\mathrm{P}(\vec{E})$ is a projective subspace if it is an image of a non-zero vector subspace $\vec{F}$ of $\vec{E}$.

Proposition 1. *Let V and W be two projective subspaces of* $\mathrm{P}(\vec{E})$*.*

- *If dim V + dim W $\geq$ dim* $\mathrm{P}(\vec{E})$*, then $V \cap W$ is not empty.*
- *Let H be a hyperplane of* $\mathrm{P}(\vec{E})$*, and let m be a point not in H. Every line through m intersects H at a unique point.*

Proof. Let $\vec{F}$ and $\vec{G}$ be the vector subspaces of $\vec{E}$ from which V and W were deduced, i.e. $V = \mathrm{P}(\vec{F})$, and $W = \mathrm{P}(\vec{G})$. The statement can be translated into vector subspaces as

$$\begin{aligned} &(\dim\, \vec{F} - 1) + (\dim\, \vec{G} - 1) \geq (\dim\, \vec{E} - 1) \\ \implies &\dim\, \vec{F} + \dim\, \vec{G} \geq \dim\, \vec{E} + 1 \end{aligned}$$

We can use the linear algebraic properties to further deduce that:

$$\dim \vec{F} + \dim \vec{G} = \dim(\vec{F} + \vec{G}) + \dim(\vec{F} \cap \vec{G}) \leq \dim \vec{E} + \dim(\vec{F} \cap \vec{G})$$

Therefore,

$$\dim\, (\vec{F} \cap \vec{G}) \geq 1$$

This can be translated back into projective geometry to conclude that $V \cap W$ is not empty.

Now, to prove the second property, let $\vec{J}$ be the vector hyperplane of which H is image of. The point m is the image of a line ℓ in $\vec{E}$, not contained in the hyperplane $\vec{J}$. The assertion, translated in terms of linear algebra, is that any plane $\vec{P}$ containing ℓ meets $\vec{J}$ along a unique line. Since ℓ is not in $\vec{J}$, we have $\vec{P} + \vec{J} = \vec{E}$. Hence,

$$\begin{aligned}\dim(\vec{P} \cap \vec{J}) &= \dim \vec{P} + \dim \vec{J} - \dim(\vec{P} + \vec{J}) \\ &= 2 + \dim \vec{E} - 1 - \dim \vec{E} = 1\end{aligned}$$

■

4.1.2 Projective Transformation

Definition 8. Let $\vec{E}$ and $\vec{E}'$ be two vector subspaces, and $p\colon \vec{E} \setminus \{0\} \to \mathrm{P}(\vec{E})$ and $p'\colon \vec{E}' \setminus \{0\} \to \mathrm{P}(\vec{E}')$ be the two projections. A *projective transformation* $g\colon \mathrm{P}(\vec{E}) \to \mathrm{P}(\vec{E}')$ is a mapping such that there exists a linear isomorphism $f\colon \vec{E} \to \vec{E}'$ with $p' \circ f = g \circ p$.

Proposition 2. *The set of projective transformations from* $\mathrm{P}(\vec{E})$ *to itself,* $\mathrm{PGL}(\vec{E})$*, is a group under composition.*

Proof. From the definitions, the projective transformation that descends from identity map of $\vec{E}$ forms the identity of the group. For any projective tranformation g that descends from a linear isomorphism f, the transformation g' that descends from f^{-1} will act as its inverse. Since functional composition obeys associativity, $\mathrm{PGL}(\vec{E})$ is a group under composition. ■

4.1.3 Homogeneous Coordinates and Projective Frames

Given a basis of vector space $\vec{E}$, the vectors in $\vec{E}$ can be descirbed by their coordinates with respect to the basis.

Definition 9. A point m in $\mathrm{P}(\vec{E})$ can be described by the non-zero vector that generates the line m. In a n-dimensional projective space $\mathrm{P}(\vec{E})$, the $(n+1)$ tuples $[x_1 : \cdots : x_{n+1}]$ and $[x'_1 : \cdots : x'_{n+1}]$ represent the same point iff there exists a non-zero scalar λ such that $x_i = \lambda x'_i$ for all i.

In a projective space $\mathrm{P}(\vec{E})$ with dimension n, we actually need $n+2$ points to uniquely determine the basis of the underlying space $\vec{E}$, which will be proved in the next lemma. It will also justify the next definition.

Definition 10. If $\vec{E}$ is a vector space of dimension $n+1$, a *projective frame* of $\mathrm{P}(\vec{E})$ is a set of $n+2$ points $(m_0, \ldots, m_{n+1})$ such that $m_1, \ldots, m_{n+1}$ are the images of the vectors $\vec{e}_1, \ldots, \vec{e}_{n+1}$ in a basis of $\vec{E}$, and m_0 is the image of $\vec{e}_1 + \cdots + \vec{e}_{n+1}$.

Lemma 1. *Let* $(m_0, \ldots, m_{n+1})$ *be a projective frame of* $\mathrm{P}(\vec{E})$*. If the two bases of* $\vec{E}$ $(\vec{e}_1, \ldots, \vec{e}_{n+1})$ *and* $(\vec{e'}_1, \ldots, \vec{e'}_{n+1})$ *are such that* $p(\vec{e}_i) = p(\vec{e'}_i) = m_i$ *and* $p(\vec{e}_1 + \cdots + \vec{e}_{n+1}) = p(\vec{e'}_1 + \cdots + \vec{e'}_{n+1}) = m_0$*, then they are proportional.*

Proof. Consider the points m_i of $\mathrm{P}(\vec{E})$. Since the vectors $\vec{e_i}$ and $\vec{e'}_i$ both generate the line m_i, $\vec{e_i} = \lambda_i \vec{e'}_i$ for some non-zero λ_i. Using the $(n+2)$-th point, we can conclude that

$$(\vec{e}_1 + \cdots + \vec{e}_{n+1}) = \lambda(\vec{e'}_1 + \cdots + \vec{e'}_{n+1})$$

Thus,

$$\lambda_1 \vec{e}_1 + \cdots + \lambda_{n+1} \vec{e}_{n+1} = \lambda(\vec{e}_1 + \cdots + \vec{e}_{n+1})$$

As we are dealing with a basis, $\lambda_i = \lambda$. Thus two bases are proportional. ■

Proposition 3. *Let* $\mathrm{P}(\vec{E})$ *and* $\mathrm{P}(\vec{E'})$ *be two projective spaces of dimension* n. *Any projectivve mapping from* $\mathrm{P}(\vec{E})$ *to* $\mathrm{P}(\vec{E'})$ *maps a projective frame of* $\mathrm{P}(\vec{E})$ *onto a projective frame of* $\mathrm{P}(\vec{E'})$.

4.2 Fundamental Theorem of Projective Geometry

Theorem 12 (Fundamental Theorem of Projective Geometry)**.** *Let* $(a_1, \ldots, a_{n+1})$ *and* $(b_1, \ldots, b_{n+1})$ *be two sets of points in* $\mathrm{P}(\mathbb{R}^n)$ *such that none of the* a_i *and* b_i *belong to the projective subspace defined by* $n-1$ *of the others in their respective sets. Then there exists a unique projective transformation* $f\colon \mathrm{P}(\mathbb{R}^n) \to \mathrm{P}(\mathbb{R}^n)$ *such that,* $f(a_i) = b_i$ *for all* $i = 1, \ldots, n+1$.

Proof. The set of points (a_i) and (b_i) are both projective frames of $\mathrm{P}(\mathbb{R}^n)$. Let $(\vec{e}_1, \ldots, \vec{e}_n)$, $(\vec{e'}_1, \ldots, \vec{e'}_n) \in \mathbb{R}^n$ be the basis that generate the frames (a_i) and (b_i) respectively. We know that there exists a unique isomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $f(\vec{e_i}) = \vec{e'}_i$. The projective transformation g that descends from f will map the first frame to the second.

To prove the uniqueness: let f and f' two such projcetive transformations. The projective transormation $g^{-1} \circ g'$ from $\mathrm{P}(\mathbb{R}^n)$ into itself keeps the frame invariant. Thus it is the identity transformation. ■

Theorem 13 (Desargues's Theorem)**.** *Let* $\triangle ABC$ *and* $\triangle A'B'C'$ *be triangles in* $\mathbb{R}^2$ *such that the lines* AA', BB', *and* CC' *meet at point* U. *Let* BC *and* $B'C'$ *meet at* P, CA *and* $C'A'$ *at* Q, *and* AB *and* $A'B'$ *at* R. *Then* P, Q, *and* R *are collinear.*

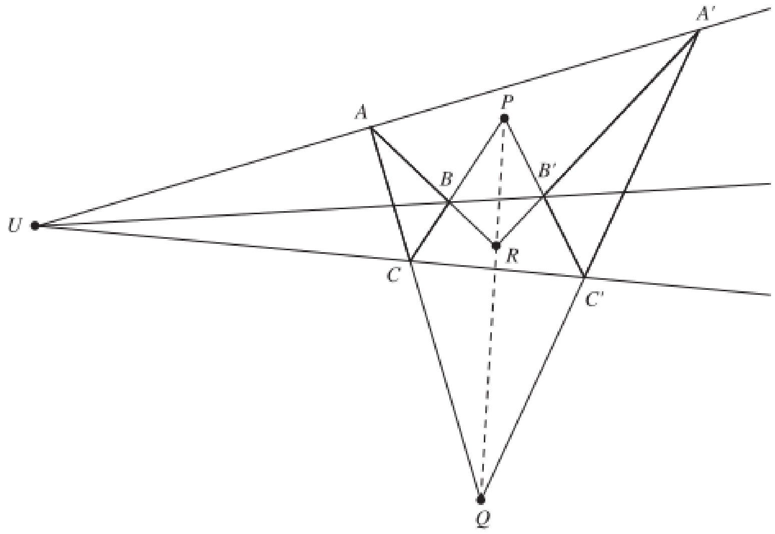


Figure 4.1 : Figure from [BEG12].

Proof. We will prove the theorem for the special case where $A = [1 : 0 : 0]$, $B = [0 : 1 : 0]$, $C = [0 : 0 : 1]$, and $U = [1 : 1 : 1]$. From the fundamental theorem of projective geometry, we know that it will be congruent to any other configuration. We can use the fact that projective congruence preserves projrctive properties, to deduce that the theorem holds in general.

The line AU has the equation $y = z$. Since A' lies on AU, it must have coordinates $[a : b : b]$, where $b \neq 0$, since $A' \neq A$. We can also wirte $A' = [p : 1 : 1]$, where $p = a/b$. Similary, $B' = [1 : q : 1]$, and $C' = [1 : 1 : r]$.

Now to find the point P, we find the equation of the line BC.

$$\begin{vmatrix} x & y & z \\ 1 & q & 1 \\ 1 & 1 & r \end{vmatrix} = 0 \implies (qr - 1)x - (r - 1)y + (1 - q) = 0$$

Substituting $x = 0$ in the equation for the line $B'C'$, we get $(r - 1)y = (1 - q)z$, which immplies $P = [0 : 1 - q : r - 1]$. Similarly we find that $Q = [1 - p : 0 : r - 1]$, and $R = [1 - p : q - 1 : 0]$.

To check the collinearity of P, Q, and R:

$$\begin{vmatrix} 0 & 1-q & r-1 \\ 1-p & 0 & r-1 \\ 1-p & q-1 & 0 \end{vmatrix} = -(1-q)(1-p)(1-r) + (r-1)(1-p)(q-1) = 0$$

i.e P, Q, and R are collinear. ■

Proposition 4. *There is a unique projective conic through any given set of five points, no three of which are collinear.*

Proof. By the fundamental theorem of projective geometry, there exists a projective transformation t which maps the four out of given five points to the points $[1 : 0 : 0]$, $[0 : 1 : 0]$, $[0 : 0 : 1]$ and $[1 : 1 : 1]$. Let $[a : b : c]$ be the image of the fifth point under t. Since projective transformations preserve collinearity, no three of the five points are collinear, and also it can be deduced that a, b and c are non-zero, since if it were so, it would be collinear with other two points.

Let the conic that passes through these 5 points be of the form

$$Ax^2 + Bxy + Cy^2 + Fxz + Gyz + Hz^2$$

By substituting the points $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$, the equation can be reduced to the form

$$Bxy + Fxz + Gyz = 0$$

Since the porjective concic also passes through $[1 : 1 : 1]$ and $[a : b : c]$, we get the equations

$$B + F + G = 0$$

and

$$Bab + Fac + Gbc = 0$$

Solving these simultaneous equations, we get

$$F = -G\frac{ab - bc}{ab - ac}$$

and

$$B = -G\frac{ac - bc}{ac - ab}$$

It follows that the conic is of the form

$$-G\frac{ac - bc}{ac - ab}xy - G\frac{ab - bc}{ab - ac}xz + Gyz = 0$$

or

$$c(a - b)xy + b(c - a)xz + a(b - c)yz = 0$$

Since the conic is uniquely determined by the fifth point, it follows that it is unique.

■

Remark. **The Standard Projective Conic**

The projective conic $E = \{[x : y : z] : xy + yz + zx = 0\}$ is called the standard projective conic. It passes through the traingle of reference formed by the points $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$. This fact can be used to simplify calculations involving projcetive conics.

All the points on the conic except than $[1 : 0 : 0]$ can be parameterized as $[t^2 + t : t + 1 : -t]$, where $t \in \mathbb{R}$. All points on $\vec{E}$ satisfy $xy + yz + zx = 0$. Suppose $y \neq 0$, let $t = x/y$. Then $x = ty$, and so

$$\begin{aligned} &(ty)y + yz + z(ty) = 0 \\ \implies &ty + (t + 1)z = 0 \\ \implies &y = -\frac{t + 1}{t}z, x = -(t + 1)z \end{aligned}$$

Thus the point has homogeneous coordinates $\left[-(t + 1)z : -\frac{t+1}{t}z : z\right]$, which can be rewritten in the form $[t(t + 1) : t + 1 : -t]$. This also happens to hold for the point $[0 : 0 : 1]$, where $y = 0$.

Proposition 5. *Let E_1 and E_2 be non-degenerate conics that pass through the points P_1, Q_1, R_1 and P_2, Q_2, R_2 respectively. Then there exists a projective transformation t which maps E_1 to E_2 such that $t(P_1) = P_2$, $t(Q_1) = Q_2$, and $t(R_1) = R_2$.*

Proof. We prove this result by proving that for any conic E_1, there exists a transormation t_1 which maps it to the standard conic $xy + yz + zx = 0$ such that $t_1(P_1) = [1 : 0 : 0]$, $t_2(Q_1) = [0 : 1 : 0]$, and $t_3(R_1) = [0 : 0 : 1]$ for any $P_1, Q_1, R_1 \in E$.

Let f be a transformation that maps P_1 to $[1 : 0 : 0]$, Q_1 to $[0 : 1 : 0]$, and R_1 to $[0 : 0 : 1]$. It will map the conic E_1 into a conic E' of the form

$$Fxy + Gyz + Hzy = 0$$

for some $F, G, H \in \mathbb{R}$. Divide the equation by FGH to rewrite E' in the form

$$\frac{xy}{GH} + \frac{yz}{FH} + \frac{zx}{FG} = 0$$

Now, let g be the transformation of the form $g([x : y : z]) = A[x : y : z] \, \forall \, [x : y : z] \in \mathrm{P}(\mathbb{R}^3)$ where A is a 3×3 matrix such that

$$A = \begin{bmatrix} \frac{1}{H} & 0 & 0 \\ 0 & \frac{1}{G} & 0 \\ 0 & 0 & \frac{1}{F} \end{bmatrix}$$

Then, g maps E' to the standrd conic $xy + yz + zx = 0$, leaving P, Q, and R unchanged. Let $t_1 = g \circ f$. Similarly, let t_2 be the function that maps the conic E_2 to the standard conic such that $t_2(P_2) = [1 : 0 : 0]$, $t_2(Q_2) = [0 : 1 : 0]$, and $t_2(R_2) = [0 : 0 : 1]$ for any $P_2, Q_2, R_2 \in E_2$.

The composite function $t = t_2^{-1} \circ t_1$ maps E_1 to E_2 such that $t(P_1) = P_2$, $t(Q_1) = Q_2$, and $t(R_1) = R_2$, as required. ■

Theorem 14 (Pascal's Theorem)**.** *Let A, B, C, A', B', and C' be six distinct points on a non-degenerate projective conic. Let BC and $B'C$ intersect at P, CA' and $C'A$ at Q, and AB' at R. The points P, Q, and R are collinear.*

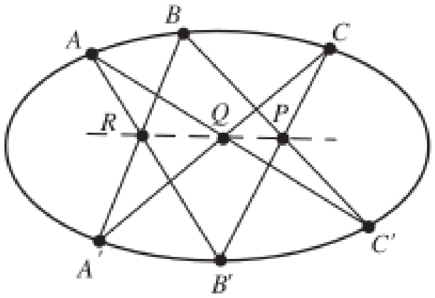


Figure 4.2 : Figure from [BEG12].

Proof. We know that any non-degenerate conic can pe transformed to the standard conic. Let the conic be in the standard form $xy+yz+zx = 0$, with $A = [1 : 0 : 0]$, $B = [0 : 1 : 0]$, and $C = [0 : 0 : 1]$. Let the point $A' = [a^2 + a : a + 1 : -a]$, $B' = [b^2 + b : b + 1 : -b]$, and $C' = [c^2 + c : c + 1 : -c]$, for some $a, b, c \in \mathbb{R}$

The line BC' has the equation $x = -(c + 1)z$, and the line $B'C$ has the equation $x = by$. The point P lies on both of these lines. Hence it has the homogeneous coordinates $[b(c+1) : c+1 : -b]$. Similarly, $Q = [a(c+1) : c+1 : -c]$, and $R = [b(a+1) : b+1 : -b]$.

To check their collinearity, evaluate the determinant:

$$\begin{vmatrix} b(c+1) & c+1 & -b \\ a(c+1) & c+1 & -c \\ b(a+1) & b+1 & -b \end{vmatrix}$$

Which, after some row operations, simplifies to be equal to 0. Hence, the points P, Q, and R are collinear. ■

Proof adapted from [BEG12].

4.3 Cross-Ratio

Definition 11. Let a, b, c and d be four points on a projective line D. There exists a unique map from D to $K \cup \{\infty\}$ that maps a to ∞, b to 0, and c to 1. The image of d under this projective mapping is called the *cross-ratio* of the points (a, b, c, d), and denoted $[a, b, c, d]$.

Proposition 6. *Let a_1, a_2, a_3, and a_4 be four points on the line D (the first three being distinct) and a'_1, a'_2, a'_3, and a'_4 be four points on another line D' (satisfying the same assumption). There exists a projective transformation $f \colon D \to D'$ such that $f(a_i) = a'_i$ iff $[a_1, a_2, a_3, a_4] = [a'_1, a'_2, a'_3, a'_4]$.*

Proof. Assume f is a projective mapping that sends a_i to a'_i. Let g and g' be functions such that $[a_1, a_2, a_3, a_4] = g(a_4)$, and $[a'_1, a'_2, a'_3, a'_4] = g'(a'_4)$. $g' \circ f$ is a function, which maps a_1 to ∞, a_2 to 0, and a_3 to 1. But such function is unique. Hence, $g = g' \circ f$, which implies that $g(a_4) = g'(a'_4)$. That is,

$$[a_1, a_2, a_3, a_4] = [a'_1, a'_2, a'_3, a'_4]$$

■

Remark. **Formulae for cross-ratio**
Let a, b, and c be four points on the affine line, the first three being distinct. Then

$$[a, b, c, d] = \frac{(d-b)(c-a)}{(d-a)(c-b)}$$

Also, since the points a and b are distinct, c and d can be written as

$$c = \alpha a + \beta b$$
$$d = \gamma a + \delta b$$

Then the cross-ratio

$$[a, b, c, d] = \frac{\gamma\beta}{\alpha\delta}$$

Proposition 7. *If a, b, c, and d are four distinct collinear points, then the following equalities hold:*

$$[a, b, c, d] + [a, c, b, d] = 1$$
$$[b, a, c, d] = [a, b, c, d]^{-1}$$
$$[a, b, d, c] = [a, b, c, d]^{-1}$$

Proof. Let f be the function that defines the cross-ratio, such that $[a, b, c, d] = f(d)$, and let f' be a function such that $f'(x) = 1 - f(x)\ \forall x$. The composite function $f' \circ f$ maps a to ∞, b to 0, and c to 1. But the function that defines the cross-ratio $[a, c, b, d]$ also maps a to ∞, b to 0, and c to 1. Since such function is unique,

$$[a, c, b, d] = f' \circ f(d)$$
$$\implies [a, c, b, d] = 1 - [a, b, c, d]$$

Let g be a function such that $g(x) = \frac{1}{f(x)}\ \forall\, x$. The composite function $g \circ f$ maps a to 0, b to ∞, and c to 1. Thus it is the function that defines the cross-ratio $[b, a, c, d]$. That is,

$$\begin{aligned} [b, a, c, d] &= g \circ f(d) \\ \implies [b, a, c, d] &= \frac{1}{f(d)} \\ &= [a, b, c, d]^{-1} \end{aligned}$$

Let h be a function such that $h(x) = \frac{f(x)}{f(d)}\ \forall\, x$. The composite function $h \circ f$ maps d to 1, keeping the images of a and b unchanged. Hence, it is the defining function of the cross-ratio $[a, b, d, c]$. That is,

$$\begin{aligned} [a, b, d, c] &= h \circ f(c) \\ \implies [a, b, d, c] &= \frac{f(c)}{f(d)} \\ &= [a, b, c, d]^{-1} \end{aligned}$$

■

Remark. If $[a, b, c, d] = k$, the 24 cross-ratios obtained by permuting the four points take six values in general:

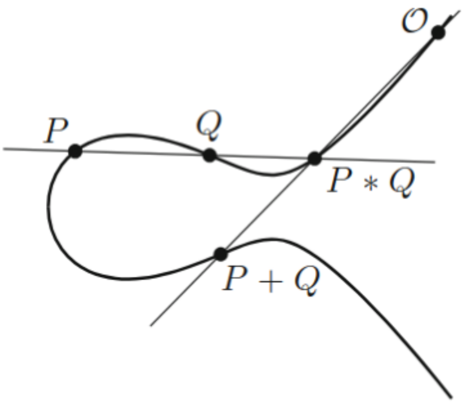
$$k \ , \ \frac{1}{k} \ , \ 1 - k \ , \ 1 - \frac{1}{k} \ , \ \frac{1}{1-k} \ , \ \frac{k}{k-1}$$

4.4 Group Law on Cubics

Throughout this section, a projective cubic curve means a non-degenerate, non-singular, non-empty projective cubic curve (i.e. its points are solutions of a cubic polynomial in the coordinates) in a two dimensional projective space. The following proof is adapted from [ST15].

Given a projective cubic curve $\mathcal{C}$ with a given point O on it. The addition law is defined as follows:

To add P and Q, take the third intersection point $P * Q$, join it to O by a line, and then take the third intersection point to be $P + Q$. In other words, set $P + Q := O * (P * Q)$ In case of $P = Q$, the line passing through P and Q is taken to be the tangent to $\mathcal{C}$ at P.



In short, we consider a point to intersect a line twice if it is tangent to the curve and thrice if the point is an inflection point.
Note that

$$P * Q = R \iff Q * R = P \iff R * P = Q$$

Now, we verify that the above addition rule with the set of points on $\mathcal{C}$ does indeed form a group.

Closure

First, we see that the set is closed under the operation $*$. From Bèzout's theorem [ST15, Thm. A.1], we can say that given two points on $\mathcal{C}$, there is a third point that intersects with the curve and line through the previous point which proves $\mathcal{C}$ is closed under $*$. Thus, for any two points P and Q that lie on $\mathcal{C}$, $P + Q := O * (P * Q)$ also lies on $\mathcal{C}$.

Identity

For any P, we have

$$P + O = O * (O * P) = P$$

which shows that O is the identity element and it belongs to $\mathcal{C}$.

Inverse

Let $S := O * O$. For any point Q, consider $Q + (Q * S)$ as

$$Q + (Q * S) = O * (Q * (Q * S)) = O * S = O$$

$(Q * S)$ exists if S exists which is true due to Bèzout's theorem [ST15, Thm. A.1]. If O is an inflection point, then $S = O$. Thus, the inverse of Q is $(Q * S)$ which lies on $\mathcal{C}$.

Associativity

We define the following lines:

$$\begin{aligned} &l_1\colon \text{Passes through } Q,\ R,\ Q * R && m_1\colon \text{Passes through } P,\ Q,\ P * Q \\ &l_2\colon \text{Passes through } O,\ P * Q,\ P + Q && m_2\colon \text{Passes through } O,\ Q * R,\ Q + R \\ &l_3\colon \text{Passes through } P,\ Q + R && m_3\colon \text{Passes through } R,\ P + R \end{aligned}$$

Because the curve is in the projective plane, the point of intersection of lines l_3 and m_3 always exists; let that point be T. Now, we consider the cubic curve obtained by multiplying the equations of l_1,l_2 and l_3 as L. Similarily, we consider the curve obtained from m_1, m_2, m_3 as M. They will meet at 9 points due to Bèzout's theorem [ST15, Thm. A.1] which are:

$$O,\ P,\ Q,\ R,\ P * Q,\ Q * R,\ P + Q,\ Q + R,\ T$$

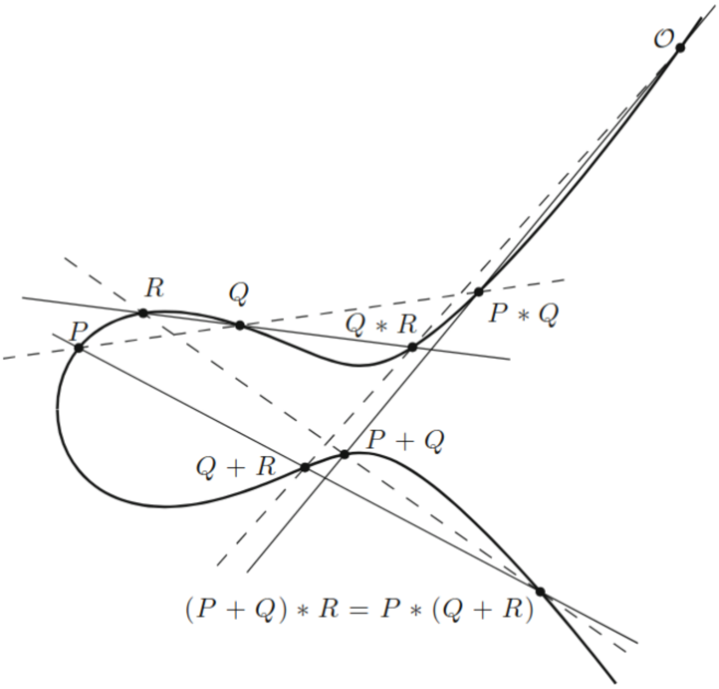


Figure 4.3 : Solid lines represents L while dotted lines represents M

But we see that the cubic curve $\mathcal{C}$ passes through all of these points except T. From the Cayley-Bacharach theorem [ST15, Thm. A.3], if two cubic curves L and M intersect at 9 points and another cubic curve $\mathcal{C}$ passes through 8 of those points then it passes through the ninth point as well. Hence, T lies on $\mathcal{C}$.

Consider the points of intersection between $\mathcal{C}$ and l_3. P and $Q + R$ lie on both of them, thus the third point of intersection will be $P * (Q + R)$, which happens to be T.

Similarly, looking at the points of intersection between $\mathcal{C}$ and m_3, we see that the point T also happens to be $(P + Q) * R$.

Because they are the same point, we have that:

$$(P + Q) * R = P * (Q + R)$$

From which we can conclude that

$$(P + Q) + R = P + (Q + R)$$

Thus, the points of $\mathcal{C}$ form a group under the operation $+$.

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