Reference Formulae

Focus : $(ae, 0)$
Directrix : $x = \frac{a}{e}$

Parabola $(e = 1)$

Equation :

$$y^2 = 4ax$$

Parametric form : $(at^2, 2at)$

Ellipse $(0 < e < 1)$

Equation :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Parametric form : $(a \cos t, b \sin t)$

Hyperbola $(e > 1)$

Equation :

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Parametric form : $(a \sec t, b \tan t)$

Group law on Parabola

Given any parabola, there exists an affine transformation that takes it to the curve $y = x^2$.

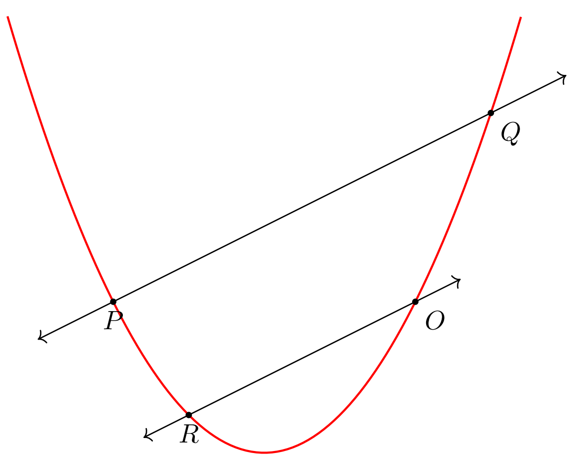


Figure 1: $R = P \oplus Q$

We define the parametric coordinates of the point R as (r, r^2). We compare the slopes of the two lines PQ and OR to obtain the co-ordinates of R.

$$\frac{r^2 - o^2}{r - o} = \frac{p^2 - q^2}{p - q}$$
$$r = p + q - o$$

We define a homomorphism from the points on the parabola to R as $\phi((x, x^2)) = x - o$. The map that is defined is a bijection hence it is an isomorphism. The curve shown in the figure is $\mathbb{R}^2$ however the algebra performed remains the same if the field is changed to $\mathbb{C}^2$.

Solving for curves in finite fields

We first investigate the solution set of a curve when working with finite field $\mathbb{Z}_p$.

$$\mathcal{C} = \{(x_1, x_2, \cdots, x_n) \in \mathbb{Z}_p^n \mid \textit{condition}\} \subseteq \mathbb{Z}_p^n$$

We notice that $\mathbb{Z}_p^n$ contains a finite number of points (p^n) and so will V. So it is a valid approach to just verify which points out of these will satisfy the condition.

We now see the solution for one such problem

$$V = \{(x, y, z) \in \mathbb{Z}_p^3 \; | x^2 + y^2 = z^2\}$$

We take a different approach to the problem. We set z as a parameter and plot the various curves for different values of z. Now the problem is simplified to two variables for each value of z. We see that we can embed $\mathbb{Z}_p^2$ in $\mathbb{R}^2$ such that $\mathbb{Z}_p^2 \subset \mathbb{R}^2$.

For every $z \in \mathbb{Z}_p$, Define $\mathcal{V}_z = \{(x, y) \; | x^2 + y^2 = z^2\}$

Notice that:

$$V = \bigcup_{z \in \mathbb{Z}_p} \mathcal{V}_z \cap \mathbb{Z}_p^2$$

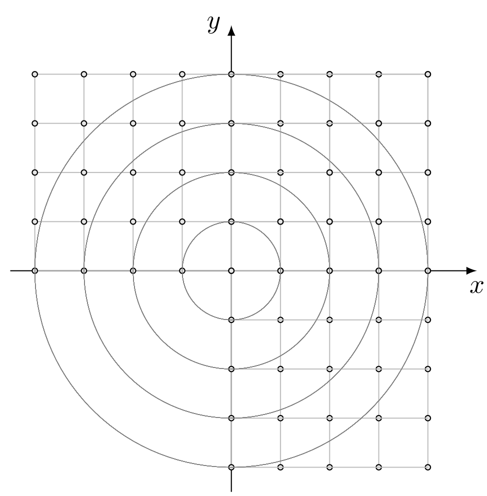


Figure 2: The figure represents the $\mathbb{Z}_5$ solutions

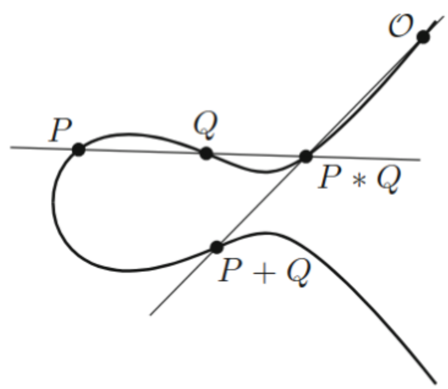
Desargues' Theorem

There are two triangles ΔABC and ΔXYZ in in three dimensional space. If the lines AX,BY and CZ meet at a point, then the points formed from joining lines AB,BC,AC with PQ,QR,PR respectively are collinear.

Group Law on Cubics

We have a projective cubic curve $\mathcal{C}$ with a given point O on it. The addition law is defined as follows:

To add P and Q, take the third intersection point $P * Q$, join it to O by a line, and then take the third intersection point to be $P+Q$. In other words, set $P+Q = O*(P*Q)$ In case of $P = Q$, the line passing through P and Q is taken to be the tangent to $\mathcal{C}$ at P.



In short, we consider a point to intersect a line twice if it is tangent to the curve and thrice if the point is an inflection point.

Notice that:

$$P * Q = R \Longleftrightarrow Q * R = P \Longleftrightarrow R * P = Q$$

Now we verify that the above addition rule with the set of points on $\mathcal{C}$ does indeed form a group.

Closure

We first see that the set is closed under the operation $*$. From Bezout's theorem we can say that given two points on $\mathcal{C}$, there is a third point that intersects with the curve and line through the previous point which proves $\mathcal{C}$ is closed under $*$. Thus, for any two points P and Q that lie on $\mathcal{C}$, $P + Q := O * (P * Q)$ also lies on $\mathcal{C}$.

Identity

For any P,we have

$$P + O = O * (O * P) = P$$

Which shows that O is the identity element and it belongs to $\mathcal{C}$

Inverse

Let $S := O * O$. For any point Q, consider $Q + (Q * S)$

$$Q + (Q * S) = O * (Q * (Q * S)) = O * S = O$$

$(Q * S)$ exists if S exists which it must due to Bezout's theorem. If O is an inflection point then $S = O$. Thus, the inverse of Q is $(Q * S)$ which lies on $\mathcal{C}$

Associativity

We define the following sets of lines:

$$\begin{aligned}
l_1 &: \text{Passes through } Q, R, Q * R \\
l_2 &: \text{Passes through } O, P * Q, P + Q \\
l_3 &: \text{Passes through } P, Q + R \\
m_1 &: \text{Passes through } P, Q, P * Q \\
m_2 &: \text{Passes through } O, Q * R, Q + R \\
m_3 &: \text{Passes through } R, P + R
\end{aligned}$$

Because the curve is in the projective plane, the point of intersection of lines l_3 and m_3 always exists and let that point be T. Now we consider two cubic curves:

$$L : l_1 l_2 l_3 \text{ and } M : m_1 m_2 m_3$$

The two cubics meet at at 9 points:

$$O, P, Q, R, P * Q, Q * R, P + Q, Q + R, T$$

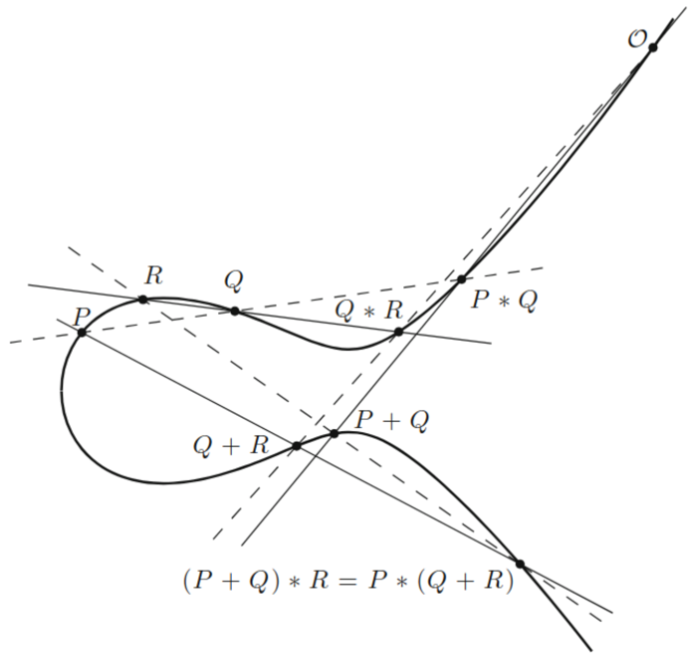


Figure 3: Solid lines represents L while dotted lines represents M

But we see that the cubic curve $\mathcal{C}$ passes through all these points except T. From the Cayley-Bacharach theorem, we have that if two cubic curves L and M intersect at 9 points and another cubic curve $\mathcal{C}$ passes through 8 of those points then it passes through the ninth. Hence we can say that T lies on $\mathcal{C}$.

Consider the points of intersection between $\mathcal{C}$ and l_3. P and $Q + R$ lie on both thus the third point of intersection will be $P * (Q + R)$ which happens to be T.

Similarly, looking at the points of intersection between $\mathcal{C}$ and m_3 we see that point T also happens to be $(P + Q) * R$.

Because they are the same point, we have that:

$$(P + Q) * R = P * (Q + R)$$

From which we can conclude

$$(P + Q) + R = P + (Q + R)$$