Realizing Non-Physical Actions through Hermitian-Preserving Map Exponentiation

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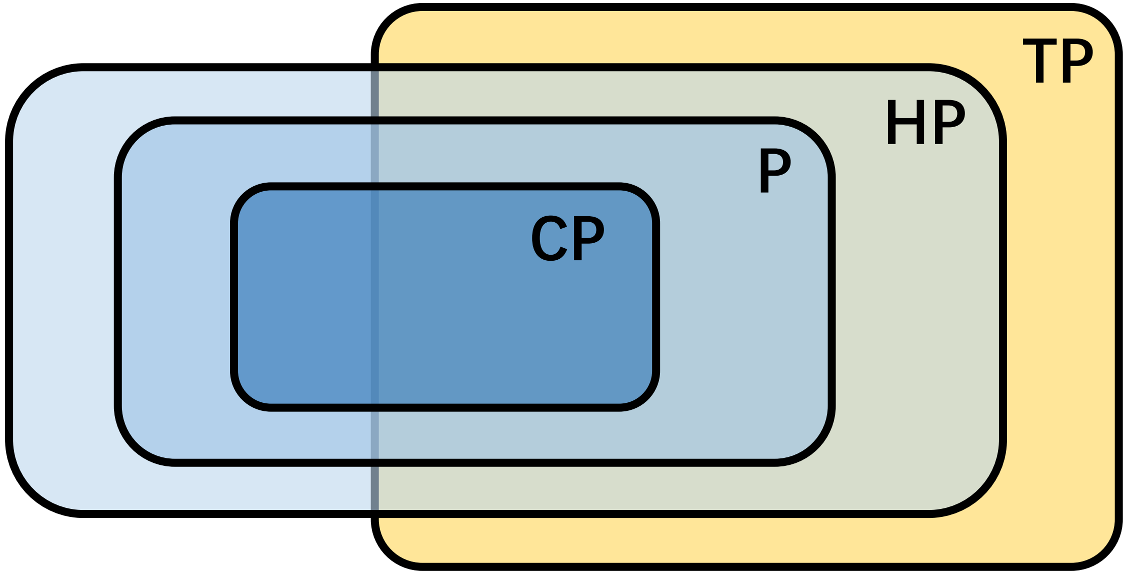
Outline

- Motivation and Background
- The HME Algorithm
- Performance Analysis
- Application: Entanglement Detection and Quantification
- Application: Noiseless States Recovery
- Conclusion & Outlook

Background

- Quantum mechanics: coherence and entanglement, which could be explored to showcase potential advantages.
- CPTP constraint still restrict some tasks in quantum information
  - Entanglement Detection and Quantification
  - Error Mitigation



Background

- Quantum mechanics: coherence and entanglement, which could be explored to showcase potential advantages.
- CPTP constraint still restrict some tasks in quantum information
  - Entanglement Detection and Quantification
  - Error Mitigation
- Existing attempts for realizing non-physical maps are not effective
  - Structural Approximation
  - N-copy Extension
  - Petz Recovery Map
  - Probabilistic Error Cancellation

Motivation

- How to avoid CPTP constraints?
- How to make the Hermitian matrix physical?
- We can change the carrier of quantum states!
- Hermitian-Preserving Map Exponentiation (HME): "Exponentiate" $\mathcal{N}(\cdot)$ to get $e^{-i\mathcal{N}(\cdot)t}$

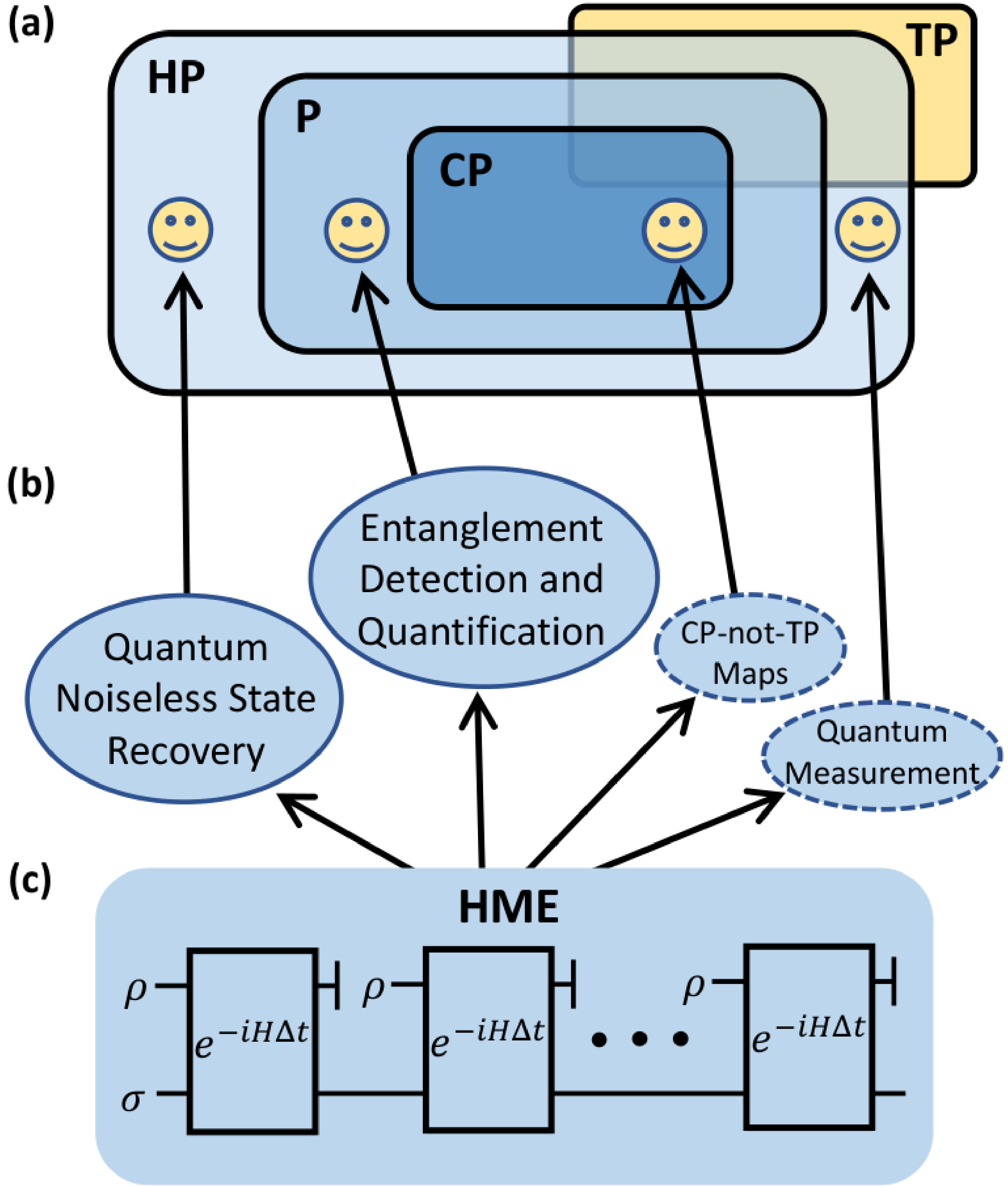
$$\rho \mapsto e^{-i\mathcal{N}(\rho)t}$$

- This new map contains all the information of $\mathcal{N}(\cdot)$
- Density Matrix Exponentiation (DME):
  - Principal Component Analysis

$$\rho \mapsto e^{-i\rho t}$$

Algorithm Overview

- Resource Requirement:
  - Quantum Memory
  - Joint Evolution
  - Efficient State Reset
- Process:
  - Prepare the first state to ρ
  - Jointly evolve the whole system using $e^{-iH\Delta t}$
  - Repeat
- Result: the evolution of $e^{-i\mathcal{N}(\rho)t}$
  - $\mathcal{N}$: Hermitian-preserving

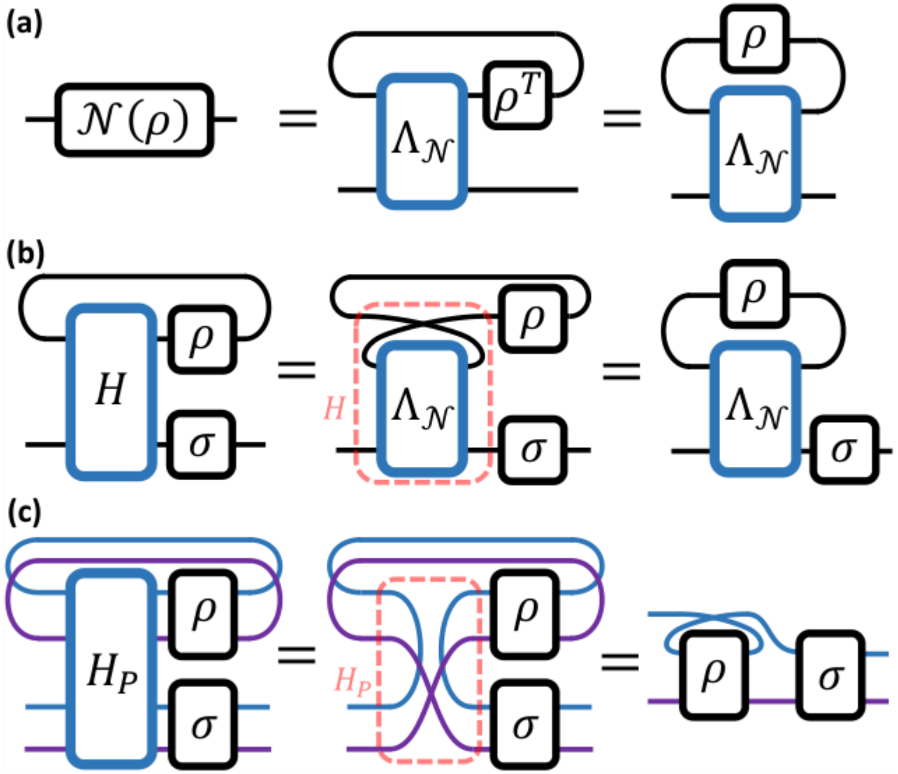


Validation

- Consider one step of the sequential operations.

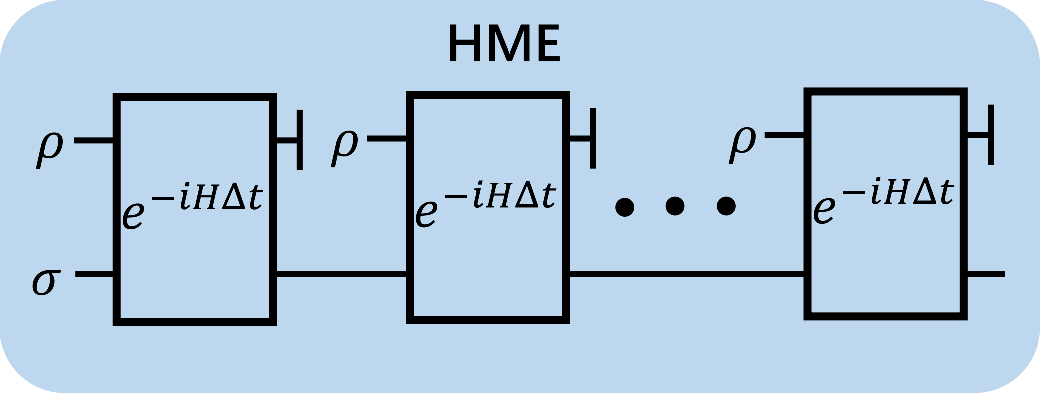
$$\mathrm{Tr}_1\left[e^{-iH\Delta t}(\rho\otimes\sigma)e^{iH\Delta t}\right]=e^{-i\mathcal{N}(\rho)\Delta t}\sigma e^{i\mathcal{N}(\rho)\Delta t}+\mathcal{O}(\Delta t^2)$$

$$H=\Lambda_{\mathcal{N}}^{T_1}$$



Performance Analysis – Upper Bound

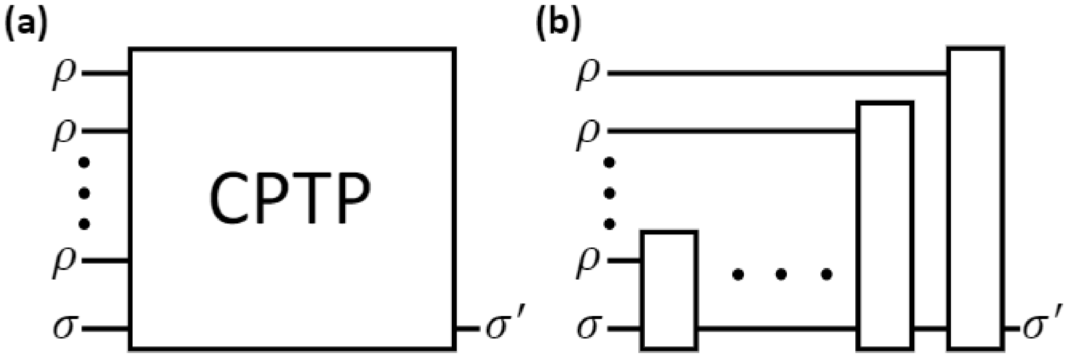
- To ensure $\|\mathcal{U}_t - \mathcal{Q}_t\|_\diamond \le \epsilon$, the number of inputting states is bounded by
$$\mathcal{O}(\epsilon^{-1}\|H\|_\infty^2 t^2)$$
- $\mathcal{O}(\epsilon^{-1}\|H\|_\infty^2 t^2)$ copies of ρ are enough for realizing the controlled-$e^{-i\mathcal{N}(\rho)t}$ evolution.



- Covers the complexity of DME when setting $H = S$.
- Infinite norm of H is an important quantity.
- The upper bound is over-estimated for some cases.

Performance Analysis – Lower Bound

- This lower bound is not tight.
- For some cases, it meets the upper bound
  - Inverse map of local amplitude damping noise.
  - Identity map, DME



Theorem 4 (Lower bound of sample complexity). *Let $\mathcal{N} \in T(\mathcal{H}, \mathcal{K})$ be a Hermitian-preserving map and set $0 < \epsilon \leq 1/6$, and $t \geq \frac{15\pi\epsilon}{4R_*}$. The minimum number of ρ needed to realize the evolution of $e^{-i\mathcal{N}(\rho)t}$ using protocols shown in Fig. 3(a) with ϵ accuracy in diamond distance satisfies*

$$f_{\mathcal{N}}(\epsilon, t) \geq \Omega\left(\epsilon^{-1} R_*^2 t^2\right), \tag{9}$$

where $R_* := \max_{A \in \mathscr{F}} R[\mathcal{N}(A)]$. $R[\cdot] = \lambda_{max}(\cdot) - \lambda_{min}(\cdot)$ denotes the spectral gap defined as the difference between the largest and the lowest eigenvalues of the processed matrix. The feasible region $\mathscr{F}$ is defined as

$$\mathscr{F} = \Big\{A \in L(\mathcal{H}) : A^\dagger = A, \mathrm{Tr}(A) = 0, \\ \|A\|_1 = 1, \left[\mathcal{N}(A^+), \mathcal{N}(A^-)\right] = 0\Big\}, \tag{10}$$

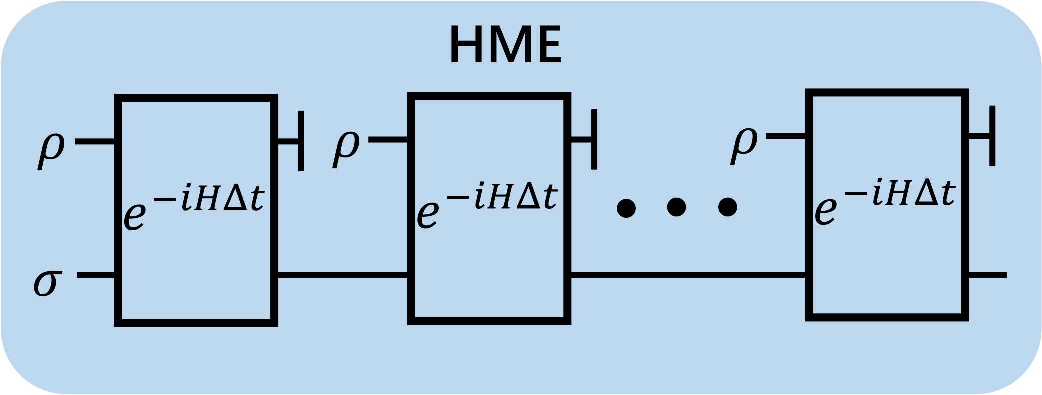
where A^+ and A^- are the positive and negative parts of A.

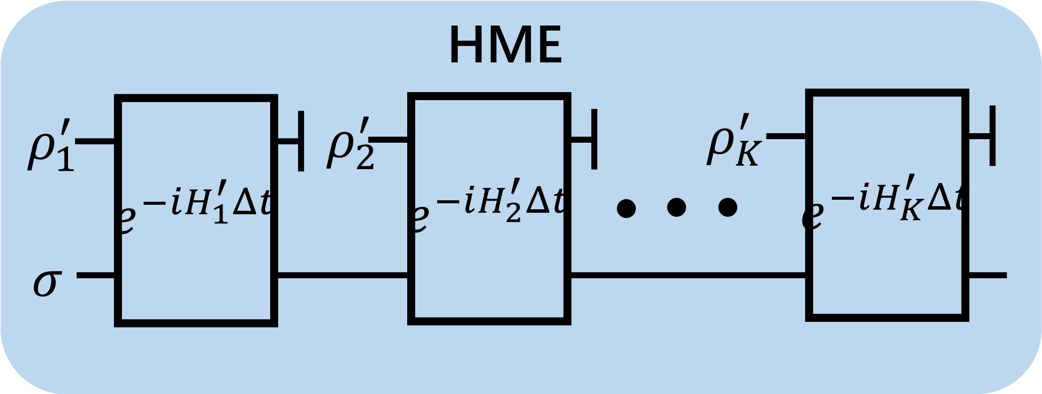
Performance Analysis - Robustness

- Considering the errors from states preparation and Hamiltonian evolution,

$$\|\mathcal{Q}_t' - \mathcal{Q}_t\|_\diamond \leq 4t(D_H + \|H\|_\infty D_S)$$

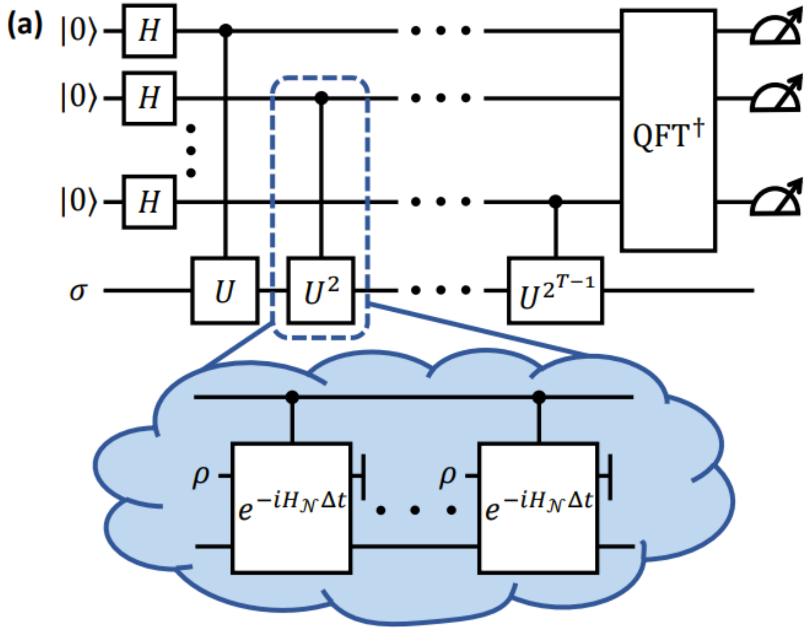
where $D_H = \frac{1}{K}\sum_{k=1}^{K}\|H_k' - H\|_\infty$ and $D_S = \frac{1}{K}\sum_{k=1}^{K}\|\rho_k' - \rho\|_\infty$





Application: Entanglement Detection

- Quantum Phase Estimation + HME + Positive Maps
- Entanglement detection is a fundamentally difficult task.
- Exponential speedup for some entanglement detection tasks.



Liu P, Liu Z, Chen S, et al. Fundamental limitation on the detectability of entanglement[J]. Physical Review Letters, 2022, 129(23): 230503.

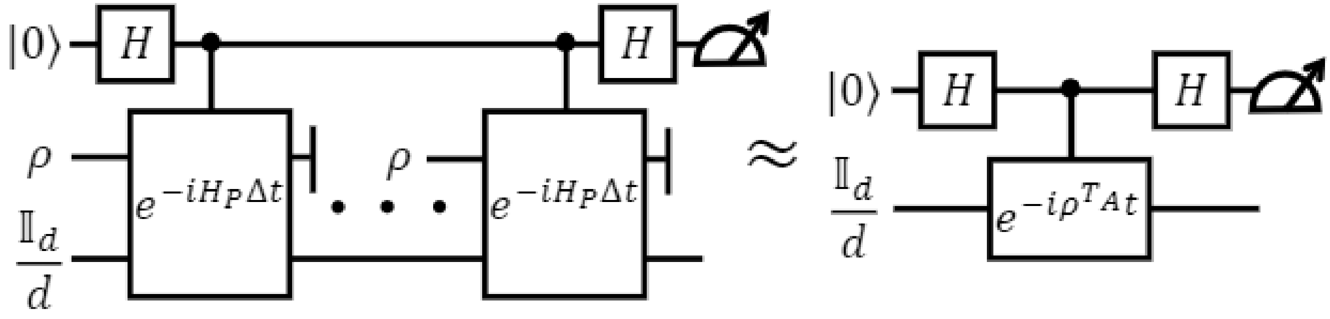
Application: Entanglement Quantification

- Negativity is an important entanglement measure, while no efficient measurement protocols.
  - Tomography.

$$N(\rho) = \frac{\left\|\rho^{T_A}\right\|_1 - 1}{2}$$

- Combining Hadamard Test + HME + Fourier Expansion, we derive a negativity estimation protocol.

$$\left\|\rho^{T_A}\right\|_1 = \frac{\pi}{2}d - \sum_{l=1}^{\infty} \frac{4}{\pi(2l-1)^2} \operatorname{Tr}\left[\cos\left((2l-1)\rho^{T_A}\right)\right].$$



$$\operatorname{Tr}\left[(X \otimes \mathbb{I}_d)\text{C-}e^{-i\rho^{T_A}t}\left(|+\rangle\langle+| \otimes \frac{\mathbb{I}_d}{d}\right)\text{C-}e^{i\rho^{T_A}t}\right]$$
$$= \frac{1}{2d}\operatorname{Tr}\left[e^{-i\rho^{T_A}t} + e^{i\rho^{T_A}t}\right] = \frac{1}{d}\operatorname{Tr}\left[\cos\left(\rho^{T_A}t\right)\right],$$

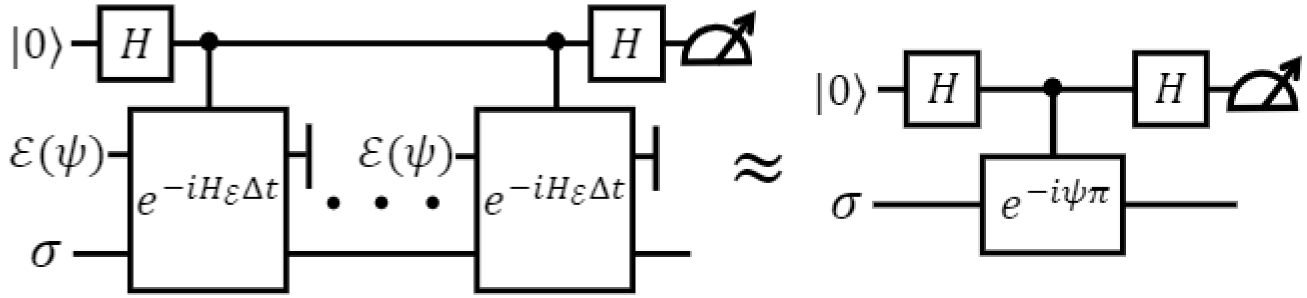
Application: Entanglement Quantification

- The sample complexity of negativity estimation has an exponential lower bound $\Omega(\epsilon^{-2}d)$ even using joint operations.
- To make sure $\left|\widehat{N}(\rho) - N(\rho)\right| \leq \epsilon$ with success probability at least $1-\delta$, the sample complexity scales as $\tilde{O}(\log(\delta^{-1})\epsilon^{-3}d^2 d_A \|\rho^{T_A}\|_1)$
- Compared with some trivial methods, like tomography, this protocol has an exponential advantage.

$$|N(\rho_1) - N(\rho_2)| \leq \frac{\sqrt{d}}{2}\|\rho_1 - \rho_2\|_1 \rightarrow \Omega(\epsilon^{-2}d^4)$$

Application: Noiseless State Recovery

- Suppose ψ is the noiseless state and $\mathcal{E}(\psi)$ is the noisy state, HME can realize the evolution of $e^{-i\mathcal{E}^{-1}(\mathcal{E}(\psi))t} = e^{-i\psi t}$.
- Combined with a simple quantum phase estimation circuit, we can recover the noiseless state.



- One needs $\mathcal{O}(\epsilon^{-1}F^{-2}\|H_{\mathcal{E}^{-1}}\|_{\infty}^{2})$ copies of $\mathcal{E}(\psi)$ to recover ψ to ϵ accuracy, where $F = \langle\psi|\sigma|\psi\rangle$.

Application: Noiseless State Recovery

- QNSR is somewhere between QEC and QEM

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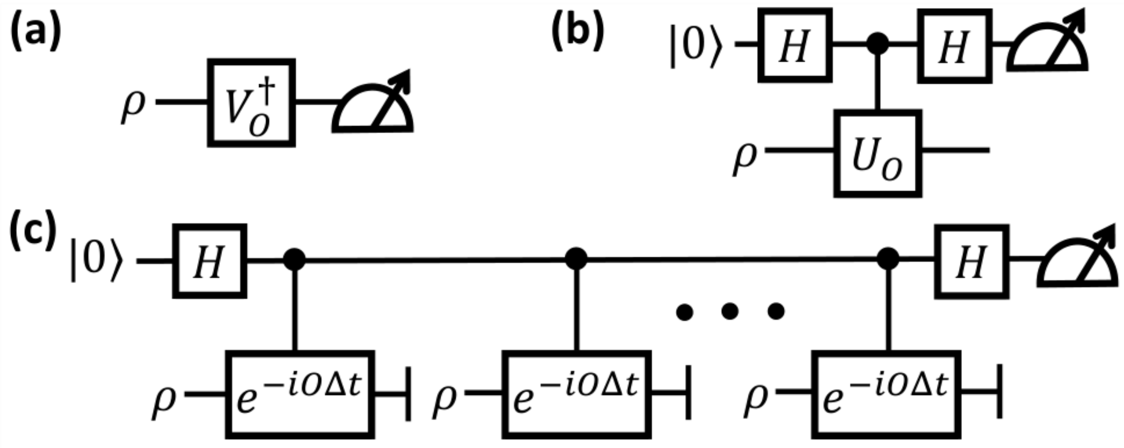
	QEC	QEM	QNSR
Main Technique	Encoding and Decoding	Statistical Methods	HME and Quantum Phase Estimation
State of Processing	Noiseless	Noisy	Noisy
Target	State Recovery	Expectation Value Recovery	State Recovery
Qubit Overhead	High	Low	Moderate
Sample Complexity	Polynomial	Exponential	Exponential
General Noise	No	Yes	Yes

- For an noise channel $\mathcal{E}$ which is the tensor product of single-qubit noises, the sample complexity for realizing $e^{-i\mathcal{E}^{-1}(\rho)t}$ is bounded by $\epsilon^{-1}2^{\Omega(n)}t^2$, even with joint operations.

Other Applications



- Expectation Values Measurement
  - $\mathrm{Tr}(O\rho)$ is also an HP map.
  - HME realizes $e^{-i\mathrm{Tr}(O\rho)t|1\rangle\langle 1|}$.
  - Thus HME encodes the expectation value on ancilla qubit: $\frac{1}{\sqrt{2}}(\,|0\rangle + |1\rangle) \rightarrow \frac{1}{\sqrt{2}}(\,|0\rangle + e^{-i\mathrm{Tr}(O\rho)t}|1\rangle)$
  - HME can also encode multiple values simultaneously.
- Quantum Algorithm
  - HME realizes $e^{-iP\rho P^\dagger t}$ for any matrix P.
  - Linear Combination of Unitaries
  - Quantum Imaginary Time Evolution

Conclusion & Outlook

- HME has advantages compared with single-copy strategies, as it utilizes quantum memory.
- HME saves the classical computing resources.

- Other applications?
- Combine with other quantum algorithms?
- Other ways to realize non-physical maps?

Thanks