The Tensor Renormalization Group - A efficient numerical method

The Tensor Renormalization Group

Caleb Cook
Department of Physics, Harvard University, Cambridge MA 02138
(Dated: May 15, 2015)

numerical methods

1

Exact diagonalization

Deal with small size systems

2

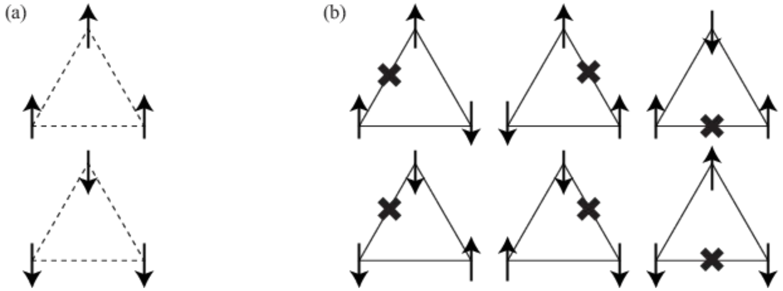
density matrix renormalization group

1-D system

3

Quantum monte carlo

Sign problem [fermion systems and frustrated spin systems]

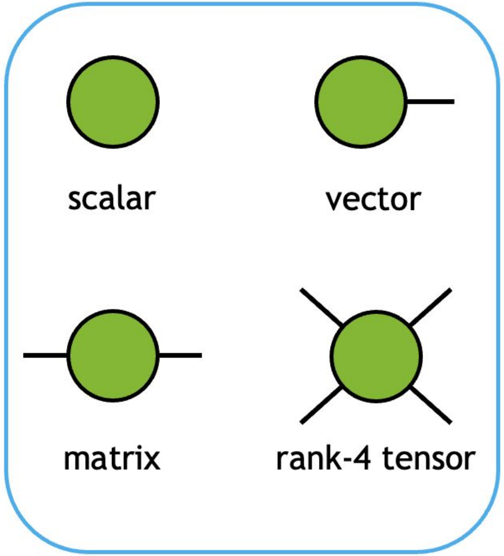


4

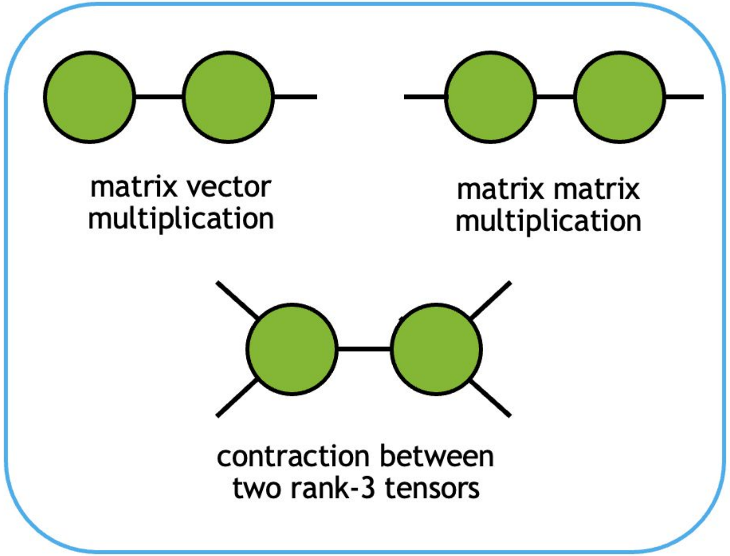
Tensor renormalization group

M. Levin and C. P. Nave. Tensor renormalization group approach to two-dimensional classical lattice models. Phys. Rev. Lett., 99:120601, Sep 2007.

What is tensor ?

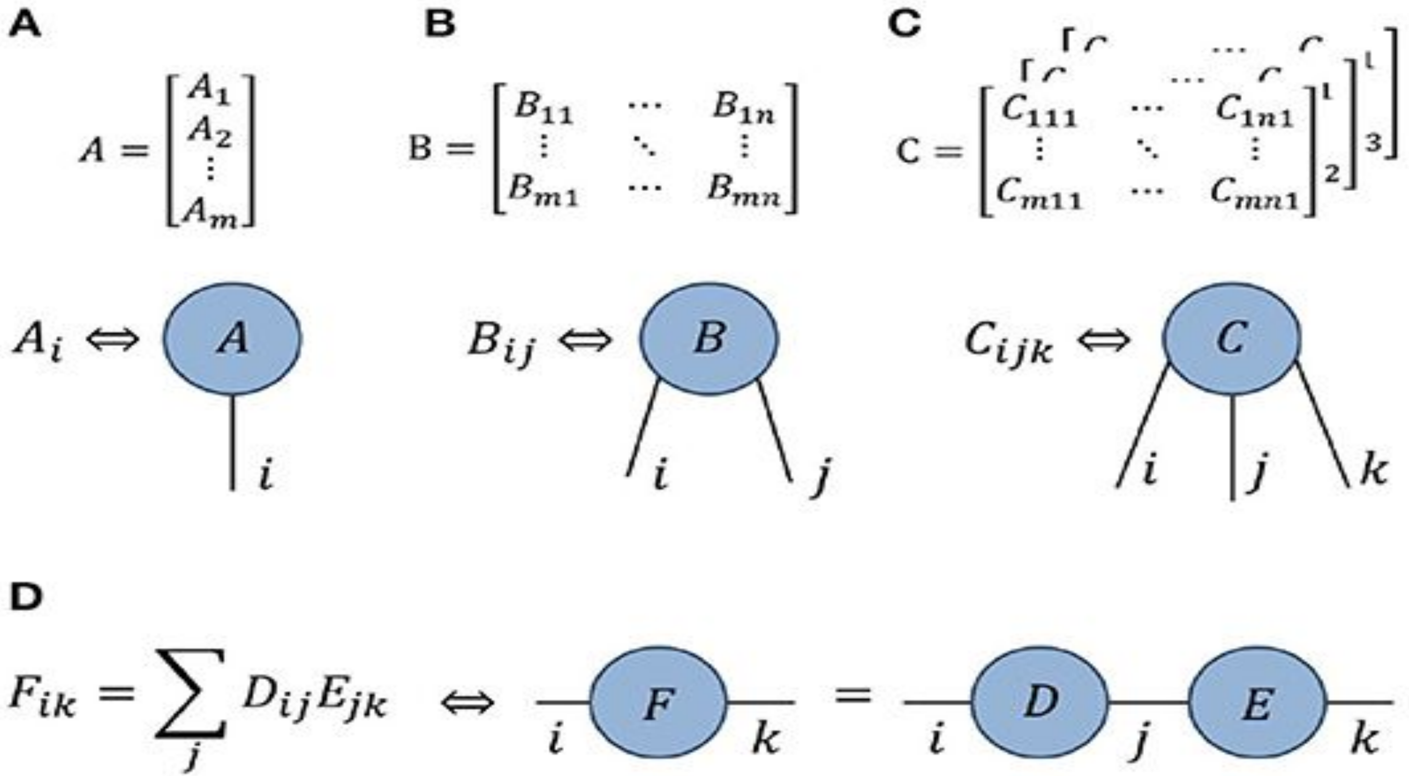


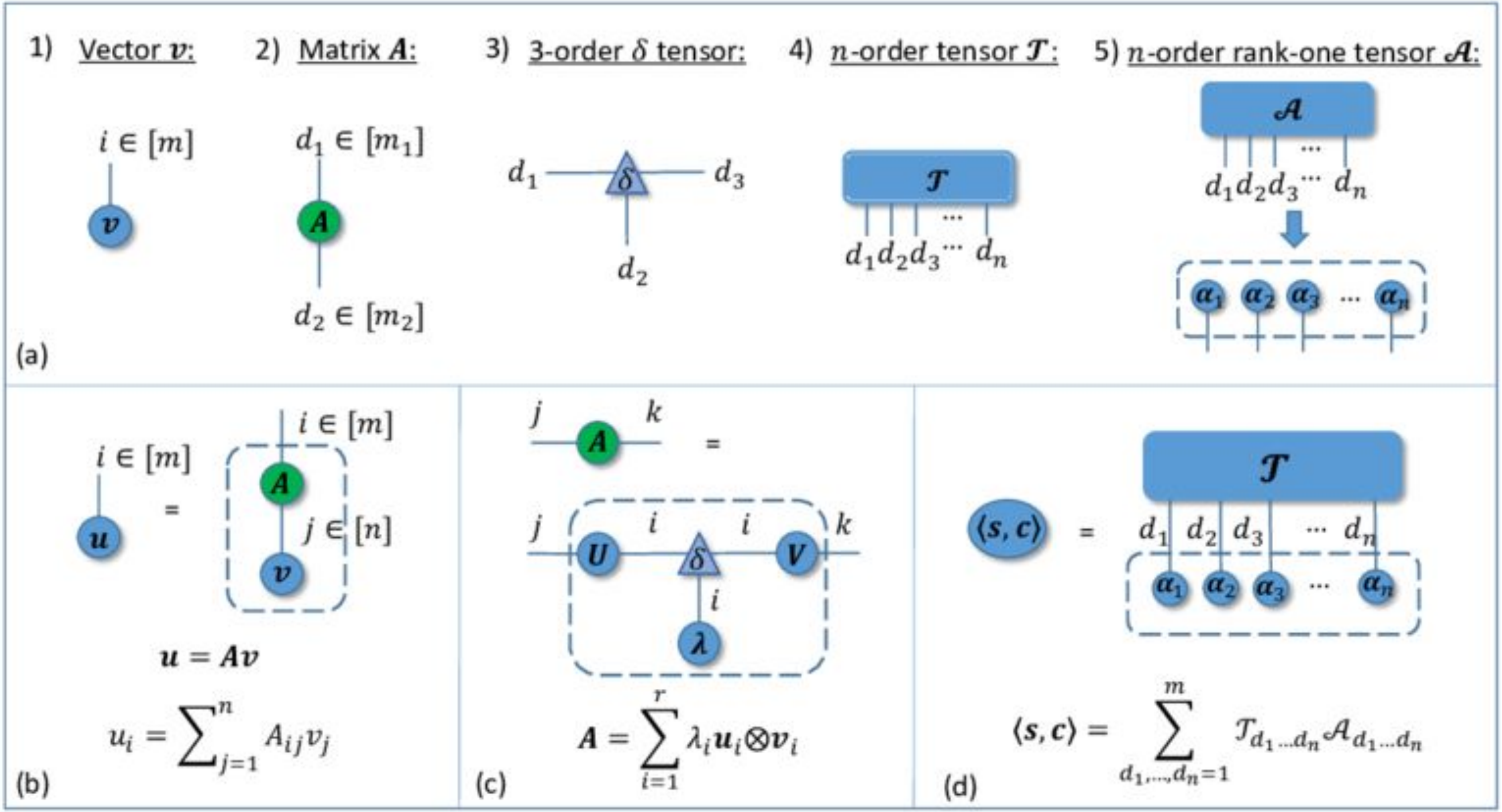
tensor diagrams



tensor contraction diagrams

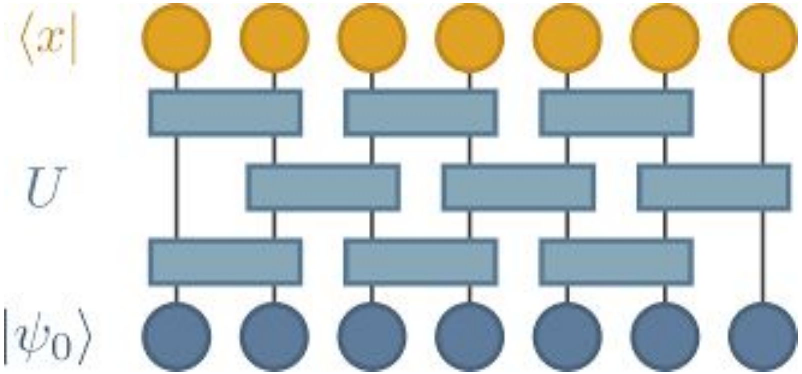
What is tensor ?



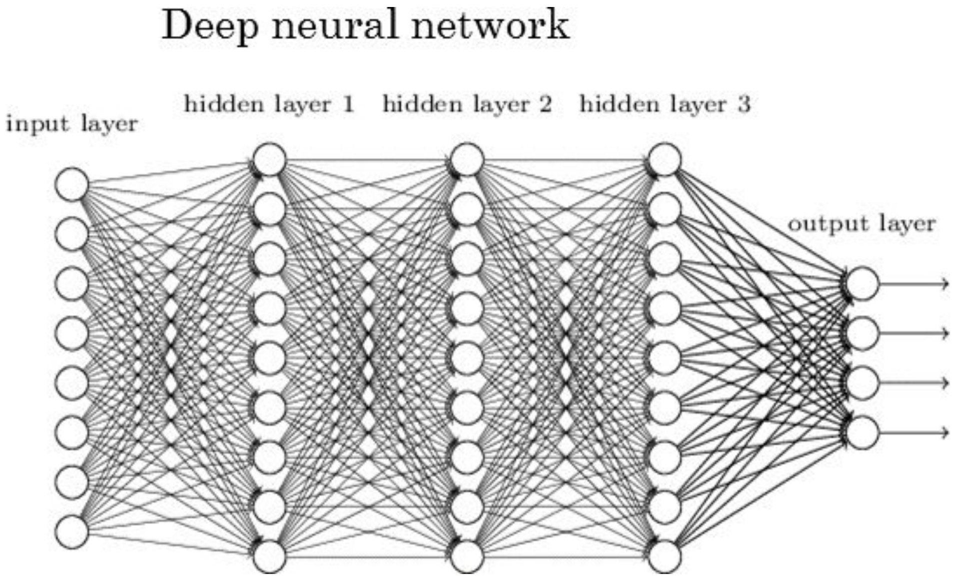


1) Vector $\boldsymbol{v}$:
2) Matrix $\boldsymbol{A}$:
3) 3-order δ tensor:
4) n-order tensor $\mathcal{T}$:
5) n-order rank-one tensor $\mathcal{A}$:
$i \in [m]$
$d_1 \in [m_1]$
$d_2 \in [m_2]$
d_1 d_2 d_3
$d_1 d_2 d_3 \cdots d_n$
α_1 α_2 α_3 ... α_n
(a)
$i \in [m]$
$j \in [n]$
$\boldsymbol{u} = A\boldsymbol{v}$
$u_i = \sum_{j=1}^{n} A_{ij} v_j$
(b)
j k i i i
$\boldsymbol{A} = \sum_{i=1}^{r} \lambda_i \boldsymbol{u}_i \otimes \boldsymbol{v}_i$
(c)
$\langle \boldsymbol{s}, \boldsymbol{c} \rangle$ =
d_1 d_2 d_3 ... d_n
$\langle \boldsymbol{s}, \boldsymbol{c} \rangle = \sum_{d_1,\ldots,d_n=1}^{m} \mathcal{T}_{d_1 \ldots d_n} \mathcal{A}_{d_1 \ldots d_n}$
(d)

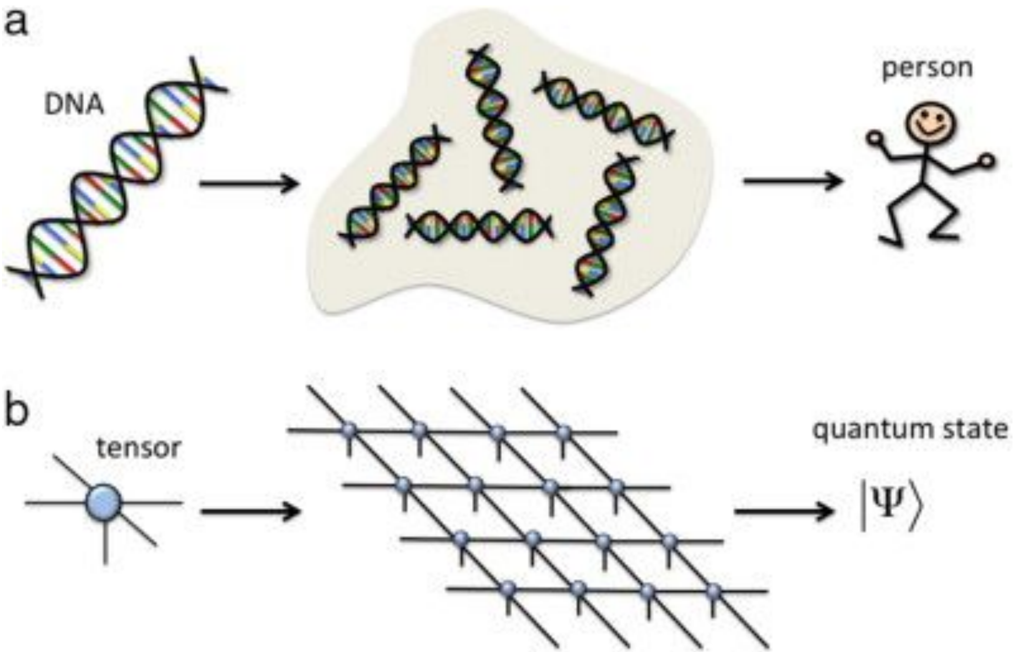
tensor network

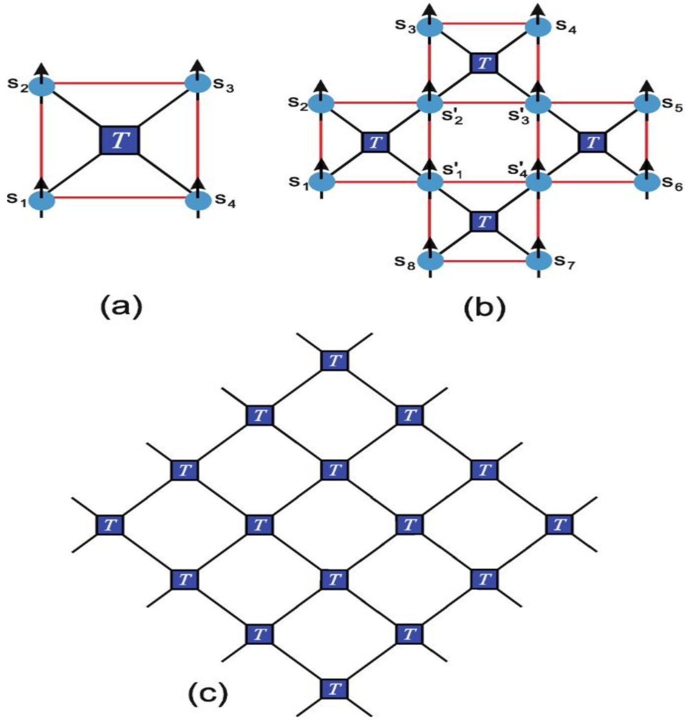


Quantum circuit

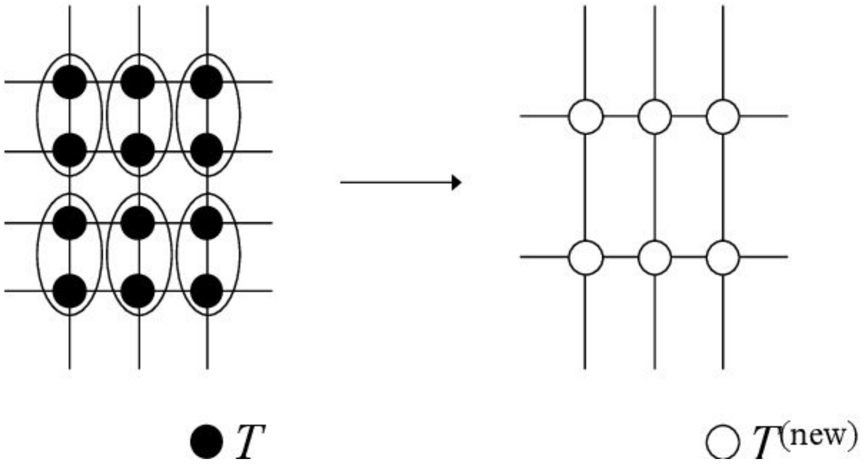


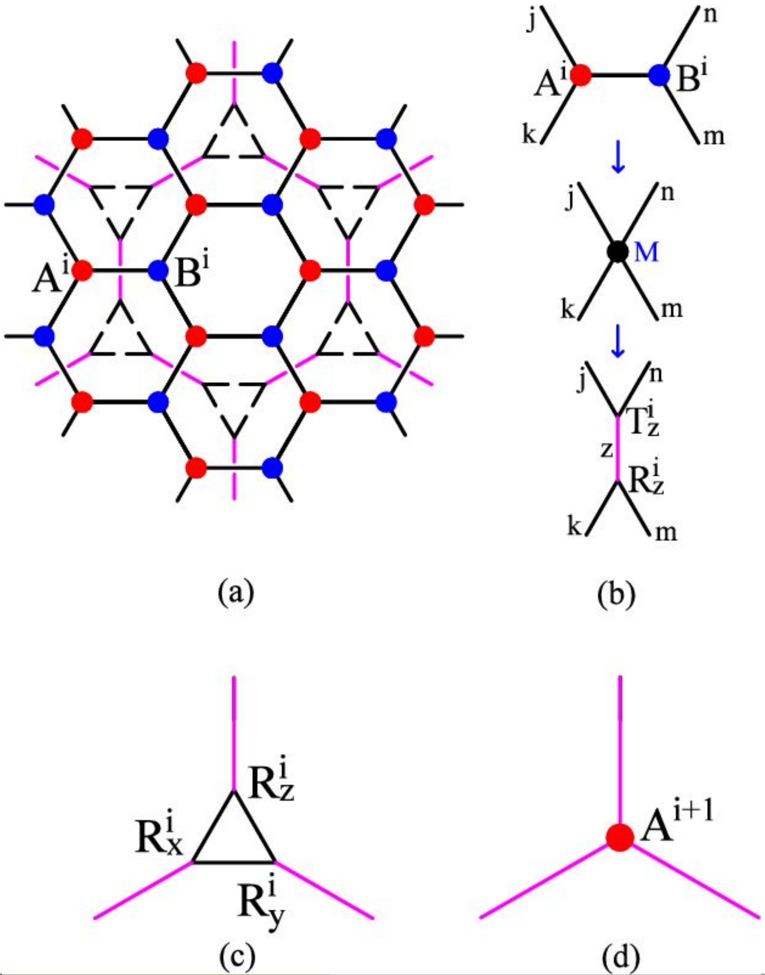
Tensor network represent quantum state and hamiltonian





Examples of TRG

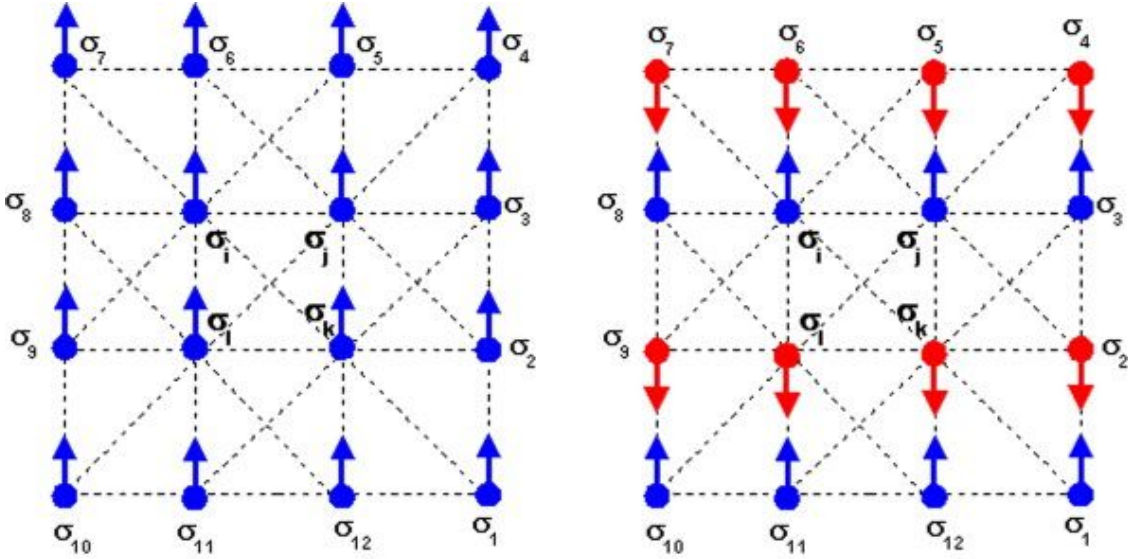




TRG on a square lattice Ising model

Ising model

$$H = -J \sum_{\langle ij \rangle} s_i s_j$$



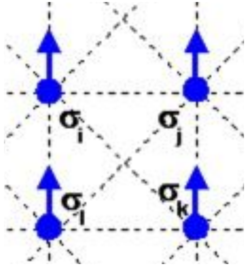
(a) ferromagnetic state

(b) superantiferromagnetic state

$$Z = \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr} \prod_{\square_{ijkl}} e^{\beta J (s_i s_j + s_j s_k + s_k s_l + s_l s_i)/2}$$

Switch spin operator to bond variables

$$\sigma_{ij} = s_i s_j = \pm 1$$



$$Z = \mathrm{Tr} \prod_{\langle ij \rangle} \delta(\sigma_{ij} - s_i s_j) \prod_{\Box_{ijkl}} e^{\beta J(\sigma_{ij}+\sigma_{jk}+\sigma_{kl}+\sigma_{li})/2}$$

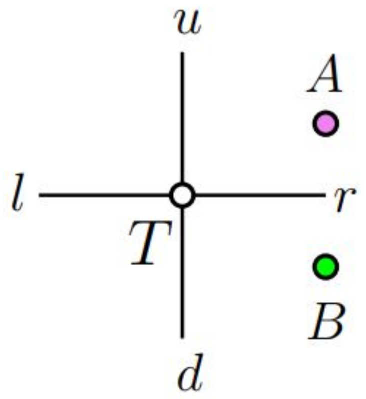
The product of bonds around a loop has value −1 is impossible.

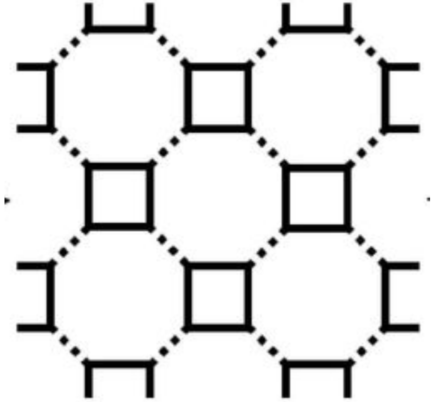
$$Z = \mathrm{Tr} \prod_{\Box_{ijkl}} \frac{1 + \sigma_{ij}\sigma_{jk}\sigma_{kl}\sigma_{li}}{2} e^{\beta J(\sigma_{ij}+\sigma_{jk}+\sigma_{kl}+\sigma_{li})/2}$$

How to represent partition function as tensor network

$$Z = \mathrm{Tr} \prod_{\square_{ijkl}} \frac{1 + \sigma_{ij}\sigma_{jk}\sigma_{kl}\sigma_{li}}{2} e^{\beta J(\sigma_{ij}+\sigma_{jk}+\sigma_{kl}+\sigma_{li})/2}$$

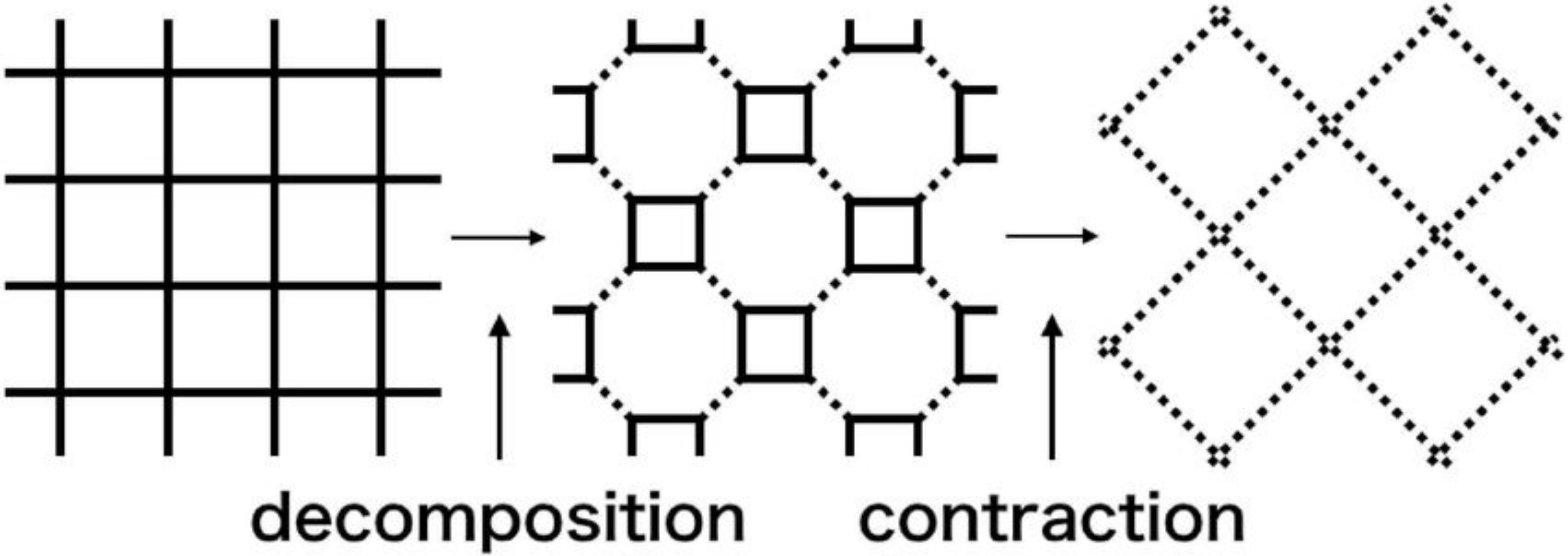
$$Z = \mathrm{tTr}\left[\otimes_{p=1}^{N} T\right]$$

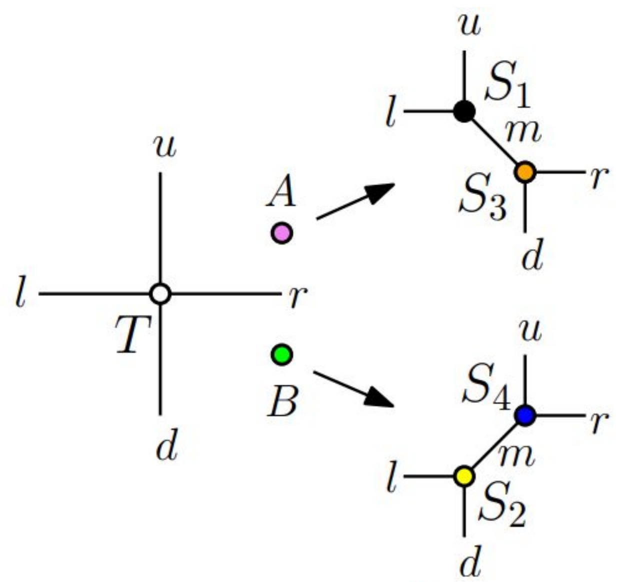




$$T_{r_p u_p l_p d_p} = \frac{1 + \sigma_r^p \sigma_u^p \sigma_l^p \sigma_d^p}{2} e^{\beta J(\sigma_r^p+\sigma_u^p+\sigma_l^p+\sigma_d^p)/2}$$

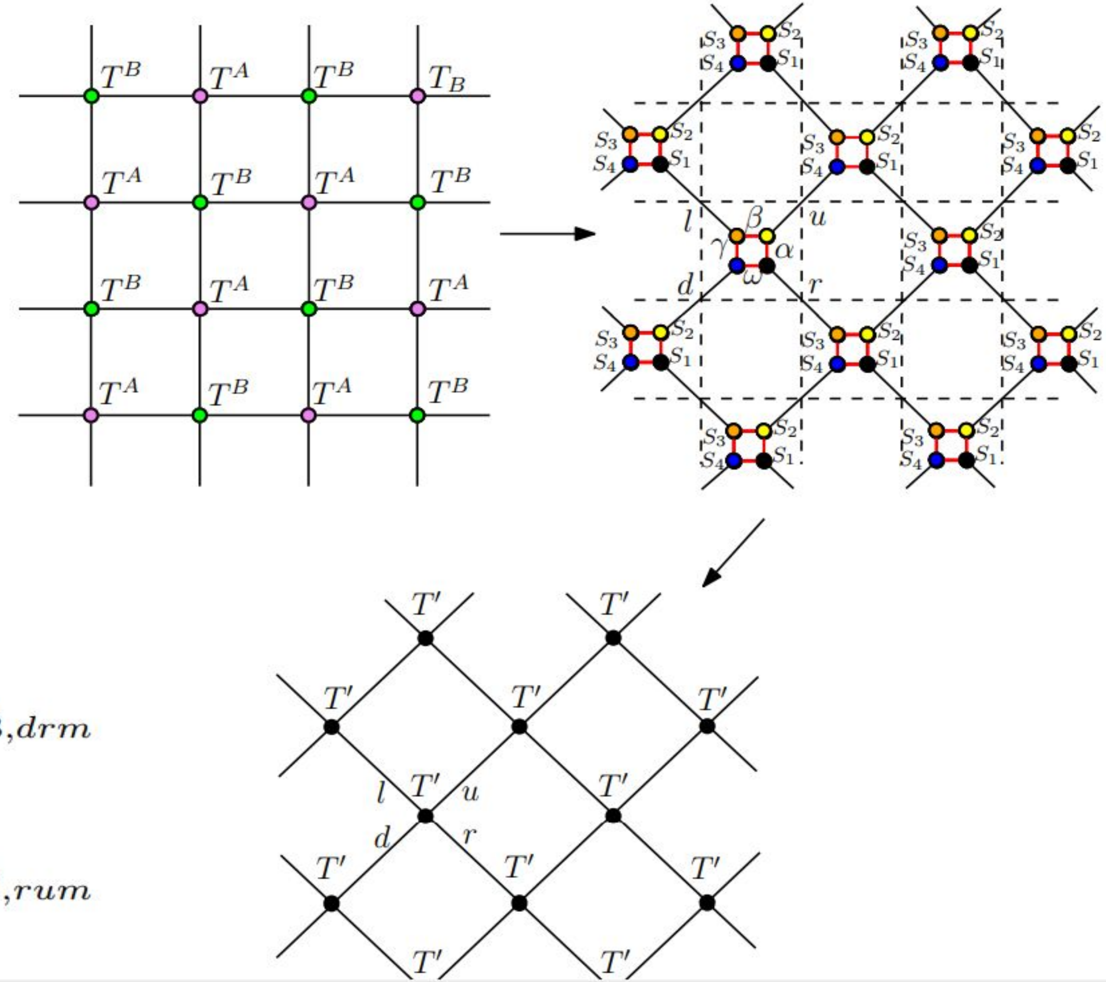
Tensor renormalization group of square lattice



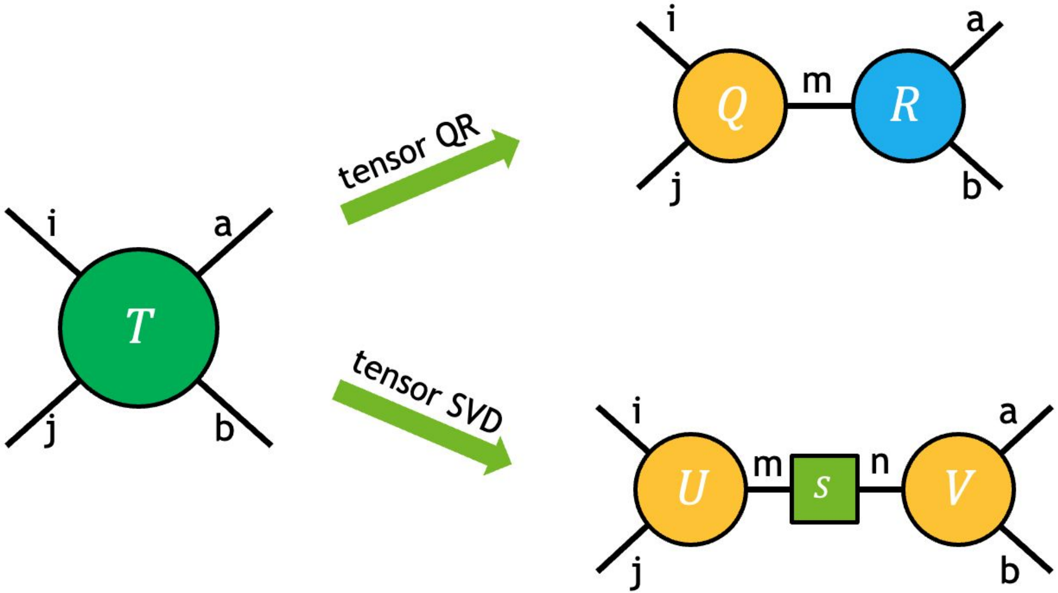


$$T^{A}_{ruld} = \sum_{m=1}^{D^2} S_{1,ulm} S_{3,drm}$$

$$T^{B}_{ruld} = \sum_{m=1}^{D^2} S_{2,ldm} S_{4,rum}$$

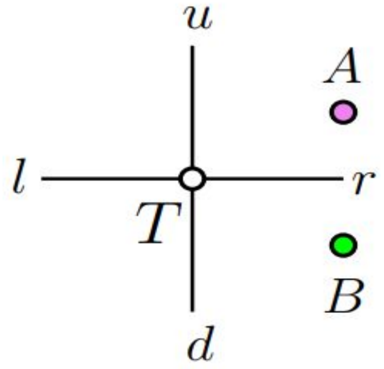


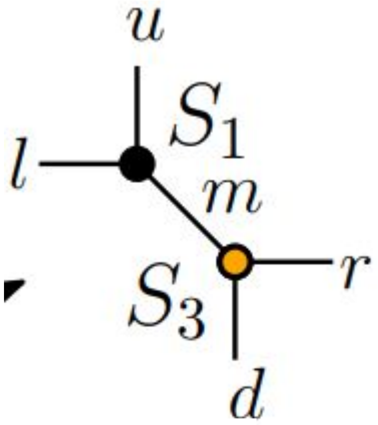
SVD decomposition

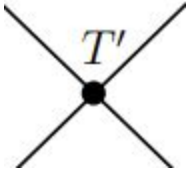


$$M = U\Sigma V^{\dagger}$$

$$T^A_{ruld} = M_{lu,rd} \quad D^2 \times D^2 \text{ matrix} \qquad T^A_{ruld} = \sum_{m=1}^{D^2} S_{1,ulm} S_{3,drm}$$

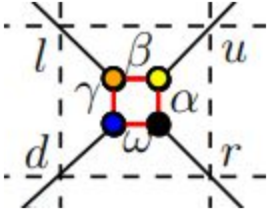




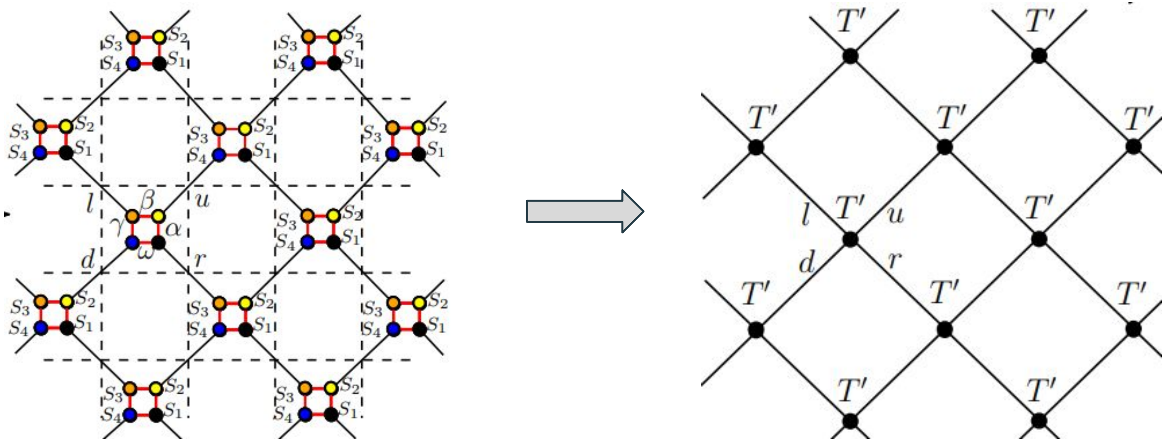


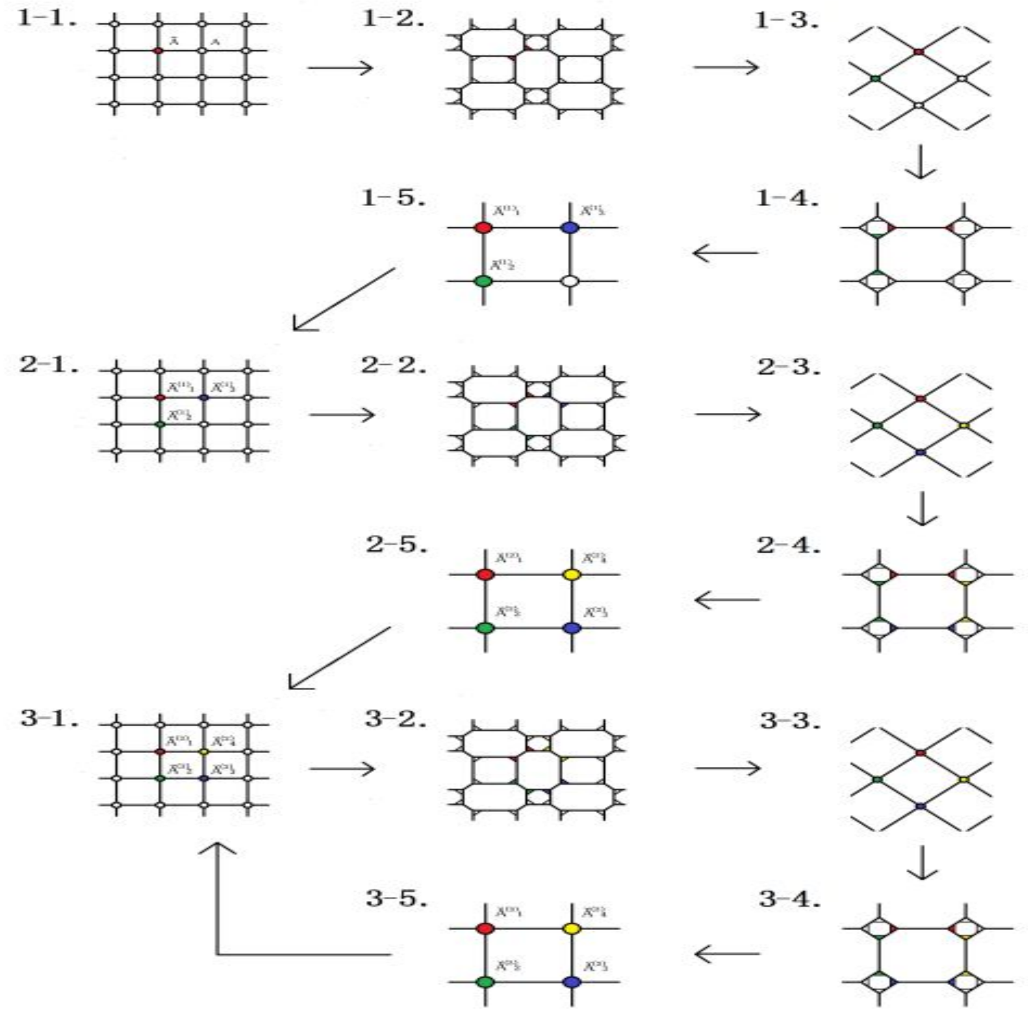
$$T^A_{ruld} \approx \sum_{m=1}^{D'} \underbrace{\left(\sqrt{\lambda_m} U_{lu,m}\right)}_{S_{1,ulm}} \underbrace{\left(\sqrt{\lambda_m} V^{\dagger}_{m,rd}\right)}_{S_{3,drm}}$$

only kept a fixed number D’ = min(D2 , Dcut) of the largest eigenvalues λm

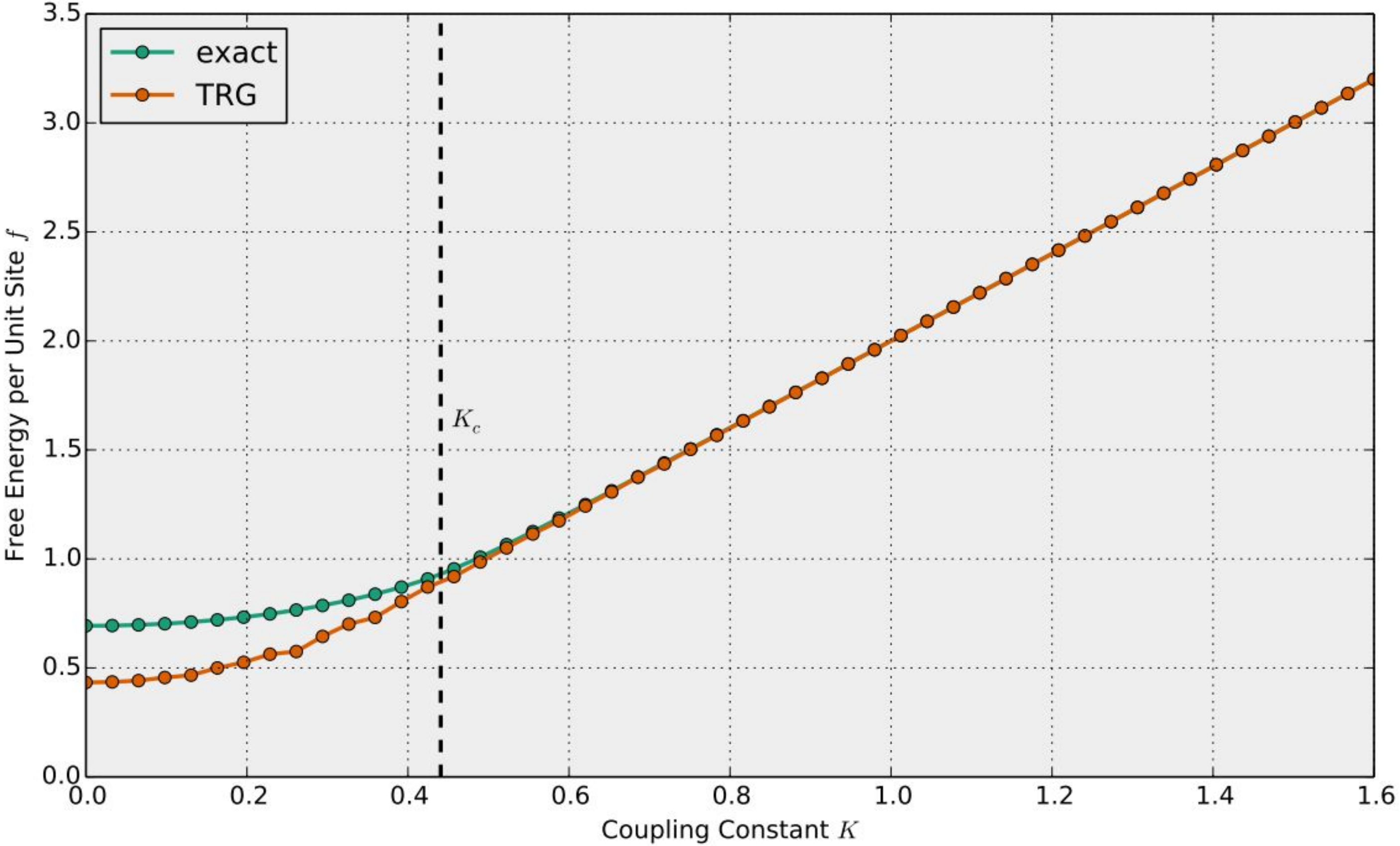


$$T'_{ruld} = \sum_{\alpha,\beta,\gamma,\omega=1}^{D} S_{1,\omega\alpha r} S_{2,\alpha\beta u} S_{3,\beta\gamma l} S_{4,\gamma\omega d}$$





Renormalization Group flow



The TRG was applied 3 times with Dcut = 6. The accuracy of the TRG method is seen to decrease by several orders of magnitude as the critical point Kc is approached.

Limitation of TRG

Near criticality quantum states under the classical-quantum lattice model mapping become gapless ground states [more entangled than their gapped counterparts]

the truncated SVD decomposition failing to accurately represent the original tensor T near criticality

The 2D square lattice Ising model experiences a phase transition and therefore becomes critical at the value

$$K_c = \frac{1}{2} \ln\left(1 + \sqrt{2}\right) \approx 0.441$$

Improve TRG $\longrightarrow$ SRG

take into account the ”environment” lattice when performing SVD composition tensors node in the rewiring step

TRG : minimizes the truncation error of the local matrices M

SRG: minimizes the truncation error of entire partition function

– improve the accuracy of the TRG by several orders of magnitude

$$Z = \mathrm{tTr}\left[M M_e\right]$$

Insight of renormalization group

Every partition function [local interaction] —> tensor network model

Exact sum of a general high-dimensional tensor network is a #P-complete problem

Transfer matrix VS coarse graining

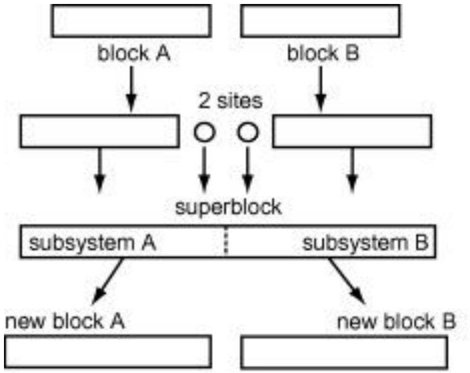
spin-blocking prescription – Kadanoff [1]

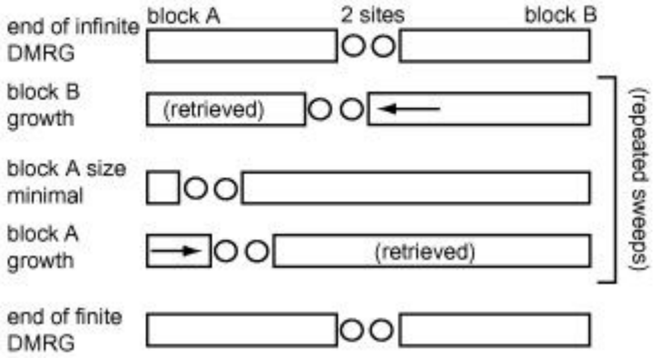
Wilson [2] the path to non-perturbative approaches based on coarse-graining a lattice

tensor renormalization group [3] a more general RG approach for classical lattice models

The tensor renormalization group may been seen as a generalization of the density matrix renormalization group (DMRG) method, introduced by White [4] to study **the ground state of Heisenberg spin chains.**

DMRG





matrix product states + Wilsonian RG

Steps:dividing the system into blocks and truncating basis states at each step

an error-minimizing way by maximizing the entanglement entropy at each step

λ are the the eigenvalues of the reduced (post-truncation) density matrix.

$$S = -\sum_l \lambda_l \ln \lambda_l$$

lack of ”minus sign” problem.

Why truncation can be precise？

Although approximate due to these truncations, the DMRG is an extraordinarily precise method in one dimensional systems.

the surface of a lattice model contains just two points and does not grow with system size.

In higher dimensions,exponential growth of the matrix dimension at each iteration is required to faithfully represent a quantum state, making the algorithm intractable.

Efficiency of dmrg

the surface of a lattice model contains just two points and does not grow with system size.

As the entanglement entropy scales with this surface area, [area law] only a small matrix dimension is necessary for a matrix product state to accurately represent a quantum state.

In higher dimensions,exponential growth of the matrix dimension at each iteration is required to faithfully represent a quantum state, making the algorithm intractable.

Thank you