

Exercise 1

(a) Using Theorem 3.6:

$$\Pr_{T \mathcal{D}^m} (\forall h \in \mathcal{H} : |err_T(h) - err_D(h)| \leq \epsilon) > 1 - \delta$$

$$\Pr_{T \mathcal{D}^m} (\forall h \in \mathcal{H} : |err_T(h) - err_D(h)| \leq \epsilon) > 0.9$$

$$\Rightarrow \delta = 0.1$$

$$m \geq \frac{1}{2\epsilon^2} \log \left(\frac{2|\mathcal{H}|}{\delta} \right)$$

$$143 \geq \frac{1}{2\epsilon^2} \log \left(\frac{2 \cdot 2^3}{0.1} \right)$$

$$143 \geq \frac{1}{2\epsilon^2} (\log(2^4) - \log(0.1))$$

$$143 \geq \frac{1}{2\epsilon^2} (4 - \log(0.1))$$

$$\epsilon^2 \geq \frac{(4 - \log(0.1))}{1432}$$

$$|\epsilon| \geq \sqrt{\frac{(4 - \log(0.1))}{286}}$$

$$\Rightarrow \epsilon \geq \sqrt{\frac{(4 - \log(0.1))}{286}}$$

$$\epsilon \geq \sqrt{\frac{(4 - \log(0.1))}{286}}$$

$$\Pr_{T \mathcal{D}^m} (\forall h \in \mathcal{H} : |err_T(h) - err_D(h)| \leq \epsilon) > 0.9$$

$$\Pr_{T \mathcal{D}^m} \left(\forall h \in \mathcal{H} : |0.03 - err_D(h)| \leq \sqrt{\frac{(4 - \log(0.1))}{286}} \right) > 0.9$$

$$\Rightarrow err_D(h) \leq 0.03 + \sqrt{\frac{(4 - \log(0.1))}{286}} \simeq 0.05560114718 \simeq 0.06$$

(b) Using Theorem 3.4:

$$\Pr_{T \mathcal{D}^m} (\forall h \in \mathcal{H} : \text{if } h \text{ is consistent with } T, \text{ then } err_D(h) \leq \epsilon) > 1 - \delta$$

$$\Pr_{T \mathcal{D}^m} (\forall h \in \mathcal{H} : \text{if } h \text{ is consistent with } T, \text{ then } err_D(h) \leq 0.01) > 0.9$$

$$\Rightarrow \epsilon = 0.01, \delta = 0.1$$

$$m \geq \frac{1}{\epsilon} \ln \left(\frac{|\mathcal{H}|}{\delta} \right)$$

$$m \geq \frac{1}{0.01} \ln \left(\frac{2^3}{0.1} \right)$$

$$m \geq 100(\ln(3) - \ln(0.1)) \sim 100 \cdot 3.40119738166 = 340.1197$$

$$\Rightarrow m \geq 341$$

Exercise 2

(a)

(b)

Exercise 3

- (a) The VC-Dimension is 3. To prove this, let us prove some properties which every Y , that shatters \mathcal{H} , must hold.

Notation:

- $a_i, b_i \in \mathbb{R}, i \in \mathbb{N}$
- $y_{i,j} \in Y, i, j \in \mathbb{N} \rightarrow y_{i,j} = (a_i, b_j)$ (analog for $y'_{i,j} \in Y'$)

Property 1:

$$\neg \exists y_{i,j} \exists y'_{i',j'} \exists y''_{i'',j''} (a_i \neq a_{i'} \wedge a_i \neq a_{i''} \wedge a_{i'} \neq a_{i''} \wedge b_i \neq b_{i'} \wedge b_i \neq b_{i''} \wedge b_{i'} \neq b_{i''})$$

In words: There cannot be 3 vectors in Y that have neither their a nor b values in common.

Let's prove this by counterexample.

Let $Y' = \{(a_1, b_1), (a_2, b_2), (a_3, b_3)\} \subseteq Y$. There exists no $h_{a,b} \in \mathcal{H}$, such that $Y' \subseteq S_{h_{a,b}}$. Therefore, $Y' = S_{h_{a,b}} \cap Y$ does not hold.

Property 2:

$$\neg \exists y_{i,j} \exists y'_{i',j'} \exists y''_{i'',j''} (y_{i,j} \neq y'_{i',j'} \wedge y_{i,j} \neq y''_{i'',j''} \wedge y'_{i',j'} \neq y''_{i'',j''}) \wedge ((a_i = a_{i'} \wedge a_i = a_{i''}) \vee (b_i = b_{i'} \wedge b_i = b_{i''}))$$

In words: There cannot be 3 vectors in Y that have their a or b values in common.

Let's prove this by counterexample.

Let $Y' = \{(a_1, b_1), (a_1, b_2)\} \subseteq \{(a_1, b_1), (a_1, b_2), (a_1, b_3)\} \subseteq Y$ (analog when the b -components are equal). Then there exists no $h_{a,b} \in \mathcal{H}$ so that $Y' \subseteq S_{h_{a,b}}$ but $(a_1, b_3) \notin S_{h_{a,b}}$. This is true, since the only functions $h_{a,b}$ with $Y' \subseteq S_{h_{a,b}}$ are where $a = a_1$, thus also $(a_1, b_3) \notin S_{h_{a,b}}$, which is a contradiction.

Property 3:

$$\neg \exists y_{i,j} \exists y'_{i',j'} \exists y''_{i'',j''} (y_{i,j} \neq y'_{i',j'} \wedge y_{i,j} \neq y''_{i'',j''} \wedge y'_{i',j'} \neq y''_{i'',j''}) \wedge (a_i = a_{i'} \wedge b_{i'} = b_{i''})$$

In words: There cannot be a vector containing of an a -component that also exists in another vector and a b -component that also exists in another vector.

Let's prove this by counterexample.

Let $Y' = \{(a_1, b_2)\} \subseteq \{(a_1, b_1), (a_1, b_2), (a_2, b_2)\} \subseteq Y$. For all $h_{a,b} \in \mathcal{H}$ with $Y' \subseteq S_{h_{a,b}}$ $a = a_1 \vee b = b_2$ must hold. However, for such $h_{a,b}$ $(a_1, b_1) \in S_{h_{a,b}}$ or $(a_2, b_2) \in S_{h_{a,b}}$ would also hold. This is a contradiction.

Thus, the only valid form of Y must be:

$$Y := \{(a_1, b_1), (a_1, b_2), (a_2, b_2)\}, a_1 \neq a_2 \wedge b_1 \neq b_2 \wedge b_1 \neq b_3 \wedge b_2 \neq b_3$$

We can prove that Y shatters \mathcal{H} :

- $Y' = \emptyset$:

We can use h_{a_0, b_0} with $a_0 \notin \{a_1, a_2\}, b_0 \notin \{b_1, b_2, b_3\}$. Thus, $Y \not\subseteq S_{h_{a_0, b_0}}$. So, $Y' = \emptyset = S_{h_{a_0, b_0}} \cap Y$.

- $Y' = \{(a_1, b_i)\}, i \in \{1, 2\}$:
We can use h_{a_0, b_i} with $a_0 \notin \{a_1, a_2\}$. Thus, $Y' = \emptyset = S_{h_{a_0, b_i}} \cap Y$.
- $Y' = \{(a_2, b_3)\}$:
We can use h_{a_2, b_3} . Thus, $Y' = \emptyset = S_{h_{a_2, b_3}} \cap Y$.
- $Y' = \{(a_1, b_1), (a_1, b_2)\}$:
We can use h_{a_1, b_1} . Thus, $Y' = \emptyset = S_{h_{a_1, b_1}} \cap Y$.
- $Y' = \{(a_1, b_i), (a_2, b_3)\}, i \in \{1, 2\}$:
We can use h_{a_2, b_i} . Thus, $Y' = \emptyset = S_{h_{a_2, b_i}} \cap Y$.
- $Y' = Y$:
We can use h_{a_1, b_3} . Thus, $Y' = \emptyset = S_{h_{a_1, b_3}} \cap Y$.

So, the VC-Dimension is at least 3. Let's now prove, that no Y with $|Y| > 3$ shatters \mathcal{H} .

For now, let Y be $Y := \{(a_1, b_1), (a_1, b_2), (a_2, b_2)\}$, $a_1 \neq a_2 \wedge b_1 \neq b_2 \wedge b_1 \neq b_3 \wedge b_2 \neq b_3$ (which we proved was the only valid form of Y with $|Y| = 3$). We can now try to add any $y_{i,j}$ to Y :

- Let's try to add $y_{1,j} = (a_1, b_j)$, $j \in \mathbb{N}$. This would violate **Property 1**. Thus, we cannot add any such $y_{1,j}$.
- Let's try to add $y_{2,j} = (a_2, b_j)$, $j \in \mathbb{N}$. Since $(a_2, b_3) \in Y$, $b_j \in \mathbb{R} \setminus \{b_3\}$. Then the condition in the remark would not hold for $Y' = Y$. Thus, we cannot add any such $y_{2,j}$.
- Let's try to add $y_{3,4} = (a_3, b_4)$, $a_3 \in \mathbb{R} \setminus \{a_1, a_2\}$, $b_4 \notin \{b_1, b_2, b_3\}$. This would violate **Property 1**. Thus, we cannot add any such $y_{3,4}$.
- Let's try to add $y_{3,j} = (a_3, b_j)$, $j \in \{1, 2\}$, $a_3 \in \mathbb{R} \setminus \{a_1, a_2\}$. Then, we cannot find any $h_{a,b} \in \mathcal{H}$ for $Y' = Y$, so that the condition in the remark holds. Thus, we cannot add any such $y_{3,j}$.
- Let's try to add $y_{3,j} = (a_3, b_3)$, $a_3 \in \mathbb{R} \setminus \{a_1, a_2\}$. Then, for $Y' = \{(a_1, b_2), (a_2, b_3), (a_3, b_3)\}$ we cannot find any $h_{a,b} \in \mathcal{H}$, for which $Y' \subseteq S_{h_{a,b}}$ but $(a_1, b_1) \notin S_{h_{a,b}}$. Thus, we cannot add any such $y_{3,j}$.

So, we cannot add any element to Y . Thus, all Y that shatter \mathcal{H} must be at most $|Y| \leq 3$. Therefore, the VC Dimension is 3.

(b)

Exercise 4

(a)

(b)

Exercise 5

(a) Iteration s=1 of 3:

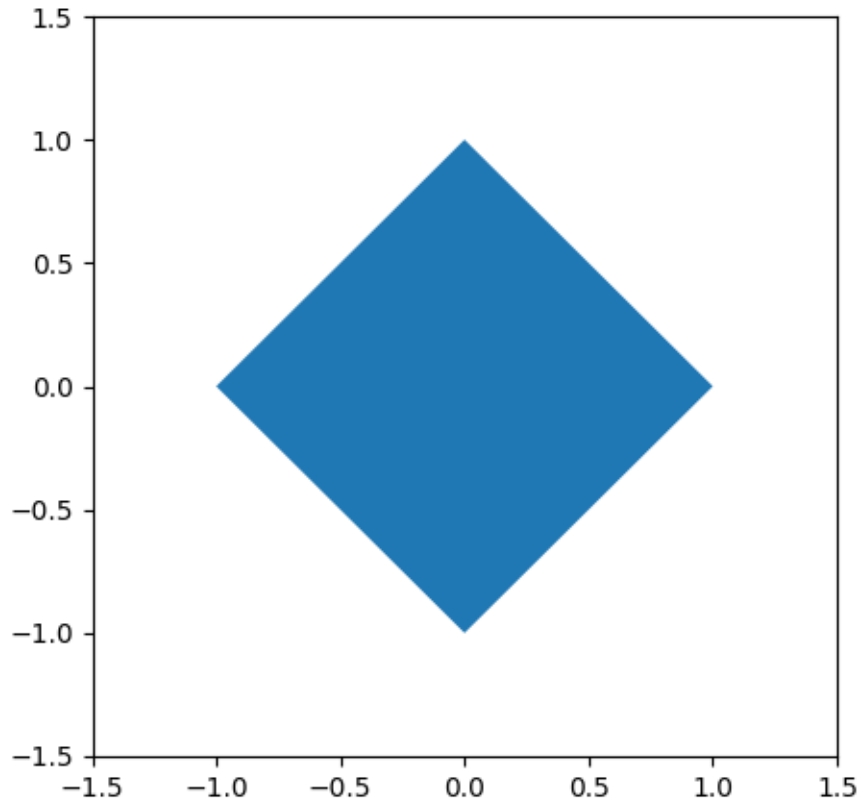
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2
3 Weights:
4 w_1^1 = 1.0
5 w_2^1 = 1.0
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6   w_3^1 = 1.0
7
8 Probabilities:
9   p_1^1 = 0.33
10  p_2^1 = 0.33
11  p_3^1 = 0.33
12
13 Iteration s=2 of 3:
14
15 Weights:
16   w_1^2 = 2.83
17   w_2^2 = 1.0
18   w_3^2 = 1.0
19
20 Probabilities:
21   p_1^2 = 0.46
22   p_2^2 = 0.27
23   p_3^2 = 0.27
24
25 Iteration s=3 of 3:
26
27 Weights:
28   w_1^3 = 2.83
29   w_2^3 = 8.48
30   w_3^3 = 1.0
31
32 Probabilities:
33   p_1^3 = 0.28
34   p_2^3 = 0.51
35   p_3^3 = 0.21
36
37
38 Final Weights:
39   w_1^4 = 2.83
40   w_2^4 = 8.48
41   w_3^4 = 5.32
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(b) This is due to the fact, that $p_3^{(3)} \neq p_1^{(1)}$.

Exercise 6



(a) (i)

(ii) Similarly to how in $l = 2$ the “corners” of the unit circle are the 2 unit vectors and their negations (so 4 in total), the “corners” of the unit circle in $l = 3$ are the 3 unit vectors and their negations (so 6 in total). Combined with the edges and facing connecting them they make for a “diamond” shape.

(b) $\text{vol}(B_1^2) = (\sqrt{1^2 + 1^2})^2 = 2$ and $\text{vol}(B_1^3) = 2 \cdot \frac{(\sqrt{1^2 + 1^2})^2 \cdot 1}{3} = \frac{4}{3}$

(c) Cover B_1^l by $2k$ cylinders. The thickness of the cylinders is $t := \frac{1}{k}$. Thus, the radius of the i th cylinder above (or below) is $r_i := 1 - (i - 1) \cdot t$. Therefore, the volume of the i th

cylinder is $t \cdot r_i^{l-1} \cdot \text{vol}(B_1^{l-1})$. Thus:

$$\begin{aligned}
 \text{vol}(B_1^l) &\leq 2 \sum_{i=1}^k t \cdot r_i^{l-1} \cdot \text{vol}(B_1^{l-1}) \\
 &= \left(2 \sum_{i=1}^k \frac{1}{k} \left(1 - \frac{i-1}{k} \right)^{l-1} \right) \cdot \text{vol}(B_1^{l-1}) \\
 &= \left(2 \sum_{i=0}^{k-1} \frac{1}{k} \left(1 - \frac{i}{k} \right)^{l-1} \right) \cdot \text{vol}(B_1^{l-1}) \\
 &= \left(2 \frac{1}{k} \left(1^{l-1} + \left(1 - \frac{1}{k} \right)^{l-1} + \dots + \left(1 - \frac{k-1}{k} \right)^{l-1} \right) \right) \cdot \text{vol}(B_1^{l-1}) \\
 &= \left(2 \left(\frac{k^{l-1}}{k^l} + \left(\frac{(k-1)^{l-1}}{k^l} \right) + \dots + \left(\frac{1^{l-1}}{k^l} \right) + \left(\frac{0^{l-1}}{k^l} \right) \right) \right) \cdot \text{vol}(B_1^{l-1}) \\
 &= \left(2 \underbrace{\sum_{i=0}^k \left(\frac{i^{l-1}}{k^l} \right)}_{:=S} \right) \cdot \text{vol}(B_1^{l-1})
 \end{aligned}$$

We can use the ratio test on the series S :

$$\lim_{k \rightarrow \infty} \left| \frac{\left(\frac{(i+1)^{l-1}}{k^l} \right)}{\left(\frac{i^{l-1}}{k^l} \right)} \right| = \left| \frac{i+1}{i} \right|^{l-1} = \left| \left(1 + \frac{1}{i} \right)^{l-1} \right| > 1$$

Huh? Something is wrong here

Appendix