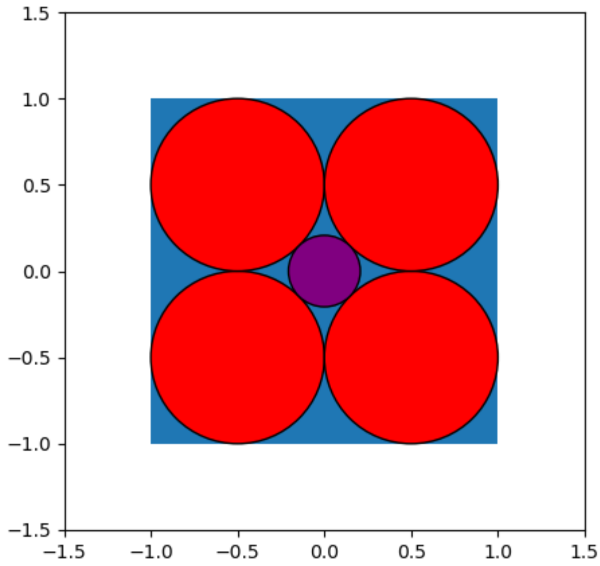
Exercise 1

(a)

Situation for l=2 and s=2:

$$Q_{2,2} = \{[x_1 x_2]^T \in \mathbb{R}^2 \mid |x_i| \leq 1 \text{ for all } i = 1,2\} = [-1,1]^2$$



(b)

First, let's calculate the radius of the inner hyperball for any $l \in \mathbb{N}, s \in \mathbb{R}_{>0}$:
The distance from the center of the inner hyperball (equal to the center of the hypercube) to the center of one of the 2^l outter hyperballs (doesn't matter which one) can be calculated the following:

$$d := \sqrt{l \cdot \left(\frac{s}{4}\right)^2}$$

Thus, the radius of the inner hyperball is equal to:

$$r := d - \frac{s}{4} = \sqrt{l \cdot \left(\frac{s}{4}\right)^2} - \frac{s}{4}$$

Now, we simply must solve the following inequality to find an $l \in \mathbb{N}$ for an arbitrary but fixed $s \in \mathbb{R}_{>0}$ such that $B(Q_{l,s}) \not\subseteq Q_{l,s}$:

$$\begin{aligned}
\frac{s}{2} &< r \\
\frac{s}{2} &< d - \frac{s}{4} \\
\frac{s}{2} &< \sqrt{l \cdot \left(\frac{s}{4}\right)^2} - \frac{s}{4} \\
\frac{3 \cdot s}{4} &< \sqrt{l} \cdot \frac{s}{4} \\
3 &< \sqrt{l} \\
l &> 9
\end{aligned}$$

Exercise 2

(a)

Let's first find all eigenvalues of A_c:

$$\{\lambda \in \mathbb{R} \mid \det(A_c - \lambda \cdot I_3) = 0\}$$
$$\{\lambda \in \mathbb{R} \mid \det \begin{bmatrix} 2-\lambda & 0 & c \\ 0 & 1-\lambda & 0 \\ c & 0 & 1-\lambda \end{bmatrix} = 0\}$$
$$\{\lambda \in \mathbb{R} \mid -\lambda^3 + 4 \cdot \lambda^2 + c^2 \cdot \lambda - 5 \cdot \lambda + 2 - c^2\}$$
$$\{\lambda \in \mathbb{R} \mid (\lambda - 1) \cdot (-\lambda^2 + 3\dot{\lambda} + c^2 - 2)\}$$
$$\{1, \frac{3 - \sqrt{4 \cdot c^2 + 1}}{2}, \frac{3 + \sqrt{4 \cdot c^2 + 1}}{2}\} =: \Lambda_{A_c}$$

So, if for all $\lambda \in \Lambda_{A_c}$ it must hold that $\lambda \geq 0$, we must set c such that the following inequality holds:

$$\frac{3 - \sqrt{4 \cdot c^2 + 1}}{2} \geq 0$$
$$3 - \sqrt{4 \cdot c^2 + 1} \geq 0$$
$$3 \geq \sqrt{4 \cdot c^2 + 1}$$
$$9 \geq 4 \cdot c^2 + 1$$
$$8 \geq 4 \cdot c^2$$
$$2 \geq c^2$$
$$c \in (-\sqrt{2}, \sqrt{2}) \subseteq \mathbb{R}$$

(b)

Using Theorem 5.19, we create an orthogonal matrix $U \in \mathbb{R}^{n \times n}$ with the columns being the eigenvectors of A. We can now use Theorem 5.20 to create the diagonal matrix $\Lambda \in \mathbb{R}^{n \times n}$ where the eigenvalue-entry $\lambda_{i,i}$ corresponds to eigenvector in the column i of U.
A simple verification:

$$\begin{aligned} A &= U \Lambda U^\top \\ A &= U \Lambda U^{-1} \\ AU &= U \Lambda \\ Av_i &= v_i \lambda_{i,i} \qquad , \forall i \in [n], v_i := col_i(U) \\ Av_i &= \lambda_{i,i} v_i \qquad , \forall i \in [n], v_i := col_i(U) \end{aligned}$$

Since A is positive semi-definite, all entries $\lambda_{i,i} \geq 0$. Thus, we can create a diagonal matrix Λ' consisting of the entries $\lambda'_{i,i} := \sqrt{\lambda_{i,i}}$. Thus, $\Lambda = \Lambda'\Lambda'^{\top}$. Let $B := UU\Lambda'$. Now:

$$\begin{aligned}
A &= U\Lambda U^{\top} \\
A &= U(\Lambda'\Lambda'^{\top})U^{\top} \\
A &= U(\Lambda'\Lambda'^{\top})U^{\top} \\
A &= (U\Lambda')(\Lambda'^{\top}U^{\top}) \\
A &= (U\Lambda')(U\Lambda')^{\top} \\
A &= BB^{\top}
\end{aligned}$$

(c)

Since A is symmetric, the following holds $\forall x, y \in R^n$:

$$\langle Ax, y\rangle = \langle x, Ay\rangle$$

Let $v_1 \in E_1, v_2 \in E_2$. Then $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$.

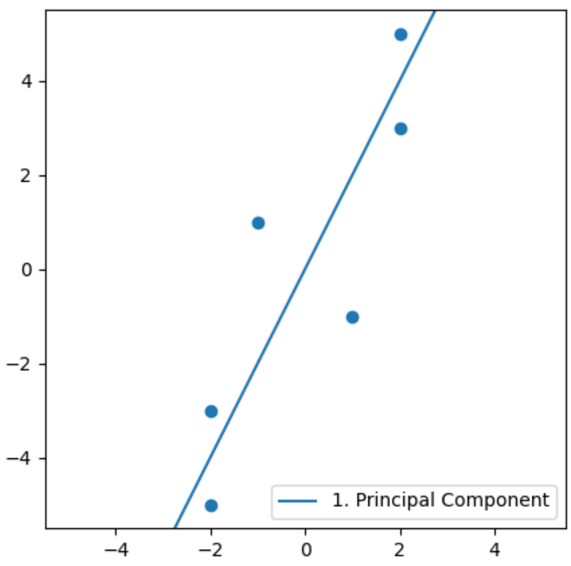
$$\begin{aligned}
0 &= \langle Av_1, v_2\rangle - \langle v_1, Av_2\rangle \\
&= \langle \lambda_1 v_1, v_2\rangle - \langle v_1, \lambda_2 v_2\rangle \\
&= (\lambda_1 - \lambda_2)\langle v_1, v_2\rangle
\end{aligned}$$

Since $\lambda_1 \neq \lambda_2$ it follows, that $\langle v_1, v_2\rangle = 0$.

Exercise 3

(a)

Plot and estimate first principal component $(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$



(b)

$$
\begin{aligned}
C &\coloneqq A^\top A \\
&= \begin{bmatrix} -2 & -2 & -1 & 1 & 2 & 2 \\ -5 & -3 & 1 & 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} -2 & -5 \\ -2 & 3 \\ -1 & 1 \\ 1 & -1 \\ 2 & 3 \\ 2 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 18 & 18 \\ 18 & 70 \end{bmatrix}
\end{aligned}
$$

(c)

View code in the appendix for the calculations.
The eigenvector we got is: $\begin{pmatrix} 0.298 \\ 0.954 \end{pmatrix}$

(d)

Well, they both point in the same general direction with a slope greater than 1. Our slope came out to be at 2, since eye-balling stuff tends to round to integer values. The more optimal slope however seems to be even greater than $3 < \frac{0.954}{0.298}$.

Exercise 4

(a)

$$
S \coloneqq \begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}
$$

$$
D \coloneqq \begin{bmatrix}
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3
\end{bmatrix}
$$

$$L \coloneqq D - S = \begin{bmatrix} 3 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 4 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 4 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & 3 \end{bmatrix}$$

(b)

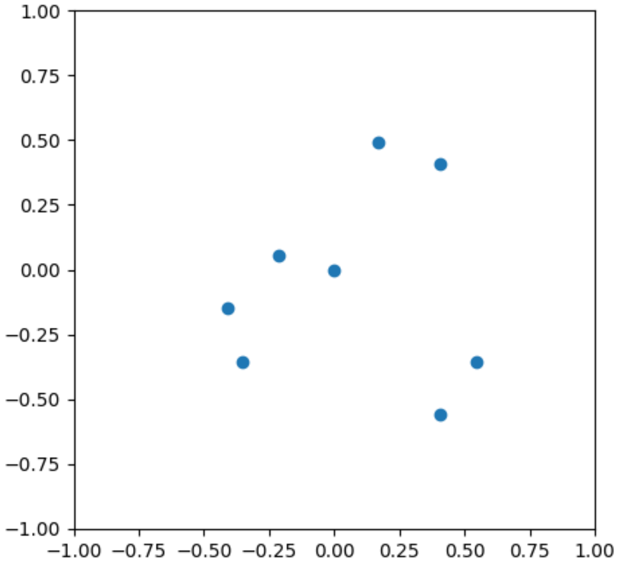
View code in the appendix for the calculations.

$$\lambda_1 = 0.0$$
$$u_1 = \begin{pmatrix} -0.354 \\ 0.169 \\ -0.408 \\ 0.408 \\ 0.548 \\ -0.214 \\ 0.408 \\ -0.002 \end{pmatrix}$$
$$\lambda_2 = 0.657$$
$$u_2 = \begin{pmatrix} -0.354 \\ 0.493 \\ -0.149 \\ -0.558 \\ -0.358 \\ 0.056 \\ 0.408 \\ -0.002 \end{pmatrix}$$

(c)

The eigenvalue λ_1 is extremely close to 0. Therefore, $\lambda_1 u_1$ is very close to 0, so also Au_1 must be very close to 0.

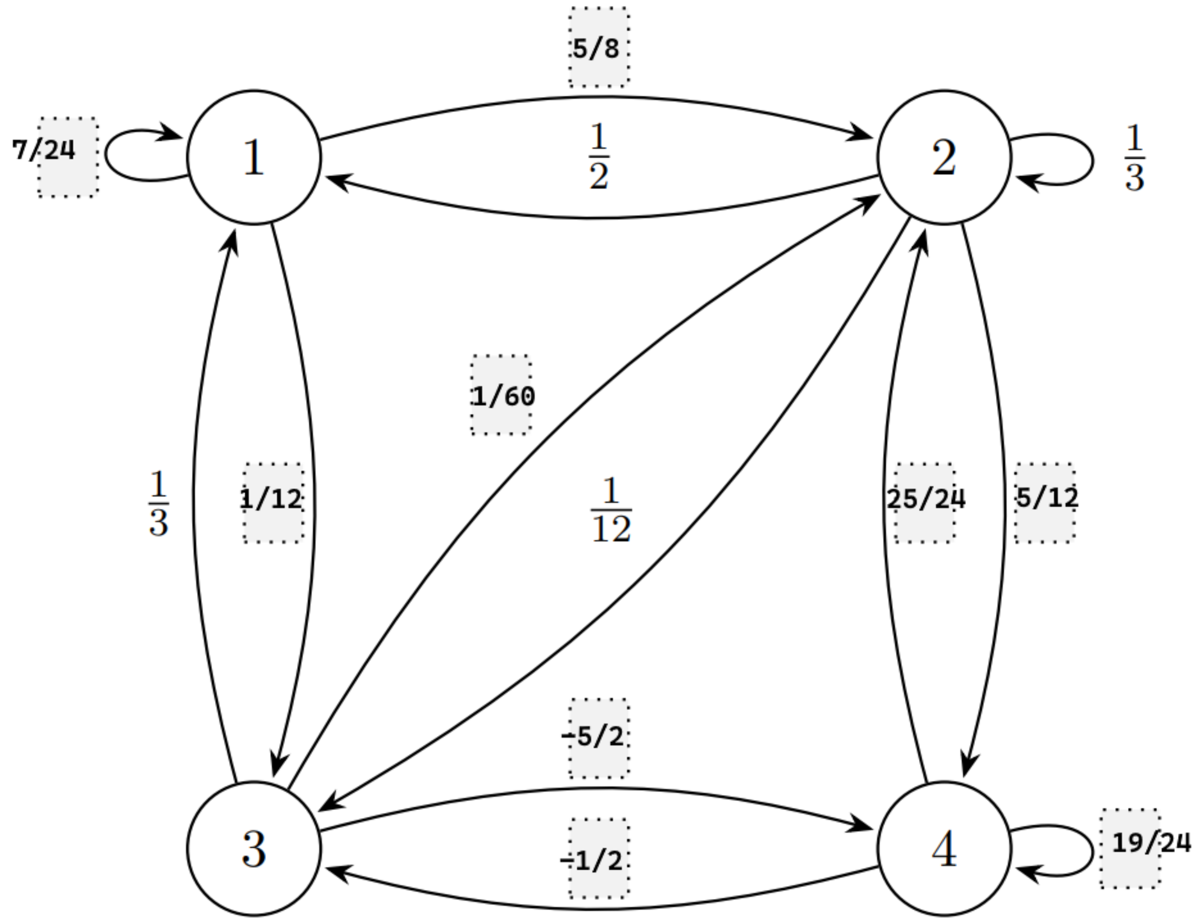
(d)



(e)

The Spectral Clustering Algorithm returns the clusters computed using the k-means clustering algorithm on the points $((u_1)_i, (u_2)_i), i = 1, \ldots, 8$. Therefore, depending on the starting centroids, we might see a cluster containing only the two bottom right points, and another containing the remaining points.

Exercise 5



$$\begin{aligned}
\pi_1 &= \pi_1 q_{11} + \pi_2 q_{21} + \pi_3 q_{31} \\
\frac{4}{12} &= \frac{4}{12} q_{11} + \frac{5}{12}\frac{1}{2} + \frac{1}{12}\frac{1}{3} \\
4 &= 4q_{11} + 5\frac{1}{2} + 1\frac{1}{3} \\
4 &= 4q_{11} + \frac{17}{6} \\
\frac{7}{6} &= 4q_{11} \\
q_{11} &= \frac{7}{24}
\end{aligned}$$

$$\begin{aligned}
\pi_1 q_{12} &= \pi_2 q_{21} \\
\frac{4}{12} q_{12} &= \frac{5}{12}\frac{1}{2} \\
4q_{12} &= 5\frac{1}{2} \\
q_{12} &= \frac{5}{8}
\end{aligned}$$

$$
\begin{aligned}
\pi_1 q_{13} &= \pi_3 q_{31} \\
\frac{4}{12} q_{13} &= \frac{1}{12}\frac{1}{3} \\
4 q_{13} &= \frac{1}{3} \\
q_{13} &= \frac{1}{12}
\end{aligned}
$$

$$
\begin{aligned}
\pi_2 q_{23} &= \pi_3 q_{32} \\
\frac{5}{12} q_{23} &= \frac{1}{12}\frac{1}{12} \\
5 q_{23} &= \frac{1}{12} \\
q_{23} &= \frac{1}{60}
\end{aligned}
$$

$$
\begin{aligned}
\pi_2 &= \pi_1 q_{12} + \pi_2 q_{22} + \pi_3 q_{32} + \pi_4 q_{42} \\
\frac{5}{12} &= \frac{4}{12}\frac{7}{24} + \frac{5}{12}\frac{1}{3} + \frac{1}{12}\frac{1}{12} + \frac{2}{12} q_{42} \\
5 &= 4\frac{7}{24} + 5\frac{1}{3} + 1\frac{1}{12} + 2 q_{42} \\
5 &= \frac{35}{12} + 2 q_{42} \\
\frac{25}{12} &= 2 q_{42} \\
q_{42} &= \frac{25}{24}
\end{aligned}
$$

$$
\begin{aligned}
\pi_2 q_{24} &= \pi_4 q_{42} \\
\frac{5}{12} q_{24} &= \frac{2}{12}\frac{25}{24} \\
5 q_{24} &= 2\frac{25}{24} \\
q_{24} &= \frac{5}{12}
\end{aligned}
$$

THIS MUST BE WRONG FOR SURE: q_{43} cannot be negative, I believe

$$
\begin{aligned}
\pi_3 &= \pi_1 q_{13} + \pi_2 q_{23} + \pi_4 q_{43} \\
\frac{1}{12} &= \frac{4}{12}\frac{1}{12} + \frac{5}{12}\frac{1}{12} + \frac{2}{12} q_{43} \\
1 &= 4\frac{1}{12} + 5\frac{1}{3} + 2q_{43} \\
1 &= 2 + 2q_{43} \\
-1 &= 2q_{43} \\
q_{43} &= -\frac{1}{2}
\end{aligned}
$$

THIS MUST BE WRONG FOR SURE: $|q_{34}|$ cannot be greater than 1, I believe

$$
\begin{aligned}
\pi_3 q_{34} &= \pi_4 q_{43} \\
\frac{1}{12} q_{34} &= \frac{5}{12} \cdot -\frac{1}{2} \\
1 q_{34} &= 5 \cdot -\frac{1}{2} \\
q_{34} &= -\frac{5}{2}
\end{aligned}
$$

$$
\begin{aligned}
\pi_4 &= \pi_2 q_{24} + \pi_3 q_{34} + \pi_4 q_{44} \\
\frac{2}{12} &= \frac{5}{12}\frac{5}{12} + \frac{1}{12} \cdot -\frac{5}{2} + \frac{2}{12} q_{44} \\
2 &= 5\frac{5}{12} + 1 \cdot -\frac{5}{2} + 2q_{44} \\
2 &= -\frac{5}{12} + 2q_{44} \\
\frac{19}{12} &= 2q_{44} \\
q_{44} &= \frac{19}{24}
\end{aligned}
$$

Appendix

Code for Exercise 3

```
import functools
import math


def length(v):
    return math.sqrt(functools.reduce(lambda a, b: a + b, map(lambda v_i:
  v_i ** 2, v)))


```

```
def truncate(number, decimals=0):
    """
    Returns a value truncated to a specific number of decimal places.
    """
    if not isinstance(decimals, int):
        raise TypeError("decimal places must be an integer.")
    elif decimals < 0:
        raise ValueError("decimal places has to be 0 or more.")
    elif decimals == 0:
        return math.trunc(number)

    factor = 10.0 ** decimals
    return math.trunc(number * factor) / factor


def matrix_times_vector(A, v):
    Av = list()
    for i in range(len(A)):
        assert(len(A[i]) == len(v))
        Av.append(functools.reduce(
            lambda a, b: a + b, map(lambda item: A[i][item[0]] * item[1],
    enumerate(v))))
    return Av


def power_iteration(A, x):
    x_length = length(x)
    v = tuple(map(lambda x_i: x_i / x_length, x))
    old_trunc_v = tuple(map(lambda v_i: truncate(v_i, 3), v))
    while True:

        Av = matrix_times_vector(A, v)
        Av_length = length(Av)
        v = tuple(map(lambda v_i: v_i / Av_length, Av))

        # check if does not converge no more
        new_trunc_v = tuple(map(lambda v_i: truncate(v_i, 3), v))
        if(old_trunc_v == new_trunc_v):
            break
        old_trunc_v = new_trunc_v
    return v


if __name__ == '__main__':
    C = [
        [18, 18],
        [18, 70]
    ]
    v = (1,1)
    v_power = power_iteration(C,v)
    print(tuple(map(lambda v_i: round(v_i, 3), v_power)))
```

Code for Exercise 4

```
import numpy as np
from numpy import linalg as LA

L = np.array([
    [3, 0, -1, -1, -1, 0, 0, 0],
    [0, 2, -1, -1, 0, 0, 0, 0],
    [-1, -1, 3, -1, 0, 0, 0, 0],
```

```
    [-1, -1, -1, 4, 0, 0, 0, -1],
    [-1, 0, 0, 0, 4, -1, -1, -1],
    [0, 0, 0, 0, -1, 3, -1, -1],
    [0, 0, 0, 0, -1, -1, 2, 0],
    [0, 0, 0, -1, -1, -1, 0, 3]
])

eigenvalues, eigenvectors = LA.eig(L)

eigen_pairs = [(eigenvalues[i], eigenvectors[i]) for i in range(len(
   eigenvalues))]
eigen_pairs.sort(key=lambda tuple: tuple[0])

eigen_pairs = list(map(lambda tuple: (round(tuple[0], 3), np.round(tuple[1],
    3)), eigen_pairs))

print(eigen_pairs[0])
print(eigen_pairs[1])
```