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Exercise 1

(a) Using Theorem 3.6:

$$
\begin{aligned}
&\Pr_{T \;\mathcal{D}^m}\left(\forall h \in \mathcal{H} : |err_T(h) - err_D(h)| \leq \epsilon\right) > 1 - \delta \\
&\Pr_{T \;\mathcal{D}^m}\left(\forall h \in \mathcal{H} : |err_T(h) - err_D(h)| \leq \epsilon\right) > 0.9 \\
&\Rightarrow \delta = 0.1
\end{aligned}
$$

$$
\begin{aligned}
m &\geq \frac{1}{2\epsilon^2} \log\left(\frac{2|\mathcal{H}|}{\delta}\right) \\
143 &\geq \frac{1}{2\epsilon^2} \log\left(\frac{2 \cdot 2^3}{0.1}\right) \\
143 &\geq \frac{1}{2\epsilon^2} (\log(2^4) - \log(0.1)) \\
143 &\geq \frac{1}{2\epsilon^2} (4 - \log(0.1)) \\
\epsilon^2 &\geq \frac{(4 - \log(0.1))}{143\dot{2}} \\
|\epsilon| &\geq \sqrt{\frac{(4 - \log(0.1))}{286}} \\
\Rightarrow \epsilon &\geq \sqrt{\frac{(4 - \log(0.1))}{286}} \\
\epsilon &\geq \sqrt{\frac{(4 - \log(0.1))}{286}}
\end{aligned}
$$

$$
\begin{aligned}
&\Pr_{T \;\mathcal{D}^m}\left(\forall h \in \mathcal{H} : |err_T(h) - err_D(h)| \leq \epsilon\right) > 0.9 \\
&\Pr_{T \;\mathcal{D}^m}\left(\forall h \in \mathcal{H} : |0.03 - err_D(h)| \leq \sqrt{\frac{(4 - \log(0.1))}{286}}\right) > 0.9 \\
&\Rightarrow err_D(h) \leq 0.03 + \sqrt{\frac{(4 - \log(0.1))}{286}} \simeq 0.05560114718 \simeq 0.06
\end{aligned}
$$

(b) Using Theorem 3.4:

$$
\begin{aligned}
&\Pr_{T \;\mathcal{D}^m}\left(\forall h \in \mathcal{H} : \text{if } h \text{ is consistent with } T, \text{ then } err_D(h) \leq \epsilon\right) 1 - \delta \\
&\Pr_{T \;\mathcal{D}^m}\left(\forall h \in \mathcal{H} : \text{if } h \text{ is consistent with } T, \text{ then } err_D(h) \leq 0.01\right) 0.9 \\
&\Rightarrow \epsilon = 0.01, \delta = 0.1
\end{aligned}
$$

$$
\begin{aligned}
&m \geq \frac{1}{\epsilon} \ln\left(\frac{|\mathcal{H}|}{\delta}\right) \\
&m \geq \frac{1}{0.01} \ln\left(\frac{2^3}{0.1}\right) \\
&m \geq 100(\ln(3) - \ln(0.1)) \sim 100 \cdot 3.40119738166 = 340.1197 \\
&\Rightarrow m \geq 341
\end{aligned}
$$

Exercise 2

Using slide 3.26.

(a) The dimension of the instance space is $l = a \cdot v$. Such a decision scheme can be described using $|h|_\Delta \in O(n(\log n + \log l)) = O(n(\log n + \log(a \cdot v))) = O(n(\log n + \log a + \log v))$ bits.

(b) $n = 10, a = 5, v = 5$??????????????????

Exercise 3

(a) The VC-Dimension is 3. To prove this, let us prove some properties which every Y, that shatters $\mathcal{H}$, must hold.

Notation:

- $a_i, b_i \in \mathbb{R}, i \in \mathbb{N}$
- $y_{i,j} \in Y, i, j \in \mathbb{N} \to y_{i,j} = (a_i, b_j)$ (analog for $y'_{i,j} \in Y'$)

Property 1:

$$\neg \exists y_{i,j} \exists y_{i',j'} \exists y_{i'',j''} (a_i \neq a_{i'} \land a_i \neq a_{i''} \land a_{i'} \neq a_{i''} \land b_i \neq b_{i'} \land b_i \neq b_{i''} \land b_{i'} \neq b_{i''})$$

In words: There cannot be 3 vectors in Y that have neither their a nor b values in common.
Let's prove this be counterexample.
Let $Y' = \{(a_1, b_1), (a_2, b_2), (a_3, b_3)\} \subseteq Y$. There exists no $h_{a,b} \in \mathcal{H}$, such that $Y' \subseteq S_{h_{a,b}}$. Therefore, $Y' = S_{h_{a,b}} \cap Y$ does not hold.

Property 2:

$$\neg \exists y_{i,j} \exists y_{i',j'} \exists y_{i'',j''} (y_{i,j} \neq y_{i',j'} \land y_{i,j} \neq y_{i'',j''} \land y_{i',j'} \neq y_{i'',j''}) \land ((a_i = a_{i'} \land a_i = a_{i''}) \lor (b_i = b_{i'} \land b_i = b_{i''}))$$

In words: There cannot be 3 vectors in Y that have their a or b values in common.
Let's prove this be counterexample.
Let $Y' = \{(a_1, b_1), (a_1, b_2)\} \subseteq \{(a_1, b_1), (a_1, b_2), (a_1, b_3)\} \subseteq Y$ (analog when the b-components are equal). Then there exists no $h_{a,b} \in \mathcal{H}$ so that $Y' \subseteq S_{h_{a,b}}$ but $(a_1, b_3) \notin S_{h_{a,b}}$. This is true, since the only functions $h_{a,b}$ with $Y' \subseteq S_{h_{a,b}}$ are where $a = a_1$, thus also $(a_1, b_3) \notin S_{h_{a,b}}$, which is a contradiction.

Property 3:

$$\neg \exists y_{i,j} \exists y_{i',j'} \exists y_{i'',j''} (y_{i,j} \neq y_{i',j'} \land y_{i,j} \neq y_{i'',j''} \land y_{i',j'} \neq y_{i'',j''}) \land (a_i = a_{i'} \land b_{i'} = b_{i''})$$

In words: There cannot be a vector containing of an a-component that also exists in another vector and a b-component that also exists in another vector.
Let's prove this be counterexample.
Let $Y' = \{(a_1, b_2)\} \subseteq \{(a_1, b_1), (a_1, b_2), (a_2, b_2)\} \subseteq Y$. For all $h_{a,b} \in \mathcal{H}$ with $Y' \subseteq S_{h_{a,b}}$ $a = a_1 \lor b = b_2$ must hold. However, for such $h_{a,b}$ $(a_1, b_1) \in S_{h_{a,b}}$ or $(a_2, b_2) \in S_{h_{a,b}}$ would also hold. This is a contradiction.

Thus, the only valid form of Y must be (or analog when two b-components are equal):

$$Y := \{(a_1, b_1), (a_1, b_2), (a_2, b_2)\}, a_1 \neq a_2 \land b_1 \neq b_2 \land b_1 \neq b_3 \land b_2 \neq b_3$$

We can prove that Y shatters $\mathcal{H}$:

- $Y' = \emptyset$:
  We can use h_{a_0,b_0} with $a_0 \notin \{a_1, a_2\}, b_0 \notin \{b_1, b_2, b_3\}$. Thus, $Y \notin S_{h_{a_0,b_0}}$. So, $Y' = \emptyset = S_{h_{a_0,b_0}} \cap Y$.
- $Y' = \{(a_1, b_i)\}$, $i \in \{1, 2\}$:
  We can use h_{a_0,b_i} with $a_0 \notin \{a_1, a_2\}$. Thus, $Y' = \emptyset = S_{h_{a_0,b_i}} \cap Y$.
- $Y' = \{(a_2, b_3)\}$:
  We can use h_{a_2,b_3}. Thus, $Y' = \emptyset = S_{h_{a_0,b_i}} \cap Y$.
- $Y' = \{(a_1, b_1), (a_1, b_2)\}$:
  We can use h_{a_1,b_1}. Thus, $Y' = \emptyset = S_{h_{a_0,b_i}} \cap Y$.
- $Y' = \{(a_1, b_i), (a_2, b_3)\}$, $i \in \{1, 2\}$:
  We can use h_{a_2,b_i}. Thus, $Y' = \emptyset = S_{h_{a_0,b_i}} \cap Y$.
- $Y' = Y$:
  We can use h_{a_1,b_3}. Thus, $Y' = \emptyset = S_{h_{a_0,b_i}} \cap Y$.

So, the VC-Dimension is at least 3. Let's now prove, that no Y with $|Y| > 3$ shatters $\mathcal{H}$.

For now, let Y be $Y := \{(a_1, b_1), (a_1, b_2), (a_2, b_2)\}$, $a_1 \neq a_2 \wedge b_1 \neq b_2 \wedge b_1 \neq b_3 \wedge b_2 \neq b_3$ (which we proved was the only valid form of Y with $|Y| = 3$). We can now try to add any $y_{i,j}$ to Y:

- Let's try to add $y_{1,j} = (a_1, b_j)$, $j \in \mathbb{N}$. This would violate **Property 1**. Thus, we cannot add any such $y_{1,j}$.
- Let's try to add $y_{2,j} = (a_2, b_j)$, $j \in \mathbb{N}$. Since $(a_2, b_3) \in Y$, $b_j \in \mathbb{R} \setminus \{b_3\}$. Then the condition in the remark would not hold for $Y' = Y$. Thus, we cannot add any such $y_{2,j}$.
- Let's try to add $y_{3,4} = (a_3, b_4)$, $a_3 \in \mathbb{R} \setminus \{a_1, a_2\}$, $b_4 \notin \{b_1, b_2, b_3\}$. This would violate **Property 1**. Thus, we cannot add any such $y_{3,4}$.
- Let's try to add $y_{3,j} = (a_3, b_j)$, $j \in \{1, 2\}$, $a_3 \in \mathbb{R} \setminus \{a_1, a_2\}$. Then, we cannot find any $h_{a,b} \in \mathcal{H}$ for $Y' = Y$, so that the condition in the remark holds. Thus, we cannot add any such $y_{3,j}$
- Let's try to add $y_{3,j} = (a_3, b_3)$, $a_3 \in \mathbb{R}\setminus\{a_1, a_2\}$. Then, for $Y' = \{(a_1, b_2), (a_2, b_3), (a_3, b_3)\}$ we cannot find any $h_{a,b} \in \mathcal{H}$, for which $Y' \subseteq S_{h_{a,b}}$ but $(a_1, b_1) \notin S_{h_{a,b}}$. Thus, we cannot add any such $y_{3,j}$.

So, we cannot add any element to Y. Thus, all Y that shatter $\mathcal{H}$ must be at most $|Y| \leq 3$. Therefore, the VC Dimension is 3.

(b) The VC-Dimension is ∞.
Let be $Y \subseteq \mathfrak{X} = \Sigma^*$. For any $Y' \subseteq Y$ we can choose $L := Y'$. Thus, $Y' = S_{h_L}$, so also $S_{h_L} \subseteq Y$. So $Y \cap S_{h_L} = S_{h_L} = Y'$. So Y can also be infinite in size.

Exercise 4

(a)

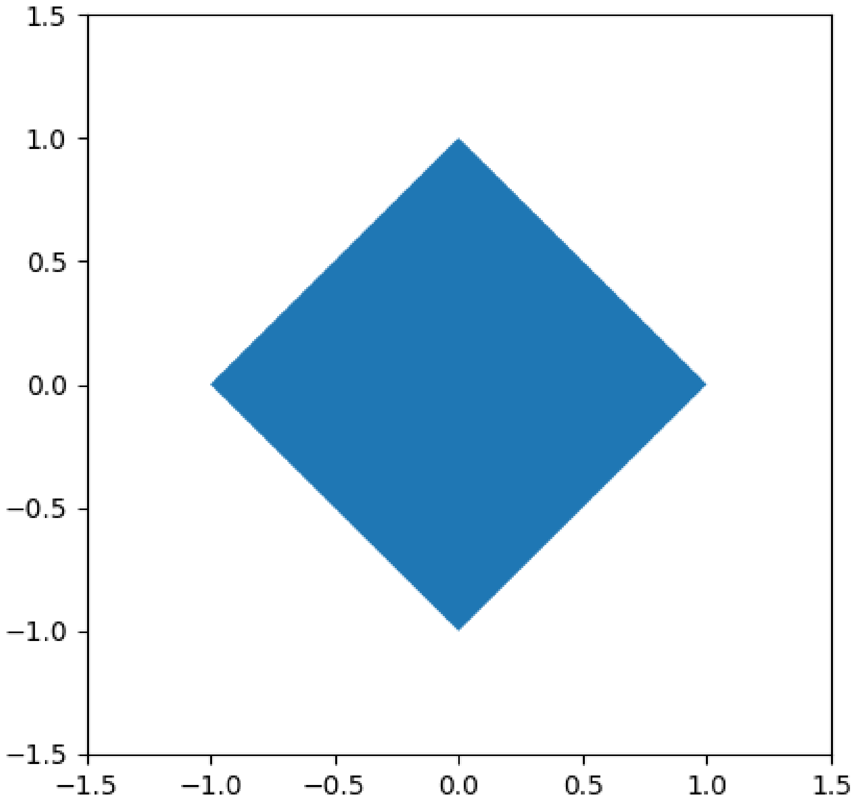
(b)

Exercise 5

(a)
```
Iteration s=1 of 3:

Weights:
  w_1^1 = 1.0
  w_2^1 = 1.0
  w_3^1 = 1.0

Probabilities:
  p_1^1 = 0.33
  p_2^1 = 0.33
  p_3^1 = 0.33

Iteration s=2 of 3:

Weights:
  w_1^2 = 2.83
  w_2^2 = 1.0
  w_3^2 = 1.0

Probabilities:
  p_1^2 = 0.46
  p_2^2 = 0.27
  p_3^2 = 0.27

Iteration s=3 of 3:

Weights:
  w_1^3 = 2.83
  w_2^3 = 8.48
  w_3^3 = 1.0

Probabilities:
  p_1^3 = 0.28
  p_2^3 = 0.51
  p_3^3 = 0.21


Final Weights:
  w_1^4 = 2.83
  w_2^4 = 8.48
  w_3^4 = 5.32
```

(b) This is due to the fact, that $p_3^{(3)} \neq p_1^{(1)}$.

Exercise 6



(a) (i)

(ii) Similarly to how in $l = 2$ the "corners" of the unit circle are the 2 unit vectors and their negations (so 4 in total), the "corners" of the unit circle in $l = 3$ are the 3 unit vectors and their negations (so 6 in total). Combined with the edges and facing connecting them they make for a "diamond" shape.

(b) $vol(B_1^2) = (\sqrt{1^2 + 1^2})^2 = 2$ and $vol(B_1^3) = 2 \cdot \frac{(\sqrt{1^2+1^2})^2 \cdot 1}{3} = \frac{4}{3}$

(c) Cover B_1^l by $2k$ cylinders. The thickness of the cylinders is $t := \frac{1}{k}$. Thus, the radius of the ith cylinder above (or below) is $r_i := 1 - (i-1) \cdot t$. Therefore, the volume of the ith cylinder is $t \cdot r_i^{l-1} \cdot vol(B_1^{l-1})$. Thus:

$$\begin{aligned}
vol(B_1^l) &\leq 2\sum_{i=1}^{k} t \cdot r_i^{l-1} \cdot vol(B_1^{l-1}) \\
&= \left(2\sum_{i=1}^{k} \frac{1}{k}\left(1 - \frac{i-1}{k}\right)^{l-1}\right) \cdot vol(B_1^{l-1}) \\
&= \left(2\sum_{i=0}^{k-1} \frac{1}{k}\left(1 - \frac{i}{k}\right)^{l-1}\right) \cdot vol(B_1^{l-1}) \\
&= \left(2\frac{1}{k}\left(1^{l-1} + (1-\frac{1}{k})^{l-1} + \cdots + (1 - \frac{k-1}{k})^{l-1}\right)\right) \cdot vol(B_1^{l-1}) \\
&= \left(2\left(\frac{k^{l-1}}{k^l} + (\frac{(k-1)^{l-1}}{k^l}) + \cdots + (\frac{1^{l-1}}{k^l}) + (\frac{0^{l-1}}{k^l})\right)\right) \cdot vol(B_1^{l-1}) \\
&= \left(2\underbrace{\sum_{i=0}^{k}\left(\frac{i^{l-1}}{k^l}\right)}_{:=S}\right) \cdot vol(B_1^{l-1})
\end{aligned}$$

We can use the ratio test on the series S:

$$\lim_{k\to\infty} \left|\frac{\left(\frac{(i+1)^{l-1}}{k^l}\right)}{\left(\frac{i^{l-1}}{k^l}\right)}\right| = \left|\frac{i+1}{i}^{l-1}\right| = \left|(1+\frac{1}{i})^{l-1}\right| > 1$$

Huh? Something is wrong here

Appendix