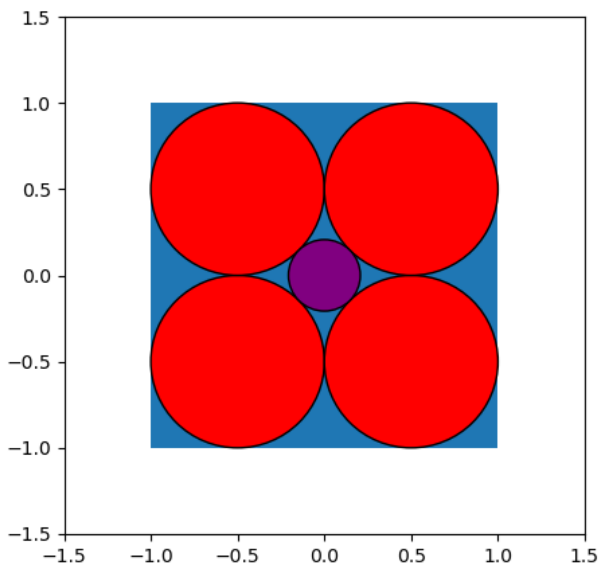
Exercise 1

(a)

Situation for l=2 and s=2:

$$Q_{2,2} = \{[x_1 x_2]^T \in \mathbb{R}^2 \mid |x_i| \leq 1 \text{ for all } i = 1, 2\} = [-1, 1]^2$$



(b)

First, let's calculate the radius of the inner hyperball for any $l \in \mathbb{N}, s \in \mathbb{R}_{>0}$:
The distance from the center of the inner hyperball (equal to the center of the hypercube) to the center of one of the 2^l outter hyperballs (doesn't matter which one) can be calculated the following:

$$d := \sqrt{l \cdot \left(\frac{s}{4}\right)^2}$$

Thus, the radius of the inner hyperball is equal to:

$$r := d - \frac{s}{4} = \sqrt{l \cdot \left(\frac{s}{4}\right)^2} - \frac{s}{4}$$

Now, we simply must solve the following inequality to find an $l \in \mathbb{N}$ for an arbitrary but fixed $s \in \mathbb{R}_{>0}$ such that $B(Q_{l,s}) \not\subseteq Q_{l,s}$:

$$\begin{aligned}
\frac{s}{2} &< r \\
\frac{s}{2} &< d - \frac{s}{4} \\
\frac{s}{2} &< \sqrt{l \cdot \left(\frac{s}{4}\right)^2} - \frac{s}{4} \\
\frac{3 \cdot s}{4} &< \sqrt{l} \cdot \frac{s}{4} \\
3 &< \sqrt{l} \\
l &> 9
\end{aligned}$$

Exercise 2

(a)

Let's first find all eigenvalues of A_c:

$$\{\lambda \in \mathbb{R} \mid \det(A_c - \lambda \cdot I_3) = 0\}$$
$$\{\lambda \in \mathbb{R} \mid \det \begin{bmatrix} 2-\lambda & 0 & c \\ 0 & 1-\lambda & 0 \\ c & 0 & 1-\lambda \end{bmatrix} = 0\}$$
$$\{\lambda \in \mathbb{R} \mid -\lambda^3 + 4 \cdot \lambda^2 + c^2 \cdot \lambda - 5 \cdot \lambda + 2 - c^2\}$$
$$\{\lambda \in \mathbb{R} \mid (\lambda - 1) \cdot (-\lambda^2 + 3\dot{\lambda} + c^2 - 2)\}$$
$$\{1, \frac{3 - \sqrt{4 \cdot c^2 + 1}}{2}, \frac{3 + \sqrt{4 \cdot c^2 + 1}}{2}\} =: \Lambda_{A_c}$$

So, if for all $\lambda \in \Lambda_{A_c}$ it must hold that $\lambda \geq 0$, we must set c such that the following inequality holds:

$$\begin{aligned}
\frac{3 - \sqrt{4 \cdot c^2 + 1}}{2} &\geq 0 \\
3 - \sqrt{4 \cdot c^2 + 1} &\geq 0 \\
3 &\geq \sqrt{4 \cdot c^2 + 1} \\
9 &\geq 4 \cdot c^2 + 1 \\
8 &\geq 4 \cdot c^2 \\
2 &\geq c^2 \\
c &\in (-\sqrt{2}, \sqrt{2}) \subseteq \mathbb{R}
\end{aligned}$$

(b)

Using Theorem 5.19, we create an orthogonal matrix $U \in \mathbb{R}^{n \times n}$ with the columns being the eigenvectors of A. We can now use Theorem 5.20 to create the diagonal matrix $\Lambda \in \mathbb{R}^{n \times n}$ where the eigenvalue-entry $\lambda_{i,i}$ corresponds to eigenvector in the column i of U.
A simple verification:

$$\begin{aligned}
A &= U \Lambda U^\top \\
A &= U \Lambda U^{-1} \\
AU &= U \Lambda \\
Av_i &= v_i \lambda_{i,i} & , \forall i \in [n], v_i := col_i(U) \\
Av_i &= \lambda_{i,i} v_i & , \forall i \in [n], v_i := col_i(U)
\end{aligned}$$

Since A is positive semi-definite, all entries $\lambda_{i,i} \geq 0$. Thus, we can create a diagonal matrix Λ' consisting of the entries $\lambda'_{i,i} \coloneqq \sqrt{\lambda_{i,i}}$. Thus, $\Lambda = \Lambda'\Lambda'^\top$. Let $B \coloneqq UU\Lambda'$. Now:

$$\begin{aligned}
A &= U\Lambda U^\top \\
A &= U(\Lambda'\Lambda'^\top)U^\top \\
A &= U(\Lambda'\Lambda'^\top)U^\top \\
A &= (U\Lambda')(\Lambda'^\top U^\top) \\
A &= (U\Lambda')(U\Lambda')^\top \\
A &= BB^\top
\end{aligned}$$

(c)

Since A is symmetric, the following holds $\forall x, y \in R^n$:

$$\langle Ax, y \rangle = \langle x, Ay \rangle$$

Let $v_1 \in E_1, v_2 \in E_2$. Then $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$.

$$\begin{aligned}
0 &= \langle Av_1, v_2 \rangle - \langle v_1, Av_2 \rangle \\
&= \langle \lambda_1 v_1, v_2 \rangle - \langle v_1, \lambda_2 v_2 \rangle \\
&= (\lambda_1 - \lambda_2)\langle v_1, v_2 \rangle
\end{aligned}$$

Since $\lambda_1 \neq \lambda_2$ it follows, that $\langle v_1, v_2 \rangle = 0$.

Exercise 3

(a)

(b)

(c)

(d)

Exercise 4

(a)

(b)

(c)

(d)

(e)

Exercise 5