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Exercise 03
Algorithmic Foundations of Data Science

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Exercise 1

(a) Using Theorem 3.6:

$$
\begin{aligned}
&\Pr_{T\ \mathcal{D}^m}\left(\forall h \in \mathcal{H} : |err_T(h) - err_D(h)| \leq \epsilon\right) > 1 - \delta \\
&\Pr_{T\ \mathcal{D}^m}\left(\forall h \in \mathcal{H} : |err_T(h) - err_D(h)| \leq \epsilon\right) > 0.9 \\
&\Rightarrow \delta = 0.1
\end{aligned}
$$

$$
\begin{aligned}
m &\geq \frac{1}{2\epsilon^2} \log\left(\frac{2|\mathcal{H}|}{\delta}\right) \\
143 &\geq \frac{1}{2\epsilon^2} \log\left(\frac{2 \cdot 2^3}{0.1}\right) \\
143 &\geq \frac{1}{2\epsilon^2} (\log(2^4) - \log(0.1)) \\
143 &\geq \frac{1}{2\epsilon^2} (4 - \log(0.1)) \\
\epsilon^2 &\geq \frac{(4 - \log(0.1))}{143\dot{2}} \\
|\epsilon| &\geq \sqrt{\frac{(4 - \log(0.1))}{286}} \\
\Rightarrow \epsilon &\geq \sqrt{\frac{(4 - \log(0.1))}{286}} \\
\epsilon &\geq \sqrt{\frac{(4 - \log(0.1))}{286}}
\end{aligned}
$$

$$
\begin{aligned}
&\Pr_{T\ \mathcal{D}^m}\left(\forall h \in \mathcal{H} : |err_T(h) - err_D(h)| \leq \epsilon\right) > 0.9 \\
&\Pr_{T\ \mathcal{D}^m}\left(\forall h \in \mathcal{H} : |0.03 - err_D(h)| \leq \sqrt{\frac{(4 - \log(0.1))}{286}}\right) > 0.9 \\
&\Rightarrow err_D(h) \leq 0.03 + \sqrt{\frac{(4 - \log(0.1))}{286}} \simeq 0.05560114718 \simeq 0.06
\end{aligned}
$$

(b) Using Theorem 3.4:

$$
\begin{aligned}
&\Pr_{T\ \mathcal{D}^m}\left(\forall h \in \mathcal{H} : \text{if } h \text{ is consistent with } T\text{, then } err_D(h) \leq \epsilon\right) 1 - \delta \\
&\Pr_{T\ \mathcal{D}^m}\left(\forall h \in \mathcal{H} : \text{if } h \text{ is consistent with } T\text{, then } err_D(h) \leq 0.01\right) 0.9 \\
&\Rightarrow \epsilon = 0.01, \delta = 0.1
\end{aligned}
$$

$$
\begin{aligned}
&m \geq \frac{1}{\epsilon} \ln\left(\frac{|\mathcal{H}|}{\delta}\right) \\
&m \geq \frac{1}{0.01} \ln\left(\frac{2^3}{0.1}\right) \\
&m \geq 100(\ln(3) - \ln(0.1)) \sim 100 \cdot 3.40119738166 = 340.1197 \\
&\Rightarrow m \geq 341
\end{aligned}
$$

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Exercise 2

(a)

(b)

Exercise 3

(a)

(b)

Exercise 4

(a)

(b)

Exercise 5

(a)

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Iteration s=1 of 3:

Weights:
  w_1^1 = 1.0
  w_2^1 = 1.0
  w_3^1 = 1.0

Probabilities:
  p_1^1 = 0.33
  p_2^1 = 0.33
  p_3^1 = 0.33

Iteration s=2 of 3:

Weights:
  w_1^2 = 2.83
  w_2^2 = 1.0
  w_3^2 = 1.0

Probabilities:
  p_1^2 = 0.46
  p_2^2 = 0.27
  p_3^2 = 0.27

Iteration s=3 of 3:

Weights:
  w_1^3 = 2.83
  w_2^3 = 8.48
  w_3^3 = 1.0

Probabilities:
  p_1^3 = 0.28
  p_2^3 = 0.51
  p_3^3 = 0.21

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Final Weights:
  w_1^4 = 2.83
  w_2^4 = 8.48
  w_3^4 = 5.32
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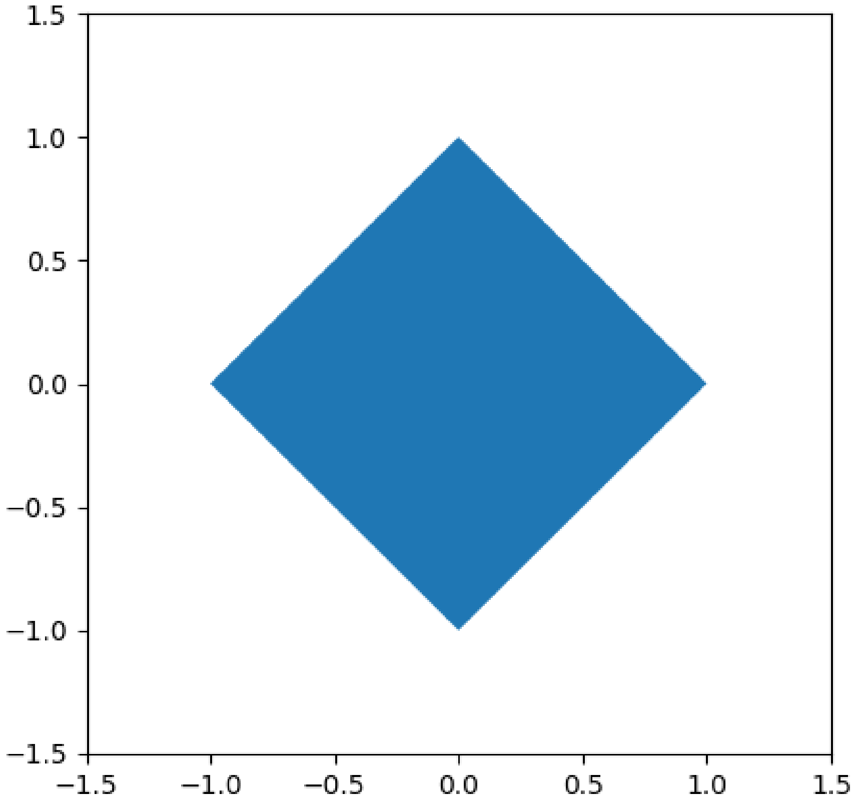
(b) This is due to the fact, that $p_3^{(3)} \neq p_1^{(1)}$.

Exercise 6



(a) (i)

(ii) Similarly to how in $l = 2$ the "corners" of the unit circle are the 2 unit vectors and their negations (so 4 in total), the "corners" of the unit circle in $l = 3$ are the 3 unit vectors and their negations (so 6 in total). Combined with the edges and facing connecting them they make for a "diamond" shape.

(b) $vol(B_1^2) = (\sqrt{1^2 + 1^2})^2 = 2$ and $vol(B_1^3) = 2 \cdot \frac{(\sqrt{1^2+1^2})^2 \cdot 1}{3} = \frac{4}{3}$

(c) Cover B_1^l by $2k$ cylinders. The thickness of the cylinders is $t := \frac{1}{k}$. Thus, the radius of the ith cylinder above (or below) is $r_i := 1 - (i-1) \cdot t$. Therefore, the volume of the ith cylinder is $t \cdot r_i^{l-1} \cdot vol(B_1^{l-1})$. Thus:

$$\begin{aligned}
vol(B_1^l) &\leq 2\sum_{i=1}^{k} t \cdot r_i^{l-1} \cdot vol(B_1^{l-1}) \\
&= \left(2\sum_{i=1}^{k} \frac{1}{k}\left(1-\frac{i-1}{k}\right)^{l-1}\right) \cdot vol(B_1^{l-1}) \\
&= \left(2\sum_{i=0}^{k-1} \frac{1}{k}\left(1-\frac{i}{k}\right)^{l-1}\right) \cdot vol(B_1^{l-1}) \\
&= \left(2\frac{1}{k}\left(1^{l-1} + (1-\frac{1}{k})^{l-1} + \dots + (1-\frac{k-1}{k})^{l-1}\right)\right) \cdot vol(B_1^{l-1}) \\
&= \left(2\left(\frac{k^{l-1}}{k^l} + (\frac{(k-1)^{l-1}}{k^l}) + \dots + (\frac{1^{l-1}}{k^l}) + (\frac{0^{l-1}}{k^l})\right)\right) \cdot vol(B_1^{l-1}) \\
&= \left(2\underbrace{\sum_{i=0}^{k}\left(\frac{i^{l-1}}{k^l}\right)}_{:=S}\right) \cdot vol(B_1^{l-1})
\end{aligned}$$

We can use the ratio test on the series S:

$$\lim_{k\to\infty} \left|\frac{\left(\frac{(i+1)^{l-1}}{k^l}\right)}{\left(\frac{i^{l-1}}{k^l}\right)}\right| = \left|\frac{i+1}{i}^{l-1}\right| = |(1+\frac{1}{i})^{l-1}| > 1$$

Huh? Something is wrong here

Appendix