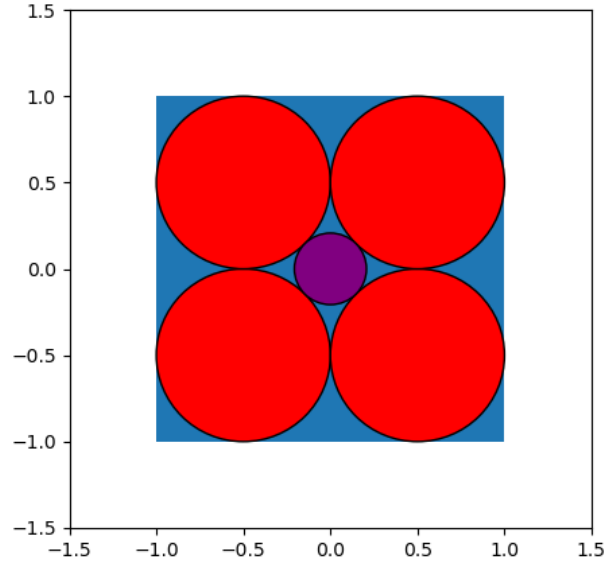


Exercise 1

(a)

Situation for $l=2$ and $s=2$:

$$Q_{2,2} = \{[x_1 x_2]^T \in \mathbb{R}^2 \mid |x_i| \leq 1 \text{ for all } i = 1, 2\} = [-1, 1]^2$$



(b)

First, let's calculate the radius of the inner hyperball for any $l \in \mathbb{N}$, $s \in \mathbb{R}_{>0}$:

The distance from the center of the inner hyperball (equal to the center of the hypercube) to the center of one of the 2^l outer hyperballs (doesn't matter which one) can be calculated the following:

$$d := \sqrt{l \cdot \left(\frac{s}{4}\right)^2}$$

Thus, the radius of the inner hyperball is equal to:

$$r := d - \frac{s}{4} = \sqrt{l \cdot \left(\frac{s}{4}\right)^2} - \frac{s}{4}$$

Now, we simply must solve the following inequality to find an $l \in \mathbb{N}$ for an arbitrary but fixed $s \in \mathbb{R}_{>0}$ such that $B(Q_{l,s}) \not\subseteq Q_{l,s}$:

$$\begin{aligned}
 \frac{s}{2} &< r \\
 \frac{s}{2} &< d - \frac{s}{4} \\
 \frac{s}{2} &< \sqrt{l \cdot \left(\frac{s}{4}\right)^2} - \frac{s}{4} \\
 \frac{3 \cdot s}{4} &< \sqrt{l} \cdot \frac{s}{4} \\
 3 &< \sqrt{l} \\
 l &> 9
 \end{aligned}$$

Exercise 2

(a)

Let's first find all eigenvalues of A_c :

$$\begin{aligned}
& \{\lambda \in \mathbb{R} \mid \det(A_c - \lambda \cdot I_3) = 0\} \\
& \{\lambda \in \mathbb{R} \mid \det \begin{bmatrix} 2 - \lambda & 0 & c \\ 0 & 1 - \lambda & 0 \\ c & 0 & 1 - \lambda \end{bmatrix} = 0\} \\
& \{\lambda \in \mathbb{R} \mid -\lambda^3 + 4 \cdot \lambda^2 + c^2 \cdot \lambda - 5 \cdot \lambda + 2 - c^2\} \\
& \{\lambda \in \mathbb{R} \mid (\lambda - 1) \cdot (-\lambda^2 + 3\lambda + c^2 - 2)\} \\
& \left\{1, \frac{3 - \sqrt{4 \cdot c^2 + 1}}{2}, \frac{3 + \sqrt{4 \cdot c^2 + 1}}{2}\right\} =: \Lambda_{A_c}
\end{aligned}$$

So, if for all $\lambda \in \Lambda_{A_c}$ it must hold that $\lambda \geq 0$, we must set c such that the following inequality holds:

$$\begin{aligned}
\frac{3 - \sqrt{4 \cdot c^2 + 1}}{2} &\geq 0 \\
3 - \sqrt{4 \cdot c^2 + 1} &\geq 0 \\
3 &\geq \sqrt{4 \cdot c^2 + 1} \\
9 &\geq 4 \cdot c^2 + 1 \\
8 &\geq 4 \cdot c^2 \\
2 &\geq c^2 \\
c &\in (-\sqrt{2}, \sqrt{2}) \subseteq \mathbb{R}
\end{aligned}$$

(b)

Using Theorem 5.19, we create an orthogonal matrix $U \in \mathbb{R}^{3 \times 3}$ with the columns being the eigenvectors of A . We can now use Theorem 5.20 to create the diagonal matrix $\Lambda \in \mathbb{R}^{3 \times 3}$ where the eigenvalue-entry $\lambda_{i,i}$ corresponds to eigenvector in the column i of U .

A simple verification:

$$\begin{aligned}
A &= U \Lambda U^\top \\
A &= U \Lambda U^{-1} \\
AU &= U \Lambda \\
Av_i &= v_i \lambda_{i,i} \quad , \forall i \in [3], v_i := \text{col}_i(U) \\
Av_i &= \lambda_{i,i} v_i \quad , \forall i \in [3], v_i := \text{col}_i(U)
\end{aligned}$$

Since A is positive semi-definite, all entries $\lambda_{i,i} \geq 0$. Thus, we can create a diagonal matrix Λ' consisting of the entries $\lambda'_{i,i} := \sqrt{\lambda_{i,i}}$. Thus, $\Lambda = \Lambda' \Lambda'^\top$. Let $B := U \Lambda'$. Now:

$$A = U \Lambda U^\top$$

$$A = U (\Lambda' \Lambda'^\top) U^\top$$

$$A = U (\Lambda' \Lambda'^\top) U^\top$$

$$A = (U \Lambda') (\Lambda'^\top U^\top)$$

$$A = (U \Lambda') (U \Lambda')^\top$$

$$A = B B^\top$$

(c)

Exercise 3

(a)

(b)

(c)

(d)

Exercise 4

(a)

(b)

(c)

(d)

(e)

Exercise 5