Question 2

From the task description we get: $P(a \mid r) = 0.3$
$P(o \mid r) = 0.4$
$P(l \mid r) = 0.3$
$P(a \mid g) = 0.3$
$P(o \mid g) = 0.3$
$P(l \mid g) = 0.4$
$P(a \mid b) = 0.5$
$P(o \mid b) = 0.5$
$P(l \mid b) = 0$

We can now determine the following probabilities:
$P(a \cap r) = P(a \mid r) \cdot P(r) = 0.3 \cdot 0.2 = 0.06$
$P(o \cap r) = P(o \mid r) \cdot P(r) = 0.4 \cdot 0.2 = 0.08$
$P(l \cap r) = P(l \mid r) \cdot P(r) = 0.3 \cdot 0.2 = 0.06$
$P(a \cap g) = P(a \mid g) \cdot P(g) = 0.3 \cdot 0.6 = 0.18$
$P(o \cap g) = P(o \mid g) \cdot P(g) = 0.3 \cdot 0.6 = 0.18$
$P(l \cap g) = P(l \mid g) \cdot P(g) = 0.4 \cdot 0.6 = 0.24$
$P(a \cap b) = P(a \mid b) \cdot P(b) = 0.5 \cdot 0.2 = 0.1$
$P(o \cap b) = P(o \mid b) \cdot P(b) = 0.5 \cdot 0.2 = 0.1$
$P(l \cap b) = P(l \mid b) \cdot P(b) = 0 \cdot 0.2 = 0$

(a)

$P(a) = P(a \cap r) + P(a \cap g) + P(a \cap b) = 0.06 + 0.18 + 0.1 = 0.34$
$P(o) = P(o \cap r) + P(o \cap g) + P(o \cap b) = 0.08 + 0.18 + 0.1 = 0.36$
$P(l) = P(l \cap r) + P(l \cap g) + P(l \cap b) = 0.06 + 0.24 + 0 = 0.3$

Thus, the probability of selecting an apple is 0.34.

(b)

$P(r \mid a) = \frac{P(a \cap r)}{P(a)} = \frac{0.06}{0.34} = 0.176$
$P(g \mid a) = \frac{P(a \cap g)}{P(a)} = \frac{0.18}{0.34} = 0.529$
$P(b \mid a) = \frac{P(a \cap b)}{P(a)} = \frac{0.1}{0.34} = 0.294$
$P(r \mid o) = \frac{P(o \cap r)}{P(o)} = \frac{0.08}{0.36} = 0.222$
$P(g \mid o) = \frac{P(o \cap g)}{P(o)} = \frac{0.18}{0.36} = 0.5$
$P(b \mid o) = \frac{P(o \cap b)}{P(o)} = \frac{0.1}{0.36} = 0.278$
$P(r \mid l) = \frac{P(l \cap r)}{P(l)} = \frac{0.06}{0.3} = 0.2$
$P(g \mid l) = \frac{P(l \cap g)}{P(l)} = \frac{0.24}{0.3} = 0.8$
$P(b \mid l) = \frac{P(l \cap b)}{P(l)} = \frac{0}{0.3} = 0$

Thus, if we observe that the selected fruit is in fact an orange, the probability that it came from the green box is 0.5.

Question 3

Error: classify(x) is not a function, iff $p(C_i \mid x) = p(C_j \mid x)$ for some $i \neq j$. Thus, we assume that $p(C_i \mid x) \neq p(C_j \mid x)$ for all $i \neq j$.

(a)

For some x, let $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, N\} \setminus \{i\}$, such that C_i is the correct classification of x. Let $j \in \{1, \ldots, N\}$:

$$\begin{aligned}
R(C_j \mid x) &= \sum_{k \in \{1,\ldots,N\} \setminus \{j\}} L_{k,j} p(C_k \mid x) + L_{j,j} p(C_j \mid x) + L_{rej,j} p(C_{rej} \mid x) \\
&= \sum_{k \in \{1,\ldots,N\} \setminus \{j\}} l_s p(C_k \mid x) + 0 \cdot p(C_j \mid x) + l_r p(C_{rej} \mid x) \\
&= \sum_{k \in \{1,\ldots,N\} \setminus \{j\}} l_s p(C_k \mid x) + l_r p(C_{rej} \mid x)
\end{aligned}$$

$$\begin{aligned}
R(C_{rej} \mid x) &= \sum_{k \in \{1,\ldots,N\}} L_{k,j} p(C_k \mid x) + L_{rej,j} p(C_{rej} \mid x) \\
&= \sum_{k \in \{1,\ldots,N\}} l_s p(C_k \mid x) + l_r p(C_{rej} \mid x)
\end{aligned}$$

TODO: Finish this question. No fucking idea bro

(b)

If $l_r = 0$, then $1 - \frac{l_r}{l_s} = 1$. Thus, we then only classify x to category C_j iff $p(C_j \mid x) = 1$ (so we are certain). If that is not the case (uncertain), then we reject. As $l_r = 0$, there is no punishment for rejecting. Therefore, the expected loss for such a function is always 0.

(c)

If $l_r > l_s$, then we never reject using classify(x). Similarly, during risk minimization, we never use a strategy that rejects, as making a substitution error yields a lesser loss.

Question 4

Note that all measurements are greater than 0, thus $g(x)$ is always 1.

$$\begin{aligned}
\frac{\partial p(x \mid \theta)}{\partial \theta} &= 2x \cdot \theta \cdot \exp(-\theta x) + x \cdot \theta^2 \cdot (-x)\exp(-\theta x) \\
&= 2x \cdot \theta \cdot \exp(-\theta x) - x^2 \cdot \theta^2 \cdot \exp(-\theta x) \\
&= x\theta \exp(-\theta x) \cdot (2 - x\theta)
\end{aligned}$$

Let:

$$f(x, \theta) := \frac{\frac{\partial p(x|\theta)}{\partial \theta}}{p(x \mid \theta)} = \frac{x\theta \exp(-\theta x) \cdot (2 - x\theta)}{\theta^2 x \exp(-\theta x)}$$

Assuming $\theta \neq 0$, we can simplify this to:

$$f(x, \theta) = \frac{2 - x\theta}{\theta}$$

We can reorder this derivative of the log likelihood to:

$$\begin{aligned}
-\sum_{n=1}^{N} f(x_n, \theta) &= -\sum_{n=1}^{N} \frac{2 - x_n\theta}{\theta} \\
&= -\frac{2}{\theta} + \sum_{n=1}^{N} x_n
\end{aligned}$$

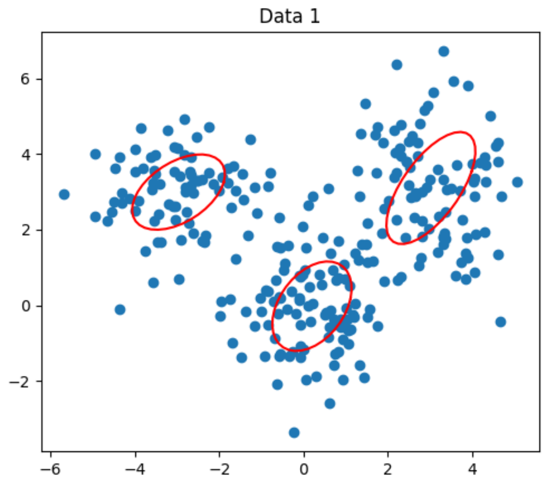
Setting this to 0 yields:

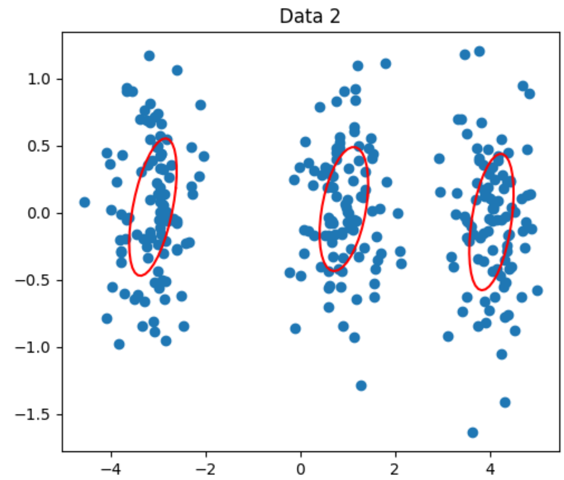
$$\begin{aligned}
0 &= -\frac{2}{\theta} + \sum_{n=1}^{N} x_n \\
\sum_{n=1}^{N} x_n &= \frac{2}{\theta} \\
\theta &= \frac{2}{\sum_{n=1}^{N} x_n}
\end{aligned}$$

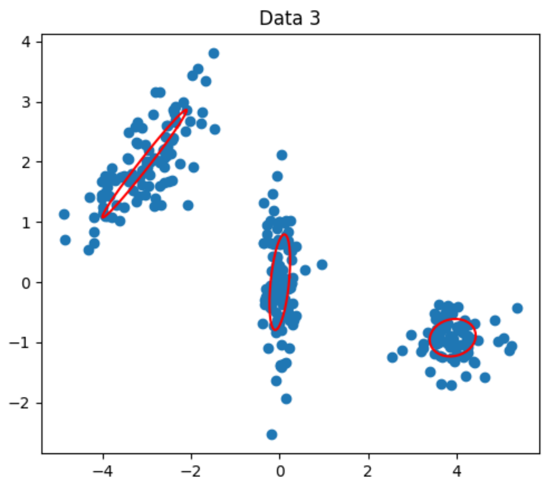
Thus, the maximum likelihood estimate $\widetilde{\theta}$ for the given measurements is: $\widetilde{\theta} = \frac{2}{\sum_{n=1}^{N} x_n}$

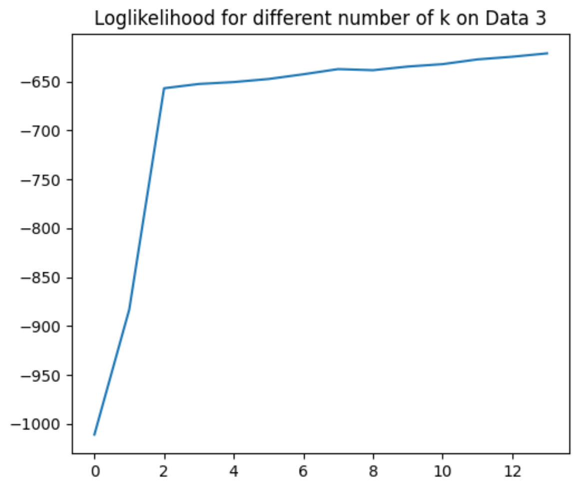
Question 6

(f)





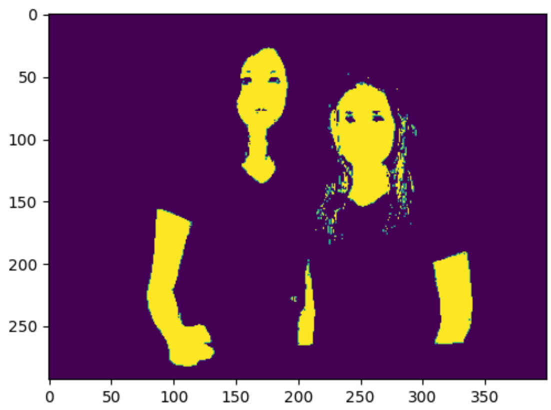




We can see, that in all three data sets there are 3 clusters (more so in data2 and data3 than in data1). Therefore, $K = 3$ clusters is a very suitable number of clusters for this application.

$K = 2$ yields very under fitting cluster parameters, even for data1. For $K > 3$ we get clusters that are "overlapping" each other, so we see overfitting.

(g)



Increasing K here improves accuracy on the training data set, thus reducing generalization. For $K = 9$, a model is produced that separates the colors in the test image better than the default of $K = 3$. Increasing K further might lead to poor accuracy on other test data. Since the colors are there dimensional, and we are looking at two classifications (skin and no-skin), intuitively speaking, $3 * 2$ clusters should be a good starting point for finding a suitable number of clusters.

Increasing the threshold *theta* increases the required confidence, that a given pixel color represents skin. This leads to a more conservative classification (so rather non-skin than skin), which is more robust to noise, however, can also be less generalized. We increased *theta* to 3 and got the above result.