Question 2

From the task description we get: $P(a \mid r) = 0.3$
$P(o \mid r) = 0.4$
$P(l \mid r) = 0.3$
$P(a \mid g) = 0.3$
$P(o \mid g) = 0.3$
$P(l \mid g) = 0.4$
$P(a \mid b) = 0.5$
$P(o \mid b) = 0.5$
$P(l \mid b) = 0$

We can now determine the following probabilities:
$P(a \cap r) = P(a \mid r) \cdot P(r) = 0.3 \cdot 0.2 = 0.06$
$P(o \cap r) = P(o \mid r) \cdot P(r) = 0.4 \cdot 0.2 = 0.08$
$P(l \cap r) = P(l \mid r) \cdot P(r) = 0.3 \cdot 0.2 = 0.06$
$P(a \cap g) = P(a \mid g) \cdot P(g) = 0.3 \cdot 0.6 = 0.18$
$P(o \cap g) = P(o \mid g) \cdot P(g) = 0.3 \cdot 0.6 = 0.18$
$P(l \cap g) = P(l \mid g) \cdot P(g) = 0.4 \cdot 0.6 = 0.24$
$P(a \cap b) = P(a \mid b) \cdot P(b) = 0.5 \cdot 0.2 = 0.1$
$P(o \cap b) = P(o \mid b) \cdot P(b) = 0.5 \cdot 0.2 = 0.1$
$P(l \cap b) = P(l \mid b) \cdot P(b) = 0 \cdot 0.2 = 0$

(a)

$P(a) = P(a \cap r) + P(a \cap g) + P(a \cap b) = 0.06 + 0.18 + 0.1 = 0.34$
$P(o) = P(o \cap r) + P(o \cap g) + P(o \cap b) = 0.08 + 0.18 + 0.1 = 0.36$
$P(l) = P(l \cap r) + P(l \cap g) + P(l \cap b) = 0.06 + 0.24 + 0 = 0.3$

Thus, the probability of selecting an apple is 0.34.

(b)

$P(r \mid a) = \frac{P(a \cap r)}{P(a)} = \frac{0.06}{0.34} = 0.176$
$P(g \mid a) = \frac{P(a \cap g)}{P(a)} = \frac{0.18}{0.34} = 0.529$
$P(b \mid a) = \frac{P(a \cap b)}{P(a)} = \frac{0.1}{0.34} = 0.294$
$P(r \mid o) = \frac{P(o \cap r)}{P(o)} = \frac{0.08}{0.36} = 0.222$
$P(g \mid o) = \frac{P(o \cap g)}{P(o)} = \frac{0.18}{0.36} = 0.5$
$P(b \mid o) = \frac{P(o \cap b)}{P(o)} = \frac{0.1}{0.36} = 0.278$
$P(r \mid l) = \frac{P(l \cap r)}{P(l)} = \frac{0.06}{0.3} = 0.2$
$P(g \mid l) = \frac{P(l \cap g)}{P(l)} = \frac{0.24}{0.3} = 0.8$
$P(b \mid l) = \frac{P(l \cap b)}{P(l)} = \frac{0}{0.3} = 0$

Thus, if we observe that the selected fruit is in fact an orange, the probability that it came from the green box is 0.5.

Question 3

Error: classify(x) is not a function, iff $p(C_i \mid x) = p(C_j \mid x)$ for some $i \neq j$. Thus, we assume that $p(C_i \mid x) \neq p(C_j \mid x)$ for all $i \neq j$.

(a)

For some x, let $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, N\} \setminus \{i\}$, such that C_i is the correct classification of x. Let $j \in \{1, \dots, N\}$:

$$\begin{aligned} R(C_j \mid x) &= \sum_{k \in \{1,\dots,N\} \setminus \{j\}} L_{k,j} p(C_k \mid x) + L_{j,j} p(C_j \mid x) + L_{rej,j} p(C_{rej} \mid x) \\ &= \sum_{k \in \{1,\dots,N\} \setminus \{j\}} l_s p(C_k \mid x) + 0 \cdot p(C_j \mid x) + l_r p(C_{rej} \mid x) \\ &= \sum_{k \in \{1,\dots,N\} \setminus \{j\}} l_s p(C_k \mid x) + l_r p(C_{rej} \mid x) \end{aligned}$$

$$\begin{aligned} R(C_{rej} \mid x) &= \sum_{k \in \{1,\dots,N\}} L_{k,j} p(C_k \mid x) + L_{rej,j} p(C_{rej} \mid x) \\ &= \sum_{k \in \{1,\dots,N\}} l_s p(C_k \mid x) + l_r p(C_{rej} \mid x) \end{aligned}$$

TODO: No idea how to continue

(b)

If $l_r = 0$, then $1 - \frac{l_r}{l_s} = 1$. Thus, we then only classify x to category C_j iff $p(C_j \mid x) = 1$ (so we are certain). If that is not the case (uncertain), then we reject. As $l_r = 0$, there is no punishment for rejecting. Therefore, the expected loss for such a function is always 0.

(c)

If $l_r > l_s$, then we never reject using classify(x). Similarly, during risk minimization, we never use a strategy that rejects, as making a substitution error yields a lesser loss.

Question 4

Note that all measurements are greater than 0, thus $g(x)$ is always 1.

$$\begin{aligned}\frac{\partial p(x \mid \theta)}{\partial \theta} &= 2x \cdot \theta \cdot \exp(-\theta x) + x \cdot \theta^2 \cdot (-x) \exp(-\theta x) \\ &= 2x \cdot \theta \cdot \exp(-\theta x) - x^2 \cdot \theta^2 \cdot \exp(-\theta x) \\ &= x\theta \exp(-\theta x) \cdot (2 - x\theta)\end{aligned}$$

Let:

$$f(x, \theta) := \frac{\frac{\partial p(x|\theta)}{\partial \theta}}{p(x \mid \theta)} = \frac{x\theta \exp(-\theta x) \cdot (2 - x\theta)}{\theta^2 x \exp(-\theta x)}$$

Assuming $\theta \neq 0$, we can simplify this to:

$$f(x, \theta) = \frac{2 - x\theta}{\theta}$$

We can reorder this derivative of the log likelihood to:

$$\begin{aligned}-\sum_{n=1}^{N} f(x_n, \theta) &= -\sum_{n=1}^{N} \frac{2 - x_n\theta}{\theta} \\ &= -\frac{2}{\theta} + \sum_{n=1}^{N} x_n\end{aligned}$$

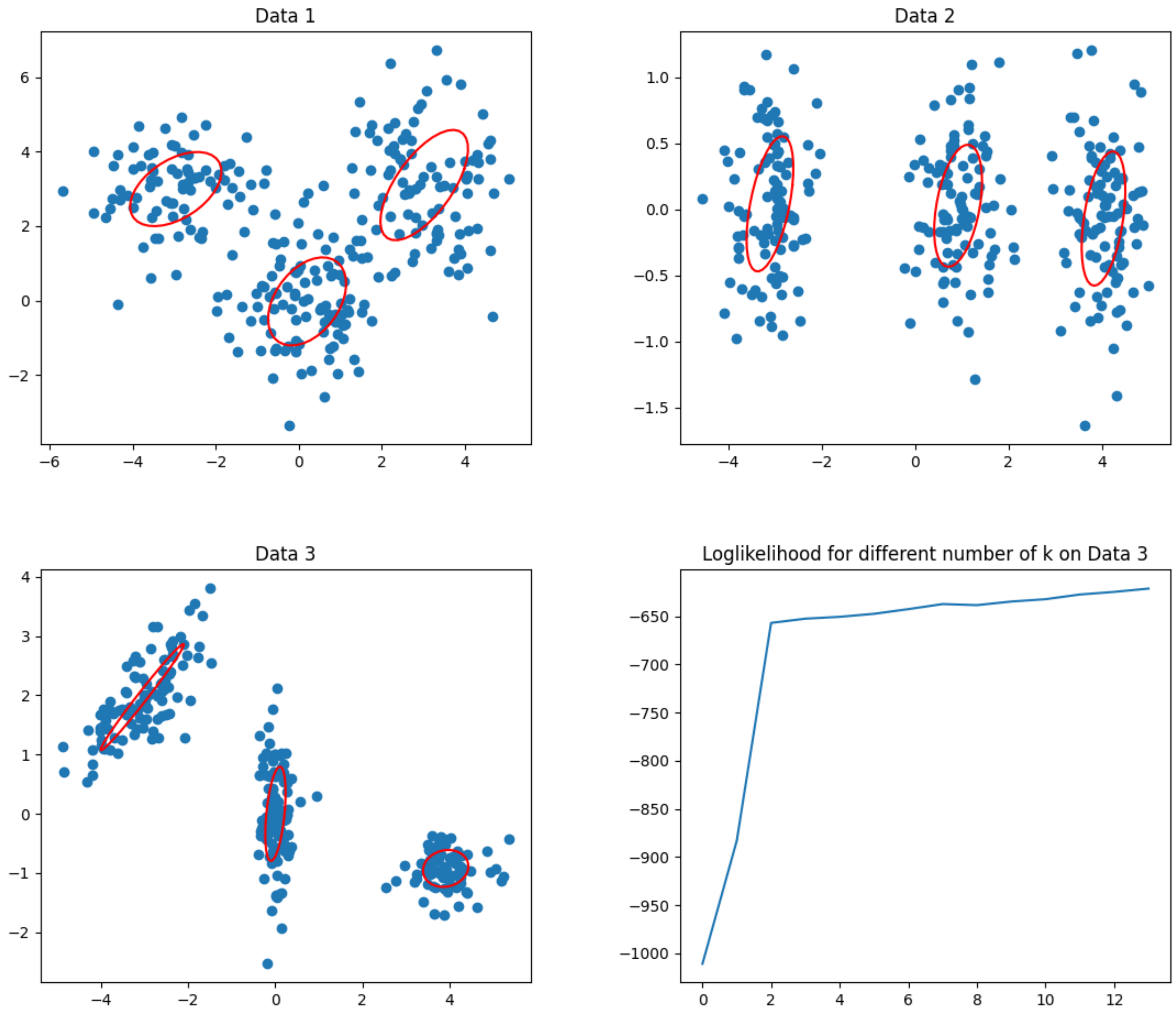
Setting this to 0 yields:

$$\begin{aligned}0 &= -\frac{2}{\theta} + \sum_{n=1}^{N} x_n \\ \sum_{n=1}^{N} x_n &= \frac{2}{\theta} \\ \theta &= \frac{2}{\sum_{n=1}^{N} x_n}\end{aligned}$$

Thus, the maximum likelihood estimate $\widetilde{\theta}$ for the given measurements is: $\widetilde{\theta} = \frac{2}{\sum_{n=1}^{N} x_n}$

Question 6

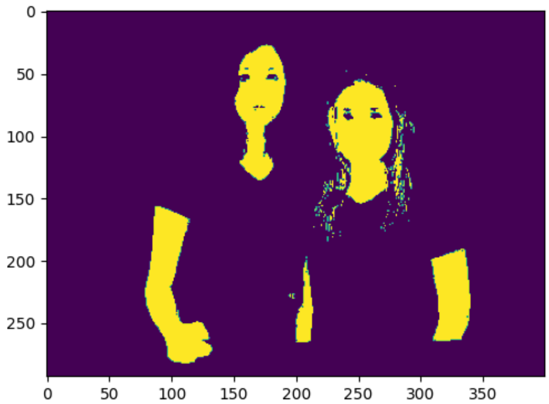
(f)



We can see, that in all three data sets there are 3 clusters (more so in data2 and data3 than in data1). Therefore, $K = 3$ clusters is a very suitable number of clusters for this application.

$K = 2$ yields very under fitting cluster parameters, even for data1. For $K > 3$ we get clusters that are "overlapping" each other, so we see overfitting.

(g)



Increasing K here improves accuracy on the training data set, thus reducing generalization. For $K = 9$, a model is produced that separates the colors in the test image better than the default of $K = 3$. Increasing K further might lead to poor accuracy on other test data. Since the colors are there dimensional, and we are looking at two classifications (skin and no-skin), intuitively speaking, $3 * 2$ clusters should be a good starting point for finding a suitable number of clusters.

Increasing the threshold *theta* increases the required confidence, that a given pixel color represents skin. This leads to a more conservative classification (so rather non-skin than skin), which is more robust to noise, however, can also be less generalized. We increased *theta* to 3 and got the above result.

(h)

More images using the above parameters:

From the first two images we can learn that the algorithm is very confident when it comes to pixels similar to skin color, however, cannot distinguish between foreground and background. The last two landscape images are mostly detected as not-skin besides a few artifacts.