Component Bound Branching in a Branch-and-Price Framework

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Abstract

This master thesis integrates the component bound branching rule, proposed by Vanderbeck et al. [1, 2], into the branch-price-and-cut solver GCG. This rule, similarly to Vanderbeck's generic branching scheme [3], exclusively operates within the Dantzig-Wolfe reformulated problem, where branching decisions generally have no corresponding actions in the original formulation. The current GCG framework requires modifications for such branching rules, especially within the pricing loop, as seen in Vanderbeck's method implementation. These rules also fail to utilize enhancements like dual value stabilization.

A significant contribution of this thesis is the enhancement of the GCG architecture to facilitate the seamless integration of new branching rules that operate solely on the reformulated problem. This allows these rules to benefit from current and future improvements in the branch-price-and-cut framework, including dual value stabilization, without necessitating alterations to the branching rule itself.

The thesis proposes an interface to manage constraints in the master problem that lack counterparts in the original formulation. These constraints require specific modifications to the pricing problems to ensure their validity in the master. The 'generic mastercut' interface, tightly integrated into the GCG solver, spans the pricing loop, column generation, and dual value stabilization. Enhancements to the existing branching rule interface complement this integration, enabling effective utilization of the generic mastercuts.

Finally, the component bound branching rule will be implemented using this new interface and evaluated on a set of benchmark instances. Its performance will be benchmarked against the existing Vanderbeck branching rule, offering a practical comparison of both approaches.

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Chapter 1

Introduction

Chapter 2

Preliminaries

In this preliminary chapter we will provide a brief rundown of theorems and algorithms on which the techniques described in later chapters, such as Column Generation in Section 3.1, are building upon. Understanding these concepts is essential to understanding the theory later presented. If, however, one is familiar with these, we invite the reader to skip ahead to Chapter 3.

2.1 Polyhedron Representation

Definition 2.1. *Given k points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in \mathbb{R}^n$, any $\boldsymbol{x} = \sum_{i=1}^{k} \alpha_i \boldsymbol{x}_i$ is a **conic combination** of the $\boldsymbol{x}_i$, iff $\forall i \in \{1, \ldots, k\}.\alpha_i \geq 0$.*

Definition 2.2. *Given k points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in \mathbb{R}^n$, any $\boldsymbol{x} = \sum_{i=1}^{k} \alpha_i \boldsymbol{x}_i$ is a **convex combination** of the $\boldsymbol{x}_i$, iff $\sum_{i=1}^{k} \alpha_i = 1 \wedge \forall i \in \{1, \ldots, k\}.\alpha_i \geq 0$.*

The set of all convex combinations of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ is therefore defined as:

$$\operatorname{conv}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k) \coloneqq \{\sum_{i=1}^{k} \alpha_i \boldsymbol{x}_i \mid \sum_{i=1}^{k} \alpha_i = 1 \wedge \forall i \in \{1, \ldots, k\}.\alpha_i \geq 0\}$$

Corollary 2.1. *The intersection of two convex sets is convex.*

Definition 2.3. *Let $\mathcal{P}$ be a convex set. A point $\boldsymbol{p} \in \mathcal{P}$ is an **extreme point** of $\mathcal{P}$ if there is no non-trivial convex combination of any two points in $\mathcal{P}$ expressing $\boldsymbol{p}$, i.e.*

$$\forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{P}.\forall \alpha \in \mathbb{R}_+ \backslash \{0\}.\boldsymbol{x}_1 \neq \boldsymbol{x}_2 \implies \boldsymbol{p} \neq \alpha \boldsymbol{x}_1 + (1 - \alpha)\boldsymbol{x}_2$$

Definition 2.4. *Let $\mathcal{P}$ be a convex set. A vector $\boldsymbol{r} \in \mathbb{R}_0^n \backslash \{0\}$ is a **ray** of $\mathcal{P}$ iff $\forall \boldsymbol{x} \in \mathcal{P}.\forall \beta \in \mathbb{R}_+.\boldsymbol{x} + \beta \boldsymbol{r} \in \mathcal{P}$.*

The span of rays $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_k \in \mathbb{R}_+^n$ we denote as:

$$\operatorname{rayspan}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_k) \coloneqq \bigcup_{i=1}^{k} \{\omega \boldsymbol{r}_i \mid \omega \in \mathbb{R}_+\}$$

Definition 2.5. *A ray $\boldsymbol{r}$ of $\mathcal{P}$ is an **extreme ray** of $\mathcal{P}$ if there is no non-trivial conic combination of any two rays in $\mathcal{P}$ expressing $\boldsymbol{r}$, i.e.*

$$\forall \boldsymbol{r}_1, \boldsymbol{r}_2 \in \mathcal{P}. \forall \alpha_1, \alpha_2, \beta \in \mathbb{R}_+ \backslash \{0\}. \boldsymbol{r}_1 \neq \beta \boldsymbol{r}_2 \implies \boldsymbol{r} \neq \alpha_1 \boldsymbol{r}_1 + \alpha_2 \boldsymbol{r}_2$$

Definition 2.6. *A **hyperplane** $\mathcal{H} \subset \mathbb{R}^n$ of a n-dimensional space is a subspace of dimension $n-1$, and can therefore be described using a vector $\boldsymbol{f} \in \mathbb{R}^n$ and a scalar $f \in \mathbb{R}$ as $\mathcal{H} = \{\boldsymbol{x} \mid \boldsymbol{f}^\intercal \boldsymbol{x} = f\}$.*

Corollary 2.2. *Any hyperplane is a convex set.*

Proof. Let $\mathcal{H} = \{\boldsymbol{x} \mid \boldsymbol{f}^\intercal \boldsymbol{x} = f\}$ be a hyperplane. Let $k \in \mathbb{N}$, $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in \mathcal{H}$. For any $\alpha_1, \ldots, \alpha_k \in \mathbb{R}_+$ with $\sum_{i=1}^k$:

$$\begin{aligned} \boldsymbol{f}^\intercal \left(\sum_{i=1}^k \alpha_i \boldsymbol{x}_i \right) &= \sum_{i=1}^k \alpha_i \boldsymbol{f}^\intercal \boldsymbol{x}_i \\ &= \sum_{i=1}^k \alpha_i \cdot f \\ &= f \cdot \sum_{i=1}^k \alpha_i \\ &= f \end{aligned}$$

Therefore, the convex combination $\sum_{i=1}^k \alpha_i \boldsymbol{x}_i$ is in the hyperplane $\mathcal{H}$. □

Definition 2.7. *A **halfspace** is the set above or below a hyperplane. A halfspace is open if the points on the hyperplane are excluded, otherwise closed.*

Corollary 2.3. *Any halfspace is a convex set.*

Proof. Let $\mathcal{H}^+ = \{\boldsymbol{x} \mid \boldsymbol{f}^\intercal \boldsymbol{x} > f\}$ be an open halfspace (analogous for $\mathcal{H}^- = \{\boldsymbol{x} \mid \boldsymbol{f}^\intercal \boldsymbol{x} < f\}$, and for the closed halfspaces). Let $k \in \mathbb{N}$, $\boldsymbol{x_1}, \ldots, \boldsymbol{x_k} \in \mathcal{H}$. For any $\alpha_1, \ldots, \alpha_k \in \mathbb{R}_+$ with $\sum_{i=1}^k$:

$$\begin{aligned} \boldsymbol{f}^\intercal \left(\sum_{i=1}^k \alpha_i \boldsymbol{x}_i \right) &= \sum_{i=1}^k \alpha_i \boldsymbol{f}^\intercal \boldsymbol{x}_i \\ &> \sum_{i=1}^k \alpha_i \cdot f \\ &= f \cdot \sum_{i=1}^k \alpha_i \\ &= f \end{aligned}$$

Therefore, the convex combination $\sum_{i=1}^k \alpha_i \boldsymbol{x}_i$ is in the halfspace $\mathcal{H}$. □

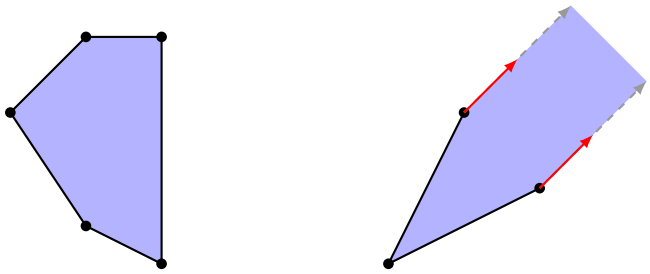

Figure 2.1: Illustration of the Minkowski-Weyl theorem. The left figure shows a fully encapsulated polyhedron which can be represented only by its extreme points. Unbounded polyhedra, such as the one on the right, require extreme rays, drawn in red, to be described completely.

Definition 2.8. *A **polyhedron** $\mathcal{P} \subseteq \mathbb{R}^n$ is defined by the intersection of a set of closed halfspaces, i.e. $\mathcal{P} \coloneqq \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\}$, with $\boldsymbol{A} \in \mathbb{R}^{m \times n}, \boldsymbol{b} \in \mathbb{R}^m$.*

By Corollaries 2.1 and 2.3, a polyhedron is also a convex set of points.

Definition 2.9. *The **Minkowski sum** of two sets P, Q is defined by:*

$$P \oplus Q \coloneqq \{\boldsymbol{p} + \boldsymbol{q} \mid \boldsymbol{p} \in P \wedge \boldsymbol{q} \in Q\}$$

Theorem 2.1 (Minkowski-Weyl)**.** *For $\mathcal{P} \subseteq \mathbb{R}^n$ the following statements are equivalent:*

1. *$\mathcal{P}$ is a polyhedron, i.e., there exists some finite matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and some vector $\boldsymbol{b} \in \mathbb{R}^m$ such that $P = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$*

2. *There exist fine vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_s \in \mathbb{R}^n$ and finite vectors $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_t \in \mathbb{R}^n_+$, such that $P = \text{conv}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_s) \oplus \text{rayspan}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_t)$*

In simple terms, the Minkowski-Weyl theorem states that any polyhedron can always be defined in two ways: either by its faces, i.e. closed halfspaces, or by its vertices and rays. Most polyhedra can be represented in this way using only their extreme points and extreme rays. Figure 2.1 illustrates this theorem on two exemplary polyhedra.

The following theorem builds upon the Minkowski-Weyl theorem to describe a polyhedron, which is represented by its extreme points $\{\boldsymbol{x}_p\}_{p \in P}$ and extreme rays $\{\boldsymbol{x}_r\}_{r \in R}$, using hyperplanes. Here, the sets P, R are used to index the extreme points and extreme rays, respectively.

Theorem 2.2 (Nemhauser-Wolsey)**.** *Consider the polyhedron $\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{Q}\boldsymbol{x} \geq \boldsymbol{b}\}$ with full row rank matrix $\boldsymbol{Q} \in \mathbb{R}^{m \times n}$, i.e. $\text{rank}(\boldsymbol{Q}) = m \leq n \wedge \mathcal{P} \neq \emptyset$.*

An equivalent description of $\mathcal{P}$ using its extreme points $\{\boldsymbol{x}_p\}_{p\in P}$ and extreme rays $\{\boldsymbol{x}_r\}_{r\in R}$ is:

$$\mathcal{P} = \left\{ \boldsymbol{x} \in \mathbb{R}^n \middle| \begin{array}{lll} \sum_{p\in P} \boldsymbol{x}_p\lambda_p + & \sum_{r\in R} \boldsymbol{x}_r\lambda_r = \boldsymbol{x} & \\ \sum_{p\in P} \lambda_p & = 1 & \\ \lambda_p & \geq 0 & \forall p \in P \\ & \lambda_r \geq 0 & \forall r \in R \end{array} \right\} \tag{2.1}$$

In the Nemhauser-Wolsey theorem, the conditions of the Minkowski-Weyl theorem are clearly encoded: the second and third lines ensure that the convex set of the extreme points are considered in the first line (Definition 2.2), the last playing a part in the span of extreme rays (Definition 2.4), and the first line being the Minkowski sum of the convex hull of extreme rays and the span of extreme rays.

2.2 Primal Simplex Algorithm

Have the following linear program in standard form:

$$\begin{array}{lll} \min & \boldsymbol{c}^\intercal \boldsymbol{x} & \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} & [\boldsymbol{\pi}] \\ & \boldsymbol{x} \geq \boldsymbol{0} & \end{array} \tag{2.2}$$

The primal simplex algorithm finds an optimal solution by moving from one extreme point of the polyhedron to the next, therefore always remaining feasible. A central part of this algorithm is the sufficient optimality condition. For a basic solution $\boldsymbol{X} = [\boldsymbol{x}_\mathcal{B}, \boldsymbol{x}_\mathcal{N}]$ at a given extreme point to be optimal, the reduced costs $\bar{c}_j \coloneqq c_j - \boldsymbol{\pi}^\intercal \boldsymbol{a}_j$ for $j \in \mathcal{N}$ must be non-negative.

This sufficient optimality condition gives rise to the **pricing problem**, which either verifies the optimality of the current basic solution, and otherwise determines the non-basic variable x_l, $l \in \mathcal{N}$ with the least reduced cost ($\bar{c}_l < 0$) to be swapped into the basis next, according to Dantzig's rule (TODO cite). Formally, this can be written as:

$$l \in \arg\min_{j\in\mathcal{N}} c_j - \boldsymbol{\pi}^\intercal \boldsymbol{a}_j \tag{2.3}$$

or as the linear program:

$$\bar{c}(\boldsymbol{\pi}) = \min_{j\in\mathcal{N}} c_j - \boldsymbol{\pi}^\intercal \boldsymbol{a}_j \tag{2.4}$$

Solving the pricing problem thus plays an integral role in the primal simplex algorithm:

Algorithm 2.1: Primal simplex algorithm with Dantzig's rule

Input: *LP* in standard form (2.2); Basic and non-basic index-sets $\mathcal{B}, \mathcal{N}$

Output: Optimal Solution $(\boldsymbol{x}, z)$

loop

  $\boldsymbol{\pi}^\intercal \leftarrow \boldsymbol{c}_{\mathcal{B}}^\intercal \boldsymbol{A}_{\mathcal{B}}^{-1}$; $\bar{\boldsymbol{b}} \leftarrow \boldsymbol{A}_{\mathcal{B}}^{-1}\boldsymbol{b}$;

  $\bar{c}_j \leftarrow c_j - \boldsymbol{\pi}^\intercal \boldsymbol{a}_j; \qquad \forall j \in \mathcal{N}$

  $l \leftarrow \underset{j \in \mathcal{N}}{\arg\min}\, \bar{c}_j$; $\bar{c}(\boldsymbol{\pi}) \leftarrow \bar{c}_l$;

  if $\bar{c}(\boldsymbol{\pi}) \geq 0$ **then**

    return $\left(\left[\bar{\boldsymbol{b}}, \boldsymbol{0}\right], \boldsymbol{c}_{\mathcal{B}}^\intercal \boldsymbol{x}_{\mathcal{B}}\right)$ *by optimality*

  end

  $\bar{\boldsymbol{a}}_l \leftarrow \boldsymbol{A}_{\mathcal{B}}^{-1}\boldsymbol{a}_l$;

  if $\bar{\boldsymbol{a}}_l \leq \boldsymbol{0}$ **then**

    return None *by unboundedness*

  end

  $s \leftarrow \underset{i \in \{1,\ldots,m\}}{\arg\min} \frac{\bar{b}_i}{\bar{a}_{il}}$; $x_l \leftarrow \frac{\bar{b}_s}{\bar{a}_{sl}}$; $\mathcal{B} \leftarrow \mathcal{B} \cup \{l\} \subseteq \{s\}$; $\mathcal{N} \leftarrow \mathcal{N} \cup \{s\} \subseteq \{l\}$;

Chapter 3

Column Generation and Branch-and-Price

3.1 Column Generation

Let us consider the following linear program, which we will henceforth call the **master problem** MP, where $c_{\boldsymbol{x}} \in \mathbb{R}, \boldsymbol{a}_{\boldsymbol{x}}, \boldsymbol{b} \in \mathbb{R}^m, \forall \boldsymbol{x} \in \mathcal{X}$:

$$\begin{aligned} z_{MP} = \min \quad & \sum_{\boldsymbol{x} \in \mathcal{X}} c_{\boldsymbol{x}} \lambda_{\boldsymbol{x}} \\ \text{s.t.} \quad & \sum_{\boldsymbol{x} \in \mathcal{X}} \boldsymbol{a}_{\boldsymbol{x}} \lambda_{\boldsymbol{x}} \geq \boldsymbol{b} \quad [\boldsymbol{\pi}] \\ & \lambda_{\boldsymbol{x}} \geq \boldsymbol{0} \qquad \forall \boldsymbol{x} \in \mathcal{X} \end{aligned} \tag{3.1}$$

Assume the number of variables is huge, i.e. a lot larger than the number of constraints ($m \ll |\mathcal{X}| < \infty$). Because of this, solving MP in a reasonable amount of time, sometimes at all, is infeasible.

We can, however, make use of a crucial property of the primal simplex algorithm: at any given vertex solution, only few variables are in the basis. Most variables are in the non-basis, and therefore have a solution value of 0. Having a solution value of 0 is equivalent to not being in the linear program at all. Therefore, the primal simplex algorithm can also function using a manageable subset of variables $\mathcal{X}' \subseteq \mathcal{X}$, finding a possibly non-optimal, yet still feasible solution for the entire optimization problem MP. We denote this master problem restricted to a subset of variables as the **restricted master problem** RMP:

$$\begin{aligned} z_{RMP} = \min \quad & \sum_{\boldsymbol{x} \in \mathcal{X}'} c_{\boldsymbol{x}} \lambda_{\boldsymbol{x}} \\ \text{s.t.} \quad & \sum_{\boldsymbol{x} \in \mathcal{X}'} \boldsymbol{a}_{\boldsymbol{x}} \lambda_{\boldsymbol{x}} \geq \boldsymbol{b} \quad [\boldsymbol{\pi}] \\ & \lambda_{\boldsymbol{x}} \geq \boldsymbol{0} \qquad \forall \boldsymbol{x} \in \mathcal{X}' \end{aligned} \tag{3.2}$$

Assuming *MP* is feasible, two important aspects of finding an optimal solution to *MP* are still missing: first, how do we find a subset $\mathcal{X}'$ of the variables, such that *RMP* stays feasible? Without this property of the set of variables, no solution of *RMP* can be found, and therefore none can be found for *MP*, which would contradict the feasibility of *MP*. Secondly, assuming a solution of *RMP* was found, possibly even optimal for the *RMP*, how could we build upon this solution to eventually find an optimal solution for *MP*?

In the following we will dive into these two questions in detail (Sections 3.1.1 and 3.1.2), making way for the final column generation algorithm (Section 3.1.3).

3.1.1 Farkas Pricing

Let us assume *MP* is feasible, but our current selection of variables $\mathcal{X}' \subset \mathcal{X}$ results in the *RMP* being infeasible. The task is now to find additional variables such that a new set $\mathcal{X}''$ with $\mathcal{X}' \subset \mathcal{X}'' \subseteq \mathcal{X}$ makes the *RMP* feasible. For this, consider Farkas' lemma:

Theorem 3.1 (Farkas' lemma)**.** *Given $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$, then exactly one of the following statements holds:*

1. $\exists \boldsymbol{x} \in \mathbb{R}^n_+ . \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}$
2. $\exists \boldsymbol{\pi} \in \mathbb{R}^n_+ . \boldsymbol{\pi}^\intercal \boldsymbol{A} \leq \boldsymbol{0} \wedge \boldsymbol{\pi}^\intercal \boldsymbol{b} > 0$

Given that the *MP* is feasible, the following must hold for the *MP* with $\boldsymbol{A} = \boldsymbol{A}_{|\mathcal{X}}$:

$$\begin{aligned} neg \exists \boldsymbol{\pi} \in \mathbb{R}^n_+ . \boldsymbol{\pi}^\intercal \boldsymbol{A} \leq \boldsymbol{0} \wedge \boldsymbol{\pi}^\intercal \boldsymbol{b} > 0 & \\ \Leftrightarrow \forall \boldsymbol{\pi} \in \mathbb{R}^n_+ . \neg \left(\boldsymbol{\pi}^\intercal \boldsymbol{A} \leq \boldsymbol{0} \wedge \boldsymbol{\pi}^\intercal \boldsymbol{b} > 0 \right) & \\ \Leftrightarrow \forall \boldsymbol{\pi} \in \mathbb{R}^n_+ . \boldsymbol{\pi}^\intercal \boldsymbol{A} > \boldsymbol{0} \vee \boldsymbol{\pi}^\intercal \boldsymbol{b} \leq 0 & \end{aligned} \tag{3.3}$$

Furthermore, from the infeasibility of *RMP* we can also derive the following statement:

$$\begin{aligned} & \left(\forall \boldsymbol{\pi} \in \mathbb{R}^n_+ . \boldsymbol{\pi}^\intercal \boldsymbol{A} > \boldsymbol{0} \vee \boldsymbol{\pi}^\intercal \boldsymbol{b} \leq 0 \right) \wedge \left(\exists \boldsymbol{\pi} \in \mathbb{R}^n_+ . \boldsymbol{\pi}^\intercal \boldsymbol{A}_{|\mathcal{X}'} \leq \boldsymbol{0} \wedge \boldsymbol{\pi}^\intercal \boldsymbol{b} > 0 \right) \\ \Rightarrow & \left(\neg \forall \boldsymbol{\pi} \in \mathbb{R}^n_+ . \boldsymbol{\pi}^\intercal \boldsymbol{b} \leq 0 \right) \wedge \left(\exists \boldsymbol{\pi} \in \mathbb{R}^n_+ . \boldsymbol{\pi}^\intercal \boldsymbol{A} > \boldsymbol{0} \right) \end{aligned} \tag{3.4}$$

Therefore, there is some variable $\boldsymbol{x} \in \mathcal{X} \backslash \mathcal{X}'$ such that its column $\boldsymbol{a}_{\boldsymbol{x}} \coloneqq \boldsymbol{A}_{|\{\boldsymbol{x}\}}$ is $\boldsymbol{\pi}^\intercal \boldsymbol{a}_{\boldsymbol{x}} > 0$ for some $\boldsymbol{\pi} \in \mathbb{R}^n_+$. If none existed, *MP* would not be feasible.

This process of finding corresponding columns $\boldsymbol{a}_{\boldsymbol{x}}$ to add to the *RMP* can be formalized as a pricing problem with cost coefficients $c_{\boldsymbol{x}} = 0$ (see Equation (2.4)). Let us denote this subproblem as the *FP-SP*:

$$\mathrm{F}(\boldsymbol{\pi}) = \min_{x \in \mathcal{X}} \; - \boldsymbol{\pi}^\intercal \boldsymbol{a}_x \tag{3.5}$$

We can add all solutions $\boldsymbol{x}$ with a solution value of $\mathrm{F}(\boldsymbol{\pi}) < 0$ to $\mathcal{X}'' \coloneqq \mathcal{X}' \cup \{\boldsymbol{x}_i\}$, adding the corresponding column $\begin{bmatrix} 0 \\ \boldsymbol{a_x} \end{bmatrix}$ to the problem, thus turning any infeasible *RMP* feasible.

3.1.2 Reduced Cost Pricing

Assume the *RMP* is feasible. Using a solver of our choice, we can now construct a solution that is optimal within the *RMP*, providing us with the dual values $\boldsymbol{\pi}$. One now must verify whether this solution is also optimal for the *MP*. For this purpose, we can utilize the pricing problem we are already familiar with from the primal simplex algorithm (see Equation (2.4)). Let us denote this subproblem as the *RCP-SP*:

$$\bar{c}(\boldsymbol{\pi}) = \min_{\boldsymbol{x} \in \mathcal{X}} c_{\boldsymbol{x}} - \boldsymbol{\pi}^\intercal \boldsymbol{a_x} \tag{3.6}$$

Note, that due to the optimality of *RMP* the reduced costs of all variables $\boldsymbol{x} \in \mathcal{X}'$ are already non-negative. If now $\bar{c}(\boldsymbol{\pi}) \geq 0$, we have also proven optimality of the current solution for the *MP*. Otherwise, if $\bar{c}(\boldsymbol{\pi}) < 0$, then there is some $\boldsymbol{x} \in \mathcal{X} \backslash \mathcal{X}'$ with $\bar{c}(\boldsymbol{\pi}) = c_{\boldsymbol{x}} - \boldsymbol{\pi}^\intercal \boldsymbol{a_x} < 0$. Similarly to how the primal simplex algorithm would then swap this variable into the basis, during column generation we add the corresponding column $\begin{bmatrix} c_{\boldsymbol{x}} \\ \boldsymbol{a_x} \end{bmatrix}$ to the *RMP*. An important property of this process is that the *RMP* remains feasible.

3.1.3 Column Generation Algorithm

The column generation algorithm can now be viewed as a variation of the primal simplex algorithm: We start to solve our problem, the *MP*, with a subset of the original variables, initialized as the empty set or by using some selection-heuristics. If this restricted master problem *RMP* is infeasible, we use Farkas pricing to find new variables to add to *RMP*, either until it is feasible, or until there are no new variables to add, proving infeasibility of *MP*. As soon as *RMP* is feasible, we solve it to optimality, using reduced cost pricing to verify whether the solution is also optimal for the *MP*. If it is, we have found the optimal solution to the *MP*. Otherwise, we add the corresponding column to the *RMP* and repeat the process.

Algorithm 3.1: Column generation algorithm

Input: *RMP* with subset $\mathcal{X}' \backslash \mathcal{X}$, *RCP-SP*, *FP-SP*
Output: Optimal Solution $(\boldsymbol{\lambda}, z)$ for the *MP*

while IsInfeasible(*RMP*) **do**
  $(\textbf{None}, \boldsymbol{\pi}) \leftarrow \text{Solve}(RMP)$;
  $(\boldsymbol{x}, \text{F}(\boldsymbol{\pi})) \leftarrow \text{Solve}(FP\text{-}SP, \boldsymbol{\pi})$;
  if $\text{F}(\boldsymbol{\pi}) \geq 0$ **then**
    return None *by MP infeasibility*
  end
  $\mathcal{X}' \leftarrow \mathcal{X}' \cup \{\boldsymbol{x}\}$;
  $\boldsymbol{A} \leftarrow \begin{bmatrix} \boldsymbol{A} & \boldsymbol{a}_{\boldsymbol{x}} \end{bmatrix}$;
end
loop
  $(\boldsymbol{\lambda}_{RMP}, \boldsymbol{\pi}) \leftarrow \text{Solve}(RMP)$;
  $(\boldsymbol{x}, \bar{c}(\boldsymbol{\pi})) \leftarrow \text{Solve}(RCP\text{-}SP, \boldsymbol{\pi})$;
  if $\bar{c}(\boldsymbol{\pi}) \geq 0$ **then**
    return $(\boldsymbol{\lambda}_{RMP}, \boldsymbol{c}_{\mathcal{B}}^{\intercal} \boldsymbol{x}_{\mathcal{B}})$ *by optimality*
  end
  $\mathcal{X}' \leftarrow \mathcal{X}' \cup \{\boldsymbol{x}\}$;
  $\boldsymbol{A} \leftarrow \begin{bmatrix} \boldsymbol{A} & \boldsymbol{a}_{\boldsymbol{x}} \end{bmatrix}$;

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3.3 Dantzig-Wolfe Reformulation for Mixed Integer Programs

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