

Component Bound Branching in a Branch-and-Price Framework

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- 6 Conclusion

- ▶ consider the following bounded integer program (IP):

$$\begin{aligned} z_{IP}^* = \min \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s. t.} \quad & \mathbf{Ax} \geq \mathbf{b} \quad [\pi_b] \\ & \mathbf{Dx} \geq \mathbf{d} \quad [\pi_d] \\ & \mathbf{x} \in \mathbb{Z}_+^n \end{aligned}$$

- ▶ we want to solve this IP using Column Generation (CG)

Dantzig-Wolfe Reformulation for IP s

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→ Apply Dantzig-Wolfe Reformulation

Dantzig-Wolfe Reformulation for *IPs*

- reformulation using discretization yields the master problem (*MP*):

$$\begin{aligned} z_{MP}^* = \min \quad & \sum_{q \in Q} c_q \lambda_q \\ \text{s. t.} \quad & \sum_{q \in Q} a_q \lambda_q \geq b \quad [\pi_b] \\ & \sum_{q \in Q} \lambda_q = 1 \quad [\pi_0] \\ & \lambda_q \in \{0, 1\} \quad \forall q \in Q \end{aligned}$$

- and the subproblem (*SP*):

$$\begin{aligned} z_{SP}^* = \min \quad & (c^\top - \pi_b^\top A) x - \pi_0 \\ \text{s. t.} \quad & Dx \geq d \quad [\pi_d] \\ & x \in \mathbb{Z}_+^n \end{aligned}$$

Branch-and-Price Algorithm:

- ① solve the relaxed MP using CG to optimality
- ② branch, if λ^* is fractional
- ③ repeat

Branch-and-Price Algorithm:

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But how to branch?

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- ▶ branching on a single λ_q is unbalanced:
 $\lambda_q \leq 0$ forbids almost nothing, $\lambda_q \geq 1$ forbids much
- branch on a set of columns $Q' \subset Q$, such that:

$$\sum_{q \in Q'} \lambda_q^* =: K \notin \mathbb{Z}$$

Component Bound Sequences

- ▶ to find such a set Q' of columns, we find a component bound sequence:

$$S := \{(x_i, \eta_i, v_i) \mid \eta_i \in \{\leq, \geq\}, v_i \in \mathbb{Z}\}$$

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- ▶ a master variable λ_q satisfies a component bound (x_i, η_i, v_i) if:

$$x_{q,i} \eta_i v_i$$

- ▶ we define Q' as the set of columns, that satisfy all component bounds in S
- ▶ there always exists a component bound sequence S such that:

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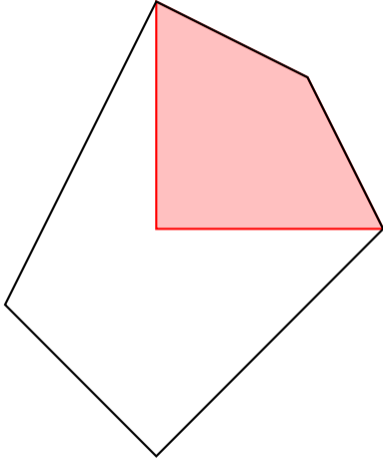
$$\sum_{q \in Q'} \lambda_q^* = K \notin \mathbb{Z}$$

- ▶ how do we enforce this?

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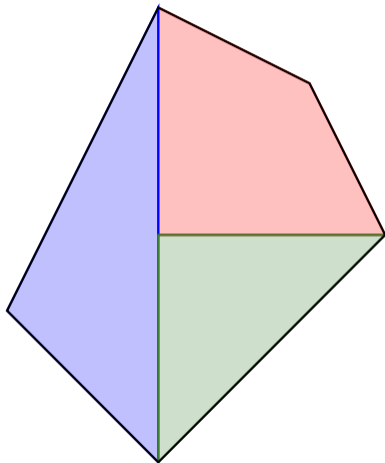
Overview of Vanderbeck's Generic Branching Scheme



- assume we have found a component bound Sequence

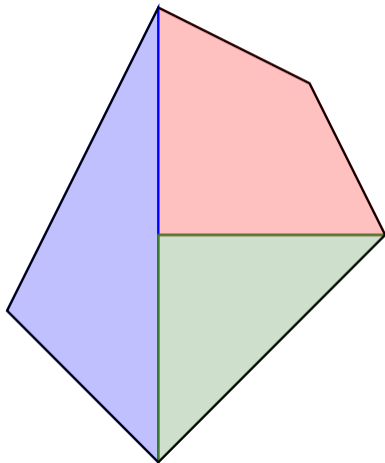
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Overview of Vanderbeck's Generic Branching Scheme



- ▶ assume we have found a component bound Sequence
$$S = \{(x_1, \geq, 1), (x_2, \geq, 2)\}$$
- ▶ create $|S| + 1 = 3$ child nodes:
 - 1 $S_1 := \{(x_1, <, 1)\}$
 - 2 $S_2 := \{(x_1, \geq, 1), (x_2, <, 2)\}$
 - 3 $S_3 := \{(x_1, \geq, 1), (x_2, \geq, 2)\}$
- ▶ search for solutions only within these regions

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- ▶ search for solutions only within these regions
- ▶ refine the regions in deeper nodes

Enforcing a Component Bound Sequence S_j in Node j

- ▶ to the MP add constraint:

$$\sum_{q \in Q'_j} \lambda_q \geq K_j$$

- ▶ move the bounds in the SP :

$$x_i \geq v_i \quad \forall (x_i, \geq, v_i) \in S_j$$

$$x_i \leq v_i \quad \forall (x_i, \leq, v_i) \in S_j$$

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Component Bound Branching Rule

Again:

- ▶ find a subset Q' of columns using a component bound sequence S :

$$S := \{(x_i, \eta_i, v_i) \mid \eta_i \in \{\leq, \geq\}, v_i \in \mathbb{Z}\}$$

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Alternative scheme:

- ▶ create binary search tree:

$$\sum_{q \in Q'} \lambda_q \leq \lfloor K \rfloor \quad [\gamma]$$

$$\sum_{q \in Q'} \lambda_q \geq \lceil K \rceil \quad [\gamma] \quad (1)$$

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- ▶ allows for solutions violating S to be found / generated

Component Bound Branching Rule

- ▶ SP determines whether a new column is in Q' , i.e., satisfies S
- ▶ create new variable $y \in \{0, 1\}$ along new constraints, such that: $y = 1 \Leftrightarrow x \in S$

Component Bound Branching Rule

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- ▶ create new variable $y \in \{0, 1\}$ along new constraints, such that: $y = 1 \Leftrightarrow \mathbf{x} \in S$

$$z_{SP}^* = \min \quad (\mathbf{c}^\top - \boldsymbol{\pi}_b^\top \mathbf{A}) \mathbf{x} - \gamma y - \pi_0$$

s. t.

$$\mathbf{D}\mathbf{x} \geq \mathbf{d}$$

$$y = 1 \Leftrightarrow \sum_{i \in S} y_i = |S|$$

$$y_i = 1 \Leftrightarrow x_i \leq v_i \quad \forall (x_i, \eta_i, v_i) \in S$$

$$y \in \{0, 1\}$$

$$y_i \in \{0, 1\} \quad \forall (x_i, \eta_i, v_i) \in S$$

$$\mathbf{x} \in \mathbb{Z}_+^n$$

Differences

Vanderbeck's Generic Branching	Component Bound Branching
tightens bounds in SP → use special algorithms	adds new variables and constraints to SP → fall back to MIP solver

Vanderbeck's Generic Branching	Component Bound Branching
tightens bounds in SP → use special algorithms	adds new variables and constraints to SP → fall back to MIP solver
refines component bounds → SP becomes faster to solve	does not refine component bounds → SP becomes slower to solve

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Master Constraints without Original Formulation

- ▶ GCG solver only allows adding original constraints, which are reformulated
- ▶ we want to add constraint MP :

$$\sum_{p \in P} f(p) \lambda_p + \sum_{r \in R} f(r) \lambda_r \leq f \quad [\gamma]$$

- ▶ with the following modification to the pricing problem:

$$\begin{aligned} z_{SP}^* = \min \quad & (\mathbf{c}^\top - \boldsymbol{\pi}_b^\top \mathbf{A}) \mathbf{x} - \gamma \mathbf{y} - \pi_0 \\ \text{s. t.} \quad & \mathbf{D} \mathbf{x} \geq \mathbf{d} \quad [\boldsymbol{\pi}_d] \\ & \mathbf{y} = \mathbf{f}(\mathbf{x}) \\ & \mathbf{x} \in \mathbb{Z}_+^n \\ & \mathbf{y} \in Y \end{aligned}$$

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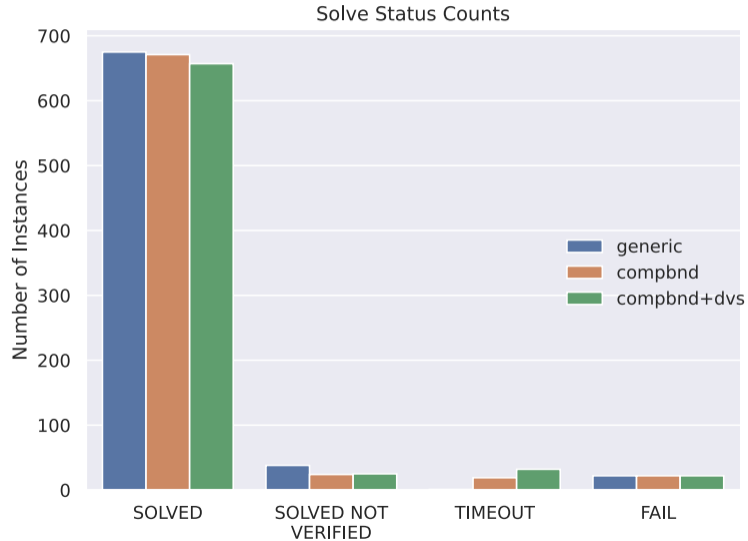
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→ new interface in GCG to create and manage such constraints

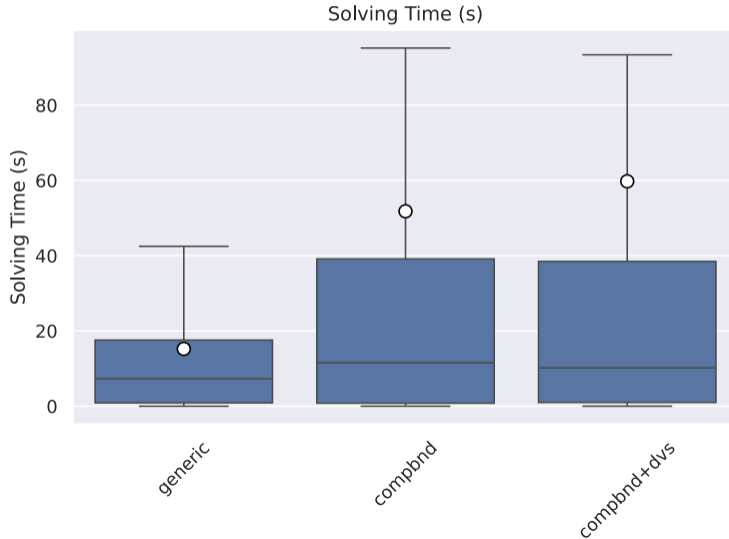
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- ▶ Vanderbeck's Generic Branching vs. Component Bound Branching Rule
- ▶ Dual Value Stabilization: Root-Only vs. Full-Tree



Evaluation



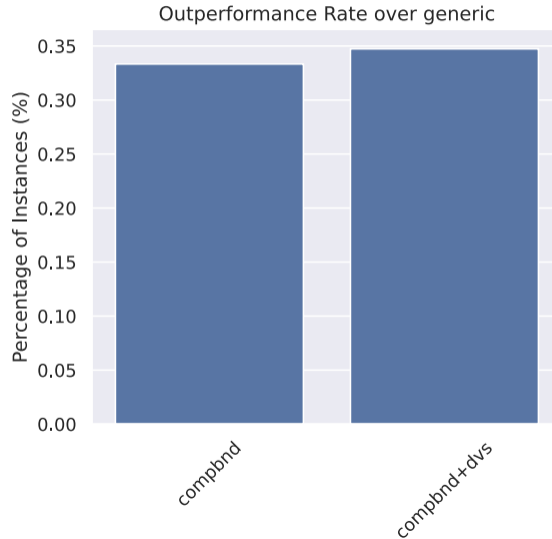


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- ▶ implemented a new interface for managing generic mastercuts in GCG
 - ▶ implemented a simple branching rule for component bound sequences using generic mastercuts
 - ▶ Vanderbeck's Generic Branching Scheme outperforms the Component Bound Branching Rule
 - ▶ implementation of the Component Bound Branching Rule showcases the potential of the new interface
- opens up new possibilities for future research (e.g., master separators)

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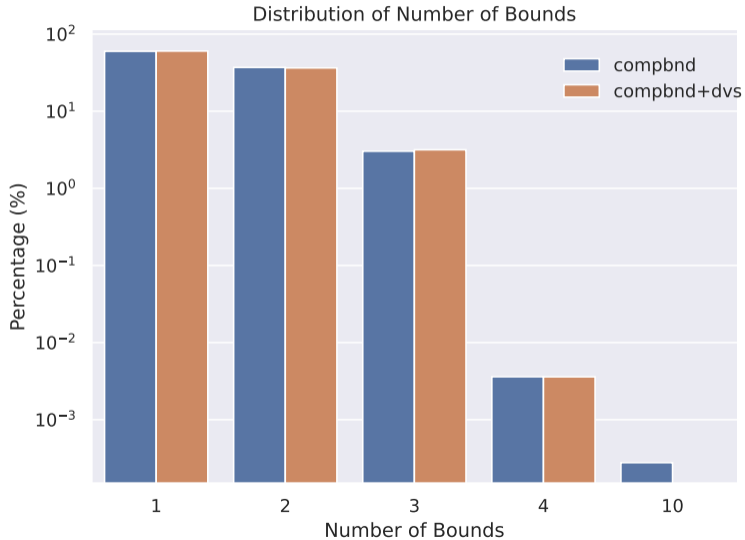
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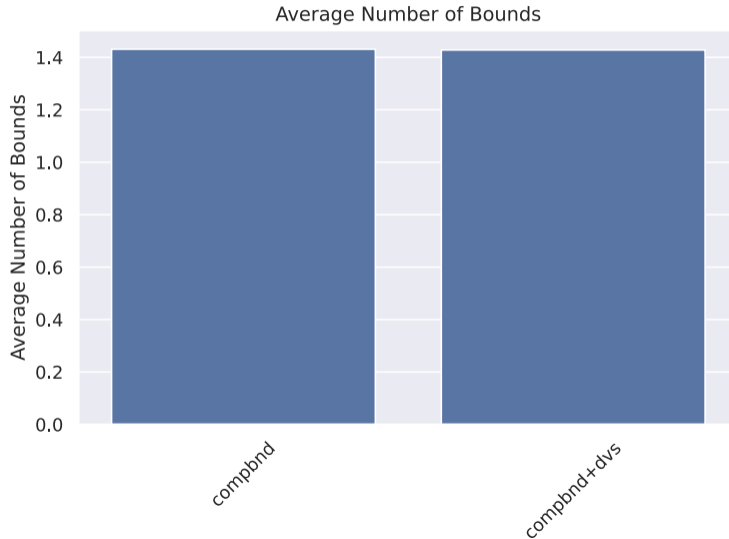
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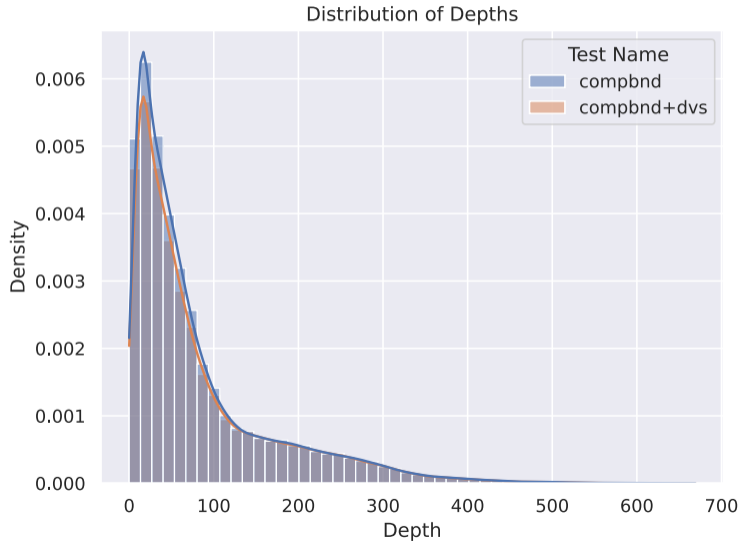
Appendix



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