Component Bound Branching in a Branch-and-Price Framework

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Dantzig-Wolfe Reformulation for *IP*s

- consider the following bounded integer program (*IP*):

$$
\begin{aligned}
z^*_{IP} = \min \quad & \boldsymbol{c}^\intercal \boldsymbol{x} & \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} & [\boldsymbol{\pi_b}] \\
& \boldsymbol{D}\boldsymbol{x} \geq \boldsymbol{d} & [\boldsymbol{\pi_d}] \\
& \boldsymbol{x} \in \mathbb{Z}^n_+ &
\end{aligned}
$$

- we want to solve this *IP* using Column Generation (CG)

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$\rightarrow$ Apply Dantzig-Wolfe Reformulation

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Dantzig-Wolfe Reformulation for *IP*s

- reformulation using discretization yields the master problem (*MP*):

$$
\begin{aligned}
z^*_{MP} = \min \quad & \sum_{q \in Q} c_q \lambda_q & & \\
\text{s. t.} \quad & \sum_{q \in Q} \boldsymbol{a}_q \lambda_q \geq \boldsymbol{b} & & [\boldsymbol{\pi_b}] \\
& \sum_{q \in Q} \lambda_q = 1 & & [\pi_0] \\
& \lambda_q \in \{0,1\} & & \forall q \in Q
\end{aligned}
$$

- and the subproblem (*SP*):

$$
\begin{aligned}
z^*_{SP} = \min \quad & \left(\boldsymbol{c}^\intercal - \boldsymbol{\pi_b}^\intercal \boldsymbol{A}\right) \boldsymbol{x} - \pi_0 & & \\
\text{s. t.} \quad & \boldsymbol{D}\boldsymbol{x} \geq \boldsymbol{d} & & [\boldsymbol{\pi_d}] \\
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Branch-and-Price

Branch-and-Price Algorithm:

1. solve the relaxed *MP* using CG to optimality
2. branch, if $\boldsymbol{\lambda}^*$ is fractional
3. repeat

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But how to branch?

- branching on a single λ_q is unbalanced:
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$\rightarrow$ branch on a set of columns $Q' \subset Q$, such that:

$$\sum_{q \in Q'} \lambda_q^* =: K \notin \mathbb{Z}$$

Component Bound Sequences - Vanderbeck *et al.* 1996

- to find such a set Q' of columns, we find a component bound sequence:

$$S \coloneqq \{(x_i, \eta_i, v_i) \mid \eta_i \in \{\leq, \geq\}, v_i \in \mathbb{Z}\}$$

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- a master variable λ_q satisfies a component bound (x_i, η_i, v_i) if:

$$x_{q,i} \ \eta_i \ v_i$$

- we define Q' as the set of columns, that satisfy all component bounds in S
- there always exists a component bound sequence S such that:

$$\sum_{q \in Q'} \lambda_q^* \in (0, 1)$$

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Component Bound Sequences

	q_1	q_2	q_3	q_4
x_{1,q_i}	1	1	2	2
x_{2,q_i}	1	2	1	2
$\lambda^*_{q_i}$	0.25	0	0.5	0.25

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Component Bound Sequences

	q_1	q_2	q_3	q_4
x_{1,q_i}	1	1	2	2
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$\lambda^*_{q_i}$	0.25	0	0.5	0.25

- Find $S = \{(x_1, \leq, 1)\}$
- $Q' = \{q_1, q_2\}$
- $\sum_{q \in Q'} \lambda^*_q = 0.25 \notin \mathbb{Z}$

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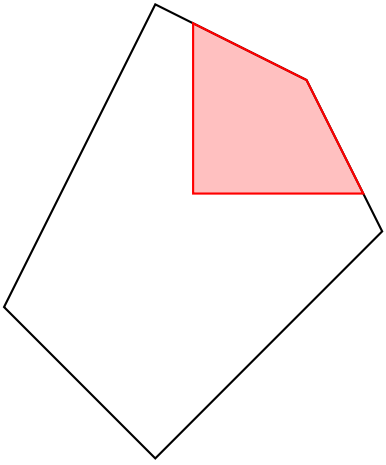
Vanderbeck's Generic Branching Scheme - Vanderbeck 2011



- assume we have found a component bound sequence
  $S = \{(x_1, \geq, 1), (x_2, \geq, 2)\}$

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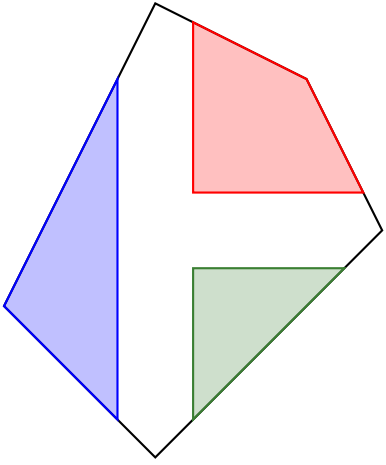
Vanderbeck's Generic Branching Scheme - Vanderbeck 2011



- assume we have found a component bound sequence
  $S = \{(x_1, \geq, 1), (x_2, \geq, 2)\}$
- create $|S| + 1 = 3$ child nodes:
  1. $S_1 \coloneqq \{(x_1, \leq, 0)\}$
  2. $S_2 \coloneqq \{(x_1, \geq, 1), (x_2, \leq, 1)\}$
  3. $S_3 \coloneqq \{(x_1, \geq, 1), (x_2, \geq, 2)\}$
- search for solutions only within these regions

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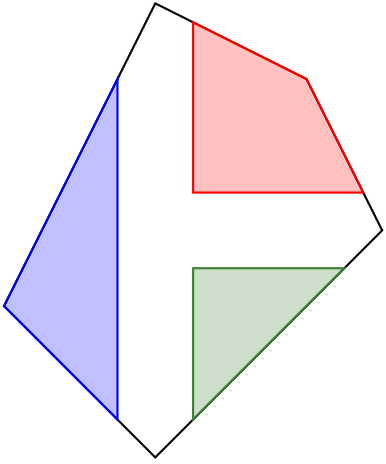
Vanderbeck's Generic Branching Scheme - Vanderbeck 2011



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  3. $S_3 \coloneqq \{(x_1, \geq, 1), (x_2, \geq, 2)\}$
- search for solutions only within these regions
- refine the regions in deeper nodes

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Enforcing a Component Bound Sequence S_j in Node j

- to the *MP* add constraint:

$$\sum_{q \in Q'_j} \lambda_q \geq 1$$

- move the bounds in the *SP*:

$$x_i \geq v_i \quad \forall\, (x_i, \geq, v_i) \in S_j$$
$$x_i \leq v_i \quad \forall\, (x_i, \leq, v_i) \in S_j$$

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Component Bound Branching Rule - Desrosiers *et al.* 2024

Again:

- find a subset Q' of columns using a component bound sequence S:

$$S \coloneqq \{(x_i, \eta_i, v_i) \mid \eta_i \in \{\leq, \geq\}, v_i \in \mathbb{Z}\}$$

- such that: $\sum_{q \in Q'} \lambda_q^* \in (0, 1)$

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Alternative scheme:

- create binary search tree:

$$\sum_{q \in Q'} \lambda_q \leq 0 \quad [\gamma] \qquad \sum_{q \in Q'} \lambda_q \geq 1 \quad [\gamma] \tag{1}$$

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$$\sum_{q \in Q'} \lambda_q \leq 0 \quad [\gamma] \qquad\qquad \sum_{q \in Q'} \lambda_q \geq 1 \quad [\gamma] \tag{1}$$

- allows for solutions violating S to be found / generated

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Component Bound Branching Rule - Desrosiers *et al.* 2024

- *SP* determines whether a new column is in Q', i.e., satisfies S
- create new variable $y \in \{0, 1\}$ along new constraints, such that: $y = 1 \Leftrightarrow \boldsymbol{x} \in S$

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z^*_{SP} = \min \quad & (\boldsymbol{c}^\intercal - \boldsymbol{\pi}_{\boldsymbol{b}}^\intercal \boldsymbol{A})\, \boldsymbol{x} - \gamma y - \pi_0 \\
\text{s.t.} \quad & \boldsymbol{D}\boldsymbol{x} \geq \boldsymbol{d} \\
& y = 1 \Leftrightarrow \sum_{i \in S} y_i = |S| \\
& y_i = 1 \Leftrightarrow x_i \leq v_i \qquad \forall\, (x_i, \leq, v_i) \in S \\
& y_i = 1 \Leftrightarrow x_i \geq v_i \qquad \forall\, (x_i, \geq, v_i) \in S \\
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& \boldsymbol{x} \in \mathbb{Z}^n_+
\end{aligned}
$$

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Differences

Vanderbeck's Generic Branching	Component Bound Branching
tightens bounds in *SP* $\rightarrow$ use special algorithms	adds new variables and constraints to *SP* $\rightarrow$ fall back to *MIP* solver

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Differences

Vanderbeck's Generic Branching	Component Bound Branching
tightens bounds in *SP* $\rightarrow$ use special algorithms	adds new variables and constraints to *SP* $\rightarrow$ fall back to *MIP* solver
refines component bounds $\rightarrow$ *SP* becomes faster to solve	does not refine component bounds $\rightarrow$ *SP* becomes slower to solve

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Master Constraints without Original Formulation

- we want to add constraint *MP*:

$$\sum_{p \in P} \mathrm{f}(p)\lambda_p + \sum_{r \in R} \mathrm{f}(r)\lambda_r \leq f \qquad [\gamma]$$

- with the following modification to the pricing problem:

$$\begin{aligned} z^*_{SP} = \min \quad & \left(\boldsymbol{c}^\intercal - \boldsymbol{\pi}_{\boldsymbol{b}}^\intercal \boldsymbol{A}\right) \boldsymbol{x} - \gamma y - \pi_0 \\ \text{s. t.} \quad & \boldsymbol{D}\boldsymbol{x} \geq \boldsymbol{d} \qquad [\boldsymbol{\pi}_{\boldsymbol{d}}] \\ & y = \mathrm{f}(\boldsymbol{x}) \\ & \boldsymbol{x} \in \mathbb{Z}^n_+ \\ & y \in Y \end{aligned}$$

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Core Evaluation Questions

- implemented generic mastercuts as a new interface in GCG, Gamrath 2010
- implemented component bound branching using this interface

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1. Vanderbeck's Generic Branching vs. Component Bound Branching Rule
2. Dual Value Stabilization: Root-Only vs. Full-Tree

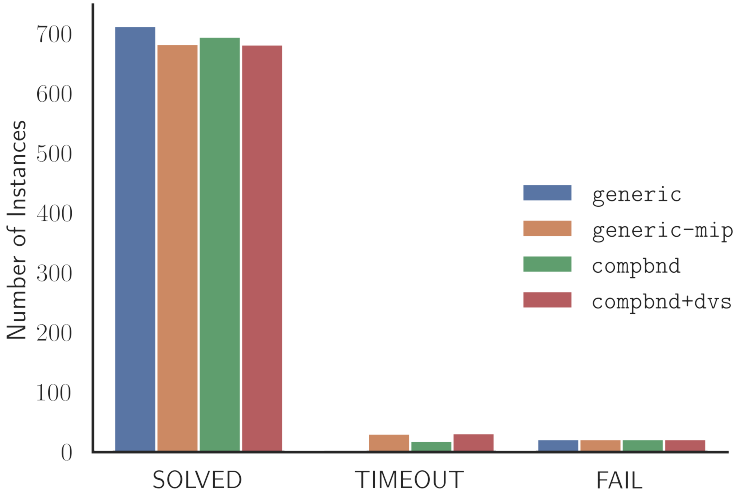
Core Evaluation Questions

- implemented generic mastercuts as a new interface in `GCG`, Gamrath 2010
- implemented component bound branching using this interface

1. Vanderbeck's Generic Branching vs. Component Bound Branching Rule
2. Dual Value Stabilization: Root-Only vs. Full-Tree

- extracted 736 instances from the structured Integer Programming Library (`strIPlib`), Bastubbe *et al.* 2025

Evaluation · Solving Status



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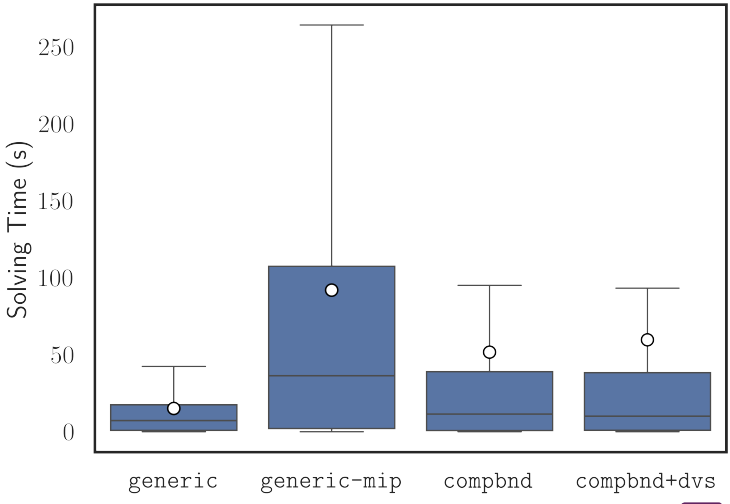
Evaluation · Solving Times



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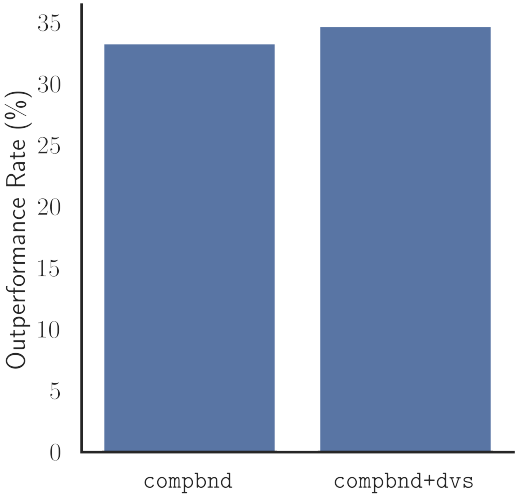
Evaluation · Outperformance Rate over `generic`



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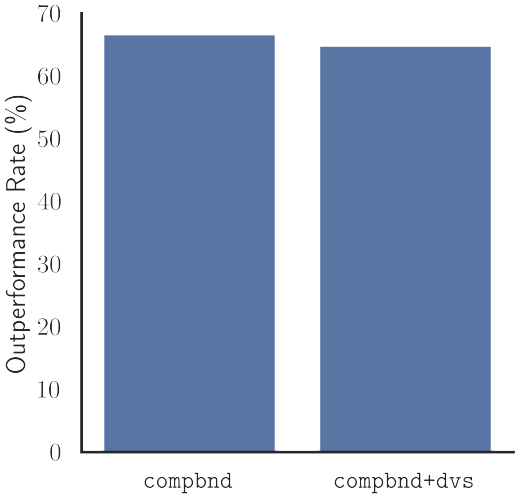
Evaluation · Outperformance Rate over `generic-mip`



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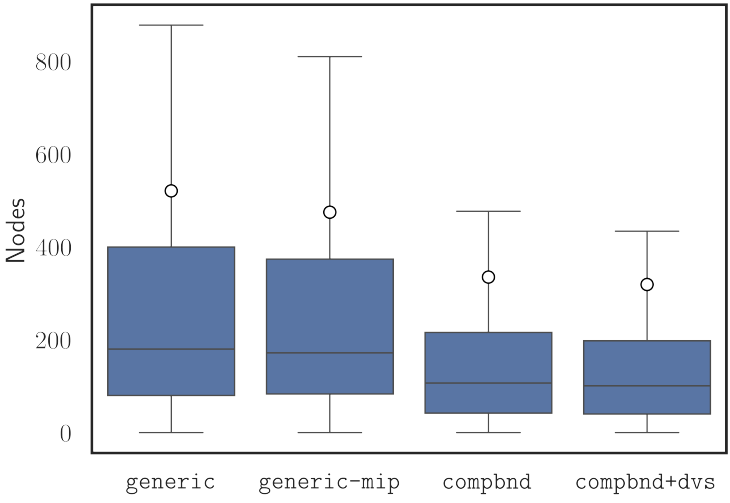
Conclusion

- GCG has a new interface for generic master constraints
- GCG has a new branching rule operating on component bound sequences

→ opens up new possibilities for future research (e.g., master separators)

Appendix



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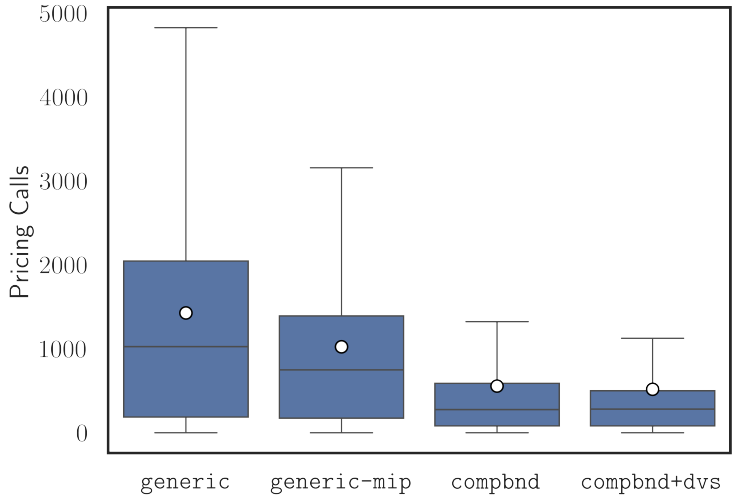

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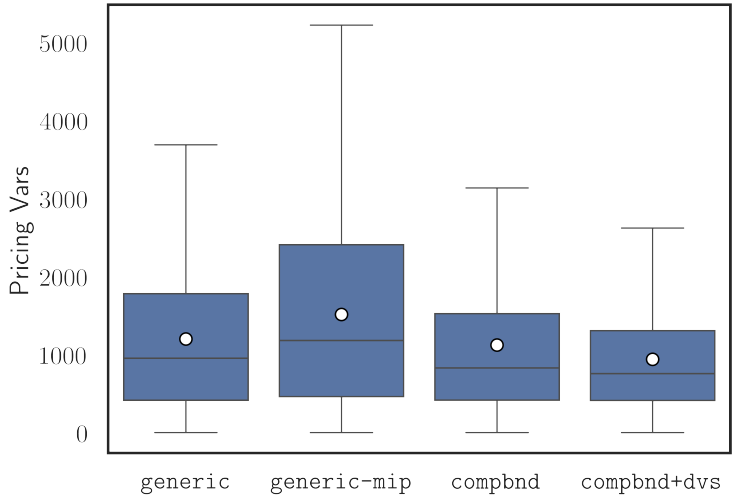

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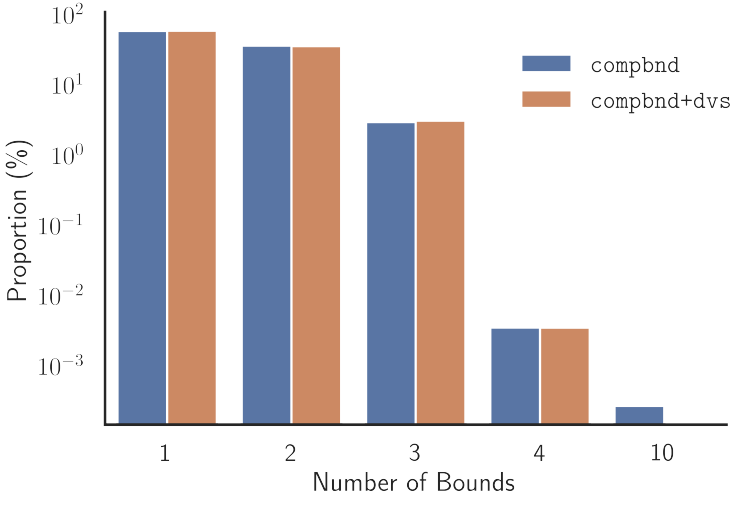

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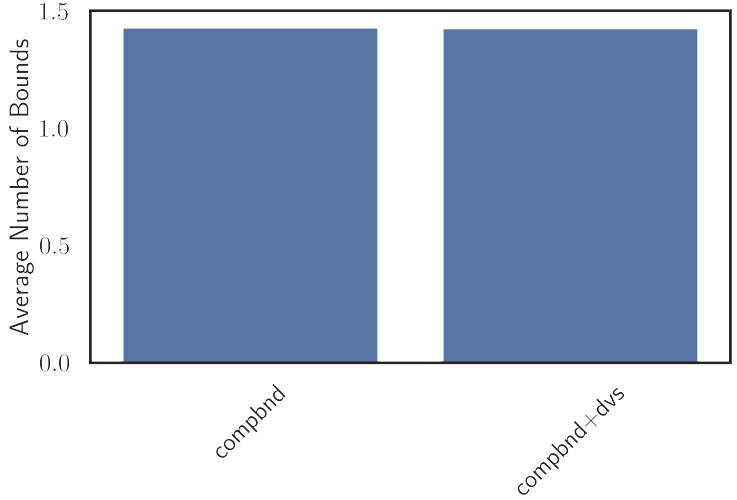

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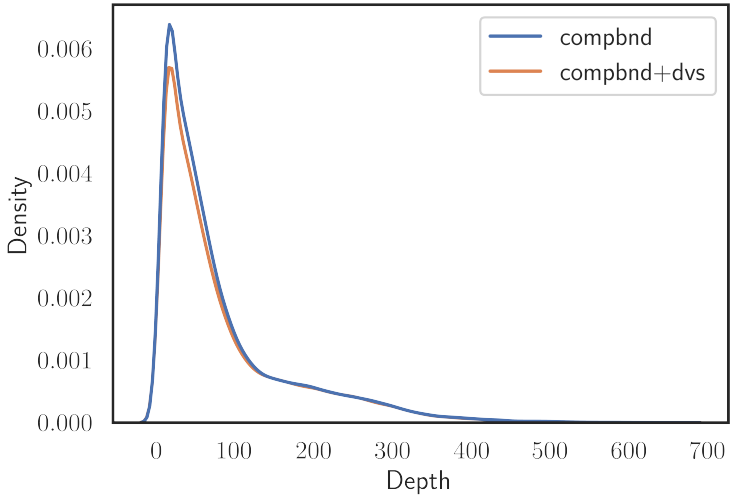

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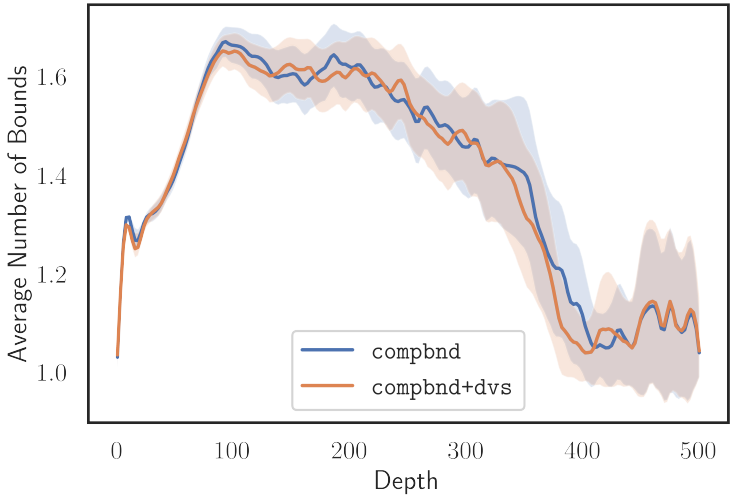

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References I

Achterberg, Tobias (2007). "Constraint integer programming".

Achterberg, Tobias, Thorsten Koch, and Alexander Martin (2005). "Branching rules revisited". *Operations Research Letters* 33.1, pp. 42–54.

Bastubbe, Michael, Marco E Lübbecke, and Jonas T Witt (2018). "A Computational Investigation on the Strength of Dantzig-Wolfe Reformulations". *17th International Symposium on Experimental Algorithms (SEA 2018)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik.

Bastubbe, Michael *et al.* (2025). "strIPlib: Structured Integer Programming Library".

Chappell, Nathan (2019). "Minkowski-Weyl Theorem".

Dantzig, George B and Mukund N Thapa (1997). *The simplex method*. Springer.

Desrosiers, Jacques *et al.* (June 2024). *Branch-and-Price*. Les Cahiers du GERAD G-2024-36. GERAD, Montréal QC H3T 2A7, Canada: Groupe détudes et de recherche en analyse des décisions, pp. 1–689. eprint: `https://www.gerad.ca/papers/G-2024-36.pdf?locale=fr`. published.

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References II

Gamrath, Gerald (2010). "Generic branch-cut-and-price". MA thesis.

Matouek, Jií and Bernd Gärtner (2007). *Understanding and using linear programming*. Vol. 1. Springer.

Pessoa, Artur *et al.* (2013). "In-out separation and column generation stabilization by dual price smoothing". *Experimental Algorithms: 12th International Symposium, SEA 2013, Rome, Italy, June 5-7, 2013. Proceedings 12*. Springer, pp. 354–365.

Pessoa, Artur *et al.* (2018). "Automation and combination of linear-programming based stabilization techniques in column generation". *INFORMS Journal on Computing* 30.2, pp. 339–360.

Ploskas, Nikolaos and Nikolaos Samaras (2014). "Pivoting rules for the revised simplex algorithm". *Yugoslav Journal of Operations Research* 24.3, pp. 321–332.

Schmickerath, Marcel (2012). "Experiments on Vanderbeck's generic Branch-and-Price scheme".

References III

Vanderbeck, François (2011). "Branching in branch-and-price: a generic scheme". *Mathematical Programming* 130, pp. 249–294.

Vanderbeck, François and Laurence A Wolsey (1996). "An exact algorithm for IP column generation". *Operations research letters* 19.4, pp. 151–159.

— (2010). *Reformulation and decomposition of integer programs*. Springer.

Witt, Jonas T (2013). "Separation of Generic Cutting Planes in Branch-and-Price". PhD thesis. Master's thesis. RWTH Aachen University.

Wolsey, Laurence A and George L Nemhauser (2014). *Integer and combinatorial optimization*. John Wiley & Sons.

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