BinomialModel

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1 The Binomial Model

The usefulness of the Binomial model depends on its ability to provide a reasonable representation of the stock price distribution.

1.1 The Random Walk Model

Imagine flipping a coin repeatedly. Let the random variable Y denote the outcome of the flip. If the coin lands a head, $Y = 1$. If the coin lands a tail, $Y = -1$. If the probability of a head is 50%, we say it is a fair coin. After n flips, with the i^{th} flip denoted Y_i, the cumulative total, Z_n, is

$$Z_n = \sum_{i=1}^{n} Y_i$$

The more times we flip, on average, the farther we will move from where we start. If you get a head on the first flip you move to $+1$, and as far as the remaining flips are concerned, *this is your new starting point*. After the second flip, you will either be at 0 or $+2$. If you are a zero it is like starting over, however, if you are at $+2$, you are starting at $+2$. Continuing in this way, your average distance form the starting point increases with the number of flips.

Another way to represent the process followed by Z_n is in terms of the *change* in Z_n:

$$Z_n - Z_{n-1} = Y_n$$

We can represent this more explicitly as

$$\text{Heads:} \quad Z_n - Z_{n-1} = +1$$

$$\text{Tails:} \quad Z_n - Z_{n-1} = -1$$

With heads the change is $+1$ and with tails the change is -1. This is a simple version of a random walk.

The idea that prices should follow a random walk was introduced by Samuelson (1965). In efficient markets, an asset price should reflect all available information. By definition, new information is a surprise. In response to new information the price is equally likely to move up or down, as with a coin flip. The price after a period of time is the initial price plus the cumulative up and down movements due to informational surprises.

1.2 Modeling Stock Prices as Random Walks

The idea that stock prices should move up or down makes sense; however, the description of a random walk in the previous section is not a satisfactory description of stock price movements. Suppose we take the random walk model literally. Assume the beginning stock price is $100, and the stock price will move up or down $1 each time we flip the coin (get an informational surprise). There are at least three problems with this model:

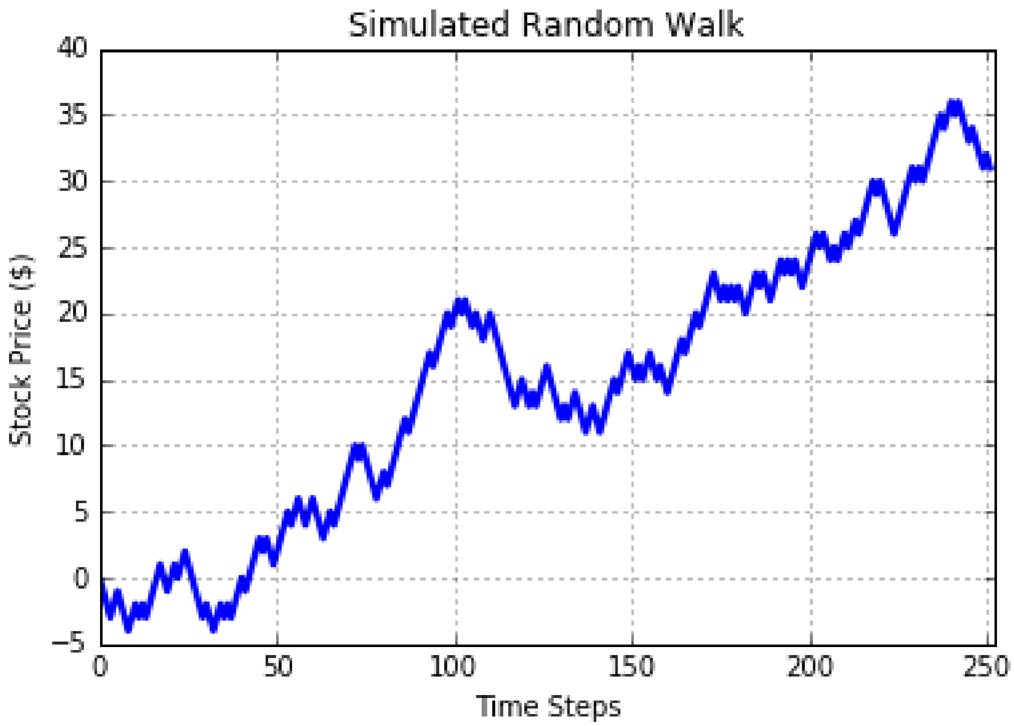
1. If by chance we get enough cumulative down movements, the stock price will become negative. Stockholders have limited liability, so a stock price should never be negative.
2. The magnitude of the move ($1) should depend upon how quickly the coin flips occur and the level of the stock price. If we flip coins once a second, $1 moves are excessive; in real life, a $100 stock will not typically have 60 $1 up or down moves in 1 minute. Also, if a $1 move is appropriate for a $100 stock, it likely isn't appropriate for a $5 stock.
3. The stock on average should have a positive return. The random walk model taken literally does not permit this.

Let's simulate a simple random walk after this model:

```
In [1]: %matplotlib inline
```

```
In [2]: import numpy as np
        import matplotlib.pyplot as plt

        def SimulateBinom(S, r, v, T, h):
            np.random.seed() #12345 #5555
            n = int(T / h)
            prob = np.random.uniform(0, 1, n)
            spot = np.zeros((n,))
            u = np.exp((r * h) + v * np.sqrt(h))
            d = np.exp((r * h) - v * np.sqrt(h))
            spot[0] = S

            for t in range(1, n):
                if prob[t] >= 0.5:
                    spot[t] = spot[t-1] + 1
                else:
                    spot[t] = spot[t-1] - 1

            return spot

        ## Main
        ## See Figure 11.5 in McDonald text on page 332
        S = 0
        r = 0.06
        v = 0.3
        T = 1.0
        h = T / 252.0
```

```
spot = SimulateBinom(S, r, v, T, h)
t = range(int(T / h))

plt.plot(spot, 'b', linewidth=2.5)
plt.title("Simulated Random Walk")
plt.xlabel("Time Steps")
plt.ylabel("Stock Price ($)")
plt.xlim((0,252))
plt.grid(True)
plt.show()
```



Here is Figure 11.5 from the textbook:

It turns out that the Binomial model is a variant of the random walk model that solves all of these problems at once. The Binomial model assumes that *continuously compounded returns are a random walk with drift*.

1.3 The Binomial Tree and Lognormality

The Binomial model stock price dynamics follows:

$$S_{t+h} = S_t e^{(r-\delta)h \pm \sigma\sqrt{h}}$$

Taking logs, we obtain

$$\ln\left(S_{t+h}/S_t\right) = (r-\delta)h \pm \sigma\sqrt{h}$$

Since $\ln(S_{t+h}/S_t)$ is the continuously compounded return from t to $t+h$, $r_{t,t+h}$, the Binomial model is simply a particular way to model the continuously compounded return. That return has two parts:

- one is $[(r-\delta)h]$
- the other is uncertain $(\pm\sigma\sqrt{h})$ (generates the up and down prices)

Let's see how the Binomial model solves all three problems with the random walk model:

1. The stock price cannot become negative. Even if we move down the Binomial tree many times in a row, the resulting large, negative, continuously compounded return will give us a positive price.
2. As stock price moves occur more frequently, h gets smaller, therefore up and down moves get smaller. By construction, annual volatility is the same no matter how many Binomial periods there are. Since returns follow a random walk, the percentage price change is the same whether the stock price is \$100 or \$5.
3. There is a $(r-\delta)h$ term, and we can choose the probability of an up move, so we can guarantee that the expected change in the stock price is positive.

1.3.1 Lognormality

- The Binomial tree approximates a lognormal distribution, which is commonly used to model stock prices.
- The lognormal distribution is the probability distribution that arises from the assumption that continuously compounded returns on the stock are normally distributed.
- With the lognormal distribution, the stock price is positive, and the distribution is skewed to the right, that is, there is a chance that extremely high stock prices will occur.

The Binomial model implicitly assigns probabilities to the various nodes:

1.4 Coding the Binomial Model for a European Option

The fact that the Binomial tree implicitly assigns probabilities to the various nodes gives us a computational strategy for implementing the Binomial model for a European option.

When we traverse the Binomial tree, we are implicitly adding up Binomial random return components of $(r-\delta)h \pm \sigma\sqrt{h}$. In the limit (as $n \to \infty$ or, the same thing, $h \to 0$), the sum of Binomial random variables is normally distributed, which means that the stock price is lognormally distributed. We will discuss this more in Chapters 18 and 20.

Suppose that a Binomial tree has n periods and the risk-neutral probability of an up move is p^*. To reach the top node, we must go up n times in a row, which occurs with a probability of $(p^*)^n$. The price at the top node is Su^n. There is only one path through the tree by which we can reach the top node. To reach the first node below the top node, we must go up $n-1$ times and down once, for a probability of $(p^*)^{n-1} \times (1-p^*)$. The price at the top node is $Su^{n-1}d$. Since the single down move can occur in any of the n periods, there are n ways this can happen. The probability of reaching the i^{th} node below the top is $(p^*)^{n-i} \times (1-p^*)^i$. The price at this node is $Su^{n-i}d^i$. The number of ways to reach this node is:

$$\text{Number of ways to reach } i^{th} \text{ node} = \frac{n!}{(n-i)!i!} = \binom{n}{i}$$

where $n! = n \times (n-1) \times \cdots \times 1$.

We can construct the implied probability distribution in the Binomial tree by plotting the stock price at each final period node, $Su^{n-i}d^i$, against the probability of reaching that node.

The following graph compares the probability distribution for a 25-period Binomial tree with the corresponding lognormal distribution.

```
In [7]: import numpy as np
        from scipy.stats import binom

        def CallPayOff(Spot, Strike):
            return np.maximum(Spot - Strike, 0.0)

        def EuropeanBinomial(S, X, r, u, d, T):
            numSteps = 2
            numNodes = numSteps + 1
            spotT = 0.0
            callT = 0.0
            pu = (np.exp(r*T) - d) / (u - d)
            pd = 1 - pu

            for i in range(numNodes):
                spotT = S * (u ** (numSteps - i)) * (d ** (i))
                callT += CallPayOff(spotT, X) * binom.pmf(numSteps - i, numSteps, p
            callPrice = callT * np.exp(-r * T)

            return callPrice

        def main():
            S = 41
            X = 40
            r = 0.08
            T = 1.0
            v = 0.30
            u = 1.2
            d = 0.8

            callPrice = EuropeanBinomial(S, X, r, u, d, T)
            print("The Two Period European Binomial Price is = {0:.4f}".format(call

        main()
```

```
The Two Period European Binomial Price is = 8.8157
```

```
In [ ]:
```