Stochastic Volatility

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In this document we review the stochastic volatility model of the Heston (1993) model. Relative to Black-Scholes we now have a bivariate system of stochastic differential equations:

with .

In the original paper, Heston was concerned with deriving a semi-closed form solution for the option pricing problem. Our concern is how to use Monte Carlo simulation to simulate from this system of equations under the risk-neutral density.

The first problem we face is how to simulate from these equations. There are a few options:

1. For the Heston model, Glasserman and Kaya (2004) show how to do this for the exact transition law of the process. It turns out though that this involves integration of a characteristic function expressed in terms of Bessel functions. It is very time consuming, and only works in the special case of Heston. That is, many other related stochastic volatility models offer no way to do exact simulation.

We therefore, turn to discretization schemes:

1. Euler discretization.
2. Milstein discretization.

See chapter 5 of Glasserman (2004).

Let's take a look at how this works:

1. Take the case of , where is a Brownian motion.
2. We simulate over the time interval , which we assume to be discretized as . Where the time increments are equally spaced with width .
3. We can integrate from to and get

This equation is the starting point for any discretization scheme. At time , the value of is known, and we wish to obtain the next value .

## The Euler Scheme

The simplest way to discretize the above integral equation is called the Euler approximation. This is equivalent to approximating the integrals using the left-point rule. Hence, the first integral is approximated as the product of the integrand at time , and the integration range

**NB:** recall the the left-point rule states that:

We can use the left-point rule since at time the value of is known.

Similarly, the second integral is approximated as

Since and are indentical in distribution where .

Thus the Euler discretization for

is

Now let's look at the Euler discretization of the Black-Scholes model:

1. We have
2. Applying the above we get:

An alternative is to work with . Then by It's Lemma we have

Euler discretization gives us

So that

where .

### Euler Scheme for Heston's Stochastic Volatility

Again, for Heston dynamics are described by the bivariate process:

The stochastic differential equation (SDE) in integral form is:

Again we use the left-point rule

where .

This gives us

To avoid negative values, we can set to . This is called the "full truncation scheme." The "reflection scheme" is an alternative, which sets to its absolute value .

The process for is:

Euler discretization leads to:

where . We end up with:

with .

### The Process for

Let . Then

Or in integral form

Euler discretization gives us

Hence, for we have

Again, to avoid negative variances we need to replace with either or .

To generate and with correlation , we do the following:

1. First generate two independent standard normals and .
2. Set .
3. Set

Let's see how to do this in Python:

import numpy as np  
rho = 0.85  
z1 = np.random.normal(size=10000)  
z2 = np.random.normal(size=10000)  
zv = z1  
zs = rho \* z1 + np.sqrt(1 - (rho \* rho)) \* z2  
np.corrcoef(zv,zs)

Now we can use Python to actually carry out the Monte Carlo simulation for a stochastic volatility model:

import numpy as np