DEPARTMENT OF ECONOMICS
FACULTY OF SOCIAL SCIENCES
OBAFEMI AWOLOWO UNIVERSITY, ILE-IFE, NIGERIA
SSC106: MATHEMATICS FOR SOCIAL SCIENCES II
RAIN SEMESTER EXAMINATION (2015/2016 SESSION)

INSTRUCTIONS:

- Attempt all questions in **Section A;**
- Answer any two questions from **Section B.**
- Show all workings clearly

Time allowed: 2 hours

SECTION A

1. *(a)* What is a function?

   (b) Distinguish between the functions:
   $y = x^b$, $b \neq 0$ and $y = b^x$, $b > 1$

2. Matrix was developed by an English mathematician called ……………….. in the year ………………

3. If $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 2^2 & 2^2 & 0 & 0 \\ 2^3 & 2^3 & 2^3 & 0 \\ 2^4 & 2^4 & 2^4 & 2^4 \end{pmatrix}$,

(a) Spur of $A =$ ....

(b) $|A| =$ ....

(c) What type of matrix is A? ....

4. Differentiate with respect to x:

(a) $y = \log(\sin x + \cos x)$;

(b) $y = ax^n + bx^m$.

5. Use Euler's theorem to determine the degree of homogeneity of the functions:

(a) $f(x, y) = x^3 + 2x^2y + y^3$;

(b) $Q = AL^{\alpha}K^{\beta}$.

6. *(a)* What is partial differentiation?

(b) $f(x, y) = x^3 + 2x^2y + 3y^2$, compute Z_{xx}, Z_{xy}, Z_{yx} and Z_{yy}.

7. *(a)* What is integration?

(b) Distinguish between definite and indefinite integral.

8. Evaluate the following:

(a) $\displaystyle\int_1^5 \frac{dx}{x-1}$;

(b) $\displaystyle\int_0^1 e^{2t}\ dt$;

(c) $\displaystyle\int_1^5 \frac{2x+7}{x^2+7x-4}\ dx$;

9. Find the relative optima of the function:
$y = 4x - x^2$

10. Suppose the total cost of a firm is $120q - q^2 + 0.02q^3$ and the demand function: $P = 114 - 0.25q$, obtain MC and MR.

SECTION B

1. *(a)* Outline the allied calculus conditions for the optimization of the function:

$$y = f(x)$$

(b) Given the Total Revenue (TR) and Total Cost (TC) functions are:

$$TR(q) = 24q - 3q^2$$

and

$$TC(q) = 100 + 4q + 2q^2$$

(i) Determine the level of output that maximizes its profit;

(ii) Test whether the firm actually maximizes profit at this level of output.

(c) Max $Z = xy + 2x$;

x, y subject to the constraint:

$$4x + 2y = 60$$

2. *(a)* A matrix can be classified using:

(i) ……………………….

(ii) ……………………….

(iii) ……………………….

With appropriate examples; list two types of matrices under each classification.

(b) Given:

$$7x - y - z = 0$$
$$10x - 2y + z = 8$$
$$6x + 3y - 2z = 7$$

Find x, y and z using matrix inverse approach.

OR

(c) Given:

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & 9 \\ 7 & 6 & 1 \end{pmatrix} \text{ and } E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

Show that $|EA| = |E||A|$.

3. *(a)* Outline any three techniques for integrating complex functions.

(b) Using any of the techniques outline in *(a)* above; find:

(i) $$\int \frac{3x - 5}{(x-1)(x-2)}\, dx$$

(ii) $$\int (3x+5)^{10}\, dx$$

(iii) $$\int_1^3 4x^3 e^{x^4}\, dx$$

SOLUTION TO THE PAST QUESTIONS

All topics needed here have been thoroughly treated through and through, all you need to do is to sit down and answer all these questions like a joke, you are not expected to start reading from here but from the topics, you make a joke of yourself when you read the concept of any subject starting from the side of the past question; more so, this text isn't a past question but a complete guide to lead you to a straight A in your SSC106 exam.

Hence, keep calm and check the solutions to these past questions one-by-one;

GOOD LUCK AND GOD'S BEST!

SOLUTION TO THE SSC106 EXAMINATION 2015/2016 ACADEMIC SESSION

The instruction is you answer all questions in the **Section A** and only two questions from Section B, we'll be answering everything in both sections in this book though; keep calm and move with me;

Remember every singular solution here, you can get the full concept by reading this book and not the past question section exclusively;

SECTION A

Question 1

(a) What is a function?

(b) Distinguish between the functions: $y = x^b, b \neq 0$ and $y = b^x, b > 1$

(a)

A function is a mathematical relationship between sets of inputs and a set of permissible outputs with each input related to one output. It is a rule which satisfies a particular type of relation between two or more variables.

(b)

This is a difference that was stated clearly in the notes; Here;

In:

$$y = x^b, b \neq 0$$

The above is a power function; where x is raised to a constant power, b.

In:

$$y = b^x, b > 1$$

While the above is an exponential functions, where a constant, b is raised to the power of x, the independent variable.

Question 2

Matrix was developed by an English mathematician called **James Sylvester in the year 1850.**

Question 3

(a)

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 2^2 & 2^2 & 0 & 0 \\ 2^3 & 2^3 & 2^3 & 0 \\ 2^4 & 2^4 & 2^4 & 2^4 \end{pmatrix}$$

Here is a matrix;

The spur (or trace) of a matrix is the sum of elements on its main diagonal. Hence;

$$\text{Spur of } A = 2 + 2^2 + 2^3 + 2^4$$

$$\text{Spur of } A = 2 + 4 + 8 + 16$$

$$\text{Spur of } A = 30$$

(b)

In matrices, the determinant of a 4×4 matrix should be really hell, but for this type of matrix, a triangular matrix, the determinant is equivalent to the product of the elements of its main diagonal, hence, here;

$$|A| = 2 \times 2^2 \times 2^3 \times 2^4$$

$$|A| = 2^{1+2+3+4} = 2^{10}$$

$$|A| = 1024$$

(c)

The matrix is a triangular matrix, more specifically; it is a **lower triangular matrix.**

Ensure you write the more specific definition though.

Question 4

Differentiate with respect to x:

(a) $y = \log(\sin x + \cos x)$;

(b) $y = ax^n + bx^m$.

(a)

A substitution is needed;

$$u = \sin x + \cos x$$

$$\frac{du}{dx} = \cos x + (-\sin x)$$

$$\frac{du}{dx} = \cos x - \sin x$$

Hence,

$$y = \log u$$

Log without base is log to base 10;

$$y = \log_{10} u$$

By rules;

$$\frac{dy}{du} = \frac{1}{u \ln 10}$$

Hence,

Chain rule;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Hence,

$$\frac{dy}{dx} = \frac{1}{u \ln 10} \times (\cos x - \sin x)$$

Return u;

$$\frac{dy}{dx} = \frac{\cos x - \sin x}{(\sin x + \cos x)\ln 10}$$

(b)

$$y = ax^n + bx^m$$

Since we're differentiating with respect to x, both n and m will be nothing but constant, hence, this is a case of constant power.

Here;

$$\frac{dy}{dx} = n \times ax^{n-1} + m \times bx^{m-1}$$

$$\frac{dy}{dx} = nax^{n-1} + mbx^{m-1}$$

Question 5

Use Euler's theorem to determine the degree of homogeneity of the functions:

(a) $f(x, y) = x^3 + 2x^2y + y^3$;

(b) $Q = AL^\alpha K^\beta$.

What is Euler's theorem?

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$$

Where n is the degree of homogeneity; Hence;

(a)

$$f(x, y) = x^3 + 2x^2y + y^3$$

$$\frac{\partial f}{\partial x} = 3x^{3-1} + 2 \times 2x^{2-1}y + 0$$

$$\frac{\partial f}{\partial x} = 3x^2 + 4xy$$

$$\frac{\partial f}{\partial y} = 0 + 1 \times 2x^2y^{1-1} + 3y^{3-1}$$

$$\frac{\partial f}{\partial y} = 2x^2 + 3y^2$$

Hence,

$$x(3x^2 + 4xy) + y(2x^2 + 3y^2) = nf(x, y)$$

Expand!

$$3x^3 + 4x^2y + 2x^2y + 3y^3 = n(x^3 + 2x^2y + y^3)$$

Simplify;

$$3x^3 + 6x^2y + 3y^3 = n(x^3 + 2x^2y + y^3)$$

Factorization can be done on the left hand side;

$$3(x^3 + 2x^2y + y^3) = n(x^3 + 2x^2y + y^3)$$

Hence, by comparison;

$$n = 3$$

(b)

$$Q = AL^{\alpha}K^{\beta}$$

Here,

It's not well stated, however, we'll make some assumptions: that Q is a multivariate function of L and K with A, α and β being constants.

Hence, Euler's theorem here will be;

$$LQ_L + KQ_K = nQ$$

Where Q_L and Q_K are the derivatives of Q with respect to L and K respectively, I expect you to know that!

Hence,

$$Q_L = \alpha \times AL^{\alpha-1}K^{\beta}$$

$$Q_L = \alpha AL^{\alpha-1}K^{\beta}$$

$$Q_K = \beta \times AL^{\alpha}K^{\beta-1}$$

$$Q_K = \beta AL^{\alpha}K^{\beta-1}$$

Hence,

$$L\left(\alpha AL^{\alpha-1}K^{\beta}\right) + K\left(\beta AL^{\alpha}K^{\beta-1}\right) = nQ$$

Hence,

$$L\left(\alpha AL^{\alpha-1}K^{\beta}\right) + K\left(\beta AL^{\alpha}K^{\beta-1}\right) = n\left(AL^{\alpha}K^{\beta}\right)$$

Indices!!!!!!!

$$\alpha AL^{\alpha-1+1}K^{\beta} + \beta AL^{\alpha}K^{\beta-1+1} = n\left(AL^{\alpha}K^{\beta}\right)$$

Simplify;

$$\alpha AL^{\alpha}K^{\beta} + \beta AL^{\alpha}K^{\beta} = n\left(AL^{\alpha}K^{\beta}\right)$$

Hence, $AL^{\alpha}K^{\beta}$ is common, factorize!

$$AL^{\alpha}K^{\beta}(\alpha + \beta) = n\left(AL^{\alpha}K^{\beta}\right)$$

$$(\alpha + \beta)AL^{\alpha}K^{\beta} = n\left(AL^{\alpha}K^{\beta}\right)$$

By comparison;

$$n = \alpha + \beta$$

Question 6

(a) What is partial differentiation?

(b) $f(x, y) = x^3 + 2x^2y + 3y^2$, compute Z_{xx}, Z_{xy}, Z_{yx} and Z_{yy}.

Very very simple!

(a)

Partial differentiation is the differentiation of multivariable functions which yields several partial derivatives (differential coefficients) where a partial derivative of a function of several variables (the multivariate functions) is its derivative with respect to one of the variables that constitute the function with other variables temporarily kept constant.

(b)

$$f(x, y) = x^3 + 2x^2y + 3y^2$$

Little error in the question, we are told to calculate Z_{xx}, Z_{xy}, Z_{yx} and Z_{yy}; however, the function isn't even stated in terms of Z; hence, here, we'll just assume the question is:

$$Z = x^3 + 2x^2y + 3y^2$$

Hence,

$$Z_x = 3 \times x^{3-1} + 4 \times x^{2-1}y + 0$$

$$Z_x = 3x^2 + 4xy$$

And

$$Z_y = 0 + 1 \times 2x^2y^{1-1} + 2 \times 3y^{2-1}$$

$$Z_y = 2x^2 + 6y$$

Hence, going for the second order partial derivatives;

$$Z_{xx} = (Z_x)_x$$

$$Z_{xx} = \frac{\partial Z}{\partial x}(3x^2 + 4xy)$$

$$Z_{xx} = 2 \times 3x^{2-1} + 1 \times 4x^{1-1}y$$

$$Z_{xx} = 6x + 4y$$

Also

$$Z_{xy} = (Z_x)_y$$

$$Z_{xy} = \frac{\partial Z}{\partial y}(3x^2 + 4xy)$$

$$Z_{xy} = 0 + 1 \times 4xy^{1-1}$$

$$Z_{xy} = 4x$$

Then;

$$Z_{yx} = \left(Z_y\right)_x$$

$$Z_{yx} = \frac{\partial Z}{\partial x}(2x^2 + 6y)$$

$$Z_{yx} = 2 \times 2x^{2-1} + 0$$

$$Z_{yx} = 4x$$

Lastly;

$$Z_{yy} = \left(Z_y\right)_y$$

$$Z_{yy} = \frac{\partial Z}{\partial y}(2x^2 + 6y)$$

$$Z_{yy} = 0 + 1 \times 6y^{1-1}$$

$$Z_{yy} = 6$$

Question 7

(a) What is integration?

(b) Distinguish between definite and indefinite integral.

(a)

Integration is the process of finding a function from its derivative. It is the process of finding the area under a curve for any given function.

(b)

The difference between the indefinite integral and the definite integral is that the **indefinite integral** gives the general form of the anti-derivative of a function while the **definite integral** gives the area under a curve between two given points and is the value gotten by evaluating the integral from the two limits. The **indefinite integral** contains an arbitrary constant while the **definite integral** doesn't contain an arbitrary constant.

Question 8

Evaluate the following:

(a) $\displaystyle\int_1^5 \frac{dx}{x-1}$;

(b) $\displaystyle\int_0^1 e^{2t}\, dt$;

(c) $\displaystyle\int_1^5 \frac{2x+7}{x^2+7x-4}\, dx$;

(a)

$$\int_1^5 \frac{dx}{x-1}$$

Simple case of integration!

$$\int \frac{dx}{x-1}$$

Here, substitution!

$$u = x - 1$$

$$\frac{du}{dx} = 1$$

$$dx = du$$

Hence,

$$\int \frac{du}{u}$$

By rules;

$$\ln u + C$$

Return u;

$$\ln(x - 1) + C$$

Definite;

$$\int_1^5 \frac{dx}{x - 1}$$

Hence, we have;

$$[\ln(x - 1)]_1^5$$

Since the arbitrary constant is always not needed in the definite integral;

Hence,

$$\ln(5-1)-\ln(1-1)$$

We have;

$$\ln 4 - \ln 0$$

Unfortunately; ln 0 doesn't exist and hence, we'll be leaving our answer like that.

(b)

$$\int_0^1 e^{2t}\ dt$$

Simple case of integration!

$$\int e^{2t}\,dt$$

Here, substitution!

$$u = 2t$$

$$\frac{du}{dx} = 2$$

$$dx = \frac{du}{2}$$

Hence,

$$\int e^u \frac{du}{2}$$

$$\frac{1}{2}\int e^u du$$

By rules;

$$\frac{1}{2}(e^u) + C$$

Return u;

$$\frac{1}{2}e^{2t} + C$$

Definite;

$$\int_0^1 e^{2t}\, dt$$

Hence, we have;

$$\left[\frac{1}{2}e^{2t}\right]_0^1$$

Since the arbitrary constant is always not needed in the definite integral;

Hence,

$$\frac{1}{2}e^{2\times1} - \frac{1}{2}e^{2\times0}$$

We have;

$$\frac{1}{2}(e^{2} - e^{0})$$

$$\frac{1}{2}(7.389 - 1)$$

$$3.195$$

e^2 was gotten from a calculator, e^0 is 1.

(c)

$$\int_{1}^{5} \frac{2x + 7}{x^2 + 7x - 4}\,dx$$

Another case of substitution, a type:

$$\frac{f'(x)}{f(x)}$$

$$\int \frac{2x + 7}{x^2 + 7x - 4}\,dx$$

Hence,

$$u = x^2 + 7x - 4$$

$$\frac{du}{dx} = 2x + 7$$

Hence,

$$dx = \frac{du}{2x + 7}$$

We have;

$$\int \frac{2x + 7}{u} \frac{du}{2x + 7}$$

$2x + 7$ cancels out;

$$\int \frac{du}{u}$$

By rules;

$$\ln u + C$$

Return u;

$$\ln(x^2 + 7x - 4) + C$$

Definite;

$$\int_1^5 \frac{2x + 7}{x^2 + 7x - 4}\, dx$$

Hence, we have;

$$[\ln(x^2 + 7x - 4)]_1^5$$

Since the arbitrary constant is always not needed in the definite integral;

Hence,

$$\ln(5^2 + 7(5) - 4) - \ln(1^2 + 7(1) - 4)$$

We have;

$$\ln 56 - \ln 4$$

From our calculator, we have:

$$4.0254 - 1.3863 = 2.6391$$

Question 9

Find the relative optima of the function:
$y = 4x - x^2$

Finding the relative optima, we're to completely optimize the function and determine the real nature.

$$y = 4x - x^2$$

$$\frac{dy}{dx} = 1 \times 4x^{1-1} - 2x^{2-1}$$

$$\frac{dy}{dx} = 4 - 2x$$

At stationary point;

$$\frac{dy}{dx} = 0$$

Hence,

$$4 - 2x = 0$$

$$x = 2$$

To determine the nature;

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\frac{d^2y}{dx^2} = 0 - 1 \times 2x^{1-1}$$

$$\frac{d^2y}{dx^2} = -2$$

Hence, since the second derivative is negative, the point $x = 2$ is a maximum point.

As for the maximum value, substitute $x = 2$ into the main function.

$$y = 4x - x^2$$

$$y = 4(2) - (2)^2$$

$$y = 4$$

Hence, the relative optima is $y = 4$, corresponding to $x = 2$ (a maximum value)

Question 10

Suppose the total cost of a firm is $120q - q^2 + 0.02q^3$ and the demand function: $P = 114 - 0.25q$, obtain MC and MR.

Total cost, $C(q)$, demand function, P

From formula, the revenue $R(q)$ is:

$$R(q) = P \times q$$

Hence,

$$R(q) = (114 - 0.25q)q$$

$$R(q) = 114q - 0.25q^2$$

MC, is the marginal cost, $C'(q)$

$$C(q) = 120q - q^2 + 0.02q^3$$

Hence,

$$\begin{aligned} C'(q) &= MC \\ &= 120q^{1-1} - 2 \times q^{2-1} + 3 \times 0.02q^{3-1} \end{aligned}$$

Hence,

$$\boldsymbol{MC = 120 - 2q + 0.06q^2}$$

MR, is the marginal revenue, $R'(q)$

$$R(q) = 114q - 0.25q^2$$

Hence,

$$R'(q) = MR = 114q^{1-1} - 2 \times 0.25q^{2-1}$$

Hence,

$$\boldsymbol{MR = 114 - 0.5q}$$

SECTION B

Larger questions over here, let's solve them all in a bit.

Question 1

(a) Outline the allied calculus conditions for the optimization of the function:

$$y = f(x)$$

(b) Given the Total Revenue (TR) and Total Cost (TC) functions are:

$$TR(q) = 24q - 3q^2$$

and

$$TC(q) = 100 + 4q + 2q^2$$

(iii) Determine the level of output that maximizes its profit;

(iv) Test whether the firm actually maximizes profit at this level of output.

(c) Max $Z = xy + 2x$;

x, y subject to the constraint:

$$4x + 2y = 60$$

(a)

The conditions are in your notes;

For stationary points on a given function, $y = f(x)$; where:

$$\frac{dy}{dx} = 0 \quad \text{or} \quad f'(x) = 0$$

If;

$$\frac{d^2y}{dx^2} > 0, f''(x) > 0$$

Then the stationary point is **a minimum point.**
If;

$$\frac{d^2y}{dx^2} < 0, f''(x) < 0$$

Then the stationary point is **a maximum point.**

If;

$$\frac{d^2y}{dx^2} = 0, f''(x) = 0$$

Then the stationary point is **a point of inflexion**.

(b)

(i)

$$TR(q) = 24q - 3q^2$$

$$TC(q) = 100 + 4q + 2q^2$$

We want the maximum profit, and since we already have the revenue and cost functions; we know that at maximum profit, the marginal cost is equal to the marginal revenue.

Hence,

$$TR(q) = 24q - 3q^2$$

We have:

$$MR(q) = 1 \times 24q^{1-1} - 2 \times 3q^{2-1}$$

$$MR(q) = 24 - 6q$$

Also;

$$TC(q) = 100 + 4q + 2q^2$$

We have;

$$MC(q) = 0 + 1 \times 4q^{1-1} + 2 \times 2q^{2-1}$$

$$MC(q) = 4 + 4q$$

At maximum profit;

$$MR = MC$$

Hence,

$$24 - 6q = 4 + 4q$$

Solving for q;

$$q = 2$$

Hence, maximum profit is at $q = 2$.

(ii)

To test if profit is maximized, let's see if the profit is positive at its maximum level;
The profit function is given by:

$$P = TR - TC$$

$$P(q) = 24q - 3q^2 - (100 + 4q + 2q^2)$$

$$P(q) = 24q - 3q^2 - 100 - 4q - 2q^2$$

$$P(q) = 20q - 5q^2 - 100$$

Hence, solving for $q = 2$

$$P(2) = 20(2) - 5(2^2) - 100$$

$$P(2) = 40 - 20 - 100$$

$$P(2) = -80$$

Hence, profit is not actually maximized at that output level. This is because even at maximum profit, the profit is yielding a negative value. You should be quite familiar with that!

(c)

Max $Z = xy + 2x$;
x, y subject to the constraint:

$$4x + 2y = 60$$

Here, the objective function is:

$$xy + 2x$$

The constraint function is:

$$4x + 2y - 60 = 0$$

Note that it has been equated to zero;

Hence, using the method of Lagrangean multiplier; we have;

$$\mathcal{L}(x, y, \lambda) = xy + 2x - \lambda(4x + 2y - 60)$$

Take the first order partials;

$$\mathcal{L}_x = 1 \times x^{1-1}y + 1 \times 2x^{1-1} - \lambda(1 \times 4x^{1-1} + 0 - 0)$$

$$\mathcal{L}_x = y + 2 - 4\lambda$$

$$\mathcal{L}_y = 1 \times x \times y^{1-1} + 0 - \lambda(0 + 2 \times y^{1-1} - 0)$$

$$\mathcal{L}_y = x - 2\lambda$$

$$\mathcal{L}_\lambda = 0 + 0 - 1 \times \lambda^{1-1}(4x + 2y - 60)$$

$$\mathcal{L}_\lambda = -4x - 2y + 60$$

Equate each to zero;

$$y + 2 - 4\lambda = 0$$

$$y - 4\lambda = -2 \ldots \ldots \ldots (1)$$

$$x - 2\lambda = 0 \ldots \ldots \ldots (2)$$

$$-4x - 2y + 60 = 0$$

$$-2x - y = -30 \ldots \ldots \ldots (3)$$

(1) and (2) can be easily sorted out;

$$\mathbf{2} \times (\mathbf{2}): 2x - 4\lambda = 0$$

From (1);

$$y + 2 = 4\lambda$$

From (2);

$$2x = 4\lambda$$

Hence,

From (1) and (2);

$$y + 2 = 2x$$

Since both are equal to 4λ

Hence,

$$-2x + y = -2 \ldots\ldots\ldots (4)$$

Solving simultaneously with (3);

$$-2x - y = -30 \ldots\ldots\ldots (3)$$

$$-2x + y = -2 \ldots\ldots\ldots (4)$$

Adding;

$$-4x = -32$$

$$x = 8$$

Hence, from (3);

$$-2x - y = -30$$

Hence,

$$-2(8) - y = -30$$

$$y = 14$$

Hence, the optimal point is $x = 8$ and $y = 14$

From (2);

$$2x = 4\lambda$$

$$2(8) = 4\lambda$$

$$\lambda = 4$$

Hence, to maximize Z, find its value at the optimal point;

$$Z = xy + 2x$$

Hence, maximum Z is:

$$Z = 8(14) + 2(8)$$

$$Z = 128$$

Question 2

(a) A matrix can be classified using:

(i) ……………………….

(ii) ……………………….

(iii) ……………………….

With appropriate examples; list two types of matrices under each classification.

(b) Given:

$$7x - y - z = 0$$
$$10x - 2y + z = 8$$
$$6x + 3y - 2z = 7$$

Find x, y and z using matrix inverse approach.

OR

(c) Given:

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & 9 \\ 7 & 6 & 1 \end{pmatrix} \text{ and } E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

Show that $|EA| = |E||A|$.

(a)

A summary of the different ways we classified matrices; the three ways a matrix can be classified are:

- The relationship between its rows and columns;
- The arrangement of its elements
- A derived relationship with itself

Examples of each classification;

- The relationship between its rows and columns;

Examples are many, we need two;

  - ✓ Rectangular matrix
  - ✓ Square matrix
- The arrangement of its elements
  - ✓ Null matrix
  - ✓ Identity matrix
- A derived relationship with itself
  - ✓ Orthogonal matrix
  - ✓ Idempotent matrix

DONE!

You're told to pick one question between (b) and (c), let's see the solution to both though;

(b)

$$7x - y - z = 0$$
$$10x - 2y + z = 8$$
$$6x + 3y - 2z = 7$$

They're already arranged so let's write out the matrix equation;

$$\begin{pmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & -2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 7 \end{pmatrix}$$

$$\text{Let } A = \begin{pmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & -2 \end{pmatrix}$$

Hence; we have;

$$\begin{vmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & -2 \end{vmatrix}$$

$$7\begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} - (-1)\begin{vmatrix} 10 & 1 \\ 6 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 10 & -2 \\ 6 & 3 \end{vmatrix}$$

$$7[(-2)(-2) - (1)(3)] + 1[(10)(-2) - (1)(6)] \\ - 1[(10)(3) - (-2)(6)]$$

$$|A| = 7(1) + (-26) - 42 = -61$$

You noticed we were told to use the method of matrix inverse right?

Let's proceed to find the minors and cofactors of all the elements; as we know, it's minors first;

Let's find the minor elements;

$$min(7) = \begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} \qquad min(-1) = \begin{vmatrix} 10 & 1 \\ 6 & -2 \end{vmatrix}$$

$$min(-1) = \begin{vmatrix} 10 & -2 \\ 6 & 3 \end{vmatrix} \qquad min(10) = \begin{vmatrix} -1 & -1 \\ 3 & -2 \end{vmatrix}$$

$$min(-2) = \begin{vmatrix} 7 & -1 \\ 6 & -2 \end{vmatrix} \qquad min(1) = \begin{vmatrix} 7 & -1 \\ 6 & 3 \end{vmatrix}$$

$$min(6) = \begin{vmatrix} -1 & -1 \\ -2 & 1 \end{vmatrix} \qquad min(3) = \begin{vmatrix} 7 & -1 \\ 10 & 1 \end{vmatrix}$$

$$min(-2) = \begin{vmatrix} 7 & -1 \\ 10 & -2 \end{vmatrix}$$

Hence,
The matrix of minors is;

$$\text{minor}(A) = \begin{pmatrix} 1 & -26 & 42 \\ 5 & -8 & 27 \\ -3 & 17 & -4 \end{pmatrix}$$

From the cofactor matrix sign notation below:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Hence,
The matrix of cofactors is;

$$\text{cofactor}(A) = \begin{pmatrix} 1 & 26 & 42 \\ -5 & -8 & -27 \\ -3 & -17 & -4 \end{pmatrix}$$

We have the cofactor matrix, we can straightforward have the adjoint matrix;

The adjoint is the transpose of the cofactor matrices:

$$adj(A) = [\text{cofactor}(A)]^T$$

$$\text{adj}(A) = \begin{pmatrix} 1 & 26 & 42 \\ -5 & -8 & -27 \\ -3 & -17 & -4 \end{pmatrix}^T$$

$$\text{adj}(A) = \begin{pmatrix} 1 & -5 & -3 \\ 26 & -8 & -17 \\ 42 & -27 & -4 \end{pmatrix}$$

We have evaluated the determinant already and hence;

$$A^{-1} = \frac{1}{23}\begin{pmatrix} 1 & -5 & -3 \\ 26 & -8 & -17 \\ 42 & -27 & -4 \end{pmatrix}$$

Back to the matrix equation;

To solve the equation; pre-multiply both sides of this equation by A^{-1}

$$\begin{pmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & -2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 7 \end{pmatrix}$$

We have;

$$A^{-1}\begin{pmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$= -\frac{1}{61}\begin{pmatrix} 1 & -5 & -3 \\ 26 & -8 & -17 \\ 42 & -27 & -4 \end{pmatrix}\begin{pmatrix} 0 \\ 8 \\ 7 \end{pmatrix}$$

The left hand side is reduces completely since A^{-1} multiplying A will yield the identity matrix; we start expanding the right hand side multiplication;

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$= \frac{1}{23}\begin{pmatrix} (1)(0) + (-5)(8) + (-3)(7) \\ (26)(0) + (-8)(8) + (-17)(7) \\ (42)(0) + (-27)(8) + (-4)(7) \end{pmatrix}$$

The left hand side simply yields the matrix of coefficients since it is multiplied by an identity matrix;

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{61}\begin{pmatrix} -61 \\ -183 \\ -244 \end{pmatrix}$$

Hence,

Expanding the matrices and applying the matrix equality rule;

$$x = -\frac{61}{-61} = 1$$

$$y = -\frac{183}{-61} = 3$$

$$z = -\frac{244}{-61} = 4$$

(c)

Given:

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & 9 \\ 7 & 6 & 1 \end{pmatrix} \text{ and } E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

Show that $|EA| = |E||A|$.

This is the rule of the determinant of a product being equal to the product of their determinants, a similar question is in your note, it'll equally lengthy and that's why it's an option against the simultaneous equation in (b), let's do it though;

$$EA = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & 9 \\ 7 & 6 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0(1) + (1)(2) + 0(7) & 0(3) + (1)(4) + 0(6) & 0(2) + (1)(9) + 0(1) \\ 1(1) + 0(2) + 0(7) & 1(3) + 0(4) + 0(6) & 1(2) + 0(9) + 0(1) \\ 0(1) + 0(2) + 1(7) & 0(3) + 0(4) + 1(6) & 0(2) + 0(9) + 1(1) \end{pmatrix}$$

Kindly zoom that!

$$EA = \begin{pmatrix} 2 & 4 & 9 \\ 1 & 3 & 2 \\ 7 & 6 & 1 \end{pmatrix}$$

Hence, we have EA;

$$|EA| = \begin{vmatrix} 2 & 4 & 9 \\ 1 & 3 & 2 \\ 7 & 6 & 1 \end{vmatrix}$$

$$|EA| = 2\begin{vmatrix} 3 & 2 \\ 6 & 1 \end{vmatrix} - (4)\begin{vmatrix} 1 & 2 \\ 7 & 1 \end{vmatrix} + 9\begin{vmatrix} 1 & 3 \\ 7 & 6 \end{vmatrix}$$

$$|EA| = 2(-9) - 4(-13) + 9(-15)$$

$$|EA| = -18 + 52 - 135 = -101$$

Then, proving the second identity; $|E||A|$, we take the determinants separately and multiply them;

$$|A| = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 4 & 9 \\ 7 & 6 & 1 \end{vmatrix}$$

$$|A| = 1\begin{vmatrix} 4 & 9 \\ 6 & 1 \end{vmatrix} - (3)\begin{vmatrix} 2 & 9 \\ 7 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 4 \\ 7 & 6 \end{vmatrix}$$

$$|A| = 1[(4)(1) - (9)(6)] - 3[(2)(1) - (9)(7)] + 2[(2)(6) - (4)(7)]$$

$$|A| = 1(-50) - 3(-61) + 2(-16) = 101$$

$$|E| = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$|E| = 0\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} - (1)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$$

$$|E| = 0 - 1[(1)(1) - (0)(0)] + 0$$

$$|E| = -1(1) = -1$$

Hence;

$$|E||A| = -1 \times 101 = -101$$

Hence, we have confirmed what we need to confirm since both are equal to -101.

Question 3

(a) Outline any three techniques for integrating complex functions.

(b) Using any of the techniques outline in *(a)* above; find:

(i) $$\int \frac{3x - 5}{(x - 1)(x - 2)}\, dx$$

(ii) $$\int (3x + 5)^{10}\, dx$$

(iii) $$\int_1^3 4x^3 e^{x^4}\, dx$$

(a)

There are several techniques for integrating functions. However, since the (b) part of the questions states that we should use the techniques mentioned in (a) to solve the given problem, let's list the methods that'll solve the problems in (b)

(i) Integration by partial fractions
(ii) Integration by substitution
(iii) Integration by parts

(b)

Note carefully that both (ii) and (iii) are solved using substitution, hence, the method (iii) in (a) was just put to complete the list of three methods.

(i)

$$\int \frac{3x-5}{(x-1)(x-2)}\,dx$$

Clearly a case of integration by partial fraction!

$$\frac{3x-5}{(x-1)(x-2)} \equiv \frac{A}{x-1} + \frac{B}{x-2}$$

Clear;

$$3x-5 = A(x-2) + B(x-1)$$

Put $x = 1$

$$3(1) - 5 = A(1-2) + B(1-1)$$

Hence,

$$A = 2$$

Also; put $x = 2$

$$3(2) - 5 = A(2-2) + B(2-1)$$

Hence,

$$B = 1$$

$$\frac{3x-5}{(x-1)(x-2)} \equiv \frac{2}{x-1} + \frac{1}{x-2}$$

Hence,

$$\int \frac{3x-5}{(x-1)(x-2)}\,dx = \int \left(\frac{2}{x-1} + \frac{1}{x-2}\right)\,dx$$

Integration of sums;

$$\int \frac{2}{x-1}\,dx + \int \frac{1}{x-2}\,dx$$

Slight substitutions needed;

$$\int \boldsymbol{\frac{2}{x-1}\,dx}$$

$$u = x - 1$$

$$\frac{du}{dx} = 1$$

$$dx = du$$

$$\int \frac{2}{u}\, du = 2\int \frac{du}{u}$$

$$2(\ln u)$$

$$2\ln(x - 1)$$

$$\boldsymbol{\int \frac{1}{x - 2}\, dx}$$

$$z = x - 2$$

$$\frac{dz}{dx} = 1$$

$$dx = dz$$

$$\int \frac{1}{z}\, dz$$

$$(\ln z)$$

$$\ln(x - 2)$$

Hence,

Integral is:

$$2\ln(x-1)+\ln(x-2)+C$$

The arbitrary constant is added just once!

(b)

$$\int (3x+5)^{10}\,dx$$

Linear substitution!

$$u=3x+5$$

$$\frac{du}{dx}=3$$

$$dx=\frac{du}{3}$$

$$\int u^{10}\frac{du}{3}=\frac{1}{3}\int u^{10}\,du$$

By algebraic power rule;

$$\frac{1}{3}\left[\frac{u^{10+1}}{10+1}\right]$$

$$\frac{u^{11}}{33}$$

Return u;

$$\frac{(3x+5)^{11}}{33}+C$$

(c)

$$\int_1^3 4x^3 e^{x^4}\,dx$$

Substitution! A case of:

$$f'(x)g[f(x)]$$

$$\int 4x^3 e^{x^4}\,dx$$

Here;

$$u = x^4$$

$$\frac{du}{dx} = 4x^3$$

$$dx = \frac{du}{4x^3}$$

$$\int 4x^3 e^u \frac{du}{4x^3}$$

$4x^3$ cancels out;

$$\int e^u \, du$$

Straight exponent rule!

$$e^u$$

Return u;

$$e^{x^4} + C$$

Definite integral;

$$\left[e^{x^4}\right]_1^3$$

Arbitrary constant not needed in definite integral!

$$e^{3^4} - e^{1^4} = e^{81} - e^1$$

From calculator;

$$1.506 \times 10^{35} - 2.718$$

Extremely ridiculously large value! The definite integral is:

$$1.506 \times 10^{35}$$

Don't be surprised, subtracting 2.718 from such extremely large value makes no effect, **except for extremely large significant figures**!

!!!!!