INTEGRATION

The basic idea of integration which is also known as integral calculus is the second part of the concept of calculus, calculus as you have already been told is an interesting concept; however, integration, the second part of calculus is a quite tricky concept which in honesty is less straightforward than the concept of differentiation; however, without much stress, following the concept of this book bit by bit and one by one is all you need to understand it even though not the complete concept of integration but what you need basically as a student in the faculty of administration or as a student in the faculty of social science.

So, by definition, ***integration is the reverse process of differentiation, the process of finding a function from its derivative;***

Now, differentiation which we've properly seen entailed the finding of the derivative of a given function, integration being the reverse process is finding the function from a given derivative. It's important I reinstate the fact that integration is far from a straightforward process.

Finding integrals of functions involve a process that attempts to retrace the steps we took from finding the derivatives of functions to bring back the derivatives of such functions. Unfortunately however, ☹☹☹ there is no straightforward way to do that or rules as it were, that are used to find integrals of functions.

Hence, without any form of deceit, the process of integration involves several and I mean a whole lot of several manipulations, hence, you really need to read this chapter very calmly, and with a whole lot of open mind, I hope that is clear? Trust me, if you read it well, and pay cool attention, you should be able to handle every form of integration that comes at least in the SSC106 way. However, the whole idea of integration is a very vast concept and even your MTH counterparts in the science and technology faculties definitely cannot integrate all functions.

You needn't be scared though, the concept of integration in this course is no big deal, I need to encourage you over and over before we'll get started as I am aware that many are afraid of the dreadful concept of integration; follow me to take it just soft though.

As much as I said there are no fixed rules for integration, it doesn't mean there are no rules at all. In comparing with differentiation, the formula for differentiating $\sin x$ can differentiate all functions involving sin and only needs to be combined with chain rule (and/or product rule) to differentiate any function involving sin. However, the same cannot be said of integration.

All is well. No more stories;

So… … … …

So, let's see notations in the concept of integration. The integral sign known as the integral of a certain function is the long S symbol which is shown below; let me scatter an array of the symbol for you to see them in a dope manner; ☺☺☺

$$\int \int \int \int \int \int \int \int \int \int \int \int \int \int \int$$

That's the integral sign you've seen scattered above, I hope you like that dope arrangement?

Yeah, so that's the integral function anyway; so briefly, let's see how, what it takes and what it really means.

Now, let $g(x)$ be the derivative with respect to x of the function $f(x)$; then from our studies of differentiation which shouldn't be a strange concept to you at all;

$$\frac{d}{dx}[f(x)] = g(x)$$

Now, given $g(x)$; in the need to get $f(x)$ back, we need the concept of integral calculus and there, our dope elongated S sign comes in; so it is applied thus;

From the derivative of $f(x)$ shown above,

$$df(x) = g(x)dx$$

Apply the integral sign;

$$\int df(x) = \int g(x)dx$$

Now, the basic role of integral calculus is to return you back to the main function and hence, removing the differential coefficient (derivative) sign from a situation;

Straight; on the left hand side; the integral removes the d (which is the derivative sign) from $f(x)$ to leaves it as $f(x)$.

Hence, on the right hand side, the dx multiplying the $g(x)$ shows that the function is meant to be integrated with respect to x, basically, only the term on the right hand side is the elongated S (the integral sign) shown since it's operation on the left hand side is to simply remove the d from $f(x)$.

So, if $g(x)$ is the derivative of $f(x)$;

Then,

$$f(x) = \int g(x)dx$$

Using the normal derivative notations for functions; then $f'(x)$ is the derivative of $f(x)$;

Hence,

$$f(x) = \int f'(x)dx$$

If the independent and dependent variable form is used where y is the dependent variable and x is the independent variable; then we'll have it that;

$$\frac{dy}{dx} = g(x)$$

Then,

$$dy = g(x)dx$$

And after integration;

$$y = \int g(x)dx$$

So, we have said so so much stories, the stories are so good anyway, let's move quickly to the main – the main.

So, let's look at **the arbitrary constant.**

If you recall in the concept of differentiation; when a constant is differentiated, the derivative is zero. Let us explain this with a practical example.

So, consider the derivative of the function;

$$y = x^3 - 5$$

$$\frac{dy}{dx} = 3x^2$$

Now, we see that 5 has been softly lost in the differential coefficient of y. Now from what we know about integration; we know that we can get y from the differential coefficient by:

$$y = \int 3x^2\,dx$$

Now, before we even know how to integrate this, we already know the integral of $3x^2$ is x^3 since we differentiated x^3 at first to get $3x^2$***(relax, we'll still see how it is integrated but let's learn about arbitrary constants first).*** Hence, when this is integrated, there is no evidence whatsoever, that the constant that was there before is -5, if we weren't privy to the function that was differentiated; then there is no way to establish what was in the original function, hence, when integrating, that is a great ignorance everyone suffers, the ignorance of the constant that was actually there.

So that introduces the concept of **the arbitrary constants.** Since we are not and cannot be aware of the constant that was there initially, then what we do is to *kuku* add any random constant to our integration answer;

If you do check your dictionary, you'll see that arbitrary means a random stuff, hence, that's the idea of arbitrary constant. Hence, in the integral of $3x^2$; we simply say the integral is:

$$x^3 + C$$

Where C is the arbitrary constant;

Let's see manipulations of arbitrary constants.

Now, if we have $2, 3, 4$ or any amounts of arbitrary constants, the sum, difference, product or quotient of them in any order gives birth to another arbitrary constant and hence, you don't get an integral sunk with arbitrary constants, only one arbitrary constant is needed in every integral because any mathematical operation on arbitrary constants gives birth to another arbitrary constant.

Also, the logarithm, exponent and other natural operations on arbitrary constants all give birth to arbitrary constants, so basically, one arbitrary constant is needed in integration.

Examples of what we just explained about manipulations involving arbitrary constants are given below:

Now, given that A, B, C are all arbitrary constants;

Then, $A + B - C$ is another arbitrary constant somewhere somehow and hence, instead of clustering three arbitrary constants in a question, then it could be summarized as an arbitrary constant, say D.

What of the situation of having e^B, still the same thing, as B is an arbitrary constant, then its exponent is also one arbitrary constant somewhere, and hence, e^B can be summarized by an arbitrary constant, say E.

Other examples like $\log A$, $\ln C$, $e^A \log B$ and several other manipulations of arbitrary constants all yield another arbitrary constant somewhere and hence, can be summarized as an arbitrary constant.

That's it about arbitrary constant! That's a very important introduction anyway.

So, moving to the aspect of integration proper now, as I have said, that in integration, there are no rules as it were, there are just manipulations and ways the function to be integrated is bent in such a way that it can be differentiated from some sets of values that are called **standard integrals.**

STANDARD INTEGRALS

As said already, the basic of idea of integration is the reverse process of differentiation. So from every of the differential rule we considered, we can derive a set of standard integrals from there, and from there, build on the standard integrals.

Now, of course, the first thing to consider is the power rule, I mean the constant power rule, you know that already;

Now we know that in the power rule of differentiation, the derivative is gotten by first multiplying the whole thing by the power and

then reducing it by 1 (subtracting 1 from that power). Now, to reverse this process, we start from where we stopped, we simply add 1 to the power we subtracted and we also reverse the multiplication process by dividing the whole thing by the power we have restored;
Hence,

$$\int x^n = \frac{x^{n+1}}{n+1}$$

So here, the power is reversed first before the multiplication is reversed, I hope that was clear, I trust it was; I guess we'll need five or more examples on this; I guess you'd read properly the chapter of mathematical operations before coming here, you should've read it before reading differentiation actually, so in case you jumped here, you'd better go back there now. It's the first part of this book in the introduction aspect.

So, find the following integrals.

- $\int x^{\frac{3}{2}}\, dx$

- $\int x^{-\frac{7}{3}}\, dx$

So let's solve these two simple questions, very simple for that matter;

- $\int x^{\frac{3}{2}}\, dx$

So, for this one; we simply apply the rule that we add 1 to the power and dividing the whole thing by the result of adding one to the power; let's do that here;

$$\int x^{\frac{3}{2}}\, dx = \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}$$

$$\frac{x^{\frac{5}{2}}}{\frac{5}{2}} = x^{\frac{5}{2}} \times \frac{2}{5} = \frac{2x^{\frac{5}{2}}}{5} + C$$

It's that simple, just follow the process serially;

Let's see the next one and move on, we still have some things to deal with here;

- $\int x^{-\frac{7}{3}}\,dx$

We simply do the same thing, any time we have a constant power, nothing else in it;

$$\int x^{\frac{3}{2}}\,dx = \frac{x^{-\frac{7}{3}+1}}{-\frac{7}{3}+1}$$

$$\frac{x^{-\frac{4}{3}}}{-\frac{4}{3}} = x^{-\frac{4}{3}} \times \frac{3}{-4} = -\frac{3}{4x^{\frac{4}{3}}} + C$$

The negative power law of indices was applied there, and that's it!

Notice our arbitrary constant is added in both answers, yes, that's how it always will be; you have to add it, yes, it is compulsory, every singular time you integrate, because you aren't aware if the constant was there or not, and no one is aware, so you have to add it, or your answer will be meaningless! It's that serious!

So, let's introduce another basic rule;

We have the concept of a constant multiplying a function to be integrated, for example; the form;

ax^n where a is a constant.

A constant just like in differentiation doesn't affect the integral of a function, hence, it is taken OUT of the integral while the function is then integrated, full stop!

Cool, so let's continue with our examples;

Let's find some more integrals;

- $\int 7x^{\frac{3}{4}}\,dx$
- $\int \frac{4}{3}x^{-\frac{2}{5}}\,dx$
- $\int -\frac{3}{4x^{\frac{4}{3}}}\,dx$
- $\int -\frac{dx}{7\sqrt[3]{x}}$

- $\int -\frac{2}{3}\,dx$

Okay, we have quite some questions over here; let's gently work on them one by one.

- $\int 7x^{\frac{3}{4}}\,dx$

Now, let's see how the rule we just introduced, the constant here multiplying the function is 7, so what we do is given thus:

$$\int 7x^{\frac{3}{4}}\,dx = 7\int x^{\frac{3}{4}}\,dx$$

The 7 is taken out, we evaluate the integral and multiply it by the 7 we've taken out, the fact is that it wasn't going to affect the integral so bring it out of the integral.

Hence, we now apply our power rule here and complete the whole stuff.

We have:

$$7\left[\frac{x^{\frac{3}{4}+1}}{\frac{3}{4}+1}\right] = 7\left[\frac{x^{\frac{7}{4}}}{\frac{7}{4}}\right]$$

$$7 \times x^{\frac{7}{4}} \times \frac{4}{7} = 4x^{\frac{7}{4}} = \frac{4}{x^{\frac{7}{4}}} = \frac{4}{\sqrt[4]{x^7}} + C$$

Let's move faster;

- $\int \frac{4}{3} x^{-\frac{2}{5}}\, dx$

The constant here multiplying the function is $\frac{4}{3}$, so what we do is to take it out and evaluate the integral on a norm ☺!

$$\int \frac{4}{3} x^{-\frac{2}{5}}\, dx = \frac{4}{3} \int x^{-\frac{2}{5}}\, dx$$

After taking it out, we can now find our integral and complete it after;

$$\frac{4}{3}\left[\frac{x^{-\frac{2}{5}+1}}{-\frac{2}{5}+1}\right] = \frac{4}{3}\left[\frac{x^{\frac{3}{5}}}{\frac{3}{5}}\right]$$

$$\frac{4}{3} \times x^{\frac{3}{5}} \times \frac{5}{3} = \frac{20}{9} x^{\frac{3}{5}} = \frac{20}{9x^{\frac{3}{5}}} = \frac{20}{9\sqrt[5]{x^3}} + C$$

FASTER;

- $\int -\frac{3}{4x^{\frac{4}{3}}} dx$

Before discussing the constant, like in differentiation, we have to make all the powers as a power in the numerator; hence, we have to apply the negative power rule over here to simplify the power to a numerator. Hence, we have:

$$\int -\frac{3}{4x^{\frac{4}{3}}} dx = \int -\frac{3x^{-\frac{4}{3}}}{4} dx$$

So, the constant here multiplying the function is $-\frac{3}{4}$, so what we do is to take it out and evaluate the integral on a norm ☺!

$$\int -\frac{3x^{-\frac{4}{3}}}{4} dx = -\frac{3}{4} \int x^{-\frac{4}{3}} dx$$

After taking it out, we can now find our integral and complete it after;

$$-\frac{3}{4}\left[\frac{x^{-\frac{4}{3}+1}}{-\frac{4}{3}+1}\right] = -\frac{3}{4}\left[\frac{x^{-\frac{1}{3}}}{-\frac{1}{3}}\right]$$

$$-\frac{3}{4} \times x^{-\frac{1}{3}} \times \frac{3}{-1} = \frac{9}{4}x^{-\frac{1}{3}} = \frac{9}{4x^{\frac{1}{3}}} = \frac{9}{4\sqrt[3]{x}} + C$$

More… … …,

- $\int -\frac{dx}{7\sqrt[3]{x}}$

This is something you're going to be experiencing several times in integration, where dx is the numerator, it's nothing to be scared about though. As dx is the denominator, it simply means 1 is the numerator, sure you know dx is a product, throwback to where we explained the concept of integration, we see dx been multiplied to the right hand side, hence, dx here means $1 \times dx$; hence, we can simplify this to:

$$\int -\frac{dx}{7\sqrt[3]{x}} = \int -\frac{1}{7\sqrt[3]{x}}\,dx$$

We can now convert our roots to fractional powers and then apply the negative power law to make it a power in the numerator, and of course, we'll also bring the constant $\left(-\frac{1}{7}\right)$ out of the integral.

$$-\frac{1}{7}\int \frac{1}{x^{\frac{1}{3}}}\,dx = -\frac{1}{7}\int x^{-\frac{1}{3}}\,dx$$

We have performed all operations including taking the constant out, we can now find our integral and complete it after;

$$-\frac{1}{7}\left[\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}\right] = -\frac{1}{7}\left[\frac{x^{\frac{2}{3}}}{\frac{2}{3}}\right]$$

$$-\frac{1}{7}\times x^{\frac{2}{3}}\times\frac{3}{2} = -\frac{3}{14}x^{\frac{2}{3}} = \frac{3x^{\frac{2}{3}}}{14} = \frac{3\sqrt[3]{x^2}}{14} + C$$

Kk, finally... ... ...,

- $\int -\frac{2}{3}dx$

Lwkmd! I guess you're still searching for the x term that we're finding the integral right! Just take a chill pill and check this;

Unlike in differentiation where derivative of constants is zero, integration isn't so, remember that differentiating $2x$ will give you 2 right? Hence, constants are **fully eligible** for integration.

So don't get it mixed up, this is different from the rule of bringing out the constant when it is multiplying a function; that still stands. However, this case here is a constant standing on its own. As a matter of fact, we'll still be bringing the constant out so just chill.

To integrate this, since dx is multiplying the integral, we are integrating with respect to x; we consider the constant as multiplying x^0; now you know $x^0 = 1$ since anything raised to power zero is 1, hence, nothing has changed; let's do that and see how it turns out.

$$\int -\frac{2}{3}dx = \int -\frac{2}{3}x^0 dx$$

So let's take the constant out, we now have a function of x, we can now find our integral and complete it after;

$$-\frac{2}{3}\int x^0 dx = -\frac{2}{3}\left[\frac{x^{0+1}}{0+1}\right] = -\frac{2}{3}\left[\frac{x^1}{1}\right]$$

$$-\frac{2}{3}x = -\frac{2x}{3} + C \text{ (hence, that's our answer!)}$$

So, x comes out as a power of 1 which of course is what is expected. The integral of dx is x as we saw in the beginning that applying the integral to dy simply removes the d to leave it as y.

So, there's just one aspect of this power stuff that I've not mentioned, let me attack you with it in a question.

Find the integral;

$$\int \frac{3}{4x}dx$$

Cool, let's see this:

So, as usual let's bring out constant out, and find our integral fast!

$$\int \frac{3}{4x} dx = \frac{3}{4} \int \frac{1}{x} dx$$

But we also need to make the power at the numerator, using the negative power law of indices.
We have:

$$\frac{3}{4} \int x^{-1} dx$$

So we apply the power rule thus;

$$\frac{3}{4} \left[\frac{x^{-1+1}}{-1+1} \right] = \frac{3}{4} \left[\frac{x^{0}}{0} \right] = \frac{3}{4} \left[\frac{1}{0} \right]$$

Lol, so where are we going *bayii*??? Please we need to cancel these jargons before we are misled! ☹☹☹

$$\cancel{\frac{3}{4} \left[\frac{x^{-1+1}}{-1+1} \right] = \frac{3}{4} \left[\frac{x^{0}}{0} \right] = \frac{3}{4} \left[\frac{1}{0} \right]}$$

So, let's clear this! Why was it wrong in the first instance, yes, this is it!

$\frac{1}{0}$ is an undefined fraction, dividing by zero makes **any fraction undefined**. Hence, this isn't a value!

Now, throwback to the time you read differentiation, that's probably not long ago; Can you remember the derivative of the natural log ($\log_e x$) of any variable (say x)? You should, it is $\frac{1}{x}$

And so what's the moral lesson in that??? YES! The moral lesson is that in the power rule of integration, there is **a big exception,** when that power is -1 which makes the fraction, in variable of x to be $\frac{1}{x}$, the integral is simply $\ln x$ or in log base form, $\log_e x$.

Hence; as a rule;

$$\int \frac{1}{x} dx = \ln x$$

Hence, back to the question we were solving;

$$\frac{3}{4}\int x^{-1}dx$$

The integral will simply be:

$$\frac{3}{4}[\ln x] = \frac{3}{4}\ln x + C$$

I hope you've been observing from the background, how we've been adding the arbitrary constant, it's very important we add it to **every final answer** of an integration situation, you don't go adding it in the middle of the integration, it is added after the whole thing!

THE INTEGRAL OF SUMS AND DIFFERENCES

There is this cool rule that we'll chip in just now, **the integral of sums and differences**; it states that when the function to be integrated is a function made up of sums and differences of separate functions; the integral is the sum of the individual integrals;

That's a similar language to the derivative of sums and differences; that's by the way anyway.

So, in symbolic notation, if:

$$y = \int [f(x) + g(x) + h(x)]dx$$

Then, we have that;

$$y = \int f(x)dx + \int g(x)dx + \int h(x)dx$$

So before we start, you don't go ahead adding several arbitrary constants, we simply add one arbitrary constant after the whole solution.

So, let's see some few examples before we move on!

So, let's find some more integrals;

- $\int (x^4 - 3x^2 + 2x + 7)\, dx$

- $\int \left(x^\alpha + x^\beta - x^\gamma\right) dx$

- $\int \frac{(x^3 - x^2 + 2x)}{\sqrt{x}} dx$

- $\int \frac{(2x^n + 3x^p)}{x^5} dx$

So… … … SOLUTION!

- $\int (x^4 - 3x^2 + 2x + 7) dx$

So here from our rule of integrating sums, this is how we'll do it, split the sums and differences into different integrals with respect to x, our integral becomes:

$$\int x^4 dx - \int 3x^2 dx + \int 2x\, dx + \int 7\, dx$$

So here, we apply the integral rule separately, we'll take everything one by one, we'll apply the law as we have seen above several times; bringing out the constant and doing everything necessary in applying the integral rule;

Hence, let's bring out the constants first!

$$\int x^4 dx - 3\int x^2 dx + 2\int x\, dx + 7\int x^0\, dx$$

Let's now apply the integral rule of power rule!

$$\left[\frac{x^{4+1}}{4+1}\right] - 3\left[\frac{x^{2+1}}{2+1}\right] + 2\left[\frac{x^{1+1}}{1+1}\right] + 7\left[\frac{x^{0+1}}{0+1}\right]$$

$$\frac{x^5}{5} - \frac{3x^3}{3} + \frac{2x^2}{2} + \frac{7x^1}{1}$$

$$\frac{x^5}{5} - x^3 + x^2 + 7 + C$$

Never forget that arbitrary constant!

Let's move on... ... ...,

- $\int \left(x^\alpha + x^\beta - x^\gamma\right) dx$

So here from our rule of integrating sums, this is how we'll do it, split the sums and differences into different integrals with respect to x, our integral becomes:

$$\int x^{\alpha} dx + \int x^{\beta} dx - \int x^{\gamma} dx$$

So here, we apply the integral rule separately, we'll take everything but each, we'll apply the law as we have seen above several times; bringing out the constant and doing everything necessary in applying the integral rule; Here, there isn't any constant to bring out so we simply go into applying the integral rule, we have constants as α, β, γ are constants, obviously you should know they should be constants as we are integrating with respect to x.

Let's now apply the integral

$$\left[\frac{x^{\alpha+1}}{\alpha+1}\right] + \left[\frac{x^{\beta+1}}{\beta+1}\right] - \left[\frac{x^{\gamma+1}}{\gamma+1}\right]$$

Basically, there isn't any simplification as it were that can be done to this, let's find any simplification for it though, expressing the denominators as inverse fractions;

$$\frac{1}{\alpha+1}(x^{\alpha+1}) + \frac{1}{\beta+1}\left(x^{\beta+1}\right) - \frac{1}{\gamma+1}(x^{\gamma+1}) + C$$

So, that's the integral!!! The arbitrary constant **MUST BE THERE!**

Alright?

- $\int \frac{(x^3 - x^2 + 2x)}{\sqrt{x}} dx$

So here, before anything concerning our rule of integrating sums, we have a key simplification to do over here;

We experienced this during our studies of differentiation, we'll split this fraction, into separate terms, you know when the denominator is one, it can be split on each of the terms on the numerator;

$$\int \left(\frac{x^3}{\sqrt{x}} - \frac{x^2}{\sqrt{x}} + \frac{2x}{\sqrt{x}}\right) dx$$

Convert the square root to power $\frac{1}{2}$ and split your integrals based on the rule of sums and differences.

$$\int \left(\frac{x^3}{x^{\frac{1}{2}}}\right) dx - \int \left(\frac{x^2}{x^{\frac{1}{2}}}\right) dx + \int \left(\frac{2x}{x^{\frac{1}{2}}}\right) dx$$

So, we'll simplify the indicial expressions by subtracting the powers at the denominator from the powers at the numerator.

$$\int x^{3-\frac{1}{2}}\, dx - \int x^{2-\frac{1}{2}}\, dx + \int 2x^{1-\frac{1}{2}}\, dx$$

Bring out the constants from the integral where necessary and find the integral;

$$\int x^{\frac{5}{2}}\, dx - \int x^{\frac{3}{2}}\, dx + 2\int x^{\frac{1}{2}}\, dx$$

Applying the power rule of integration;

$$\left[\frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1}\right] - \left[\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right] + 2\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]$$

$$\left[\frac{x^{\frac{7}{2}}}{\frac{7}{2}}\right] - \left[\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right] + 2\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]$$

$$\left[x^{\frac{7}{2}} \times \frac{2}{7}\right] - \left[x^{\frac{5}{2}} \times \frac{2}{5}\right] + 2\left[x^{\frac{3}{2}} \times \frac{2}{3}\right]$$

$$\left[\frac{2x^{\frac{7}{2}}}{7}\right] - \left[\frac{2x^{\frac{5}{2}}}{5}\right] + 2\left[\frac{2x^{\frac{3}{2}}}{3}\right]$$

So, we can either leave the fractions in fractional form or convert them to root form, whichever one you're doing, *sha* don't forget the arbitrary constant.

$$\frac{2\sqrt{x^7}}{7} - \frac{2\sqrt{x^5}}{5} + \frac{4\sqrt{x^3}}{3} + C$$

If the root form is problematic, please kindly leave it in fractional form *edakun*.

Next! I guess that's the last example;

- $\displaystyle\int \frac{(2x^n + 3x^p)}{x^5}\,dx$

So here, before anything concerning our rule of integrating sums, this is another splitting something, we'll split this fraction, into separate terms, you know when the denominator is one, it can be split on each of the terms on the numerator;

$$\int \left(\frac{2x^n}{x^5} + \frac{3x^p}{x^5}\right) dx$$

Separate the integrals;

$$\int \left(\frac{2x^n}{x^5}\right) dx + \int \left(\frac{3x^p}{x^5}\right) dx$$

So, we'll simplify the indicial expressions by subtracting the powers at the denominator from the powers at the numerator.

$$\int 2x^{n-5}\, dx + \int 3x^{p-5}\, dx$$

Bring out the constants from the integral where necessary and find the integral;

$$2\int x^{n-5}\, dx + 3\int x^{p-5}\, dx$$

Applying the power rule of integration; we must take the powers $(n-5)$ and $(p-5)$ into brackets as an entity, because the power must be worked on as an entity. $(n-5)$ and $(p-5)$ are undoubtedly constants.

$$2\left[\frac{x^{(n-5)+1}}{(n-5)+1}\right]+3\left[\frac{x^{(p-5)+1}}{(p-5)+1}\right]$$

$$2\left[\frac{x^{n-5+1}}{n-5+1}\right]+3\left[\frac{x^{p-5+1}}{p-5+1}\right]$$

$$2\left[\frac{x^{n-4}}{n-4}\right]+3\left[\frac{x^{p-4}}{p-4}\right]$$

$$\frac{2}{n-4}(x^{n-4})+\frac{3}{p-4}(x^{p-4})+C$$

We brought out $(n-4)$ and $(p-4)$ out as constants to express them as inverse fractions since they're constants. And of course our arbitrary constant is very very important.

Let's do this;

- If:

$$\frac{dP}{dx} = 11 - 2x$$

find P as a function of x given that $P = 4$ when $x = 1$.

$$\frac{dP}{dx} = 11 - 2x$$

To find, P in terms of x; we evaluate it thus;

$$dP = (11 - 2x)dx$$

Take integral of both sides; of course you already know the LHS will simply cancel out the differential coefficient, we now have:

$$\int dP = \int (11 - 2x)dx$$

We'll split out integral on the RHS and continue as usual:

$$P = \int 11\, dx - \int 2x\, dx$$

We bring out our constants as usual, you know all these already!

$$P = 11 \int x^0 \, dx - 2 \int x \, dx$$

$$P = 11 \left[\frac{x^{0+1}}{0+1} \right] - 2 \left[\frac{x^{1+1}}{1+1} \right]$$

$$P = 11 \left[\frac{x^1}{1} \right] - 2 \left[\frac{x^2}{2} \right]$$

$$P = 11x - x^2 + C$$

We have added our arbitrary constant; however, we have extra information that'll take it from being just an arbitrary constant to a proper fixed constant.

So, from here;

$$P = 11x - x^2 + C$$

Since, $P = 4$ when $x = 1$

Hence, substitute for P and x to find appropriately the true value of C.

$$4 = 11(1) - (1)^2 + C$$
$$4 = 11 - 1 + C$$
$$C = 4 - 10 = -6$$

Hence, we can find the true value of the arbitrary constant if and only if we have extra information.

Hence; finally;

$$\boldsymbol{P = 11x - x^2 - 6}$$

So, okay;

We've already seen the integral of x^n and the exception of $\frac{1}{x}$ as well; there are standard forms for a quite vast range of differential coefficients which we didn't touch in our studies of differential calculus owing to the scope of this course;

So here are the rest!

- $$\int e^x \, dx = e^x + C$$

- $$\int a^x \, dx = \frac{a^x}{\ln a} + C$$

- $$\int \sin x \, dx = -\cos x + C$$

- $$\int \cos x \, dx = \sin x + C$$

- $$\int \sec^2 x \, dx = \tan x + C$$

So, the standard integrals above were derived from the facts below; they're nothing new, they're just a direct opposite process of the differential coefficients we derived in the study of differentiation.

- $$\frac{d}{dx}(e^x + C) = e^x$$

  It follows that:

  $$\int e^x \, dx = e^x + C$$

- $$\frac{d}{dx}(a^x + C) = a^x \ln a$$

Hence, dividing by $\ln a$; we can get the value for the integral of a^x exclusively;

$$\int a^x \, dx = \frac{a^x}{\ln a} + C$$

- $$\frac{d}{dx}(\cos x + C) = -\sin x$$

  Hence, the negative of $\cos x$ will give the integral of $\sin x$

  $$\int \sin x \, dx = -\cos x + C$$

- $$\frac{d}{dx}(\sin x + C) = \cos x$$

  Hence, the integral of $\cos x$ will be:

  $$\int \cos x \, dx = \sin x + C$$

- $$\frac{d}{dx}(\tan x + C) = \sec^2 x$$

Hence, the integral of $\sec^2 x$ will be;

$$\int \sec^2 x\, dx = \tan x + C$$

The derivatives of $\sec x$, $\operatorname{cosec} x$ and $\cot x$ yield complex derivatives which are $\sec x \tan x$, $-\cot x \operatorname{cosec} x$, $-\operatorname{cosec}^2 x$ respectively and hence, we didn't include them in the standard integrals above, they aren't mostly used, not even in SSC106; for the sake of formalities though, we have such standard integrals given below:

- $$\frac{d}{dx}(\sec x + C) = \sec x \tan x$$

  Hence, the integral of $\sec x \tan x$is:

  $$\int \sec x \tan x\, dx = \sec x + C$$

- $$\frac{d}{dx}(\operatorname{cosec} x + C) = -\cot x \operatorname{cosec} x$$

  Hence, the negative of $\operatorname{cosec} x$ will give the integral of $\cot x \operatorname{cosec} x$

$$\int \cot x \operatorname{cosec} x \, dx = -\operatorname{cosec} x + C$$

- $$\frac{d}{dx}(\cot x + C) = -\operatorname{cosec}^2 x$$

  Hence, the negative of $\cot x$ will give the integral of $\operatorname{cosec}^2 x$;

$$\int \operatorname{cosec}^2 x \, dx = -\cot x + C$$

There's nothing much to deal with in the following standard integrals above, they're simply as they are, just like there is nothing worth it dwelling on derivatives rules (we simply go ahead to use them in conjunction with chain rule); in the same way, one thing we need to do now is to delve into the aspect of **integration by substitution;**

Before that though, let's just see a few random questions; love it right?

You should, let's find the following integrals.

- $\int \left(x^2 + \sin x + \sqrt{x}\right) dx$

- $\int (e^x - \cos x)\, dx$

- $\int (6 \sec^2 x - 3^x)\, dx$

So… … … We have the first one thus,

- $\int \left(x^2 + \sin x + \sqrt{x}\right) dx$

So here from our rule of integrating sums, we split the sums into different integrals with respect to x, our integral becomes, you know that already:

$$\int x^2 dx + \int \sin x \, dx + \int \sqrt{x}\, dx$$

So here, we apply the integral rule separately, we'll take everything one by one, we'll apply the law as we have seen above several times; doing everything necessary in applying the integral rule, you can see the standard integral of $\sin x$ showing up, so that's it that way; you directly integrate that and apply power rule to the remaining two, there isn't any constant to bring out in this situation;

$$\int x^2 dx + \int \sin x \, dx + \int x^{\frac{1}{2}} dx$$

$$\left[\frac{x^{2+1}}{2+1}\right] + (-\cos x) + \left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]$$

$$\frac{x^3}{3} - \cos x + \frac{x^{\frac{3}{2}}}{\frac{3}{2}}$$

$$\frac{x^3}{3} - \cos x + x^{\frac{3}{2}} \times \frac{2}{3}$$

We can rearrange and turn the fractional powers to root forms and of course, you should know by now that the arbitrary constant MUST BE ADDED;

$$\frac{x^3}{3} + \frac{2\sqrt{x^3}}{3} - \cos x + C$$

Moving on,

- $\int (e^x - \cos x)dx$

So here from our rule of integrating sums, we split the sums into different integrals with respect to x, our integral becomes, you know that already:

$$\int e^x dx - \int \cos x \, dx$$

So here, we apply the integral rule separately, we'll take everything one by one, these two are cases of standard integrals, you can see the standard integral of e^x and $\cos x$ showing up, so that's it that way; you directly integrate the both; there isn't any constant to bring out in this situation;

$$\int e^x dx - \int \cos x \, dx$$

$$e^x - \sin x + C$$

The arbitrary constant must be included;

The third one,

- $\int (6\sec^2 x - 3^x)dx$

So here from our rule of integrating sums, we split the sums into different integrals with respect to x, our integral becomes, you know that already:

$$\int 6\sec^2 x\,dx - \int 3^x dx$$

So here, we apply the integral rule separately, we'll take everything one by one, these two are cases of standard integrals, you can see the standard integral of $\sec^2 x$ and 3^x showing up; you directly integrate the both; we'll bring out the constant in $6\sec^2 x$; nothing to bring out in 3^x please, 3 is not an independent constant here, it is part of the function.

$\sec^2 x$ is a standard integral

3^x is a standard integral of the form a^x where $a = 3$ where we'll simply apply the standard integral rule; hence, we have:

$$6\int \sec^2 x\,dx - \int 3^x dx$$

$$6\tan x - \frac{a^x}{\ln 3} + C$$

And of course, the arbitrary constant is there!

INTEGRATION BY SUBSTITUTION

So, now::: the aspect of **integration by substitution** is the aspect that looks like function of a function in the study of differential calculus; however, with function of functions as straightforward as multiplying and multiplying even in times when it reaches the range of three substitutions, it is not as easy as that here; there are serious restrictions in doing this here;

To integrate a function such as $(2x+3)^5$; you cannot simply apply the integral rule of x^n here as obviously you can see that $(2x+3)$ and x are two different things, just like in the aspect of differentiation where we explained that $(2x+3)$ isn't the most basic way x can be expressed. So this is how it is done in this case; unlike function of function in differentiation where we multiply by the derivative of the function in it, we have to classify this one into separate types of the functions that could be appear as other functions;

The first case is the case of **linear functions** where the function is a function of a linear function.

Linear Functions;

Such situations occur in the form where the function that the function depends on is a linear function, that is in the form: $ax + b$, here the power of x is 1, hence, it is a linear function.

Such examples include;

$\sin(ax + b)$, $\cos(ax + b)$, $(ax + b)^n$, e^{ax+b}, a^{ax+b} and other forms of standard integrals where x isn't the basic value, but is a linear function

This is how it is done, we also make substitutions, however, let's take it slowly before we move into cases where it isn't a linear function.

Consider the integral;

$$\int f(ax+b)dx$$

Where $f(ax+b)$ is any of the listed functions which are standard integrals, or at most standard integrals multiplied by a constant;

Consider the substitution;

$$u = ax + b$$

$$\frac{du}{dx} = a \text{ w}$$

which is a constant value;

From this, we can make dx expressed in terms of du by going thus;

$$adx = du$$

And hence,

$$dx = \frac{du}{a}$$

So, bringing this back into the integral, we have;

$$\int f(u)\frac{du}{a}$$

This is after making the substitutions;

$$u = ax + b$$

and

$$dx = \frac{du}{a}$$

We can substitute for dx since from what we have established from the beginning of this, it is a product multiplying the function to be integrated and it is eliminated after **successful integration**.

Hence, since a is a constant; we bring it out from the integral;

$$\frac{1}{a}\int f(u)du$$

Hence, we can apply the rule of the function of u to it and then return the substitution u into it, this is done softly, and hence, just like in differentiation, the substitution we borrowed initially is returned in place;

So, the above integral isn't one formula I am giving you, I do not encourage dealing with formulas per say especially in integration, follow due process at all times, it is better slow than it being wrong, trying to apply formulas could end up in error; so understanding of the above process is important, you may want to go back to check it again!

So, let's take examples under the integral of the function of linear functions.

Before that though, let's just see a few random questions; love it right?

You should, let's find the following integrals.

- $\int 4(3x+10)^{11}\,dx$

- $\int -\frac{3}{4}\sin\left(7-\frac{1}{3}x\right)dx$

- $\int \frac{1}{3}e^{\left(2-\frac{3}{2}x\right)}\,dx$

- $\int 3^{2x+1}\,dx$

- $\int -\frac{2}{7}\cos(7x-4)\,dx$

- $\int \frac{3}{5}\sec^2(4-3x)\,dx$

So, we have quite some functions here; but they're no big deal, *fast fast*, we're going to handle these questions in a split second;

So… … … We have the first one thus,

- $\int 4(3x+10)^{11}\,dx$

So here, this is in the form x^n but since it isn't basic, we'll make a substitution since we can see that $(3x+10)$ is a linear function ***(You should recognize linear functions with ease, we took them in our study of functions)***:

Hence, substitute;

$$u = 3x + 10$$

to make this a basic function; hence, with

$$u = 3x + 10$$

$$\frac{du}{dx} = 1 \times 3 \times x^{1-1} + 0 = 3$$

$$\text{Hence, here, } dx = \frac{du}{3}$$

Make substitutions for $(3x + 10)$ and dx in the integral;

$$\int 4(u)^{11} \frac{du}{3}$$

We can now bring our constants out;

$$\frac{4}{3} \int u^{11} \, du$$

We now apply the integral;

$$\frac{4}{3} \left[\frac{u^{11+1}}{11 + 1} \right] = \frac{4}{3} \left[\frac{u^{12}}{12} \right] = \frac{1}{9} u^{12}$$

Hence, we return the u we borrowed for the sake of integration into this and of course add the normal – the arbitrary constants:

$$\frac{1}{9}(3x + 10)^{12} + C$$

Next!

- $\int -\frac{3}{4}\sin\left(7 - \frac{1}{3}x\right)dx$

So here, this is in the form $\sin x$ but since it isn't basic, we'll make a substitution $u = 7 - \frac{1}{3}x$ to make this a basic function since it is a linear function; hence, with:

$$u = 7 - \frac{1}{3}x$$

$$\frac{du}{dx} = 0 - 1 \times \frac{1}{3} \times x^{1-1}$$

$$\frac{du}{dx} = -\frac{1}{3}$$

Hence, here, $dx = -3du$

Make substitutions for $\left(7-\frac{1}{3}x\right)$ and dx in the integral;

$$\int -\frac{3}{4}\sin u \times -3du$$

We can now bring our constants out;

$$-\frac{3}{4}\times -3\int \sin u\, du$$

We now apply the integral of $\sin u$ which is a standard integral, we have;

$$\frac{9}{4}[-\cos u] = -\frac{9}{4}\cos u$$

Hence, we return the u we borrowed for the sake of integration into this and of course add the normal – the arbitrary constants:

$$-\frac{9}{4}\cos\left(7-\frac{1}{3}x\right)+C$$

Quick! Quick! Quick!

- $\int \frac{1}{3} e^{\left(2-\frac{3}{2}x\right)}\, dx$

So here, this is in the form e^{x} but since it isn't basic, we'll make a substitution:

$$u = 2 - \frac{3}{2}x$$

to make this a basic function; hence, with

$$u = 2 - \frac{3}{2}x$$

$$\frac{du}{dx} = 0 - 1 \times \frac{3}{2} \times x^{1-1}$$

$$\frac{du}{dx} = -\frac{3}{2}$$

Hence, here, $dx = -\frac{2}{3}du$

Make substitutions for $\left(2 - \frac{3}{2}x\right)$ and dx in the integral;

$$\int \frac{1}{3} e^{u} \times -\frac{2}{3} du$$

We can now bring our constants out;

$$\frac{1}{3} \times -\frac{2}{3} \int e^{u} du$$

We now apply the integral of e^{u} which is a standard integral, we have;

$$-\frac{2}{9}[e^{u}]$$

Hence, we return the u we borrowed for the sake of integration into this and of course add the normal – the arbitrary constants:

$$-\frac{2}{9} e^{\left(2-\frac{3}{2}x\right)} + C$$

Faster please!

- $\int 3^{2x+1} dx$

So here, this is in the form a^x where $a = 3$, but since it isn't basic, we'll make a substitution:

$$u = 2x + 1$$

to make this a basic function; hence, with:

$$u = 2x + 1$$

$$\frac{du}{dx} = 1 \times 2 \times x^{1-1}$$

$$\frac{du}{dx} = 2$$

$$\text{Hence, here, } \; dx = \frac{du}{2}$$

Make substitutions for $(2x + 1)$ and dx in the integral;

$$\int 3^u \times \frac{du}{2}$$

We can now bring our constants out;

$$\frac{1}{2} \int 3^u \, du$$

We now apply the integral of a^u which is a standard integral, where $a = 3$, we have;

$$\frac{1}{2}\left[\frac{3^u}{\ln 3}\right]$$

Hence, we return the u we borrowed for the sake of integration into this and of course add the normal – the arbitrary constants:

$$\frac{3^{2x+1}}{2\ln 3} + C$$

More speed!

- $\int -\frac{2}{7}\cos(7x - 4)\, dx$

So here, this is in the form $\cos x$ but since it isn't basic, we'll make a substitution:

$$u = 7x - 4$$

to make this a basic function; hence, with

$$u = 7x - 4$$

$$\frac{du}{dx} = 1 \times 7 \times x^{1-1} + 0$$

$$\frac{du}{dx} = 7$$

Hence, here, $dx = \frac{du}{7}$

Make substitutions for $(7x - 4)$ and dx in the integral;

$$\int -\frac{2}{7}\cos(u) \times \frac{du}{7}$$

We can now bring our constants out;

$$-\frac{2}{7} \times \frac{1}{7}\int \cos u \, du$$

We now apply the integral of $\cos u$ which is a standard integral, we have;

$$-\frac{2}{49}[\sin u]$$

Hence, we return the u we borrowed for the sake of integration into this and of course add the normal – the arbitrary constants:

$$-\frac{2}{49}\sin(7x-4)+C$$

I guess that's the last one, we still need more speed though!

- $\int \frac{3}{5}\sec^2(4-3x)\,dx$

So here, this is in the form $\sec^2 x$ but since it isn't basic, we'll make a substitution:

$$u=4-3x$$

to make this a basic function; hence, with

$$u=4-3x$$

$$\frac{du}{dx}=0-1\times 3\times x^{1-1}=-3$$

Hence, here, $dx = -\frac{du}{3}$

Make substitutions for $(4 - 3x)$ and dx in the integral;

$$\int \frac{3}{5}\sec^2(u) \times -\frac{du}{3}$$

We can now bring our constants out;

$$\frac{3}{5} \times -\frac{1}{3}\int \sec^2 u \, du$$

We now apply the integral of $\sec^2 u$ which is a standard integral, we have;

$$-\frac{1}{5}[\tan u]$$

Hence, we return the u we borrowed for the sake of integration into this and of course add the normal – the arbitrary constants:

$$-\frac{1}{5}\tan(4 - 3x) + C$$

So, just these last two under the function of a linear function;

So, we have this integral.

- $\int \frac{3}{4(2x-7)^5} dx$

To evaluate this integral; we'll first of make the substitution since we have a linear function;

$$u = 2x - 7$$

$$\frac{du}{dx} = 1 \times 2 \times x^{1-1} - 0$$

$$\frac{du}{dx} = 2$$

$$\text{Hence, here, } dx = \frac{du}{2}$$

Make substitutions for $(2x - 7)$ and dx in the integral;

$$\int \frac{3}{4(u)^5}\frac{du}{2}$$

We can now bring our constants out;

$$\frac{3}{4} \times \frac{1}{2}\int \frac{1}{u^5}\,du$$

We'll convert the denominator power to u^{-5} in the numerator,

$$\frac{3}{4} \times \frac{1}{2}\int u^{-5}\,du$$

We now apply the integral of u^n which is a standard integral, we have;

$$\frac{3}{8}\left[\frac{u^{-5+1}}{-5+1}\right] = \frac{3}{8}\left[\frac{u^{-4}}{-4}\right] = -\frac{3}{32}u^{-4}$$

Hence, we return the u we borrowed for the sake of integration into this and of course add the normal – the arbitrary constants:

$$-\frac{3}{32}(2x-7)^{-4}+C$$

Convert the negative power to the denominator;

$$-\frac{3}{32(2x-7)^{4}}+C$$

Also, let’s see this:

So, we have this integral.

- $\int \frac{4}{7(7-3x)}dx$

To evaluate this integral; we’ll first of make the substitution since we have a linear function;

$$u=7-3x$$

$$\frac{du}{dx}=0-1\times 3\times x^{1-1}$$

$$\frac{du}{dx}=-3$$

$$\text{Hence, here, } dx = -\frac{du}{3}$$

Make substitutions for $(7 - 3x)$ and dx in the integral;

$$\int \frac{4}{7(u)} \times -\frac{du}{3}$$

We can now bring our constants out;

$$\frac{4}{7} \times -\frac{1}{3} \int \frac{1}{u} du$$

We now apply the integral of $\frac{1}{u}$ which is a standard integral, that is equal to $\ln u$, used that as a refreshing process in case you'd forgotten about it; we have;

$$\frac{-4}{21} [\ln u]$$

Hence, we return the u we borrowed for the sake of integration into this and of course add the normal – the arbitrary constants:

$$-\frac{4}{21}\ln(7-3x)+C$$

Now, you know the basic idea of **substitution in integration**, but watch it, that's just very straightforward when it is a linear function, when it isn't linear, it isn't just like that, it involves opening your eyes more than this; in this first section, ***you just have to open your eyes to see that it is a linear function in an integral, that's all you need, follow the regular steps and you'll be done!***

However, there are other cases of substitution;

Integration by Substitution;

Consider situations when we have integrals in the form:

$$\int \frac{f'(x)}{f(x)}dx$$

$$\int f'(x)\,f(x)dx$$

$$\int f'(x)\,g[f(x)]dx$$

Let's see these three typical situations and let's see if we can have 20 examples or thereabouts; yeah, it's that serious, twenty! Once again, it is easy to give you formulas but I won't wanna do that, that isn't healthy to ensuring you really understand what is going on here!

Case 1:

$$\int \frac{f'(x)}{f(x)}\,dx$$

Now, obviously, this we appear as two functions dividing each other and hence, not a standard integral at all and nothing can be done directly, but like we did the other time, we make the substitution thus;

$$u = f(x)$$

Then:

$$\frac{du}{dx} = f'(x)$$

So, we have:

$$dx = \frac{du}{f'(x)}$$

Still looks more like the process in linear functions, keep watching though.

So, returning these substitutions back into the integral; and making substitutions for $f(x)$ and dx, where:

$$u = f(x)$$

$$dx = \frac{du}{f'(x)}$$

We have:

$$\int \frac{f'(x)}{f(x)}\,dx = \int \frac{f'(x)}{u}\frac{du}{f'(x)}$$

So, hence, $f'(x)$ cancels off; and we're where we're going! After cancelling off, we're left with something decent like this;

$$\int \frac{1}{u}\,du$$

This now has been reduced to a standard integral and thus is:

$$\ln u$$

We return what u actually is and we're left with:

$$\ln f(x) + C$$

That's the typical CASE 1.

Case 2;

$$\int f'(x)\,f(x)dx$$

Again, since $f'(x)$ is still at the numerator (assuming the denominator is 1), we still make the same substitution;

$$u = f(x)$$

Then:

$$\frac{du}{dx} = f'(x)$$

So, we have:

$$dx = \frac{du}{f'(x)}$$

So, returning these substitutions back into the integral; and making substitutions for $f(x)$ and dx, where:

$$u = f(x)$$

$$dx = \frac{du}{f'(x)}$$

We have:

$$\int f'(x)\, f(x) dx = \int f'(x)\, [u] \frac{du}{f'(x)}$$

So, hence, $f'(x)$ cancels off; and we're where we're going! After cancelling off, we're left with something very decent like this;

$$\int u\, du$$

This is simply a power function rule; the integral is;

$$\frac{u^{1+1}}{1+1} + C = \frac{u^2}{2} + C$$

We return what u actually is and we're left with:

$$\frac{[f(x)]^2}{2} + C$$

That's the typical CASE 2.

So, let's have **CASE 3:**

$$\int f'(x)\, g[f(x)]dx$$

Here, once again, we have a quite tricky situation here, where g is a function of $f(x)$, the interesting part of this however is that; we're still making the same substitution;

We have;

$$u = f(x)$$

Then:

$$\frac{du}{dx} = f'(x)$$

So, we have:

$$dx = \frac{du}{f'(x)}$$

So, returning these substitutions back into the integral; and making substitutions for $f(x)$ and dx, where:

$$u = f(x)$$

$$dx = \frac{du}{f'(x)}$$

We have:

$$\int f'(x)\, g[f(x)]dx = \int f'(x)\, g[u]\frac{du}{f'(x)}$$

Hence, here, $f'(x)$ cancels out and we're left with a more decent integral thus;

$$\int g[u]\,du$$

So, g will be a function such as an exponential, trigonometric or whatever, but it is expected to be a standard integral, and hence, the integral is gotten and u is put back in place as $f(x)$.

So, in these three typical cases of substitution, these are the major rules;

- Firstly, the first thing you have to notice is if you can find any two functions related by their derivative in the integral, such situations could include, finding two functions in the integral like:

- $(2x + 3)$ and $(x^2 + 3x - 6)$; here, $2x + 3$ is the derivative of $(x^2 + 3x - 6)$
- $sin\, x$ and $\cos x$
- e^x and e^x

- $log_e\, x$ and $\frac{1}{x}$
- And many other similar situations;

Also, when the only difference between the **derivative and its function** pair is only a constant (multiplying either of the derivatives or the function); examples are given below;

- $3x$ and $4x^2 + 3$; here, the derivative of $(4x^2 + 3)$ is $8x$, hence, 3 and 8 are the only difference between the derivative of the main function and the other function.
- $\sin x$ and $\cos x$; the derivative of $\cos x$ is $-\sin x$ and hence, -1 is the only difference between the two.
- $4x + 6$ which can be factorized as $2(2x + 3)$ and $x^2 + 3x + 4$; the derivative of $(x^2 + 3x + 4)$ is $2x + 3$ and hence, 2 is the only difference relating them as derivative and function pair.

- Another very important rule is that as you can see in all the above three cases;

  The $f'(x)$, which is the derivative must always be on the numerator in the case of a fraction; I guess that's it, let's now begin! 20 questions yeah!

Let's list the problems we have to tackle out now!

Let's find the following integrals. We'll break each down properly and hence, you'll understand them.

- $\int \frac{2x+3}{x^2+3x-5} dx$
- $\int \frac{6x^2}{x^3-4} dx$
- $\int \cot x \, dx$
- $\int \tan x \, dx$

- $$\int \frac{4x-8}{x^2-4x+5}\,dx$$

- $$\int \frac{x-3}{x^2-6x+2}\,dx$$

- $$\int \tan x \sec^2 x \, dx$$

- $$\int \sin x \cos x \, dx$$

- $$\int \frac{\ln x}{x}\,dx$$

- $$\int (x^2+7x-4)(2x+7)\,dx$$

- $$\int x\sqrt{1+x^2}\,dx$$

- $$\int e^{\cos x} \sin x \, dx$$

- $$\int (2x-5)a^{x^2-5x+7}\,dx$$

- $\int (2ax + b)(ax^2 + bx)^7 \, dx$
- $\int \frac{\cos\theta}{1 + \sin\theta} \, d\theta$
- $\int \frac{\tan x}{\log\cos x} \, dx$
- $\int \frac{1}{x \log x} \, dx$
- $\int x^2 (x^3 - 5)^3 \, dx$
- $\int \frac{\sin x - \cos x}{\sin x + \cos x} \, dx$
- $\int x^3 e^{x^4} \, dx$

So… … … Quite some questions, a whole twenty and guess what, we're *gonna* solve everything!

Quickly, let's begin, I'm sure you still have all the case studies, case 1, case 2 and case 3 at the back of your head, so let's begin!

NUMBER ONE:

- $\int \frac{2x+3}{x^2+3x-5}\,dx$

So here, after seeing the two functions; the function in the denominator is of a higher degree than the numerator, hence, we suspect that the numerator is its derivative, after suspecting however, we need to check it out for real and not only by suspecting.

$$\frac{d}{dx}(x^2+3x-5)$$
$$= 2 \times x^{2-1} + 1 \times 3 \times x^{1-1} + 0$$

$$\frac{d}{dx}(x^2+3x-5) = 2x+3$$

Hence, our suspicion is true! So we proceed to the substitution process;

This is a case of:

$$\frac{f'(x)}{f(x)}$$

Here;

$$f(x) = x^2 + 3x - 5$$
$$f'(x) = 2x + 3$$

As we saw when explaining the different cases, we know that we are substituting for $f(x)$

$$u = x^2 + 3x - 5$$

We have checked above already:

$$\frac{du}{dx} = 2x + 3$$

Hence,

$$dx = \frac{du}{2x + 3}$$

Substitute back into the integral, substitute for $(x^2 + 3x - 5)$ and for dx

$$\int \frac{2x + 3}{u} \times \frac{du}{2x + 3}$$

$(2x + 3)$ cancels out, we now have a decent situation;

$$\int \frac{1}{u} du$$

Hence, our standard integral tells us that this is:

$$\ln u + C$$

We have our arbitrary constant already, hence, we return the value of u in place, the integral is:

$$\ln(x^2 + 3x - 5) + C$$

EASY! Right?

Let's keep going on. Quite a long way to go!

- $\int \frac{6x^2}{x^3 - 4} dx$

So here, after seeing the two functions; the function in the denominator is of a higher degree than the numerator, hence, we suspect that the numerator is its derivative, after suspecting however, we need to check it out for real and not only by suspecting.

$$\frac{d}{dx}(x^3 - 4) = 3 \times 1 \times x^{3-1} - 0$$

$$\frac{d}{dx}(x^3 - 4) = 3x^2$$

Hence, our suspicion is quite true but not directly true, but since it's just the constants that differ (as you can see the derivative is $3x^2$ and the numerator is $6x^2$, the difference is only 3 and 6), we can successfully implement the process! Let's proceed to the substitution process;

This is a case of:

$$\frac{f'(x)}{f(x)}$$

As we saw when explaining the different cases, we know that we are substituting for $f(x)$

$$u = x^3 - 4$$

We have checked above already:

$$\frac{du}{dx} = 3x^2$$

Hence,

$$dx = \frac{du}{3x^2}$$

Substitute back into the integral, substitute for $(x^3 - 4)$ and for dx

$$\int \frac{6x^2}{u} \times \frac{du}{3x^2}$$

x^2 cancels out, we now have a decent situation;

$$\int \frac{2}{u} du$$

We bring out the constant and apply the integral;

$$2 \int \frac{1}{u} du$$

Hence, our standard integral tells us that this is:

$$2(\ln u) + C$$

We have our arbitrary constant already, hence, we return the value of u in place, the integral is:

$$2 \ln(x^3 - 4) + C$$

Next!

- $\int \cot x \, dx$

Here, we have $\cot x$; from trigonometric identities;

$$\cot x = \frac{\cos x}{\sin x}$$

Hence;
The integral is:

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx$$

So here, we have two functions which are derivatives with respect to each other; We know the derivative of $\cos x$ is $-\sin x$ and the derivative of $\sin x$ is $\cos x$ and both are derivatives of the other, only the -1 symbol differentiates derivative of $\cos x$ and hence, we'll simply substitute for the one in the denominator since the denominator is taken as the function in the Case 1 situation and is substituted for.

$$u = \sin x$$

$$\frac{du}{dx} = \cos x$$

Hence,

$$dx = \frac{du}{\cos x}$$

So, we'll substitute back into the integral, substitute for $(\sin x)$ and for dx

$$\int \frac{\cos x}{u} \times \frac{du}{\cos x}$$

$\cos x$ cancels out, we now have a decent situation;

$$\int \frac{1}{u} du$$

No constant to bring out so hence, we apply the integral; our standard integral tells us that this is:

$$\ln u + C$$

We have our arbitrary constant already, hence, we return the value of u in place, the integral is:

$$\ln \sin x + C$$

Note that this is also a Case 1 situation;

Next!

- $\int \tan x \, dx$

Here, we have $\tan x$; from trigonometric identities;

$$\tan x = \frac{\sin x}{\cos x}$$

Hence;
The integral is:

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$$

Just like in the very previous example, we established that $\cos x$ and $\sin x$ are derivatives of each other, hence, here; we'll substitute for $\cos x$ since it is at the denominator;

$$u = \cos x$$

$$\frac{du}{dx} = -\sin x$$

Hence,

$$dx = -\frac{du}{\sin x}$$

So, we'll substitute back into the integral, substitute for $(\cos x)$ and for dx

$$\int \frac{\sin x}{u} \times -\frac{du}{\sin x}$$

$\sin x$ cancels out, we now have a decent situation;

$$\int -\frac{1}{u} du$$

We bring out the constant -1, and after, we apply the integral; our standard integral tells us that this is:

$$-1 \int \frac{1}{u} du$$

$$-\ln u + C$$

We have our arbitrary constant already, hence, we return the value of u in place, the integral is:

$$-\ln \cos x + C$$

This is a Case 1 situation.

So, the next question;

- $\int \frac{4x-8}{x^2-4x+5}dx$

So here, after seeing the two functions; the function in the denominator is of a higher degree than the numerator, hence, we suspect that the numerator is its derivative, after suspecting however, we need to check it out for real and not only by suspecting.

$$\frac{d}{dx}(x^2-4x+5)$$
$$= 2 \times 1 \times x^{2-1} - 1 \times 4 \times x^{1-1} + 0$$

$$\frac{d}{dx}(x^2-4x+5) = 2x-4$$

Now, look at this situation carefully, it'll seem as if this will not be a Case 1 situation; cos the derivative of the denominator and the function at the denominator do not look alike, and once they're not the same, our substitution will make no sense as it will not cancel out and you'll end up with two variables in a single integral and hence, it'll be a meaningless integral.

But wait a minute, let's access our numerator very well.

$$4x - 8$$

Don't you think it can be factorized; fine let's factorize it, 4 can be factored out for us to be left with:

$$4(x - 2)$$

Still we don't have the derivative of the denominator showing up, I guess this question isn't under this section of substitution anyway! NO! Calm down!

Factorize the numerator with 2, 2 is factored out leaving us with:

$$2(2x - 4)$$

I guess the term in the bracket is the derivative of the denominator right, hence, they're much related and differ with merely a constant, and hence, we can successfully proceed with our substitution.

Hence, our suspicion is still quite true but not directly true, but since it's just the constants that differ, we can successfully implement the process! Let's proceed to the substitution process;

$$u = x^2 - 4x + 5$$

We have checked above already:

$$\frac{du}{dx} = 2x - 4$$

Hence,

$$dx = \frac{du}{2x - 4}$$

Substitute back into the integral, substitute for $(2x - 4)$ and for dx

$$\int \frac{4x - 8}{u} \times \frac{du}{2x - 4}$$

Factorize the numerator as we have established at first!

$$\int \frac{2(2x - 4)}{u} \times \frac{du}{2x - 4}$$

So, $(2x - 4)$ then cancels out! Leaving us with a decent integral;

$$\int \frac{2}{u} du$$

We bring out the constant and apply the integral;

$$2\int \frac{1}{u} du$$

Hence, our standard integral tells us that this is:

$$2(\ln u) + C$$

We have our arbitrary constant already, hence, we return the value of u in place, the integral is:

$$2\ln(x^2 - 4x + 5) + C$$

Of course, this is a Case 1 situation!

We really need to move faster!

- $\displaystyle\int \frac{x-3}{x^2 - 6x + 2} dx$

This is very similar to the just previous question; however, it's like this!

Here, after seeing the two functions; the function in the denominator is of a higher degree than the numerator, hence, we suspect that the numerator is its derivative, after suspecting however, we need to check it out for real and not only by suspecting.

$$\frac{d}{dx}(x^2 - 6x + 2)$$
$$= 2 \times 1 \times x^{2-1} - 1 \times 6 \times x^{1-1} + 0$$
$$\frac{d}{dx}(x^2 - 6x + 2) = 2x - 6$$

So remember all the factorization saga in the previous question, the only thing here is that we'll be factorizing the derivative of the denominator to adjust it to the numerator, let's make the substitution straight! They're actually related because if you factorize the derivative of the denominator which is $2x - 6$, you'll have $2(x - 3)$ which is perfectly in terms of the numerator, the constant, 2, the only difference!

Hence, we can successfully proceed with our substitution.

$$u = x^2 - 6x + 2$$

We have checked above already:

$$\frac{du}{dx} = 2x - 6$$

Hence,

$$dx = \frac{du}{2x - 6}$$

Substitute back into the integral, substitute for $(2x - 4)$ and for dx

$$\int \frac{x - 3}{u} \times \frac{du}{2x - 6}$$

Factorize the derivative of the denominator as we have established above;

$$\int \frac{x - 3}{u} \times \frac{du}{2(x - 3)}$$

So, $(x - 3)$ then cancels out! Leaving us with a decent integral;

$$\int \frac{1}{2u} du$$

We bring out the constant and apply the integral;

$$\frac{1}{2}\int \frac{1}{u}\,du$$

Hence, our standard integral tells us that this is:

$$\frac{1}{2}(\ln u) + C$$

We have our arbitrary constant already, hence, we return the value of u in place, the integral is:

$$\frac{1}{2}\ln(x^2 - 6x + 2) + C$$

Of course, this again, is a Case 1 situation!

NEXT! We really need to move faster!

- $\displaystyle\int \tan x \sec^2 x\, dx$

Hmm, yeah, these are two functions; but one is a derivative of the other. By what we know about derivatives;

$$\sec^2 x \text{ is the derivative of } \tan x$$

Hence, we have a **CASE 2** situation here, product of a function and its derivative;

Hence, we'll proceed with the substitution, you know we'll be substituting for the function and not its derivative, hence; here; we have:

$$u = \tan x$$

It follows that:

$$\frac{du}{dx} = \sec^2 x$$

Hence,

$$dx = \frac{du}{\sec^2 x}$$

Substitute back into the integral, substitute for $(\tan x)$ and for dx

$$\int u \sec^2 x \frac{du}{\sec^2 x}$$

So, $\sec^2 x$ then cancels out! Leaving us with a decent integral;

$$\int u \, du$$

There is no constant to bring out, hence, we apply the integral, the integral is;

$$\left[\frac{u^{1+1}}{1+1}\right] + C$$

We have;

$$\frac{u^2}{2} + C$$

We have our arbitrary constant already, hence, we return the value of u in place, the integral is:

$$\frac{(\tan x)^2}{2} + C$$

This can be written as:

$$\frac{\tan^2 x}{2} + C$$

Of course, I've mentioned it, this is a Case 2 situation!

NEXT! More speed please!

- $\int \sin x \cos x \, dx$

These two close friends have come again; they're derivatives with respect to each other, in this case, since they're both in the numerator, we can use any of them for our substitution. This is because;

If we substitute for $\cos x$

The derivative will be $-\sin x$ which will eventually cancel out;

If we substitute for $\sin x$

The derivative will be $\cos x$ which will still cancel out either way;

Well, by the way, this obviously is a Case 2 situation.

I think I'm a bigger fan of $\sin x$ so I'll go for it to apply the substitution!

$$u = \sin x$$

It follows that:

$$\frac{du}{dx} = \cos x$$

Hence,

$$dx = \frac{du}{\cos x}$$

Substitute back into the integral, substitute for $(\sin x)$ and for dx

$$\int u \cos x \frac{du}{\cos x}$$

So, $\cos x$ then cancels out! Leaving us with a decent integral;

$$\int u\, du$$

There is no constant to bring out, hence, we apply the integral, the integral is;

$$\left[\frac{u^{1+1}}{1+1}\right] + C$$

We have;

$$\frac{u^2}{2} + C$$

We have our arbitrary constant already, hence, we return the value of u in place, the integral is:

$$\frac{(\sin x)^2}{2} + C$$

This can be written as:

$$\frac{\sin^2 x}{2} + C$$

Well, for the fans of $\boldsymbol{\cos x}$

If we'd used the substitution of $\cos x$; we'd have gotten a quite different answer for our integral which will be;

$$-\frac{\cos^2 x}{2} + C$$

You can check that by substituting for $\cos x$ instead;

Both are equally integrals of the given integral we needed, it's just a peculiarity between sin and cos functions in calculus, you can of course try it, differentiate both answers, you'll end up in $\sin x \cos x$ as your derivative which confirms that the both integrals are correct!

Of course, I've mentioned it, this is a Case 2 situation!

NEXT!

- $\int \frac{\ln x}{x} dx$

This looks pretty tricky! It's threatening to violate the rule that we must always substitute for the denominator, this is understandable though;

This is because, the function $\ln x$ which we all know its derivative is $\frac{1}{x}$ which is obviously an inverse value as its derivative and hence, during substitution for its differential coefficient, it won't cancel out as the rest are doing; let's see this and I'll explain better!

Obviously we know we're to substitute for $\ln x$; hence, this is what we have!

$$u = \ln x$$

It follows that:

$$\frac{du}{dx} = \frac{1}{x}$$

Hence, if you simplify well, we have that:

$$dx = xdu$$

Let's still substitute back into the integral before I explain, substitute for $(\ln x)$ and for dx

$$\int \frac{u}{x} xdu$$

Obviously, x will cancel out leaving us with udu, a perfect integral;

Now, imagine if x was at the numerator; we'll be having this integral;

$\int \boldsymbol{u\,x^2du}$ ***which is as meaningless as anything,***

You can't have two variables in an integral, hence, all these aspects of substitution, the canceling out aspect that leaves us with one variable is the most important part of the substitution rule!

So, here, since it's a correct question, x is at the denominator, and it'll cancel out soft! Leaving us with a decent integral;

To make this finally completely clear; it is a case 2 situation and not a case one situation and hence, both pairs (function and derivative are multiplying each other and none is at the top as it were)

$$f(x) = \ln x$$

$$f'(x) = \frac{1}{x}$$

Hence;

$$f'(x)f(x) = \frac{1}{x} \times \ln x$$

If both are made a uniform fraction;

$$f'(x)f(x) = \frac{\ln x}{x}$$

Hence, it is a case 2 situation; let's continue from where we stopped before the explanation session;

$$\int \frac{u}{x} x du = \int u \, du$$

There is no constant to bring out, hence, we apply the integral, the integral is;

$$\left[\frac{u^{1+1}}{1+1}\right] + C$$

We have;

$$\frac{u^2}{2} + C$$

We have our arbitrary constant already, hence, we return the value of u in place, the integral is:

$$\frac{(\ln x)^2}{2} + C$$

As said already, this is a Case 2 situation!

NEXT! More speed please!

- $\int (x^2 + 7x - 4)(2x + 7)\, dx$

Alright, we have two functions with one a higher degree than the other, both multiplying each other, we have to start our suspicion right away; let's confirm our suspicion though;

$$\frac{d}{dx}\left(x^2 + 7x - 4\right)$$
$$= 2 \times 1 \times x^{2-1} - 1 \times 7 \times x^{1-1} + 0$$

$$\frac{d}{dx}(x^2 + 7x - 4) = 2x + 7$$

Our suspicion is softly confirmed; it didn't bother us with the headache of factorization; so, let's apply our substitution straight!

$$u = x^2 + 7x - 4$$

It follows that:

$$\frac{du}{dx} = 2x + 7$$

Hence,

$$dx = \frac{du}{2x + 7}$$

Substitute back into the integral, substitute for $(x^2 + 7x - 4)$ and for dx

$$\int u(2x + 7)\frac{du}{2x + 7}$$

So, $(2x + 7)$ then cancels out! Leaving us with a decent integral;

$$\int u\,du$$

There is no constant to bring out, hence, we apply the integral, the integral is;

$$\left[\frac{u^{1+1}}{1+1}\right] + C$$

We have;

$$\frac{u^2}{2} + C$$

We have our arbitrary constant already, hence, we return the value of u in place, the integral is:

$$\frac{(x^2 + 7x - 4)^2}{2} + C$$

NEXT!

Right, this is Case 3 situation, I'm sure if any of the cases seemed complicated, it's the Case 3 but here, you'll understand it perfectly!

- $\int x\sqrt{1+x^2}\,dx$

Alright, we have two functions; x and $\sqrt{1+x^2}$; the second function however, is a function of another function $(1+x^2)$ which is a degree higher that the second function. Let's first confirm if there is a derivative relationship between the two!

$$\frac{d}{dx}(1+x^2) = 0 + 1 \times 2 \times x^{2-1}$$

$$\frac{d}{dx}(1+x^2) = 2x$$

Well, it's still a good one, the only difference between the derivative and the first function is 2 so they can properly cancel out; however, what is the difference between this Case 3 and the other first two cases?

What we're substituting for here, goes into another function, hence, after successful cancelation, we'll still have another integral to work on.

Here... ... ...

$$u = 1 + x^2$$

You can see this is quite different from where we'll take the whole of the second term (or the denominator) as a whole;

$$\frac{du}{dx} = 2x$$

Hence,

$$dx = \frac{du}{2x}$$

Substitute back into the integral, substitute for $(1 + x^2)$ and for dx

$$\int x\sqrt{u}\,\frac{du}{2x}$$

So, x then cancels out! Leaving us with this integral;

$$\int \frac{\sqrt{u}}{2}\,du$$

We bring out the constant, and after, we apply the integral, the integral is;

$$\frac{1}{2}\int \sqrt{u}\, du$$

We convert the square root to fractional power and apply the power rule of integration;

$$\frac{1}{2}\int u^{\frac{1}{2}}\, du$$

$$\frac{1}{2}\left[\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right] = \frac{1}{2}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]$$

$$\frac{1}{2} \times u^{\frac{3}{2}} \times \frac{2}{3} = \frac{4}{3}u^{\frac{3}{2}}$$

We can simplify into root form;

$$\frac{4}{3}\sqrt{u^3} + C$$

We won't forget our arbitrary constant, we've added it already, hence, we return the value of u in place, the integral is:

$$\frac{4}{3}\sqrt{(1 + x^2)^3} + C$$

Next! We move on, another case 3 situation;

- $\int e^{\cos x} \sin x \, dx$

Alright, we have two functions; $e^{\cos x}$ and $\sin x$; the first function however, is a function of another function $(\cos x)$ which we know is a very close friend with $\sin x$. So, we'll simply make the substitution for $\cos x$;

$$u = \cos x$$

You can also see that this is quite different from where we'll take the whole of the second term (or the denominator) as a whole;

$$\frac{du}{dx} = -\sin x$$

Hence,

$$dx = -\frac{du}{\sin x}$$

Substitute back into the integral, substitute for $(\cos x)$ and for dx

$$\int e^u \sin x \times -\frac{du}{\sin x}$$

So, $\sin x$ then cancels out! Leaving us with this integral;

$$\int -e^u \, du$$

We bring out the constant(-1), and after, we apply the integral, the integral is;

$$-1 \int e^u \, du$$

Apply the exponential standard integral rule!

$$-1[e^u] + C$$

$$-e^u + C$$

We won't forget our arbitrary constant, we've added it already, hence, we return the value of u in place, the integral is:

$$-e^{\cos x} + C$$

Next, another case 3 situation!

- $\int (2x-5)a^{x^2-5x+7}\,dx$

Alright, we have two functions; $(2x-5)$ and a^{x^2-5x+7}; the second function however, is a function of another function (x^2-5x+7) which is a degree higher that the second function. Let's first confirm if there is a derivative relationship between the two!

$$\frac{d}{dx}(x^2-5x+7)$$
$$= 2 \times 1 \times x^{2-1} - 1 \times 5 \times x^{1-1} + 0$$

$$\frac{d}{dx}(x^2-5x+7) = 2x-5$$

Fine, this is dope now, directly, the first term is the derivative of the function the second function depends on, hence, our substitution is going on smoothly!

$$u = x^2-5x+7$$

$$\frac{du}{dx} = 2x-5$$

Hence,

$$dx = \frac{du}{2x - 5}$$

Substitute back into the integral, substitute for $(x^2 - 5x + 7)$ and for dx

$$\int (2x - 5)a^u \frac{du}{2x - 5}$$

So, $(2x - 5)$ then cancels out! Leaving us with this integral;

$$\int a^u \, du$$

There's no constant to bring out, so we'll apply the integral, the standard integral is;

$$\left[\frac{a^u}{\ln a}\right] + C$$

We won't forget our arbitrary constant, we've added it already, hence, we return the value of u in place, the integral is:

$$\frac{a^{x^2-5x+7}}{\ln a} + C$$

Cool, let's just continue;

Next, another case 3 situation!

- $\int (2ax + b)(ax^2 + bx)^7 \, dx$

We have two functions; $(2ax + b)$ and $(ax^2 + bx)^7$; the second function however, is a function of another function $(ax^2 + bx)$ which is a degree higher that the second function. Let's first confirm if there is a derivative relationship between the two!

$$\frac{d}{dx}(ax^2 + bx) = 2 \times 1 \times x^{2-1} - 1 \times b \times x^{1-1}$$

$$\frac{d}{dx}(ax^2 + bx) = 2ax + b$$

Hence, dope, directly, the first term is the derivative of the function the second function depends on, hence, our substitution is going on smoothly!

$$u = ax^2 + bx$$

$$\frac{du}{dx} = 2ax + b$$

Hence,

$$dx = \frac{du}{2ax + b}$$

Substitute back into the integral, substitute for $(ax^2 + bx)$ and for dx;

$$\int (2ax + b)(u)^7 \frac{du}{2ax + b}$$

So, $(2ax + b)$ then cancels out! Leaving us with this integral;

$$\int u^7 \, du$$

There's no constant to bring out, so we'll apply the integral by power rule, the integral is;

$$\left[\frac{u^{7+1}}{7+1}\right] + C = \frac{u^8}{8} + C$$

We won't forget our arbitrary constant, we've added it already, hence, we return the value of u in place, the integral is:

$$\frac{(ax^2 + bx)^8}{8} + C$$

We're almost done, I'm sure you're loving the way you're just understanding it!

Next… … … …

- $\displaystyle\int \frac{\cos\theta}{1 + \sin\theta}\, d\theta$

Alright, we have two functions; $\cos\theta$ and $(1 + \sin\theta)$; Now, the good friends, $\cos x$ and $\sin x$ seemed not yet to agree in this, however, they agree over here.

$$\frac{d}{dx}(1 + \sin\theta) = 0 + \cos\theta$$

Hence,

$$\frac{d}{dx}(1 + \sin\theta) = \cos\theta$$

Hence, dope, directly, the numerator is the derivative of the function the second function depends on, hence, our substitution is going on smoothly!

The only thing that seemed to stand in the way of the function and derivative relationship between $\sin\theta$ and $\cos\theta$ is merely 1 added to it, which dissolved to zero.

$$u = 1 + \sin\theta$$

$$\frac{du}{d\theta} = \cos\theta$$

Hence,

$$d\theta = \frac{du}{\cos\theta}$$

Substitute back into the integral, substitute for $(1 + \sin\theta)$ and for $d\theta$;

$$\int \frac{\cos\theta}{u} \times \frac{du}{\cos\theta}$$

So, $(\cos\theta)$ then cancels out! Leaving us with this integral;

$$\int \frac{1}{u} du$$

There's no constant to bring out, so we'll apply the integral, a standard integral;

$$\ln u + C$$

We won't forget our arbitrary constant, we've added it already, hence, we return the value of u in place, the integral is:

$$\ln(1 + \sin\theta) + C$$

Moving on… … …

- $\displaystyle\int \frac{\tan x}{\log\cos x} dx$

Alright, we have two functions; $\tan x$ and $(\log\cos x)$; the denominator however here is looking tricky, we got NO CHOICE; Let us just verify this, it isn't showing any green light at all, as in like at all.

However, you should start suspecting that of logarithm functions that involves the reciprocal of the function, before any suspecting, let's just confirm this soft!

$$\frac{d}{dx}(\log\cos x) = \frac{d}{dx}(\log_{10}\cos x)$$

When there's no base, the log base is base 10.

The integral substitution will be

$$u = \log_{10}\cos x$$

We need the derivative of this, but this is a function of a function, hence, we need to make the substitution (*so you mean function of function followed us here)*;

$$z = \cos x$$

$$\frac{dz}{dx} = -\sin x$$

Hence,

$$u = \log_{10} z$$

From differentiation;

$$\frac{du}{dz} = \frac{1}{z\ln 10}$$

From chain rule;

$$\frac{du}{dx} = \frac{du}{dz} \times \frac{dz}{dx}$$

$$\frac{du}{dx} = \frac{1}{z \ln 10} \times -\sin x = -\frac{\sin x}{z \ln 10}$$

Return $z = \cos x$

$$\frac{du}{dx} = -\frac{\sin x}{\cos x \ln 10}$$

But from trigonometry,

$$\frac{\sin x}{\cos x} = \tan x$$

I'm sure you know that identity, hence;

$$\frac{du}{dx} = -\frac{\tan x}{\ln 10}$$

Hence, we have now seen the derivative of the denominator, now, we can see $-\frac{\tan x}{\ln 10}$ and hence, the derivative of the denominator is fully related to the numerator, this is because the only difference between them is the constant $-\frac{1}{\ln 10}$, and hence, we can legibly use substitution for this!

Hence,

$$u = \log_{10} \cos x$$

$$\frac{du}{dx} = -\frac{\tan x}{\ln 10}$$

Hence, after rearranging;

$$dx = -\frac{du \ln 10}{\tan x}$$

Substitute back into the integral, substitute for $(\log_{10} \cos x)$ and for dx;

$$\int \frac{\tan x}{u} \times -\frac{du \ln 10}{\tan x}$$

So, $(\tan x)$ then cancels out! Leaving us with this integral;

$$\int \frac{-\ln 10}{u} du$$

Bring out the constant and after, we'll apply the integral, a standard integral;

$$-\ln 10 \int \frac{1}{u} du$$

Hence, from our standard integral;

$$-\ln 10\,(\ln u) + C$$

We won't forget our arbitrary constant, we've added it already, hence, we return the value of u in place, the integral is:

$$-\ln 10\,(\ln(\log \cos x)) + C$$

We could substitute for the true value of $\ln 10$ to have:

$$-2.303(\ln(\log \cos x)) + C$$

Quite a big integral but the question is still kay, it is a proper CASE 1 situation!

Moving on… … …

- $\int \frac{1}{x \log x} dx$

Here, we can see that the derivative of $\log x$ is $\frac{1}{x \ln 10}$ since $\log x$ implies $\log_{10} x$;

Hence, the derivative of $\log x$ and the second function, x are very related, since $\frac{1}{x \ln 10}$ is inverse, this should cancel out with x in the denominator;

So, let's make our substitution.

$$u = \log_{10} x$$

$$\frac{du}{dx} = \frac{1}{x \ln 10}$$

Hence, after rearranging;

$$dx = x \ln 10 \, du$$

Substitute back into the integral, substitute for $(\log x)$ and for dx;

$$\int \frac{1}{xu} \times x \ln 10 \, du$$

So, x then cancels out! Leaving us with this integral;

$$\int \frac{1}{u} \ln 10 \, du$$

Bring out the constant and after, we'll apply the integral, a standard integral;

$$\ln 10 \int \frac{1}{u} du$$

Hence, from our standard integral;

$$\ln 10 \, (\ln u) + C$$

We won't forget our arbitrary constant, we've added it already, hence, we return the value of u in place, the integral is:

$$\ln 10 \, (\ln(\log x)) + C$$

We could substitute for the true value of $\ln 10$ to have:

$$2.303(\ln(\log x)) + C$$

We're almost done!

There is something you need to know about questions like this in Case 1 to Case 3; when it seems confusing like this, you can test for the substitution on a rough paper and see if it'll possibly cancel out.

Moving on… … …

- $\int x^2(x^3 - 5)^3\, dx$

Here, we can see two functions; and of course a very big suspicion that the first function (x^2) is at least, in terms of the derivative of the function inside the second function $(x^3 - 5)$, let's confirm that anyways;

$$\frac{d}{dx}(x^3 - 5) = 3 \times 1 \times x^{3-1} - 0$$

$$\frac{d}{dx}(x^3 - 5) = 3x^2$$

Hence, only the constant 3 distinguishes them, hence, we can successfully proceed with our substitution!

$$u = x^3 - 5$$

As we have seen above;

$$\frac{du}{dx} = 3x^2$$

Hence, after rearranging;

$$dx = \frac{du}{3x^2}$$

Substitute back into the integral, substitute for $(x^3 - 5)$ and for dx;

$$\int x^2(u)^3 \times \frac{du}{3x^2}$$

So, x^2 then cancels out! Leaving us with this integral;

$$\int u^3 \times \frac{du}{3}$$

Bring out the constant and after, we'll apply the integral, a standard integral;

$$\frac{1}{3}\int u^3\, du$$

Hence, we integrate using the power rule;

$$\frac{1}{3}\left[\frac{u^{3+1}}{3+1}\right] = \frac{1}{3}\left[\frac{u^4}{4}\right]$$

$$\frac{1}{3} \times \frac{1}{4} \times u^4$$

We won't forget our arbitrary constant, we've added it already, hence, we return the value of u in place, the integral is:

$$\frac{1}{12}[(x^3 - 5)^4] + C$$

We're almost done!

Smiles; the examples are almost over my dear, you should be excited that we're already taking almost all the possible forms questions can be set under substitution;

- $\int \frac{\sin x - \cos x}{\sin x + \cos x} dx$

Here, we can see two functions dividing each other; It's already established that $\sin x$ and $\cos x$ are close friends and hence, since both the numerator and the denominator are in form of sums of both $\sin x$ and $\cos x$, we can proceed to substituting for our denominator with all joy and gladness, it is very most likely, the derivative of the numerator will be somewhat in terms of the numerator, let's check it out!

Hence, out substitution;

$$u = \sin x + \cos x$$

These are two standard derivatives;

$$\frac{du}{dx} = \cos x - (\sin x)$$

$$\frac{du}{dx} = \cos x - \sin x$$

When we compare this derivative with the numerator, we may get disappointed because we have the numerator as $\sin x - \cos x$ which is quite different from $\cos x - \sin x$ as subtraction isn't a commutative operation; but let's think it first,

Can we try factorization for the derivative, $\frac{du}{dx}$?

Let's try; factorizing -1

$$\frac{du}{dx} = \cos x - \sin x = -1(-\cos x + \sin x)$$

$$\frac{du}{dx} = -(\sin x - \cos x)$$

YESSS! We now have the numerator but with the exception of the constant -1, who cares? -1 is merely a constant.

Hence, here:

$$dx = -\frac{du}{\sin x - \cos x}$$

Hence, we proceed to take the substitution to the integral; we can now substitute back into the integral, substitute for $(\sin x + \cos x)$ and for dx;

$$\int \frac{\sin x - \cos x}{u} \times -\frac{du}{\sin x - \cos x}$$

So, $(\sin x - \cos x)$ then cancels out! Leaving us with this integral;

$$\int \frac{1}{u} \times -du$$

Bring out the constant and after, we'll apply the integral, a standard integral;

$$-1 \int \frac{1}{u} du$$

Hence, from our standard integral;

$$-1(\ln u) + C$$

We won't forget our arbitrary constant, we've added it already, hence, we return the value of u in place, the integral is:

$$-\ln(\sin x + \cos x) + C$$

FINALLY!!!

Wow, so after twenty examples; we're in the final question for integration by substitution now, both keep calm dear, there are still two basic rules; the **integration by parts** and the **integration by partial fractions.**

However, before we start getting excited, we've not solved the last question.

Smiles; we're finally here, the twentieth example.

- $\int x^3 e^{x^4}\, dx$

Here, we can see two functions; and of course a very big suspicion that the first function (x^3) is at least, in terms of the derivative of the function inside the second function (x^4), let's confirm that anyways;

$$\frac{d}{dx}(x^4) = 4 \times 1 \times x^{4-1}$$

$$\frac{d}{dx}(x^4) = 4x^3$$

Hence, only the constant 3 distinguishes them, hence, we can successfully proceed with our substitution!

$$u = x^4$$

As we have seen above;

$$\frac{du}{dx} = 4x^3$$

Hence, after rearranging;

$$dx = \frac{du}{4x^3}$$

Substitute back into the integral, substitute for (x^4) and for dx;

$$\int x^3 e^u \times \frac{du}{4x^3}$$

So, x^3 then cancels out! Leaving us with this integral;

$$\int e^u \times \frac{du}{4}$$

Bring out the constant and after, we'll apply the integral, a standard integral;

$$\frac{1}{4}\int e^u \, du$$

Hence, this is a standard integral;

$$\frac{1}{4}e^u + C$$

We won't forget our arbitrary constant, we've added it already, hence, we return the value of u in place, the integral is:

$$\frac{1}{4}e^{x^4} + C$$

Fine that's it about substitution in integration; I believe you have firmly gotten this concept? Yes, you should, we have comprehensively dealt with over 30 questions already.

So, without any further ado, we'll go to the **integration by parts**;

INTEGRATION BY PARTS

We saw in the concept of differential calculus how a product of two functions is differentiated; the integration by part rule of integration could've also been called the product rule of integration but for its restrictions. You can clearly see in the previous section on substitutions that several examples we did, virtually all of them are products and quotients, hence, that confirms the restriction in integration, like I said earlier in the introduction of this topic integration, it is far from being straightforward; therefore, one needs to settle down when attacking integration problems. So, why this long story?

It's of utmost importance I give this story, some called the integration by part rule the **product rule,** however, as it were, it is not really a product rule like in differential calculus where for every two functions multiplying each other, the product rule of differentiation is set to solve the problem.

In integration here, the integration by parts method is used to solve to products when one isn't the derivative of the other where the substitution method will be used, in most cases, in integration by parts, one of the functions can be easily integrated and all that, however, not all products can be sorted out by the integration by parts formula! Before I jump to that, let's see the integration by parts formula;

From the product rule of differentiation; we know that for a given product of two functions;

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

Now, from the integration of sums and differences rule, we see that to integrate the above with respect to x, we'll be having;

$$\int \frac{d}{dx}(uv)dx = \int v\frac{du}{dx}dx + \int u\frac{dv}{dx}dx$$

So, the left hand side is reduced to uv as the integral sign normally cancels off the differential coefficient sign;

Hence, we'll be having;

$$uv = \int v\frac{du}{dx}dx + \int u\frac{dv}{dx}dx$$

Hence, we can now isolate one of the integrals to be left with something like this;

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

That is the integration by parts formula;

If you are the irritable type like me, you'll wonder what sense is it in reducing the integral situation when another integration is still involved in the situation, well, you're right, the integration by parts doesn't eliminate the fact that a product is being integrated, however, ***except we are unfortunate,*** the product that the integration by parts formula yields is normally easier to integrate than the original products;

So, how is the integration by parts formula used? Cool, for a given integration situation where we need to have a product of two functions and firstly after checking well, we discovered that none is the derivative of the other in any form, we then known that substitution cannot be made; we'll then have:

$$\int f(x)g(x)\,dx$$

Now, we take one of the functions as u

We take the other function as: $\frac{dv}{dx}$

As you can see; I haven't used the first and the second functions but one and the other, this is because there are necessary things to consider in the assigning of u and $\frac{dv}{dx}$;

So, after successful assignation of u and $\frac{dv}{dx}$, we:

Differentiate u with respect to x to get $\frac{du}{dx}$

Integrate $\frac{dv}{dx}$ with respect to x to get v

With that we'll have all the necessary parameters for the integration by parts formula and we'll substitute everything necessary in the integration by parts formula.

So right here, there is another short rule, the choice of assignment of the products to u and $\frac{dv}{dx}$;

The choice of u is the most important as after u is taken, the other is simply taken as $\frac{dv}{dx}$

So, the priority conditions for u are as follows:

- If the product involves one log or ln function, then the logarithm function is taken as u as there isn't any standard integrand for log functions, hence, it is taken as u;
- One there isn't a log function, the king that'll take the value of u becomes the function in form of a power of x, that is, the power functions of the form, x^n
- When both logarithm and powers of x are absent, the exponential function becomes the king that'll take the value of u, that is, the form e^x or a^x

So, let's begin with some examples! I think we should have quite some examples here;

- Find the integral:

$$\int x \sin x \, dx$$

Right, so we have two different functions, x and $\sin x$; from our rule of priority; only the log function can stop the power function of x functions from taking the place of u and hence, since the log function isn't here, we take our substitutions thus;

$$u = x$$

And hence,

$$\frac{dv}{dx} = \sin x$$

So, we'll deal with each separately, we have to differentiate u with respect to x and to integrate $\frac{dv}{dx}$ with respect to x as well.

$$u = x$$

Hence,

$$\frac{du}{dx} = 1 \times x^{1-1} = 1$$

And:

$$\frac{dv}{dx} = \sin x$$

Hence,

$$dv = \sin x \, dx$$

Integrate!

$$\int dv = \int \sin x \, dx$$

Here, straight from a standard integral

$$v = -\cos x$$

Hence, we have all we need; rush to the integration by parts formula making the appropriate substitutions for **all terms**:

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$$\int x \sin x \, dx = x(-\cos x) - \int (-\cos x)(1) dx$$

Hence, simplifying further, we have

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx$$

I believe you can see the essence of the integration by parts reduction; we have reduced it to the sum of a term $(-x\cos x)$ and a very simple integral $(\int \cos x\, dx)$ which can be integrated easily;

So, let's evaluate this integral we have in our reduced form;

$$\int \cos x\, dx$$

This is a standard integral;
We have the integral as:

$$\sin x$$

Take this value back to where we stopped.

$$\int x \sin x\, dx = -x\cos x + \sin x$$

Therefore, after proper rearrangement, and of course, the very important arbitrary constant that we mustn't forget, we have our integral as:

$$\int x \sin x\, dx = \sin x - x\cos x + C$$

Next!

- Find the integral;

$$\int x \log x \, dx$$

Right, so we have two different functions, x and $\log x$; from our rule of priority; we know that logarithm functions are the overall kings for taking the place of u and hence, we take our substitutions thus;

$$u = \log x$$

And hence,

$$\frac{dv}{dx} = x$$

So, we'll deal with each separately, we have to differentiate u with respect to x and to integrate $\frac{dv}{dx}$ with respect to x as well.

$$u = \log x$$

Since there isn't a base;

$$u = \log_{10} x$$

Hence,

$$\frac{du}{dx} = \frac{1}{x \ln 10}$$

And:

$$\frac{dv}{dx} = x$$

Hence,

$$dv = xdx$$

Integrate!

$$\int dv = \int xdx$$

Here, straight from the power integral rule:

$$v = \left[\frac{x^{1+1}}{1+1}\right] = \frac{x^2}{2}$$

Hence, we have all we need; rush to the integration by parts formula making the appropriate substitutions for **all terms**:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int x \log x \, dx = (\log x)\left(\frac{x^2}{2}\right) - \int \left(\frac{x^2}{2}\right)\left(\frac{1}{x \ln 10}\right)dx$$

Hence, simplifying further, we have

$$\int x \log x \, dx = \frac{x^2 \log x}{2} - \int \left(\frac{x^2}{2x \ln 10} \right) dx$$

We have reduced the integral to the sum of a term and another integral which should be integrated easily;

So, let's evaluate this integral we have in our reduced form;

$$\int \left(\frac{x^2}{2x \ln 10} \right) dx$$

$\ln 10$ is one hopeless constant, kick it out of the integral,

$$\frac{1}{\ln 10} \int \left(\frac{x^2}{2x} \right) dx$$

Simplify the powers of x

$$\frac{1}{\ln 10} \int \left(\frac{x}{2} \right) dx$$

Oops, there still remain a hopeless constant there, kick it out as well;

$$\frac{1}{2\ln 10}\int xdx$$

We can now integrate easily by the power rule;

$$\frac{1}{2\ln 10}\times\left[\frac{x^{1+1}}{1+1}\right]=\frac{1}{2\ln 10}\times\left[\frac{x^2}{2}\right]$$

We have:

$$\frac{x^2}{4\ln 10}$$

Take this value back to where we stopped.

$$\int x\log x\,dx=\frac{x^2\log x}{2}-\frac{x^2}{4\ln 10}$$

No needed for rearrangement, we have our integral as this after adding the arbitrary constant, forgetting it is a taboo:

$$\int x\log x\,dx=\frac{x^2\log x}{2}-\frac{x^2}{4\ln 10}+C$$

Next!

- Find the integral;

$$\int x^2\, e^{3x} dx$$

Right, so we have two different functions, x^2 and e^{3x}; from our rule of priority; only the log function can stop the power of x from taking the place of u and hence, since the log function isn't here, we take our substitutions thus;

$$u = x^2$$

And hence,

$$\frac{dv}{dx} = e^{3x}$$

So, we'll deal with each separately, we have to differentiate u with respect to x and to integrate $\frac{dv}{dx}$ with respect to x as well.

$$u = x^2$$

Hence,

$$\frac{du}{dx} = 2 \times x^{2-1} = 2x$$

And:

$$\frac{dv}{dx} = e^{3x}$$

Hence,

$$dv = e^{3x}dx$$

Integrate!

$$\int dv = \int e^{3x}dx$$

$$v = \int e^{3x}dx$$

In the integral of e^{3x}, recall that; $3x$ is a linear function and hence, we can easily substitute,

$$\text{Put } a = 3x$$

$$\frac{da}{dx} = 3 \times x^{1-1} = 3$$

Hence,

$$dx = \frac{da}{3}$$

Hence,

$$v = \int e^{a} \times \frac{da}{3}$$

Bring out the constant,

$$v = \frac{1}{3}\int e^{a}\, da$$

This is a standard integral;

$$v = \frac{1}{3}e^{a}$$

Return $a = 3x$

$$v = \frac{1}{3}e^{3x}$$

This isn't the final answer, just a means to the answer, so no arbitrary constant is added yet, till the end of the solution, every arbitrary constant in all these gather to one arbitrary constant at the end.

Hence, we have all we need; rush to the integration by parts formula making the appropriate substitutions for **all terms**:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int x^2 e^{3x} dx = (x^2)\left(\frac{1}{3}e^{3x}\right) - \int \left(\frac{1}{3}e^{3x}\right)(2x)dx$$

Hence, simplifying further, we have

$$\int x^2 e^{3x} dx = \frac{x^2 e^{3x}}{3} - \int \left(\frac{2xe^{3x}}{3}\right) dx$$

We have reduced the integral to the sum of a term and another integral which should be integrated easily;

So, let's evaluate this integral we have in our reduced form;

$$\int \left(\frac{2xe^{3x}}{3}\right) dx$$

Kick out the constant out of the integral,

$$\frac{2}{3}\int (xe^{3x})dx$$

Wawu, I'm sorry; this is another case of integration by parts.

We'll do it anyways!

x will take the priority substitution for u; hence, we have slight variable changes though, u and v have been used above, hence, we'll be using;

$$u_1 = x$$

And hence,

$$\frac{dv_1}{dx} = e^{3x}$$

We'll deal with each separately, we must still take it calmly to avoid errors; we have to differentiate u_1 with respect to x and to integrate $\frac{dv_1}{dx}$ with respect to x as well.

$$u_1 = x$$

Hence,

$$\frac{du_1}{dx} = 1 \times x^{1-1} = 1$$

And:

$$\frac{dv_1}{dx} = e^{3x}$$

Hence,

$$dv_1 = e^{3x}dx$$

Integrate!

$$\int dv_1 = \int e^{3x}dx$$

$$v_1 = \int e^{3x} dx$$

We already have this same integral solved above; hence,

$$v_1 = \frac{1}{3} e^{3x}$$

Hence, we have all we need; rush to the integration by parts formula making the appropriate substitutions for **all terms**:

$$\int u_1 \frac{dv_1}{dx} dx = u_1 v_1 - \int v_1 \frac{du_1}{dx} dx$$

$$\frac{2}{3} \int (xe^{3x}) dx = \frac{2}{3} \left[(x) \left(\frac{1}{3} e^{3x} \right) - \int \left(\frac{1}{3} e^{3x} \right) (1) dx \right]$$

Hence, simplifying further, we have

$$\frac{2}{3} \int (xe^{3x}) dx = \frac{2}{3} \left[\frac{xe^{3x}}{3} - \int \frac{e^{3x}}{3} dx \right]$$

$$\frac{2}{3} \int (xe^{3x}) dx = \frac{2}{3} \left(\frac{xe^{3x}}{3} \right) - \frac{2}{3} \left(\int \frac{e^{3x}}{3} dx \right)$$

Evaluate the integral, I’m sorry this is getting into something else, you can perfectly understand though if only you want to, it’s long but still step by step;

$$\int \frac{e^{3x}}{3} dx$$

Constant brought out!

$$\frac{1}{3} \int e^{3x} dx$$

We already solved for the integral of e^{3x} above, hence, we have:

$$\frac{1}{3}\left[\frac{1}{3} e^{3x}\right] = \frac{1}{9} e^{3x}$$

It’s now down to a gradual returning process; return this value into our integral;

$$\frac{2}{3} \int (xe^{3x}) dx = \frac{2}{3}\left(\frac{xe^{3x}}{3}\right) - \frac{2}{3}\left(\int \frac{e^{3x}}{3} dx\right)$$

Fix it in:

$$\frac{2}{3}\int(xe^{3x})dx = \frac{2}{3}\left(\frac{xe^{3x}}{3}\right) - \frac{2}{3}\left(\frac{1}{9}e^{3x}\right)$$

Simplifying;

$$\frac{2}{3}\int(xe^{3x})dx = \frac{2}{9}xe^{3x} - \frac{2}{27}e^{3x}$$

It’s another returning process;

$$\int x^2e^{3x}dx = \frac{x^2e^{3x}}{3} - \int\left(\frac{2xe^{3x}}{3}\right)dx$$

Fix it in:

$$\int x^2e^{3x}dx = \frac{x^2e^{3x}}{3} - \left[\frac{2}{9}xe^{3x} - \frac{2}{27}e^{3x}\right]$$

Expanding and simplifying:

$$\int x^2e^{3x}dx = \frac{x^2e^{3x}}{3} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x}$$

e^{3x} can be factored out to bring us to our final answer and hence, it’s time to bring in our arbitrary constant.

$$\int x^2 e^{3x} dx = e^{3x}\left[\frac{x^2}{3} - \frac{2x}{9} + \frac{2}{27}\right] + C$$

Quite some question, go through it carefully and make use of your pen and paper to help you go along this!

Next!

- Find the integral;

$$\int x^2 \ln x \, dx$$

I'm pretty sure you aren't excited to see x^2 again given the pretty lengthy work in the previous example; however, the logarithm function is quite some friendly function in integrating by parts since it is the king in substituting for u. Here:

$$u = \ln x$$

And hence,

$$\frac{dv}{dx} = x^2$$

So, we'll deal with each separately, we have to differentiate u with respect to x and to integrate $\frac{dv}{dx}$ with respect to x as well.

$$u = \ln x$$

Hence, standard derivative,

$$\frac{du}{dx} = \frac{1}{x}$$

And:

$$\frac{dv}{dx} = x^2$$

Hence,

$$dv = x^2 dx$$

Integrate!

$$\int dv = \int x^2 dx$$

Here, straight from the power integral rule:

$$v = \left[\frac{x^{2+1}}{2+1}\right] = \frac{x^3}{3}$$

Hence, we have all we need; rush to the integration by parts formula making the appropriate substitutions for **all terms**:

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$$\int x^2 \ln x \, dx = (\ln x)\left(\frac{x^3}{3}\right) - \int \left(\frac{x^3}{3}\right)\left(\frac{1}{x}\right) dx$$

Hence, simplifying further, we have

$$\int x^2 \ln x \, dx = \frac{x^3 \ln x}{3} - \int \left(\frac{x^3}{3x}\right) dx$$

We have reduced the integral to the sum of a term and another integral which should be integrated easily;

So, let's evaluate this integral we have in our reduced form;

$$\int \left(\frac{x^3}{3x}\right) dx$$

$\frac{1}{3}$ is one hopeless constant, kick it out of the integral,

$$\frac{1}{3}\int \left(\frac{x^3}{x}\right) dx$$

Simplify the powers of x

$$\frac{1}{3}\int x^2 dx$$

We can now integrate easily by the power rule;

$$\frac{1}{3}\times\left[\frac{x^{2+1}}{2+1}\right]=\frac{1}{3}\times\left[\frac{x^3}{3}\right]=\frac{1}{9}x^3$$

Take this value back to where we stopped.

$$\int x^2 \ln x\, dx=\frac{x^3 \ln x}{3}-\int\left(\frac{x^3}{3x}\right)dx$$

Hence;

$$\int x^2 \ln x\, dx=\frac{x^3 \ln x}{3}-\frac{1}{9}x^3$$

No serious rearrangement, we have our integral as this after adding the arbitrary constant, forgetting it is a taboo; we can also factorize x^3 over there:

$$\int x^2 \ln x\, dx=x^3\left(\frac{\ln x}{3}-\frac{1}{9}\right)+C$$

So, let’s take this last example under the integration by part; let’s visit that unfriendly guy that is fond of elongating our work ☺☺☹☹

Find the integral:

- Find the integral;

$$\int x^2 \cos 2x \, dx$$

Right, so we have two different functions, x and $\cos 2x$; from our rule of priority; only the log function can stop the power of x from taking the place of u and hence, since the log function isn’t here, we take our substitutions thus;

$$u = x^2$$

And hence,

$$\frac{dv}{dx} = \cos 2x$$

So, we’ll deal with each separately, we have to differentiate u with respect to x and to integrate $\frac{dv}{dx}$ with respect to x as well.

$$u = x^2$$

Hence,

$$\frac{du}{dx} = 2 \times x^{2-1} = 2x$$

And:

$$\frac{dv}{dx} = \cos 2x$$

Hence,

$$dv = \cos 2x \, dx$$

Integrate!

$$\int dv = \int \cos 2x \, dx$$

We need a quick substitution for this; it's a function of a lineate function;

$$\text{Put } a = 2x$$

$$\frac{da}{dx} = 2 \times x^{1-1} = 2$$

Hence,

$$dx = \frac{da}{2}$$

Hence,

$$v = \int \cos a \times \frac{da}{2}$$

Bring out the constant,

$$v = \frac{1}{2}\int \cos a \ da$$

This is a standard integral;

$$v = \frac{1}{2}\sin a$$

Return $a = 2x$;

$$v = \frac{1}{2}\sin 2x$$

Hence, we have all we need; rush to the integration by parts formula making the appropriate substitutions for **all terms**:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int x^2 \cos 2x \, dx$$

$$= (x^2)\left(\frac{1}{2}\sin 2x\right) - \int \left(\frac{1}{2}\sin 2x\right)(2x)dx$$

Hence, simplifying further, we have

$$\int x^2 \cos 2x \, dx = \frac{x^2 \sin 2x}{2} - \int \left(\frac{2x \sin 2x}{2}\right) dx$$

We have reduced the integral to the sum of a term and another integral which should be integrated easily;

So, let's evaluate this integral we have in our reduced form;

$$\int \left(\frac{2x \sin 2x}{2}\right) dx$$

2 cancels out, we have:

$$\int (x \sin 2x) dx$$

Lol, you know this is an extended question, no need to complain, we knew it'll take time from the onset, this is another case of integration by parts.

x will take the priority substitution for u; hence, we have slight variable changes though, u and v have been used above, hence, we'll be using;

$$u_1 = x$$

And hence,

$$\frac{dv_1}{dx} = \sin 2x$$

We'll deal with each separately, we must still take it calmly to avoid errors; we have to differentiate u_1 with respect to x and to integrate $\frac{dv_1}{dx}$ with respect to x as well.

$$u_1 = x$$

Hence,

$$\frac{du_1}{dx} = 1 \times x^{1-1} = 1$$

And:

$$\frac{dv_1}{dx} = \sin 2x$$

Hence,

$$dv_1 = \sin 2x\, dx$$

Integrate!

$$\int dv_1 = \int \sin 2x\, dx$$

$$v_1 = \int \sin 2x\, dx$$

We already have this integral solved above; hence,

$$v_1 = \frac{1}{2}\sin 2x$$

Hence, we have all we need; rush to the integration by parts formula making the appropriate substitutions for **all terms**:

$$\int u_1 \frac{dv_1}{dx} dx = u_1 v_1 - \int v_1 \frac{du_1}{dx} dx$$

$$\int (x \sin 2x) dx = (x)\left(\frac{1}{2}\sin 2x\right) - \int \left(\frac{1}{2}\sin 2x\right)(1) dx$$

Hence, simplifying further, we have

$$\int (x \sin 2x) dx = \frac{x \sin 2x}{2} - \int \frac{\sin 2x}{2} dx$$

Evaluate the integral;

$$\int \frac{\sin 2x}{2} dx$$

Constant brought out!

$$\frac{1}{2}\int \sin 2x \, dx$$

We already solved for the integral of $\sin 2x$ above, hence, we have:

$$\frac{1}{2}\left[\frac{\sin 2x}{2}\right] = \frac{1}{4}\sin 2x$$

It's now down to a gradual returning process; return this value into our integral;

$$\int (x \sin 2x) dx = \frac{x \sin 2x}{2} - \int \frac{\sin 2x}{2} dx$$

Fix it in:

$$\int (x \sin 2x) dx = \frac{x \sin 2x}{2} - \frac{1}{4}\sin 2x$$

Simplifying;

$$\int (x \sin 2x) dx = \frac{1}{2} x \sin 2x - \frac{1}{4}\sin 2x$$

It's another returning process;

$$\int x^2 \cos 2x \, dx = \frac{x^2 \sin 2x}{2} - \int \left(\frac{2x \sin 2x}{2}\right) dx$$

$$\int x^2 \cos 2x\, dx = \frac{x^2 \sin 2x}{2} - \int (x \sin 2x) dx$$

Fix it in:

$$\int x^2 \cos 2x\, dx$$
$$= \frac{x^2 \sin 2x}{2} - \left[\frac{1}{2} x \sin 2x - \frac{1}{4} \sin 2x\right]$$

Expanding and simplifying:

$$\int x^2 \cos 2x\, dx$$
$$= \frac{1}{2} x^2 \sin 2x - \frac{1}{2} x \sin 2x + \frac{1}{4} \sin 2x$$

$\sin 2x$ can be factored out to bring us to our final answer and hence, it's time to bring in our arbitrary constant.

$$\int x^2 \cos 2x\, dx = \sin 2x \left[\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right] + C$$

Right there, that rounds up integration by parts, twenty examples aren't needed this time around, as you can see, the whole process revolves around the same thing, and hence, you just need to follow it carefully. You are expected to know which types of functions take precedence in substituting for u and that'll be all about issues with integration by parts.

Now, the next section, even though a lengthy one, is quite very easy, this is because it's more straightforward and easy to spot at first sight.

It is the method of integration by **partial fraction**; and **polynomial division**.

Both are the same as polynomial division occurs when we have an improper fraction;

A BIG QUESTION THOUGH! Do you still remember polynomial division and partial fractions which were both treated in SSC105, if 'NO,' this then, is a highway to nowhere!

Well, let's have a quick recap!

Polynomial division and Partial Fractions;

Polynomial division occurs when we need to divide a higher polynomial by a lower one; the process is very a simple one, let's see polynomial division or division in general anyway and then, see how goes;

Alright, let's start with an example and see how division is done by method of long division;

- Divide $x^3 - 2x^2 + 5x - 1$ by $x - 1$

Let's introduce our long division sign;

$$x - 1\overline{\smash{)}x^3 - 2x^2 + 5x - 1}$$

Take the first values to divide each other; the first value of the dividend divided by the first value of divisor; mind you, dividend and divisor is given thus;

$$\begin{array}{r} \text{quotient} \\ \text{divisor}\overline{\smash{)}\text{dividend}} \end{array}$$

Here, the first term of the divisor dividing the first term of the dividend is: $\frac{x^3}{x} = x^2$; write x^2 on top of the long division which forms part of the quotient;

$$\begin{array}{r l}
 & x^2 \\
x - 1 & \overline{\big)\, x^3 - 2x^2 + 5x - 1}
\end{array}$$

Next, we multiply **the whole of the divisor** by the term(s) in the quotient and write it below the dividend and make the subtraction operation; Here, that'll be:

$$x^2(x - 1) = x^3 - x^2$$

We have the subtraction operation, i.e., subtracting what is on top from what is below, the subtraction sign is shown, the subtraction operation is thus:

$$(x^3 - 2x^2 + 5x - 1) - (x^3 - x^2)$$
$$= -x^2 + 5x - 1$$

The x^3 term goes off and we're left with

$$\begin{array}{r l}
 & x^2 \\
x - 1 & \overline{\big)\, x^3 - 2x^2 + 5x - 1} \\
 & \underline{-x^3 - x^2 } \\
 & - x^2 + 5x - 1
\end{array}$$

Again after the subtraction, we're gradually breaking down the dividend, again we take the division of the new first terms again, the first term of the result of subtraction and the first term of the divisor; here, it'll be:

$$-\frac{x^2}{x} = -x$$

Next, we multiply **the whole of the divisor** by the new most recent term in the quotient $\left(-x\right)$ and write it below the new dividend;
Here, that'll be:

$$-x(x-1) = -x^2 + x$$

Write it down again and we'll have the subtraction operation again,

$$(-x^2 + 5x - 1) - (-x^2 + x) = 4x - 1$$

The x^2 term goes off and we're left with

$$\begin{array}{r|l} & \quad x^2 - x \\ \hline x - 1 & \quad x^3 - 2x^2 + 5x - 1 \\ & -x^3 - x^2 \\ \cline{2-2} & \qquad - x^2 + 5x - 1 \\ & \quad - - x^2 + x \\ \cline{2-2} & \qquad\qquad 4x - 1 \end{array}$$

Again after the subtraction, we're making progress breaking down the dividend, again we take the division of the new first terms again, the first term of the result of subtraction and the first term of the divisor; here, it'll be:

$$\frac{4x}{x} = 4$$

Next, we multiply **the whole of the divisor** by the new most recent term in the quotient (4) and write it below the new dividend;
Here, that'll be:

$$4(x - 1) = 4x - 4$$

Write it down again and we'll have the subtraction operation again,

$$(4x - 1) - (4x - 4) = 3$$

$$\begin{array}{r|r}
 & x^2 - x + 4 \\ \cline{2-2}
x - 1 & x^3 - 2x^2 + 5x - 1 \\
 & -x^3 - x^2 \qquad\qquad \\ \cline{2-2}
 & - x^2 + 5x - 1 \\
 & - - x^2 + x \qquad \\ \cline{2-2}
 & 4x - 1 \\
 & -4x - 4 \\ \cline{2-2}
 & 3
\end{array}$$

Hence, we're through with our division operation and we're left with 3 which obviously isn't divisible by x which is the first term of the divisor.

Hence, in our division operation; our quotient is given by:

Quotient: $x^2 - x + 4$ while our remainder is 3;

I'm sure you understood that; let's try another example; it'll get better; but before then, let's get some things right here;

Let's see another example;

- Divide: $3x^3 - 4x^2 + x - 7$ by $x^2 + 2x + 1$

Let's introduce our long division sign;

$$x^2 + 2x + 1\sqrt{3x^3 - 4x^2 + x - 7}$$

Take the first values to divide each other; the first value of the dividend divided by the first value of divisor;

As usual, the first term of the divisor dividing the first term of the dividend is: $\frac{3x^3}{x^2} = 3x$; write x on top of the long division which forms part of the quotient;

$$\begin{array}{r} 3x \qquad\qquad\qquad\quad \\ x^2 + 2x + 1\overline{\big)\,3x^3 - 4x^2 + x - 7} \end{array}$$

Next, we multiply **the whole of the divisor** by the term in the quotient and write it below the dividend and make the subtraction operation; Here, that'll be:

$$3x(x^2 + 2x + 1) = 3x^3 + 6x^2 + 3x$$

We have the subtraction operation, i.e., subtracting what is on top from what is below, the subtraction sign is shown, the subtraction operation is thus:

$$(3x^3 - 4x^2 + x - 7) - (3x^3 + 6x^2 + 3x)$$
$$= -10x^2 - 2x - 7$$

The x^3 term goes off and we're left with

$$
\begin{array}{r|l}
 & \quad 3x \\
\cline{2-2}
x^2+2x+1 & \quad 3x^3 - 4x^2 + x - 7 \\
 & -3x^3 + 6x^2 + 3x \\
\cline{2-2}
 & \qquad -10x^2 - 2x - 7
\end{array}
$$

Again after the subtraction, the dividend is reducing, again we take the division of the new first terms again, the first term of the result of subtraction and the first term of the divisor; and write it amongst the quotient, here, it'll be:

$$-\frac{10x^2}{x^2} = -10$$

Next, we multiply **the whole of the divisor** by the new most recent term in the quotient (-10) and write it below the new dividend;
Here, that'll be:

$$-10(x^2+2x+1) = -10x^2 - 20x - 10$$

Write it down again and we'll have the subtraction operation again,

$$(-10x^2 - 2x - 7) - (-10x^2 - 20x - 10)$$
$$= 18x + 3$$

The x^2 term goes off and we're left with

$$
\begin{array}{r|l}
 & \quad 3x - 10 \\
\hline
x^2 + 2x + 1 & 3x^3 - 4x^2 + x - 7 \\
 & -3x^3 + 6x^2 + 3x \\
\cline{2-2}
 & \quad -10x^2 - 2x - 7 \\
 & - \; -10x^2 - 20x - 10 \\
\cline{2-2}
 & \qquad 18x + 3
\end{array}
$$

Hence, we're through with our division operation as we're left with $18x + 3$ which obviously isn't divisible by x^2 which is the first term of the divisor. Hence, we can't go further from here, the division is done.

Hence, in our division operation; our quotient is given by:

Quotient: $3x - 10$ while our remainder is $18x + 3$

I'm sure you're understanding it better, let's try another example; it'll get better; let's get some things right here;

Let's see examples of some other strange looking types of division;

- Divide y^2 by $y-1$

So, this isn't looking like what needs long division at first sight but that's just what we'll be doing here, place your dividend and divisor where they should be;

Let's analyze it with normal basic mathematics;

$$\frac{y^2}{y-1}$$

If this was by normal division, nothing can be done, only when the denominator is singular that this can be split into the numerators, hence, long division is what will be used;

Let's introduce our long division sign;

$$y-1\sqrt{y^2}$$

Take the first values to divide each other; the first value of the dividend divided by the first value of divisor; just like any normal long division;

As usual, the first term of the divisor dividing the first term of the dividend is: $\frac{y^2}{y} = y$; write y on top of the long division which forms part of the quotient;

$$\begin{array}{r} y \\ y-1\sqrt{y^2} \end{array}$$

Next, we multiply **the whole of the divisor** by the term in the quotient and write it below the dividend and make the subtraction operation; Here, that'll be:

$$y(y-1) = y^2 - y$$

We have the subtraction operation, i.e., subtracting what is on top from what is below, the subtraction sign is shown, the subtraction operation is thus:

$$(y^2) - (y^2 - y) = y$$

The y^2 term goes off and we're left with

$$\begin{array}{r|l} & \quad y \\ \hline y-1 & \quad y^2 \\ & -y^2-y \\ \hline & \qquad\; y \end{array}$$

Again after the subtraction, the dividend is reducing, again we take the division of the new first terms again, the first term of the result of subtraction and the first term of the divisor; and write it amongst the quotient, here, it'll be:

$$\frac{y}{y} = 1$$

Next, we multiply **the whole of the divisor** by the new most recent term in the quotient (1) and write it below the new dividend;
Here, that'll be:

$$1(y - 1) = y - 1$$

Write it down again and we'll have the subtraction operation again,

$$(y) - (y - 1) = 1$$

The y term goes off and we're left with:

$$
\begin{array}{r|l}
 & \quad y+1 \\
\hline
y-1 & y^2 \\
 & -y^2-y \\
\cline{2-2}
 & \qquad y \\
 & \qquad -y-1 \\
\cline{2-2}
 & \qquad\qquad 1
\end{array}
$$

Hence, we’re through with our division operation as we’re left with 1 which obviously isn’t divisible by y which is the first term of the divisor. Hence, we can’t go further from here, the division is done.

Hence, in our division operation; our quotient is given by:

Quotient: $y + 1$ while our remainder is 1

That’s it about refreshing you on polynomial division;

The whole idea is to keep applying the rule till you can divide no more; that’s a good refreshing on division of polynomial, let’s be refreshed on partial fractions too!

A recap on Partial Fractions;

Very interesting topic! Just quite unfortunate it's not a topic in SSC106 where we'd have taken proper exegesis on it and dissolve it, we'll just be taking a refreshment on it here;

So, consider the desire to add these fractions below;

$$\frac{2}{x-3}+\frac{6}{2x+5}$$

Let's add these quickly; I'm sure you remember how fractions are added;

$$\frac{2(2x+5)+6(x-3)}{(x-3)(2x+5)}$$

$$\frac{4x+10+6x-18}{(x-3)(2x+5)}=\frac{10x-8}{(x-3)(2x+5)}$$

So, consider this new fraction;

$$\frac{10x-8}{(x-3)(2x+5)}$$

Is it somewhat possible to bring back this fraction to where we started from? Of course YES! That is the concept of partial fractions; if I can't remember anything else, you were thought in SSC105, all the same, we'll revisit it.

How to break down a partial fraction;

$$\frac{10x-8}{(x-3)(2x+5)} \equiv \frac{A}{x-3} + \frac{B}{2x+5}$$

When we have two or more linear factors at the denominator (you should know the meaning of linear factors, we mentioned it even in this topic under integration by linear substitution); so, when we have two or more linear functions, they're broken down into separate fractions where each factor at the separate denominators of each new fraction and their numerators constants A, B, C and so on; that's how it is;

Another example below; for linear factors;

$$\frac{2x-13}{(x-3)(2x+5)(x+2)} \equiv \frac{A}{x-3} + \frac{B}{2x+5} + \frac{C}{x+2}$$

A case where we have repeated factors at the denominator; we'll have a situation like this;

$$\frac{3x^2+2}{(x-1)(x-3)^2} \equiv \frac{A}{x-1} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

In the case of repeated factors at the denominator, we still maintain our rule of splitting them one by one, the only difference is that we'll be taking the repeated factor from the power of 1 till the last power it is represented;

To explain that further, see what is below;

$$\frac{x^2+5}{(x-3)^3} \equiv \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3}$$

In the case of having a quadratic equation that cannot be factorized into two linear factors such as quadratic expressions like: $\boldsymbol{x^2+2x+2}$, $\boldsymbol{x^2+5x+7}$; such equations are sorted out thus;

$$\frac{2x-3}{(x+5)(x^2+2x+2)} \equiv \frac{A}{x-5} + \frac{Bx+C}{x^2+2x+2}$$

The partial fraction split normally but when the denominator is this non-factorable quadratic equation, it'll be expressed as a linear equation and two variables are sought.

Further example is shown below; the numerator of the non-factorable denominator is expressed as a linear factor; the form $(Ax + B)$, $(Bx + C)$ and so on, dependent on the variables available on used; as exemplified by the two examples below;

$$\frac{4x+3}{(x^2+5x+7)(x-2)^2} \equiv \frac{Ax+B}{x^2+5x+7} + \frac{C}{x-2} + \frac{D}{(x-2)^2}$$

$$\frac{2x-3}{(x+5)(x^2+2x+2)} \equiv \frac{A}{x-5} + \frac{Bx+C}{x^2+2x+2}$$

So, that's the whole thing of splitting into partial fractions, the only type remaining is the type of improper fractions; we'll treat that soon; let's see how partial fractions are sorted out to find the values of these constants we've broken the fractions into;

Let's start with our first example where we know the initial separate fractions, that is, the compound fraction we got after adding the two fractions when we started;

$$\frac{10x-8}{(x-3)(2x+5)} \equiv \frac{A}{x-3} + \frac{B}{2x+5}$$

We'll multiply through by the denominator of the major fraction; $(x-3)(2x+5)$ and clear out every needed clearing, we're left with;

$$10x - 8 = A(2x+5) + B(x-3)$$

So, this is how it is done in this most basic way; we substitute terms for x in such a way that they give the best results; for example, we notice that if we equate each factor to zero and substitute for that value, we get values that'll leave us with one unknown left.

Here,

$$\text{Let } 2x + 5 = 0$$

Here,

$$x = -\frac{5}{2}$$

Then, substitute in the main equation

$$10\left(-\frac{5}{2}\right) - 8 = A\left(2\left(-\frac{5}{2}\right) + 5\right) + B\left(-\frac{5}{2} - 3\right)$$

$$-25 - 8 = A(-5 + 5) + B\left(-\frac{11}{2}\right)$$

$$-33 = -\frac{11B}{2}$$

Here,

$$B = 6$$

Again, we saw how we successfully destroyed A, it is equally possible to destroy B by equating the factor with B to zero as well;

$$\text{Let } x - 3 = 0$$

Hence,

$$x = 3$$

Then, substitute in the main equation

$$10(3) - 8 = A(2(3) + 5) + B(3 - 3)$$

$$30 - 8 = A(6 + 5) + B(0)$$

$$22 = 11A$$

Here,

$$A = 2$$

We have found our two values for A and B and we're good!

$$\frac{10x-8}{(x-3)(2x+5)} \equiv \frac{2}{x-3} + \frac{6}{2x+5}$$

Head back to the initial equation and see if they're the same. Of course they will, you can go do the checking anyways.

So, trust me, for most cases where partial fractions will be used for integration in the SSC106 way, they'll mostly include fractions of two linear factors, let's still check some other examples though;

- Break into partial fractions;

$$\frac{4x+1}{x^2-x-2}$$

Now, we have a quadratic expression in our denominator, we can factorize it though;

$$x^2-x-2$$

$-2 \times x^2 = -2x^2$
$-2x + x = -x$
$-2x \times x = -2x^2$

$$x^2-2x+x-2$$

$$x(x-2)+1(x-2)$$
$$(x-2)(x+1)$$

Quadratic factorization has been treated in the introductory basic chapter; we now replace the expression with the factorized form and we can now break it down freely into two fractions with two different denominators;

$$\frac{4x+1}{(x-2)(x+1)} \equiv \frac{A}{x-2}+\frac{B}{x+1}$$

We'll multiply through by the denominator of the major fraction; $(x-2)(x+1)$ and clear out every needed clearing, we're left with;

$$4x+1 = A(x+1)+B(x-2)$$

We'll be substituting for x to get a destroyed situation that'll leave us with B alone left and solve. Here,

$$\text{Let } x+1=0$$

Here,

$$x=-1$$

Then, substitute in the main equation

$$4(-1)+1 = A(-1+1)+B(-1-2)$$

$$-4 + 1 = A(0) + B(-3)$$

$$-3 = -3B$$

Here,

$$B = 1$$

Again, we saw how we successfully destroyed A, it is equally possible to destroy B by equating the factor with B to zero as well;

$$\text{Let } x - 2 = 0$$

Hence,

$$x = 2$$

Then, substitute in the main equation

$$4(2) + 1 = A(2 + 1) + B(2 - 2)$$

$$9 = A(3) + B(0)$$

$$9 = 3A$$

Here,

$$A = 3$$

We have found our two values for A and B and we're good!

$$\frac{4x + 1}{(x - 2)(x + 1)} \equiv \frac{3}{x - 2} + \frac{1}{x + 1}$$

Let's see case of repeated factors;

- Express in partial fractions;

$$\frac{1}{(x-1)(x+2)^2}$$

Right,
This is a case of repeated factors; we know how we'll be expanding this;

$$\frac{1}{(x-1)(x+2)^2} \equiv \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

We take the normal process of clearing the fractions by multiplying through by the denominator of the major fraction and simplify; we'll have:

$$1 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$$

We start our destruction process;

We'll be substituting for x to get variables destroyed that'll leave us with one variable alone left and solve.
Here,

$$\text{Let } x - 1 = 0$$

Here,

$$x = 1$$

Then, substitute in the main equation

$$1 = A(1+2)^2 + B(1-1)(1+2) + C(1-1)$$

$$1 = A(3)^2 + B(0)(3) + C(0)$$

$$1 = 9A$$

Here,

$$A = \frac{1}{9}$$

Next, we make the next destruction,

$$\text{Let } x + 2 = 0$$

Hence,

$$x = -2$$

Then, substitute in the main equation

$$1 = A(-2+2)^2 + B(-2-1)(-2+2) + C(-2-1)$$

$$1 = A(0)^2 + B(-3)(0) + C(-3)$$

$$1 = -3C$$

Here,

$$C = -\frac{1}{3}$$

Now, as it were, there is nothing to destroy again and we still haven't found B, so what will be done here is to put $x = 0$ to get a simpler equation in terms of A, B and C and then do the job;

$$\text{Let } x = 0$$

Substitute;

$$1 = A(0+2)^2 + B(0-1)(0+2) + C(0-1)$$

$$1 = A(2)^2 + B(-1)(2) + C(-1)$$

$$1 = 4A - 2B - C$$

Now, substitute for $A = \frac{1}{9}$ and $C = -\frac{1}{3}$

$$1 = 4\left(\frac{1}{9}\right) - 2B - \left(-\frac{1}{3}\right)$$

$$1 = \frac{4}{9} + \frac{1}{3} - 2B$$

$$1 - \frac{4}{9} - \frac{1}{3} = -2B$$

$$\frac{9-4-3}{9} = -2B$$

$$\frac{2}{9} = -2B$$

Here;

$$B = -\frac{1}{9}$$

Now, the whole idea of substituting zero is no magic, we can get infinite numbers of equations in terms of A, B and C by substituting any values for x; but even from the beginning, we substitute values that make our work easier, hence, here, putting $x = 0$ makes our work easier for ease of expansion and we get an equation in A, B and C and solve; putting $x = -1, 2$, 3 and other values of x will equally give us equations in A, B and C and will equally be solved to give this same value of B we have, they may only take a longer process;

We'll now fix our values in the expansion;

$$\frac{1}{(x-1)(x+2)^2} \equiv \frac{\frac{1}{9}}{x-1} + \frac{-\frac{1}{9}}{x+2} + \frac{-\frac{1}{3}}{(x+2)^2}$$

After successful manipulation of fractions; we have;

$$\frac{1}{(x-1)(x+2)^2} \equiv \frac{1}{9(x-1)} - \frac{1}{9(x+2)} - \frac{1}{3(x+2)^2}$$

Let's see another;

- Resolve;

$$\frac{x^2+1}{x(x+1)^2}$$

Here, in breaking down, we must be conscious of the fact that x is equally a factor on its own and it'll be the denominator of one of the partial fractions;

$$\frac{x^2+1}{x(x+1)^2} \equiv \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

The repeated factor still takes its effect;

We take the normal process of clearing the fractions by multiplying through by the denominator of the major fraction and simplify; we'll have:

$$x^2+1 = A(x+1)^2 + Bx(x+1) + Cx$$

We start our destruction process;

We'll be substituting for x to get variables destroyed that'll leave us with one variable alone left and solve.

Here,

$$\text{Let } x + 1 = 0$$

Here,

$$x = -1$$

Then, substitute in the main equation

$$(-1)^2 + 1$$
$$= A(-1+1)^2 + B(-1)(-1+1) + C(-1)$$

$$1 + 1 = A(0)^2 + B(-1)(0) - C$$

$$2 = -C$$

Here,

$$C = -2$$

Next, we make the next destruction,

$$\text{Let } x = 0$$

We use $x = 0$, because x is one of the factors and hence, it'll be part of the destruction substitution; Hence,

$$x = 0$$

Then, substitute in the main equation

$$(0)^2 + 1 = A(0+1)^2 + B(0)(0+1) + C(0)$$

$$0 + 1 = A(1)^2 + B(0)(1) + 0$$

$$1 = A$$

Here,

$$A = 1$$

Now, as it were, there is nothing to destroy again and we still haven't found B, so what will be done here is to make another substitution to get an equation in terms of A, B and C and then do the job; now, $x = 0$ has already been used in the destruction process, hence, here, we'll have another substitution; let's choose anything, but it's better small values; hence

$$\text{Let } x = 1$$

Substitute;

$$(1)^2 + 1 = A(1+1)^2 + B(1)(1+1) + C(1)$$

$$1 + 1 = A(2)^2 + B(1)(2) + C(1)$$

$$2 = 4A + 2B + C$$

Now, substitute for $A = 1$ and $C = -2$

$$2 = 4(1) + 2B + (-2)$$

$$2 = 4 - 2 + 2B$$

$$2 - 4 + 2 = 2B$$

$$0 = 2B$$

Here;

$$B = 0$$

Now, you can see that substituting zero is no magic, we used $x = 1$ here and still got our answer; we can get infinite numbers of equations in terms of A, B and C by substituting any values for x; but we substitute values that make our work easier, hence, We'll now fix our values in the expansion;

$$\frac{x^2 + 1}{x(x+1)^2} \equiv \frac{1}{x} + \frac{0}{x+1} + \frac{-2}{(x+1)^2}$$

The middle term finds its way out, it is equal to zero

$$\frac{x^2 + 1}{x(x+1)^2} \equiv \frac{1}{x} - \frac{2}{(x+1)^2}$$

Let's see just one example in non-factorable quadratic equation, you know we're not treating partial fraction but integration, this is just a recap sort of;

- Resolve

$$\frac{x^2 - 2x - 9}{(x^2 + x + 6)(x+1)}$$

Analyzing $(x^2 + x + 6)$, we can see that it can't be factorized; so let's move on and treat it like we're supposed to treat it.

$$\frac{x^2 - 2x - 9}{(x^2 + x + 6)(x + 1)} \equiv \frac{Ax + B}{x^2 + x + 6} + \frac{C}{x + 1}$$

So, let's start our very normal process; clear the fraction by multiplying through by $(x^2 + x + 6)(x + 1)$, we have:

$$\begin{aligned} &x^2 - 2x - 9 \\ &\qquad = (Ax + B)(x + 1) + C(x^2 + x + 6) \end{aligned}$$

We start our destruction process;

We'll be substituting for x to get a destroyed situation that'll leave us with one variable alone left and solve.

Here,

$$\text{Let } x + 1 = 0$$

Here,

$$x = -1$$

Then, substitute in the main equation

$$(-1)^{2} - 2(-1) - 9 = (A(-1) + B)\big((-1) + 1\big) + C((-1)^2 + (-1) + 6)$$

$$1 + 2 - 9 = (-A + B)(0) + C(1 - 1 + 6)$$

$$-6 = 0 + C(6)$$

$$-6 = 6C$$

Here,

$$C = -1$$

Next, we make the next destruction but ops, there's no more destruction possible as $(x + 1)$ is the only linear factor in the denominators available, hence, let's substitute for $x = 0$,

$$\text{Let } x = 0$$

Hence,

$$x = 0$$

Then, substitute in the main equation

$$(0)^{2} - 2(0) - 9 = (A(0) + B)\big((0) + 1\big) + C((0)^2 + (0) + 6)$$

$$-9 = (B)(1)^2 + C(6)$$
$$-9 = B + 6C$$

Here, since $C = -1$;

$$-9 = B + 6(-1)$$
$$B = -3$$

Here, let's make another substitution, we need another equation to find A;

$$\text{Let } x = 1$$

Substitute;

$$(1)^2 - 2(1) - 9$$
$$= (A(1) + B)\big((1) + 1\big)$$
$$+ C((1)^2 + (1) + 6)$$

$$1 - 2 - 9 = (A + B)(2) + C(1 + 1 + 6)$$

$$-10 = 2A + 2B + 8C$$

Dividing through by 2;

$$-5 = A + B + 4C$$

But $C = -1, B = -3$; hence;

$$-5 = A + (-3) + 4(-1)$$

$$-5 = A - 3 - 4$$

Here;

$$A = 2$$

So, we bring our three values back into the partial fractions and we have;

$$\frac{x^2 - 2x - 9}{(x^2 + x + 6)(x + 1)} \equiv \frac{2x + (-3)}{x^2 + x + 6} + \frac{-1}{x + 1}$$

$$\frac{x^2 - 2x - 9}{(x^2 + x + 6)(x + 1)} \equiv \frac{2x - 3}{x^2 + x + 6} - \frac{1}{x + 1}$$

So, the only aspect remaining is the aspect of improper fractions which involves the polynomial division we just treated;

We use polynomial divisions when the degree of the variable in the numerator is equal to or greater than the degree of the variable in the denominator.

Examples of improper fractions are;

$$\frac{x^2+2x+2}{x^2-4x+3}, \qquad \frac{x^3+2x-1}{x^2+5x+6}, \qquad \frac{x^2-2x}{x+1}$$

All the above, the highest degree of x in the numerator is either equal to (as in the first fraction) or greater than (as in the second and third fractions) the highest degree in the denominator. This is resolved by dividing the numerator by the denominator; getting the quotient and remainder and resolving the partial fraction thus;

$$\text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}$$

So, the $\frac{\text{Remainder}}{\text{Divisor}}$ fraction is then broken into partial fractions by the normal rule of substitution we've taken above;

So, let's break those examples above into partial fractions;

- Number 1;

$$\frac{x^2+2x+2}{x^2-4x+3}$$

This is an improper fraction so **we divide the numerator by the denominator** and not vice-versa! Please head back to the explanation of the division of polynomial we just took before this section in case you skipped it, you're not advised to skip anything in the course of reading this text though;

$$
\begin{array}{r|l}
 & \quad 1 \\
\hline
x^2 - 4x + 3 & \quad x^2+2x+2 \\
 & -x^2-4x+3 \\
\cline{2-2}
 & \qquad\quad 6x-1
\end{array}
$$

Thus, it's a pretty short division, we can't go ahead since $6x$ cannot be divided by x^2 to yield a polynomial expression, polynomials involve only positive powers;

Hence, we have the partial fraction, in the format we just discussed, as;

$$1 + \frac{6x - 1}{x^2 - 4x + 3}$$

We now break this into partial fractions;

$$\frac{6x - 1}{x^2 - 4x + 3}$$

Factorize the denominator;

$$x^2 - 4x + 3$$

$3 \times x^2 = 3x^2$

$-3x - x = -4x$

$-3x \times -x = 3x^2$

$$x^2 - 3x - x + 3$$

$$x(x - 3) - 1(x - 3)$$

$$(x - 3)(x - 1)$$

We have:

$$\frac{6x - 1}{(x - 3)(x - 1)} \equiv \frac{A}{x - 3} + \frac{B}{x - 1}$$

Then, do this just normally;

Multiply through by the denominator of the major fraction; $(x - 3)(x - 1)$ and clear out every needed clearing, we're left with;

$$6x - 1 = A(x - 1) + B(x - 3)$$

We'll be substituting for x to get a destroyed situation that'll leave us with B alone left and solve.

Here,

$$\text{Let } x - 1 = 0$$

Here,

$$x = 1$$

Then, substitute in the main equation

$$6(1) - 1 = A(1 - 1) + B(1 - 3)$$

$$5 = A(0) + B(-2)$$

$$5 = -2B$$

Here,

$$B = -\frac{5}{2}$$

Next, destroy B;

$$\text{Let } x - 3 = 0$$

Hence,

$$x = 3$$

Then, substitute in the main equation

$$6(3) - 1 = A(3 - 1) + B(3 - 3)$$

$$17 = A(2) + B(0)$$

$$17 = 2A$$

Here,

$$A = \frac{17}{2}$$

We have found our two values for A and B and we're good!

$$\frac{6x-1}{(x-3)(x-1)} \equiv \frac{\frac{17}{2}}{x-3} + \frac{-\frac{5}{2}}{x-1}$$

We manipulate our fractions, let me expand it well for you this time around, didn't do that in the first case we had fractional answers;

$$\frac{6x-1}{(x-3)(x-1)} = \left(\frac{17}{2} \times \frac{1}{x-3}\right) + \left(-\frac{5}{2} \times \frac{1}{x-1}\right)$$

$$\frac{6x-1}{(x-3)(x-1)} \equiv \frac{17}{2(x-3)} - \frac{5}{2(x-1)}$$

Therefore, the whole partial fraction is;

$$\frac{x^2+2x+2}{x^2-4x+3} \equiv 1 + \frac{17}{2(x-3)} - \frac{5}{2(x-1)}$$

The next example;

$$\frac{x^3+2x-1}{x^2+5x+6}$$

Again, this is an improper fraction so we do the usual;

$$
\begin{array}{r|l}
 & \quad x + 5 \\
\hline
x^2 + 5x + 6 & \quad x^3 + 2x - 1 \\
 & -x^3 - 5x^2 + 6x \\
\cline{2-2}
 & \qquad 5x^2 - 4x - 1 \\
 & \quad -5x^2 + 25x + 30 \\
\cline{2-2}
 & \qquad\qquad -29x - 31
\end{array}
$$

It’s also quite a short division, we can’t go ahead since $-29x$ cannot be divided by x^2 to yield a polynomial expression, polynomials involve only positive powers; notice these subtractions though;

$$x^3 + 2x - 1 - (x^3 - 5x^2 + 6x)$$
$$= 5x^2 - 4x - 1$$

$$5x^2 - 4x - 1 - (5x^2 + 25x + 30)$$
$$= -29x - 31$$

Hence, we have the partial fraction as;

$$(x + 5) + \frac{-29x - 31}{x^2 + 5x + 6}$$

From what we know about factorizing -1. We have;

$$(x+5)+\frac{-(29x+31)}{x^2+5x+6}$$

Finally;

$$(x+5)-\frac{(29x+31)}{x^2+5x+6}$$

We now break this into partial fractions;

$$\frac{(29x+31)}{x^2+5x+6}$$

Factorize the denominator;

$$x^2+5x+6$$

$6\times x^2=6x^2$
$3x+2x=5x$
$3x\times 2x=6x^2$

$$x^2+3x+2x+6$$
$$x(x+3)+2(x+3)$$
$$(x+3)(x+2)$$

We have:

$$\frac{29x+31}{(x+3)(x+2)}\equiv\frac{A}{x+3}+\frac{B}{x+2}$$

Then, do this just normally;

Multiply through by the denominator of the major fraction; $(x+3)(x+2)$ and clear out every needed clearing, we're left with;

$$29x+31=A(x+2)+B(x+3)$$

We'll be substituting for x to get a destroyed situation that'll leave us with B alone left and solve. Here,

$$\text{Let } x+2=0$$

Here,

$$x=-2$$

Then, substitute in the main equation

$$29(-2)+31=A(-2+2)+B(-2+3)$$

$$-58+31=A(0)+B(1)$$

$$-27=B$$

Here,

$$B=-27$$

Next, destroy B;

$$\text{Let } x+3=0$$

Hence,

$$x=-3$$

Then, substitute in the main equation

$$29(-3) + 31 = A(-3 + 2) + B(-3 + 3)$$

$$-56 = A(-1) + B(0)$$

$$-56 = -A$$

Here,

$$A = 56$$

We have found our two values for A and B and we're good!

$$\frac{29x + 31}{(x + 3)(x + 2)} \equiv \frac{56}{x + 3} + \frac{-27}{x + 2}$$

We manipulate our fractions, let me expand it well for you this time around, didn't do that in the first case we had fractional answers;

$$\frac{29x + 31}{(x + 3)(x + 2)} \equiv \frac{56}{x + 3} - \frac{27}{x + 2}$$

Therefore, the whole partial fraction is;

$$\frac{x^3 + 2x - 1}{x^2 + 5x + 6} \equiv x + 5 - \left(\frac{56}{x + 3} - \frac{27}{x + 2}\right)$$

$$\frac{x^3 + 2x - 1}{x^2 + 5x + 6} \equiv x + 5 - \frac{56}{x+3} + \frac{27}{x+2}$$

LAST!

- Resolve into partial;

$$\frac{x^2 - 2x}{x + 1}$$

An improper fraction! DIVIDE!

$$\begin{array}{r|l} & \quad x - 3 \\ \hline x + 1 & \quad x^2 - 2x \\ & -x^2 + x \\ \hline & \qquad -3x \\ & \quad - -3x - 3 \\ \hline & \qquad\qquad 3 \end{array}$$

It's also quite a short division, we can't go ahead since 3 cannot be divided by x to yield a polynomial expression, polynomials involve only positive powers; notice these subtractions though;

$$x^2 - 2x - (x^2 + x) = -3x$$

$$-3x - (-3x - 3) = 3$$

Hence, we have the partial fraction as;

$$(x - 3) + \frac{3}{x + 1}$$

Now, we'd have proceeded to breaking $\frac{3}{x+1}$ into partial fraction but the denominator is already linear, hence, nothing else to break, so the partial fraction is simply given thus;

$$\frac{x^2 - 2x}{x + 1} \equiv (x - 3) + \frac{3}{x + 1}$$

So, that's a cool reminder of partial fractions, we can now proceed to the integration using partial fractions;

It's pretty simple; the main work is the breaking down the compound fractions into partial fractions and applying the integral of sums and differences; let's takes some examples;

Take it simple; let's evaluate these integrals;

INTEGRATION BY PARTIAL FRACTIONS

- $\displaystyle\int \frac{1}{(x-1)(x+2)^2}\,dx$
- $\displaystyle\int \frac{1}{(x-1)(x+2)^2}\,dx$
- $\displaystyle\int \frac{10x-8}{(x-3)(2x+5)}\,dx$
- $\displaystyle\int \frac{x+1}{x^2-3x+2}\,dx$
- $\displaystyle\int \frac{x^2+1}{(x+2)^3}\,dx$
- $\displaystyle\int \frac{3x^2}{x+1}\,dx$
- $\displaystyle\int \frac{x^2}{x-1}\,dx$
- $\displaystyle\int \frac{1}{1+e^{-x}}\,dx$

Cool, *asap*, let's finish these!

$$\int \frac{1}{(x-1)(x+2)^2}dx$$

So, we'll simply split this into partial fractions and solve the integration problems separately; I won't spend ages explaining partial fractions or polynomial divisions here, they've been thoroughly dealt with;

$$\frac{1}{(x-1)(x+2)^2} \equiv \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

The normal process of clearing the fractions;

$$1 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$$

Our destruction process;

$$\text{Let } x-1=0$$

Here,

$$x = 1$$

Then, substitute in the main equation

$$1 = A(1+2)^2 + B(1-1)(1+2) + C(1-1)$$

$$1 = A(3)^2 + B(0)(3) + C(0)$$

$$1 = 9A$$

Here,

$$A = \frac{1}{9}$$

The next destruction,

$$\text{Let } x + 2 = 0$$

Hence,

$$x = -2$$

Then, substitute in the main equation

$$1 = A(-2 + 2)^2 + B(-2 - 1)(-2 + 2) + C(-2 - 1)$$

$$1 = A(0)^2 + B(-3)(0) + C(-3)$$

$$1 = -3C$$

Here,

$$C = -\frac{1}{3}$$

Nothing to destroy again and we still haven't found B, Put $x = 0$ to get a simpler equation in terms of A, B and C and then do the job;

$$\text{Let } x = 0$$

Substitute;

$$1 = A(0 + 2)^2 + B(0 - 1)(0 + 2) + C(0 - 1)$$

$$1 = A(2)^2 + B(-1)(2) + C(-1)$$

$$1 = 4A - 2B - C$$

Since $A = \frac{1}{9}$ and $C = -\frac{1}{3}$

$$1 = 4\left(\frac{1}{9}\right) - 2B - \left(-\frac{1}{3}\right)$$

$$1 = \frac{4}{9} + \frac{1}{3} - 2B$$

$$1 - \frac{4}{9} - \frac{1}{3} = -2B$$

$$\frac{9 - 4 - 3}{9} = -2B$$

$$\frac{2}{9} = -2B$$

Here;

$$B = -\frac{1}{9}$$

We'll now fix our values in the expansion;

$$\frac{1}{(x-1)(x+2)^2} \equiv \frac{\frac{1}{9}}{x-1} + \frac{-\frac{1}{9}}{x+2} + \frac{-\frac{1}{3}}{(x+2)^2}$$

After successful manipulation of fractions; we have;

$$\frac{1}{(x-1)(x+2)^2} \equiv \frac{1}{9(x-1)} - \frac{1}{9(x+2)} - \frac{1}{3(x+2)^2}$$

Hence, our integral becomes;

$$\int \frac{1}{(x-1)(x+2)^2}\,dx$$
$$= \int \left(\frac{1}{9(x-1)} - \frac{1}{9(x+2)} - \frac{1}{3(x+2)^2}\right) dx$$

$$\int \left(\frac{1}{9(x-1)} - \frac{1}{9(x+2)} - \frac{1}{3(x+2)^2}\right) dx$$

From the integrals of sums and differences; we'll have:

$$\int \frac{1}{9(x-1)}\,dx - \int \frac{1}{9(x+2)}\,dx - \int \frac{1}{3(x+2)^2}\,dx$$

Take the integrals separately;

$$\int \frac{1}{9(x-1)}dx$$

The three integrals are linear substitution cases;

Put $u = x - 1$;

$$\frac{du}{dx} = 1$$

Hence,

$$dx = du$$

We have;

$$\int \frac{1}{9u}du = \frac{1}{9}\int \frac{1}{u}du = \frac{1}{9}(\ln u)$$

We have;

$$\frac{1}{9}\ln(x-1)$$

Next!

$$\int \frac{1}{9(x+2)}dx$$

Put $z = x + 2$;

$$\frac{dz}{dx} = 1$$

Hence,

$$dx = dz$$

We have;

$$\int \frac{1}{9z} dz = \frac{1}{9} \int \frac{1}{z} dz = \frac{1}{9} (\ln z)$$

We have;

$$\frac{1}{9} \ln(x + 2)$$

NEXT!

$$\int \frac{1}{3(x + 2)^2} dx$$

Put $z = x + 2$; it's still same substitution so z can still be used;

$$\frac{dz}{dx} = 1$$

Hence,

$$dx = dz$$

We have;

$$\int \frac{1}{3z^2} dz = \frac{1}{3} \int \frac{1}{z^2} dz = \frac{1}{3} \int z^{-2}\, dz$$

$$\frac{1}{3}\left[\frac{z^{-2+1}}{-2+1}\right]=\frac{1}{3}\left[\frac{z^{-1}}{-1}\right]=-\frac{1}{3}\left(\frac{1}{z}\right)=-\frac{1}{3z}$$

We have;

$$-\frac{1}{3(x+2)}$$

Finally, let's combine everything;

$$\int\frac{1}{9(x-1)}dx-\int\frac{1}{9(x+2)}dx-\int\frac{1}{3(x+2)^2}dx$$

$$\frac{1}{9}\ln(x-1)-\frac{1}{9}\ln(x+2)--\frac{1}{3(x+2)}$$

$$\frac{1}{9}\ln(x-1)-\frac{1}{9}\ln(x+2)+\frac{1}{3(x+2)}+C$$

Of course, our arbitrary constant is added; these three separate integrals are nothing special, they're just linear functions substitutions which we've thorough dealt with in this chapter;

NEXT:::

$$\int\frac{10x-8}{(x-3)(2x+5)}dx$$

Partial fraction! Break it down;

$$\frac{10x-8}{(x-3)(2x+5)} \equiv \frac{A}{x-3} + \frac{B}{2x+5}$$

Clear out every needed clearing, we're left with;

$$10x - 8 = A(2x+5) + B(x-3)$$

Here,

$$\text{Let } 2x + 5 = 0$$

Here,

$$x = -\frac{5}{2}$$

Then, substitute in the main equation

$$10\left(-\frac{5}{2}\right) - 8 = A\left(2\left(-\frac{5}{2}\right) + 5\right) + B\left(-\frac{5}{2} - 3\right)$$

$$-25 - 8 = A(-5+5) + B\left(-\frac{11}{2}\right)$$

$$-33 = -\frac{11B}{2}$$

Here,

$$B = 6$$

Again,

$$\text{Let } x - 3 = 0$$

Hence,

$$x = 3$$

Then, substitute in the main equation

$$10(3) - 8 = A(2(3) + 5) + B(3 - 3)$$

$$30 - 8 = A(6 + 5) + B(0)$$

$$22 = 11A$$

Here,

$$A = 2$$

We have found our two values for A and B and we're good!

$$\frac{10x - 8}{(x-3)(2x+5)} \equiv \frac{2}{x-3} + \frac{6}{2x+5}$$

$$\int \frac{10x - 8}{(x-3)(2x+5)} dx = \int \left(\frac{2}{x-3} + \frac{6}{2x+5} \right) dx$$

From the integrals of sums and differences; we'll have:

$$\int \frac{10x - 8}{(x-3)(2x+5)} dx$$

$$= \int \frac{2}{x-3} dx + \int \frac{6}{2x+5} dx$$

Take the integrals separately;

$$\int \frac{2}{x-3} dx$$

Linear substitutions;
Put $u = x - 3$;

$$\frac{du}{dx} = 1$$

Hence,

$$dx = du$$

We have;

$$\int \frac{2}{u} du = 2 \int \frac{1}{u} du = 2(\ln u)$$

We have;

$$2 \ln(x-3)$$

Next!

$$\int \frac{6}{2x+5} dx$$

Put $z = 2x + 5$;

$$\frac{dz}{dx} = 2$$

Hence,

$$dx = \frac{dz}{2}$$

We have;

$$\int \frac{6}{z}\frac{dz}{2} = \frac{6}{2}\int \frac{1}{z}dz = 3(\ln z)$$

We have;

$$3\ln(2x + 5)$$

Finally, let's combine everything;

$$\int \frac{2}{x - 3}dx + \int \frac{6}{2x + 5}dx$$

$$2\ln(x - 3) + 3\ln(2x + 5) + C$$

Of course, our arbitrary constant is added; these two separate integrals are again nothing special, they're just linear functions substitutions which we've thorough dealt with in this chapter;

NEXT EXAMPLE;

$$\int \frac{x+1}{x^2-3x+2}dx$$

Partial fraction! Break it down;
Factorize the denominator;

$$x^2-3x+2$$

$2 \times x^2 = 2x^2$
$-2x - x = -3x$
$-2x \times -x = 2x^2$

$$x^2-2x-x+2$$
$$x(x-2)-1(x-2)$$
$$(x-2)(x-1)$$

Hence, we have;

$$\int \frac{x+1}{(x-2)(x-1)}dx \equiv \frac{A}{x-2}+\frac{B}{x-1}$$

Clear out every needed clearing, we're left with;

$$x+1 = A(x-1)+B(x-2)$$

Here,

$$\text{Let } x-1=0$$

Here,

$$x = 1$$

Then, substitute in the main equation

$$1 + 1 = A(1 - 1) + B(1 - 2)$$

$$2 = A(0) + B(-1)$$

$$2 = -B$$

Here,

$$B = -2$$

Again,

$$\text{Let } x - 2 = 0$$

Hence,

$$x = 2$$

Then, substitute in the main equation

$$2 + 1 = A(2 - 1) + B(2 - 2)$$

$$3 = A(1) + B(0)$$

$$3 = A$$

Here,

$$A = 3$$

We have found our two values for A and B and we're good!

$$\frac{x+1}{(x-2)(x-1)} \equiv \frac{3}{x-2} + \frac{-2}{x-1}$$

$$\frac{x+1}{(x-2)(x-1)}dx \equiv \frac{3}{x-2} - \frac{2}{x-1}$$

$$\int \frac{x+1}{(x-2)(x-1)}dx \equiv \int \left(\frac{3}{x-2} - \frac{2}{x-1}\right)dx$$

From the integrals of sums and differences; we'll have:

$$\int \frac{x+1}{(x-2)(x-1)}dx \equiv \int \frac{3}{x-2}dx - \int \frac{2}{x-1}dx$$

Take the integrals separately;

$$\int \frac{3}{x-2}dx$$

Put $u = x - 2$;

$$\frac{du}{dx} = 1$$

Hence,

$$dx = du$$

We have;

$$\int \frac{3}{u} du = 3 \int \frac{1}{u} du = 3(\ln u)$$

We have;

$$3 \ln(x-2)$$

Next!

$$\int \frac{2}{x-1} dx$$

Put $z = x - 1$;

$$\frac{dz}{dx} = 1$$

Hence,

$$dx = dz$$

We have;

$$\int \frac{2}{z} dz = 2 \int \frac{1}{z} dz = 2(\ln z)$$

We have;

$$2 \ln(x-1)$$

Finally, let's combine everything;

$$\int \frac{3}{x-2} dx - \int \frac{2}{x-1} dx$$

$$3\ln(x-2) - 2\ln(x-1) + C$$

Of course, our arbitrary constant is added; these two separate integrals are again nothing special, they're just linear functions substitutions which we've thorough dealt with in this chapter;

$$\int \frac{x^2+1}{(x+2)^3}\,dx$$

A case of repeated factors; we know how to break the partial fraction;

$$\frac{x^2+1}{(x+2)^3} \equiv \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$$

Clear out every needed clearing, we're left with;

$$x^2 + 1 = A(x+2)^2 + B(x+2) + C$$

Here,

$$\text{Let } x + 2 = 0$$

Here,

$$x = -2$$

Then, substitute in the main equation

$$(-2)^2 + 1 = A(-2+2)^2 + B(-2+2) + C$$
$$4 + 1 = A(0)^2 + B(0) + C$$
$$5 = C$$

Here,

$$C = 5$$

No more destructible substitution, let's start the random substitution, we'll start from $x = 0$

Let $x = 0$

Then, substitute in the main equation

$$(0)^2 + 1 = A(0+2)^2 + B(0+2) + C$$
$$1 = A(2)^2 + B(2) + C$$
$$1 = 4A + 2B + C$$

But $C = 5$

$$1 = 4A + 2B + 5$$
$$4A + 2B = -4$$

Clear by dividing through by 2;

$$2A + B = -2$$

That's equation 1;

$$\text{Put } x = 1$$

Then, substitute in the main equation

$$(1)^2 + 1 = A(1+2)^2 + B(1+2) + C$$
$$1 + 1 = A(3)^2 + B(3) + C$$
$$2 = 9A + 3B + C$$

But $C = 5$

$$2 = 9A + 3B + 5$$
$$9A + 3B = -3$$

Clear by dividing through by 3;

$$3A + B = -1$$

That's equation 2;

Solve eq(1) and eq(2) simultaneously;

$$2A + B = -2$$
$$3A + B = -1$$

Subtracting (2) from (1);

$$-A = -1$$
$$A = 1$$

Put $A = 1$ in eq(1);

$$2A + B = -2$$
$$2(1) + B = -2$$
$$B = -4$$

We have found our two values for A, B and C and we're good!

$$\frac{x^2 + 1}{(x + 2)^3} \equiv \frac{1}{x + 2} + \frac{-4}{(x + 2)^2} + \frac{5}{(x + 2)^3}$$

$$\frac{x^2 + 1}{(x + 2)^3} \equiv \frac{1}{x + 2} - \frac{4}{(x + 2)^2} + \frac{5}{(x + 2)^3}$$

The integral becomes;

$$\int \frac{x^2 + 1}{(x + 2)^3}\,dx \equiv \int \left(\frac{1}{x + 2} - \frac{4}{(x + 2)^2} + \frac{5}{(x + 2)^3}\right) dx$$

$$\int \left(\frac{1}{x + 2} - \frac{4}{(x + 2)^2} + \frac{5}{(x + 2)^3}\right) dx$$

From integral of sums and differences; we have;

$$\int \frac{1}{x+2}dx - \int \frac{4}{(x+2)^2}dx + \int \frac{5}{(x+2)^3}dx$$

Take the integrals separately;

$$\int \frac{1}{x+2}dx$$

$$\frac{dz}{dx} = 1$$

Hence,

$$dx = dz$$

We have;

$$\int \frac{1}{z}dz = (\ln z)$$

We have;

$$\ln(x+2)$$

Next!

$$\int \frac{4}{(x+2)^2}dx$$

Put $z = x + 2$; it's still same substitution so z can still be used;

$$\frac{dz}{dx} = 1$$

Hence,

$$dx = dz$$

We have;

$$\int \frac{4}{z^2} dz = 4 \int \frac{1}{z^2} dz = 4 \int z^{-2} dz$$

$$4\left[\frac{z^{-2+1}}{-2+1}\right] = 4\left(\frac{z^{-1}}{-1}\right) = -4z^{-1} = -\frac{4}{z}$$

We have;

$$-\frac{4}{x+2}$$

NEXT!

$$\int \frac{5}{(x+2)^3} dx$$

Put $z = x + 2$; it's still same substitution so z can still be used;

$$\frac{dz}{dx} = 1$$

Hence,

$$dx = dz$$

We have;

$$\int \frac{5}{z^3} dz = 5 \int \frac{1}{z^3} dz = 5 \int z^{-3}\, dz$$

$$5\left[\frac{z^{-3+1}}{-3+1}\right] = 5\left[\frac{z^{-2}}{-2}\right] = -\frac{5}{2}\left(\frac{1}{z^2}\right) = -\frac{5}{2z^2}$$

We have;

$$-\frac{5}{2(x+2)^2}$$

Finally, let's combine everything;

$$\int \frac{1}{x+2}dx - \int \frac{4}{(x+2)^2}dx + \int \frac{5}{(x+2)^3}dx$$

$$\ln(x+2) - \left(-\frac{4}{x+2}\right) + \left(-\frac{5}{2(x+2)^2}\right)$$

$$\ln(x+2) + \frac{4}{x+2} - \frac{5}{2(x+2)^2} + C$$

Of course, our arbitrary constant is added; these three separate integrals are nothing special, they're just linear functions substitutions which we've thorough dealt with in this chapter;

NEXT;

$$\int \frac{3x^2}{x+1}dx$$

This is clearly a case of improper fractions; let's see how it'll go; divide!

$$
\begin{array}{r|l}
 & 3x - 3 \\
\hline
x + 1 & \quad 3x^2 \\
 & -3x^2 + 3x \\
\cline{2-2}
 & \qquad\quad -3x \\
 & \quad - \quad -3x - 3 \\
\cline{2-2}
 & \qquad\qquad\qquad 3
\end{array}
$$

It's also quite a short division, we can't go ahead since 3 cannot be divided by x to yield a polynomial expression, polynomials involve only positive powers; notice these subtractions though;

$$3x^2 - (3x^2 + 3x) = -3x$$

$$-3x - (-3x - 3) = 3$$

Hence, we have the partial fraction as;

$$(3x - 3) + \frac{3}{x + 1}$$

Now, we'd have proceeded to breaking $\frac{3}{x+1}$ into partial fraction but the denominator is already linear, hence, nothing else to break, so the partial fraction is simply given thus;

$$\frac{3x^2}{x+1} \equiv (x-3) + \frac{3}{x+1}$$

And the integral becomes;

$$\int \frac{3x^2}{x+1}dx = \int \left(x - 3 + \frac{3}{x+1}\right) dx$$

From sums and differences;

$$\int \frac{3x^2}{x+1}dx = \int x\,dx - \int 3\,dx + \int \frac{3}{x+1}dx$$

Take the integrals separately;

$$\int x\,dx = \left[\frac{x^{1+1}}{1+1}\right] = \frac{x^2}{2}$$

$$\int 3\,dx = \int 3x^0\,dx = 3\int x^0\,dx$$

$$3\left[\frac{x^{0+1}}{0+1}\right] = 3x$$

$$\int \frac{3}{x+1}\,dx$$

Put $z = x + 1$;

$$\frac{dz}{dx} = 1$$

Hence,

$$dx = dz$$

We have;

$$\int \frac{3}{z}\,dz = 3\int \frac{1}{z}\,dz = 3(\ln z)$$

We have;

$$3\ln(x+1)$$

Finally, let's combine everything;

$$\int x\,dx - \int 3\,dx + \int \frac{3}{x+1}\,dx$$

$$\frac{x^2}{2} - (3x) + 3\ln(x+1)$$

$$\frac{x^2}{2} - 3x + 3\ln(x+1) + C$$

Of course, our arbitrary constant is added; these three separate integrals are nothing special, they're just linear functions substitutions and some other, the most basic integrals; which we've thoroughly dealt with in this chapter;

NEXT!

$$\int \frac{x^2}{x-1}\,dx$$

Just like the just previous example;

This is clearly a case of improper fractions; let's see how it'll go; divide!

$$\begin{array}{r|l} & \quad x+1 \\ \hline x-1 & x^2 \\ & -x^2-x \\ \cline{2-2} & \qquad x \\ & \qquad -x-1 \\ \cline{2-2} & \qquad\quad 1 \end{array}$$

It's also quite a short division, we can't go ahead since 1 cannot be divided by x to yield a polynomial expression, polynomials involve only positive powers; notice these subtractions though;

$$x^2 - (x^2 - x) = x$$

$$x - (x - 1) = 1$$

Hence, we have the partial fraction as;

$$(x + 1) + \frac{1}{x - 1}$$

Now, we'd have proceeded to breaking $\frac{1}{x-1}$ into partial fraction but the denominator is already linear, hence, nothing else to break, so the partial fraction is simply given thus;

$$\frac{x^2}{x - 1} \equiv (x + 1) + \frac{1}{x - 1}$$

And the integral becomes;

$$\int \frac{x^2}{x - 1} dx = \int \left(x + 1 + \frac{1}{x - 1}\right) dx$$

From sums and differences;

$$\int \frac{x^2}{x - 1} dx = \int x \, dx + \int 1 \, dx + \int \frac{1}{x - 1} dx$$

Take the integrals separately;

$$\int x\,dx = \left[\frac{x^{1+1}}{1+1}\right] = \frac{x^2}{2}$$

$$\int 1\,dx = \int 1x^0\,dx = 1\int x^0\,dx$$

$$1\left[\frac{x^{0+1}}{0+1}\right] = x$$

$$\int \frac{1}{x-1}\,dx$$

Put $z = x - 1$;

$$\frac{dz}{dx} = 1$$

Hence,

$$dx = dz$$

We have;

$$\int \frac{1}{z}\,dz = (\ln z)$$

We have;

$$\ln(x-1)$$

Finally, let's combine everything;

$$\int x\,dx + \int 1\,dx + \int \frac{1}{x-1}\,dx$$

$$\frac{x^2}{2} + x + \ln(x-1) + C$$

Of course, our arbitrary constant is added; these three separate integrals are nothing special, they're just linear functions substitutions and some other, the most basic integrals; which we've thoroughly dealt with in this chapter;

Let's see the last example before we cap the whole of integration up;

- Evaluate this integral;

$$\int \frac{1}{1+e^{-x}}\,dx$$

Now, this is obviously looking impossible but can you please calm down first; if we substitute for the denominator, you can't still cancel off so it'll be a baseless substitution; let's try polynomial division;

Let's just try; 1 divided by $1 + e^{-x}$

$$\begin{array}{r|l} & \quad 1 \\ \hline 1+e^{-x} & \quad 1 \\ & -1+e^{-x} \\ \hline & \quad -e^{-x} \end{array}$$

Let me break down the division process;

1 dividing $1 = 1$; hence, write 1 as part of the quotient and multiply the whole divisor and write;

$$1 \times (1 + e^{-x}) = 1 + e^{-x}$$

Subtract it;

$$1 - (1 + e^{-x}) = -e^{-x}$$

Nothing need be done again, we have what we need as when we have it here; $-e^{-x}$ divided by the divisor will yield a better integral; if we go on, it'll keep recurring;

Hence, we have the partial fraction as;

$$1 + \frac{-e^{-x}}{1 + e^{-x}} = 1 - \frac{e^{-x}}{1 + e^{-x}}$$

$$\frac{1}{1 + e^{-x}} \equiv 1 - \frac{e^{-x}}{1 + e^{-x}}$$

And the integral becomes;

$$\int \frac{1}{1+e^{-x}}\,dx = \int \left(1 - \frac{e^{-x}}{1+e^{-x}}\right) dx$$

From sums and differences;

$$\int \frac{1}{1+e^{-x}}\,dx = \int 1\,dx - \int \frac{e^{-x}}{1+e^{-x}}\,dx$$

Take the integrals separately;

$$\int 1\,dx = \int 1x^0\,dx = 1\int x^0 dx$$

$$\left[\frac{x^{0+1}}{0+1}\right] = x$$

$$\int \frac{e^{-x}}{1+e^{-x}}\,dx$$

Put $z = 1 + e^{-x}$;

$$\frac{dz}{dx} = 0 + e^{-x} \times -1 \quad \text{(Chain rule occurs here)}$$

$$\frac{dz}{dx} = -e^{-x}$$

Hence,

$$dx = -\frac{dz}{e^{-x}}$$

We have;

$$\int \frac{e^{-x}}{z} \times -\frac{dz}{e^{-x}} = \int -\frac{1}{z} dz = -\int \frac{1}{z} dz$$

We have;

$$-\ln(z) = -\ln(1 + e^{-x})$$

Finally, let's combine everything;

$$\int 1\, dx - \int \frac{e^{-x}}{1 + e^{-x}}\, dx$$

$$x - (-\ln(1 + e^{-x}))$$

$$x + \ln(1 + e^{-x}) + C$$

Of course, our arbitrary constant is added; these separate integrals are nothing special, they're just substitutions, recall in the substitution when the numerator is expressed as a derivative of the denominator at worse, they're related by a constant term which can be eliminated from the integral, that is what occurred in the second integral; we've thoroughly dealt with all these in this chapter;

We’ve treated the last part of the integration aspect **in the SSC106 way**; it’s all you need though integration is far wider than that;

Before we draw the curtain, I want us to tackle some few questions;

- Evaluate the integral;

$$\int \frac{e^x - 2x^2 + xe^x}{x^2 e^x} dx$$

We’ll have to split this fraction into the denominators separately;

$$\int \left(\frac{e^x}{x^2 e^x} - \frac{2x^2}{x^2 e^x} + \frac{xe^x}{x^2 e^x} \right) dx$$

We’ll clear the fractions separately;

$$\int \left(\frac{1}{x^2} - \frac{2}{e^x} + \frac{1}{x} \right) dx$$

And turn the integral to separate integrals;

$$\int \frac{1}{x^2}dx - \int \frac{2}{e^x}dx + \int \frac{1}{x}dx$$

Take them separately;

$$\int x^{-2}dx$$

$$\left[\frac{x^{-2+1}}{-2+1}\right] = \frac{x^{-1}}{-1} = -\frac{1}{x}$$

$$\int \frac{2}{e^x}dx$$

$$2\int \frac{1}{e^x}dx = 2\int e^{-x}\,dx$$

Put $u = -x$

$$\frac{du}{dx} = -1$$

$$dx = -du$$

We have;

$$2\int e^u \times -du = -2\int e^u\,du$$

$$-2e^{u} = -2e^{-x}$$

$$\int \frac{1}{x} dx$$

This is a standard integral straight;

$$\ln x$$

Combine the separate integrals;

$$\int \frac{1}{x^2} dx - \int \frac{2}{e^x} dx + \int \frac{1}{x} dx$$

$$-\frac{1}{x} - (-2e^{-x}) + \ln x$$

$$\ln x + 2e^{-x} - \frac{1}{x} + C$$

Of course, your arbitrary constant must be added;

- Evaluate the integral;

$$\int \left(x - \frac{1}{x}\right)^2 dx$$

We'll have to expand this;

$$\int \left(x^2 - 1 - 1 + \frac{1}{x^2}\right) dx$$

$$\int \left(x^2 + \frac{1}{x^2} - 2\right) dx$$

And turn the integral to separate integrals;

$$\int x^2 dx + \int \frac{1}{x^2} dx - \int 2dx$$

Take them separately;

$$\int x^2 dx$$

$$\left[\frac{x^{2+1}}{2+1}\right] = \frac{x^3}{3}$$

$$\int \frac{1}{x^2} dx = \int x^{-2} dx$$

$$\left[\frac{x^{-2+1}}{-2+1}\right] = \frac{x^{-1}}{-1} = -\frac{1}{x}$$

$$\int 2dx = \int 2x^0 dx = 2\int x^0 dx$$

$$2\left[\frac{x^{0+1}}{0+1}\right] = 2\left(\frac{x^1}{1}\right) = 2x$$

Combine the separate integrals;

$$\int x^2 dx + \int \frac{1}{x^2} dx - \int 2dx$$

$$\frac{x^3}{3} + \left(-\frac{1}{x}\right) - (2x)$$

$$\frac{x^3}{3} - \frac{1}{x} - 2x + C$$

Of course, your arbitrary constant must be added;

- Evaluate the integral;

$$\int \frac{1}{x^2} e^{-\frac{1}{x}} dx$$

From here; let's make the substitution;

$$u = -\frac{1}{x} = -x^{-1}$$

$$\frac{du}{dx} = -1 \times -1 \times x^{-1-1} = 1 \times x^{-2} = \frac{1}{x^2}$$

It's actually a case of:

$$f'(x)g[f(x)]$$

Hence,

$$dx = du \times x^2$$

The integral becomes;

$$\int \frac{1}{x^2} e^u \, du \times x^2$$

x^2 cancels out;

$$\int e^u \, du$$

The is a standard integral;

$$e^u$$

Replace u
We have;

$$e^{-\frac{1}{x}} + C$$

And of course the arbitrary constant is added;

When in case of substitution, I'm sure you keep wondering how we know one-time that a substitution is to be made, of course, sometimes by mere sight, we can see it; however, sometimes, you can do well to test the substitution on a separate sheet before drawing your conclusion to see if the substitution will actually cancel out to leave us with one variable or not;

- Evaluate the integral;

$$\int (x+1)(x^2+2x)^3 dx$$

Alright, this is another substitution something, we can see that the second function is of just one higher degree than the first function; and hence, it is done this way; let's try and substitute for the second function;

$$u = x^2 + 2x$$

Here;

$$\frac{du}{dx} = 2 \times x^{2-1} + 2 \times x^{1-1} = 2x + 2$$

Now, the first function is $x+1$ but don't you think if we factorize this derivative of the second function, we could have something interesting?

$$2x+2=2(x+1)$$

Hence, we have the derivative in terms of the first function;

$$\frac{du}{dx}=2(x+1)$$

Hence,

$$dx=\frac{du}{2(x+1)}$$

Back to the integral;

$$\int (x+1)(u)^3\times\frac{du}{2(x+1)}$$

$x+1$ cancels off;

$$\int \frac{u^3}{2}du=\frac{1}{2}\int u^3 du$$

$$\frac{1}{2}\left[\frac{u^{3+1}}{3+1}\right]=\frac{1}{2}\left(\frac{u^4}{4}\right)=\frac{u^4}{8}$$

Return the substitution;

$$\frac{(x^2 + 2x)^4}{8} + C$$

Our arbitrary constant is never forgotten at the end of the solution;

Let's see this question from your past questions;

- Let:

$$U = \int \frac{\sin x}{a \sin x + b \cos x} dx$$

  and

$$V = \int \frac{\cos x}{a \sin x + b \cos x} dx$$

  find:

  (i) $aU + bV$;
  (ii) $aV - bU$.

In great honesty, I can't explain why such tough question is set for you in SSC106 but I guess it's been set about three times now, let's just thrash it in a moment now; however, it is possible to spend about 2 hours trying to make headway to no avail in this question, thank God you're seeing the question now;

$$U = \int \frac{\sin x}{a \sin x + b \cos x} dx$$

$$V = \int \frac{\cos x}{a \sin x + b \cos x} dx$$

Now, trying to solve for U or for V separately, you'll be here for the next six hours or more without a solution; no matter the substitution, whether $\sin x$ or $\cos x$ cannot cancel this and leave you with just one variable;

The test this question brings is to test your knowledge on the properties of integration; the addition and subtraction of integration and also the aspect of constants in integration; now, let's take this as gradually as possible;

$$aU + bV$$

Go ahead and multiply them;

$$aU = a \int \frac{\sin x}{a \sin x + b \cos x} dx$$

Take the a inside the integral, if it was inside, we could take it outside, so now, let's do the reverse;

$$aU = \int \frac{a(\sin x)}{a\sin x + b\cos x}dx$$

Similarly;

$$bV = b\int \frac{\cos x}{a\sin x + b\cos x}dx$$

Take the b inside the integral, if it was inside, we could take it outside, so now, let's do the reverse;

$$bV = \int \frac{b(\cos x)}{a\sin x + b\cos x}dx$$

Is there progress yet? Maybe! Let's just proceed;

$aU + bV$ will be given by:

$$\int \frac{a(\sin x)}{a\sin x + b\cos x}dx + \int \frac{b(\cos x)}{a\sin x + b\cos x}dx$$

So, another most basic integral rule; these are split integral; they could be together before they are split, let's bring them together as if we were there before;

$$\int \left(\frac{a(\sin x)}{a \sin x + b \cos x} + \frac{b(\cos x)}{a \sin x + b \cos x} \right) dx$$

Let's add that fraction within; the denominators are the same so we can add them straight with one common denominator;

$$\int \left(\frac{a(\sin x) + b(\cos x)}{a \sin x + b \cos x} \right) dx$$

$$\int \left(\frac{a \sin x + b \cos x}{a \sin x + b \cos x} \right) dx$$

Are you seeing what I'm seeing in that fraction; the numerator and the denominator are the same; hence, cancel them; we have the simplest integral ever!

$$\int (1)\, dx = \int 1x^0 dx = 1 \int x^0 dx$$

$$1 \left[\frac{x^{0+1}}{0+1} \right] = x + C$$

Hence, $$aU + bV = x + C$$

The question was actually little of asking you about integration laws but the properties of integration; we'll be treating the second part just like this;

$$aV - bU$$

$$aV = a\int \frac{\cos x}{a\sin x + b\cos x}dx$$

Take the a inside the integral, if it was inside, we could take it outside, so now, let's do the reverse;

$$aV = \int \frac{a\cos x}{a\sin x + b\cos x}dx$$

$$bU = b\int \frac{\sin x}{a\sin x + b\cos x}dx$$

Take the b inside the integral, if it was inside, we could take it outside, so now, let's do the reverse;

$$bV = \int \frac{b\sin x}{a\sin x + b\cos x}dx$$

Is there progress yet? There should be, just like the previous example;

$aV - bU$ will be given by:

$$\int \frac{a\cos x}{a\sin x + b\cos x}\,dx - \int \frac{b\sin x}{a\sin x + b\cos x}\,dx$$

So, another most basic integral rule; these are split integral; they could be together before they are split, let's bring them together as if we were there before;

$$\int \left(\frac{a\cos x}{a\sin x + b\cos x} - \int \frac{b\sin x}{a\sin x + b\cos x}\right) dx$$

Let's subtract that fraction within; the denominators are the same so we can add them straight with one common denominator;

$$\int \left(\frac{a\cos x - b\sin x}{a\sin x + b\cos x}\right) dx$$

It's not looking to cancel each other as the previous part; however, checking the denominator, the numerator could be its derivative;

Let's see;

$$z = a\sin x + b\cos x$$

$$\frac{dz}{dx} = a\cos x + b(-\sin x)$$

$$\frac{dz}{dx} = a\cos x - b\sin x$$

That obviously is the numerator; hence; here is a case of;

$$\frac{f'(x)}{f(x)}$$

$$dx = \frac{dz}{a\cos x - b\sin x}$$

Hence, we have;

$$\int \left(\frac{a\cos x - b\sin x}{z}\right) \times \frac{dz}{a\cos x - b\sin x}$$

$a\cos x - b\sin x$ cancels out;

$$\int \left(\frac{1}{z}\right) \times dz = \ln z$$

Return the substitution;

We have;

$$\ln(a \sin x + b \cos x) + C$$

Of course the arbitrary constant cannot be forgotten, we have that;

$$aV - bU = \ln(a \sin x + b \cos x) + C$$

- Find the integral;

$$\int \log_e x \, dx$$

Lol, someone is still expecting this to be a standard integral, no please! Stop there! It isn't any standard integral, we'll be solving this in a very strange way! Integration by parts!

Lol, where is the product that we are integrating by parts now??? Never mind, you'll see it now. It's done this way!

We express $\log_e x$ as $(1 \times \log_e x)$

At least you agree multiplying by 1 changes nothing! Let's move on.

We have now successfully made it a product forcefully!

Now, we now have the integration by part, as you know, in a product, the logarithm is always the first to take the place of u since it has no integral; hence;

$$\int 1 \times \log_e x \, dx$$

Put

$$u = \log_e x$$

Standard derivative;

$$\frac{du}{dx} = \frac{1}{x}$$

Also;

$$\frac{dv}{dx} = 1$$

Integrate!

$$\int dv = \int 1 dx$$

1 is same as x^0;

$$\int dv = \int x^0 dx$$

Here, straight from the power integral rule:

$$v = \left[\frac{x^{0+1}}{0+1}\right] = x$$

Hence, we have all we need; rush to the integration by parts formula making the appropriate substitutions for **all terms**:

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$$\int 1 \times \log_e x \, dx = \log_e x \, (x) - \int x \left(\frac{1}{x}\right) dz$$

Simplifying further;

$$\int 1 \times \log_e x \, dx = x \log_e x - \int 1 dx$$

We have reduced the integral to the sum of a term and another integral which should be integrated easily;

So, let's evaluate this integral we have in our reduced form;

$$\int 1dx$$

Soft work!

$$\int 1dx = \int x^0 dx$$

We can now integrate easily by the power rule;

$$\left[\frac{x^{0+1}}{0+1}\right] = x$$

Hence,
Finally the integral of $\log_e x$ is:

$$\int 1 \times \log_e x\, dx = x \log_e x - \int 1dx$$

$$\int \log_e x\, dx = x \log_e x - x$$

Since;

$$\int 1 \times \log_e x\, dx = \int \log_e x\, dx$$

Lol, you sure know mathematics isn't one subject that is okay mentally, nonetheless, that is how logarithm functions are integrated, let's see another case of logarithm.

- Find the integral;

$$\int \log_e(2x+3)\,dx$$

This is a case of linear substitution, hence, we have;

$$z = 2x + 3$$

$$\frac{dz}{dx} = 2$$

And

$$dx = \frac{dz}{2}$$

Hence,

We have:

$$\int \log_e z \frac{dz}{2}$$

Bringing the constant out;

$$\frac{1}{2}\int \log_e z\,dz$$

We can now integrate $\log_e z$ using integration by parts, the same as the previous example, just variable change.

Multiply by 1;

$$\frac{1}{2}\int 1 \times \log_e z \, dz$$

Facing the integral squarely now;

$$\int 1 \times \log_e z \, dz$$

Put;

$$u = \log_e z$$

Standard derivative;

$$\frac{du}{dz} = \frac{1}{z}$$

Also;

$$\frac{dv}{dz} = 1$$

Integrate!

$$\int dv = \int 1 dz$$

1 is same as z^0;

$$\int dv = \int z^0 dz$$

Here, straight from the power integral rule:

$$v = \left[\frac{z^{0+1}}{0+1}\right] = z$$

Hence, we have all we need; rush to the integration by parts formula making the appropriate substitutions for **all terms**:

$$\int u\frac{dv}{dz}dz = uv - \int v\frac{du}{dz}dz$$

Note the integration by parts formula changes to dz since we are having a new variable now;

$$\int 1 \times \log_e z\, dz = \log_e z\,(z) - \int z\left(\frac{1}{z}\right)dz$$

Simplifying further;

$$\int 1 \times \log_e z\, dz = z\log_e z - \int 1dz$$

We have reduced the integral to the sum of a term and another integral which should be integrated easily;

So, let's evaluate this integral we have in our reduced form;

$$\int 1dz$$

$$\int 1dz = \int z^0 dz$$

We can now integrate easily by the power rule;

$$\left[\frac{z^{0+1}}{0+1}\right] = z$$

Hence,
Finally the integral of $\log_e z$ is:

$$\int 1 \times \log_e z \, dz = z \log_e z - \int 1dz$$

$$\int \log_e z \, dz = z \log_e z - z$$

Since;

$$\int 1 \times \log_e z \, dz = \int \log_e z \, dz$$

Hence;
Tracking back; we had a substitution where $z = (2x + 3)$;

$$\frac{1}{2}\int 1 \times \log_e z \, dz = \frac{1}{2}\left(z \log_e z - z\right)$$

Hence; substituting back z, we have the final integral as:

$$\int \log_e(2x+3)\,dx = \frac{1}{2}[(2x+3)\log_e(2x+3) - (2x+3)]$$

THE DEFINITE INTEGRAL

The concept of definite integrals isn't a difficult concept at all, it is the process of taking integrals with limits; any function given is integrated using the very normal process in the rules we've discussed above and then the limits are applied; a definite integral is different from an indefinite integral, interestingly, we're able to eliminate the arbitrary constant when definite integrals are taken; finding definite integrals are pretty easy, it merely involves just substituting values and subtraction, that's easy, basically, we're through with the concept of integration already;

The integral of $f(x)$ from $x = a$ to $x = b$ is given by:

$$\int_b^a f(x)\,dx$$

To evaluate this, the integral is taken as normally; then, the a and then b is substituted for x in the indefinite integral since it's with respect to x and then the latter subtracted from the former.

Hence, let's explain with examples;

- Evaluate:

$$\int_{-2}^{3} (x^3 - x^2)dx$$

So, at first, we'll integrate it normally first, then we'll take the limits from a to b;

$$\left[\frac{x^{3+1}}{3+1}\right] - \left[\frac{x^{2+1}}{2+1}\right] + C$$

$$\frac{x^4}{4} - \frac{x^3}{3} + C$$

Then place the limits

$$\left[\frac{x^4}{4} - \frac{x^3}{3} + C\right]_{-2}^{3}$$

The limits are written as above;
Then subtract the result of substituting for the lower limit from that of substituting for the upper limit;

$$\left(\frac{(3)^4}{4} - \frac{(3)^3}{3} + C\right) - \left(\frac{(-2)^4}{4} - \frac{(-2)^3}{3} + C\right)$$

$$\frac{81}{4} - \frac{27}{3} + C - \left(\frac{16}{4} - \left(-\frac{8}{3}\right) + C\right)$$

$$\frac{81}{4} - \frac{27}{3} + C - \left(\frac{16}{4} + \frac{8}{3} + C\right)$$

$$\frac{81}{4} - \frac{27}{3} + C - \frac{16}{4} - \frac{8}{3} - C$$

So, no matter how, the arbitrary constant cancels out, and hence, when having definite integrals, we don't bother ourselves with the arbitrary constants, I just put it here to make sure you understand why we leave it out; hence, the definite integral is given by:

$$\int_{-2}^{3}(x^3 - x^2)dx = \frac{81}{4} - \frac{27}{3} - \frac{16}{4} - \frac{8}{3}$$

$$\int_{-2}^{3}(x^3 - x^2)dx = \frac{3(81) - 4(27) - 3(16) - 4(8)}{12}$$

$$\int_{-2}^{3}(x^3 - x^2)dx = \frac{243 - 108 - 48 - 32}{12}$$

$$\int_{-2}^{3}(x^3 - x^2)dx = \frac{55}{12}$$

It's that simple! We were through with integration long before now, this is just an additional exercise but it's equally very important.

So the basic difference is that:

- where the indefinite integral gives the general form of the anti-derivative of a function, the definite integral gives the area under a curve between two given points and is the value gotten by evaluating the integral from the two limits.

- the indefinite integral also contains an arbitrary constant while a definite integral doesn’t contain an arbitrary constant.

Let’s see two more examples and we’re done!

- Evaluate

$$\int_{30}^{60} \sin 3x + 3$$

Normally, this integral above should be represented as:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 3x + 3$$

However, the notation of π is a science trigonometry notation and hence, I’ll spare you that and use the normal angle notation; hence; here;

$$\int_{30}^{60} \sin 3x + 3$$

We'll take the integral normally first;

$$\int \sin 3x \, dx + \int 3dx$$

$$\int \sin 3x \, dx$$

Linear substitution;

$$u = 3x$$

$$\frac{du}{dx} = 3$$

Hence;

$$dx = \frac{du}{3}$$

We have:

$$\int \sin u \frac{du}{3} = \frac{1}{3}\int \sin u \, du$$

Standard integral;

$$\frac{1}{3}(-\cos u)$$

Return u;

$$-\frac{1}{3}\cos 3x$$

$$\int 3dx = 3\int x^0\, dx$$

$$3\left[\frac{x^{0+1}}{0+1}\right] = 3x$$

Hence; we have the integral as:

$$-\frac{1}{3}\cos 3x + 3x$$

No need for arbitrary constant, we want to evaluate the definite integral;

$$\left[-\frac{1}{3}\cos 3x + 3x\right]_{30}^{60}$$

$$\left(-\frac{1}{3}\cos(3\times 60) + 3(60)\right) - \left(-\frac{1}{3}\cos(3\times 30) + 3(30)\right)$$

$$\left(-\frac{1}{3}\cos 180 + 180\right) - \left(-\frac{1}{3}\cos 90 + 90\right)$$

$$\cos 180 = -1$$
$$\cos 90 = 0$$

Hence;

We have;

$$-\frac{1}{3}(1) + 180 + \frac{1}{3}(0) - 90$$

$$90 - \frac{1}{3} = \frac{269}{3}$$

INTEGRATION WAS INTERESTING RIGHT? I'M GLAD YOU NOW UNDERSTAND IT, HAVE SOME FEEDBACK, CHECK THE ABOUT PAGE TO SEND ME A MESSAGE!