APPLICATION OF CALCULUS TO ECONOMICS AND MANAGEMENT SCIENCES

Calculus (differentiation and integration) find several and loads of uses in the world of economics, the social sciences and even in the administration world in general.

The teaching of calculus amongst social science and administration students seem a burden; however, after this chapter, I believe you will see beyond doubts that the learning of mathematics in the Faculty of Social Sciences and the Faculty of Administration is in no way unnecessary – The reason you're offering SSC106!

We covered concepts of maximum and minimum values in the applications of differential calculus. Such concepts and lots more will help us as we delve into several concepts in economic analysis that use both differential and integral calculus for optimization.

The several use of $\frac{dy}{dx}$ and that overbearing long S $\rightarrow \int$ seemed unnecessary to you as a student of your faculty, However, let's see as they're applied to your own field! Beyond anything you've learnt in this course, make it a priority that you understand this beyond the exam point as it is very important in life and business.

MARGINAL FUNCTIONS

One of the applications of derivatives in a real world situation is in the area of marginal analysis. Marginal analysis uses the derivative (or rate of change) to determine the rate at which a particular quantity is increasing or decreasing. Marginal functions cover all forms of functions, cost functions, profit functions, average cost functions, revenue functions and every other form of functions. Anything that can be expressed as a function has its corresponding marginal function.

No matter which function we are dealing with, the word "marginal" indicates to us that we need to find the derivative of the function. For example, if we are asked to find the marginal cost function then we need to find the derivative of the cost function. ***That seemed quite straightforward right?*** If you are asked to find the marginal revenue function, then you find the derivative of the revenue function. Sure you remember derivative right? That $\frac{dy}{dx}$ stuff;

Let's rush to an example quickly!

EXAMPLE 1:

If a company's total cost function is defined as $\boldsymbol{C(x) = 0.00002x^3 - 0.02x^2 + 400x + 50000}$, find the marginal cost function and evaluate it when $\boldsymbol{x = 200}$.

Now, it's this way;

The cost function is:

$$C(x) = 0.00002x^3 - 0.02x^2 + 400x + 50000$$

Now, we seek to find the marginal cost function, recall that we just discussed that for whichever function, the corresponding marginal function is the derivative of the original function.

Recall how to find derivatives of this kind of function right? You should!

Also, the derivative of functions in the form $C(x)$ is in the form $C'(x)$, ***we sure discussed that extensively while treating differentiation,*** also in this case, the marginal cost function, the derivative of $C(x)$ is also denoted as $C'(x)$.

* Recall that some books may denote it as $C^1(x)$, however, $C'(x)$ will be used in the course of this!

$$C'(x) = \frac{d}{dx}(0.00002x^3 - 0.02x^2 + 400x + 5000)$$

Here, $C'(x) = 0.00006x^2 - 0.04x + 400$

The marginal function is as simple as that → $C'(x)$

We were asked to evaluate it at $x = 200$;

Hence;

$$C'(200) = 0.00006(200)^2 - 0.04(200) + 400$$
$$C'(200) = 0.00006(40000) - 0.04(200) + 400$$
$$C'(200) = 2.4 - 8 + 400$$
$$C'(200) = 394.4$$

Now we've seen what marginal functions are; so what is the essence of marginal functions; let's have a very short explanation:

Marginal functions, whether for revenues, costs or profit functions as the case may be give the approximate change for the next unit of such function at a given value, ***didn't get that, no worries, you'll get it now!***

In the example above, the marginal cost function at 200 was 394.4. It simply means that the cost of producing **the next unit** which will be the 201st unit is 394.4;

The marginal cost function means adding one extra unit to the 200 units to make the production 201 units will cost an extra amount. Hence, an extra approximate 394.4 will be spent in producing 201 units instead of 200.

Before we continue, **a cost function is a mathematical relationship used to relate how production expenses will differ at different output levels (units of output produced). It is a function of the units of output produced.**

PEN THAT DOWN;

A cost function usually consists of **a fixed cost** and a **variable cost**.

In:

$$C(x) = 0.00002x^3 - 0.02x^2 + 400x + 50000$$

the variable costs are the first three terms of the function while the fixed cost is 5000. The variable costs change with the quantity of goods produced whilst the 5000 is constantly added regardless of the quantity of goods produced.

Back to where we were, let's see from **Example one** what marginal cost function is;

The cost of producing 200 units will be given by $C(200)$;

$$C(x) = 0.00002x^3 - 0.02x^2 + 400x + 50000$$

$$C(200) = 0.00002(200)^3 - 0.02(200)^2 + 400(200) + 50000$$

$$C(200) = 160 - 800 + 80000 + 50000$$

$$C(200) = 129360$$

The cost of producing 201 units will be given by $C(201)$;

$$C(201) = 0.00002(201)^3 - 0.02(201)^2 + 400(201) + 50000$$

$$C(201) = 162.41202 - 808.02 + 80400 + 50000$$

$$C(201) = 129754.392$$

Now, let's evaluate the difference between the costs:

$$C(201) - C(200) = 129754.392 - 129360$$
$$C(201) - C(200) = 394.392$$

The difference is the same value gotten when the marginal cost function was evaluated at $x = 200$;

Hence, the marginal cost function at any unit gives the increase in cost in producing 1 extra unit at that unit.

That's just for the sake of your knowledge to know what a marginal function really means such that <u>you're not just learning blankly</u>, you simply need to evaluate the derivative when you're asked for a marginal function. You don't go ahead proving the difference in real exam situations; However, an understanding of this will let you cope when a question is twisted for you, such as asking for the marginal function indirectly. We'll still see it in the next example.

Let's also quickly consider the relationship between the marginal functions and the change in the units of a function.

Every function depends on a certain unit, like the cost function just treated above depends on the quantity (or output), other functions depend on something else as well. Some other functions that'll be seen in the course of the chapter such as revenue functions also depend on the quantity (output).

In normal differentiation; we have the relationship below;

$$\Delta y = \frac{dy}{dx}\Delta x$$

Where;

$$\Delta y = \text{Change in } y$$
$$\Delta x = \text{Change in } x$$

Marginal functions in general tell us how a particular quantity (cost, profit, revenue, etc.) in forms of a function increases or decreases. Hence, the marginal function is the rate of change of that function. It is the $\frac{dy}{dx}$ of that function.

Hence, if we have the change in the unit any function depends on, we can have the change in the whole function itself, using the formula shown above.

Let’s explain that;

For example, in a cost function. If we have a change in the quantity produced (that is the unit the cost function depends on), we can get the change in the total cost, using the formula.

Hence, for any function, $f(x)$, the marginal function is $f'(x)$ and we have the change in x as Δx, and the corresponding the change in the function itself, $\Delta f(x)$ is given by:

$$\Delta f(x) = f'(x)\Delta x$$

Example;

- Find the effect of:
  - ✓ an increase by 2 in the quantity of units produced;
  - ✓ a decrease by 2 in the quantity of units produced;

  on the cost of producing a product whose cost function is defined by:

$$TC = 0.00002q^3 - 0.02q^2 + 400q + 50000$$

Now, for an increase, the change in the unit is positive, and for a decrease, the change is negative.

Hence, for an increase of 2, we have: $\Delta q = 2$
And for a decrease of 2, we have: $\Delta q = -2$

We have our cost function as:

$$TC = 0.00002q^3 - 0.02q^2 + 400q + 50000$$

And hence, the marginal cost, MC, is given by:

$$MC = \frac{d}{dq}(0.00002q^3 - 0.02q^2 + 400q + 50000)$$

$$MC = 0.00006q^2 - 0.04q + 400$$

Hence, to find the effect on the total cost, we have, for an increase of 2;

$$\Delta TC = MC \times \Delta q$$

Hence;

$$\Delta TC = (0.00006q^2 - 0.04q + 400)(2)$$

$$\Delta TC = \ 0.00012q^2 - 0.08q + 800$$

Hence, the total cost changes by $(\mathbf{0.00012}\boldsymbol{q}^{\mathbf{2}} - \mathbf{0.08}\boldsymbol{q} + \mathbf{800})$ for an increase of 2 in the quantity of unit produced.

Note strongly that actual value of the change in the total cost will be dependent on the quantity produced.

Hence, for example, if 30 quantities are produced;

The effect of producing 2 more units will be given by substituting 30 in the ΔTC at a change of 2 units in the unit.

$$\Delta TC(30) = \ 0.00012(30)^2 - 0.08(30) + 800$$

$$\Delta TC(30) = 797.708$$

The effect of producing 2 more units will be different if it is evaluated when the quantity produced is 60;

$$\Delta TC(60) = 0.00012(60)^2 - 0.08(60) + 800$$

$$\Delta TC(60) = 795.632$$

Also, for a decrease, we'll be evaluating it as:

$$\Delta q = -2$$

$$\Delta TC = MC \times \Delta q$$

Hence;

$$\Delta TC = (0.00006q^2 - 0.04q + 400)(-2)$$

$$\Delta TC = -0.00012q^2 + 0.08q - 800$$

Hence, the total cost changes by $(\boldsymbol{-0.00012q^2 + 0.08q - 800})$ for a decrease of 2 in the quantity of unit produced.

Hence, for example, if 30 quantities are produced;

The effect of producing 2 less units will be given by substituting 30 in the ΔTC at a change of -2 units in the unit.

$$\Delta TC(30) = -0.00012(30)^2 + 0.08(30) - 800$$

$$\Delta TC(30) = -797.708$$

Similarly, if 60 units are produced; the effect of producing 2 less units will be given by substituting 60 in the ΔTC at a change of -2 units in the unit.

$$\Delta TC(60) = -0.00012(60)^2 + 0.08(60) - 800$$

$$\Delta TC(60) = -795.632$$

The formula isn't fixed only to cost functions, each function that has marginal functions which we'll still be seeing below all obey the same formula.

Let's find an extra example here:

EXAMPLE 2:

If the marginal revenue of goods produced is by Adekunle's enterprise is defined by $R'(x) = -0.06x + 750$, evaluate the total revenue when the company produces 50 units of the goods.

Marginal has shown up again!

The marginal rule! The marginal function of any function is the derivative of that given function!

Now, we have the marginal function; we need the original function! Common **calculus logic of oppositeness** means we'd integrate it (the marginal function) to get the original function!

It's that SIMPLE! What then do we have after integrating? We have the revenue function, $R(x)$.

Hence,

$$R(x) = \int R'(x)\,dx = \int (-0.06x + 750)\,dx$$

$$R(x) = \int (-0.06x)\,dx + \int (750)\,dx$$

I'm sure you remember the integration thing right? Don't hesitate to remind yourself a little.

$$R(x) = \frac{(-0.06x^2)}{2} + \frac{(750x)}{1}$$

$$R(x) = -0.03x^2 + 750x + C$$

Remember the arbitrary constant is always added anywhere you see integration happening!

Now, no information is given as to how we're to get the value of C, for the sake of that, the standard thing to do is to assume the value of the constant is zero; hence, we have;

$$R(x) = -0.03x^2 + 750x$$

Now, we have the revenue function;

Now we were asked for the total revenue when 50 units of the goods are produced! That's easy!

The revenue function is usually a function of units of goods produced! Hence, find the revenue function at $x = 50$ which gives $R(50)$.

$$R(50) = -0.03(50)^2 + 750(50)$$

$$R(50) = -75 + 37500$$

$$R(50) = 37425$$

Hence, the revenue realized when 50 goods is produced is 37425.

"…In a nutshell, from the function, differentiate it to get the corresponding marginal function. And from the marginal function, integrate it to get the original function."

Another very important thing to know is that apart from $C'(x)$, $R'(x)$ for representing the marginal functions of cost and revenue; $MC(x)$, $MR(x)$ could also be used, widely based on the text, or the personnel involved.

In exam situations however, you wouldn't need to bother much about that, what will be used will be given.

So, marginal cost functions can be represented as:

Marginal cost function: $C'(x)$, $MC(x)$, MC

Marginal revenue function: $R(x)$, $MR(x)$, MR

Meanwhile… … … **A revenue function is a mathematical function that gives the total amount realized relative to the total number of units of goods produced.**

The Revenue function is a function of the quantity of goods produced.

A simple term to quickly look into again is **the average function**.

The average function is simply given by dividing the function by the unit by which it depends on. Like the marginal function, the average function exists for every function; such as cost functions, the revenue function, the profit functions and so on.

EXAMPLE 3:

If a company's total cost function is defined as $C(x) = 0.00002x^3 - 0.02x^2 + 400x + 50000$, find the average cost function, and the marginal average cost when $x = 200$.

Like I've said, the average function is gotten simply dividing the function by the unit which it depends on:

Here, The average cost function, $\bar{C}(x)$

Oops, I've not mentioned that; the average function is denoted by a bar on the original function, $\overline{\boldsymbol{C}}(\boldsymbol{x})$ for average cost function, $\overline{\boldsymbol{R}}(\boldsymbol{x})$ for average revenue function and so on. It could also be denoted as $AC(x)$ for the average cost function and $AR(x)$ for the average revenue function… ***Like that–like that, just play around it.***

Hence, back to **EXAMPLE 3;**

The average function is simply given by dividing the function by the unit by which it depends on. Hence;

$\bar{C}(x) = \frac{C(x)}{x}$; simply input the cost function:

$$\bar{C}(x) = \frac{0.00002x^3 - 0.02x^2 + 400x + 50000}{x}$$

Remember how to simplify this easily? Distribute the singular denominator on each denominator;

$$\bar{C}(x) = 0.00002x^2 - 0.02x + 400 + \frac{50000}{x}$$

As simple as that! That's the average cost function. However, there's still one guy there that hasn't been resolved! → and the marginal average cost when $x = 200$

Lol… You know everything marginal, just find the derivative!

Here, we need the marginal average cost function!

We have the average cost function already: $\bar{C}(x)$

The marginal average cost function is: $\bar{C}'(x)$:::

$$\bar{C}'(x) = \frac{d}{dx}[C(x)]$$

$$\bar{C}'(x) = \frac{d}{dx}\left(0.00002x^2 - 0.02x + 400 + \frac{50000}{x}\right)$$

$$\bar{C}'(x) = 0.00004x - 0.02 - \frac{50000}{x^2}$$

You should fully know how to differentiate now, derivative of 400 is zero;

Now, evaluating the marginal average cost at $x = 200$ is simply solving for $\bar{C}'(200)$

$$\bar{C}'(x) = 0.00004(200) - 0.02 - \frac{50000}{(200)^2}$$

$$\bar{C}'(x) = 0.008 - 0.02 - 1.25 = -1.262$$

Same way you'll find anything average, whether revenue, profit, production or any other form of functions!

Now, dealing with the **revenue function** proper, we'll be introducing some terms very quickly now. Just stay right here!

The revenue function can be gotten in when the price of items is known. It is of common experience that prices (in real life) varies with the quantity of good bought, hence, the ***price per unit,*** normally denoted by p can be expressed in terms of the quantity. Permit me to name it ***the price function.***

A price function is dependent again on the quantity of goods bought and hence, is a function of the quantity of goods bought.

An example of a price function is $\boldsymbol{p = 200 - 0.04q}$ where q is the quantity of goods bought (or demanded).

Now, back to the revenue function, a revenue function can be expressed in terms of the ***price per unit,*** p by multiplying p by the quantity of goods bought (or demanded), q.

Therefore, $R(q) = p \times q = pq$

Lol…Surprised? You don't need to be, $R(q)$ is the same as the $R(x)$ we've been going through all along, just that then, the quantity of goods was expressed as x but now, it is expressed as q.

Hence, $R(x) = p \times x = px$

Such little things could throw one into confusion, isn't it? But you know now!

Hence, let's hop into **Example 4:**

EXAMPLE 4: A firm produces two products x and y and joint total cost function, TC, is given by:

$$TC = 6x^2 + 3x + 1.5xy + 3y^2 + 2y + 50$$

(i) Determine the marginal cost function for each of the products.

(ii) Evaluate the marginal cost when $y = 3$ and $x = 5$

The definition of marginal functions will not change because it's a case of multivariable functions, we still need the derivatives and hence, we'll be taking **partial derivatives** to find the marginal cost for each separate product (since it is a multivariable case).

To find the marginal cost function for each product; we will be taking the partial derivative of the cost function with respect to x and y;

$$MC(x) = \frac{\partial}{\partial x}(6x^2 + 3x + 1.5xy + 3y^2 + 2y + 50)$$

$$MC(x) = 2 \times 6x^{2-1} + 1 \times 3x^{1-1} + 1 \times 1.5x^{1-1} \times y + 0 + 0 + 0$$

$$MC(x) = 12x + 3 + 1.5y$$

$$MC(y) = \frac{\partial}{\partial y}(6x^2 + 3x + 1.5xy + 3y^2 + 2y + 50)$$

$$MC(y) = 0 + 0 + 1 \times 1.5x \times y^{1-1} + 2 \times 3 \times y^{2-1} + 1 \times 2y^{1-1} + 0$$

$$MC(y) = 1.5x + 6y + 2$$

Those are the separate marginal cost functions for each product;

To evaluate the marginal costs at the given points, we'll simply substitute as usual in functions.

$$MC(x) = 12x + 3 + 1.5y$$

At $y = 3$ and $x = 5$

$$MC(x) = 12(5) + 3 + 1.5(3)$$

$$MC(x) = 60 + 3 + 4.5 = 67.5$$

$$MC(y) = 1.5x + 6y + 2$$

At $y = 3$ and $x = 5$

$$MC(y) = 1.5(5) + 6(3) + 2$$

$$MC(y) = 7.5 + 18 + 2 = 27.5$$

Hence, the marginal cost of product x is 67.5 and the marginal cost for y is 27.5, both at $y = 3$ and $x = 5$

EXAMPLE 5: Given that the relationship between the unit price in dollars and the quantity demanded, x is given by the equation: $p = -0.03x + 750$ where $0 \leq x \leq 25000$; find $MR(3000)$.

Here, this is as simple as you like it.

The revenue function, $R(x)$ is given by the product of p and x; since, x is used as the quantity of goods bought (demanded) here; we have the price function given here:

Therefore, $R(x) = px = (-0.03x + 750)x$

This is the simplest expansion process;

$R(x) = -0.03x^2 + 750x$;

Now, we're told to find $MR(3000)$ but easy first, we have to evaluate $MR(x)$ itself first – the marginal revenue function, that derivative something!

$$MR(x) = \frac{d}{dx}[R(x)] = \frac{d}{dx}(-0.03x^2 + 750x)$$

$$MR(x) = -0.06x + 750$$

Now, evaluating it at $x = 3000$;

$$MR(3000) = -0.06(3000) + 750$$

$$MR(3000) = -180 + 750 = 570$$

Therefore, the sale of the 3001^{st} unit of that product will produce revenue of 570.

By the way, this guy here i.e. where $\mathbf{0 \leq x \leq 25000}$ was just for formalities, it doesn't affect you in anyway. Kindly leave it alone.

Cool, we're moving into **the profit function!**

In layman language, revenue is the money at hand collected when goods are sold; however it is well known that the whole of that money isn't the profit made as **the goods weren't produced for free**; hence, to get the profit actually, the cost of production is subtracted first! ***In market language, you ask, how much did we spend (cost), and how much did we sell it (revenue), afterwards, we now ask ourselves how much we gained (profit) by subtracting how much we spent (cost) from how much we sold it (revenue).***

Same thing here!!! The profit function is the difference between the revenue function and the cost function. Of course, for a business to be called a business, the revenue should exceed the cost else that'll make it a useless business.

Therefore, $P(x) = R(x) - C(x)$!

EXAMPLE 6: Given a company's weekly demand is $p = 500 - 0.05q$ where p is the unit price in naira and q is the quantity of goods demanded. The cost function is $C(q) = 0.00002q^3 - 0.03q^2 + 300q + 78000$; find the profit made when 1000 units of the goods are sold.

Straight-off! The revenue function is given by:

$$R(q) = p \times q$$

$$R(q) = (500 - 0.05q) \times q$$

$$R(q) = 500q - 0.05q^2$$

Get the profit function by the relationship:

$$P(q) = R(q) - C(q)$$

$$P(q) = 500q - 0.05q^2 - (0.00002q^3 - 0.03q^2 + 300q + 78000)$$

I'm sure you know the importance of that bracket; the minus affects all of the cost function and not only the first term; hence, opening the bracket:

$$P(q) = 500q - 0.05q^2 - 0.00002q^3 + 0.03q^2 - 300q - 78000$$

$$P(q) = -0.00002q^3 - 0.02q^2 + 200q - 78000$$

Let's evaluate the profit at sales of 1000 units of goods;

$$P(1000) = -0.00002(1000)^3 - 0.02(1000)^2 + 200(1000) - 78000$$

$$P(1000) = -0.00002(1000)^3 - 0.02(1000)^2 + 200(1000) - 78000$$

$$P(1000) = -20000 - 20000 + 200000 - 78000$$

$$P(1000) = 82000$$

Hence, 82,000 is made when 1000 units of the goods are sold.

We're good to go; let's move briefly into something of extreme importance;

The Production function is a mathematical relationship that relates the quantity of produce produced (production outputs) to the quantities of physical inputs (the factors of production).

Usually, it could be dependent on two or more variables (such as labour, capital, etc). However, mostly within the scope of this course (SSC106), the production function is expressed as a function dependent on one variable:

Now, I said it's of extreme importance as the production function is also represented as $P(L)$ or $P(K)$ as the case may be; hence, P, the same used in the profit function, is also used. However, the two are two quite different things, hence, watch it!

An example of a production function is given thus: $P(L) = 3L^2 - \frac{L}{3} + 100$;

Let's have an example:

EXAMPLE 7: Given that at various levels of labour, L, the production quantity of a company is given by: $P(L) = 500 + 14L^2 - \frac{L^3}{4}$; find the company's average product produced when 30 labourers are available. Evaluate also the marginal product at the same labour input.

Now, we have:

$$P(L) = 500 + 14L^2 - \frac{L^3}{4}$$

Now recall what we discussed about average functions, it is the function itself divided by the unit it depends on; hence, to find the average production here; divide the production function by the unit it depends on, labour, L.

$$\bar{P}(L) = AP(L) = \frac{P(L)}{L}$$

Note that average production could be represented by either of $\bar{P}(L)$ or $AP(L)$.

Therefore,

$$AP(L) = \frac{\left(500 + 14L^2 - \frac{L^3}{4}\right)}{L}$$

$$AP(L) = \frac{500}{L} + 14L - \frac{L^2}{4}$$

To find the average quantity of products produced when 30 workers are available; evaluate $AP(30)$

$$AP(L) = \frac{500}{30} + 14(30) - \frac{(30)^2}{4}$$

$$AP(L) = 16.67 + 420 - 225 = 211.67$$

Hence, an **approximate** average of 212 units of the products is produced per labourer when 30 labourers are available.

The marginal production function; find the derivative of the production function:

$$P'(L) = MP(L) = \frac{d}{dL}[P(L)]$$

$$MP(L) = \frac{d}{dL}\left(500 + 14L^2 - \frac{L^3}{4}\right)$$

$$MP(L) = 28L - \frac{3L^2}{4}$$

The marginal product at same labourers, i.e. 30 labourers is $MP(30)$

$$MP(30) = 28(30) - \frac{3(30)^2}{4}$$

$$MP(30) = 28(30) - \frac{3(30)^2}{4}$$

$$MP(30) = 840 - 675 = 165$$

Hence, for an additional 1 labourer to 30 labourers, more 165 units of the products will be produced!

I think that's all about the first part… We'll still need the concepts learnt above though in the next section! However, let's see another very key function.

The consumption function (sometimes called the aggregate consumption function) is a function that gives the relationship between total consumption and gross income.

The consumption function is a function of the income (disposable) of the particular case study and hence, it depends on the disposable income.

Further studies on the consumption function gives more information on the consumption function in the field of economics. However, we have two key things to calculate from the consumption function.

The marginal propensity to consume (MPC) and the marginal propensity to save (MPS). The relationship between MPC and MPS is that:

$$MPC + MPS = 1$$

The relationship between MPC and MPS is simply due to the fact that anything not consumed is automatically saved (whether deliberately or not) and hence, they add up together to make up the whole income.

The first derivative of the consumption gives the marginal propensity to consume (MPC), the marginal propensity to save is then calculated by:

$$MPS = 1 - MPC$$

We have that:

$$MPC = \frac{d}{dx}\big(C(x)\big)$$

Where;

$C(x)$ is the consumption function; **not the cost function in this case;**

Let's see this example;

EXAMPLE 8: The aggregate consumption function for a community is given by:

$$C(x) = 400 + 5\sqrt{x}$$

Where $C(x) =$ total consumption; and

$x =$ disposable income of the community.

Find the marginal propensity to consume (MPC) and marginal propensity to save (MPS) when: $x = 100$.

Right here; we have the consumption function; hence, we can get the MPC;

$$C(x) = 400 + 5\sqrt{x}$$
$$C(x) = 400 + 5x^{\frac{1}{2}}$$

The marginal function is given by the derivative:

$$MPC = C'(x) = 0 + \frac{1}{2} \times 5x^{\frac{1}{2}-1}$$

$$C'(x) = \frac{5}{2}x^{-\frac{1}{2}} = \frac{5}{2} \times x^{-\frac{1}{2}}$$

$$MPC = \frac{5}{2x^{\frac{1}{2}}}$$

Hence,
To evaluate the MPC at the required point; $x = 100$, we have:

MPC at $x = 100$;

$$MPC(100) = \frac{5}{2(100)^{\frac{1}{2}}} = \frac{5}{2\sqrt{100}}$$

$$MPC(100) = \frac{5}{2 \times 10} = \frac{1}{4}$$

From the relationship, we can get MPS;

$$MPS = 1 - MPC$$

$$MPS = 1 - \frac{1}{4} = \frac{3}{4}$$

Hence, at $x = 100$; the marginal propensity to consume is 0.25 while the marginal propensity to save is 0.75.

NEXT!

DEMAND AND SUPPLY FUNCTIONS, EQUILIBRIUM, ELASTICITY, MINIMIZATION AND MAXIMIZATION

The demand function relates the quantity of goods demanded to the price of the goods. It is a function of the price per unit product.

The supply function relates the quantity of goods supplied to the price of the goods. It is also a function of the price per unit product.

Those are two simple definitions of demand and supply.

Equilibrium price (of demand and supply) refers to the price where the quantity of goods demanded is equal to the quantity of goods supplied.

Examples of demand and supply functions are given below:

$q_d = 2000 - 5p$; a demand function;

$q_s = -500 + 45p$; for supply function.

Suppose the above demand and supply functions are for a particular product; the equilibrium price will be given by:

$q_d = q_s$; equilibrium for demand and supply

Hence, for the examples of demand and supply functions shown above, the equilibrium price will be given by:

$2000 - 5p = -500 + 45p$;

Solving for the price, p,

$2500 = 50p$;

$p = 50$. Hence, at a price 50 is the price (equilibrium price) where the demand is equal to supply.

The supply function is very much passive in usage hence; the demand function is the most important function actively used. Hence, most times, the demand function is expressed independently as q; Let's see what the demand function is proper now.

Recall the price function in the previous section? Yeah. The demand function had been showing up since then, you didn't just notice it. Recall the price function was a function of quantity demanded, q, yes, that q, is the demand function.

(Check Example 5 to see the word demand mentioned blatantly).

Now, that is why the price per unit (what I called the price function is called the **inverse-demand function).**

Hence, for the price function $p = 200 - 0.04q$, the corresponding demand function is gotten by making q the subject.

$$p = 200 - 0.04q$$

$$200 - p = 0.04q$$

$$\frac{200}{0.04} - \frac{p}{0.04} = \frac{0.04q}{0.04}$$

$$q = 5000 - 25p$$

Hence, that establishes the relationship between the demand function and the price per unit of products; they're two functions inversely related; basically, the both represents the same function.

The demand function and the revenue function are in no way unrelated.

With the demand function, a function of the price per unit;

Recall that the revenue function is also gotten from the product of the price per unit and the quantity of goods demanded;

Now, the average revenue is derived (as for every other function) by dividing the revenue function by the unit it depends on (which is the quantity of goods demanded).

Taking the quantity as q;

$$\bar{R}(q) = AR(q) = \frac{R(q)}{q}$$

But $R(q) = p \times q$ i.e. price multiplied by quantity

Hence;

$$AR(q) = \frac{p \times q}{q}$$

of course q cancels out;

We're left with:

$$AR(q) = p$$

Therefore, the average revenue generated is equal to the price per unit of the goods; hence, the average revenue function and the demand function are also two functions inversely related; basically, the both represents the same function.

Hence, with a demand function, the revenue function can be easily established by getting the inversed demand function and applying $R(q) = pq$;

Example is given below:

Please as said in the introduction of this book, ensure you have a pen while reading, many terms have been introduced that need you to take note of each of them, they've been expressed as simply as possible for you here.

EXAMPLE 9: Given the demand function for goods x is given by $q = 5000 - 20p$; find the total revenue and marginal revenue functions.

The demand function here is given by

$q = 5000 - 20p$;

Note that as I said, in most cases the demand function is expressed as q without any subscript as it is often used actively.

Now, $q = 5000 - 20p$; make p the subject of the relationship;

$20p = 5000 - q$; dividing through by 20 gives;

$$p = 250 - \frac{q}{20}$$

Now,

$$R(q) = pq$$

Hence,

$$R(q) = \left(250 - \frac{q}{20}\right) q$$

$$R(q) = 250q - \frac{q^2}{20}$$

That gives the revenue function.

The marginal revenue; that should be nothing strange anymore;

$$R'(q) = MR(q) = \frac{d}{dq}\left(250q - \frac{q^2}{20}\right)$$

$$MR(q) = 250 - \frac{2q}{20} = 250 - \frac{q}{10}$$

Thus; the marginal revenue is given.

PRICE ELASTICITY

Price elasticity (or point elasticity) is a phenomenon that compares the relative effect of price change and quantity demanded (or supplied) on each other.

Elasticity is basically defined by a ratio of the percentage change in quantity demanded (or supplied) to the percentage change in the unit price.

Like in the previous section, the price elasticity of demand finds immense use in business optimization far more than the price elasticity of supply; however, both the demand and supply functions have elasticities and are calculated in the same way.

Just calm down as many things are dished out to your book now, keep the calm, you'll understand it fully in due course; avoid getting discouraged in the middle of this. Keep writing!

The initial definition of elasticity as a ratio of change in quantity to change in price is usually used when raw values are given or readings are taken from a graph, however, for the scope of SSC106, functions are usually given; as seen already, we've dealt with a quite large range of functions.

Hence, let's see how the elasticities of demand (and supply) functions is derived from demand (and supply) functions.

For a given function, q which is a function of price; the point elasticity is given by

$$P.E = \frac{pdq}{qdp}$$

The above product can be broken into two terms that can be gotten independently from a given function;

$$P.E = \left(\frac{dq}{dp}\right) \times \left(\frac{p}{q}\right)$$

$\left(\frac{dq}{dp}\right)$ is the derivative of the demand (or supply) function with respect to p and $\left(\frac{p}{q}\right)$ are two corresponding values of p and q in the function.

Let's relax from the whole stress above of several equations with this example; it'll do a whole lot in helping you understand this properly;

EXAMPLE 10: Find the point elasticities of the demand and supply functions given by $q_d = 2000 - 5p$ and $q_s = -500 + 45p$ respectively. Evaluate their values at their equilibrium prices.

Cool, let's do the last thing first, let's get the values of p (and the corresponding q) to be used, we're going to be solving for their equilibrium price since we're evaluating the elasticities at their equilibrium price.

$$q_d = q_s$$

$2000 - 5p = -500 + 45p;$

Solving for the price, p,

$2500 = 50p$;
$p = 50$.

Hence, $p = 50$ is the price at which we'll be evaluating the price elasticities for both the demand and the supply functions.

Hence, we'll also solve for the corresponding values of q.

For the demand function;

$q_d = 2000 - 5p$; at $p = 50$;
$q_d = 2000 - 5(50)$;
$q_d = 2000 - 250$;
$q_d = 1750$.

For the supply function;

$q_s = -500 + 45p$; at $p = 50$;
$q_s = -500 + 45(50)$;
$q_s = -500 + 2250$;
$q_s = 1750$.

Well I only allowed us to solve it all the way, at equilibrium price; the quantity demanded and supplied are equal. Their values however will differ at other prices though.

Now, to find their different price elasticities; take each function on its own and find their separate derivatives with respect to p;

For the demand function;

$q_d = 2000 - 5p$;　　　　　　$q_s = -500 + 45p$;

Differentiating q_d and q_s respectively with respect to p;

$\frac{dq_d}{dp} = -5$　　　　　　$\frac{dq_s}{dp} = 45$

Now; let's get the price elasticity of demand (PED) and the price elasticity of supply (PES) for the respective functions at $p = 50$;

$PED = \left(\frac{dq_d}{dp}\right) \times \left(\frac{p}{q}\right)$; now p and q have corresponding values as $p = 50$; $q = 1750$.

Hence, we have PED thus:

$$PED = -5 \times \left(\frac{50}{1750}\right) = -\frac{1}{7} = -0.143$$

$PES = \left(\frac{dq_s}{dp}\right) \times \left(\frac{p}{q}\right)$; now p and q have corresponding values as $p = 50; q = 1750.$

Hence, we have PES thus:

$$PES = 45 \times \left(\frac{50}{1750}\right) = \frac{9}{7} = 1.286$$

As for the game lovers and football lovers, PES means point elasticity of supply and not Pro Evolution Soccer! ☺☺☺

Hence, we have seen how to find the respective point elasticities of demand and supply; they actually involve same process.

Now, let's see this question;

EXAMPLE 11: Find y in terms of x when it is given that the elasticity of y with respect to x is a constant b.

Now, such a weird question; we can't find the function, and they're asking you for elasticity? Who does that?

Lol, I actually lifted this question from your past question, and I also initially did think this question was faulty; however, that's a common reason to failure to solving mathematics questions. **It is profitable to attempt solving a question first before jumping into conclusions.** Now, let's start this question:

So, the elasticity of y with respect to x is a constant b…

Demand and supply functions are functions of price and hence, we have their elasticities with respect to their price, p.

Now, the point elasticity of y with respect to x will be given by:

$$PEy = \left(\frac{dy}{dx}\right) \times \left(\frac{x}{y}\right)$$

Got it? You should, still the same thing as the previous ones; just slight variable change as it is now y with respect to x instead of q with respect to p; note also that we have used PEy to denote point elasticity of y;

We're told $PEy = b$; ***since the elasticity of y with respect to x is a constant b***

Hence,

$$b = \left(\frac{dy}{dx}\right) \times \left(\frac{x}{y}\right)$$

So where do we go from here now?

C'mon, this is a differential equation, we can solve to express y in terms of x by **separating the variables.** That's a simple way mathematics questions bow, analyze them gradually!

Now, since b is a constant, this is a simple first order differential equation.

Separating variables;

$$\frac{bdx}{x} = \frac{dy}{y}$$

Evaluate their separate integrals;

$$b\int \frac{dx}{x} = \int \frac{dy}{y}$$

I'm sure you know b is constant and can be left out of the integral.

Integrating… … … … … … on each side.

$$b \ln x = \ln y$$

Our focus is making y the subject so let's settle down and do some manipulations, we did similar manipulations while treating differential equations.

$$\ln y = b \ln x$$

***Take natural exponents of both sides;

$$e^{\ln y} = e^{b \ln x}$$

Natural exponent of a natural log cancels out, that gives:

$$y = e^{b \ln x}$$

We can also simplify the right hand side by separating the powers into double powers as shown below;

$$y = \left(e^{\ln x}\right)^{b}$$

The exponent and the log once again cancels out; thus:

$$y = x^{b}$$

***We have simplified by taking the natural logarithm of both sides; however, from another side of the coin, we could also simplify in another way by dividing both sides by $\ln x$

$$\ln y = b \ln x$$

Here;

$$\log_e y = b \log_e x$$

From log rules; $c \log_b a = \log_b a^c$

Hence;

$$\log_e y = \log_e x^b$$

Equating the log numbers;

$$y = x^b$$

I guess that's it about elasticity; I'm sure you loved this last question…!

OPTIMAL MAXIMIZATION AND MINIMIZATION

The sole aim of every business by a sane human being is to make as much money as possible from such business and to reduce as much stress (and money spent) as possible. Hence, there are rules to maximize the amount of money one can make from a business.

We shall consider three concepts: the concepts of **revenue maximization**, **profit maximization** and **cost minimization**.

Revenue Maximization;

There are processes by which revenue can be maximized; let's cut it short. Let's list out condition for maximizing revenue.

Revenue is maximized if:

i) the marginal revenue function is equal to zero;
ii) the absolute value of the point elasticity of demand equals to unity (1).

Two simple rules to maximizing the revenue are given above: ***I'm assuming you're writing all these down!***

The number one rule above shouldn't be anything strange. We have already seen in previous chapters that a function is maximized (or minimized) where its first derivative is zero; we have also learnt the method of the second derivative test in determining whether functions are maximum or minimum. Here, we have already comprehensively known that marginal functions are the first derivative of such functions; hence, we can see that when the marginal revenue is zero, the first derivative is zero.

From the nature of revenue functions, the stationary values, which are the maximum (or minimum) values, are always maximum. Hence, no second derivative test is needed in solving for the maximum revenue.

The second rule involving PED is merely derived from the relationship of the point elasticity of demand and the revenue function.

Let's see examples of maximizing revenue using the two basic rules we have highlighted already.

EXAMPLE 12:

Suppose that the demand for good x is given by the equation; $x = 10,000 - 10p$.

(i) Derive an equation for the inverse demand function, $p(x)$.

(ii) Derive formulas for the total revenue and marginal revenue functions (as functions of x).

(iii) Find the price and quantity combination that maximizes total revenue.

(iv) Calculate the price elasticity of demand for the price-quantity combination you found in part (iii).

Okay, in this case, the quantity of the demand is denoted as x, as opposed to the q that has been used for a while, yeah, let's do this.

The first part, (i) requires we get the inverse demand function, $p(x)$, that is, the price per unit function which we have discussed already as a function of the quantity;

We have denoted as previously as simply p; it is denoted here as $p(x)$ here; Do it.

$x = 10,000 - 10p$; make p the subject;

$10000 - x = 10p$; rearrange and divide by p

$$\frac{10p}{10} = \frac{10000}{10} - \frac{x}{10}$$

$$p = 1000 - \frac{x}{10}$$

That is the price per unit function; the inverse demand function.

Next, (ii); we're told to find the total revenue function and the marginal revenue function:

This shouldn't be a problem at all:

$R(x) = p \times x$;

$$R(x) = \left(1000 - \frac{x}{10}\right)x$$

$$R(x) = 1000x - \frac{x^2}{10}$$

As simple as that!

Straight, we can find the marginal revenue function: I don't think that needs any explanation anymore:

$$R'(x) = MR(x) = \frac{d}{dx}[R(x)]$$

$$MR(x) = \frac{d}{dx}\left(1000x - \frac{x^2}{10}\right)$$

$$MR(x) = 1000 - \frac{x}{5}$$

The marginal revenue function is as simple as that; nothing is new actually yet.

Now, to the new part, (iii); we're told to find the price and corresponding quantity (the price and quantity combination) that maximizes the revenue.

The first rule says it all, the marginal revenue is equal to zero at maximum revenue; hence, equate the marginal revenue to zero and solve for the required quantity!

$MR(x) = 0$; for maximum revenue

$$1000 - \frac{x}{5} = 0$$

Solve for x, multiplying through by 5;

$$5000 - x = 0$$

Hence, $x = 5000$. The revenue is maximized when 5000 quantity of goods are sold.

To get the corresponding price, move to the price function:

$$p = 1000 - \frac{x}{10}$$

Plug in $x = 5000$;

$$p = 1000 - \frac{5000}{10}$$

$$p = 1000 - 500 = 500$$

Hence, the corresponding price is 500.

Lastly, we're told to evaluate PED at $(p, x) = (500, 5000)$; *Lol*, I hope that didn't confuse you? That means corresponding price and quantity just like x and y coordinates.

Those are the combinations we got in (c)

To evaluate the PED; head back to the initial demand function;

$$x = 10,000 - 10p$$

Find $\frac{dx}{dp}$ and fix it in the price elasticity demand formula and fix $(p, x) = (500, 5000)$ for $\left(\frac{p}{x}\right)$;

$$\frac{dx}{dp} = -10$$

Hence, we have that PED at the desired point with x representing q this time around in the formula;

$$PED = \left(\frac{dx}{dp}\right) \times \left(\frac{p}{x}\right)$$

$$PED = -10 \times \left(\frac{500}{5000}\right)$$

Simplifying this gives:

$$PED = -1$$

Now, as you can see that 1 is showing up but it is negative… Hence, that is why the second condition for maximizing revenue explains that the **absolute value** of price elasticity of demand is equal to unity (1).

First of all, unity means 1. Secondly, absolute values refers to the positive value of any number; hence, 1 and -1 could be two different numbers, but their absolute values are the same. Similarly, -2 and 2 have the same absolute values of 2 and so on.

It is a matter of common experience from the nature of demand functions that the PED is in most cases negative; hence, the absolute value is used during revenue maximization. The price and quantity combination used from (c) where the price and quantity combination for the maximum revenue and hence, we definitely must have the PED as -1 whose absolute value is 1.

Profit Maximization;

Revenue maximization and profit maximization are two different things entirely. We have already explained the difference between revenue and profit in this book.

That revenue is maximized doesn't mean profit has been maximized; that is something that happens in live economics.

Hence, let's move to profit maximization.

Profit is maximized if the marginal cost is equal to the marginal revenue. Did you get that? Yeah, it's very simple.

Therefore, $\boldsymbol{MC = MR}$, for maximum profit

I don't think that's too difficult to see, let's stroll to the next example:

But oops, before we stroll to the next example; let's get something clear;

Like in the revenue function, profit is maximized when the marginal profit is equal to zero. This is the same with the explanation in the revenue function that explains the maximized values of functions when the first derivative is zero. That shouldn't be new.

Now, we have that the marginal profit function is zero when the profit is maximized!

$P'(q) = MP(q) = 0;$ Profit is maximized;

However, hope you can remember this very vividly, for the profit function:

$$P(q) = R(q) - C(q)$$

From the rule of derivative of sums (and differences); the derivative of a function expressed as sums of other functions is the sum of the individual derivatives of the separate functions; hence, we have that:

$$P'(q) = R'(q) - C'(q)$$

That's clear right? The derivative of sums rule;

Now, $R'(q)$ is the marginal revenue right? YES! Also, $C'(q)$ is the marginal cost right? YES!

Now, it means the marginal profit, MP is given by:

$$MP = \text{marginal revenue} - \text{marginal cost}$$

Now, we know that at maximum profit, the marginal profit is zero;

Hence, it follows that:

$\boldsymbol{MR - MC = 0}$; at maximum profit.

This is a simple mathematical expression that'll follow that:

$$MR = MC$$

at maximum profit; Hence, the initial rule we said is provable mathematically. Now, don't cheat yourself by thinking this prove isn't necessary, question **1***(c)* in 2002/2003 Rain Semester Examination could be of interest to you.

I think we can now stroll freely to the next example.

EXAMPLE 13: Given the average cost function of a good is:

$$\bar{C}(q) = \frac{100}{q} - \frac{1}{10} + \frac{q}{20}$$

and the average revenue function for the same good is given by:

$$\bar{R}(q) = 80 - \frac{q}{10}$$

Determine the level of output that maximizes profit; confirm if the firm actually maximizes profit at this price.

As we have seen from the rules of determining maximum profit, we need both the cost and revenue functions first.

$$\bar{C}(q) = \frac{100}{q} - \frac{1}{10} + \frac{q}{20}$$

From the rule of average functions, the cost function is gotten by multiplying $\bar{C}(q)$ by q; recall that:

$$\bar{C}(q) = \frac{C(q)}{q}$$

Hence;

$$C(q) = \bar{C}(q) \times q$$

Therefore, we have our cost function, $C(q)$ as:

$$C(q) = \left(\frac{100}{q} - \frac{1}{10} + \frac{q}{20}\right) \times q$$

$$C(q) = 100 - \frac{q}{10} + \frac{q^2}{20}$$

Similarly, the revenue function is given by:

$$R(q) = \bar{R}(q) \times q$$

Therefore, we have our revenue function, $R(q)$ as:

$$R(q) = \left(80 - \frac{q}{10}\right) \times q$$

$$R(q) = 80q - \frac{q^2}{10}$$

Now, to evaluate our point of maximum profit; let's evaluate the marginal revenue functions and the marginal cost functions and we'll equate them.

$$R(q) = 80q - \frac{q^2}{10}$$

The marginal revenue function,

$$MR = \frac{d}{dq}[R(q)]$$

$$MR = \frac{d}{dq}\left(80q - \frac{q^2}{10}\right)$$

$$MR = 80 - \frac{2q}{10} = 80 - \frac{q}{5}$$

Also;

$$C(q) = 100 - \frac{q}{10} + \frac{q^2}{20}$$

The marginal cost function, MC is given by:

$$MC = \frac{d}{dq}[C(q)]$$

$$MC = \frac{d}{dq}\left(100 - \frac{q}{10} + \frac{q^2}{20}\right)$$

$$MC = -\frac{1}{10} + \frac{2q}{20} = -\frac{1}{10} + \frac{q}{10}$$

At maximum profit, $MR = MC$;

Hence, we have:

$$80 - \frac{q}{5} = -\frac{1}{10} + \frac{q}{10}$$

Now, we'll solve for the value of q.

$$80 + \frac{1}{10} = \frac{q}{5} + \frac{q}{10}$$

$$\frac{800 + 1}{10} = \frac{2q + q}{10}$$

$$\frac{801}{10} = \frac{3q}{10}$$

Here, $q = 267$ after solving!

Hence, profit is maximized when 267 units of the products are produced!

Now, there is an additional clause; confirm if the firm actually maximizes profit at this price.

Wow, confirm again! Yes, please! We need to confirm, because it is a matter of real life that some products even at the maximum profit attainable do not actually bring profit, those are cases when the cost function is too large relative to the revenue function; hence, even after solving for the maximum profit, it is important to check if profit is actually maximized or not; so that leads to the reason for this weird question.

So, fine, let's test it, testing is done by evaluating the total revenue and total cost at the maximum profit quantity level and comparing;

Total revenue when 267 units are produced;

$$R(q) = 80q - \frac{q^2}{10}$$

$$R(267) = 80(267) - \frac{(267)^2}{10}$$
$$R(799) = 14{,}231.1$$

Total cost when 267 units are produced;

$$C(q) = 100 - \frac{q}{10} + \frac{q^2}{20}$$

$$C(267) = 100 - \frac{267}{10} + \frac{(267)^2}{20}$$

$$C(267) = 3637.75$$

Obviously, 14,231.1 is greater than 3637.75 hence, the maximum profit in this case yields a positive value i.e. when the cost is subtracted from the revenue! Therefore, profit is actually maximized in this given product which has maximum profit when 267 units of products are produced!

> ***Generally, if you have the profit function (which can be easily gotten by subtracting the cost function from the revenue function), you needn't evaluate the revenue and cost functions, you just evaluate the profit function at the maximum profit point and see if it yields a positive value.***

I guess that's it, the business is a wrong business when even at maximum profit, **the profit function yields a negative value**; a negative value of profit denotes a loss and hence, when the maximum profit possible is a loss, such product is either disregarded or the revenue increased by increasing the price of the product.

You'll see an example of such product in **Question 1*(a),* Section B *of* Rain Semester SSC106 exam, 2015/2016 session (Kindly check it now).**

We don't need another example; just ensure you followed that step-by-step.

Up to cost minimization;

Cost minimization: Cost minimization is also a good factor to consider in businesses. Cost is minimized (and maximized) as well when the marginal cost is zero; Now, only foolhardiness will cause someone to maximize cost, hence, cost minimization is the required and desired phenomenon in economic optimization.

Now, the rule for cost minimization is that the first derivative of cost function, the marginal cost is zero.

$$MC = 0$$

At minimum cost;

However, since some cost functions are cubic functions, the second derivative test that we learnt in function optimization is required in finding the minimum cost.
Example… … …

EXAMPLE 13: Find the optimal units of quantity produced if the cost of the product is to be minimized when the cost function for the product is:

$$C(x) = 0.02x^3 - 0.5x^2 + 4x + 50000$$

Now, the marginal cost is given by

$$C'(x) = \frac{d}{dx}(0.02x^3 - 0.5x^2 + 4x + 5000)$$

$$C'(x) = 0.06x^2 - x + 4$$

Now, at minimum cost, the marginal cost is zero;

Therefore;

$$0.06x^2 - x + 4 = 0$$

We need to solve this quadratic equation;

Multiply through by 100;

$$6x^2 - 100x + 400 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-100) \pm \sqrt{(-100)^2 - 4(6)(400)}}{2(6)}$$

$$x = \frac{100 \pm \sqrt{10000 - 9600}}{12}$$

$$x = \frac{100 \pm \sqrt{400}}{12} = \frac{100 \pm 20}{12}$$

After solving, we get two values of x thus:

$$x = 10 \qquad \text{or} \qquad x = 6\frac{2}{3}$$

Then, the second derivative test to determine the maximum and minimum:

$$C'(x) = 0.06x^2 - x + 4$$

$$C''(x) = \frac{d}{dx}[C'(x)] = \frac{d}{dx}(0.06x^2 - x + 4)$$

$$C''(x) = 0.12x - 1$$

Now, when $x = 10$;

$$C''(x) = 0.12x - 1$$

$$C''(10) = 0.12(10) - 1 = 1.2 - 1$$

$$C''(10) = 0.2; \text{at } x = 10$$

$C''(x)$ is greater than zero. Hence, **it is a minimum point**;

Also, when $x = 6\frac{2}{3}$;

$$C''(x) = 0.12\left(6\frac{2}{3}\right) - 1$$

$$C''\left(6\frac{2}{3}\right) = 0.12\left(6\frac{2}{3}\right) - 1 = 0.8 - 1$$

$$C''\left(6\frac{2}{3}\right) = -0.2 \text{ ; at } x = 6\frac{2}{3}$$

$C''(x)$ is less than zero. Hence, it is a **maximum point;**

For a sane human being, the cost needs to be minimized as possible; hence the optimal point needed is the minimum cost point which corresponds to when 10 units of the goods are produced.

Let's see this example; there are times that getting the minimum cost is much-easier and straightforward.

EXAMPLE 14: Given the average cost of a produce is:

$$AC(q) = \frac{100}{q} - 4 + 0.5q$$

Obtain the optimum number of goods produced when cost is minimized;

Fine... ... ... Over here, the cost function needs to be gotten from the average function relation;

The cost function, $C(q)$ is given by;

$$C(q) = \bar{C}(q) \times q$$

Where $\bar{C}(q)$ is the average cost function, $AC(q)$

Here:

$$C(q) = \left(\frac{100}{q} - 4 + 0.5q\right) \times q$$

$$C(q) = 100 - 4q + 0.5q^2$$

Now, this is a much simpler quadratic cost function that'll easily yield one result for the optimal cost values;

At minimum cost, $C'(q) = 0$;

Here,

$$C'(q) = \frac{d}{dq}[C(q)] = \frac{d}{dq}(100 - 4q + 0.5q^2)$$

$$C'(q) = -4 + q$$

At minimum; $-4 + q = 0$; Hence, $q = 4$;

Hence, cost is minimized when 4 units of the product is produced.

A slight needless check; find the second derivative;

$C'(q) = -4 + q;$

$$C''(q) = \frac{d}{dq}[C'(q)] = \frac{d}{dq}(-4 + q)$$

$$C''(q) = 1$$

1 is obviously greater than zero confirming that the optimal point is a minimum point.

CONSTRAINED ECONOMIC OPTIMIZATION

This draws the curtain in this super-interesting concept in SSC106 which deals with a wide range of applications of calculus to economic analysis, I'm sure you've been enjoying every bit of it and you are now fully aware that mathematics is not in any way unnecessary in the field of social sciences and administration.

Oops, am I closing this so soon? This is also a very important aspect, optimization in economics when the optimization need is constrained.

As a matter of fact, most real life problems are constrained problems. It's mostly not possible to be producing 10 units, 4 units in large scale businesses, hence, mostly, a constraint is put to still optimize whilst obeying a certain constraint.

I could do well to start concluding the chapter that early as this is a very easy concept. Recall the aspect of the applications of differentiation and the aspect of constrained optimization; it is nothing different at all from this; however, this well be only more fun as you will see practical uses of the concept you learnt in this; I'm sure you must have read applications of differentiation before heading here, you definitely will as you won't want to be seeing the Lagrangean multiplier when you haven't seen what it means initially.

There are two types of optimization we'll be seeing under the constrained type of economic optimization which are:

- Utility maximization; and
- Cost minimization.

So, I don't think any introduction is needed; let's begin.

UTILITY MAXIMIZATION;

The utility is a term related to the consumer of products; the utility refers to the benefits and advantages a consumer enjoys from using a product. It is the desire of every consumer to maximize utility as much as possible.

The idea of utility to be taken in this context is maximizing the utility of a consumer on two given products when the consumer is subject to a budget constraint.

Now, we aren't going to be looking at the method of direct substitution at all. The Lagrangean multiplier method will be used solely for this aspect of constrained optimization problem as it is the most commonly used in constrained economic analysis.

Now, in this case, the utility function is taken as the objective function while the budget constraint is taking as the constraint of optimization.

The budget constraint; The budget constraint refers to the limit through which the consumer can acquire the desired products he(she) wants to acquire. Basically, the budget of any consumer is solely dependent on his(her) income and hence, budget constraint is evaluated in terms of the income. The prices of the products desired by a consumer are also the components that make up the budget constraint of a consumer.

Example, if the income of a consumer is ₦5000 and he desires to purchase eggs at ₦30 each, bread at ₦300 each and milk at ₦50 each; the consumer's budget constraint is such that the total money spent by the consumer is less than #5000, **or at worse the consumer exhausts (finishes) his whole money.** At least, he can't spend more than he (she) has.

In the example above, if he will by a number of eggs, b number of bread and c number of milk, the budget constraint will be given by:

$$30a + 300b + 50c \leq 5000$$

Furthermore, the budget constraint will be given by

$$30a + 300b + 50c = 5000$$

when he exhausts all his income, the budget constraint with the equality sign is usually used.

Hence, the budget constraint is given by equating the sum of the products of the quantity and price of the individual products to the total income accruable by the consumer.

That's it about budget constraints; finding utility functions is beyond this course and hence, the utility functions are always given in constrained optimization problems involving consumer utility.

We know how to use the Lagrangean multiplier method to maximize functions subject to constraint functions, hence, we'll move straight to working constraint optimization problems in consumer utility.

EXAMPLE 16: The utility function, U of a consumer consuming two commodities X and Y is given by $U(X, Y) = X^2 - Y^2 + XY - 5X$ subject to the budget constraint $X - 2Y = 6$. Evaluate the values of X and Y for maximum utility.

The first thing is to establish our objective function and the constraint function and form our Lagrangean equation; in this question however, the both are readily given and we do not need any extra computation to find our budget constraint; therefore, work starts immediately. Please call that Lagrangean multiplier guy and let's start;

Cool, here is it: $\boldsymbol{\lambda}$

We form our Lagrangean equation thus:

We express our budget constraint **equated to zero** either way;

$$X - 2Y - 6 = 0 \text{ or even } 6 - X + 2Y = 0$$

any of the two is allowed; you've understood that already while we treated function optimization.

$$L(X, Y, \lambda) = X^2 - Y^2 + XY - 5X - \lambda(X - 2Y - 6)$$

Now, evaluate the first order partials with respect to X, Y and λ and equate them to zero; you surely remember partial differentiation, it's in this book!

$$L_X = 2X + Y - 5 - \lambda = 0;$$

$L_Y = -2Y + X + 2\lambda = 0;$

$L_\lambda = -X + 2Y + 6 = 0;$

We can simplify all the equations to form three simultaneous equations thus:

$2X + Y - \lambda = 5$ ... ...(1)
$X - 2Y + 2\lambda = 0$ ... ...(2)
$-X + 2Y = 6$ ... ...(3)

Now, this is an equation in three unknowns, this can be solved in straightforward simultaneous equations algebra. However, since simultaneous equations isn't in anyway in SSC106 and the matrix method of solving simultaneous equations has been treated in this particular course, then it's not bad if you solve this using Crammer's rule which we'll be doing shortly.

We have the matrix equation thus:

$$\Delta = \begin{vmatrix} 2 & 1 & -1 \\ 1 & -2 & 2 \\ -1 & 2 & 0 \end{vmatrix}$$

Now, zero is used as the coefficient of λ in equation (3) since it isn't present.

$$\Delta = 2\begin{vmatrix} -2 & 2 \\ 2 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix}$$

$$\Delta = 2(-2 \times 0 - 2 \times 2) - 1(1 \times 0 - 2 \times -1) - 1\big(1 \times 2 - (-2 \times -1)\big)$$

$$\Delta = 2(-4) - 1(2) - 1(0) = -10$$

On to $\Delta_X = \begin{vmatrix} 5 & 1 & -1 \\ 0 & -2 & 2 \\ 6 & 2 & 0 \end{vmatrix}$

I'm assuming all these processes aren't strange to you;

$$\Delta_X = 5\begin{vmatrix} -2 & 2 \\ 2 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & 2 \\ 6 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & -2 \\ 6 & 2 \end{vmatrix}$$

$$\Delta_X = 5(-2 \times 0 - 2 \times 2) - 1(0 \times 0 - 2 \times 6) - 1\big(0 \times 2 - (-2 \times 6)\big)$$

$$\Delta_X = 5(-4) - 1(-12) - 1(12) = -20$$

On to $\Delta_Y = \begin{vmatrix} 2 & 5 & -1 \\ 1 & 0 & 2 \\ -1 & 6 & 0 \end{vmatrix}$

$$\Delta_Y = 2\begin{vmatrix} 0 & 2 \\ 6 & 0 \end{vmatrix} - 5\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ -1 & 6 \end{vmatrix}$$

$$\Delta_Y = 2(0 \times 0 - 2 \times 6) - 5(1 \times 0 - 2 \times -1) - 1(1 \times 6 - 0 \times -1)$$

$$\Delta_Y = 2(-12) - 5(2) - 1(6) = -40$$

On to $\Delta_\lambda = \begin{vmatrix} 2 & 1 & 5 \\ 1 & -2 & 0 \\ -1 & 2 & 6 \end{vmatrix}$;

$$\Delta_\lambda = 2\begin{vmatrix} -2 & 0 \\ 2 & 6 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ -1 & 6 \end{vmatrix} + 5\begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix}$$

$$\Delta_\lambda = 2(-2 \times 6 - 0 \times 2) - 1(1 \times 6 - 0 \times -1) + 5\big(1 \times 2 - (-2 \times -1)\big)$$

$$\Delta_\lambda = 2(-12) - 1(6) + 5(0) = -30$$

Therefore, we have;

$$X = \frac{\Delta_X}{\Delta} = \frac{-20}{-10} = 2$$

$$Y = \frac{\Delta_Y}{\Delta} = \frac{-40}{-10} = 4$$

$$\lambda = \frac{\Delta_\lambda}{\Delta} = \frac{-30}{-10} = 3$$

Hence, the consumer's utility is maximized when 2 quantities of X and 4 quantities of Y are purchased.

Now, the Crammer's method seems long but it isn't per say, if you can cope with solving the equations with algebra, it's cool and you're free, but I have chosen Crammer's method since it has been taught in this same course and it's universal for everybody.

We need another example though:

EXAMPLE 17: Given the utility function $U(x, y) = x^{\frac{2}{3}}y^{\frac{1}{3}}$ of a consumer for two given commodities is given by $U(x, y)$. If the prices of both x and y are ₦100 and the income of the consumer is ₦300000, find the optimal purchase for maximum utility.

Here;

Let's find the budget constraint of the consumer first; where x is the quantity of the first commodity and y is the quantity of the second commodity; we have the budget constraint as has been explained in the beginning of this since their prices are ₦100:

$$100x + 100y = 300000$$

We'll now express it equated to zero as:

$$100x + 100y - 300000 = 0$$

for our Lagrangean equation, hence, we'll form our Lagrangean equation thus:

$$L(x, y, \lambda) = x^{\frac{2}{3}}y^{\frac{1}{3}} - \lambda(100x + 100y - 300000)$$

Now, evaluate the first order partials with respect to x, y and λ and equate them to zero; you surely remember partial differentiation, it's in this book!

$$L_X = \frac{2}{3}x^{-\frac{1}{3}}y^{\frac{1}{3}} - 100\lambda = 0$$

$$L_Y = \frac{1}{3}x^{\frac{2}{3}}y^{-\frac{2}{3}} - 100\lambda = 0$$

$$L_\lambda = -100x - 100y + 300000 = 0$$

$\frac{2}{3}x^{-\frac{1}{3}}y^{\frac{1}{3}} - 100\lambda = 0$ … … … …(1)

$\frac{1}{3}x^{\frac{2}{3}}y^{-\frac{2}{3}} - 100\lambda = 0$ … … … …(2)

$-100x - 100y + 300000 = 0$ … … … …(3)

Now, Crammer's rule cannot be used over here, and why? The terms are not expressed in the same way as you can see $x^{-\frac{1}{3}}$ in (1), $x^{\frac{2}{3}}$ in (2) and normal x in (3):

We need to evaluate this using some careful manipulations, go along with me carefully.

From (1); $100\lambda = \frac{2}{3}x^{-\frac{1}{3}}y^{\frac{1}{3}}$

From (2); $100\lambda = \frac{1}{3}x^{\frac{2}{3}}y^{-\frac{2}{3}}$

Since both are equal to 100λ, the both can be equated!

$$\frac{2}{3}x^{-\frac{1}{3}}y^{\frac{1}{3}} = \frac{1}{3}x^{\frac{2}{3}}y^{-\frac{2}{3}}$$

Clear fractions by multiplying through by 3;

$$2x^{-\frac{1}{3}}y^{\frac{1}{3}} = x^{\frac{2}{3}}y^{-\frac{2}{3}}$$

divide through $x^{-\frac{1}{3}}y^{\frac{1}{3}}$ and evaluate powers of x and y separately.

$$2 = \frac{x^{\frac{2}{3}}y^{-\frac{2}{3}}}{x^{-\frac{1}{3}}y^{\frac{1}{3}}}$$

$$2 = x^{\frac{2}{3}-\left(-\frac{1}{3}\right)} \times y^{-\frac{2}{3}-\frac{1}{3}}$$

Recall:

$$\frac{a^m}{a^n} = a^{m-n}$$

Hence,

$$2 = x^1 \times y^{-1}$$

$$2 = x \times \frac{1}{y}$$

Hence,

$$x = 2y$$

Now, this can be inputted into (3);

$-100x - 100y + 300000 = 0$ … … …(3)

$$x = 2y$$

Hence:

$$-100(2y) - 100y + 300000 = 0$$

$$300y = 300000$$

$$y = 1000$$

Since,

$$x = 2y$$

$$x = 2 \times 1000 = 2000$$

Hence, the optimal purchase for maximum utility is purchasing 2000 units of x and 1000 units of y.

This question has disturbed you with indices a bit; well, you did indices in SSC105 actually!

You'll see more questions in the past questions section!

Now, on to the final phase – Cost minimization with constraints!

COST MINIMIZATION WITH CONSTRAINTS;

We've seen how cost is minimized; however, there exists a little different optimization issue that'll involve a certain constraint to the minimization of cost. Such constraints could involve when a certain number of goods may be produced which will force the cost minimization values to still fit in to the certain limit of the number of goods that must be produced.

The cost as we know is no more a new phenomenon and we've seen many cost functions and hence, we need no definition of the cost function.

The cost function in this case is the objective function to be optimized; in this case, it'll be minimized. As explained and we know already, sanity requires that cost is minimized as much as possible.

Again, we aren't going to be looking at the method of direct substitution at all. The Lagrangean multiplier method will be used solely for this aspect of constrained optimization problem as it is the most commonly used in constrained economic analysis.

Optimization of real life functions in economics requires beyond merely finding the actual minimum value as the minimum values could (and in most cases) are values which produce relatively very small amount of goods, and hence, cost minimization is mostly done with an aspect in mind that a least amount of goods must be produced. Such optimizations hence help producers in producing goods in varying quantity to minimize cost while still ensuring a certain amount of goods are produced.

The total units constrain; The constraint function in this case of optimization is the total amount (or least amount) of the units (quantity) of the goods that must be produced.

Unconstrained optimization mostly yields a minute amount of the units of goods and hence, mostly, in real life, optimization is done with constraints to ensure the units of products produced aren't too small.

Hence, if the products x and y are to be produced, and we at least [or a total of] 100 products in all, the total units constraints will be given by:

$$x + y = 100$$

The equation is quite basic as it implies that the total sum of the units of x and y is 100.

Hence, that's it about cost minimization with constraints, once the equation is put in the Lagrangean form, it can be fully solved as it is just what we are used to solving already!

EXAMPLE 18: A firm producing two goods x and y has the total cost function given as:

$$C(x, y) = 8x^2 - xy + 12y^2$$

If the firm is bound by contract to produce a combination of two goods totaling 42;

(i) Write out the objective and the constraint functions;

(ii) Derive the equations of the first order conditions for the firm's constrained optimization problems.

(iii) Use Crammer's rule to solve for the optimal values of x and y.

This has been gently extracted from your past questions. There aren't much ways the question on constrained cost minimization can be set. Hence;

Since we seek to optimize the cost while we still obey the constraint; but we must first of all extract the constraint;

As stated in the first part of the question, we are to find the objective and constraint functions.

As explained already; the objective function is always the cost function which is to be minimized; hence; the objective function is:

$$C(x, y) = 8x^2 - xy + 12y^2$$

For the constraint function, we are told the total goods must be 42, hence, we'll be having:

$$x + y = 42$$

To derive the first order conditions as in the second part of the question, we'll write our Lagrangean function; sure you know how to fix all that;

$$\mathcal{L}(x, y, \lambda) = 8x^2 - xy + 12y^2 - \lambda(x + y - 42)$$

Take the first order partials;

$$\mathcal{L}_x = 2 \times 8x^{2-1} - 1 \times x^{1-1} \times y + 0 - \lambda(1 \times x^{1-1} + 0 - 0)$$

$$\mathcal{L}_x = 16x - y - \lambda$$

$$\mathcal{L}_y = 0 - 1 \times x \times y^{1-1} + 2 \times 12y^{2-1} - \lambda(0 + 1 \times y^{1-1} - 0)$$

$$\mathcal{L}_y = -x + 24y - \lambda$$

$$\mathcal{L}_\lambda = 0 - 0 + 0 - 1 \times \lambda^{1-1}(x + y - 42)$$

$$\mathcal{L}_\lambda = -x - y + 42$$

For the first order conditions, equate each to zero;

$$16x - y - \lambda = 0 \ldots\ldots\ldots (1)$$

$$-x + 24y - \lambda = 0 \ldots\ldots\ldots (2)$$

$$-x - y + 42 = 0$$

$$-x - y = -42 \ldots\ldots\ldots (3)$$

As for the third part of the question, we're told to resolve this using the Crammer's rule, like I've said severally, you can always use Crammer's rule even when not specified in case you can't cope with simultaneous equations in three variables;

Here, we have three equations;

$$16x - y - \lambda = 0 \ldots\ldots\ldots (1)$$

$$-x + 24y - \lambda = 0 \ldots\ldots\ldots (2)$$

$$-x - y = -42 \ldots\ldots\ldots (3)$$

The equivalent matrix determinant needed here is:

$$\Delta = \begin{vmatrix} 16 & -1 & -1 \\ -1 & 24 & -1 \\ -1 & -1 & 0 \end{vmatrix}$$

Evaluating this determinant;

$$\begin{aligned} \Delta = 16[(24)(0) - (-1)(-1)] \\ - (-1)[(-1)(0) - (-1)(-1)] \\ + (-1)[(-1)(-1) - (24)(-1)] \end{aligned}$$

$$\Delta = 16(-1) + 1(-1) - 1(25)$$

$$\Delta = -16 - 1 - 25 = -42$$

To evaluate Δ_x, we replace the column of x with the column matrix of answers;

$$\Delta_x = \begin{vmatrix} 0 & -1 & -1 \\ 0 & 24 & -1 \\ -42 & -1 & 0 \end{vmatrix}$$

Evaluating this determinant;

$$\begin{aligned} \Delta_x = 0[(24)(0) - (-1)(-1)] \\ - (-1)[(0)(0) - (-1)(-42)] \\ + (-1)[(0)(-1) - (24)(-42)] \end{aligned}$$

$$\Delta_x = 0(-1) + 1(-42) - 1(1008)$$

$$\Delta_x = 0 - 42 - 1008 = -1050$$

To evaluate Δ_y, we replace the column of y with the column matrix of answers;

$$\Delta_y = \begin{vmatrix} 16 & 0 & -1 \\ -1 & 0 & -1 \\ -1 & -42 & 0 \end{vmatrix}$$

Evaluating this determinant;

$$\begin{aligned}\Delta_y = 16[(0)(0) - (-1)(-42)] \\ - 0[(-1)(0) - (-1)(-1)] \\ + (-1)[(-1)(-42) - (0)(-1)]\end{aligned}$$

$$\Delta_y = 16(-42) + 0(-1) - 1(42)$$

$$\Delta_y = -714$$

To evaluate Δ_λ, we replace the column of λ with the column matrix of answers;

$$\Delta_z = \begin{vmatrix} 16 & -1 & 0 \\ -1 & 24 & 0 \\ -1 & -1 & -42 \end{vmatrix}$$

Evaluating this determinant;

$$\begin{aligned} \Delta_\lambda = 16[(24)(-42) - (0)(-1)] \\ - (-1)[(-1)(-42) - (0)(-1)] \\ + 0[(-1)(-1) - (-1)(24)] \end{aligned}$$

$$\Delta_\lambda = 16(-1008) + 1(42) - 0(25)$$

$$\Delta_\lambda = -16086$$

Hence, we have our x, y and λ values as;

$$x = \frac{\Delta_x}{\Delta} = \frac{-1050}{-42} = 25$$

$$y = \frac{\Delta_y}{\Delta} = \frac{-714}{-42} = 17$$

$$\lambda = \frac{\Delta_\lambda}{\Delta} = \frac{-16086}{-42} = 383$$

Hence, the cost is minimized when a total of 25 units of x and 17 units of y are produced.

Obviously, $25 + 17 = 42$; which makes it clear that the constraint was obeyed.

Such a nice topic! That draws the curtain on practical applications of calculus in real life economics and management sciences!