DEPARTMENT OF ECONOMICS
FACULTY OF SOCIAL SCIENCES
OBAFEMI AWOLOWO UNIVERSITY,
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SSC106: MATHEMATICS FOR SOCIAL SCIENCES II

RAIN SEMESTER EXAMINATION
(2003/2004 SESSION)

INSTRUCTIONS:

- *****
- Show all workings clearly

Time allowed: 2 hours

1. Distinguish between the following pairs of concepts using relevant examples;
   (a) Harmonic function and Young's theorem
   (b) Euler's theorem and Jacobian theorem;
   (c) Monotonic function and polynomial function;
   (d) Definite integral and indefinite integral.

2. *(a)* For the following system of simultaneous equations, find each of the following:

(i) The coefficient matrix of the system of the equations;

(ii) The value of its determinant;

(iii) The minors and cofactors of the coefficient matrix.

(iv) The solution values of the unknowns; $(X_i = 1,2,3)$

$$7X_1 - X_2 - X_3 = 0$$
$$10X_1 - 2X_2 + X_3 = 8$$
$$6X_1 + 3X_2 + 2X_3 = 7$$

(b) If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

(i) Obtain A' and AA'

(ii) What is the nature of matrix is AA'.

3. The utility function U of a consumer consuming two commodities X and Y, the prices of the commodities and the income (M) of the consumer is as shown below:

$U = (X + 2)(Y + 1);$

$P_X = 2, P_Y = 5, M = 51.$

(a) Write out the Lagrangean function;

(b) Find the marginal utility function for each of the commodities X and Y.

(c) Use the Lagrangean multiplier method to optimize the utility function.

(d) Determine the level of utility of the consumer.

4. *(a)* Find $f'(x)$ and $f''(x)$ in each of the following:

(i) $f(x) = x^3 - 3x^2 + 4$

(ii) $f(x) = (x + 1)^3$

(b) The revenue from the sales of a given item is:

$$R(x) = 50x + \frac{x^2}{20}$$

Obtain the marginal revenue function and determine the venue of the marginal revenue when $x = 10$

(c) Differentiate each of the following functions:

(i) $y = (6x^2 - 2)(4x + 1)$

(ii) $y = x^2(4x + 6)$

(iii) $y = \frac{x^2}{1+x^3}$

(iv) $y = (x^2 + 3)^5$

(v) $y = e^{x^2+3x+4}$

5. *(a)* Distinguish between differential and integral calculus.

(b) Integrate each of the following:

(i) $\frac{5x^4}{x^5+16}$

(ii) $(2ax + b)(ax^2 + bx)^7$

(iii) $\log_e 2x$

(c) A firm operates such that its marginal cost function and the price of its product are given by:

$$MC = f(q) = -15 + q;$$
$$P = f(q) = 2.5 - 0.75q;$$

Where q is the level of output.

(i) Obtain the profit function of the firm;

(ii) At what level of output is the profit maximized;

(iii) Show that the level of output in (ii) above actually signifies a point of maximum profit;

(iv) Determine the profit of the firm.

SOLUTION TO THE PAST QUESTIONS

All topics needed here have been thoroughly treated through and through, all you need to do is to sit down and answer all these questions like a joke, you are not expected to start reading from here but from the topics, you make a joke of yourself when you read the concept of any subject starting from the side of the past question; more so, this text isn't a past question but a complete guide to lead you to a straight A in your SSC106 exam.

Hence, keep calm and check the solutions to these past questions one-by-one;

GOOD LUCK AND GOD'S BEST!

SOLUTION TO THE SSC106 EXAMINATION 2003/2004 ACADEMIC SESSION

I don't seem to find the instruction in this exam year; all the same, we'll be answering everything though;

Remember every singular solution here, you can get the full concept by reading this book and not the past question section exclusively;

Question 1

Distinguish between the following pairs of concepts using relevant examples;

(a) Harmonic function and Young's theorem

(b) Euler's theorem and Jacobian theorem;

(c) Monotonic function and polynomial function;

(d) Definite integral and indefinite integral.

Number one is basically theoretical;

Well, we've been told distinguish between concepts using relevant examples; while some of them aren't direct opposite of each other ***as it***

were, the only way to handle them is to define each term and distinguish it as much as possible;

(a)

A harmonic function is a continuous twice differentiable function whose sum of all its direct second order derivatives is equal to zero; while the **Young's theorem** says that for a continuous twice differentiable function, the second order mixed partial derivatives are equal. **A huge difference is that** the harmonic function doesn't hold for all twice differentiable functions while the Young's theorem holds for all twice differentiable functions.

For a harmonic function; $f(x, y)$;

$$f_{xx} + f_{yy} = 0$$

The Young's theorem for the function; $f(x, y)$

$$f_{xy} = f_{yx}$$

(b)

Euler's theorem states that the sum of the product of the arguments and their first order partial derivatives of a function is equal to the degree of homogeneity of a function multiplying the function; while the **Jacobian theorem** states that; The Jacobian determinant for a set of functions will be equal to zero for all values of the arguments of the functions if and only if the functions are (linearly or nonlinearly) dependent.

(c)

A monotonic function is a function that is either strictly non-increasing or non-decreasing, it is not the both; **polynomial functions** however are functions in positive integral powers of an independent variable. Polynomial functions unlike monotonic functions are both increasing and decreasing over different ranges of the independent variable;

(d)

Definite integral gives the area under the curve of the graph of a given function while an **indefinite integral** gives the general form of the anti-derivative of a function. A very obvious difference also is that indefinite integrals contain an arbitrary constant while a definite integral doesn't contain an arbitrary constant.

$$\int f(x)\,dx \text{ is an indefinite integral}$$

$$\int_b^a f(x)\,dx \text{ is a definite integral}$$

Question 2

(a) For the following system of simultaneous equations, find each of the following:

(i) The coefficient matrix of the system of the equations;

(ii) The value of its determinant;

(iii) The minors and cofactors of the coefficient matrix.

(iv) The solution values of the unknowns; $(X_i = 1,2,3)$

$$7X_1 - X_2 - X_3 = 0$$
$$10X_1 - 2X_2 + X_3 = 8$$
$$6X_1 + 3X_2 + 2X_3 = 7$$

(b) If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

(i) Obtain A' and AA'

(ii) What is the nature of matrix is AA'.

Right, this is a matrix question, not difficult <u>but easy to make mistakes</u>; be careful;

For the (a) part, reading from (i) to (iv), we should be using the matrix multiplication method since it involves finding the cofactors of all the elements; hence, to resolve this equation; hence; we have;

(a)

(i)

From the equations; we can write the matrix form of the equation system;

$$7X_1 - X_2 - X_3 = 0$$
$$10X_1 - 2X_2 + X_3 = 8$$
$$6X_1 + 3X_2 + 2X_3 = 7$$

$$\begin{pmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & 2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 7 \end{pmatrix}$$

(ii)

We need the determinant of this matrix;
Let

$$A = \begin{pmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & 2 \end{pmatrix}$$

Hence; we have;

$$|A| = \begin{vmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & 2 \end{vmatrix}$$

$$7\begin{vmatrix} -2 & 1 \\ 3 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 10 & 1 \\ 6 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 10 & -2 \\ 6 & 3 \end{vmatrix}$$

$$7[(-2)(2) - (1)(3)] + 1[(10)(2) - (1)(6)] + 1[(10)(3) - (-2)(6)]$$

$$|A| = 7(-7) + (14) + 42 = 7$$

(iii)

Let’s find the minors and cofactors of all the elements; as we know, it’s minors first;

Let's find the minor elements;

$$min(7) = \begin{vmatrix} -2 & 1 \\ 3 & 2 \end{vmatrix} \qquad min(-1) = \begin{vmatrix} 10 & 1 \\ 6 & 2 \end{vmatrix}$$

$$min(-1) = \begin{vmatrix} 10 & -2 \\ 6 & 3 \end{vmatrix} \qquad min(10) = \begin{vmatrix} -1 & -1 \\ 3 & 2 \end{vmatrix}$$

$$min(-2) = \begin{vmatrix} 7 & -1 \\ 6 & 2 \end{vmatrix} \qquad min(1) = \begin{vmatrix} 7 & -1 \\ 6 & 3 \end{vmatrix}$$

$$min(6) = \begin{vmatrix} -1 & -1 \\ -2 & 1 \end{vmatrix} \qquad min(3) = \begin{vmatrix} 7 & -1 \\ 10 & 1 \end{vmatrix}$$

$$min(2) = \begin{vmatrix} 7 & -1 \\ 10 & -2 \end{vmatrix}$$

Hence,

The matrix of minors is;

$$\text{minor}(A) = \begin{pmatrix} -7 & 14 & 42 \\ 1 & 20 & 27 \\ -3 & 17 & -4 \end{pmatrix}$$

From the cofactor matrix rule below:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Hence,

The matrix of cofactors is;

$$\text{cofactor}(A) = \begin{pmatrix} -7 & -14 & 42 \\ -1 & 20 & -27 \\ -3 & -17 & -4 \end{pmatrix}$$

(iv)

To find the solution values; we'll simply complete the process in (iii) and find the adjoint and hence inverse of the matrix and compute the remaining;

$$adj(A) = [\text{cofactor}(A)]^T$$

$$\text{adj}(A) = \begin{pmatrix} -7 & -14 & 42 \\ -1 & 20 & -27 \\ -3 & -17 & -4 \end{pmatrix}^T$$

$$\text{adj}(A) = \begin{pmatrix} -7 & -1 & -3 \\ -14 & 20 & -17 \\ 42 & -27 & -4 \end{pmatrix}$$

We have evaluated the determinant in (i); hence;

$$A^{-1} = \frac{1}{7}\begin{pmatrix} -7 & -1 & -3 \\ -14 & 20 & -17 \\ 42 & -27 & -4 \end{pmatrix}$$

To solve the equation; pre-multiply both sides of this equation by A^{-1}

$$\begin{pmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & 2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 7 \end{pmatrix}$$

We have;

$$A^{-1}\begin{pmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & 2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} -7 & -1 & -3 \\ -14 & 20 & -17 \\ 42 & -27 & -4 \end{pmatrix}\begin{pmatrix} 0 \\ 8 \\ 7 \end{pmatrix}$$

The left hand side is reduces completely since A^{-1} multiplying A will yield the identity matrix; we start expanding the right hand side multiplication;

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} (-7)(0) + (-1)(8) + (-3)(7) \\ (-14)(0) + (20)(8) + (-17)(7) \\ (42)(0) + (-27)(8) + (-4)(7) \end{pmatrix}$$

The left hand side simply yields the matrix of coefficients since it is multiplied by an identity matrix;

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} -29 \\ 41 \\ -244 \end{pmatrix}$$

Hence, expanding the matrices;

$$X_1 = -\frac{29}{7}$$

$$X_2 = \frac{41}{7}$$

$$X_3 = -\frac{244}{7}$$

(b)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

We're told to obtain A' which from our matrix studies, we know is the transpose of A;

$$A' = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

We're to also obtain AA'; hence, we have matrix multiplication; this will be quite tedious;

$$AA' = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

$$\begin{pmatrix} (1)(1)+(2)(2)+(3)(3) & (1)(4)+(2)(5)+(3)(6) & (1)(7)+(2)(8)+(3)(9) \\ (4)(1)+(5)(2)+(6)(3) & (4)(4)+(5)(5)+(6)(6) & (4)(7)+(5)(8)+(6)(9) \\ (7)(1)+(8)(2)+(9)(3) & (7)(4)+(8)(5)+(9)(6) & (7)(7)+(8)(8)+(9)(9) \end{pmatrix}$$

Sorry for the small font in the expansion, kindly zoom it;

$$AA' = \begin{pmatrix} 14 & 32 & 50 \\ 32 & 77 & 122 \\ 50 & 122 & 194 \end{pmatrix}$$

(ii)

Of course, we treated this in the book, the product of a matrix and its transpose is a **symmetric matrix;** that is, a matrix whose row elements are equal to its corresponding column elements, hence, such a matrix is equal to its transpose.

Hence, AA' is such that:

$$AA' = (AA')'$$

Question 3

Quite some lengthy question;

The utility function U of a consumer consuming two commodities X and Y, the prices of the commodities and the income (M) of the consumer is as shown below:

$U = (X + 2)(Y + 1)$;
$P_X = 2, P_Y = 5, M = 51$.

(a) Write out the Langragean function;
(b) Find the marginal utility function for each of the commodities X and Y.
(c) Use the Langragean multiplier method to optimize the utility function.
(d) Determine the level of utility of the consumer.

A case of utility maximization; to write the Lagrangean function, we know that the utility function is the objective function to be optimized.

Here; to get the budget constraint; the quantities are X and Y with respective prices 2 and 5, total income is 51; therefore; from what we've learnt, the budget constraint is;

$$2X + 5Y = 51$$

Express this equated to zero;

$$2X + 5Y - 51 = 0$$

Hence, we can now write our Lagrangean function

$$\mathcal{L}(X, Y, \lambda) = (X + 2)(Y + 1) - \lambda(2X + 5Y - 51)$$

$$\mathcal{L}(X, Y, \lambda) = XY + X + 2Y + 1 - \lambda(2X + 5Y - 51)$$

Taking the first order partial derivatives with respect to X, Y and λ

$$\mathcal{L}_X = X^{1-1}Y + X^{1-1} + 0 + 0 - \lambda(2X^{1-1} + 0 - 0)$$

$$\mathcal{L}_X = Y + 1 - 2\lambda$$

$$\mathcal{L}_Y = XY^{1-1} + 0 + 2Y^{1-1} + 0 - \lambda(0 + 5Y^{1-1} - 0)$$

$$\mathcal{L}_Y = X + 2 - 5\lambda$$

$$\mathcal{L}_\lambda = 0 + 0 + 0 + 0 - \lambda^{1-1}(2X + 5Y - 51)$$

$$\mathcal{L}_\lambda = -2X - 5Y + 51$$

(c)

To do the optimization, we'll be following the regular method of the Lagrangean multiplier;

Equating all partials to zero;

$$\mathcal{L}_X = 0$$
$$\mathcal{L}_Y = 0$$
$$\mathcal{L}_\lambda = 0$$

$$Y + 1 - 2\lambda = 0 \dots \dots \dots (1)$$
$$X + 2 - 5\lambda = 0 \dots \dots \dots (2)$$
$$-2X - 5Y + 51 = 0 \dots \dots \dots (3)$$

From (1);

$$Y + 1 = 2\lambda \dots \dots \dots (4)$$

From (2);

$$X + 2 = 5\lambda \dots \dots \dots (5)$$

$$\mathbf{5} \times (\mathbf{4}): 5(Y + 1 = 2\lambda)$$

$$5Y + 5 = 10\lambda \dots \dots \dots (6)$$

$$\mathbf{2} \times (\mathbf{5}):\ 2(X + 2 = 5\lambda)$$

$$2X + 4 = 10\lambda \dots \dots \dots (7)$$

$(6) - (7)$;

$$5Y + 5 - (2X + 4) = 10\lambda - 10\lambda$$

$$5Y + 5 - 2X - 4 = 0$$

Hence,

$$5Y - 2X = -1 \ldots \ldots \ldots (8)$$

Add (3) and (8);

$$-2X - 5Y = -51 \ldots \ldots \ldots (3)$$
$$5Y - 2X = -1 \ldots \ldots \ldots (8)$$
$$-4X = -52$$

Hence,

$$X = 13$$

From (8);

$$5Y - 2X = -1$$
$$5Y - 2(13) = -1$$
$$5Y = 25$$
$$Y = 5$$

Hence, the optimal values are $X = 13$ and $Y = 5$

Let's find the value of the Lagrange multiplier for the sake of full marks ☺

$$2X + 4 = 10\lambda$$
$$2(13) + 4 = 10\lambda$$

Here;

$$\lambda = 3$$

Please, please, please and please! Use the method of Crammer's rule if the simultaneous equation solving looks too vague and tedious for you.

(d)

To determine the utility level is like evaluating the optimal value which we learnt that we'll head back to the objective function;

$$U = (X + 2)(Y + 1)$$

Hence,

$$U = (13 + 2)(5 + 1) = 15 \times 6 = 90$$

Hence, the optimum utility of the consumer is 90.

Question 4

Let's keep moving

(a) Find $f'(x)$ and $f''(x)$in each of the following:

(i) $f(x) = x^3 - 3x^2 + 4$

(ii) $f(x) = (x + 1)^3$

(b) The revenue from the sales of a given item is:

$$R(x) = 50x + \frac{x^2}{20}$$

Obtain the marginal revenue function and determine the venue of the marginal revenue when $x = 10$

(c) Differentiate each of the following functions:

(vi) $y = (6x^2 - 2)(4x + 1)$

(vii) $y = x^2(4x + 6)$

(viii) $y = \frac{x^2}{1+x^3}$

(ix) $y = (x^2 + 3)^5$

(x) $y = e^{x^2+3x+4}$

Quite some questions here but straightforward;

(a)

You know the concept of higher derivatives, hence;

(i)

$$f(x) = x^3 - 3x^2 + 4$$

$$f'(x) = \frac{d}{dx}(x^3 - 3x^2 + 4)$$

$$f'(x) = 3 \times x^{3-1} - 2 \times 3x^{2-1} + 0$$

$$f'(x) = 3x^2 - 6x$$

$$f''(x) = \frac{d}{dx}\left(f'(x)\right) = \frac{d}{dx}(3x^2 - 6x)$$

$$f''(x) = 2 \times 3x^{2-1} - 1 \times 6x^{1-1}$$

$$f''(x) = 6x - 6 = 6(x - 1)$$

(ii)

$$f(x) = (x + 1)^3$$

$$f'(x) = \frac{d}{dx}[(x + 1)^3]$$

For this; substitution; $u = x + 1$

$$\frac{du}{dx} = 1$$

Hence,

$$f(x) = u^3$$

$$\frac{df(x)}{du} = 3u^{3-1} = 3u^2$$

$$\frac{df(x)}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$f'(x) = 3u^2 \times 1 = 3u^2$$

Put $u = x + 1$

$$\frac{df(x)}{dx} = 3(x+1)^2$$

$$f'(x) = 3 \times x^{3-1} - 2 \times 3x^{2-1} + 0$$

$$f'' = \frac{d}{dx}\left(f'(x)\right) = \frac{d}{dx}\left[3(x+1)^2\right]$$

Same substitution; $u = x + 1$

$$\frac{du}{dx} = 1$$

Hence,

$$f'(x) = 3u^2$$

$$\frac{df'(x)}{du} = 2 \times 3u^{2-1} = 6u$$

$$\frac{df'(x)}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$f''(x) = 3u^2 \times 1 = 6u$$

Put $u = x + 1$

$$f''(x) = \frac{df'(x)}{dx} = 6(x + 1)$$

(b)

From the revenue function, differentiate it to get the marginal revenue;

$$R(x) = 50x + \frac{x^2}{20}$$

$$R'(x) = \frac{d}{dx}\left(50x + \frac{x^2}{20}\right)$$

$$R'(x) = 1 \times 50x^{1-1} + 2 \times \frac{x^{2-1}}{20}$$

$$R'(x) = 50 + \frac{x}{10}$$

To evaluate this at $x = 10$

$$R'(10) = 50 + \frac{10}{10} = 50 + 1 = 51$$

(c)

(i)

$$y = (6x^2 - 2)(4x + 1)$$

Product rule;

$$u = 6x^2 - 2$$

$$\frac{du}{dx} = 12x$$

$$v = 4x + 1$$

$$\frac{dv}{dx} = 4$$

Product rule;

$$y = uv$$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = (4x + 1)(12x) + (6x^2 - 2)(4)$$

$$\frac{dy}{dx} = 48x^2 + 12x + 24x^2 - 8$$

$$\frac{dy}{dx} = 72x^2 + 12x - 8$$

(ii)

$$y = x^2(4x + 6)$$

Product rule;

$$u = x^2$$

$$\frac{du}{dx} = 2x$$

$$v = 4x + 6$$

$$\frac{dv}{dx} = 4$$

Product rule;

$$y = uv$$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = (4x + 6)(2x) + (x^2)(4)$$

$$\frac{dy}{dx} = 8x^2 + 12x + 4x^2$$

$$\frac{dy}{dx} = 12x^2 + 12x = 12x(x + 1)$$

(iii)

$$y = \frac{x^2}{1 + x^3}$$

Quotient rule;

$$u = x^2$$

$$\frac{du}{dx} = 2x$$

$$v = 1 + x^3$$

$$\frac{dv}{dx} = 3x^2$$

Product rule;

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(1 + x^3)(2x) - (x^2)(3x^2)}{(1 + x^3)^2}$$

$$\frac{dy}{dx} = \frac{2x + 2x^4 - 3x^4}{(1 + x^3)^2}$$

$$\frac{dy}{dx} = \frac{2x - x^4}{(1 + x^3)^2}$$

(iv)

$y = (x^2 + 3)^5$

Using the substitution; $u = x^2 + 3$

$$\frac{du}{dx} = 2x$$

Hence,

$$y = u^5$$

$$\frac{dy}{du} = 5u^{5-1} = 5u^4$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = 5u^4 \times 2x = 10xu^4$$

Put $u = x^2 + 3$

$$\frac{dy}{dx} = 10x(x^2 + 3)^4$$

(v)

$$y = e^{x^2+3x+4}$$

Using the substitution; $u = x^2 + 3x + 4$

$$\frac{du}{dx} = 2x + 3$$

Hence,

$$y = e^u$$

$$\frac{dy}{du} = e^u$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = e^u \times (2x + 3) = (2x + 3)e^u$$

Put $u = x^2 + 3x + 4$

$$\frac{dy}{dx} = (2x + 3)e^{x^2+3x+4}$$

Question 5

(a) Distinguish between differential and integral calculus.

(b) Integrate each of the following:

(i) $\frac{5x^4}{x^5+16}$

(ii) $(2ax + b)(ax^2 + bx)^7$

(iii) $\log_e 2x$

(c) A firm operates such that its marginal cost function and the price of its product are given by:

$$MC = f(q) = -15 + q;$$
$$P = f(q) = 2.5 - 0.75q;$$

Where q is the level of output.

(i) Obtain the profit function of the firm;

(ii) At what level of output is the profit maximized;

(iii) Show that the level of output in (ii) above actually signifies a point of maximum profit;

(iv) Determine the profit of the firm.

(a)

To differentiate between differential and integral calculus, it's simple, you can check the definition in the note; **however, let's differentiate them in simpler terms;**

Differential and integral calculus are two opposite processes; differential calculus is the aspect of calculus that deals with finding the derivative of a function which could mean the gradient of the curve of the function or the rate of change of the function. Integral calculus on the other hand is the opposite of the process of differential calculus, the process of finding the anti-derivative of a function, in definite situation; it is the area under of a curve between two points on the curve.

(b)

(i)

$$\frac{5x^4}{x^5 + 16}$$

To integrate this, we have

$$\int \frac{5x^4}{x^5 + 16} dx$$

This is a case of

$$\int \frac{f'(x)}{f(x)} dx$$

Hence, put $u = x^5 + 16$

$$\frac{du}{dx} = 5x^{5-1} + 0 = 5x^4$$

Hence,

$$dx = \frac{du}{5x^4}$$

We have;

$$\int \frac{5x^4}{u} \times \frac{du}{5x^4}$$

Hence, $5x^4$ cancels out;

$$\int \frac{1}{u} du$$

From integral rules; this is:

$$[\ln u] + C$$

Return $u = x^5 + 16$

$$\ln(x^5 + 16) + C$$

(ii)

$$(2ax + b)(ax^2 + bx)^7$$

To integrate this, we have

$$\int (2ax + b)(ax^2 + bx)^7 dx$$

This is a case of

$$\int f'(x)g[f(x)]dx$$

Hence, put $u = ax^2 + bx$

$$\frac{du}{dx} = 2 \times ax^{2-1} + 1 \times bx^{1-1}$$

$$\frac{du}{dx} = 2ax + b$$

Hence,

$$dx = \frac{du}{2ax + b}$$

We have;

$$\int (2ax + b)(u)^7 \times \frac{du}{2ax + b}$$

Hence, $2ax + b$ cancels out;

$$\int u^7 du$$

From integral rules; this is:

$$\left[\frac{u^{7+1}}{7+1}\right] + C = \frac{u^8}{8} + C$$

Return $u = ax^2 + bx$

$$\frac{(ax^2 + bx)^8}{8} + C$$

(iii)

$$\log_e 2x$$

A case of substitution;

$$z = 2x$$

$$\frac{dz}{dx} = 2 \times x^{1-1} = 2$$

Hence,

$$dx = \frac{dz}{2}$$

We have;

$$\int \log_e z \times \frac{dz}{2}$$

Bring the constant out;

$$\frac{1}{2}\int \log_e z\, dz$$

From integration by parts, we'll see the light as to how to integrate logarithm functions, express $\log_e z$ as multiplied by 1, this was treated in the note; this is:

$$\frac{1}{2}\int 1 \times \log_e z\, dz$$

Facing the integral squarely now;

$$\int 1 \times \log_e z\, dz$$

Integration by parts;
Put

$$u = \log_e z$$

Standard derivative;

$$\frac{du}{dz} = \frac{1}{z}$$

Also;

$$\frac{dv}{dz} = 1$$

Integrate!

$$\int dv = \int 1 dz$$

Here, straight;

$$v = z$$

Hence, we have all we need; rush to the integration by parts formula making the appropriate substitutions for **all terms**:

$$\int u \frac{dv}{dz} dz = uv - \int v \frac{du}{dz} dz$$

$$\int 1 \times \log_e z \, dz = \log_e z \, (z) - \int z \left(\frac{1}{z}\right) dz$$

Simplifying further;

$$\int 1 \times \log_e z \, dz = z \log_e z - \int 1 dz$$

We have reduced the integral to the sum of a term and another integral which should be integrated easily;

So, let's evaluate this integral we have in our reduced form;

$$\int 1 dz = z$$

Hence,
Finally the integral of $\log_e z$ is:

$$\int \log_e z \, dx = z \log_e z - z$$

Since;

$$\int 1 \times \log_e z \, dz = \int \log_e z \, dz$$

But then, we're far from been done! Firstly, z is actually:

$$z = 2x$$

Also;
This is the real value of the integral, we only focused squarely on the integral:

$$\frac{1}{2} \int 1 \times \log_e z \, dz$$

Hence,
Returning everything to normal, we'll be having;

$$\frac{1}{2} \int 1 \times \log_e z \, dz = \frac{1}{2} \left(z \log_e z - z \right)$$

Hence;
We have:

$$\frac{1}{2}(2x\log_e 2x - 2x)$$

Expanding the bracket;

$$\boldsymbol{x\log_e 2x - x}$$

That's our final integral; referring back to the notes on integration by parts, this is nothing too difficult, just notice carefully that in the integral of

$$\int 1 \times \log_e z\, dz$$

The integration by parts formula changes from $\frac{du}{dx}$ and $\frac{dv}{dx}$ to $\frac{du}{dz}$ and $\frac{dv}{dz}$ since we're now working with respect to z owing to the substitution before everything was returned to normal.

(c)

(i)

$MC = f(q) = -15 + q;$
$P = f(q) = 2.5 - 0.75q;$

We have our marginal cost function and hence, we can find our cost function by integrating it, you know that fully;

$$C(q) = \int MC = \int (-15 + q)dq$$

$$C(q) = -\frac{15q^{0+1}}{0+1} + \frac{q^{1+1}}{1+1} + C$$

$$C(q) = -15q + \frac{q^2}{2} + C$$

Now, appropriately, we should be given the fixed cost to find C so since we aren't given the fixed cost, then we assume $C = 0$; hence,

$$C(q) = -15q + \frac{q^2}{2}$$

We also have the price;

$$P = f(q) = 2.5 - 0.75q$$

The revenue function is given by the product of p and the quantity (level of output)

$$R(q) = p \times q = (2.5 - 0.75q)q$$

$$R(q) = 2.5q - 0.75q^2$$

Hence, we can now easily find the profit function;

$$P(q) = R(q) - C(q)$$

$$P(q) = 2.5q - 0.75q^2 - \left(-15q + \frac{q^2}{2}\right)$$

$$P(q) = 2.5q - 0.75q^2 + 15q - \frac{q^2}{2}$$

$$P(q) = -1.75q^2 + 17.5q$$

(ii)

When profit is maximized; the marginal profit function is zero; let's find the marginal profit, the derivative of the profit function;

$$P'(q) = \frac{d}{dq}(-1.75q^2 + 17.5q)$$

$$P'(q) = 2 \times -1.75q^{2-1} + 1 \times 17.5q^{1-1}$$

$$P'(q) = -3.5q + 17.5$$

At maximum profit;

$$P'(q) = 0$$

Hence,

$$-3.5q + 17.5 = 0$$

$$3.5q = 17.5$$

$$q = \frac{17.5}{3.5} = 5$$

Hence, maximum profit occurs at $q = 5$

(iii)

To show it; show that the profit is greater than zero at this level of maximum profit (i.e. positive); rush back to the profit function;

$$P(q) = -1.75q^2 + 17.5q$$

$$P(5) = -1.75(5)^2 + 17.5(5)$$

$$P(5) = -1.75(25) + 17.5(5)$$

$$P(5) = -43.75 + 87.5 = 43.75$$

Hence, since the profit is positive at the maximum profit point, the profit is actually maximized!

(iv)

We already done that in (iii) while checking; the profit as calculated in (iii) above is: 43.75

That's it about SSC106 2003/2004 SESSION. TRUST YOU CAN BEAR WITNESS THAT EVERYTHING HAS BEEN TAUGHT IN DETAILS IN THIS BOOK!!!