MATRICES

So, let's have a short introduction to this; to assure you from the start, the concept of matrices is definitely nothing difficult. You should believe that easily as it's a concept you've started treating from SSC105. As a matter of fact, there is basically nothing much to understand in the studies of matrices ***but there are many things to be remembered*** and of course, most importantly, it is very important to know that **there are many mistakes possible to be made on matrices.** So as it were, I'd like to reinstate the fact that there is virtually nothing really needed to be understood in matrices; let's leave too much stories alone and face the main thing;

> ***Beyond any topic, you need your pen and book to be your companion in this topic. There are many points you need to take note of as we dropped them gradually.***

On a historical note, the term "matrix" (Latin for womb was derived from *mater*–mother) was coined by **James Sylvester** in the year **1850.** He understood matrices as an object giving rise to a number of determinants which are called *"minors"* today.

A matrix (matrices for plural) is a rectangular array of numbers, symbols or expressions usually arranged in grid (rows and columns).

The numbers, symbols or expressions in a matrix are called **entries** or its **elements**. The horizontal lines of entries are called **rows** and vertical lines of entries are called **columns**.

Most commonly, elements of matrices are scalar elements (mostly numbers).

Let’s consider this matrix below to explain fully the stories explained above:

$$\begin{array}{rc} \begin{array}{r}\mathbf{R1}\longrightarrow\\ \mathbf{R2}\longrightarrow\\ \mathbf{R3}\longrightarrow\end{array} & \begin{pmatrix}2 & 1 & 3 & 6\\ 1 & 2 & -1 & 0\\ -5 & -4 & -4 & 1\end{pmatrix} \\ & \begin{array}{cccc}\uparrow & \uparrow & \uparrow & \uparrow\\ \mathbf{C1} & \mathbf{C2} & \mathbf{C3} & \mathbf{C4}\end{array}\end{array}$$

Hence, here, $R1$, $R2$, $R3$ implies the row 1, row 2 and row 3 of the matrix which are the horizontal lines of the entries; $C1$, $C2$, $C3$, $C4$ implies the column 1, column 2, column 3 and column 4 of the matrix which are the vertical lines of the entries. **Matrices are usually enclosed in brackets.**

The size of a matrix;

The size of a matrix is defined by the number of rows and columns that it contains. A matrix with m rows and n columns is called an $m \times n$ matrix or m-by-n matrix, m and n are called its dimensions. **The size of a matrix is also called the order of the matrix.** Examples are given below:

$$\begin{pmatrix} 2 & -3 & 1 \\ 5 & 0 & -3 \end{pmatrix} \rightarrow 2 \times 3 \text{ matrix}$$

The matrix above is a 2×3 matrix since it has two rows and three columns.

$$\begin{pmatrix} 2 & -1 \\ 4 & 2 \\ 5 & -3 \end{pmatrix} \rightarrow 3 \times 2 \text{ matrix}$$

The matrix above is a 3×2 matrix since it has three rows and two columns.

$$\begin{pmatrix} 1 & -2 & 3 \end{pmatrix} \rightarrow 1 \times 3 \text{ matrix}$$

The matrix above is a 1×3 matrix since it has two rows and three columns.

The notations in entries of a matrix;

The entries (or elements) of a matrix are denoted by their position.

Matrices are usually denoted by uppercase letters (capital letters) such as shown below:

$$\boldsymbol{A} = \begin{pmatrix} -2 & -1 \\ 4 & 2 \\ 5 & -3 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} 2 & -3 & 1 \\ 5 & 0 & 3 \end{pmatrix}$$

For a matrix with an uppercase letters such as $\boldsymbol{A}$ in the matrix above, corresponding lowercase letters with the rows and columns as subscripts are used to represent entries in a matrix; Ways they're represented varies; some are listed below for a matrix $\boldsymbol{A}$

$$a_{ij},\ a_{i,j},\ \ A(i,j), \text{ and etc}$$

Where i and j are the corresponding row and column position.

A matrix element is completely defined by its corresponding row and column.

In the matrix $\boldsymbol{A}$:

$$\boldsymbol{A} = \begin{pmatrix} -2 & -1 \\ 4 & 2 \\ 5 & -3 \end{pmatrix}$$

The a_{ij} notation is the most used notation and hence, the notation that'll be used in the course of this book. Hence, in $\boldsymbol{A}$ above:

-2 is defined by a_{11} since its position is defined by the row 1 and column 1 of the matrix.

-1 is defined by a_{12} since its position is defined by the row 1 and column 2 of the matrix.

4 is defined by a_{21} since its position is defined by the row 2 and column 1 of the matrix.

2 is defined by a_{22} since its position is defined by the row 2 and column 2 of the matrix.

5 is defined by a_{31} since its position is defined by the row 3 and column 1 of the matrix.

-3 is defined by a_{32} since its position is defined by the row 3 and column 2 of the matrix.

Let's consider the matrix $\boldsymbol{B}$, here the lowercase *b* will be used in the entries notation.

$$\boldsymbol{B} = \begin{pmatrix} 2 & -3 & 1 \\ 5 & 0 & 3 \end{pmatrix}$$

2 is defined by a_{11} since its position is defined by the row 1 and column 1 of the matrix.

-3 is defined by a_{12} since its position is defined by the row 1 and column 2 of the matrix.

1 is defined by a_{13} since its position is defined by the row 1 and column 3 of the matrix.

5 is defined by a_{21} since its position is defined by the row 2 and column 1 of the matrix.

0 is defined by a_{22} since its position is defined by the row 2 and column 2 of the matrix.

3 is defined by a_{23} since its position is defined by the row 2 and column 3 of the matrix.

There are special types of matrices based on their size;

A row matrix is a matrix that has just one row and any number of columns:

$\begin{pmatrix} 2 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -3 & 2 \end{pmatrix}$ are examples of row matrices.

A row matrix is also called a **row vector.**

A column matrix is a matrix that has just one column and any number of rows.

$\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$ are examples of column matrices.

A column matrix is also called a **column vector.**

A square matrix is an extremely important type of matrix that has a host of applications in the world of matrices.

A square matrix is a matrix that has equal number of rows and columns.

Matrices of the order $2 \times 2,\ 3 \times 3,\ 4 \times 4$ and a host of other matrices are all types of square matrices.

$$\begin{bmatrix} 2 & 5 & 7 & 12 \\ 1 & -4 & 8 & 10 \\ -2 & 0 & 8 & -3 \\ 3 & -2 & 16 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -8 & 2 \\ 3 & 3 & -4 \\ 1 & -6 & 5 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix}$$

are examples of $4 \times 4,\ 3 \times 3$ and 2×2 matrices respectively, all of which are square matrices.

In essence, we have the definition of a rectangular matrix.

A rectangular matrix is a matrix that has unequal number of rows and columns.

Hence, matrices of order 3×2, 2×3, 4×2, 3×1 and etc are all examples of rectangular matrices. Such matrices look like rectangles since one side (either the row or the column) is always longer than the other.

Sure we'll still come back to square matrices but for now, let's see some manipulations under entries of matrices.

- Construct a matrix A given that A is a matrix of the 2×3 order; if A is given by:

$$\boldsymbol{a_{ij} = 3i^2 - 4j}$$

Kk, here, we have a simple question on entries of matrices manipulation, the entries part looks irrelevant but here is a question for you to solve;

Here, we have to construct a 2×3 matrix with a rule that the elements are given by an equation;

We have: $a_{ij} = 3i^2 - 4j$

Hence, $a_{11} = 3(1)^2 - 4(1) = 3 - 4 = -1$

$$a_{12} = 3(1)^2 - 4(2) = 3 - 8 = -5$$

$$a_{13} = 3(1)^2 - 4(3) = 3 - 12 = -9$$

$$a_{21} = 3(2)^2 - 4(1) = 12 - 4 = 8$$

$$a_{22} = 3(2)^2 - 4(2) = 12 - 8 = 4$$

$$a_{23} = 3(2)^2 - 4(3) = 12 - 12 = 0$$

Now, note that we know the level of entries to be restricted to since we've been given the order of

the matrix to be constructed; You could guide yourself with a diagram to know the restrictions of how many entries you are to construct.
So, right here;

We have: $A = \begin{bmatrix} -1 & -5 & -9 \\ 8 & 4 & 0 \end{bmatrix}$

That's the matrix we need to construct;

NOW! DO NOT go ahead thinking all matrices have a rule that their entries follow, this is only to test your understanding of the concept of matrices entries; many matrices do not follow any rule for their elements.

- Construct $\left[b_{ij}\right]_{3\times 3}$ if $b_{ij} = \frac{i}{i+j}$

Cool,
Now there is quite some reasoning to embark on here; now, the whole rules are simply summarized in one term:

$$\left[b_{ij}\right]_{3\times 3}$$

It means we're to construct a matrix B (the uppercase of b) of size 3×3 with the given rule.

$$b_{ij} = \frac{i}{i+j}$$

Hence, here;

$$a_{11} = \frac{1}{1+1} = \frac{1}{2}$$

$$a_{12} = \frac{1}{1+2} = \frac{1}{3}$$

$$a_{13} = \frac{1}{1+3} = \frac{1}{4}$$

$$a_{21} = \frac{2}{2+1} = \frac{2}{3}$$

$$a_{22} = \frac{2}{2+2} = \frac{2}{4} = \frac{1}{2}$$

$$a_{23} = \frac{2}{2+3} = \frac{2}{5}$$

$$a_{31} = \frac{3}{3+1} = \frac{3}{4}$$

$$a_{32} = \frac{3}{3+2} = \frac{3}{5}$$

$$a_{33} = \frac{3}{3+3} = \frac{3}{6} = \frac{1}{2}$$

Now, we have quite some values, that is because it's a 3×3 matrix we're talking about here; Hence, our matrix B is given by:

$$B = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{2}{3} & \frac{1}{2} & \frac{2}{5} \\ \frac{3}{4} & \frac{3}{5} & \frac{1}{2} \end{pmatrix}$$

- Consider the matrix below:

$$A = \begin{pmatrix} 1 & - & - \\ - & 0 & - \\ -3 & - & -6 \end{pmatrix}$$

If $a_{12} = -a_{13}$; $a_{23} - a_{21} = a_{11}$;

$-3a_{32} = a_{33}$; $a_{12} = -a_{31}$; $a_{21} = 4$

Find completely the matrix A.

So okay, we have the original matrix with some things deleted; let's have our rules one by one.

Statement 1

$a_{12} = -a_{13}$; None of these two are known yet.

Statement 2

$a_{23} - a_{21} = a_{11}$; From our matrix shown, you can see glaringly that: $\boldsymbol{a_{11} = 1}$

Hence, $a_{23} - a_{21} = 1$

Statement 3
$-3a_{32} = a_{33}$; Again, from our matrix given, you can see glaringly that, $\boldsymbol{a_{33} = -6}$; Hence, we have;

$-3a_{32} = -6$; divide both by -3; $\boldsymbol{a_{32} = 2}$

Statement 4
$a_{12} = -a_{31}$; From the matrix, it is obvious that;
$\boldsymbol{a_{31} = -3}$;

Hence, $\boldsymbol{a_{12} = -(-3) = 3}$;

From **Statement 1:**
$a_{12} = -a_{13}$;
From the above, we got that: $a_{12} = 3$; hence,

$3 = -a_{13}$; hence,
$\boldsymbol{a_{13} = -3}$

Statement 5
$\boldsymbol{a_{21} = 4}$;

From statement 2: $a_{23} - a_{21} = 1$;

$a_{23} - 4 = 1$;
$\boldsymbol{a_{23} = 1 + 4 = 5}$

We can completely express the matrix now;

$$A = \begin{pmatrix} 1 & 3 & -3 \\ 4 & 0 & 5 \\ -3 & 2 & -6 \end{pmatrix}$$

Equality of matrices;

Two matrices can only be equal **if and only if** they are of the same size (order). It is given that for two matrices A and B of the same order to be equal, then for every i, j;

$$a_{ij} = b_{ij}$$

Example;

$A = \begin{pmatrix} 2 & 3 \\ 8 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 1+1 & 3 \\ \frac{16}{2} & 9-2 \end{pmatrix}$

Then, after simplification of B;

We have: $\begin{pmatrix} 2 & 3 \\ 8 & 7 \end{pmatrix}$; with every element in B equal to the elements in A; then $A = B$

There are better questions under matrices equality but firstly, let us rush some concepts and take the questions together, I guess that's fine?

SUM AND SCALAR MULTIPLICATIONS OF MATRICES;

Two matrices can be added ***if and only if the two matrices are of the same size.*** As usual, same size matrices imply same rows and same columns.

No two matrices with different sizes can be added.

Matrices are added by adding the corresponding elements with each other; for two matrices with the same size, the element occupying the $R1, C1$ position is the added matrix is the sum of the $R1, C1$ of each of the matrices to be added.

Matrices addition is not limited to two matrices alone, as many matrices as possible can be added.

Addition and subtraction of matrices follow the same rule; however, during subtraction of matrices, the order of the operation must be taken into consideration: Just like every mathematics operation; $2 - 3$ is different from $3 - 2$ so hence, the same applies here.

So, let's see this example;

- Given the two matrices A and B below:

$$A = \begin{pmatrix} 1 & -3 & -1 \\ 4 & 2 & 6 \\ 1 & 0 & 3 \end{pmatrix} \text{ and } B = \begin{pmatrix} -2 & 5 & 2 \\ 3 & -2 & 1 \\ 2 & 1 & -1 \end{pmatrix}$$

Find:

(i) $A + B$

(ii) $A - B$

(iii) $B - A$

So, quick!

$$A + B = \begin{pmatrix} 1 & -3 & -1 \\ 4 & 2 & 6 \\ 1 & 0 & 3 \end{pmatrix} + \begin{pmatrix} -2 & 5 & 2 \\ 3 & -2 & 1 \\ 2 & 1 & -1 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 1 + (-2) & -3 + 5 & -1 + 2 \\ 4 + 3 & 2 + (-2) & 6 + 1 \\ 1 + 2 & 0 + 1 & 3 + (-1) \end{pmatrix}$$

$A + B = \begin{pmatrix} -1 & 2 & 1 \\ 7 & 0 & 7 \\ 3 & 1 & 2 \end{pmatrix}$; As simple as that, add each element to their corresponding positions in the elements of the matrices to be added.

$$A - B = \begin{pmatrix} 1 & -3 & -1 \\ 4 & 2 & 6 \\ 1 & 0 & 3 \end{pmatrix} - \begin{pmatrix} -2 & 5 & 2 \\ 3 & -2 & 1 \\ 2 & 1 & -1 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 1-(-2) & -3-5 & -1-2 \\ 4-3 & 2-(-2) & 6-1 \\ 1-2 & 0-1 & 3-(-1) \end{pmatrix}$$

$$A - B = \begin{pmatrix} 3 & -8 & -3 \\ 1 & 4 & 5 \\ -1 & -1 & 4 \end{pmatrix}$$

So, we are conscious that the elements of the matrix on the right of the minus sign are subtracted from the corresponding elements of the matrix on the left of the minus sign.

Now, let's evaluate $B - A$;

$$B - A = \begin{pmatrix} -2 & 5 & 2 \\ 3 & -2 & 1 \\ 2 & 1 & -1 \end{pmatrix} - \begin{pmatrix} 1 & -3 & -1 \\ 4 & 2 & 6 \\ 1 & 0 & 3 \end{pmatrix}$$

$$B - A = \begin{pmatrix} -2-1 & 5-(-3) & 2-(-1) \\ 3-4 & -2-2 & 1-6 \\ 2-1 & 1-0 & -1-3 \end{pmatrix}$$

$$B - A = \begin{pmatrix} -3 & 8 & 3 \\ -1 & -4 & -5 \\ 1 & 1 & -4 \end{pmatrix}$$

Notice the nature of the elements of $B - A$ relative to $A - B$; notice all elements in $(B - A)$ are the negative form of $(A - B)$;
It follows that;

$$(B - A) = -(A - B)$$

That'll lead us to the concept of **scalar multiplication of matrices;**

When a matrix is multiplied by a scalar, the scalar goes ahead to multiply all the elements of the matrix. A scalar by the way, refers basically to a number. Hence, if a matrix below, A is multiplied by a scalar, k, then k multiplies all the elements of the matrix; Scalar multiplication doesn't in any way tamper with the size of a matrix, it only affects the elements of the matrix.

For a more expanded definition; I can remember seeing a question asking you to define what a vector is in your past question.

So, a scalar, by definition is a quantity with a magnitude by with no direction.

While, a vector is the opposite of a scalar, it is a quantity that has both magnitude and direction.

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$kA = k\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} ka & kb & kc \\ kd & ke & kf \\ kg & kh & ki \end{pmatrix}$$

So, in the issue of matrix subtraction;

$(A - B)$ is -1 multiplied by $(B - A)$ and vice-versa.

$$(A - B) = -1(B - A)$$
$$(B - A) = -1(A - B)$$

However, $-1(A) = -A$

Hence;

$$(A - B) = -(B - A)$$
$$(B - A) = -(A - B)$$

So, let's see more on scalar multiplication and addition operations in matrices; Examples are given below:

- Given the four matrices A, B, C, D below:

$$A = \begin{bmatrix} 3 & 0 & -2 \\ 4 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -11 & 1 & 12 \\ 7 & -2 & 13 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 0 \\ -2 & 4 \\ 3 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 4 \\ -2 & 2 \\ 3 & 10 \end{bmatrix}$$

Evaluate the following,

(i) $2A + B$

(ii) $3A + 2B$

(iii) $3A - 2B$

(iv) $2C + 3D$

(v) $2C - D$

(vi) $2A + D$

(vii) $2B - 3C$

(i) $2A + B$; we got to evaluate $2A$ first from the principle of scalar multiplication.

$$2A = 2\begin{bmatrix} 3 & 0 & -2 \\ 4 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2(3) & 2(0) & 2(-2) \\ 2(4) & 2(3) & 2(1) \end{bmatrix}$$

$$2A = \begin{bmatrix} 6 & 0 & -4 \\ 8 & 6 & 2 \end{bmatrix}$$

$$2A + B = \begin{bmatrix} 6 & 0 & -4 \\ 8 & 6 & 2 \end{bmatrix} + \begin{bmatrix} -11 & 1 & 12 \\ 7 & -2 & 13 \end{bmatrix}$$

$$2A + B = \begin{bmatrix} 6 + (-11) & 0 + 1 & -4 + 12 \\ 8 + 7 & 6 + (-2) & 2 + 13 \end{bmatrix}$$

$$2A + B = \begin{bmatrix} -5 & 1 & 8 \\ 15 & 4 & 15 \end{bmatrix}$$

(ii) $3A + 2B$; we need to evaluate both $3A$ and $2B$

$$3A = 3\begin{bmatrix} 3 & 0 & -2 \\ 4 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 3(3) & 3(0) & 3(-2) \\ 3(4) & 3(3) & 3(1) \end{bmatrix}$$

$$3A = \begin{bmatrix} 9 & 0 & -6 \\ 12 & 9 & 3 \end{bmatrix}$$

$$2B = 2\begin{bmatrix} -11 & 1 & 12 \\ 7 & -2 & 13 \end{bmatrix} = \begin{bmatrix} 2(-11) & 2(1) & 2(12) \\ 2(7) & 2(-2) & 2(13) \end{bmatrix}$$

$$2B = \begin{bmatrix} -22 & 2 & 24 \\ 14 & -4 & 26 \end{bmatrix}$$

Hence, we have $3A + 2B$

$$3A + 2B = \begin{bmatrix} 9 & 0 & -6 \\ 12 & 9 & 3 \end{bmatrix} + \begin{bmatrix} -22 & 2 & 24 \\ 14 & -4 & 26 \end{bmatrix}$$

$$3A + 2B = \begin{bmatrix} 9 + (-22) & 0 + 2 & -6 + 24 \\ 12 + 14 & 9 + (-4) & 3 + 26 \end{bmatrix}$$

$$3A + 2B = \begin{bmatrix} -13 & 2 & 18 \\ 26 & 5 & 29 \end{bmatrix}$$

(iii) $3A - 2B$; we already have both $3A$ and $2B$, hence,

$$3A - 2B = \begin{bmatrix} 9 & 0 & -6 \\ 12 & 9 & 3 \end{bmatrix} - \begin{bmatrix} -22 & 2 & 24 \\ 14 & -4 & 26 \end{bmatrix}$$

$$3A - 2B = \begin{bmatrix} 9 - (-22) & 0 - 2 & -6 - 24 \\ 12 - 14 & 9 - (-4) & 3 - 26 \end{bmatrix}$$

$$3A - 2B = \begin{bmatrix} 31 & -2 & -30 \\ -2 & 13 & -23 \end{bmatrix}$$

(iv) $2C + 3D$, we need to evaluate both $2C$ and $3D$;

$$2C = 2\begin{bmatrix} 3 & 0 \\ -2 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2(3) & 2(0) \\ 2(-2) & 2(4) \\ 2(3) & 2(1) \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ -4 & 8 \\ 6 & 2 \end{bmatrix}$$

$$3D = 3\begin{bmatrix} 1 & 4 \\ -2 & 2 \\ 3 & 10 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(4) \\ 3(-2) & 3(2) \\ 3(3) & 3(10) \end{bmatrix} = \begin{bmatrix} 3 & 12 \\ -6 & 6 \\ 9 & 30 \end{bmatrix}$$

$$2C + 3D = \begin{bmatrix} 6 & 0 \\ -4 & 8 \\ 6 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 12 \\ -6 & 6 \\ 9 & 30 \end{bmatrix}$$

$$2C + 3D = \begin{bmatrix} 6+3 & 0+12 \\ -4+(-6) & 8+6 \\ 6+9 & 2+30 \end{bmatrix} = \begin{bmatrix} 9 & 12 \\ 2 & 14 \\ 15 & 32 \end{bmatrix}$$

(v) $2C - D$; we already have $2C$ and D is known already, hence,

$$2C - D = \begin{bmatrix} 6 & 0 \\ -4 & 8 \\ 6 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ -2 & 2 \\ 3 & 10 \end{bmatrix}$$

$$2C - D = \begin{bmatrix} 6-1 & 0-4 \\ -4-(-2) & 8-2 \\ 6-3 & 2-10 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -2 & 6 \\ 3 & -8 \end{bmatrix}$$

(vi) $2A + D$; **NO SIRE! NO MA'AM!** Those two matrices are not of the same order (size), 2 is a scalar multiplying A doesn't change its order, hence, the matrix is still a 2×3 matrix and hence cannot be added to a 3×2 matrix.

(vii) Same ordeal with (vi)

- Solve the equation;

$$3\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 12\begin{pmatrix} -\frac{1}{2} \\ \frac{3}{4} \end{pmatrix}$$

This tests our knowledge of addition, scalar multiplication and equality of matrices;
So, we'll expand the various scalar multiplications first;

$$\begin{pmatrix} 3x \\ 3y \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 12\left(-\frac{1}{2}\right) \\ 12\left(\frac{3}{4}\right) \end{pmatrix}$$

$$\begin{pmatrix} 3x \\ 3y \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -6 \\ 9 \end{pmatrix}$$

Perform the operations of subtraction on the LHS:

$$\begin{pmatrix} 3x - 1 \\ 3y - 2 \end{pmatrix} = \begin{pmatrix} -6 \\ 9 \end{pmatrix}$$

From the rule of matrix equality; the corresponding matrix positions are equal to each other.

$$3x - 1 = -6$$
$$3x = -6 + 1$$
$$3x = -5$$

Hence,

$$x = -\frac{5}{3}$$

$$3y - 2 = 9$$
$$3y = 9 + 2$$
$$3y = 11$$

Hence,

$$y = \frac{11}{3}$$

- Solve the equation:

$$x\binom{3}{2} - y\binom{-4}{7} = 3\binom{2}{4}$$

Alright, here, the scalars multiplying the matrices are x and y, expand by the rules of scalar multiplication on the LHS and RHS:

$$\binom{3x}{2x} - \binom{-4y}{7y} = \binom{2(3)}{4(3)}$$

Use the subtraction rule on the LHS while you keep expanding;

$$\begin{pmatrix} 3x - (-4y) \\ 2x - 7y \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} 3x + 4y \\ 2x - 7y \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$$

From the rule of equality of matrices, we have:

$$3x + 4y = 6$$
$$2x - 7y = 12$$

This is a simultaneous equation on a small scale;
From the first equation;

$$3x + 4y = 6$$

$$x = \frac{6 - 4y}{3}$$

Put in the second equation;

$$2x - 7y = 12$$

$$2\left(\frac{6 - 4y}{3}\right) - 7y = 12$$

Clear by multiplying through by 3

$$2(6-4y)-(7y\times 3)=12\times 3$$
$$12-8y-21y=36$$
$$-29y=24$$

$$y=-\frac{24}{29}$$

Here, since; $y=-\frac{24}{29}$

$$x=\frac{6-4\left(-\frac{24}{29}\right)}{3}=\left[6+\frac{96}{29}\right]\times\frac{1}{3}$$

$$x=\frac{270}{29}\times\frac{1}{3}=\frac{90}{29}$$

Hence,

$$x=\frac{90}{29}$$

$$y=-\frac{24}{29}$$

Transpose of a matrix (Matrix transposition):

The transpose of a matrix is gotten by *reflecting the elements of a matrix along the main diagonal by flipping the matrix over its diagonal.*

Lol. That was big grammar and also quite unnecessary too! However, you could learn that for a *posh* definition for you in your exam should in case you are asked for the definition, you know SSC106 questions and *define this and that, differentiate between this and that* are close friends.

That was on a lighter note anyway, the transpose of a matrix is gotten by interchanging the rows and columns of a matrix; i.e. write the rows of the matrix as the column of the new matrix (the transpose) and write the column of the matrix as the row of the new matrix (the transpose);
Now, for a matrix, A, the transpose of the matrix is given by A^T. However, there are other notations for the transpose of a matrix such as:

$$A', \qquad A^{tr}, \qquad {}^{t}A \quad \text{or} \quad A^t$$

Some books also denote this notation, A' as A^1.

The commonest two however are A^T and A'.

For a transpose matrix, the order of the original matrix is **always directly reversed.** Hence, the transpose of a matrix A of order $m \times n$ is A^T which will be of order $n \times m$.

Example; Find the A^T and B^T respectively in the matrices below;

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 5 & -7 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Here,

A^T will be gotten by turning all the rows into columns, the matrix will fall in place after that operation; either of the two is done; you either turn columns into rows or rows into columns.

$$A^T = \begin{bmatrix} 1 & -2 & 5 \\ -2 & 3 & -7 \\ 1 & 1 & -3 \end{bmatrix},$$

here, $(1 \ -2 \ \ 1)$ which were in row 1 in the original matrix are moved to the column 1 of the new transpose matrix.

$(-2 \quad 3 \quad 1)$ which were in row 2 in the original matrix are moved to the column 2 of the new transpose matrix.

$(5 \quad -7 \quad -3)$ which were in row 3 in the original matrix are moved to the column 3 of the new transpose matrix.

Now, the order of A is 3×3 so even after reversal, the order of the transpose will still be 3×3.

That's how transpose is done;

In $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

B^T will be gotten by turning all the rows into columns, the matrix will fall in place after that operation; either of the two is done; you either turn columns into rows or rows into columns.

$$B^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Here, $(1 \quad 2 \quad 3)$ which were in row 1 in the original matrix are moved to the column 1 of the new transpose matrix.

$(4 \quad 5 \quad 6)$ which were in row 2 in the original matrix are moved to the column 2 of the new transpose matrix.

Now, the order of B is 2×3, hence, after reversal, the order of the transpose is truncated to the 3×2 order.

The transpose of a matrix was introduced by the British Mathematician Arthur Cayley in 1858.

Taking the transpose of a matrix simply returns the matrix to its original state; meaning:

$$(A^T)^T = A$$

You should see the above obviously, the columns that were truncated will be returned in place during the second operation of transposition.

Other properties of transpose matrix are given by:

$$(A + B)^T = A^T + B^T$$

<u>The transpose of a sum is equal to the sum of their individual transpose.</u>

$(cA)^T = cA^T$; where c is scalar;

<u>A transpose of a matrix multiplied by a scalar is equal to the scalar multiplied by the transpose of the matrix.</u>

$$(AB)^T = B^T A^T$$

Also, a very important property of transposes, the transpose of a product is the reverse product of their transposes;

We've not taken the concept of matrix multiplication and it'll be taken in the next section, AB is the product of matrices A and B.

Here are few types of matrices that are related to their relationships with their transpose.

A square matrix that is equal to its transpose is called a **symmetric matrix.**

A square matrix that is equal to the negative value of its transpose is called a **skew-symmetric matrix.**

Other relationships with the transpose of a matrix exist but we'll see them later in the course of this.

Let's move quickly.

Some other special types of matrices:

Many matrices, especially square matrices have several types based on several matrices; the symmetric and skew-symmetric matrices above are examples of types of square matrices;

The main diagonal of a matrix is the line which runs from the top left corner to the bottom right corner strictly in a square matrix.

For instance, the main diagonal on a 4×4 matrix contain the elements;

a_{11}, a_{22}, a_{33} and the a_{44} entries.

In the following matrices; the main diagonal is ruled out;

$$\begin{bmatrix} \cancel{2} & 5 & 7 & 12 \\ 1 & \cancel{-4} & 8 & 10 \\ -2 & 0 & \cancel{8} & -3 \\ 3 & -2 & 16 & \cancel{-1} \end{bmatrix},$$

$$\begin{bmatrix} \cancel{1} & -8 & 2 \\ 3 & \cancel{3} & -4 \\ 1 & -6 & \cancel{5} \end{bmatrix} \text{and} \begin{bmatrix} \cancel{1} & 2 \\ -3 & \cancel{2} \end{bmatrix}$$

Now, with knowledge of the main diagonal; we now have several types of matrices;

A diagonal matrix is a matrix with all elements **"not" on the main diagonal** equal to zero.

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

The two matrices above are examples of diagonal matrices.

A triangular matrix is a matrix that has only zero elements **above (or below) but not both above and below** the main diagonal.

Triangular matrices are of two types, the upper and lower triangular matrices;

The **upper triangular matrices** have elements **below the main diagonal** equal to zero;

The **lower triangular matrices** have elements **above the main diagonal** equal to zero;

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & -3 & 2 \\ 0 & 0 & 5 \end{bmatrix} \text{ is an upper triangular matrix.}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & -3 & 0 \\ 2 & 2 & 5 \end{bmatrix}$$ is a lower triangular matrix.

An identity matrix is a square matrix, a diagonal matrix but a special type which has all the elements on the main diagonal equal to unity i.e. they're equal to 1. Identity matrices have a wild range of use in matrices due to their special property.

Examples of identity matrices are shown below;

$$\boldsymbol{I_4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\boldsymbol{I_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \boldsymbol{I_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

As seen above, the identity matrix is represented by the uppercase letter ***I*** and hence, no other matrix is (or should be) represented by I.

Occasionally, the order of the identity matrix is represented by a subscript on the letter I as shown in the matrices above, mostly however, it is denoted as ***I*, an identity matrix of the order *n*.**

The above [identity] matrices are called identity matrices of a certain order, n where n is the number of rows (or columns) in the matrix.

The above are identity matrices of the order 4, 3 and 2 respectively. They are also called the 4×4, 3×3 and 2×2 identity matrix respectively.

Generally, when compared with natural algebra, identity matrices serve as the number 1.

I guess I skipped one matrix up there; it is called a **null matrix** or **a zero matrix**. It is a matrix [of any order] that has **its entire elements as zero.** A null matrix doesn't have to be a square matrix; null matrices exist for matrices of all sizes;

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

All the above are null matrices of the 2×2, 3×2, 2×3 and 1×3 order respectively from left to right.

Null matrices are usually denoted by the **symbol 0.** The order of the null matrices is also at times denoted as subscripts under the zero or softly, they're denoted as **0** and the order mentioned afterwards. Examples:

$$\mathbf{0}_{3\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and } \mathbf{0}_{2\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Generally, when compared with natural algebra, null matrices serve as the number 0, hence, they have a very negligible amount of uses.

Let's have a few refreshing exercises on what we've learnt so far, before we move to the aspects of matrix multiplication, determinants, inverses and all.

EXERCISES

1. Consider the matrix, $Q = \begin{bmatrix} \frac{1}{3} & 6 & -\frac{1}{2} \\ 0 & -13 & \frac{1}{7} \\ 0 & 0 & -1 \end{bmatrix}$
   (i) State the order and size of Q
   (ii) State q_{21}, q_{13}, q_{33}

(iii) Is Q' upper triangular, lower triangular or neither? Q' is the transpose of Q

2. Construct $\left[a_{ij}\right]_{2\times 3}$ where $a_{ij} = -2i + 3j$

3. Find the 3×3 matrix, B such that $b_{ij} = (-1)^{i+j}(i^2 + j^2)$

4. Solve the matrix equation;

$$\begin{bmatrix} 2x & 7 \\ 7 & 2y \end{bmatrix} = \begin{bmatrix} y & 7 \\ 7 & y \end{bmatrix}$$

5. Let

$$A = \begin{bmatrix} 2 & 1 \\ 3 & -3 \end{bmatrix} \quad B = \begin{bmatrix} -6 & -5 \\ 2 & -3 \end{bmatrix}$$

$$C = \begin{bmatrix} -2 & -1 \\ -3 & 3 \end{bmatrix} \qquad 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Compute the following:

(i) $2A + 3(B + C)$

(ii) $\frac{1}{2}(A) - 2(B + 2C)$

(iii) $(B - 2A^T)^T$

(iv) $-3(B - 20_{2\times 2})$

6. In matrices A, B, C and D above; verify if or not the following are true where k, k_1 and k_2 are scalars;

(i) $k(A + B) = kA + kB$
(ii) $(k_1 + k_2)A = k_1A + k_2A$
(iii) $k_1(k_1A) = (k_1k_2)A$
(iv) $0A = 0$
(v) $(kA)^T = kA^T$

7. Solve for x and y:

$$3\begin{pmatrix} x \\ y \end{pmatrix} - 3\begin{pmatrix} -2 \\ 4 \end{pmatrix} = 4\begin{pmatrix} 6 \\ -2 \end{pmatrix}$$

8. Solve for x, y and z.

$$\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + 2\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -10 \\ -24 \\ 14 \end{pmatrix}$$

So, below are the answers to the above questions, ensure you solve them first though; do yourself a whole lot of good by attempting and solving correctly everything.

1. For the matrix Q
   (i) Size and order imply same thing; it is of the 3×3 size or order.
   (ii) $q_{21} = 0, q_{13} = -\frac{1}{2}, q_{33} = -1$
   (iii) After transposing Q, it becomes a lower triangular matrix, note that Q in itself is an upper triangular matrix.

2. $\begin{pmatrix} 1 & 4 & 7 \\ -1 & 2 & 5 \end{pmatrix}$

3. $\begin{pmatrix} 2 & -5 & 10 \\ -5 & 8 & -13 \\ 10 & -13 & 18 \end{pmatrix}$

4. After matrix equality; we have two equations;

$$2x = y$$
$$2y = y$$

$2x = y$; fine, nice equation;

BUT

$2y = y$; this is only possible if and only if $y = 0$;

hence, x also equal to 0 since $2x = y$

$$x = y = 0$$

5. For the matrices A, B, C, D;

(i) $\begin{pmatrix} -20 & -16 \\ 3 & -6 \end{pmatrix}$

(ii) $\begin{pmatrix} 21 & \frac{29}{2} \\ \frac{19}{2} & -\frac{15}{2} \end{pmatrix}$

(iii) $\begin{pmatrix} -10 & 0 \\ -11 & 3 \end{pmatrix}$, remember you are taking the transpose of A first before taking the transpose of your overall answer.

(iv) $\begin{pmatrix} 18 & 15 \\ -6 & 9 \end{pmatrix}$

6. For the matrices A, B, C, D;
   - (i) true;
   - (ii) true;
   - (iii) true;
   - (iv) true;
   - (v) true; (i) to (v) are general properties of matrix addition and subtraction.

7. $x = 6, y = \frac{4}{3}$

8. $x = -6,\ y = -14,\ z = 1$

Right, up you tackled all those successfully, it's time to move to the next frame; they're very big concepts but just like every form of matrices; there's basically nothing too big to understand in them.

MATRIX MULTIPLICATION

The aspect of matrix multiplication deals with a more real case where two matrices are multiplied. Unlike the case of scalar multiplication, it involves a situation where two matrices are expressed as a product. We've seen other operations within matrices such as addition and subtraction and we've seen how they relate to real numbers such as the commutative laws in matrix addition and lot more, we now expand our scope to the aspect of matrix multiplication.

Now, unlike in real numbers, the order of operation is a matter of urgent attention in the case of matrices; in real numbers, 49×17 and 17×49 will have the same value; however, that is different in matrices; hence, we have the **pre-multipliers** and the **post-multipliers**; the pre-multiplier multiplies another matrix and is on the left side of the multiplication operation while the post-multiplier is being multiplied and is on the right side of the multiplication operation.

Given two matrices A and B;

In the operation of $A \times B$ which is same as AB;
A is the pre-multiplier while B is the post-multiplier.

If the order of operation is now switched and changed to $B \times A$ which is BA; then B is the pre-multiplier while A is the post-multiplier.

Unlike matrix addition, not only directly equal matrices can be multiplied, the rule of matrix multiplication goes thus;

For two matrices to be multiplied; the number of columns in the pre-multiplier must be equal to the number of rows in the post-multiplier; this rule is called **the conformability condition for matrix multiplication.**

$A \times B$ is a valid multiplication if and only if the number of columns in matrix A is equal to the number of rows in matrix B.

Same for $B \times A$ to be valid, it is if and only the number of columns in matrix B is equal to the number of rows in matrix A.

So, after all these stories; how are two matrices actually multiplied?

Kk, good question, let's see this now;

Now, we are told the number of columns in the pre-multiplier must be equal to the number of rows in the post-multiplier. Let's see this example with a matrix, P which is of the order, 2×3 and another matrix, Q which is of the order, 3×2.

We have:

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \\ q_{31} & q_{32} \end{bmatrix}$$

Let's try to evaluate the matrix product PQ:

$$PQ = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \\ q_{31} & q_{32} \end{bmatrix}$$

Row 1 is taken in the pre-multiplying matrix to come together with the column 1 of the post multiplying matrix; they form the $R1, C1$ element of the new product matrix.

Now, from the conformability condition, rows in the pre-multiplier will have the same number of elements as the columns in the post multiplier;

Hence, here, $R1, C1$ of the product matrix is given by correspondingly multiplying elements together and summing them up;

$$(p_{11})(q_{11}) + (p_{12})(q_{21}) + (p_{13})(q_{31})$$

The above sum forms one element in the product matrix.

The first operation for the (row 1 column 1) element is given below: the elements crossed are multiplied correspondingly and added together;

$$\begin{bmatrix} \cancel{p_{11}} & \cancel{p_{12}} & \cancel{p_{13}} \\ p_{21} & p_{22} & p_{23} \end{bmatrix} \begin{bmatrix} \cancel{q_{11}} & q_{12} \\ \cancel{q_{21}} & q_{22} \\ \cancel{q_{31}} & q_{32} \end{bmatrix}$$

Now, the same row 1 in the pre-multiplier is now taken to meet the column 2 of the post-multiplier for the $(R1, C2)$ of the product matrix and the same process of adding correspondingly is repeated.

Hence, here, row 1, column 1 of the product matrix is given by;

$$(p_{11})(q_{12}) + (p_{12})(q_{22}) + (p_{13})(q_{32})$$

The above sum forms one element in the product matrix.

The second operation for the (row 1 column 2) element is given below: the elements crossed are multiplied correspondingly and added together;

$$\begin{bmatrix} \cancel{p_{11}} & \cancel{p_{12}} & \cancel{p_{13}} \\ p_{21} & p_{22} & p_{23} \end{bmatrix} \begin{bmatrix} q_{11} & \cancel{q_{12}} \\ q_{21} & \cancel{q_{22}} \\ q_{31} & \cancel{q_{32}} \end{bmatrix}$$

Now, the row 1 in the pre-multiplier has no column in the post-multiplier to pair with again, hence, row 2 picks up the responsibility and starts with the column 1 all over again.

Row 2 in the pre-multiplier is now taken to meet the column 1 of the post-multiplier for the $(R2, C1)$ of the product matrix and the same process of adding correspondingly is repeated.

Hence, here, row 2, column 1 of the product matrix is given by;

$$(p_{21})(q_{11}) + (p_{22})(q_{21}) + (p_{23})(q_{32})$$

The above sum forms one element in the product matrix.

The second operation for the (row 2 column 1) element is given below: the elements crossed are multiplied correspondingly and added together;

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \\ q_{31} & q_{32} \end{bmatrix}$$

Also, the same row 2 in the pre-multiplier is now taken to meet the column 2 of the post-multiplier for the $(R2, C2)$ of the product matrix and the same process of adding correspondingly is repeated.

Hence, here, row 2, column 2 of the product matrix is given by;

$$(p_{21})(q_{12}) + (p_{22})(q_{22}) + (p_{23})(q_{32})$$

The above sum forms one element in the product matrix.

The second operation for the (row 1 column 2) element is given below: the elements crossed are multiplied correspondingly and added together;

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \\ q_{31} & q_{32} \end{bmatrix}$$

Hence, $P \times Q$ is a 2×2 matrix since the above four entries are the entries of the product matrix.

Now, the matrix P has the order 2×3 matrix and the matrix Q has the order 3×2; the columns in the pre-multiplier and the rows in the post-multiplier condition has been meant. For the finished product matrix; it has the order thus; the rows in the pre-multiplier by the columns in the multiplier; hence, in PQ above, it is a 2×2 order matrix.

As a matter of rule, if P is of the order $m \times n$ and Q is of the order $n \times p$;
Then: PQ will be of order $m \times p$.

TURN YOUR SCREEN TO LANDSCAPE MODE (just turn without switching it to auto-rotate mode)

WE HAVE SOME EXPANSIONS TO DEAL WITH SHORTLY!

Let’s see more practical examples over here;

Evaluate AB if matrices A and B are given below:

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 4 & -1 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & 2 & 6 \\ 7 & 3 & 0 \\ -4 & 5 & 1 \end{pmatrix}$$

We have our product thus:

$$AB = \begin{pmatrix} 3 & 2 & 1 \\ 4 & -1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 & 6 \\ 7 & 3 & 0 \\ -4 & 5 & 1 \end{pmatrix}$$

Here, row 1 in the pre-multiplier will join hands with columns 1, 2 and 3 in the post-multiplier, row 2 will do the same, joining hands with columns 1, 2 and 3 in the post-multiplier; each done one after the other.

$$AB = \begin{pmatrix} 3(3)+2(7)+1(-4) & 3(2)+2(3)+1(5) & 3(6)+2(0)+1(1) \\ 4(3)+(-1)(7)+2(-4) & 4(2)+(-1)(3)+2(5) & 4(6)+(-1)(0)+2(1) \end{pmatrix}$$

$$AB = \begin{pmatrix} 19 & 17 & 19 \\ -3 & 15 & 26 \end{pmatrix}$$

Let’s see some other examples;

Given that

$$A = \begin{pmatrix} 1 & 4 & -3 \\ 2 & -3 & 4 \\ 5 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -1 & 2 \\ 4 & 1 & -3 \\ 2 & -3 & 5 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 3 & 0 \\ 4 & -2 & -1 \\ 3 & 1 & -2 \end{pmatrix}$$

Evaluate:

- $(AC)^T$
- $C^T A^T$
- $(A + B)(A + C)$

To get $(AC)^T$, we’ll evaluate AC first;

$$AC = \begin{pmatrix} 1 & 4 & -3 \\ 2 & -3 & 4 \\ 5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 4 & -2 & -1 \\ 3 & 1 & -2 \end{pmatrix}$$

Here, row 1 in the pre-multiplier will join hands with columns 1, 2 and 3 in the post-multiplier; row 2 will do the same, joining hands with columns 1, 2 and 3 in the post-multiplier; row 3 will do the same, joining hands with columns 1, 2 and 3 in the post-multiplier; each done one after the other.

$$AC = \begin{pmatrix} 1(1)+4(4)+(-3)(3) & 1(3)+4(-2)+(-3)(1) & 1(0)+4(-1)+(-3)(-2) \\ 2(1)+(-3)(4)+(4)(3) & 2(3)+(-3)(-2)+(4)(1) & 2(0)+(-3)(-1)+(4)(-2) \\ 5(1)+(0)(4)+(1)(3) & 5(3)+(0)(-2)+(1)(1) & 5(0)+(0)(-1)+(1)(-2) \end{pmatrix}$$

$$AC = \begin{pmatrix} 8 & -8 & 2 \\ 2 & 16 & -5 \\ 8 & 16 & -2 \end{pmatrix}$$

Hence, $(AC)^T$ is:

$$(AC)^T = \begin{pmatrix} 8 & 2 & 8 \\ -8 & 16 & 16 \\ 2 & -5 & -2 \end{pmatrix}$$

To find $C^T A^T$, we need to find C^T and A^T first; transpose is that simple thing we've learnt already.

$$C^T = \begin{pmatrix} 1 & 4 & 3 \\ 3 & -2 & 1 \\ 0 & -1 & -2 \end{pmatrix} \text{ and } A^T = \begin{pmatrix} 1 & 2 & 5 \\ 4 & -3 & 0 \\ -3 & 4 & 1 \end{pmatrix}$$

Hence, we have $C^T A^T$ as:

$$C^T A^T = \begin{pmatrix} 1 & 4 & 3 \\ 3 & -2 & 1 \\ 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 5 \\ 4 & -3 & 0 \\ -3 & 4 & 1 \end{pmatrix}$$

Hence, we have the product $C^T A^T$ expanded thus:

$$\begin{pmatrix} 1(1)+4(4)+(3)(-3) & 1(2)+4(-3)+(3)(4) & 1(5)+4(0)+(3)(1) \\ 3(1)+(-2)(4)+(1)(-3) & 3(2)+(-2)(-3)+(1)(4) & 3(5)+(-2)(0)+(1)(1) \\ 0(1)+(-1)(4)+(-2)(-3) & 0(2)+(-1)(-3)+(-2)(4) & 0(5)+(-1)(0)+(-2)(1) \end{pmatrix}$$

$$C^T A^T = \begin{pmatrix} 8 & 2 & 8 \\ -8 & 16 & 16 \\ 2 & -5 & -2 \end{pmatrix}$$

Notice the equality of $(AC)^T$ and $C^T A^T$, that gives a rule in matrix multiplication and transpose which has been stated in the previous section when transpose of matrices was been treated.

For every matrices A, B;

$$(AB)^T = B^T A^T$$

The transpose of a product of two matrices is equal to the product of the individual transposes in the reverse order.

- $(A+B)(A+C)$

We need to evaluate $(A+B)$ and $(A+C)$ first;

$$(A+B) = \begin{pmatrix} 1 & 4 & -3 \\ 2 & -3 & 4 \\ 5 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 3 & -1 & 2 \\ 4 & 1 & -3 \\ 2 & -3 & 5 \end{pmatrix}$$

$$A+B = \begin{pmatrix} 1+3 & 4+(-1) & -3+2 \\ 2+4 & -3+1 & 4+(-3) \\ 5+2 & 0+(-3) & 1+5 \end{pmatrix}$$

$$(A+B) = \begin{pmatrix} 4 & 3 & -1 \\ 6 & -2 & 1 \\ 7 & -3 & 6 \end{pmatrix}$$

$$(A + C) = \begin{pmatrix} 1 & 4 & -3 \\ 2 & -3 & 4 \\ 5 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 0 \\ 4 & -2 & -1 \\ 3 & 1 & -2 \end{pmatrix}$$

$$A + C = \begin{pmatrix} 1+1 & 4+3 & -3+0 \\ 2+4 & -3+(-2) & 4+(-1) \\ 5+3 & 0+3 & 1+(-2) \end{pmatrix}$$

$$(A + C) = \begin{pmatrix} 2 & 7 & -3 \\ 6 & -5 & 3 \\ 8 & 3 & -1 \end{pmatrix}$$

$$(A + B)(A + C) = \begin{pmatrix} 4 & 3 & -1 \\ 6 & -2 & 1 \\ 7 & -3 & 6 \end{pmatrix} \begin{pmatrix} 2 & 7 & -3 \\ 6 & -5 & 3 \\ 8 & 3 & -1 \end{pmatrix}$$

We have the product $(A + B)(A + C)$ expanded thus;

$$\begin{pmatrix} 4(2)+3(6)+(-1)(8) & 4(7)+3(-5)+(-1)(3) & 4(-3)+3(3)+(-1)(1) \\ 6(2)+(-2)(6)+(1)(8) & 6(7)+(-2)(-5)+(1)(3) & 6(-3)+(-2)(3)+(1)(1) \\ 7(2)+(-3)(6)+(6)(8) & 7(7)+(-3)(-5)+(6)(3) & 7(-3)+(-3)(3)+(6)(1) \end{pmatrix}$$

Hence, $(A+B)(A+C)$;

$$(A+B)(A+C) = \begin{pmatrix} 18 & 3 & -4 \\ 8 & 55 & -23 \\ 44 & 82 & -24 \end{pmatrix}$$

Let's have this simple one;

- If $f(x) = x^2 - 3x + 3$, find $f(A)$ in:

$$A = \begin{pmatrix} 1 & 4 & -3 \\ 2 & -3 & 4 \\ 5 & 0 & 1 \end{pmatrix}$$

So, okay; we know how functions roll, we replace x with A in the function to evaluate $f(A)$

In normal functions, we simply add the constants to the remaining. For example;

$$f(x) = x^2 - 3x + 3$$

$f(3)$ will be:

$$f(3) = (3)^2 - 3(3) + 3 = 3$$

However,
In matrices, how do we add 3 to end up in matrix form? This is how it is done. We have stated that the identity matrix acts as the number 1, hence, to express a constant in matrix form, we'll simply express it as a product of 3(1) where the 1 will be the identity matrix that corresponds to the order of the matrix we are working on in the function.

Hence, in this example, since A is a 3×3 matrix, we'll be making use of the 3×3 identity matrix.

Hence,

$$f(x) = x^2 - 3x + 3$$

Since A is a matrix;

$$f(A) = A^2 - 3A + 3I$$

$$f(A) = \begin{pmatrix} 1 & 4 & -3 \\ 2 & -3 & 4 \\ 5 & 0 & 1 \end{pmatrix}^2 - 3\begin{pmatrix} 1 & 4 & -3 \\ 2 & -3 & 4 \\ 5 & 0 & 1 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We'll evaluate each of them one by one;

$$\begin{pmatrix} 1 & 4 & -3 \\ 2 & -3 & 4 \\ 5 & 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 4 & -3 \\ 2 & -3 & 4 \\ 5 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 & -3 \\ 2 & -3 & 4 \\ 5 & 0 & 1 \end{pmatrix}$$

We have the product thus expanded;

$$A^2=\begin{pmatrix} 1(1)+4(2)+(-3)(5) & 1(4)+4(-3)+(-3)(0) & 1(-3)+4(4)+(-3)(1) \\ 2(1)+(-3)(2)+(4)(5) & 2(4)+(-3)(-3)+(4)(0) & 2(-3)+(-3)(4)+(4)(1) \\ 5(1)+(0)(2)+(1)(5) & 5(4)+(0)(-3)+(1)(0) & 5(-3)+(0)(4)+(1)(1) \end{pmatrix}$$

$$A^2 = \begin{pmatrix} -6 & -8 & 2 \\ 16 & 17 & -14 \\ 10 & 21 & -14 \end{pmatrix}$$

$3A$ is simply a scalar product;

$$3A = 3\begin{pmatrix} 1 & 4 & -3 \\ 2 & -3 & 4 \\ 5 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3(1) & 3(4) & 3(-3) \\ 3(2) & 3(-3) & 3(4) \\ 3(5) & 3(0) & 3(1) \end{pmatrix}$$

$$3A = \begin{pmatrix} 3 & 12 & -9 \\ 6 & -9 & 12 \\ 15 & 0 & 3 \end{pmatrix}$$

As established already during the studies of identity matrices, it serves as the number 1 in natural numbers; hence, here, to evaluate 3 in matrix form, multiply 3 by the identity matrix; an identity matrix multiplied by any matrix leaves the matrix unchanged. The product below is a scalar multiplication again;

$$3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3(1) & 3(0) & 3(0) \\ 3(0) & 3(1) & 3(0) \\ 3(0) & 3(0) & 3(1) \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Hence, we have $f(A)$ completely as:

$$f(A) = \begin{pmatrix} -6 & -8 & 2 \\ 16 & 17 & -14 \\ 10 & 21 & -14 \end{pmatrix} - \begin{pmatrix} 3 & 12 & -9 \\ 6 & -9 & 12 \\ 15 & 0 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$f(A) = \begin{pmatrix} -6-3+3 & -8-12+0 & 2-(-9)+0 \\ 16-6+0 & 17-(-9)+3 & -14-12+0 \\ 10-15+0 & 21-0+0 & -14-3+3 \end{pmatrix}$$

$$f(A) = \begin{pmatrix} -6 & -20 & 11 \\ 10 & 29 & -26 \\ -5 & 21 & -14 \end{pmatrix}$$

There's no taking many examples on matrix multiplication, just follow it serially, it's very easy to make mistakes during expansion so it's better taken gently and softly;

Having gotten the knowledge of matrix multiplication; we have that the product:

$$A^T A$$

has a rule thus;
When a matrix is multiplied by its transpose, the matrix formed, $A^T A$ is a **symmetric matrix**; that is, a matrix whose row elements are equal to its corresponding column elements; now note that a symmetric matrix is first essentially a matrix which is equal to its transpose. But, a matrix is only equal to its transpose ***only and only if*** its row elements are equal to its corresponding column elements.

So, it's like this; there are basic properties of matrix multiplication;

(i) $AB \neq BA$ (Hence, matrix multiplication unlike addition isn't commutative, changing the order of operation will not yield the same result).

(ii) $A(BC) = (AB)C$

(iii) $(A + B)C = AC + BC$

(iv) $k(AB) = (kA)B = A(kB)$

(v) $AI = IA = A$; identity matrix has that exemption in matrix multiplication, either way of multiplication, it gives the same matrix.

(vi) $(AB)^T = B^T A^T$; we've seen that already;

From our basic rules; let's take note of some things;

Given four matrices; A, B, C and D; to make the following expansions:

(i) $(A + B)(C + D)$
(ii) $(A + B)^2$
(iii) $(A + B)(A - B)$

Now, we have it thus; we'll expand it just like in normal expansion but we'll take cognizance of all the properties of matrix multiplication;

(i) $(A + B)(C + D) = AC + AD + BC + BD$

Now, all the four terms are distinct and they're separate products, remember the order is very important; the pre-multiplier and the post-multiplier;

(ii) $(A + B)^2 = (A + B)(A + B) = A^2 + AB + BA + B^2$

In ordinary operations; $AB = BA$ but that is not so in matrix operations; hence, we have something quite different here; we can't add the two and the expansion is still left as it is. In normal algebraic operation, $AB + BA$ would have been $2AB$ but that isn't done over here.

(iii) $(A + B)(A - B) = A^2 - AB + BA - B^2$

This expansion in normal algebra is called the difference of two squares since $-AB$ and BA will cancel out on in normal algebra since $AB = BA$, however, that doesn't happen here as AB is not always equal to BA except in special cases such as multiplication with identity matrices;

Why the above three instances, well, I'll explain now. They are used in matrix proving since mostly, when proving is done in matrices, you use understanding of these expansions. Hence, the above cases are not meant for cases when you're given matrices to perform operations like we did in Page 54, no point expanding in that type of example as that will cost you to have four cases of multiplication instead of the one case we had by adding them first and then multiplying ***(you can check again if you seem lost),*** these expansion rules are basically meant for matrix proving to understand the relationship between AB and BA and etc.

Now, there is a very important type of matrix based on the rule of matrix multiplication which gives a special type of matrix;

A matrix which when multiplied by itself is still equal to itself, it is called an **idempotent matrix.**

Hence, for an idempotent matrix,

$$A^2 = A$$

It follows that,

$$A^3 = (A^2)(A) = A(A) = A^2 = A$$

Hence,
All positive powers of A are equal to A since $A^2 = A$; as the expansion goes all along, A keeps multiplying A and keeps yielding A. For idempotent matrix:

$$A^n = A$$

If n is a whole number;
Let's see an example of an idempotent matrix.

If $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$, find A^2.

$$A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

We have the product thus A^2 expanded thus;

$$\begin{pmatrix} 2(2)+(-2)(-1)+(-4)(1) & 2(-2)+(-2)(3)+(-4)(-2) & 2(-4)+(-2)(4)+(-4)(-3) \\ (-1)(2)+(3)(-1)+(4)(1) & -1(-2)+(3)(3)+(4)(-2) & -1(-4)+(3)(4)+(4)(-3) \\ 1(2)+(-2)(-1)+(-3)(1) & 1(-2)+(-2)(3)+(-3)(-2) & 1(-4)+(-2)(4)+(-3)(-3) \end{pmatrix}$$

$$A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

As you can see, the product of A and itself still yields the matrix itself. Another easy example of an idempotent matrix is the identity matrix.

Let's move to the very important part of matrices; the determinant of matrices;

DETERMINANTS OF MATRICES

This is a very key aspect of matrices and it delves very deep into the most major applications of the studies of matrices in mathematics and even in the general economic world; we'll deal more with square matrices in this section henceforth as only square matrices operate the rule of determinants. Let's start some brief formalities;

The determinant of a matrix is a special number that can be calculated from a square matrix. It is the scalar, the number associated to every square matrix.

Basically as it were, there isn't any good English per say to give the definition of what a determinant is hence, the weird definition above which gives it as a special number associated from a matrix.

Determinants occur almost throughout mathematics. It is used in representing coefficients of a system of linear equations, and the determinants itself can be used to solve these equations; determinants are used in the Jacobian determinants (we'll still see a brief of that in partial differentiation in this book), and many other aspects we won't need to bring here to keep stories short.

Let's begin real business;
The determinant of a given matrix, A is represented as:

$$\det(A)\,,\det A \text{ or } |A|$$

The simplest case of a determinant is the smallest square matrix, the 1×1 matrix where **the determinant is simply equal to the only element in the matrix.**

Properties of determinants

- The determinant is a real number, and not a matrix;
- The determinants can be a negative number;
- It only exists for square matrix, and hence, any other properties of matrices associated with determinants only exist for square matrix;
- The inverse of a matrix hinges on its determinant; the inverse of a matrix exists only if the determinant exists.

Let's see the second most basic determinant of a matrix, we have seen the determinant of a 1×1 matrix which is equal to the only element in the matrix; let's see an example of that;

- If $A = (4)$; find the $|A|$

Here, the determinant of the matrix being a 1×1 matrix is the only element; the determinant however, isn't expressed as a matrix but a number; hence;

$$|A| = 4$$

Now, the next square matrix is the 2×2 matrix which we'll be seeing the method of its determinant evaluation;

The determinant of the 2×2 matrix is evaluated by the **open scissors technique.**

The open scissors technique goes thus; for a given matrix A;

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The determinant, $|A| = ad - bc$

The idea of calling it a scissors technique is this; the multiplication-subtraction operation is given thus, from top left to bottom right minus top right to bottom left;

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The operation looks like an open scissors; hence, it is called the open scissors technique, the operations move from left downward to right first then subtracting the second diagonal multiplication operation.

Let’s equally see examples in this;

- Evaluate the following determinants;

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}, \begin{vmatrix} 2 & -3 \\ -2 & 1 \end{vmatrix}, \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix}, \begin{vmatrix} 5 & -1 \\ 2 & 3 \end{vmatrix}$$

Cool, let’s evaluate these determinants; the open scissors;

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (4)(1) - (2)(3) = 4 - 6 = -2$$

$$\begin{vmatrix} 2 & -3 \\ -2 & 1 \end{vmatrix} = (2)(1) - (-3)(-2) = 2 - (6) = -4$$

$$\begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} = (2)(3) - (-1)(0) = 6 - 0 = 6$$

$$\begin{vmatrix} 5 & -1 \\ 2 & 3 \end{vmatrix} = (5)(3) - (2)(-1) = 15 - (-2) = 17$$

Let’ see some terms in square matrices that we need in the course of determinants of matrices;

MINORS

Minors are related to elements of a matrix specifically;

The minor of a given element is a number which is the determinant that results from the matrix formed **when the rows and columns** that the element belongs to are deleted;

The notation for the minor of an element of a matrix is M_{ij} where as usual, i is the row and j is the column. It can also be expressed as $\min(a_{ij})$ where a matrix A is been considered. Minors of elements are like the major backbone of the determinants of matrices; let's see how minors are derived from matrices, I'll explain in full details now, you'll get other examples on shorter notes; like I said, matrices don't need much, it is basically nothing difficult to understand, mistakes are just easy to make hence you need to be very careful.

Consider the matrix below;

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

To find the minors of all elements in A;

The elements in this matrix are: a, b, c and d

$\min(a)$ is given by deleting the row and column where a is a member;

$$\min(a) = \begin{pmatrix} \cancel{a} & \cancel{b} \\ \cancel{c} & d \end{pmatrix} = |d| = d$$

$$\min(b) = \begin{pmatrix} \cancel{a} & \cancel{b} \\ c & \cancel{d} \end{pmatrix} = |c| = c$$

$$\min(c) = \begin{pmatrix} \cancel{a} & b \\ \cancel{c} & \cancel{d} \end{pmatrix} = |b| = b$$

$$\min(d) = \begin{pmatrix} a & \cancel{b} \\ \cancel{c} & \cancel{d} \end{pmatrix} = |a| = a$$

We see how we remove the rows and elements the elements belong and then we evaluate the determinants of the matrix remaining; we see an easy situation here where the matrix remaining is merely a 1×1 matrix.

Now, let's get to the more major situations; where we need the minors of a 3×3 matrix;

Consider the matrix;

$$B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

To get the various minors of the elements of B; we'll take it thus;

$$\min(a) = \begin{pmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ \cancel{d} & e & f \\ \cancel{g} & h & i \end{pmatrix} = \begin{vmatrix} e & f \\ h & i \end{vmatrix} = (ei - fh)$$

$$\min(b) = \begin{pmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & \cancel{e} & f \\ g & \cancel{h} & i \end{pmatrix} = \begin{vmatrix} d & f \\ g & i \end{vmatrix} = (di - fg)$$

$$\min(c) = \begin{pmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & e & \cancel{f} \\ g & h & \cancel{i} \end{pmatrix} = \begin{vmatrix} d & e \\ g & h \end{vmatrix} = (dh - eg)$$

$$\min(d) = \begin{pmatrix} \cancel{a} & b & c \\ \cancel{d} & \cancel{e} & \cancel{f} \\ \cancel{g} & h & i \end{pmatrix} = \begin{vmatrix} b & c \\ h & i \end{vmatrix} = (bi - ch)$$

$$\min(e) = \begin{pmatrix} a & \cancel{b} & c \\ \cancel{d} & \cancel{e} & \cancel{f} \\ g & \cancel{h} & i \end{pmatrix} = \begin{vmatrix} a & c \\ g & i \end{vmatrix} = (ai - cg)$$

$$\min(f) = \begin{pmatrix} a & b & \cancel{c} \\ \cancel{d} & \cancel{e} & \cancel{f} \\ g & h & \cancel{i} \end{pmatrix} = \begin{vmatrix} a & b \\ g & h \end{vmatrix} = (ah - bg)$$

$$\min(g) = \begin{pmatrix} \cancel{a} & b & c \\ \cancel{d} & e & f \\ \cancel{g} & \cancel{h} & \cancel{i} \end{pmatrix} = \begin{vmatrix} b & c \\ e & f \end{vmatrix} = (bf - ec)$$

$$\min(h) = \begin{pmatrix} a & \cancel{b} & c \\ d & \cancel{e} & f \\ \cancel{g} & \cancel{h} & \cancel{i} \end{pmatrix} = \begin{vmatrix} a & c \\ d & f \end{vmatrix} = (af - cd)$$

$$\min(i) = \begin{pmatrix} a & b & \cancel{c} \\ d & e & \cancel{f} \\ \cancel{g} & \cancel{h} & \cancel{i} \end{pmatrix} = \begin{vmatrix} a & b \\ d & e \end{vmatrix} = (ae - bd)$$

You can see how the rows and columns are deleted, you don't go ahead doing all these (i.e. drawing the line you're using to delete the row and column) in practical cases, this is just to show you what we're doing over here; the determinants are taken separately and hence, we have the minors of each element given as above;

The matrix of minors;

This is the matrix (a square matrix) where each element is the minor for the number in the original position.

The matrix of minors is gotten by replacing each element in a given matrix by its minor; let's see from the above matrix, B.

For the matrix B considered in the previous section;

$$B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\text{minor}(B) = \begin{pmatrix} \min(a) & \min(b) & \min(c) \\ \min(d) & \min(e) & \min(f) \\ \min(g) & \min(h) & \min(i) \end{pmatrix}$$

Hence, $\text{minor}(B)$;

$$\text{minor}(B) = \begin{pmatrix} ei - fh & di - fg & dh - eg \\ bi - ch & ai - cg & ah - bg \\ bf - ec & af - cd & ae - bd \end{pmatrix}$$

COFACTOR

A cofactor for every element in a matrix is not very different from the minors of the elements; it gives the unique sign notation of the minors of each element in a matrix; the cofactor of elements in a matrix can only differ in the sign of the minors of the matrix but cannot differ totally, there are times the cofactor of an element is the minor of the matrix itself and there are times it'll be the negative of the minor.

Now, the negative-positive sign notation is hinged on the sum of the row and column of the position;

$$a_{11} = 1 + 1 = \text{even}$$
$$a_{12} = 1 + 2 = \text{odd}$$
$$a_{13} = 1 + 1 = \text{even}$$
$$a_{21} = 2 + 1 = \text{odd}$$
$$a_{22} = 2 + 2 = \text{even}$$
$$a_{14} = 1 + 4 = \text{odd}$$

And so on; and so forth;

Every element when the sum of the row and column position is even, the sign is positive, and when the sum is odd, the sign is negative.

The sign notations follow a series of interchange of $+$ and $-$ in every square matrix while the positive sign is always starting the first element, this is because $1 + 1$ is even and the first element is always row 1, column 1;

$$1 \times 1 \text{ matrix: } (+)$$

$$2 \times 2 \text{ matrix: } \begin{pmatrix} + & - \\ - & + \end{pmatrix}$$

$$3 \times 3 \text{ matrix: } \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$4 \times 4 \text{ matrix: } \begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

Hence, cofactor matrices attach signs to minor matrices as shown above;

Hence, for the matrix, B;

$$B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\text{minor}(B) = \begin{pmatrix} ei - fh & di - fg & dh - eg \\ bi - ch & ai - cg & ah - bg \\ bf - ec & af - cd & ae - bd \end{pmatrix}$$

$$\text{cofactor}(B) = \begin{pmatrix} ei - fh & -(di - fg) & dh - eg \\ -(bi - ch) & ai - cg & -(ah - bg) \\ bf - ec & -(af - cd) & ae - bd \end{pmatrix}$$

The matrix above is called the **matrix of cofactors** of a matrix.

That's it about virtually destroying the concept of determinants;

Now, to find the determinant of a matrix, we take **any row or column** in the square matrix and multiply each element by its cofactor and add them; this rule starts from 3×3 matrix and above; we know how to find the determinants of the matrices 2×2 and 1×1 matrices which we've seen above already; however, this method for determinants of the 3×3 matrices and above is called the method of **Laplace expansion** which is the very general method of finding the determinants of all orders of matrices; interestingly, it can be used to even prove the open scissors technique for the square matrices of order 2. Let's see this;

Find the determinant of the matrices below;

$$A = \begin{pmatrix} 1 & -2 & 2 \\ 4 & 2 & -1 \\ 2 & 0 & 3 \end{pmatrix}$$

We take any row or column, I'll be showing you that briefly now using two instances, any row or column taken will yield the same answer; let's use **Row 1** and **Column 2.**

Going along Row 1; we have each element on the row multiplying its corresponding cofactor, you of course know how we find the minor elements and hence yielding the cofactor elements, the cofactor elements will be including signs in our work;

We have three elements in row 1 which are: $1, -2, 2$

By deleting their corresponding rows and columns, their minors are given thus;

$$\min(1) = \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix}$$

$$\min(2) = \begin{vmatrix} 4 & -1 \\ 2 & 3 \end{vmatrix}$$

$$\min(-2) = \begin{vmatrix} 4 & 2 \\ 2 & 0 \end{vmatrix}$$

In the cofactor sign assignment, as we have seen, the **row one** follows the $+\ -\ +$ sequence; hence, we attached these signs to the minors to get the cofactor;

The determinant of A is now found by multiplying each of these elements by their corresponding cofactor as done below:

$$|A| = 1\begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix} - (-2)\begin{vmatrix} 4 & -1 \\ 2 & 3 \end{vmatrix} + 2\begin{vmatrix} 4 & 2 \\ 2 & 0 \end{vmatrix}$$

Now, notice that we have brought the negative sign in the a_{12} cofactor element to come first **for the sake of orderliness** and as you can see (in front of -2), it is still multiplying minor; the negative sign mustn't be omitted because the cofactor sign notation must always hold else the determinant you calculate will be **totally and downright wrong**.

$$|A| = 1[(2)(3) - (-1)(0)] + 2[(4)(3) - (-1)(2)] + 2[(4)(0) - (2)(2)]$$

$$|A| = (6) + 2(14) + 2(-4) = 6 + 28 - 8$$

$$|A| = 26$$

SO NEXT!

Going along Column 2; we have each element on the column multiplying its cofactor;

We have three elements in column 2 which are: $-2, 2, 0$

By deleting their corresponding rows and columns, their minors are given thus;

$$\min(1) = \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix}$$

$$\min(2) = \begin{vmatrix} 4 & -1 \\ 2 & 3 \end{vmatrix}$$

$$\min(-2) = \begin{vmatrix} 4 & 2 \\ 2 & 0 \end{vmatrix}$$

In the cofactor sign assignment, as we have seen, the **column two** follows the $- \; + \; -$ (from top to down) sequence; hence, we attached these signs to the minors to get the cofactor;

$$|A| = -(-2)\begin{vmatrix} 4 & -1 \\ 2 & 3 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix}$$

The determinant of A is now found by multiplying each of these elements by their corresponding cofactor as done below:

Now, notice that we have brought the negative sign in the a_{12} cofactor element to come first **for the sake of orderliness** and as you can see (in front of -2), it is still multiplying minor; the same occurs too for the element 0; always remember the negative sign on the cofactor.

$$|A| = 2[(4)(3) - (-1)(2)] + 2[(1)(3) - (2)(2)] + 0$$

Notice that there is no point wasting time evaluating the cofactor for the element zero whatever value it takes, it'll end up as zero. Generally, in making choice for the row or column to go along when evaluating the determinant of a matrix, it is good to go for the row or column that has the most zeros since it saves time.

$$|A| = 2(14) + 2(-1) = 28 - 2$$

$$|A| = 26$$

Hence, wherever you go through, the determinant still stands, you just must use a complete row or a complete column; don't go there testing for two options in ideal situations (just use one and make sure you're extra careful to avoid mistakes), this is just for a proving sake; and also, never forget the cofactor sign notation as you can see the negative signs added in the multiplication involving elements a_{12} and a_{32}.

So, let’s see this;

- Using the Laplace expansion; show that for a 2×2 matrix shown below;

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Show that its determinant is given by;

$$|B| = ad - bc$$

So, let’s see; going through **Row 1**; multiplying each element by its cofactor **(you should be well used to the cofactor thing now and the sign notation and all, always let the sign be on your mind)**, we have;

$$|B| = a|d| - b|c| = ad - bc$$

Notice that in the b_{12} position, the negative sign is included.
PROVED! As short as that!

Let's just see some few more examples in determinants operation; just like everything in matrix, it is nothing difficult, you just must be careful;

Before then, let's introduce this for formalities; it is a method that is used only for the determinant of a 3×3 matrix; it is done as shown below; for a 3×3 matrix; it is called the **Sarrus rule;**

Let me give the comprehensive ***story definition*** of the Sarrus Rule; the sum of the products of three diagonal north-west to south-east lines of matrix elements, minus the sum of the products of three diagonal south-west to north-east lines of elements, when the copies of the first two columns of the matrix are written beside it as in the illustration; ***boring story, isn't it? Just know it.***

For the matrix; A;

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

To find $|A|$;

$$|A| = \begin{matrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{matrix}$$

$$|A| = (aei + bfg + cdh) - (ceg + afh + bdi)$$

We repeat the first two columns on the right side of the 3×3 matrix and evaluate the product as shown above; it is meant only for the 3×3 matrix, straight lines for the ones we are adding and dotted lines for the ones we are subtracting.;

Let's also see properties of matrix determinants before we see other examples;

- The determinant of the identity matrix which we've seen in the previous sections above is 1;

$$|I| = 1$$

- The determinant of the transpose of a matrix is equal to the determinant of the matrix.

$$|A^T| = |A|$$

- If two rows (or columns) are interchanged in a matrix, the determinant sign only changes but the absolute value doesn’t;

- If a matrix has two identical rows or columns, then the determinant of such matrix is zero;

- A square matrix which has it determinant equal to zero is called a **singular matrix**;

- For two matrices; the determinant of their product is the product of their determinants;

$$|AB| = |A||B|$$

- ***A special determinant exists for triangular matrices; the determinant is equal to the product of the elements on the main diagonal; whether an upper or a lower triangular matrix;***

There are other properties of matrices determinants but let's just take these ones as this;

See; let's just take everything together before we go on, I'm assuming strongly that you're still writing everything you're learning;

The Adjugate or Adjoint matrix;

The Adjugate or Adjoint matrix is the transpose of the matrix of cofactors of a matrix, i.e. the matrix formed by replacing each element with its cofactor;

$$\operatorname{adj}(A) = [\operatorname{cofactor}(A)]^T$$

In terms of the adjugate (adjoint) matrix; we have this equation;

$$|A|I = A[\mathrm{adj}(A)] = [\mathrm{adj}(A)]A$$

The product, either way of a matrix and its adjugate yields the product of the determinant of the matrix and the identity matrix;
From the properties of matrices which we listed above; the determinant of an identity matrix is 1;

Now, let’s derive a very interesting part of matrix from the relationship we just stated above;

$$|A|I = A[\mathrm{adj}(A)] = [\mathrm{adj}(A)]A$$

Take the first two since the third equality is same as the second

$$|A| \times I = A \times [\mathrm{adj}(A)]$$

Dividing through by $|A|$;

$$\frac{|A|I}{|A|} = \frac{A[\operatorname{adj}(A)]}{|A|}$$

$$I = \frac{A[\operatorname{adj}(A)]}{|A|}$$

Now, let's split the right hand side since it's all about multiplication;

$$I = A \times \frac{[\operatorname{adj}(A)]}{|A|}$$

Now, in ordinary arithmetic, any number that multiplies another number to yield 1 is called the inverse of such number;

For example;

$$\frac{1}{2} \text{ is the inverse of 2 since } 2 \times \frac{1}{2} = 1$$

Hence; since the identity matrix serves as the number 1 in matrices, then, here; the expression:

$$\frac{[\mathrm{adj}(A)]}{|A|} \text{ is called the inverse of a matrix}$$

Hence, **the inverse of a matrix** is the given by dividing the adjugate of the matrix by its determinant; so, since the determinant is a denominator in the inverse matrix, then, the condition for the existence of a matrix inverse is that the determinant $\neq 0$;

Therefore, for a matrix, A, where, $\det A \neq 0$; then the determinant exists.

Hence, **the inverse of a matrix doesn't exist if the determinant of the matrix is equal to zero.**

The inverse is:

$$A^{-1} = \frac{\mathrm{adj}(A)}{|A|}$$

Taking the inverse of the inverse of a matrix simply returns the matrix to its original state; meaning:

$$(A^{-1})^{-1} = A$$

Other properties of inverse of a matrix are given by:

$$(A + B)^{-1} \neq A^{-1} + B^{-1}$$

<u>The inverse of a sum **is not equal** to the sum of their individual transpose.</u>

$(cA)^{-1} = cA^{-1}$; where c is scalar;

<u>The inverse of a matrix multiplied by a scalar is equal to the scalar multiplied by the inverse of the matrix.</u>

$$(AB)^{-1} = B^{-1}A^{-1}$$

Also, a very important property of matrix inverses, <u>the inverse of a product is the equal to the product of their individual inverses in the reverse order;</u>

Let's rush this as soon as possible; let's introduce another special type of matrix; the orthogonal matrix;

An orthogonal matrix is a square matrix whose transpose is equal to its inverse; it is a special type of matrix related to both the transpose of a matrix and the inverse of a matrix;

Hence, for an orthogonal matrix; by definition, an orthogonal matrix is a square matrix whose transpose is equal to the matrix's inverse.

Hence, when the transpose of an orthogonal matrix multiplies its matrix, it yields the identity matrix.

$$AA^T = A^T A = I$$

For orthogonal matrices;

$$A^T = A^{-1}$$

Since the transpose matrix will be acting as the inverse matrix; and an inverse when multiplying its matrix yields an identity matrix;

Please let's still touch something extra; **the trace of a matrix;**

The trace of a matrix is also a unique operation for square matrices too; ***the trace of a matrix is the sum of all elements on the main diagonal (the diagonal from the upper left to the lower right);*** we already know what the main diagonal is;

The properties of the trace of a matrix for all square matrices; A, B is;

The trace of a sum of two square matrices is equal to the sum of their traces;

$$tr(A + B) = tr(A) + tr(B)$$

The trace of a matrix is equal to the trace of its transpose;

$$tr(A) = tr(A^T)$$

For two matrix product; even though $AB \neq BA$; their traces are equal;

$$tr(AB) = tr(BA)$$

The trace of a matrix is also known as the Spur of a matrix.

Let's now see plenty examples in all these we've considered;

- Show that $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ is a singular matrix;

We evaluate the determinant since the determinant of a singular matrix is zero; that is the closest test possible we can think about for this question; hence;

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = (1)(4) - (2)(2) = 4 - 4 = 0$$

Looked too short? No problem, that is the only thing that can be done about that question; next?

- Use the Laplace expansion method to show that the Sarrus method of the determinants of 3×3 matrix is valid, using the primitive matrix;

$$|X| = \begin{bmatrix} x & y & z \\ a & b & c \\ w & u & d \end{bmatrix}$$

Too much stories, the long and short of the whole thing is to find the determinant of the above matrix using the Sarrus method and show that it is also equal to the general method of Laplace expansion;

So, cool, using the Sarrus method, we repeat the first two columns on the right and apply the Sarrus diagonal operations as we saw above;

To find $|X|$;

$$|X| = \begin{matrix} x & y & z & x & y \\ a & b & c & a & b \\ w & u & d & w & u \end{matrix}$$

$$|X| = xbd + ycw + zau - (zbw + xcu + yad)$$

We can decide to expand if we wish;

$$|X| = xbd + ycw + zau - zbw - xcu - yad$$

Then using the Laplace expansion method, going along row 1, we have the determinant as;

$$|X| = x\begin{vmatrix} b & c \\ u & d \end{vmatrix} - y\begin{vmatrix} a & c \\ w & d \end{vmatrix} + z\begin{vmatrix} a & b \\ w & u \end{vmatrix}$$

$$x[bd - cu] - y[ad - cw] + z[au - bw]$$

Expanding;

$$xbd - xcu - yad + ycw + zau - zbw$$

Rearranging;

$$xbd + ycw + zau - zbw - xcu - yad$$

Hence, both methods yield the same solution which shows that the Sarrus rule is a valid method for evaluating 3×3 determinants;

- Show that the inverse of the matrix: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by;

$$\frac{1}{ab - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

So, from the rule of inverse; we first need the matrix of cofactors which of course emanates from the matrix of minors which of course we can still remember the rule of deleting the row and column an element belongs;

The minors… … … … …

$$\min(a) = |d| = d$$
$$\min(b) = |c| = c$$
$$\min(c) = |b| = b$$
$$\min(d) = |a| = a$$

Hence;

$$\text{minor}(A) = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

From the cofactor sign notations for the 2×2 matrices;

$$\begin{pmatrix} + & - \\ - & + \end{pmatrix}$$

Hence,

$$\text{cofactor}(A) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

The adjoint matrix is the transpose of the cofactor matrix; hence;

$$\operatorname{adj}(A) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}^T = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The determinant of the matrix is simply given by the open scissors;

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\text{But } A^{-1} = \frac{\operatorname{adj}(A)}{|A|}$$

Hence, here;

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Note,

The above is for clarity’s sake as the two expressions below are the same thing, placing a matrix at the numerator of a fraction won’t be clear enough hence, it is expressed as multiplying by the inverse of the determinant which doesn’t change the expression;

$$\frac{\operatorname{adj}(A)}{|A|} = \operatorname{adj}(A) \times \frac{1}{|A|}$$

- Show that the matrix B is a singular matrix;

$$B = \begin{pmatrix} 2 & 3 & -2 \\ 1 & 0 & -1 \\ -5 & 4 & 5 \end{pmatrix}$$

For a singular matrix, we need its determinant to be zero to prove it, hence, let’s check it;

$$|B| = \begin{vmatrix} 2 & 3 & -2 \\ 1 & 0 & -1 \\ -5 & 4 & 5 \end{vmatrix}$$

$$|B| = 2\begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix} - 3\begin{vmatrix} 1 & -1 \\ -5 & 5 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ -5 & 4 \end{vmatrix}$$

$$|B| = 2[(0)(5) - (-1)(4)] - 3[(1)(5) - (-1)(-5)] \\ - 2[(1)(4) - (0)(5)]$$

$$|B| = 2(4) - 3(0) - 2(4) = 8 - 0 - 8 = 0$$

Hence, it is proved that the matrix is a singular matrix;

- Given the two matrices; A and E below;

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -3 & 0 \\ 2 & 4 & 1 \end{pmatrix} \quad and \quad E = \begin{pmatrix} 1 & -3 & 2 \\ 6 & 2 & 1 \\ -3 & 2 & 1 \end{pmatrix}$$

Show that the identity $|AE| = |A||E|$ is a valid one.

This is the rule of the determinant of a product being equal to the product of their determinants, will be quite a lengthy question, let's do it though;

$$AE = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -3 & 0 \\ 2 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & -3 & 2 \\ 6 & 2 & 1 \\ -3 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1(1) + (-1)(6) + 2(-3) & 1(-3) + (-1)(2) + 2(2) & 1(2) + (-1)(1) + 2(1) \\ 2(1) + (-3)(6) + 0(-3) & 2(-3) + (-3)(2) + 0(2) & 2(2) + (-3)(1) + 0(1) \\ 2(1) + 4(6) + 1(-3) & 2(-3) + 4(2) + 1(2) & 2(2) + 4(1) + 1(1) \end{pmatrix}$$

$$AE = \begin{pmatrix} -11 & -1 & 3 \\ -16 & -12 & 1 \\ 23 & 4 & 9 \end{pmatrix}$$

Hence, we have AE;

$$|AE| = \begin{vmatrix} -11 & -1 & 3 \\ -16 & -12 & 1 \\ 23 & 4 & 9 \end{vmatrix}$$

$$|AE| = -11\begin{vmatrix} -12 & 1 \\ 4 & 9 \end{vmatrix} - (-1)\begin{vmatrix} -16 & 1 \\ 23 & 9 \end{vmatrix} + 3\begin{vmatrix} -16 & -12 \\ 23 & 4 \end{vmatrix}$$

$$|AE| = -11(-112) + 1(-167) + 3(212)$$

$$|AE| = 1232 - 167 + 636 = 1701$$

Then, proving the second identity; $|A||E|$, we take the determinants separately and multiply them;

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix}$$

$$|A| = 1\begin{vmatrix}-3 & 0\\ 4 & 1\end{vmatrix} - (-1)\begin{vmatrix}2 & 0\\ 2 & 1\end{vmatrix} + 2\begin{vmatrix}2 & -3\\ 2 & 4\end{vmatrix}$$

$$|A| = 1[(-3)(1) - (0)(4)] + 1[(2)(1) - (0)(2)] + 2[(2)(4) - (-3)(2)]$$

$$|A| = 1(-3) + 1(2) + 2(14) = 27$$

$$|E| = \begin{vmatrix}1 & -3 & 2\\ 6 & 2 & 1\\ -3 & 2 & 1\end{vmatrix}$$

$$|E| = 1\begin{vmatrix}2 & 1\\ 2 & 1\end{vmatrix} - (-3)\begin{vmatrix}6 & 1\\ -3 & 1\end{vmatrix} + 2\begin{vmatrix}6 & 2\\ -3 & 2\end{vmatrix}$$

$$|E| = 1[(2)(1) - (1)(2)] + 3[(6)(1) - (1)(-3)] + 2[(6)(2) - (2)(-3)]$$

$$|E| = 1(0) + 3(9) + 2(18) = 63$$

Hence;

$$|A||E| = 27 \times 63 = 1701$$

Hence, we have confirmed what we need to confirm;

- Show that the determinant of an orthogonal matrix is ± 1;

Cool,

This is a very tricky test of our knowledge of matrix multiplication and what an orthogonal matrix is and of course, our knowledge of properties of matrix determinants;

Let’s go; we won’t be drawing any matrix here; we’ll simply work strictly with matrix properties;

For an orthogonal matrix, A;

$$AA^T = I$$

Hence, taking determinants of both sides;

$$\det(AA^T) = \det(I)$$

From the rule we just confirmed in the just previous example; the determinant of a product is equal to the product of their determinants; hence, this is what we have here; we'll split the determinants on the LHS; ***never mind, I've used the second notation for the determinants (det) so it doesn't get too passive;***

$$\det(A \times A^T) = \det(I)$$

$$\det(A) \times \det(A^T) = \det(I)$$

Now, from the knowledge of identity matrix, the determinant of an identity matrix is 1; hence;

$$\det(I) = 1$$

$$\det(A) \times \det(A^T) = 1$$

Also from determinant properties, the determinant of a transpose is equal to the determinant of the matrix itself;

Hence,

$$\det(A) = \det(A^T)$$

Hence, the whole thing becomes;

$$\det(A) \times \det(A) = 1$$

$$(\det(A))^2 = 1$$

Taking square roots of both sides;

$$\det(A) = \sqrt{1} = \pm 1$$

I'm sure you know $\pm$ means plus or minus which means -1 or 1; we have another successful prove; **it is, in fact, a rule, that the determinant of all orthogonal matrices is $+1$ or -1**.

- The determinant of a triangular matrix is given by the product of the elements of its main diagonal; prove this using the primitive matrix:

$$|X| = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

This as a reminder is an upper triangular matrix, that's by the way though;

From the rule of the special determinants of triangular matrices which is stated in the question itself;

$$|X| = \begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{vmatrix} = a \times d \times f = adf$$

Now, using the original method of Laplace expansion, along Row 1;

$$|X| = \begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{vmatrix}$$

$$a\begin{vmatrix} d & e \\ 0 & f \end{vmatrix} - b\begin{vmatrix} 0 & e \\ 0 & f \end{vmatrix} + c\begin{vmatrix} 0 & d \\ 0 & 0 \end{vmatrix}$$

$$a[(d)(f) - (e)(0)] - b[(0)(f) - (e)(0)] + c[(0)(0) - (d)(0)]$$

$$a[df] - b[0] + c[0] = adf$$

Hence, we have proved it, the determinant of triangular matrices is the product of the elements on their main diagonal, again, it's a square matrix something;

- Find the determinant of the matrix given below;

$$B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 9 & 2 & -4 & 0 \\ 10 & 1 & 3 & 17 \end{pmatrix}$$

Lol… … … Someone's getting scared already. Are we going to sleep here today, a 4×4 matrix? Do they want to kill *somebori ni*?

Rest your nerves; as you can see, this is a triangular matrix (a lower one to be specific) and hence, the determinant is the product of the major elements in the main diagonal;

$$|B| = \begin{vmatrix} -1 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 9 & 2 & -4 & 0 \\ 10 & 1 & 3 & 17 \end{vmatrix} = (-1)(3)(-4)(17) = 204$$

EASY! And you were getting scared already!

We’ve not mentioned anything about the trace of a matrix, let’s do that now;

- If: $A = \begin{pmatrix} -1 & 0 & 6 & 1 \\ -2 & 3 & -5 & -2 \\ 9 & 3 & -4 & 3 \\ -3 & 1 & 3 & 17 \end{pmatrix}$, find spur A

Like we said, the trace of a matrix is also called the spur of the matrix; a trace is the sum of the elements on the main diagonal. Hence, here;

$$\text{spur}(A) = -1 + 3 + (-4) + 17 = 15$$

It’s that simple;

- If we have the matrix, Y, given thus;

$$Y = \begin{pmatrix} -6 & 0 & 6 & 1 \\ 0 & 3 & -5 & -2 \\ 0 & 0 & -14 & 3 \\ 0 & 0 & 0 & 11 \end{pmatrix}$$

Find the difference between $|Y|$ and $\mathrm{tr}(Y)$;

You may keep wondering the relationship between determinants and traces; however, just watch it gradually; as much as the trace of a matrix isn't limited to a particular type of matrix alone, a form of determinant is limited to a particular type of matrix alone; the product of the elements on the main diagonal to yield the determinant of triangular matrices; in the above question, we have a triangular matrix (upper);

The difference hence is that the trace of Y which is $\mathrm{tr}(Y)$ is the ***sum*** of the elements on the main diagonal of Y while the determinant, $|Y|$ is the ***product*** of the elements on the main diagonal;

$$\mathrm{tr}(Y) = -6 + 3 + (-14) + 11 = -6$$

$$|Y| = (-6)(3)(-14)(11) = 2772$$

- Find x if:

$$\begin{vmatrix} 1 & x & -4 \\ 5 & 3 & 0 \\ -2 & -4 & 8 \end{vmatrix} = 0$$

Here, we need to just evaluate the determinant as if nothing special (the x amongst the elements) is in the elements and then sort out the obstacle that emanates from the situation; here; along Row 1;

$$1\begin{vmatrix} 3 & 0 \\ -4 & 8 \end{vmatrix} - x\begin{vmatrix} 5 & 0 \\ -2 & 8 \end{vmatrix} + (-4)\begin{vmatrix} 5 & 3 \\ -2 & -4 \end{vmatrix}$$

$$1[(3)(8) - (0)(-4)] - x[(5)(8) - (0)(-2)] - 4[(5)(-4) - (3)(-2)]$$

$$1(24) - x(40) - 4(-14) = 80 - 40x$$

We're told the determinant is equal to zero; hence,

$$80 - 40x = 0$$

Here, very obviously, solving it yields;

$$x = 2$$

- Show that;

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = xy$$

Again, like the previous question; evaluate this determinant just like you will evaluate any; we'll do it and expand it ***extra carefully.***

$$1\begin{vmatrix} 1+x & 1 \\ 1 & 1+y \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 1+y \end{vmatrix} + 1\begin{vmatrix} 1 & 1+x \\ 1 & 1 \end{vmatrix}$$

$$[(1+x)(1+y)-(1)(1)]-[(1)(1+y)-(1)(1)]+[(1)(1)-(1+x)(1)]$$

$$[1+y+x+xy-1]-(1+y-1)+[1-(1+x)]$$

$$x+y+xy-y-x$$

x and y are out of the sum; the determinant is:

$$xy$$

Hence, the prove is completely shown that the determinant is equal to xy.

Let's see some manipulations under the special types of matrices;

- Given that A is an idempotent matrix; show that $I-A$ is also idempotent, where I is the identity matrix.

Right, the first think you'll probably think of is drawing a primitive matrix (sample matrix) and start some subtraction and expansion; well, probably if you are careful enough, you may arrive at an answer. However, that method is far from the appropriate way to approach this kind of a question, rather, you'll need the properties of matrices that you already know, in this case, properties of the idempotent matrix and other matrix properties. ***As I've mentioned several times, proving is mostly done using the properties of the matrix operation and not drawing matrices.***

For an idempotent matrix, the square of the matrix is also equal to itself, hence, since A is idempotent,

$$A^2 = A$$

Hence, for $I - A$, how else do we know whether $(I - A)$ is idempotent except we test for its square;

Hence, test for the square of $I - A$ to see if it'll also be equal to itself;

$$(I - A)^2 = (I - A)(I - A)$$

$$(I - A)^2 = I^2 - IA - AI + A^2$$

From matrix rules, the product **(either way)** of any matrix and the identity matrix is still the matrix itself, hence;

$$IA = AI = A$$

Also, the identity matrix is also idempotent and hence, $I^2 = I$

And from the question, A is idempotent, hence, $A^2 = A$

Hence, we have:

$$(I - A)^2 = I - A - A + A$$

Hence, $-A + A$ cancels out, leaving:

$$(I - A)^2 = I - A$$

Hence, since the square of $(I - A)$ is equal to $I - A$, it is proved that $I - A$ is also idempotent.

- If matrices A and B are idempotent, prove that:
  (i) their sum, $(A + B)$ is only idempotent if, $AB = BA = 0$
  (ii) their difference, $(A - B)$ is only idempotent if, $AB = BA = B$

Wow, looks complicated, but just like the previous question, we'll be making use of matrix properties. We're *gonna* be testing for the two matrices and see the conditions where the square will be equal to the main matrix.

N.B.: This is a piece of your past question so take note of it carefully!

(i) Now, for the first one;

We want to find the conditions such that:

$$(A + B)^2 = (A + B)$$

The above is the condition for a matrix to be idempotent, note carefully that unlike the last question, we're not making a complete prove but looking for conditions for which these matrices will be idempotent.

Consider the analogy, a lion is always a wild animal and hence, it can be proved wild. However, a dog isn't readily a wild animal but can be made wild under certain conditions, and hence, the last question was like proving that the lion is wild while in this question, we want to see the certain condition that'll make the dog wild, i.e. we're looking for conditions for which these matrices will be idempotent.

Expand $(A + B)^2$

$$(A + B)^2 = (A + B)(A + B)$$

$$(A + B)^2 = A^2 + AB + BA + B^2$$

Since A and B are idempotent matrices, we have that;

$$A^2 = A$$
$$B^2 = B$$

Hence,

$$(A + B)^2 = A + AB + BA + B$$

Is there any other further simplification that can be done? No! There is basically no relationship between AB and BA for matrices, and hence, since no simplification is possible, we do this.

For $(A + B)$ to be idempotent,

$$(A + B)^2 = (A + B)$$

Hence, we compare the right hand side of the true value of $(A + B)^2$ with the condition for it to be idempotent, hence, we have;

$(A + AB + BA + B)$ **compared with** $A + B$

Understood that? Go over it again, we know how it'll be idempotent, hence, we want to see the conditions that will force the true value of the square of the matrix to make it equal to itself. We can achieve that by equating the two together and solving.

$$A + AB + BA + B = A + B$$

Both A and B cancels out of the equation, leaving us with:

$$AB + BA = 0$$

Hence, the above is the condition for $(A + B)^2$ to be equal to AB

Simplifying the above, it is possible in two ways; solving the equation directly, we have;

$$\boldsymbol{AB = -BA}$$

Above is the first condition for $(A + B)$ to be idempotent, but we can't find this condition in the question? Back to the same equation, let's see the second way it can be resolved!

$$AB + BA = 0$$

What if both AB and BA are equal to zero (i.e. $AB = BA = 0$) their sum will be:

$$0 + 0 = 0$$

Hence, another condition is for:

$$\boldsymbol{AB = BA = 0}$$

The question states that we should prove that their sum, $(A + B)$ is only idempotent if, $AB = BA = 0$; hence; the prove is the second one;

As a matter of fact, the above are the only two conditions that can make $(A + B)$ idempotent for two idempotent matrices A and B. The second part should make it clearer.

(ii) For the second one;

We want to find the conditions such that (their difference $(A - B)$ is idempotent):

$$(A - B)^2 = (A - B)$$

Expand $(A - B)^2$

$$(A - B)^2 = (A - B)(A - B)$$
$$(A - B)^2 = A^2 - AB - BA + B^2$$

Since A and B are idempotent matrices, we have that;

$$A^2 = A$$
$$B^2 = B$$

Hence,

$$(A - B)^2 = A - AB - BA + B$$

Is there any other further simplification that can be done? No! There is basically no relationship between AB and BA for matrices, and hence, since no simplification is possible, we do this.

For $(A - B)$ to be idempotent,

$$(A - B)^2 = (A - B)$$

Hence, we compare the right hand side of the true value of $(A - B)^2$ with the condition for it to be idempotent, hence, we have;

$A - AB - BA + B$ **compared with** $A - B$

No big deal right? Equate the comparison;

$$A - AB - BA + B = A - B$$

A cancels out of the equation, leaving us with:

$$-AB - BA + B = -B$$

Rearranging,

$$2B = AB + BA$$

Hence, the above is the condition for $(A - B)^2$ to be equal to AB

Simplifying the above, it is also possible in two ways; first, solving the equation directly, we have;

$$\boldsymbol{B = \frac{AB + BA}{2}}$$

Above is the first condition for $(A - B)$ to be idempotent, but we can't find this condition in the question? Back to the same equation, let's see the second way it can be resolved!

$$2B = AB + BA$$

Also, consider a case where AB and BA are equal, we'll be having a case of:

$$AB = BA$$

Hence,

$$AB + BA = AB + AB = BA + BA$$

Hence, we'll have: since;

$$2B = AB + BA$$

It will become:

$$2B = AB + AB$$

Hence,

$$B = \frac{2AB}{2}$$

Finally,

$$B = AB$$

But, for this condition, we assumed:

$$AB = BA$$

Hence,

$$\boldsymbol{AB = BA = B}$$

The question states that we should prove that their difference, $(A - B)$ is only idempotent if, $AB = BA = B$; hence; the prove is the second one;

As a matter of fact, the above are the only two conditions that can make $(A - B)$ idempotent for two idempotent matrices A and B.

Please forgive me if that looked too hard, just calm down and settle down, you'll definitely understand it, that's just part of the toughest matrix SSC106 questions that exist!

- Let A be a square matrix. Prove that if A is idempotent, then $\det A$ is equal to 0 or 1.

You love the proving something right?
Kindly try to love it;
It's actually an interesting stuff, really!

This idempotent matrix seems to be a trouble maker, we've been proving different things from it, it's interesting though, kindly love it too!

As usual, we are not drawing anything; let's start dealing with conditions,

For A, an idempotent matrix;

$$A^2 = A$$

Since its determinants we *wanna* prove, take the determinants of both sides.

$$\det(A^2) = \det A$$

From the rule of determinants, the determinant of a product is the product of their determinants, hence,

$$\det(A \times A) = \det A$$

$$\det A \times \det A = \det A$$

Now, don't jump into cancelling off, that'd yield an incomplete solution!

Calm down, simplify and factorize!

$$[\det A]^2 = \det A$$

Don't be surprised! Anything multiplied twice is the square!

$$[\det A]^2 - \det A = 0$$

Factorize $\det A$;

$$\det A\,[\det A - 1] = 0$$

Hence, just like we solve quadratic equations,

$$\det A = 0 \quad \text{or} \quad \det A - 1 = 0$$

Solving both;

$$\det A = 0 \quad \text{or} \quad \det A = 1$$

PROVED! Short and simple!

- If A and B are two orthogonal square matrices of the same order, prove that their product, either way, is also orthogonal.

WOW! We won't be free from proving *sha*. You better calm down, this is also a piece of your past question, and you sure know you'll understand it when we're through. Normal something!

What are orthogonal matrices?

For an orthogonal matrix, A;

$$AA^T = A^T A = I$$

This implies;

$$A^T = A^{-1}$$

We want to test for their product, either way! Hence, we are to prove that the matrices AB and BA are both orthogonal.

Now, if A and B are orthogonal, it follows that:

$$A^T = A^{-1}$$
$$B^T = B^{-1}$$

To prove that AB and BA are orthogonal, we must test for the value of the transposes of AB and BA;

Hence, for AB; we test for:

$$(AB)^T$$

Now, from transpose rules; we know that the transpose of a product is the reverse product of their individual transposes:

$$(XY)^T = Y^T X^T$$

Hence,
We have:

$$(AB)^T = B^T A^T$$

From the fundamental information we have for A and B that they are orthogonal, we know that:

$$A^T = A^{-1}$$
$$B^T = B^{-1}$$

Hence, by substitution for B^T and A^T, we have;

$$(AB)^T = B^{-1}A^{-1}$$

Also, from inverse rules; we know that the inverse of a product is the reverse product of their individual inverses:

$$(XY)^{-1} = Y^{-1}X^{-1}$$

It follows that:

$$(AB)^{-1} = B^{-1}A^{-1}$$

Hence, we can hence substitute for $B^{-1}A^{-1}$ in our equation to arrive that:

$$(AB)^T = (AB)^{-1}$$

Hence, it is proved that the transpose of AB is equal to its inverse which is the condition for it to be orthogonal.

Going to the next part, proving for BA will not be different at all, same whole process!

For BA; we test for:

$$(BA)^T$$

Now, from transpose rules; we know that the transpose of a product is the reverse product of their individual transposes:

$$(AB)^T = B^T A^T$$

Hence,
We have:

$$(BA)^T = A^T B^T$$

From the fundamental information we have for A and B that they are orthogonal, we know that:

$$A^T = A^{-1}$$
$$B^T = B^{-1}$$

Hence, by substitution for A^T and B^T, we have;

$$(BA)^T = A^{-1} B^{-1}$$

Also, from inverse rules; we know that the inverse of a product is the reverse product of their individual inverses:

$$(XY)^{-1} = Y^{-1}X^{-1}$$

It follows that:

$$(BA)^{-1} = A^{-1}B^{-1}$$

Hence, we can hence substitute for $A^{-1}B^{-1}$ in our equation to arrive that:

$$(BA)^{T} = (BA)^{-1}$$

Hence, it is proved that the transpose of BA is equal to its inverse which is the condition for it to be orthogonal.

I GUESS WE'VE HAD ENOUGH OF PROVING! LET'S REFRESH OURSELVES WITH ANOTHER MAIN PART OF MATRICES.

Let's move to part of the most major applications of matrices at the elementary level, **the system of linear equations also known as simultaneous equations;**

SOLUTION OF A SYSTEM OF LINEAR EQUATIONS USING MATRIX DETERMINANTS

Matrices are useful in solving simultaneous equations in several ways; the Crammer's rule, method of reducing matrices, matrix inverse multiplication; In the context of SSC106, we'll be looking at the two methods which are these two methods:

- The Crammer's rule;
- The inverse multiplication method;

Now, there is nothing special in these two methods; we have all we need to know in these two above methods which are the determinants and matrix inverse; those are the two basic things we need in both methods so we should be through in a few moments; however, like I have said severally in this chapter, the major problem is that making mistakes are very possible, even the author of the SSC106 way makes mistakes while solving matrix questions when he rushes it, and hence, firstly, extra-carefulness and then rechecking in matrices is very key;

THE CRAMMER'S RULE;

Cool, this is the method of solving simultaneous equations using our knowledge of the determinants of matrices;

Now, let's drop the stories on the table; here are the rules of using this;

All the variables that the system of equations base on must be on the left hand side of the equation and the corresponding solutions on the right hand side of the equation; hence, in cases the equation is not properly arranged, it must be rearranged unlike in the normal case of simultaneous equations solving with algebra where it can be sorted out straight without full rearrangement, it must be well rearranged here;

I guess that's the only rule; let's see how we use the Crammer's rule with this illustration;

Consider the following system of equations;

$$\begin{aligned} ax + by &= u \\ cx + dy &= v \end{aligned}$$

We have a system of simultaneous equations on two variables, x and y with each equation having a solution on its right hand side; we'll be evaluating some determinants to solve this;

So, the first determinant we'll be finding is the determinant of the coefficients of the variables in the order they have been arranged; that is denoted as the first and basic determinant, the determinant of the whole system of equations, denoted as Δ.

Hence, in the above;

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Now, to get each variable, what we do is to replace the column each variable occupies in the system of solution with the column matrix of the solutions of the various equations that form the system, hence, that shows how serious it is to properly arrange a system of equation when dealing with Crammer's rule; hence, over here, we have:

To find x, we'll replace the column of x in the first and basic matrix with the column of the solutions as they have been correspondingly arranged; this is to find, Δ_x, the determinant of the first variable, x, hence, each variable in the equation must be given its own column in the equation such that it can be replaced by the solutions of the equations to solve for its determinant, here, we'll have;

$$\Delta_x = \begin{vmatrix} u & b \\ v & d \end{vmatrix}$$

In the same way, to find y, we need to evaluate for its determinant by replacing its column with the column of the solutions and solve for the determinants; here; we'll have;

$$\Delta_y = \begin{vmatrix} a & u \\ c & v \end{vmatrix}$$

So, to find the values of the variable that satisfy the equation; we'll have it thus;

$$x = \frac{\Delta_x}{\Delta} \qquad\qquad y = \frac{\Delta_y}{\Delta}$$

And so on for as many variables as you have; relax, for SSC106, it cannot be more than three variables;

So, however, for as many variables as possible, it's nothing different, each variable must have its column after the equation has been well arranged and then the whole thing solved for with all determinants evaluated, let's see this example;

- Solve the following system of simultaneous equations.

$$2a - 3b = 5$$
$$a - 7b = -3$$

Alright, very easy, this one has been fully arranged as it is the best way it can be arranged and hence, we can proceed to find our determinants;

We can see the coefficients of the variable which we need clearly here; the variables that determine these equations are a and b and hence, we'll be solving for them, for the first determinant, we have it thus;

$$\Delta = \begin{vmatrix} 2 & -3 \\ 1 & -7 \end{vmatrix}$$

$$\Delta = (2 \times -7) - (-3 \times 1) = -14 + 3 = -11$$

Okay, we proceed to find the determinant for the first variable; the variable, a

To find, Δ_a, we'll be replacing the column that contain the coefficients of a with the solutions of the equations;

$$\Delta_a = \begin{vmatrix} 5 & -3 \\ -3 & -7 \end{vmatrix}$$

$$\Delta_a = (5 \times -7) - -(-3 \times -3) = -35 - 9 = -44$$

To find, Δ_b, we'll be replacing the column that contain the coefficients of a with the solutions of the equations;

$$\Delta_b = \begin{vmatrix} 2 & 5 \\ 1 & -3 \end{vmatrix}$$

$$\Delta_b = (2 \times -3) - (5 \times 1) = -6 - 5 = -11$$

Notice that as we replace a column, all other columns maintain their positions, when we replace the column of x, the column of y is in place, same thing happens as we replace the column of y, the column of x is back in place! That's simple right?

It's no different when it comes to three variable systems of equations too!

So, here; we have;

$$a = \frac{\Delta_a}{\Delta} = \frac{-44}{-11} = 4$$

$$b = \frac{\Delta_b}{\Delta} = \frac{-11}{-11} = 1$$

Hence, the solution of the equation is $a = 4$ and $b = 1$;

Let's see examples in three variables;

- Solve the following system of simultaneous equations;

$$\begin{aligned} 3x + 4y + 5z &= 4 \\ 2x - 3y + 3z &= 8 \\ 2x + 2y - 4z &= 4 \end{aligned}$$

Nothing different! This one has also been fully arranged and hence, we can proceed to find our determinants;

We can see the coefficients of the variable which we need clearly here; the variables that determine these equations are x, y and z and hence, we'll be solving for them, for the first determinant, we have it thus;

We make our first determinant; Δ

$$\Delta = \begin{vmatrix} 3 & 4 & 5 \\ 2 & -3 & 3 \\ 2 & 2 & -4 \end{vmatrix}$$

$$3\begin{vmatrix} -3 & 3 \\ 2 & -4 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ 2 & -4 \end{vmatrix} + 5\begin{vmatrix} 2 & -3 \\ 2 & 2 \end{vmatrix}$$

$$3[(-3)(-4) - (3)(2)] - 4[(2)(-4) - (3)(2)] + 5[(2)(2) - (2)(-3)]$$

$$3[6] - 4[-14] + 5[10] = 124$$

For Δ_x, replace the column of x with the column matrix of the solutions, the other variables still maintain their columns here;

$$\Delta_x = \begin{vmatrix} 4 & 4 & 5 \\ 8 & -3 & 3 \\ 4 & 2 & -4 \end{vmatrix}$$

$$4\begin{vmatrix} -3 & 3 \\ 2 & -4 \end{vmatrix} - 4\begin{vmatrix} 8 & 3 \\ 4 & -4 \end{vmatrix} + 5\begin{vmatrix} 8 & -3 \\ 4 & 2 \end{vmatrix}$$

$$4[(-3)(-4) - (3)(2)] - 4[(8)(-4) - (3)(4)] + 5[(8)(2) - (4)(-3)]$$

$$4[6] - 4[-44] + 5[28] = 340$$

For Δ_y, replace the column of y with the column matrix of the solutions, the column of x is back in place as you can see;

$$\Delta_y = \begin{vmatrix} 3 & 4 & 5 \\ 2 & 8 & 3 \\ 2 & 4 & -4 \end{vmatrix}$$

$$3\begin{vmatrix} 8 & 3 \\ 4 & -4 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ 2 & -4 \end{vmatrix} + 5\begin{vmatrix} 2 & 8 \\ 2 & 4 \end{vmatrix}$$

$$3[(8)(-4) - (3)(4)] - 4[(2)(-4) - (3)(2)] + 5[(2)(4) - (8)(2)]$$

$$3[-44] - 4[-14] + 5[-8] = -116$$

For Δ_z, replace the column of z with the column matrix of the solutions;

$$\Delta_z = \begin{vmatrix} 3 & 4 & 4 \\ 2 & -3 & 8 \\ 2 & 2 & 4 \end{vmatrix}$$

$$3\begin{vmatrix} -3 & 8 \\ 2 & 4 \end{vmatrix} - 4\begin{vmatrix} 2 & 8 \\ 2 & 4 \end{vmatrix} + 4\begin{vmatrix} 2 & -3 \\ 2 & 2 \end{vmatrix}$$

$$3[(-3)(4)-(8)(2)]-4[(2)(4)-(8)(2)]+4[(2)(2)-(2)(-3)]$$

$$3[-28]-4[-8]+4[10]=-12$$

Hence, normally; we know how to find all the variables after we have successfully gotten the determinants;

Hence,

$$x=\frac{\Delta_x}{\Delta}=\frac{340}{124}=\frac{85}{31}$$

$$y=\frac{\Delta_y}{\Delta}=\frac{-116}{124}=-\frac{29}{31}$$

$$z=\frac{\Delta_z}{\Delta}=\frac{-12}{124}=-\frac{3}{31}$$

Surprised? Don't be, it's not all the time that solutions of equations are whole numbers, fractions are well allowed too!

Let's see an example on how we must firstly rearrange our equation, so it won't look as if I just feel like hammering a baseless point.

- Using Crammer's rule, solve the following system of equations;

$$2x_3 - 3x_2 - x_1 = 7$$
$$3x_1 + x_2 + 2x_3 = -5$$
$$x_2 + x_3 + 2 = 0$$

Fine, a system of equations on our hands, let's trash this out in a moment;

Now, we firstly need to rearrange these equations such that the corresponding variables come first, second and third in all the given equations and the solutions on the right hand side;

So, let's use the pattern of:

$$x_1 x_2 x_3$$

Hence, the equations are rearranged thus;

$$-x_1 - 3x_2 + 2x_3 = 7$$
$$3x_1 + x_2 + 2x_3 = -5$$
$$x_2 + x_3 = -2$$

Now, you may keep wondering *how far* with the last equation, it's simple though, the coefficient of x_1 is simply zero; got that?

Hence, we can form our determinant here, our first determinant;

$$\Delta = \begin{vmatrix} -1 & -3 & 2 \\ 3 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix}$$

$$-1\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} - (-3)\begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix}$$

$$-1[(1)(1) - (2)(1)] + 3[(3)(1) - (2)(0)] + 2[(3)(1) - (1)(0)]$$

$$-1(-1) + 3(3) + 2(3) = 16$$

For Δ_{x_1}, replace the column of x_1 with the column matrix of the solutions, the other variables still maintain their columns here;

$$\Delta_{x_1} = \begin{vmatrix} 7 & -3 & 2 \\ -5 & 1 & 2 \\ -2 & 1 & 1 \end{vmatrix}$$

$$7\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} - (-3)\begin{vmatrix} -5 & 2 \\ -2 & 1 \end{vmatrix} + 2\begin{vmatrix} -5 & 1 \\ -2 & 1 \end{vmatrix}$$

$$7[(1)(1) - (2)(1)] + 3[(-5)(1) - (2)(-2)] + 2[(-5)(1) - (1)(-2)]$$

$$7[-1] + 3[-1] + 2[-3] = -16$$

For Δ_{x_2}, replace the column of x_2 with the column matrix of the solutions, the column of x_1 is back in place as you can see;

$$\Delta_{x_2} = \begin{vmatrix} -1 & 7 & 2 \\ 3 & -5 & 2 \\ 0 & -2 & 1 \end{vmatrix}$$

$$-1\begin{vmatrix} -5 & 2 \\ -2 & 1 \end{vmatrix} - 7\begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & -5 \\ 0 & -2 \end{vmatrix}$$

$$-1[(-5)(1) - (2)(-2)] - 7[(3)(1) - (2)(0)] + 2[(3)(-2) - (-5)(0)]$$

$$-1[-1] - 7[3] + 2[-6] = -32$$

For Δ_{x_3}, replace the column of x_3 with the column matrix of the solutions;

$$\Delta_{x_3} = \begin{vmatrix} -1 & -3 & 7 \\ 3 & 1 & -5 \\ 0 & 1 & -2 \end{vmatrix}$$

$$-1\begin{vmatrix} 1 & -5 \\ 1 & -2 \end{vmatrix} - (-3)\begin{vmatrix} 3 & -5 \\ 0 & -2 \end{vmatrix} + 7\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix}$$

$$-1[(1)(-2)-(-5)(1)]+3[(3)(-2)-(-5)(0)]+7[(3)(1)-(1)(0)]$$

$$-1[3]+3[-6]+7[3]=0$$

Hence, normally; we know how to find all the variables after we have successfully gotten the determinants;

Hence,

$$x_1=\frac{\Delta_{x_1}}{\Delta}=\frac{-16}{16}=-1$$

$$x_2=\frac{\Delta_{x_2}}{\Delta}=\frac{-32}{16}=-2$$

$$x_3=\frac{\Delta_{x_3}}{\Delta}=\frac{0}{16}=0$$

Decent answers. They're definitely correct answers!

So, the next part is the section of matrix multiplication and we draw the curtain on matrices;

THE INVERSE MATRIX MULTIPLICATION METHOD

This method is very straightforward and the stories we said in the concept of Crammer's rule will do us a whole lot of good here; when using the method of matrix multiplication, the whole matrix must be well and neatly arranged as well; with each variable having its position; afterwards, the system of equations is expressed using matrix multiplication following the rule below;

For a given system of equations; the product of the matrix of coefficients and the column matrix of the corresponding variables is equal to the column matrix of the solutions to the separate simultaneous equations;

In a two-variable system like this;

$$ax + by = u$$
$$cx + dy = v$$

The equivalent matrix equation is;

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

Afterwards, from the knowledge of matrix inverses; both sides are **pre-multiplied** by the inverse of the matrix of coefficients; afterwards the matrix is eliminated from the equation leaving an identity matrix which has a determinant of 1; hence, we're left with an ideal two matrix equality, where we can see the values of the variable from the rule of equal matrices;

As explained above; we'll have;

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} u \\ v \end{pmatrix}$$

The product on the right hand side will also yield a column matrix and hence, the two matrices can be equated to find the two variables;

Examples;

- Solve the system of equations using matrix multiplication;

$$2a - 3b = 5$$
$$a - 7b = -3$$

Cool, here; we have;

$$\begin{pmatrix} 2 & -3 \\ 1 & -7 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

Nothing more; our next aim is to find the inverse of $\begin{pmatrix} 2 & -3 \\ 1 & -7 \end{pmatrix}$

$$\begin{pmatrix} 2 & -3 \\ 1 & -7 \end{pmatrix}^{-1};$$

Matrix of cofactors; minors first;

$$min(2) = |-7| = -7$$

$$min(-3) = |1| = 1$$

$$min(1) = |-3| = -3$$

$$min(-7) = |2| = 2$$

Hence,

$$\text{minor}\begin{pmatrix} 2 & -3 \\ 1 & -7 \end{pmatrix} = \begin{pmatrix} -7 & 1 \\ -3 & 2 \end{pmatrix}$$

Hence, from cofactor sign notation; we have:

$$\begin{pmatrix} + & - \\ - & + \end{pmatrix}$$

Hence,

$$\text{cofactor}\begin{pmatrix} 2 & -3 \\ 1 & -7 \end{pmatrix} = \begin{pmatrix} -7 & -1 \\ -(-3) & 2 \end{pmatrix} = \begin{pmatrix} -7 & -1 \\ 3 & 2 \end{pmatrix}$$

The adjoint is the transpose of the cofactor;

$$\text{adj}\begin{pmatrix} 2 & -3 \\ 1 & -7 \end{pmatrix} = \begin{pmatrix} -7 & -1 \\ 3 & 2 \end{pmatrix}^T = \begin{pmatrix} -7 & 3 \\ -1 & 2 \end{pmatrix}$$

The determinant of $\begin{pmatrix} 2 & -3 \\ 1 & -7 \end{pmatrix}$;

$$\begin{vmatrix} 2 & -3 \\ 1 & -7 \end{vmatrix} = (2)(-7) - (-3)(1) = -11$$

Hence, we have the inverse of $\begin{pmatrix} 2 & -3 \\ 1 & -7 \end{pmatrix}$ as;

$$\frac{1}{-11}\begin{pmatrix} -7 & 3 \\ -1 & 2 \end{pmatrix} = -\frac{1}{11}\begin{pmatrix} -7 & 3 \\ -1 & 2 \end{pmatrix}$$

Pre-multiply both sides of the equation by this we have;

$$-\frac{1}{11}\begin{pmatrix} -7 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 1 & -7 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{1}{11}\begin{pmatrix} -7 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

Expanding the right hand side using normal matrix multiplication, the left hand side yields the identity matrix since it is the inverse of the matrix;

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{1}{11}\begin{pmatrix} -7(5) + 3(-3) \\ -1(5) + 2(-3) \end{pmatrix}$$

Now, we needn't bother expanding the multiplication on the left hand side, since it's a product of the inverse of a matrix and its matrix, it'll be an identity matrix, the only issue is that we must be sure our inverse has been **correctly calculated**; hence, we have;

$$\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{1}{11}\begin{pmatrix} -44 \\ -11 \end{pmatrix}$$

The identity matrix multiplying that matrix yields the same matrix on the LHS, we then expand the RHS using scalar multiplication rule;

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -44 \times -\frac{1}{11} \\ -11 \times -\frac{1}{11} \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

From matrix equality rule;

$$a = 4 \text{ and } b = 1$$

Let’s see the more complex case of a 3-variable system, nothing complex per say, just a lengthier inverse and matrix multiplication, and that’s no issue, what is important is that you get the basic principle behind the whole thing;

- For the following system of simultaneous equations, find each of the following:

$$\begin{gathered} 7X_1 - X_2 - X_3 = 0 \\ 10X_1 - 2X_2 + X_3 = 8 \\ 6X_1 + 3X_2 + 2X_3 = 7 \end{gathered}$$

(i) The coefficient matrix of the system of the equations;

(ii) The value of its determinant;

(iii) The minors and cofactors of the coefficient matrix.

(iv) The solution values of the unknowns; $(X_i = 1,2,3)$

Too much stories, this is a very straightforward question written in four sentences; we're told to find the coefficient matrix of the system of the equations; that's the first step in solving a system of equations using matrix multiplication, that's also the first step in Crammer's rule anyway; Next, we're told to find the value of its determinant; that's basically the first step in Crammer's rule, however, it could also be the second step in matrix multiplication if we decided to start solving for our inverse matrix by finding the determinant first;

Third, this is **where the whole thing is exposed**; we're told to find the minors and cofactors of the coefficient matrix; wow; of all elements? Meaning, we're finding the full cofactor matrix of all the elements; that isn't needed in Crammer's rule and hence, it is out of here;

Lastly, to solve the equation, since we have the cofactor matrix already, then our adjoint can be gotten straight from its transpose and our inverse calculated; and hence; the full matrix multiplication rule is applied;

(i)

$$\begin{pmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & 2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 7 \end{pmatrix}$$

(ii)

$$\text{Let } A = \begin{pmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & 2 \end{pmatrix}$$

Hence; we have;

$$|A| = \begin{vmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & 2 \end{vmatrix}$$

$$7\begin{vmatrix} -2 & 1 \\ 3 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 10 & 1 \\ 6 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 10 & -2 \\ 6 & 3 \end{vmatrix}$$

$$7[(-2)(2) - (1)(3)] + 1[(10)(2) - (1)(6)] - 1[(10)(3) - (-2)(6)]$$

$$|A| = 7(-7) + (14) + (-42) = -77$$

(iii) Let's find the minors and cofactors of all the elements; as we know, it's minors first;

Let's find the minor elements;

$$min(7) = \begin{vmatrix} -2 & 1 \\ 3 & 2 \end{vmatrix} \qquad min(-1) = \begin{vmatrix} 10 & 1 \\ 6 & 2 \end{vmatrix}$$

$$min(-1) = \begin{vmatrix} 10 & -2 \\ 6 & 3 \end{vmatrix} \qquad min(10) = \begin{vmatrix} -1 & -1 \\ 3 & 2 \end{vmatrix}$$

$$min(-2) = \begin{vmatrix} 7 & -1 \\ 6 & 2 \end{vmatrix} \qquad min(1) = \begin{vmatrix} 7 & -1 \\ 6 & 3 \end{vmatrix}$$

$$min(6) = \begin{vmatrix} -1 & -1 \\ -2 & 1 \end{vmatrix} \qquad min(3) = \begin{vmatrix} 7 & -1 \\ 10 & 1 \end{vmatrix}$$

$$min(2) = \begin{vmatrix} 7 & -1 \\ 10 & -2 \end{vmatrix}$$

Hence,
The matrix of minors is;

$$\text{minor}(A) = \begin{pmatrix} -7 & 14 & 42 \\ 1 & 20 & 27 \\ -3 & 17 & -4 \end{pmatrix}$$

From the cofactor matrix sign notation below:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Hence,

The matrix of cofactors is;

$$\text{cofactor}(A) = \begin{pmatrix} -7 & -14 & 42 \\ -1 & 20 & -27 \\ -3 & -17 & -4 \end{pmatrix}$$

(iv) To solve the equation, since we have the cofactor matrix already, we can straightforward have the adjoint matrix;

The adjoint which is the transpose of the cofactor matrices and hence inverse of the matrix and compute the remaining;

$$adj(A) = [\text{cofactor}(A)]^T$$

$$\text{adj}(A) = \begin{pmatrix} -7 & -14 & 42 \\ -1 & 20 & -27 \\ -3 & -17 & -4 \end{pmatrix}^T$$

$$\operatorname{adj}(A) = \begin{pmatrix} -7 & -1 & -3 \\ -14 & 20 & -17 \\ 42 & -27 & -4 \end{pmatrix}$$

We have evaluated the determinant already and hence;

$$A^{-1} = -\frac{1}{77}\begin{pmatrix} -7 & -1 & -3 \\ -14 & 20 & -17 \\ 42 & -27 & -4 \end{pmatrix}$$

To solve the equation; pre-multiply both sides of this equation by A^{-1}

$$\begin{pmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & 2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 7 \end{pmatrix}$$

We have;

$$A^{-1}\begin{pmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & 2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = -\frac{1}{77}\begin{pmatrix} -7 & -1 & -3 \\ -14 & 20 & -17 \\ 42 & -27 & -4 \end{pmatrix}\begin{pmatrix} 0 \\ 8 \\ 7 \end{pmatrix}$$

The left hand side is reduces completely since A^{-1} multiplying A will yield the identity matrix; we start expanding the right hand side multiplication;

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} (-7)(0) + (-1)(8) + (-3)(7) \\ (-14)(0) + (20)(8) + (-17)(7) \\ (42)(0) + (-27)(8) + (-4)(7) \end{pmatrix}$$

The left hand side simply yields the matrix of coefficients since it is multiplied by an identity matrix;

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = -\frac{1}{77}\begin{pmatrix} -29 \\ 41 \\ -244 \end{pmatrix}$$

Hence, expanding the matrices and applying the matrix equality rule;

$$X_1 = \frac{29}{77}$$

$$X_2 = -\frac{41}{77}$$

$$X_3 = \frac{244}{77}$$

So, basically, there's no point filling the whole book with examples upon examples; it's basically the same whole thing, once the matrices have been properly aligned according to variables, then we are applying the same process over and over and there is nothing that can be twisted in this per say. We took *loadsa* examples in the aspect of determinants as there are many ways to twist questions in that aspect but definitely, not in a system of simultaneous equations; there is nothing to twist!

But not so fast though! Let’s still rush something quickly!

A quick glance at some quite difficult matrix questions in your past questions!

- Evaluate and comment on the nature of the matrix below:

$$\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{pmatrix}$$

For this product, the first two matrices are expanded first!

$$\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos x\,(0) + \sin x\,(1) & \cos x\,(1) + \sin x\,(0) \\ -\sin x\,(0) + \cos x\,(1) & -\sin x\,(1) + \cos x\,(0) \end{pmatrix}$$

$$\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix}$$

The product of these two is then multiplied against the third bracket to get the product:

$$\begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix}\begin{pmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{pmatrix}$$
$$= \begin{pmatrix} \sin x\,(\sin x) + \cos x\,(\cos x) & \sin x\,(-\cos x) + \cos x\,(\sin x) \\ \cos x\,(\sin x) + -\sin x\,(\cos x) & \cos x\,(-\cos x) + -\sin x\,(\sin x) \end{pmatrix}$$

$$\begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix}\begin{pmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{pmatrix}$$
$$= \begin{pmatrix} \sin^2 x + \cos^2 x & -\sin x \cos x + \sin x \cos x \\ \sin x \cos x - \sin x \cos x & -\sin^2 x - \cos^2 x \end{pmatrix}$$

Now, from trigonometric identities, $\sin^2 x + \cos^2 x = 1$

Also, $-\sin x \cos x + \sin x \cos x$ and the second similar expression $\sin x \cos x - \sin x \cos x$ also cancels out!

Also, from $-\sin^2 x - \cos^2 x$; factoring -1 yields:

$$-1(\sin^2 x + \cos^2 x) = -1(1) = -1$$

Hence, the final matrix is:

$$\begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix} \begin{pmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We're told to comment on the nature of the matrix. Well, it's not very easy to fish out the nature of this matrix. It looks like the identity matrix, I but it isn't, it isn't $-I$ either since negating the identity matrix will yield -1 in the first element as well, hence, the matrix is neither I nor $-I$. Well, then, let's check for its transpose and inverse and see what relationship occurs, we could see an idempotent, orthogonal or any other type, let's check it out.

Let:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Obviously, the matrix above is a symmetric matrix since the transpose of the matrix is equal to the matrix itself. However, I am sensing further relationship exists here. Let's check for the inverse of the matrix.

For a 2×2 matrix, the adjoint is given [straight by the 2×2 matrix rule] by:

$$\operatorname{adj} A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The determinant;

$$|A| = (-1)(1) - (0)(0) = -1$$

Hence,

$$A^{-1} = \frac{1}{-1}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -1\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence,

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hence, once again, it is obvious the inverse of the matrix is equal to the transpose of the matrix;

$$A^T = A^{-1}$$

The matrix hence is also orthogonal.

In essence,
The nature of the product is both symmetric and orthogonal.

If I skipped it in the note, I'm mentioning it now. The determinant of an orthogonal matrix is always either equal to 1 or -1. I actually mentioned it though; I just checked it, as it was even a solved example (Page 112). However, the principle is not the other way round; not all matrices with determinants of 1 or -1 are orthogonal matrices.

- Find the spur and determinant of M^T if:

$$M = \begin{bmatrix} 2 & 4 & 8 & 16 \\ 0 & 4 & 8 & 16 \\ 0 & 0 & 8 & 16 \\ 0 & 0 & 0 & 16 \end{bmatrix}$$

This is a basically fundamental question. The only thing I want to point out here is keeping you from making the mistake of working on the matrix M, you are told to find the spur and determinant of M^T and hence, you're expected to find M^T first!

$$M^T = \begin{bmatrix} 2 & 4 & 8 & 16 \\ 0 & 4 & 8 & 16 \\ 0 & 0 & 8 & 16 \\ 0 & 0 & 0 & 16 \end{bmatrix}^T$$

$$M^T = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 8 & 8 & 8 & 0 \\ 16 & 16 & 16 & 16 \end{bmatrix}$$

Hence, M was an upper triangular matrix, M^T now is a lower triangular matrix and hence, we still have a triangular matrix to work on.

The spur of the matrix, which is equal to its trace, is equal to the sum of the elements on its main diagonal, hence, we have:

$$\text{spur}(M^T) = 2 + 4 + 8 + 16 = 30$$

Since it's a triangular matrix, the determinant is the product of the elements on the main diagonal. Otherwise, finding the determinant of a 4×4 matrix would've been hell, and is actually not in the SSC106 way. Hence, we have:

$$|M^T| = 2 \times 4 \times 8 \times 16 = 1024$$

Finally, we have thoroughly gone through the concept of matrices. Well, on a concluding note, in a pretty continuous manner of a flow of thoughts, we have taken a very lot of types of matrices. Hence, let's take a glossary of types over here.

NEXT PAGE PLEASE!

MATRIX CLASSIFICATION

A matrix can be classified **based on three criteria:**

(i) **the relationship between its rows and its columns:** many a times, we defined matrices based on how their columns and rows are related which could include equal number of rows and columns (square matrices) and other relationships. The examples of such matrix classifications and their examples are stated below:

- **A row matrix:** is a matrix that has just one row and any number of columns:
- **A column matrix:** is a matrix that has just one column and any number of rows.

- **A square matrix:** is a matrix that has equal number of rows and columns.
- **A rectangular matrix:** is a matrix that has unequal number of rows and columns.

(ii) **the structure of its elements:** we also at many points in time defined matrix types by the structure of their elements, such includes situations when matrices have a special structure in which the entries of the matrix are such as the case of diagonal matrices, such matrices are easily seen and recognized at first sight. Examples of such matrix classifications and their examples are given below:

- **A diagonal matrix:** is a matrix with all elements not on the main diagonal equal to zero.
- **A triangular matrix:** is a matrix that has only zero elements above (or below) the main diagonal.
- **An identity matrix:** is a square matrix, a diagonal matrix but a special type which has all the elements on the main diagonal equal to unity.
- **A null matrix:** is a matrix that has its entire elements as zero.

(iii) **some derived relationship with itself:** idempotent, orthogonal, symmetric, lots more. Many times we saw special types of matrices which as a certain relationship with itself. Examples of such matrix classifications and their examples are given below:

- **A symmetric matrix:** A square matrix that is equal to its transpose.
- **A skew-symmetric matrix:** A square matrix that is equal to the negative value of its transpose.
- **An idempotent matrix:** is a matrix which when multiplied by itself is still equal to itself.
- **An orthogonal matrix:** is a square matrix whose transpose is equal to its inverse;

VERY IMPORTANT INFORMATION ON THE NEXT PAGE!

And you, Mr. lazy boy or Miss. Lazy girl that you want to jump to this last page to get one naughty summary, you better go back and study matrices extensively. It shouldn't be a topic you'd be missing one question in the test or exam and hence, I've given you wholesomely everything you need to squash matrices, it's up to you to make maximal use of it.

Lol, it is actually an important information; it'll be a waste if you fail any single question under matrices after this ultra-comprehensive text on matrix.

MATRIX IS REALLY AN INTERESTING TOPIC, ISN'T IT?