DEPARTMENT OF ECONOMICS
FACULTY OF SOCIAL SCIENCES
OBAFEMI AWOLOWO UNIVERSITY, ILE-IFE, NIGERIA
SSC106: MATHEMATICS FOR SOCIAL SCIENCES II
RAIN SEMESTER EXAMINATION (2014/2015 SESSION)

INSTRUCTIONS:

- Attempt all questions in **Section A;**
- Answer any two questions from **Section B.**
- Show all workings clearly

Time allowed: 2 hours

SECTION A

1. A rule which satisfies a particular type of relation between two or more variables is known as …………………

2. Given two variables X and Y, if the value of Y varies in some definite form upon the value which is assigned arbitrary to X, then the variable Y is said to be a/an …………….

3. The class of functions which describes the proportionate response or variation of the dependent variable to a proportionate variation of all the independent variables is referred to as …………………

4. If a square matrix, A is equal to its transpose, the matrix is called a ………………

5. When a square matrix, A is multiplied by itself, and gives the same matrix, it is called a/an ………………

6. A square matrix, A whose determinant is equal to zero is referred to as ………………

7. If $Y = UV$; where U and V are functions of x, then $\frac{dy}{dx}$

8. If a matrix A is a 2×4 and the matrix B is a 4×3, the dimension of the new matrix when multiplied is …………………

9. Differentiate between differential and integral calculus.

10. For what values of x are the derivatives of $\frac{1}{3}x^2$ and $x^2 + 3x$ equal?

11. The profits of a firm are determined by the function;

$$\pi = 60x + 3x^2$$

Determine the marginal profit when $x = 4$

12. Given the sales of an automobile is given as:

$$s(t) = 200e^{-0.5t}$$

Obtain the marginal sales at $t = 4$

13. Evaluate: $\int_2^4 (x - 1)dx$

14. Find: $\int 4x^3 e^{x^4} \, dx$

15. Determine the order and degree of:

(i) $$x\frac{dy}{dx} - y^2 = x^4$$

(ii) $$\frac{d^2y}{dx^2} - (x^2 - 5)\frac{dy}{dx} + xy = 0$$

(iii) $$\left(\frac{d^2y}{dx^2}\right)^2 = \left(\frac{dy}{dx}\right)^3 + 2\frac{dy}{dx} + 1$$

SECTION B

1. *(a)* Solve the following equations using Crammer's rule;

$$7A - B - C = 0$$
$$10A - 2B + C = 8$$
$$6A + 3B - 2C = 7$$

(b) If $A = \begin{pmatrix} 4 & 2 \\ 9 & 6 \end{pmatrix}$ and $k_1 = 2$ and $k_2 = 5$, verify that $k_1A + k_2A = (k_1 + k_2)A$

(c) Given that $A = \begin{pmatrix} 1 & 5x - 11 \\ 2 & x^2 - 1 \end{pmatrix}$

If $|A| = 0$, find x.

2. (a) Using the Lagrangean multiplier method, optimize the objective function;

$$Z = x^2 + 3xy - 5y^2$$

Subject to the constraint: $2x + 3y = 6$

(b) Establish whether the solution obtained in (a) above is minimum or maximum.

3. With relevant examples, clearly distinguish between the following pairs;

(i) Ordinary and partial differential equations;

(ii) Order and degree of a differential equation

(iii) Euler's theorem and Young's theorem;

4. (a) Form differential equations from:

(i) $x^2 - e^y = a$

(ii) $x^2 + y^2 - 2ax + 1$

(b) (i) Using substitution method, find:

$$\int \frac{2x + 7}{x^2 + 7x - 4} dx$$

(ii) Solve $\int \left(\frac{7}{x} + 4e^x\right) dx$

(c) Using Euler's theorem, obtain the degree of homogeneity of the function;

(i) $f(x, y) = x^2 + 2xy - 3y^2$

(ii) Show that: $f_{xy} = f_{yx}$

SOLUTION TO THE PAST QUESTIONS

All topics needed here have been thoroughly treated through and through, all you need to do is to sit down and answer all these questions like a joke, you are not expected to start reading from here but from the topics, you make a joke of yourself when you read the concept of any subject starting from the side of the past question; more so, this text isn't a past question but a complete guide to lead you to a straight A in your SSC106 exam.

Hence, keep calm and check the solutions to these past questions one-by-one;

GOOD LUCK AND GOD'S BEST!

SOLUTION TO THE SSC106 EXAMINATION 2014/2015 ACADEMIC SESSION

The instruction is you answer all questions in the **Section A** and only two questions from Section B, we'll be answering everything in both sections in this book though; keep calm and move with me;

Remember every singular solution here, you can get the full concept by reading this book and not the past question section exclusively;

SECTION A

Question 1

A rule which satisfies a particular type of relation between two or more variables is known as

…………………

A function is a mathematical relationship between sets of inputs and a set of permissible outputs with each input related to one output. **Hence, the above is a function.** Filling the above gap gives an alternative definition of a function apart from the one given in the note.

Question 2

Given two variables X and Y, if the value of Y varies in some definite form upon the value which is assigned arbitrary to X, then the variable Y is said to be a/an ……………….

The above simply defines Y to be a dependent variable since it varies based on the value assigned to X, hence, ***Y* is the dependent variable.**

Question 3

The class of functions which describes the proportionate response or variation of the dependent variable to a proportionate variation of all the independent variables is referred to as ……………………

This explains **an explicit function** since the dependent variable is proportionate (i.e. directly related) to all the independent variable(s).

Question 4

If a square matrix, A is equal to its transpose, the matrix is called a ………………

Very *very* simple! The above explains **a symmetric matrix.**

Question 5

When a square matrix, A is multiplied by itself, and gives the same matrix, it is called a/an ………………

Yet again very *very* simple! The above explains **an idempotent matrix.**

Question 6

A square matrix, A whose determinant is equal to zero is referred to as ………………

Yet again very *very* simple! **A singular matrix** has its determinant equal to zero!

Question 7

If $Y = UV$; where U and V are functions of x, then $\frac{dy}{dx}$

$$UV$$

The above implies product rule! Hence;

$$\frac{dy}{dx} = V\frac{dU}{dx} + U\frac{dU}{dx}$$

Notice we are using capitals and not small letter words because that was what was given in the question.

Question 8

If a matrix A is a 2×4 and the matrix B is a 4×3, the dimension of the new matrix when multiplied is …………………

As a matter of rule, if P is of the order $\boldsymbol{m \times n}$

and Q is of the order $\boldsymbol{n \times p}$;

Then: PQ will be of order $\boldsymbol{m \times p}$.

The above was copied from the textbook section and hence, for:
A of the 2×4 matrix and B of the 4×3 matrix, then, AB will be of the 2×3 matrix.

Question 9

Differentiate between differential and integral calculus.

To differentiate between differential and integral calculus, it's simple, you can check the definition in the note; however, let's differentiate them in simpler terms;

Differential and integral calculus are two opposite processes; differential calculus is the aspect of calculus that deals with finding the derivative of a function which could mean the gradient of the curve of the function or the rate of change of the function. Integral calculus on the other hand is the opposite of the process of differential calculus, the process of finding the anti-derivative of a function, in definite situation; it is the area under of a curve between two points on the curve.

Question 10

For what values of x are the derivatives of $\frac{1}{3}x^2$ and $x^2 + 3x$ equal?

Fine question! For what values of x are their derivatives equal? It means we firstly need their individual derivatives; hence;

$$\frac{d}{dx}\left(\frac{1}{3}x^2\right) = 2 \times \frac{1}{3}x^{2-1} = \frac{2}{3}x$$

$$\frac{d}{dx}(x^2 + 3x) = 2 \times x^{2-1} + 1 \times 3x^{1-1}$$

$$\frac{d}{dx}(x^2 + 3x) = 2x + 3$$

Hence;
We'll equate their two derivatives and solve for the value of x for which they are equal.

$$\frac{2}{3}x = 2x + 3$$

We have;

$$\frac{2x}{3} - 2x = 3$$

Hence;

$$\frac{2x - 2x(3)}{3} = 3$$

We have;

$$-4x = 9$$

Hence;

$$x = -\frac{9}{4}$$

Question 11

The profits of a firm are determined by the function;

$$\pi = 60x + 3x^2$$

Determine the marginal profit when $x = 4$

Marginal functions! Differentiate!

$$\pi' = 60x + 3x^2 = 1 \times 60x^{1-1} + 2 \times 3x^{2-1}$$

$$\pi' = 60 + 6x$$

Hence;

To find the marginal profit when $x = 4$, we have:

$$\pi'(4) = 60 + 6(4) = 84$$

Question 12

Given the sales of an automobile is given as:

$$s(t) = 200e^{-0.5t}$$

Obtain the marginal sales at $t = 4$

Marginal functions again! Differentiate!

$$s(t) = 200e^{-0.5t}$$

Substitution! Chain rule!

$$u = -0.5t$$

$$\frac{du}{dt} = 1 \times -0.5t^{1-1} = -0.5$$

Hence;

$$s = 200e^{u}$$

Exponential rule!

$$\frac{ds}{du} = 200(e^u) = 200e^u$$

Hence;

Chain rule;

$$\frac{ds}{dt} = \frac{ds}{du} \times \frac{du}{dt}$$

$$\frac{ds}{dt} = 200e^u \times -0.5$$

Hence; returning u;

$$s'(t) = -100e^{-0.5t}$$

Hence;

Marginal sales at $t = 4$ is given by:

$$s'(4) = -100e^{-0.5\times4}$$

$$s'(4) = -100e^{-2}$$

From calculator;

$$e^{-2} = 0.13533$$

Then;

$$s'(4) = -100 \times 0.13533$$

$$s'(4) = -13.533$$

Question 13

$$\text{Evaluate: } \int_{2}^{4} (x-1)dx$$

This is a case of integral of sums, we must get the indefinite integral first!

$$\int (x-1)dx = \int x\,dx - \int 1\,dx$$

$$\left[\frac{x^{1+1}}{1+1}\right] - \left[\frac{x^{0+1}}{0+1}\right]$$

We don't need an arbitrary constant since we are looking for a definite integral;

$$\frac{x^2}{2} - x$$

Input the integral limits;

$$\int_{2}^{4} (x-1)dx = \left[\frac{x^2}{2} - x\right]_{2}^{4}$$

$$\left[\frac{(4)^2}{2} - (4)\right] - \left[\frac{(2)^2}{2} - (2)\right]$$

Expanding;

$$\frac{16}{2} - 4 + \frac{4}{2} - 2$$

$$8 - 4 + 2 - 2 = 4$$

Question 14

Find: $\int 4x^3 e^{x^4}\, dx$

Substitution! A case of:

$$f'(x)g[f(x)]$$

$$\int 4x^3 e^{x^4}\, dx$$

Here;

$$u = x^4$$

$$\frac{du}{dx} = 4x^3$$

$$dx = \frac{du}{4x^3}$$

$$\int 4x^3 e^u \frac{du}{4x^3}$$

$4x^3$ cancels out;

$$\int e^u \, du$$

Straight exponent rule!

$$e^u$$

Return u;

$$e^{x^4} + C$$

Question 15

Determine the order and degree of:

(i) $x\dfrac{dy}{dx} - y^2 = x^4$

(ii) $\dfrac{d^2y}{dx^2} - (x^2 - 5)\dfrac{dy}{dx} + xy = 0$

(iii) $\left(\dfrac{d^2y}{dx^2}\right)^2 = \left(\dfrac{dy}{dx}\right)^3 + 2\dfrac{dy}{dx} + 1$

Straight! Order and degree! Very simple stuff;

(i)

$$x\frac{dy}{dx} - y^2 = x^4$$

Here, the highest derivative is $\frac{dy}{dx}$, hence the order is 1.

The highest derivative is raised to a power of 1 and hence the degree is 1.

(ii)

$$\frac{d^2y}{dx^2} - (x^2 - 5)\frac{dy}{dx} + xy = 0$$

Here, the highest derivative is $\frac{d^2y}{dx^2}$, hence the order is 2.

The highest derivative is raised to a power of 1 and hence the degree is 1.

(c)

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left(\frac{dy}{dx}\right)^3 + 2\frac{dy}{dx} + 1$$

Here, the highest derivative is $\frac{d^2y}{dx^2}$, hence the order is 2.

The highest derivative is raised to a power of 2 and hence the degree is 2.

SECTION B

Question 1

(a) Solve the following equations using Crammer's rule;

$$7A - B - C = 0$$
$$10A - 2B + C = 8$$
$$6A + 3B - 2C = 7$$

(b) If $A = \begin{pmatrix} 4 & 2 \\ 9 & 6 \end{pmatrix}$ and $k_1 = 2$ and $k_2 = 5$, verify that $k_1A + k_2A = (k_1 + k_2)A$

(c) Given that $A = \begin{pmatrix} 1 & 5x - 11 \\ 2 & x^2 - 1 \end{pmatrix}$

If $|A| = 0$, find x.

(a)

SIMPLE STUFF YOU KNOW!

Fully arranged and hence, we can proceed to find our determinants;

We make our first determinant; Δ

$$\Delta = \begin{vmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & -2 \end{vmatrix}$$

Hence; we have;

$$7\begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} - (-1)\begin{vmatrix} 10 & 1 \\ 6 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 10 & -2 \\ 6 & 3 \end{vmatrix}$$

$$7[(-2)(-2) - (1)(3)] + 1[(10)(-2) - (1)(6)] - 1[(10)(3) - (-2)(6)]$$

$$\Delta = 7(1) + (-26) - 42 = -61$$

For Δ_A, replace the column of A with the column matrix of the solutions, the other variables still maintain their columns here;

$$\Delta_A = \begin{vmatrix} 0 & -1 & -1 \\ 8 & -2 & 1 \\ 7 & 3 & -2 \end{vmatrix}$$

Hence; we have;

$$0\begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} - (-1)\begin{vmatrix} 8 & 1 \\ 7 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 8 & -2 \\ 7 & 3 \end{vmatrix}$$

$$0[(-2)(-2) - (1)(3)] + 1[(8)(-2) - (1)(7)] - 1[(8)(3) - (-2)(7)]$$

$$\Delta_A = 0(1) + (-23) - (38) = -61$$

For Δ_B, replace the column of y with the column matrix of the solutions, the column of A is back in place as you can see;

$$\Delta_B = \begin{vmatrix} 7 & 0 & -1 \\ 10 & 8 & 1 \\ 6 & 7 & -2 \end{vmatrix}$$

$$7\begin{vmatrix} 8 & 1 \\ 7 & -2 \end{vmatrix} - 0\begin{vmatrix} 10 & 1 \\ 6 & -2 \end{vmatrix} + (-1)\begin{vmatrix} 10 & 8 \\ 6 & 7 \end{vmatrix}$$

$$7[(8)(-2)-(1)(7)]-0[(10)(-2)-(1)(6)] \\ -1[(10)(7)-(8)(6)]$$

$$\Delta_B = 7(-23)+0-(22) = -183$$

For Δ_C, replace the column of C with the column matrix of the answers;

$$\Delta_z = \begin{vmatrix} 7 & -1 & 0 \\ 10 & -2 & 8 \\ 6 & 3 & 7 \end{vmatrix}$$

$$7\begin{vmatrix} -2 & 8 \\ 3 & 7 \end{vmatrix} - (-1)\begin{vmatrix} 10 & 8 \\ 6 & 7 \end{vmatrix} + 0\begin{vmatrix} 10 & -2 \\ 6 & 3 \end{vmatrix}$$

$$7[(-2)(7)-(8)(3)]+1[(10)(7)-(8)(6)] \\ +0[(10)(3)-(-2)(6)]$$

$$\Delta_C = 7(-38)+22-0 = -244$$

Hence,

$$A = \frac{\Delta_A}{\Delta} = \frac{-61}{-61} = 1$$

$$B = \frac{\Delta_B}{\Delta} = \frac{-183}{-61} = 3$$

$$C = \frac{\Delta_C}{\Delta} = \frac{-244}{-61} = 4$$

(b)

In verifying and showing, you simply establish that one thing is equal to the other.

$$A = \begin{pmatrix} 4 & 2 \\ 9 & 6 \end{pmatrix}$$

$$k_1 = 2 \ \text{ and } \ k_2 = 5$$

Verify that:

$$k_1A + k_2A = (k_1 + k_2)A$$

$$k_1A = 2\begin{pmatrix} 4 & 2 \\ 9 & 6 \end{pmatrix} = \begin{pmatrix} 8 & 4 \\ 18 & 12 \end{pmatrix}$$

$$k_2A = 5\begin{pmatrix} 4 & 2 \\ 9 & 6 \end{pmatrix} = \begin{pmatrix} 20 & 10 \\ 45 & 30 \end{pmatrix}$$

Hence;

$$k_1A + k_2A = \begin{pmatrix} 8 & 4 \\ 18 & 12 \end{pmatrix} + \begin{pmatrix} 20 & 10 \\ 45 & 30 \end{pmatrix}$$

$$\begin{pmatrix} 8+20 & 4+10 \\ 18+45 & 12+30 \end{pmatrix}$$

Hence;

$$k_1A + k_2A = \begin{pmatrix} 28 & 14 \\ 63 & 42 \end{pmatrix}$$

$$(k_1 + k_2) = 2 + 5 = 7$$

Hence;

$$(k_1 + k_2)A = 7\begin{pmatrix} 4 & 2 \\ 9 & 6 \end{pmatrix}$$

$$7\begin{pmatrix} 4 & 2 \\ 9 & 6 \end{pmatrix} = \begin{pmatrix} 7\times 4 & 7\times 2 \\ 7\times 9 & 7\times 6 \end{pmatrix}$$

$$(k_1 + k_2)A = \begin{pmatrix} 28 & 14 \\ 63 & 42 \end{pmatrix}$$

Hence;
It has been proved!

(c)

$$A = \begin{pmatrix} 1 & 5x - 11 \\ 2 & x^2 - 1 \end{pmatrix}$$

$|A| = 0$, find x.

We did something very similar in the textbook, take the determinant as though they were normal numbers and then sort it out;

$$\begin{vmatrix} 1 & 5x - 11 \\ 2 & x^2 - 1 \end{vmatrix} = (1)(x^2 - 1) - 2(5x - 11)$$

$$|A| = x^2 - 1 - 10x + 22$$

$$|A| = x^2 - 10x + 21$$

As we were told;

$$|A| = 0$$

Hence;

$$x^2 - 10x + 21 = 0$$

Simple quadratic equation;

$$x^2 - 7x - 3x + 21 = 0$$
$$x(x - 7) - 3(x - 7) = 0$$
$$(x - 3)(x - 7) = 0$$

Hence;

$$x = 3$$
$$x = 7$$

Question 2

(a) Using the Lagrangean multiplier method, optimize the objective function;

$$Z = x^2 + 3xy - 5y^2$$

Subject to the constraint: $2x + 3y = 6$

(b) Establish whether the solution obtained in (a) above is minimum or maximum.

(a)

Here, our objective function is:

$$x^2 + 3xy - 5y^2$$

Our constraint function is:

$$2x + 3y = 6$$

Express it equated to zero;

$$2x + 3y - 6 = 0$$

Let's form our Lagrangean equation;

$$\mathcal{L}(x, y, \lambda) = x^2 + 3xy - 5y^2 - \lambda(2x + 3y - 6)$$

Take the first order partials;

$$\mathcal{L}_x = 2 \times x^{2-1} + 1 \times 3x^{1-1} \times y - 0 - \lambda(2 + 0 - 0)$$

$$\mathcal{L}_x = 2x + 3y - 2\lambda$$

$$\mathcal{L}_y = 0 + 1 \times 3x \times y^{1-1} - 2 \times 5y^{2-1} - \lambda(0 + 1 \times 3y^{1-1} - 0)$$

$$\mathcal{L}_y = 3x - 10y - 3\lambda$$

$$\mathcal{L}_\lambda = 0 - 0 + 0 - 1 \times \lambda^{1-1}(2x + 3y - 6)$$

$$\mathcal{L}_\lambda = -2x - 3y + 6$$

For the first order conditions, equate each to zero;

$$2x + 3y - 2\lambda = 0 \dots\dots\dots (1)$$

$$3x - 10y - 3\lambda = 0 \dots\dots\dots (2)$$

$$-2x - 3y + 6 = 0$$

$$-2x - 3y = -6 \dots\dots\dots (3)$$

Let's solve these equations using Crammer's rule; we can use anyone we like;

Here, we have three equations;

$$2x + 3y - 2\lambda = 0 \dots\dots\dots (1)$$

$$3x - 10y - 3\lambda = 0 \ldots \ldots \ldots (2)$$

$$-2x - 3y = -6 \ldots \ldots \ldots (3)$$

The equivalent matrix determinant needed here is:

$$\Delta = \begin{vmatrix} 2 & 3 & -2 \\ 3 & -10 & -3 \\ -2 & -3 & 0 \end{vmatrix}$$

Evaluating this determinant;

$$\Delta = 2[(-10)(0) - (-3)(-3)] - (3)[(3)(0) - (-3)(-2)] + (-2)[(3)(-3) - (-10)(-2]$$

$$\Delta = 2(-9) - 3(-6) - 2(-29)$$

$$\Delta = -18 + 18 + 58 = 58$$

To evaluate Δ_x, we replace the column of x with the column matrix of answers;

$$\Delta_x = \begin{vmatrix} 0 & 3 & -2 \\ 0 & -10 & -3 \\ -6 & -3 & 0 \end{vmatrix}$$

Evaluating this determinant;

$$\Delta_x = 0[(-10)(0) - (-3)(-3)] \\ - (3)[(0)(0) - (-3)(-6)] \\ + (-2)[(0)(-3) - (-10)(-6)]$$

$$\Delta_x = 0 - 3(-18) - 2(-60)$$

$$\Delta_x = 0 + 54 + 120 = 174$$

To evaluate Δ_y, we replace the column of y with the column matrix of answers;

$$\Delta_y = \begin{vmatrix} 2 & 0 & -2 \\ 3 & 0 & -3 \\ -2 & -6 & 0 \end{vmatrix}$$

Evaluating this determinant;

$$\Delta = 2[(0)(0) - (-3)(-6)] \\ - 0[(3)(0) - (-3)(-2)] \\ + (-2)[(3)(-6) - (0)(-2)]$$

$$\Delta_y = 2(-18) - 0 - 2(-18)$$

$$\Delta = -36 - 0 + 36$$

$$\Delta_y = 0$$

To evaluate Δ_λ, we replace the column of λ with the column matrix of answers;

$$\Delta_\lambda = \begin{vmatrix} 2 & 3 & 0 \\ 3 & -10 & 0 \\ -2 & -3 & -6 \end{vmatrix}$$

Evaluating this determinant;

$$\Delta_\lambda = 2[(-10)(-6) - (0)(-3)] \\ - (3)[(3)(-6) - (0)(-2)] \\ + 0[(3)(-3) - (-10)(-2)]$$

$$\Delta_\lambda = 2(60) - 3(-18) + 0$$

$$\Delta_\lambda = 120 + 54 + 0 = 174$$

Hence, we have our x, y and λ values as;

$$x = \frac{\Delta_x}{\Delta} = \frac{174}{58} = 3$$

$$y = \frac{\Delta_y}{\Delta} = \frac{0}{58} = 0$$

$$\lambda = \frac{\Delta_\lambda}{\Delta} = \frac{174}{58} = 3$$

Hence, we have our optimal values; let us see the second part, we're asked to find the nature of the optimal values;

(ii)

Head back to the first order partials; we need that of the partials with respect to x and y;

$$\mathcal{L}_x = 2x + 3y - 2\lambda$$

$$\mathcal{L}_y = 3x - 10y - 3\lambda$$

$$\mathcal{L}_{xx} = \frac{\partial}{\partial x}(2x + 3y - 2\lambda)$$

$$\mathcal{L}_{xx} = 1 \times 2x^{1-1} + 0 - 0 = 2$$

$$\mathcal{L}_{yy} = \frac{\partial}{\partial y}(3x - 10y - 3\lambda)$$

$$\mathcal{L}_{yy} = 0 - 1 \times 10y^{1-1} - 0 = -10$$

Hence, we can see that the second order partials are different types of values; hence, like I said, finding the nature of points in the use of the Lagrangean method is beyond the scope of this book, hence, I'll solve this over again (using the method of direct substitution for us to verify the nature of the points we got.

Jump to page 59 of differential applications, I motioned it that mostly it is beyond the scope of this book to find the nature of points in Lagragean method, in that problem, we got zeros as the two direct second order partials. I didn't want to cover this under the carpet. We're using the direct substitution method now that it is hooking on the way and it is actually required in the question. Hence, when you're required to determine the nature, use the direct substitution method (except you are told specifically to use Lagrangean) since it has more assurance that yoi'll be able to determine its nature. If you're told to use Lagrangean method and to find the nature as well, then it'll be a question where either both second order partials will be positive (minimum points) or both will be negative (maximum points) – an example like that is found in Page 64. All these I'm saying will look like French if you've not read the chapter hence, go and read differential applications if you haven't read it.

We have the objective and constraint functions;

$$x^2 + 3xy - 5y^2$$

Our constraint function is:

$$2x + 3y = 6$$

From the constraint function;

$$3y = 6 - 2x$$

Hence;

$$y = \frac{6 - 2x}{3}$$

Input in the objective function!

$$x^2 + 3x\left(\frac{6 - 2x}{3}\right) - 5\left(\frac{6 - 2x}{3}\right)^2$$

Expand carefully!

$$x^2 + \frac{3x(6 - 2x)}{3} - 5\frac{(6 - 2x)^2}{3^2}$$

$$x^2 + \frac{18x - 6x^2}{3} - 5\left(\frac{36 - 24x + 4x^2}{9}\right)$$

$$x^2+\frac{18x}{3}-\frac{6x^2}{3}-5\left(\frac{36}{9}-\frac{24x}{9}+\frac{4x^2}{9}\right)$$

Expand completely!

$$x^2+6x-2x^2-20+\frac{40x}{3}-\frac{20x^2}{9}$$

Collect like terms and add them!

$$x^2-2x^2-\frac{20x^2}{9}+6x+\frac{40x}{3}-20$$

$$-\frac{29x^2}{9}+\frac{58x}{3}-20$$

Hence, we have successfully made the objective function a function with respect to one variable, we'll simply apply the allied calculus optimization conditions and finish it off, you know how we did it in the textbook!

Let the objective function be Z;

$$Z=-\frac{29x^2}{9}+\frac{58x}{3}-20$$

$$\frac{dZ}{dx} = 2 \times \frac{-29x^{2-1}}{9} + 1 \times \frac{58x^{1-1}}{3} - 0$$

$$\frac{dZ}{dx} = -\frac{58x}{9} + \frac{58}{3}$$

At stationary point;

$$\frac{dZ}{dx} = 0$$

Hence;

$$-\frac{58x}{9} + \frac{58}{3} = 0$$

Solving; multiplying through by 9;

$$-58x + 174 = 0$$

Hence;

$$x = 3$$

Since we have just one stationary point, testing for the second order conditions, we have;

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}\left(\frac{dZ}{dx}\right) = \frac{d}{dx}\left(-\frac{58x}{9} + \frac{58}{3}\right)$$

$$\frac{d^2Z}{dx^2} = 1 \times -\frac{58}{9}x^{1-1} + 0$$

$$\frac{d^2Z}{dx^2} = -\frac{58}{9}$$

Hence, since the second derivative is a constant negative value, the only negative point **a maximum point!**

Jogging back to our y value;

$$y = \frac{6 - 2x}{3}$$

But $x = 3$; hence;

$$y = \frac{6 - 2(3)}{3} = \frac{6 - 6}{3} = \frac{0}{3}$$

$$y = 0$$

Hence,

(a)

The optimum points is $x = 3$ and $y = 0$

(b)

The nature of the optimum point is that the point is a maximum point;

Further comments on this; It is obvious we got the same answer we had gotten when we used the Lagrangean method; hence, the direct substitution is easier especially when you're asked to find the nature; however, you still have to know the brief method explained in the textbook concerning finding the nature of the points when we use the Lagrangean method. There are times when the direct second order partials in the Lagrangean method are both positive and both negative; when you're asked to use the Lagrangean method and to find the nature as well, you'll definitely be given a case where both direct second order partials are of the same sign. Both positive implies that the point is a minimum point; both negative implies that it is a maximum point; as said in the textbook already!

Question 3

With relevant examples, clearly distinguish between the following pairs;

(i) Ordinary and partial differential equations;

(ii) Order and degree of a differential equation

(iii) Euler's theorem and Young's theorem;

This is a type of question I hate with immense passion! The question actually carries 15 marks hence, each is 5 marks and many at times, you probably do not know the extent your lecturer wants you to go to attain full marks! However, we'll try as much to distinguish them and show examples;

(i)

An ordinary differential equation is a differential equation involving one independent and one dependent variable. For example, an ordinary equation in y and x(with y the dependent variable and x the independent variable) will involve only differential coefficients of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and so on. Examples of ordinary differential equations are given below:

$$\frac{dy}{dx} + 5x = 5y$$

$$\frac{d^2y}{dx^2} = 3^{x+y}$$

$$\frac{d^2u}{dx^2} - x\frac{du}{dx} + u = 0$$

WHILE

A partial differential equation involves partial derivatives. It is a differential equation involving a dependent variable and more than one independent variable. For example, partial differential equations will include differential coefficients of $\frac{\partial y}{\partial x}, \frac{\partial y}{\partial t}$ in a single equation.
Examples of partial differential equations are given below:

$$\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} = x^3 - t^3$$

$$\frac{\partial^2 y}{\partial x^2} - c^2\frac{\partial^2 y}{dt^2} = x$$

(ii)

The order of a differential equation is the highest derivative involved in the differential equation. For example; in:

$$x\frac{dy}{dx} - y^2 = 0$$

The above is a differential equation of the first order since the highest derivative in the equation is $\frac{dy}{dx}$ which is a first derivative.

$$xy\frac{d^2y}{dx^2} - y^2 \sin x = 5$$

The above is a differential equation of the second order since the highest derivative in the equation is $\frac{d^2y}{dx^2}$ which is a second derivative.

WHILE

Now, the degree of a differential equation is determined after the order of the differential equation has been determined. The degree of a differential equation is the power the highest derivative has been raised to. Hence, that derivative that won the contest of the order of a differential equation, the power it is raised is called the degree of the differential equation; from the illustration below:::

$$\frac{dy}{dx} - \cos x = 0$$

The above has a degree 1 since the highest derivative, $\frac{dy}{dx}$ is raised to a power of 1.

$$\left(\frac{d^2y}{dx^2}\right)^4 - xy\frac{dy}{dx} = 2x - y$$

The above has a degree of the 4 since the highest derivative, $\frac{d^2y}{dx^2}$ is raised to a power of 4.

(iii)

Euler's theorem states that the sum of the product of the arguments and their first order partial derivatives of a function is equal to the degree of homogeneity of a function multiplying the function;
For example; if:

$f(x, y)$ is homogenous to degree of n;

Then;

$$xf_x + yf_y = nf(x,y)$$

WHILE

The Jacobian theorem states that the Jacobian determinant for a set of functions will be equal to zero for all values of the arguments of the functions if and only if the functions are (linearly or nonlinearly) dependent. For the value of the Jacobian determinant to vanish (i.e. the determinant of the Jacobian matrix being equal to zero), then the two functions must be dependent.

Question 4

(a) Form differential equations from:

(i) $x^2 - e^y = a$

(ii) $x^2 + y^2 - 2ax + 1$

(b) (i) Using substitution method, find:

$$\int \frac{2x+7}{x^2+7x-4}\,dx$$

(ii) Solve $\int \left(\frac{7}{x} + 4e^x\right) dx$

(c) Using Euler's theorem, obtain the degree of homogeneity of the function;

(i) $f(x, y) = x^2 + 2xy - 3y^2$

(ii) Show that: $f_{xy} = f_{yx}$

(a)

(i)

$$x^2 - e^y = a$$

Here, we have only one constant, this question is actually in the textbook (Page 26), I'll still solve it here though;

Implicit differentiation!

$$2x - e^y \frac{dy}{dx} = 0$$

Hence;

$$\frac{dy}{dx} = \frac{2x}{e^y}$$

The above in itself is a differential equation! No constants in the equation including derivatives hence, we need no substitution case here;

(ii)

$$x^2 + y^2 - 2ax + 1$$

Another case of implicit differentiation! We'll assume it is equated to zero since it is differential equations we're dealing with!

$$x^2 + y^2 - 2ax + 1 = 0$$

Implicit differentiation process;

$$2x + 2y\frac{dy}{dx} - 1 \times 2ax^{1-1} + 0 = 0$$

$$2x + 2y\frac{dy}{dx} - 2a = 0$$

Hence;

$$2y\frac{dy}{dx} = 2a - 2x$$

Here, since we are forming a differential equation and we can still find an arbitrary constant in the derivative equation, we'll make a the subject here, it's easy to make it the subject here than in the initial equation; hence;

$$2a = 2x + 2y\frac{dy}{dx}$$

Multiply through by $\left(\frac{1}{2}\right)$

$$\left(\frac{1}{2}\right) \times 2a = \frac{1}{2} \times 2x + \frac{1}{2} \times 2y\frac{dy}{dx}$$

$$a = x + y\frac{dy}{dx}$$

Substitute this now into the initial equation;

$$x^2 + y^2 - 2\left(x + y\frac{dy}{dx}\right)x + 1 = 0$$

Expanding;

$$x^2 + y^2 - \left(2x + 2y\frac{dy}{dx}\right)x + 1 = 0$$

Simplifying further;

$$x^2 + y^2 - 2x^2 - 2xy\frac{dy}{dx} + 1 = 0$$

Above is the needed differential equation!

(b)

(i)

$$\int \frac{2x + 7}{x^2 + 7x - 4} dx$$

Without being told, it's quite obvious it's a case of substitution!

$$u = x^2 + 7x - 4$$

Hence;

$$\frac{du}{dx} = 2x + 7$$

Hence;

$$dx = \frac{du}{2x + 7}$$

The integral hence becomes;

$$\int \frac{2x+7}{u} \times \frac{du}{2x+7}$$

$(2x+7)$ cancels out;

$$\int \frac{1}{u} du$$

Standard!

$$\ln u + C$$

Return u;

$$\ln(x^2 + 7x - 4) + C$$

(ii)

$$\text{Solve } \int \left(\frac{7}{x} + 4e^x\right) dx$$

Really not sure what we're solving, we'll simply find the integral!

$$\int \left(\frac{7}{x} + 4e^x\right) dx$$

Integral of sums, separate them!

$$\int \frac{7}{x} dx + \int 4e^x dx$$

Bring the constants out!

$$7\int \frac{1}{x}dx + 4\int e^x dx$$

We're left with standard integrals!

$$7\ln x + 4e^x + C$$

We only need one arbitrary constant!

(c)

(i)

$$f(x, y) = x^2 + 2xy - 3y^2$$

The statement of Euler's theorem of homogeneity is:

$$xf_x + yf_y = nf(x, y)$$

Hence;

$$f(x, y) = x^2 + 2xy - 3y^2$$

$$f_x = 2 \times x^{2-1} + 1 \times 2x^{1-1}y - 0$$

$$f_x = 2x + 2y$$

$$f_y = 0 + 1 \times 2xy^{1-1} - 2 \times 3y^{2-1}$$

$$f_y = 2x - 6y$$

Hence, slotting into Euler's theorem;

$$x(2x + 2y) + y(2x - 6y) = nf(x, y)$$

Expanding;

$$2x^2 + 2xy + 2xy - 6y^2 = n(x^2 + 2xy - 3y^2)$$

$$2x^2 + 4xy - 6y^2 = n(x^2 + 2xy - 3y^2)$$

2 can be factorized on the left;

$$2(x^2 + 2xy - 3y^2) = n(x^2 + 2xy - 3y^2)$$

Hence, by comparison;

$$n = 2$$

(ii)

To show;

$$f_{xy} = f_{yx}$$

We already have f_x and f_y from (i), hence, we just differentiate further partially;

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(2x + 2y)$$

$$f_{xy} = 0 + 1 \times 2y^{1-1} = 2$$

$$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(2x - 6y)$$

$$f_{yx} = 1 \times 2x^{1-1} - 0 = 2$$

Hence, obviously, the prove is complete!

DONE!