# DEPARTMENT OF ECONOMICS
# FACULTY OF SOCIAL SCIENCES
# OBAFEMI AWOLOWO UNIVERSITY, ILE-IFE, NIGERIA
# SSC106: MATHEMATICS FOR SOCIAL SCIENCES II
# RAIN SEMESTER EXAMINATION (2005/2006 SESSION)

**INSTRUCTIONS:**

- Answer all questions in Section A
- Answer only one question in Section B
- Show all workings clearly

**Time allowed: 2 hours**

## SECTION A

1. *(a)* Why is some knowledge of mathematics required in the Social Sciences?

   *(b)* (i) What is a function?

   (ii) Explain the following functions. Illustrate your answers with relevant examples and diagrams:
   - increasing function;
   - monotonic function;

- multivariate function;
- sinusoidal function;
- homogenous function.

2. *(a)* Find the spur and determinant of $M^T$ if:

$$M = \begin{bmatrix} 2 & 4 & 8 & 16 \\ 0 & 4 & 8 & 16 \\ 0 & 0 & 8 & 16 \\ 0 & 0 & 16 & 16 \end{bmatrix}$$

*(b)* If matrices $A$ and $B$ are idempotent, prove that:

(i) their sum $(A + B)$ is idempotent if and only if $AB = BA = 0$

(ii) their difference $(A - B)$ is idempotent if and only if $AB = BA = B$.

*(c)* Evaluate:

$$\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{pmatrix}$$

and comment on the nature of the product matrix.

## SECTION B

3. *(a)* An assembly plant orders $x$ units of a product needed in it's operation each time it places an order. The yearly cost on placing orders and maintaining an inventory for the product is given by:

$$f(x) = 4000 + 4x + \frac{10000}{x}$$

what size order should be placed each time if the yearly cost is to be minimized.

*(b)* State Young's theorem and apply it to the function:

$$Z_1 = x^2 + 5xy - y^2$$

*(c)* Use Euler's theorem to determine the degree of homogeneity of the function:

$$Z_2 = Ax^{\alpha}y^{\beta}$$

*(d)* Is the following function harmonic?

$$Z_3 = \sin x \sin y$$

4. *(a)* Find the relative optima, if they exist of the function:

$$y = x^3 + 3x^2 - 9x - 5$$

*(b)* (i) Define a Lagrangean function;

(ii) Using the Lagrangean multiplier method, find a rectangular consumption basket which has the largest area for a given perimeter.

5. *(a)* Find the derivative function in each of the following cases.

(i) $y = x^5 - 4x^4 + 3x^2$

(ii) $z = e^{-2t} - e^{-3t}$

(iii) $w = e^{\cos x - \sin x}$

(iv) $v = \log(at^2 + bt + c)$

*(b)* Integrate each of the following:

(i) $\dfrac{5x^4}{x^5+16}$

(ii) $(2ax + b)(ax^2 + bx)^7$

(iii) $\log_e 2x$

*(c)* Let $U = \int \frac{\sin x}{a \sin x + b \cos x} dx$ and

$V = \int \frac{\cos x}{a \sin x + b \cos x} dx$, find:

(i) $aU + bV$;

(ii) $aV - bU$.

## SOLUTION TO THE PAST QUESTIONS

All topics needed here have been thoroughly treated through and through, all you need to do is to sit down and answer all these questions like a joke, you are not expected to start reading from here but from the topics, you make a joke of yourself when you read the concept of any subject starting from the side of the past question; more so, this text isn't a past question but a complete guide to lead you to a straight A in your SSC106 exam.

Hence, keep calm and check the solutions to these past questions one-by-one;

**GOOD LUCK AND GOD'S BEST!**

# SOLUTION TO THE SSC106 EXAMINATION 2005/2006 ACADEMIC SESSION

The instruction is you answer all questions in the **Section A** and only one question from Section B, we'll be answering everything in both sections in this book though; keep calm and move with me;

Remember every singular solution here, you can get the full concept by reading this book and not the past question section exclusively;

## SECTION A

### Question 1

*(a)* Why is some knowledge of mathematics required in the Social Sciences?

*(b)* (i) What is a function?

(ii) Explain the following functions. Illustrate your answers with relevant examples and diagrams:

- increasing function;
- monotonic function;
- multivariate function;

- sinusoidal function;
- homogenous function.

(a)

(i)

A question directly from the note, a purely theoretical question, zero bit of calculation, shouldn't be an issue at all. I'll rush quickly to the note now and copy and paste the solutions right here! Let's put like five to seven since no specific number is stated that we should put:

The knowledge of mathematics is very useful in the field of Social Sciences because:

- It helps social scientists to state their research problems in specific and clear terms.
- Mathematics provide considerable insight into the way by which numerical information can be generated and presented to aid decision making in the social and management science.

- It helps in Identifying and quantifying the relationship between variables that determine the outcome of the decisions and their alternatives in the social and management science.
- It minimizes subjectivity and enhances the chance of making objective decision.
- It assists in the prediction or forecasting of future events.
- The language of mathematics is very easy to understand.
- Mathematics makes problems that could take lengthy periods to be resolved in minutes;
- With calculus, we can find the relative optima of different economical functions to easily find desired optimum results;

(b)

(i)

Another answer directly from the note, another purely theoretical which I'll again go copy and paste from the notes.

A function is a mathematical relationship between sets of inputs and a set of permissible outputs with each input related to one output.

(ii)

Yet again another theoretical question. Another 'copy and paste' situation; Bored!

- **Increasing functions** are functions which are increase in value as the value of the independent functions increase over an interval.

If $f(x)$ is increasing over the interval;

$$a \leq x \leq b$$

Then;

$$f(b) > f(a)$$

For all;

$$b > a \text{ in } a \leq x \leq b$$

Examples of increasing functions are:

$$y = 2x + 3$$

- **Monotonic functions:** are functions that are increasing or decreasing for all values of the function. And hence, not only within an interval, monotonic functions are either **strictly increasing** or **strictly decreasing** for all values of the independent variable.

Examples of monotonic functions are;

$$y = 3x + 7$$
$$y = 8 - 3x$$
$$y = e^x$$

- **The multivariate functions** are functions that have more than one independent variable, they also usually also have one dependent variable in situations where the dependent variable is shown; they're in the form;

$$Z = f(x, y)$$

Examples of multivariate functions include;

$$Z = x^2y - y^2x$$

$$y = u^3 - v^3 + 3$$

$$f(x, y) = x^5 + y^3$$

- **Sinusoidal functions** are functions that are used for describing relationships whose graphic forms are wave-like with respect to the independent variable; sinusoidal functions have a highest value which is called its **amplitude** and sinusoidal functions are also called periodic functions, sinusoidal functions repeat themselves in continuous process;

  Examples of sinusoidal functions are:

$$y = 5\cos(2x + 3)$$

$$f(x) = \sin(3x - 1)$$

- **Homogenous functions** are functions with multiplicative scaling behaviour, for a homogenous function, if all its arguments (the variables that make up the function) are multiplied by a factor; then the function is multiplied by some power of the factor;

  Examples of homogenous functions are:

$$U = x^2 + y^2$$

$$f(x, y, z) = x^5 y^2 z^3$$

***For relevant diagrams for each of the functions above, kindly find the diagrams in the notes section of this.***

## Question 2

*(a)* Find the spur and determinant of $M^T$ if:

$$M = \begin{bmatrix} 2 & 4 & 8 & 16 \\ 0 & 4 & 8 & 16 \\ 0 & 0 & 8 & 16 \\ 0 & 0 & 16 & 16 \end{bmatrix}$$

*(b)* If matrices $A$ and $B$ are idempotent, prove that:

(i) their sum $(A + B)$ is idempotent if and only if $AB = BA = 0$

(ii) their difference $(A - B)$ is idempotent if and only if $AB = BA = B$.

*(c)* Evaluate:

$$\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{pmatrix}$$

and comment on the nature of the product matrix.

This question 2 contains three questions that have been thoroughly thrashed in the note, let's go and copy and paste them here;

(a)

A basically fundamental question! As I said in the note, the only thing you must watch out for is making the mistake of working on the matrix $M$, you are told to find the spur and determinant of $M^T$ and hence, you're expected to find $M^T$ first!

$$M^T = \begin{bmatrix} 2 & 4 & 8 & 16 \\ 0 & 4 & 8 & 16 \\ 0 & 0 & 8 & 16 \\ 0 & 0 & 0 & 16 \end{bmatrix}^T$$

$$M^T = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 8 & 8 & 8 & 0 \\ 16 & 16 & 16 & 16 \end{bmatrix}$$

Hence, $M$ was an upper triangular matrix, $M^T$ now is a lower triangular matrix and hence, we still have a triangular matrix to work on.

The spur of the matrix, which is equal to its trace, is equal to the sum of the elements on its main diagonal, hence, we have:

$$\text{spur}(M^T) = 2 + 4 + 8 + 16 = 30$$

Since it's a triangular matrix, the determinant is the product of the elements on the main diagonal.

$$|M^T| = 2 \times 4 \times 8 \times 16 = 1024$$

(b)

Part of the most confusing matrix questions, however, time has been taken to explain it, I'll prefer you have understood it from the note **(Pg. 123, matrices)** and you won't need to still be checking the solution here;

(i) Now, for the first one;

We want to find the conditions such that:

$$(A + B)^2 = (A + B)$$

Expand $(A+B)^2$

$$(A+B)^2 = (A+B)(A+B)$$
$$(A+B)^2 = A^2 + AB + BA + B^2$$

Since $A$ and $B$ are idempotent matrices, we have that;

$$A^2 = A$$
$$B^2 = B$$

Hence,

$$(A+B)^2 = A + AB + BA + B$$

Is there any other further simplification that can be done? No! There is basic relationship between $AB$ and $BA$, and hence, since no simplification is possible, we do this.

For $(A+B)$ to be idempotent,

$$(A+B)^2 = (A+B)$$

Hence, we compare the right hand side of the true value of $(A+B)^2$ with the condition for it to be idempotent, hence, we have;

$(A + AB + BA + B)$ **compared with** $A + B$

$$A + AB + BA + B = A + B$$

Both $A$ and $B$ cancels out of the equation, leaving us with:

$$AB + BA = 0$$

Hence, the above is the condition for $(A + B)^2$ to be equal to $AB$

Simplifying the above, it is possible in two ways; first, solving the equation directly, we have;

$$\boldsymbol{AB = -BA}$$

Above is the first condition for $(A + B)$ to be idempotent, but we can't find this condition in the question? Back to the same equation, let's see the second way it can be resolved!

$$AB + BA = 0$$

What if both $AB$ and $BA$ are equal to zero; their sum will be:

$$0 + 0 = 0$$

Hence, another condition is for:

$$\boldsymbol{AB = BA = 0}$$

The question states that we should prove that their sum, $(A+B)$ is only idempotent if, $AB = BA = 0$; hence; the prove is the second one;

As a matter of fact, the above are the only two conditions that can make $(A+B)$ idempotent for two idempotent matrices $A$ and $B$.

(ii) For the second one;

We want to find the conditions such that (their difference $(A-B)$ is idempotent):

$$(A-B)^2 = (A-B)$$

Expand $(A-B)^2$

$$(A-B)^2 = (A-B)(A-B)$$
$$(A-B)^2 = A^2 - AB - BA + B^2$$

Since $A$ and $B$ are idempotent matrices, we have that;

$$A^2 = A$$
$$B^2 = B$$

Hence,

$$(A-B)^2 = A - AB - BA + B$$

Is there any other further simplification that can be done? No! There is basically no relationship between $AB$ and $BA$ for matrices, and hence, since no simplification is possible, we do this.

For $(A - B)$ to be idempotent,

$$(A - B)^2 = (A - B)$$

Hence, we compare the right hand side of the true value of $(A - B)^2$ with the condition for it to be idempotent, hence, we have;

$$A - AB - BA + B \textbf{ compared with } A - B$$

No big deal right? Equate the comparison!

$$A - AB - BA + B = A - B$$

$A$ cancels out of the equation, leaving us with:

$$-AB - BA + B = -B$$

Rearranging,

$$2B = AB + BA$$

Hence, the above is the condition for $(A - B)^2$ to be equal to $AB$

Simplifying the above, it is also possible in two ways; first, solving the equation directly, we have;

$$\boldsymbol{B = \frac{AB + BA}{2}}$$

Above is the first condition for $(A - B)$ to be idempotent, but we can't find this condition in the question? Back to the same equation, let's see the second way it can be resolved!

$$2B = AB + BA$$

Also, consider a case where $AB$ and $BA$ are equal, we'll be having a case of:

$$AB = BA$$

Hence,

$$AB + BA = AB + AB = BA + BA$$

Hence, we'll have: since;

$$2B = AB + BA$$

It will become:

$$2B = AB + AB$$

Hence,

$$B = \frac{2AB}{2}$$

Finally,

$$B = AB$$

But, for this condition, we assumed:

$$AB = BA$$

Hence,

$$\boldsymbol{AB = BA = B}$$

The question states that we should prove that their difference, $(A - B)$ is only idempotent if, $AB = BA = B$; hence; the prove is the second one;

As a matter of fact, the above are the only two conditions that can make $(A - B)$ idempotent for two idempotent matrices $A$ and $B$.

(c)

Kindly go to page 175 of matrices for a clearer solution to this question.

$$\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{pmatrix}$$

For this product, the first two matrices are expanded first!

$$\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \cos x\,(0) + \sin x\,(1) & \cos x\,(1) + \sin x\,(0) \\ -\sin x\,(0) + \cos x\,(1) & -\sin x\,(1) + \cos x\,(0) \end{pmatrix}$$

$$\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix}$$

The product of these two is then multiplied against the third bracket to get the product:

$$\begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix}\begin{pmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{pmatrix}$$
$$= \begin{pmatrix} \sin x\,(\sin x) + \cos x\,(\cos x) & \sin x\,(-\cos x) + \cos x\,(\sin x) \\ \cos x\,(\sin x) + -\sin x\,(\cos x) & \cos x\,(-\cos x) + -\sin x\,(\sin x) \end{pmatrix}$$

$$\begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix}\begin{pmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{pmatrix}$$
$$= \begin{pmatrix} \sin^2 x + \cos^2 x & -\sin x \cos x + \sin x \cos x \\ \sin x \cos x - \sin x \cos x & -\sin^2 x - \cos^2 x \end{pmatrix}$$

Now, from trigonometric identities,

$$\sin^2 x + \cos^2 x = 1$$

Also, $-\sin x \cos x + \sin x \cos x$ and the second similar expression $\sin x \cos x - \sin x \cos x$ also cancels out!

Also, from $-\sin^2 x - \cos^2 x$; factoring $-1$ yields:

$$-1(\sin^2 x + \cos^2 x) = -1(1) = -1$$

Hence, the final matrix is:

$$\begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix} \begin{pmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We're told to comment on the nature of the matrix. Let's check for its transpose and inverse and see what relationship occurs, we could see an idempotent, orthogonal or any other type, let's check it out.

Let:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Obviously, the matrix above is a symmetric matrix since the transpose of the matrix is equal to the matrix itself. Checking further;

For a $2 \times 2$ matrix, the adjoint is given by:

$$\operatorname{adj} A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The determinant;

$$|A| = (-1)(1) - (0)(0) = -1$$

Hence,

$$A^{-1} = \frac{1}{-1}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -1\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence,

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hence, once again, it is obvious the inverse of the matrix is equal to the transpose of the matrix;

$$A^T = A^{-1}$$

The matrix hence is also orthogonal.

In essence,
The nature of the product is both symmetric and orthogonal.

## SECTION B

### Question 3

*(a)* An assembly plant orders $x$ units of a product needed in it's operation each time it places an order. The yearly cost on placing orders and maintaining an inventory for the product is given by:

$$f(x) = 4000 + 4x - \frac{10000}{x}$$

what size order should be placed each time if the yearly cost is to be minimized.

*(b)* State Young's theorem and apply it to the function:

$$Z_1 = x^2 + 5xy - y^2$$

*(c)* Use Euler's theorem to determine the degree of homogeneity of the function:

$$Z_2 = Ax^{\alpha}y^{\beta}$$

*(d)* Is the following function harmonic?

$$Z_3 = \sin x \sin y$$

(a)

We know how to minimize the cost function; the marginal cost must be equal to zero.

Our cost function is given by this:

$$f(x) = 4000 + 4x + \frac{10000}{x}$$

Never mind that $f(x)$ is used; that's our cost function nonetheless;

$$f(x) = 4000 + 4x + 10000x^{-1}$$

The marginal cost is given by:

$$MC = f'(x)$$
$$= 0 + 1 \times 4x^{1-1} + (-1 \times 10000x^{-1-1})$$

$$MC = 4 - \frac{10000}{x^2}$$

At minimized cost, marginal cost is zero;
Hence;

$$4 - \frac{10000}{x^2} = 0$$

Solve;
Multiplying through by $x^2$;

$$4x^2 - 10000 = 0$$

Hence;

$$4x^2 = 10000$$

Divide through by 4;

$$x^2 = 2500$$

Take square roots of both sides;

$$x = \pm 50$$

$$x = 50 \quad \text{or} \quad x = -50$$

Hence, cost is minimized when 50 units of $x$ are produced; ***we know the negative value isn't taken.***

(b)

$$Z_1 = x^2 + 5xy - y^2$$

Young's theorem is:

$$\frac{\partial^2 Z_1}{\partial y \partial x} = \frac{\partial^2 Z_1}{\partial x \partial y}$$

Hence;

We'll be finding the indirect second order partial derivatives;

$$Z_1 = x^2 + 5xy - y^2$$

$$\frac{\partial Z_1}{\partial x} = 2 \times x^{2-1} + 1 \times 5x^{1-1}y - 0$$

$$\frac{\partial Z_1}{\partial x} = 2x + 5y$$

$$\frac{\partial Z_1}{\partial y} = 0 + 1 \times 5xy^{1-1} - 2 \times y^{2-1}$$

$$\frac{\partial Z_1}{\partial y} = 5x - 2y$$

Going further for the indirect partial derivatives;

$$\frac{\partial^2 Z_1}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial Z_1}{\partial x}\right) = \frac{\partial}{\partial y}(2x + 5y)$$

$$\frac{\partial^2 Z_1}{\partial y \partial x} = 0 + 1 \times 5y^{1-1}$$

$$\frac{\partial^2 Z_1}{\partial y \partial x} = 5$$

$$\frac{\partial^2 Z_1}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial Z_1}{\partial y}\right) = \frac{\partial}{\partial x}(5x - 2y)$$

$$\frac{\partial^2 Z_1}{\partial x \partial y} = 1 \times 5x^{1-1} - 0$$

$$\frac{\partial^2 Z_1}{\partial x \partial y} = 5$$

Hence;
Obviously;

$$\frac{\partial^2 Z_1}{\partial x \partial y} = \frac{\partial^2 Z_1}{\partial y \partial x}$$

(c)

$$Z_2 = Ax^{\alpha}y^{\beta}$$

Euler's theorem;
The degree of homogeneity, $n$, is:

$$x\frac{\partial Z_2}{\partial x} + y\frac{\partial Z_2}{\partial y} = nZ_2$$

Hence; solving for the first order partials;

$$Z_2 = Ax^{\alpha}y^{\beta}$$

$$\frac{\partial Z_2}{\partial x} = \alpha \times Ax^{\alpha-1}y^{\beta}$$

$$\frac{\partial Z_2}{\partial x} = \alpha Ax^{\alpha-1}y^{\beta}$$

Also;

$$\frac{\partial Z_2}{\partial y} = \beta \times Ax^{\alpha}y^{\beta-1}$$

$$\frac{\partial Z_2}{\partial y} = \beta Ax^{\alpha}y^{\beta-1}$$

Slotting into Euler's theorem;

$$x\left(\alpha Ax^{\alpha-1}y^{\beta}\right) + y\left(\beta Ax^{\alpha}y^{\beta-1}\right) = n\left(Ax^{\alpha}y^{\beta}\right)$$

Expanding the left hand side and sorting out by indices;

$$\alpha A(x^{\alpha-1} \times x)y^{\beta} + \beta Ax^{\alpha}\left(y^{\beta-1} \times y\right) = n\left(Ax^{\alpha}y^{\beta}\right)$$

$$\alpha A(x^{\alpha-1+1})y^{\beta} + \beta Ax^{\alpha}\left(y^{\beta-1+1}\right) = n\left(Ax^{\alpha}y^{\beta}\right)$$

$$\alpha Ax^{\alpha}y^{\beta} + \beta Ax^{\alpha}y^{\beta} = n\left(Ax^{\alpha}y^{\beta}\right)$$

Factorizing the LHS:

$$Ax^{\alpha}y^{\beta}(\alpha+\beta)=n\left(Ax^{\alpha}y^{\beta}\right)$$

Hence; by comparison; the degree of homogeneity, $n$ is:

$$n=\alpha+\beta$$

(d)

$$Z_3=\sin x\sin y$$

For a harmonic function:

$$(Z_3)_{xx}+(Z_3)_{yy}=0$$

Hence, we'll be finding the direct second order partial derivatives;

$$Z_3=\sin x\sin y$$

$$(Z_3)_x=(\cos x)\sin y$$

Please ensure you've read partial derivatives before coming here;

You'll look totally lost if you've not read partial derivatives;

$$(Z_3)_y = \sin x\,(\cos y)$$

As a reminder, when differentiating with respect to $x$, $y$ and in essence, $\sin y$ is nothing but a constant, same in differentiating with respect to $y$, $x$ and in essence, $\sin x$ is nothing but a constant.

Going further;

$$(Z_3)_{xx} = \frac{\partial}{\partial x}((Z_3)_x) = \frac{\partial}{\partial x}\big((\cos x)\sin y\big)$$

$$Z_{3_{xx}} = (-\sin x)\sin y$$

$$Z_{3_{xx}} = -\sin x \sin y$$

$$(Z_3)_{yy} = \frac{\partial}{\partial y}\big((Z_3)_y\big) = \frac{\partial}{\partial y}\big(\sin x\,(\cos y)\big)$$

$$Z_{3_{yy}} = \sin x\,(-\sin y)$$

$$Z_{3_{yy}} = -\sin x \sin y$$

Hence;

$$Z_{3_{xx}} + Z_{3_{yy}}$$

$$-\sin x \sin y + (-\sin x \sin y)$$

$$Z_{3_{xx}} + Z_{3_{yy}} = -2 \sin x \sin y$$

Hence, the function is not harmonic!

## Question 4

*(a)* Find the relative optima, if they exist of the function:

$$y = x^3 + 3x^2 - 9x - 5$$

*(b)* (iii) Define a Lagrangean function;

(iv) Using the Lagrangean multiplier method, find a rectangular consumption basket which has the largest area for a given perimeter.

(a)

To find the relative optima, we find the optimal points, whether maxima, minima or inflexion and go further to find the optima values.

We know the first thing to do when dealing with stationary points; we'll find their first derivatives;

$$f(x) = x^3 + 3x^2 - 9x - 5$$

$$f' = 3 \times x^{3-1} + 2 \times 3x^{2-1} - 1 \times 9x^{1-1} - 0$$

$$f'(x) = 3x^2 + 6x - 9$$

So, at the stationary point; $f'(x) = 0$
Hence, here:

$$3x^2 + 6x - 9 = 0$$

Hence, this is a quadratic equation; we'll take this quadratic equation by factorization; use the quadratic formula if you can't factorize;

$$3x^2 + 9x - 3x - 9 = 0$$

$$3x(x + 3) - 3(x + 3) = 0$$

$$(3x - 3)(x + 3) = 0$$

Hence,

$$(3x - 3) = 0 \text{ or } (x + 3) = 0$$

Break this down;

$$3x - 3 = 0$$

$$3x = 3$$

$$x = 1$$

Also,

$$x + 3 = 0$$
$$x = -3$$

Hence, we have two stationary values; hence, we need to test for their natures;

$$f'(x) = 3x^2 + 6x - 9$$

$$f''(x) = \frac{d}{dx}\left(f'(x)\right) = \frac{d}{dx}(3x^2 + 6x - 9)$$

$$f''(x) = 2 \times 3 \times x^{2-1} + 1 \times 6 \times x^{1-1} - 0$$

$$f''(x) = 6x + 6$$

Now, we know how the natures of the stationary points are gotten, we have two stationary points;

$$x = 1$$
$$x = -3$$

We test for each of the stationary points in the second derivative of the function;

$$f''(x) = 6x + 6$$

At $x = 1$;

$$f'' = 6(1) + 6 = 6 + 6 = 12$$

12 is greater than zero, hence, from our second order conditions, $x = 1$ is a minimum point.

And, similarly,

At $x = -3$;

$$f'' = 6(-3) + 6 = -18 + 6 = -12$$

$-12$ is less than zero, hence, from our second order conditions, $x = -3$ is a maximum point.

To the minimum and maximum values;

Now, we have the minimum point; $x = 1$;

Now, to get the minimum value; we'll be evaluating the value of that function at $x = 1$; now, we have evaluated the value of $x$ (the independent variable) for which the function itself is minimum.

Hence, we be evaluate $f(1)$ to find our minimum value since the corresponding minimum point is $x = 1$;

$$f(x) = x^3 + 3x^2 - 9x - 5$$

$$f(1) = (1)^3 + 3(1)^2 - 9(1) - 5$$

$$f(1) = 1 + 3 - 9 - 5$$

$$f(1) = -10$$

Hence, the **minimum value in this function is $-10$.**

Now, we also have the maximum point; $x = -3$;

Now, to get the maximum value; we'll be evaluating the value of that function at $x = -3$; now, we have evaluated the value of $x$ (the independent variable) for which the function itself is maximum.

Hence, we be evaluate $f(-3)$ to find our maximum value since the corresponding maximum point is $x = -3$;

$$f(x) = x^3 + 3x^2 - 9x - 5$$

$$f(-3) = (-3)^3 + 3(-3)^2 - 9(-3) - 5$$

$$f(-3) = -27 + 27 + 45 - 5$$

$$f(-3) = 40$$

Hence, the **maximum value in this function is 40.**

(b)

(i)

For an objective function; $f(x, y)$

And a constraint function, $g(x, y) = 0$

The Lagrangean function is given by:

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda \times g(x, y)$$

(ii)

Here, we are not given any values but we know the formulas; so who cares?

$$A = l \times b = lb$$

This is the area of the rectangular basket; we need the maximum area and hence the above is the objective function;

The perimeter is;

$$P = 2(l + b)$$

Expanding;

$$P = 2l + 2b$$

Hence, since it is a given perimeter, it means the perimeter is what the desire is meant to be fixed, hence, the constraint function is given above;

Regularly in Lagrangean functions, we'll express the constraint equaled to zero;

$$2l + 2b - P = 0$$

Hence, we'll write our Lagrangean function now with our Lagrangean multiplier involved;

$$\mathcal{L}(l, b, \lambda) = lb - \lambda(2l + 2b - P)$$

Here, our first order partials are:

$$\mathcal{L}_l = 1 \times l^{1-1}b - \lambda(1 \times 2l^{1-1} + 0 - 0)$$

$$\mathcal{L}_l = b - 2\lambda$$

$$\mathcal{L}_b = 1 \times lb^{1-1} - \lambda(0 + 1 \times 2b^{1-1} - 0)$$

$$\mathcal{L}_b = l - 2\lambda$$

$$\mathcal{L}_\lambda = 0 - 1 \times \lambda^{1-1}(2l + 2b - P)$$

$$\mathcal{L}_\lambda = -2l - 2b + P$$

**You can expand the bracket first before the partial differentiation if you seem confused;**

Solve for the first order conditions;

$$\mathcal{L}_l = b - 2\lambda = 0 \ldots\ldots\ldots (1)$$

$$\mathcal{L}_b = l - 2\lambda = 0 \ldots \ldots \ldots (2)$$

$$\mathcal{L}_\lambda = -2l - 2b + P = 0 \ldots \ldots \ldots (3)$$

From (1);

$$b - 2\lambda = 0$$

$$b = 2\lambda \ldots \ldots \ldots (4)$$

From (2);

$$l - 2\lambda = 0$$

$$l = 2\lambda \ldots \ldots \ldots (5)$$

From (4) and (5), it follows that;

$$l = b \ldots \ldots \ldots (6)$$

This is since both are equal to $2\lambda$

Put (6) into (3);

$$-2l - 2b + P = 0$$

$$-2l - 2(l) + P = 0$$

Hence,

$$-P = -4l$$

Hence,

$$l = \frac{P}{4}$$

**Note that *P* isn't a variable but a constant (the perimeter is fixed).**

Since $l = b$

$$b = \frac{P}{4}$$

From (1);

$$b - 2\lambda = 0$$

$$\frac{P}{4} - 2\lambda = 0$$

$$\lambda = \frac{P}{8}$$

Hence, the optimal values for the largest area are;;;

$$l = \frac{P}{4}$$

and the breadth,

$$b = \frac{P}{4}$$

And the largest area will be given thus:

$$A = l \times b$$

At maximum; $l = \frac{P}{4}; b = \frac{P}{4}$

Hence;

$$A = \frac{P}{4} \times \frac{P}{4} = \frac{P^2}{16}$$

And hence, the maximum area will be $\frac{P^2}{16}$ where $P$ is the given perimeter.

## Question 5

*(a)* Find the derivative function in each of the following cases.

(i) $y = x^5 - 4x^4 + 3x^2$

(ii) $z = e^{-2t} - e^{-3t}$

(iii) $w = e^{\cos x - \sin x}$

(iv) $v = \log(at^2 + bt + c)$

*(b)* Integrate each of the following:

(i) $\frac{5x^4}{x^5+16}$

(ii) $(2ax + b)(ax^2 + bx)^7$

(iii) $\log_e 2x$

*(c)* Let:

$$U = \int \frac{\sin x}{a \sin x + b \cos x} dx \text{ and}$$

$$V = \int \frac{\cos x}{a \sin x + b \cos x} dx, \text{ find:}$$

(i) $aU + bV$;
(ii) $aV - bU$.

(a)

Differentiation!
We'll be done in a jiffy! We'll be making assumptions of what we are to differentiate with respect to as they are not stated here;

(i)

$$y = x^5 - 4x^4 + 3x^2$$

With respect to $x$;
Power rule and sums;

$$\frac{dy}{dx} = 5 \times x^{5-1} - 4 \times 4x^{4-1} + 2 \times 3x^{2-1}$$

$$\frac{dy}{dx} = 5x^4 - 16x^3 + 6x$$

(ii)

$$z = e^{-2t} - e^{-3t}$$

With respect to $t$;
Sums and differences!
Each requires chain rule;

$$\frac{d}{dt}(e^{-2t})$$

Basic chain rule;
We'd have:

$$e^{-2t} \times -2 = -2e^{-2t}$$

The second;

$$\frac{d}{dt}(e^{-3t})$$

Basic chain rule;
We'd have:

$$e^{-3t} \times -3 = -3e^{-2t}$$

Hence;

$$z = e^{-2t} - e^{-3t}$$

$$\frac{dz}{dt} = \frac{d}{dt}(e^{-2t}) - \frac{d}{dt}(e^{-3t})$$

$$\frac{dz}{dt} = -2e^{-2t} - (-3e^{-3t})$$

$$\frac{dz}{dt} = 3e^{-3t} - 2e^{-2t}$$

(iii)

$$w = e^{\cos x - \sin x}$$

With respect to $x$;
Substitution!

$$u = \cos x - \sin x$$

$$\frac{du}{dx} = -\sin x - \cos x$$

Hence;

$$w = e^u$$

$$\frac{dw}{du} = e^u$$

Chain rule;

$$\frac{dw}{dx} = \frac{dw}{du} \times \frac{du}{dx}$$

$$\frac{dw}{dx} = e^u \times (-\sin x - \cos x)$$

Return $u$;

$$\frac{dw}{dx} = e^{\cos x - \sin x} \times (-\sin x - \cos x)$$

Hence, by factorization;

$$\frac{dw}{dx} = -(\sin x + \cos x)e^{\cos x - \sin x}$$

(iv)

$$v = \log(at^2 + bt + c)$$

With respect to $t$;
Substitution;

$$u = at^2 + bt + c$$

$$\frac{du}{dt} = 2 \times at^{2-1} + 1 \times bt^{1-1} + 0$$

$$\frac{du}{dt} = 2at + b$$

$$v = \log u$$

Hence;

$$v = \log_{10} u$$

$$\frac{dv}{du} = \frac{1}{u \ln 10}$$

Chain rule;

$$\frac{dv}{dt} = \frac{dv}{du} \times \frac{du}{dt}$$

$$\frac{dv}{dt} = \frac{1}{u \ln 10} \times (2at + b)$$

$$\frac{dv}{dt} = \frac{2at + b}{u \ln 10}$$

Return $u$;

$$\frac{dv}{dt} = \frac{2at + b}{(at^2 + bt + c) \ln 10}$$

These are simple stuffs that you should've been used to already!

(b)

(i)

$$\frac{5x^4}{x^5 + 16}$$

To integrate this, we have

$$\int \frac{5x^4}{x^5 + 16} dx$$

This is a case of

$$\int \frac{f'(x)}{f(x)} dx$$

Hence, put $u = x^5 + 16$

$$\frac{du}{dx} = 5x^{5-1} + 0 = 5x^4$$

Hence,

$$dx = \frac{du}{5x^4}$$

We have;

$$\int \frac{5x^4}{u} \times \frac{du}{5x^4}$$

Hence, $5x^4$ cancels out;

$$\int \frac{1}{u} du$$

From integral rules; this is:

$$[\ln u] + C$$

Return $u = x^5 + 16$

$$\ln(x^5 + 16) + C$$

(ii)

$$(2ax + b)(ax^2 + bx)^7$$

To integrate this, we have

$$\int (2ax + b)(ax^2 + bx)^7 dx$$

This is a case of

$$\int f'(x)g[f(x)]dx$$

Hence, put $u = ax^2 + bx$

$$\frac{du}{dx} = 2 \times ax^{2-1} + 1 \times bx^{1-1}$$

$$\frac{du}{dx} = 2ax + b$$

Hence,

$$dx = \frac{du}{2ax + b}$$

We have;

$$\int (2ax + b)(u)^7 \times \frac{du}{2ax + b}$$

Hence, $2ax + b$ cancels out;

$$\int u^7 du$$

From integral rules; this is:

$$\left[\frac{u^{7+1}}{7+1}\right] + C = \frac{u^8}{8} + C$$

Return $u = ax^2 + bx$

$$\frac{(ax^2 + bx)^8}{8} + C$$

(iii)

$$\log_e 2x$$

A case of substitution;

$$z = 2x$$

$$\frac{dz}{dx} = 2 \times x^{1-1} = 2$$

Hence,

$$dx = \frac{dz}{2}$$

We have;

$$\int \log_e z \times \frac{dz}{2}$$

Bring the constant out;

$$\frac{1}{2}\int \log_e z \, dz$$

**From integration by parts,** we'll see the light as to how to integrate logarithm functions, express $\log_e z$ as multiplied by 1, this was treated in the note; this is:

$$\frac{1}{2}\int 1 \times \log_e z \, dz$$

Facing the integral squarely now;

$$\int 1 \times \log_e z \, dz$$

Integration by parts;

Put

$$u = \log_e z$$

Standard derivative;

$$\frac{du}{dz} = \frac{1}{z}$$

Also;

$$\frac{dv}{dz} = 1$$

Integrate!

$$\int dv = \int 1 dz$$

Here, straight;

$$v = z$$

Hence, we have all we need; rush to the integration by parts formula making the appropriate substitutions for **all terms**:

$$\int u \frac{dv}{dz} dz = uv - \int v \frac{du}{dz} dz$$

$$\int 1 \times \log_e z \, dz = \log_e z \, (z) - \int z \left(\frac{1}{z}\right) dz$$

Simplifying further;

$$\int 1 \times \log_e z \, dz = z \log_e z - \int 1 dz$$

We have reduced the integral to the sum of a term and another integral which should be integrated easily;

So, let's evaluate this integral we have in our reduced form;

$$\int 1 dz = z$$

Hence,
Finally the integral of $\log_e z$ is:

$$\int \log_e z \, dx = z \log_e z - z$$

Since;

$$\int 1 \times \log_e z \, dz = \int \log_e z \, dz$$

But then, we're far from been done! Firstly, $z$ is actually:

$$z = 2x$$

Also;
This is the real value of the integral, we only focused squarely on the integral:

$$\frac{1}{2}\int 1 \times \log_e z\,dz$$

Hence,
Returning everything to normal, we'll be having;

$$\frac{1}{2}\int 1 \times \log_e z\,dz = \frac{1}{2}\left(z\log_e z - z\right)$$

Hence;
We have:

$$\frac{1}{2}(2x\log_e 2x - 2x)$$

Expanding the bracket;

$$\boldsymbol{x\log_e 2x - x}$$

That's our final integral; referring back to the notes on integration by parts, this is nothing too difficult, just notice carefully that in the integral of

$$\int 1 \times \log_e z\,dz$$

The integration by parts formula changes from $\frac{du}{dx}$ and $\frac{dv}{dx}$ to $\frac{du}{dz}$ and $\frac{dv}{dz}$ since we're now working with respect to $z$ owing to the substitution before everything was returned to normal.

(iii)

This has been thrashed in the notes, kindly see the explanation for details; you should have read it before coming to past questions though;

$$U = \int \frac{\sin x}{a \sin x + b \cos x} dx$$

$$V = \int \frac{\cos x}{a \sin x + b \cos x} dx$$

$$aU + bV$$

Go ahead and multiply them;

$$aU = a \int \frac{\sin x}{a \sin x + b \cos x} dx$$

Take the $a$ inside the integral, if it was inside, we could take it outside, so now, let's do the reverse;

$$aU = \int \frac{a(\sin x)}{a\sin x + b\cos x} dx$$

In same way;

$$bV = b\int \frac{\cos x}{a\sin x + b\cos x} dx$$

Take the $b$ inside the integral, if it was inside, we could take it outside, so now, let's do the reverse;

$$bV = \int \frac{b(\cos x)}{a\sin x + b\cos x} dx$$

$aU + bV$ will be given by:

$$\int \frac{a(\sin x)}{a\sin x + b\cos x} dx + \int \frac{b(\cos x)}{a\sin x + b\cos x} dx$$

So, another most basic integral rule; these are split integral; they could be together before they are split, let's bring them together as if we were there before;

$$\int \left(\frac{a(\sin x)}{a\sin x + b\cos x} + \frac{b(\cos x)}{a\sin x + b\cos x}\right) dx$$

Let's add that fraction within; the denominators are the same so we can add them straight with one common denominator;

$$\int \left(\frac{a(\sin x) + b(\cos x)}{a \sin x + b \cos x}\right) dx$$

$$\int \left(\frac{a \sin x + b \cos x}{a \sin x + b \cos x}\right) dx$$

Cancel off!

$$\int (1)\, dx = x + C$$

Hence;

$$aU + bV = x + C$$

The question was actually little of asking you about integration laws but the properties of integration; we'll be treating the second part just like this;

$$aV - bU$$

$$aV = a \int \frac{\cos x}{a \sin x + b \cos x} dx$$

Take the $a$ inside the integral, if it was inside, we could take it outside, so now, let's do the reverse;

$$aV = \int \frac{a\cos x}{a\sin x + b\cos x}dx$$

$$bU = b\int \frac{\sin x}{a\sin x + b\cos x}dx$$

Take the $b$ inside the integral, if it was inside, we could take it outside, so now, let's do the reverse;

$$bV = \int \frac{b\sin x}{a\sin x + b\cos x}dx$$

$aV - bU$ will be given by:

$$\int \frac{a\cos x}{a\sin x + b\cos x}dx - \int \frac{b\sin x}{a\sin x + b\cos x}dx$$

So, another most basic integral rule; these are split integral; they could be together before they are split, let's bring them together as if we were there before;

$$\int \left(\frac{a\cos x}{a\sin x + b\cos x} - \int \frac{b\sin x}{a\sin x + b\cos x}\right)dx$$

Let's subtract that fraction within; the denominators are the same so we can add them straight with one common denominator;

$$\int \left(\frac{a\cos x - b\sin x}{a\sin x + b\cos x}\right) dx$$

It's not looking to cancel each other as the previous part; however, checking the denominator, the numerator could be its derivative;

Let's see;

$$z = a\sin x + b\cos x$$

$$\frac{dz}{dx} = a\cos x + b(-\sin x)$$

$$\frac{dz}{dx} = a\cos x - b\sin x$$

That obviously is the numerator; hence; here, we have a case of integration by substitution;

$$dx = \frac{dz}{a\cos x - b\sin x}$$

Hence, we have;

$$\int \left(\frac{a\cos x - b\sin x}{z}\right) \times \frac{dz}{a\cos x - b\sin x}$$

$a\cos x - b\sin x$ cancels out;

$$\int \left(\frac{1}{z}\right) dz = \ln z$$

Return the substitution;

We have;

$$\ln(a\sin x + b\cos x) + C$$

Of course the arbitrary constant cannot be forgotten, we have that;

$$aV - bU = \ln(a\sin x + b\cos x) + C$$

**DONE!** This year wasn't funny *sha*. Sixty pages of solution!