FUNCTIONS

Well, basically we'll be starting with this topic as it is the simplest; the most straightforward and of course, because it has a relationship with every other topic in the context of SSC106.

Very quickly, the topic function is nothing difficult; however, basic understanding of the meaning of functions is still of extreme importance for your understanding of SSC106 in general.

A function is a mathematical relationship between sets of inputs and a set of permissible outputs with each input related to one output.

Functions find their use in virtually every aspect of life, everyday life and of course in the aspect of calculus.

In everyday life, a set of items in a store can be related to their set of possible prices; in that case, each item will lead to a corresponding price.

In economics, it may be necessary to link the cost of items to the output (the quantity produced) and hence, for every quantity produced, there is a corresponding cost. Thus, when quantities are related such that corresponding to any value of the first quantity, there is a definite value of the second, then we their relationship can be called **a function**.

In full terms, the second quantity is called a function of the first quantity.

You'll get to understand more of this as we move on; basically however, the topic function is a very interesting topic;

<u>The concept of a function;</u>

A function occurs when a set of input(s) yield a certain output and the production of each output obeys the same rule relative to the input(s) used.

From our first definition, we established functions as a relationship between sets of inputs and

permissible outputs; hence, before a mathematical relationship can be called a function, the outputs aren't taken anyhow but they obey a certain rule; this brings a very important concept in the study of functions; **the independent and dependent variable.**

As we have established already, a mathematical relationship isn't called a function unless there is a definite relationship between the set of inputs and their corresponding output.

Remember in the first illustrations given – we illustrated different costs relative to different quantities of goods produced as an example of a function. Now see this – we do not bring a cost to determine the quantity of goods to be produced; we rather check for the cost of producing a certain quantity of goods; so, here, it means the cost is determined by the quantity of goods produced, here, it is fair to say that **the cost of goods depends on the quantity of goods.**

Firstly, it is important to establish the difference between some terms such as **variables, constants** and all.

A variable is defined as a quantity that can assume different numerical values in a given model (i.e. a given situation or case study) or it is simply anything that can change.

Variables can change in different models because the value a certain variable x will take when used to represent number of lions in a zoo could be quite different from the number that'll come up if the same variable is used to represent the numbers of chickens in a poultry.

Variables can be classified mainly in two ways: they are either:

(i) Discrete or continuous;

(ii) Exogenous (independent) or endogenous (dependent)

A **discrete variable** has only fixed or known values that may be identified exactly. They occur in fixed situations, for instance, the number of eggs in a basket, number of children in a family.

A **continuous variable** that can take any value within a given range of value, for instance, the age of a person, temperature, height, etc.

In explaining the difference between the discrete and continuous variables, the number of eggs in a basket cannot change as far as the basket remains what is being talked about, same thing, as far as the same family is being referred to, the number of children remains fixed. However, concerning a person's age, the same person will take another age the next year, same thing for temperature, within a day, the temperature in the morning is different from the temperature in the afternoon, yet same city is being referred to.

Also;

An **endogenous variable** is a variable within a model (i.e. an equation) whose values are determined by other factors within the model, hence, they are also called dependent variables.

An **exogenous variable** is a variable which is independent of other variables within the model and isn't affected by any of the other variables within the model, if at all anything affects such a variable, it is an external factor outside the model.

> *Whether a variable is exogenous or endogenous is dependent on the model being considered. For example, for a farmer considering the rain amongst his determinants of the amount of harvest he gets will see the rain as an exogenous variable as he will see the rain as a gift of nature. However, for a geographer, who understands the effect environmental disrupters in amount of rainfall will see the rain as an endogenous variable that depends on other factors he works with.*

Endogenous and exogenous variables are very important concepts in the study of econometrics and indeed need further study. However, they shouldn't in any way be confused with the independent and dependent variables in **every equation.** We'll be seeing the meaning of the independent variable now.

In the cost and quantity relationship, the cost is called **the dependent variable** while the quantity is called **the independent variable**. This occurs in all forms of functions; a variable determines the value of another. The dependent variable is affected by a change in the independent variable. In our chief illustration, the cost that'll be incurred in producing 100 goods will be changed if the producer changes his mind and produces 80 goods instead.

Hence, in a mathematical function;

The independent variable is the value that is varied in a function to yield the value of the function; it is the symbol (variable) that stands for the arbitrary input in a function.

The dependent variable stands for the arbitrary outputs; it isn't determined on its own but by varying the independent variable; it gets is valued attached to a particular input in a function.

In general mathematics, the commonest symbol for the independent variable is x while the commonest symbol for the dependent variable is y, this doesn't in any way mean those are the only two possible variables usable in a function, there are loads of variables that can be used in a function, we just highlighted the commonest amongst these variables.

It is possible to have multiple independent variables in a function and (or) multiple dependent variables in a function. We'll still get back to that though.

Now, to the main aspect of this whole thing: **the representation of a function**.

Functions are represented in several ways; each way however points to the same thing, a rule that relates input to their corresponding permissive outputs;

The most common way to relate functions is the notation: $f(x)$ which is read as f of x; here the rule of the function is f and the independent variable is x;

When a dependent variable is introduced; we have;

$$y = f(x)$$

Here, we say y is a function of x, i.e. y depends on x. This is the commonest way to represent functions in mathematics;

Other ways to represent functions include graphs, ordered pairs or trios in set theory notation, arrow diagrams in form of mappings and a whole lot, for the sake of this topic though, we'll focus on the function notation and little bit of the graph representation.

So, what is the implication of the function notation? Let's see a very practical example;

$$y = x^2 + 2x$$

This is a dependent variable to independent variable notation, we're shown the rule that y obeys for every value of x; then it means that;

If $x = 2$; we'll go into the rule and substitute for x to get the corresponding value of y;

$$y = x^2 + 2x$$

At $x = 2$;

$$y = (2)^2 + 2(2) = 4 + 4 = 8$$

Hence, $y = 8$ when $x = 2$

Similarly, when $x = -1$

$$y = (-1)^2 + 2(-1) = 1 - 2 = -1$$

Hence, $y = -1$ when $x = -1$

Similarly, when $x = 100$

$$y = (100)^2 + 2(100) = 10000 + 200 = 10200$$

Hence, $y = 10200$ when $x = 100$;
The rule goes on like that till we consider the rule for another function.

Also, let's see in the function notation; where we have:

$$f(x) = 2x - 3x^3$$

Here, to find $f(2)$; we'll replace x with 2 in the function of x;

$$f(2) = 2(2) - 3(2)^3 = 4 - 24 = -20$$

Hence, $f(2) = -20$

Like I mentioned, it isn't limited to x alone neither is the rule representation limited to f alone, let's see this; g of a;

Given:

$$g(a) = a^2 - 7a + 3$$

Find; $g(1)$ and $g(-2)$;

Kk, to find, $g(1)$; replace a with 1 anywhere a appears in $g(a)$; hence, whatever is in the bracket of the rule of the function is the independent variable and is the only variable varied in the function, every other value, like 3 in the above function are constants and are added for any value of the independent variable.

In essence,

Most models usually have constant parameters. **A constant** is a quantity whose value remains constant throughout the analysis of a model (an equation). In the interpretation with graphs, the constant in an equation is the intercept of the graph of the model.

Hence,

$$g(1) = (1)^2 - 7(1) + 3 = 1 - 7 + 3 = -3$$

$$g(1) = -3$$

For $g(-2)$; replace a with -2 anywhere a appears in $g(a)$;

Hence,

$$g(-2) = (-2)^2 - 7(-2) + 3 = 4 + 14 + 3$$

$$g(-2) = 21$$

Hence, $g(-2) = 21$

Notice carefully that 3 is constantly added in both cases; it is a constant in a function and hence is added to every varying quantity in the function;

Functions can be represented with the most weird of symbols, example, in economics, the profit function are sometimes, though rarely, represented by $\pi(q)$ where π is the rule and q is the independent variable; whatever is used to represent the functions, several values of the function are easily gotten by replacing the independent variable with the arbitrary input to yield an output.

Let's round up with some examples;

- If $u = x^3 - 7x + \frac{1}{x}$, find u when $x = 2$.

- If $c'(a) = a^5 - a^3 + b$, find $c'(-3)$

- If $\pi(q) = q^2 - 2q + 18$; find $\pi(100)$

- If $y = x^2 - \frac{1}{x^2} + 4$; find y when $x = 4$

So, simple stuff;

$$u = x^3 - 7x + \frac{1}{x}$$

To find u when $x = 2$, substitutue $x = 2$ in the function.

$$u = (2)^3 - 7(2) + \frac{1}{2} = 8 - 14 + \frac{1}{2}$$

$$u = -\frac{11}{2}$$

$$c'(a) = a^5 - a^3 + b$$

To find $c'(-3)$, substitute $a = -3$ in the function since it's a function of a;

$$c'(-3) = (-3)^5 - (-3)^3 + b$$

$$c'(-3) = -243 - (-27) + b$$

$$c'(-3) = -216 + b$$

$$c'(-3) = b - 216$$

Notice we have done nothing about b, this is because the function is a function of a and hence, b is regarded as nothing else but a constant here; it isn't varying.

$$\pi(q) = q^2 - 2q + 18$$

To find $\pi(100)$; substitute $q = 100$ in the function, like I've said, anything can be used to denote the rule just like in the previous example where c' was used; hence, let's proceed in this one;

$$\pi(100) = (100)^2 - 2(100) + 18$$

$$\pi(100) = 10000 - 200 + 18 = 9818$$

$$y = x^2 - \frac{1}{x^2} + 4$$

To find y when $x = 4$, do the normal;

$$y = (4)^2 - \frac{1}{(4)^2} + 4$$

$$y = 16 - \frac{1}{16} + 4 = \frac{319}{16} = 19.9375$$

So, that's it about that, we need to move into the part that your lecturers love the most; the various types of functions;

But Ops, let's see this too, some thorough examples on working with functions.

- Find $f(-x)$ and $f(2a + 3)$ if:

$$f(x) = x^3 - x^2 + 8$$

Okay? To find $f(-x)$, replace x with $-x$ in the function, $f(x)$; it's also that basic.

$$f(-x) = (-x)^3 - (-x)^2 + 8$$

Notice I have carefully placed the substitution in brackets, as I've done in previous examples, this is because that the whole of x must be substituted for and anything taking the place must be properly put in brackets. The next example will make that clearer.

Here;

$$f(-x) = -x^3 - x^2 + 8$$

Lol, $f(2a+3)$? What's that? It's still basic, making proper use of your brackets, replace x in $f(x)$ with $(2a+3)$

$$f(2a+3) = (2a+3)^3 - (2a+3)^2 + 8$$

Hence,

$$f(2a+3) \\ = (2a+3)(4a^2+12a+9) \\ - (4a^2+12a+9) + 8$$

$$f(2a+3) \\ = 8a^3 + 36a^2 + 54a + 27 - 4a^2 \\ - 12a - 9 + 8$$

$$f(2a+3) = 8a^3 + 32a^2 + 42a + 26$$

Simple as that, it is that basic.

TYPES OF FUNCTIONS

There is practically an inexhaustible list for the different types of functions in the mathematical study of functions; however, we'll give it a try to list as many as possible types of functions that will be needful in this course.

There are several types of functions which are relative to the criteria been used in the classification.

It is possible to group all forms of functions into two separate groups; some other criteria stray around some few functions while some criteria are just limited to a very few functions. Whichever way, we'll be seeing the types of functions now.

All functions can be classified into two basic ways that are shown below:

- The implicit and explicit functions;

All mathematical functions are either implicit or explicit; they are the **primary taxonomy of functions.**

Leave English alone, taxonomy simply means classification. ☺☺☺

An explicit function is a function in which the dependent variable has been given *"explicitly"* in terms of the independent variable.

While that mayn't be an appropriate definition since explicit is still used to describe it, a close look at the meaning of the term explicitly will do the check, explicitly means *strictly* or *solely*,

Hence, when a function is expressed in such a way that the independent variable(s) is (are) solely the terms that make up the function (asides constants), then the function is an explicit function. **The direction of correspondence between the variables in explicit functions is always certain.**

An explicit function is denoted as;

$y = f(x)$; in the single variable situation

$Z = f(x, y)$; in the multivariable situation

Did I just mention something up there? Single-variable and multivariable; Yeah… we'll see that shortly.

Explicit functions are of two forms which will be explained below:

- The single variable and multivariable functions.

The single variable function has one independent variable and one dependent variable, more notably, they have one independent variable.
They're in the form;

$$y = f(x)$$

I have stated that it is more notably one independent variable because as you have seen in several functions above, there are instances when the dependent variable isn't shown in a function at all, where it is replaced with the "*function of*" notation like:

$$f(x) = 3x^5 - 9$$

Hence, examples of single-variable functions are;

$$y = x^2 - 3x + 9$$

$$y = \frac{x}{3} + \frac{1}{x}$$

$$f(x) = x^4 - 5x + \frac{3}{x}$$

The multivariable (multivariate) functions on the other hand have more than one independent variable, that is their most notable characteristics, they also usually also have one dependent variable in situations where the dependent variable is shown; they're in the form;

$$Z = f(x, y)$$

Where Z is a variable (the dependent variable) and x and y are the independent variables; multivariate functions as the name implies are not only restricted to **binary functions** (multivariate functions on two variables) but also to as many independent variables as possible;

Examples include;

$$Z = x^2y - y^2x$$

$$y = u^3 - v^3 + 3$$

$$f(x, y) = x^5 + y^3$$

$$f(x, y, z) = x^3 + y^3 + z^3$$

$$f(x_1, x_2, x_3, x_4) = 2x_1 - 3x_2 + 4(x_3)^3 + \frac{1}{x_4}$$

Hence, with the knowledge of the above, examples of explicit functions are;

$$y = ax^n + bx + c$$

$$y = 2x^6 - 7x$$

$$g(u) = u^3 - 2x + 4$$

$$Z = x^2 + 2xy - y^2$$

$$f(x, y) = 2x^2y + 3y^3$$

Also, explicit functions, the widest forms of functions, can again be classified into two other forms which are the **single-valued function** and the **multi-valued functions**:

A single-valued function has only one value for the dependent variable (or for the function in general) for any chosen value(s) of the independent variable(s). For instance; in:

$$y = x^3 - 14$$

For any chosen value of x, we only get one value of y; example, when $x = 4$, $y = 50$.

Other examples of single-valued functions are:

$$f(x, y, z) = x + y + z$$

$$Z = \sin(x^3 - 4)$$

A multi-valued function on the other hand though, has more than one value for the dependent variable for any value of the independent variable(s). A very good example is the function involving square roots, for example, in:

$$Z = \sqrt{x^2 + y^2}$$

We have two values for Z for any chosen values of the x and y, the independent variables. This is because square roots always have positive and negative values; hence;

When $x = 3$ and $y = 4$, we have that $Z = \pm 5$, i.e. $Z = 5$ or $Z = -5$.

Hence, that's it about explicit functions, they are quite wide and are the commonest types of functions.

An implicit function is a function in which the dependent variable is not expressed *"explicitly"* in terms of the independent variables(s); it is not expressed solely in terms of the independent variable(s) and hence, **in implicit functions, the direction of correspondence of variables in the function is not always certain.** Mostly such functions are equated to a constant or mostly to zero; implicit functions are mostly in the form;

$$R(x, y) = 0$$
$$f(x, y) = c$$

And so on;

Examples of implicit functions are:

$$y - x = 5$$

$$x^2 + y^2 - 1 = 0$$

$$y^4 + x^3 + 17 = 0$$

$$x^2y - 2x + z^4 = 7$$

Although some implicit functions can be converted to explicit; it is mostly not done since it at times add to the ambiguity of the functions and sometimes gives different function branch.

In;

$$x^2 + y^2 - 1 = 0$$

As an example; if converted to an explicit function (by making y the subject), we'll have:

$$y = \sqrt{1 - x^2}$$

With square roots always positive and negative values, we have two branches of y for the same value, the positive and the negative value. Hence, for this reason of discrepancy, the possible conversion of implicit functions to explicit functions sometimes give ***false values*** which do not exist for the variables in the original equation and hence, conversion of implicit functions to explicit functions is not always done.

Implicit functions shouldn't be confused with multivariate functions, in implicit functions all variables are one-sided, but in multivariate functions, a variable depends on more than one variables and the direction of dependence is certain.

Again it is reinstated, all functions are either implicit or explicit, it is the only classification that encapsulates all functions and hence, as stated already, they are called the **primary taxonomy of functions.**

Let's check out the other types of functions which will involve us taking some diagrams, however, before that, let's see some manipulations on functions.

- If:

$$f(x) = \frac{1 - x}{1 + x}$$

Show that:

$$f(-x) = \frac{1}{f(x)}$$

This is quite basic; we'll just evaluate $f(-x)$ which I believe we all already know how to thoroughly manipulate. Afterwards, we'll prove the identity that we are required to.

$$f(x) = \frac{1-x}{1+x}$$

Hence, to get to $f(-x)$, we replaced x in $f(x)$ by $-x$, that has been thoroughly explained already.

$$f(-x) = \frac{1-(-x)}{1+(-x)}$$

Expanding;

$$f(-x) = \frac{1+x}{1-x}$$

If:

$$f(x) = \frac{1-x}{1+x}$$

Then;

$$\frac{1}{f(x)} = \frac{1}{\frac{1-x}{1+x}}$$

By inverse;

$$\frac{1}{f(x)} = \frac{1+x}{1-x}$$

Hence,

Comparing $f(-x)$ and $\frac{1}{f(x)}$, it is obvious they're equal.

- If $f(x, y) = ax^2 + bxy^2 + cx^2y + dy^3$

  Find:

  ✓ $f(1,0)$
  ✓ $f(-1,1)$

Yeah, we will now see how we deal with functions in multivariable cases; we have a function dependent on both x and y;

$$f(x, y) = ax^2 + bxy^2 + cx^2y + dy^3$$

First thing first two be aware of; is that; a, b, c and d are nothing but constants in all these cases since the independent variables are x and y; hence, let's move to the questions.

- ✓ $f(1,0)$

To work with multivariable functions, in a case like this, $f(x,y)$, $f(1,0)$ will imply that, **respectively**, 1 will be taking the place of x and 0 taking the place of y. The order which the independent variables are taken in the function rule is the order we'll use whilst substituting for known values of the independent variables. Hence, since;

$$f(x,y) = ax^2 + bxy^2 + cx^2y + dy^3$$

$$f(1,0) = a(1)^2 + b(1)(0)^2 + c(1)^2(0) + d(0)^3$$

$$f(1,0) = a(1) + 0 + 0 + 0$$

Anywhere zero appears in multiplication, it scatters every other effort of every other numbers; hence;

$$f(1,0) = a$$

- ✓ $f(-1,1)$

Hence, to find $f(-1,1)$, it is as basic as substituting -1 for x and 1 for y;

$$f(x, y) = ax^2 + bxy^2 + cx^2y + dy^3$$

Hence,

$$f(-1,1) = a(-1)^2 + b(-1)(1)^2 + c(-1)^2(1) + d(1)^3$$

$$f(-1,1) = a(1) + b(-1)(1) + c(1)(1) + d(1)$$

$$f(-1,1) = a - b + c + d$$

DO THESE THREE FOR ME;

- For $f(x) = \sqrt{x^2 + 4}$; find:

  ✓ $f(2x)$
  ✓ $f(0)$

Answers;

✓ $f(2x) = \sqrt{4x^2 + 4} = 2\sqrt{x^2 + 1}$
✓ $f(0) = 2$

- For:

$$f(x) = \frac{x-1}{3x+5}$$

Find:

✓ $f\left(\frac{1}{x}\right)$

✓ $\frac{1}{f(x)}$

Answers;

✓ $f\left(\frac{1}{x}\right) = \frac{1-x}{3+5x}$

✓ $\frac{1}{f(x)} = \frac{3x+5}{x-1}$

- If $g(a,b,c) = xa^3 - xyb^2 + zc^3$

Find:

✓ $g(1,0,1)$

✓ $g(1,1,-1)$

Answers;

✓ $g(1,0,1) = x + z$

✓ $g(1,1,-1) = x - xy - z$

Let's now go into intensively defining the various types of functions.

Function types are given with their corresponding examples sketched. The $x-$ axis is conventionally the horizontal axis.

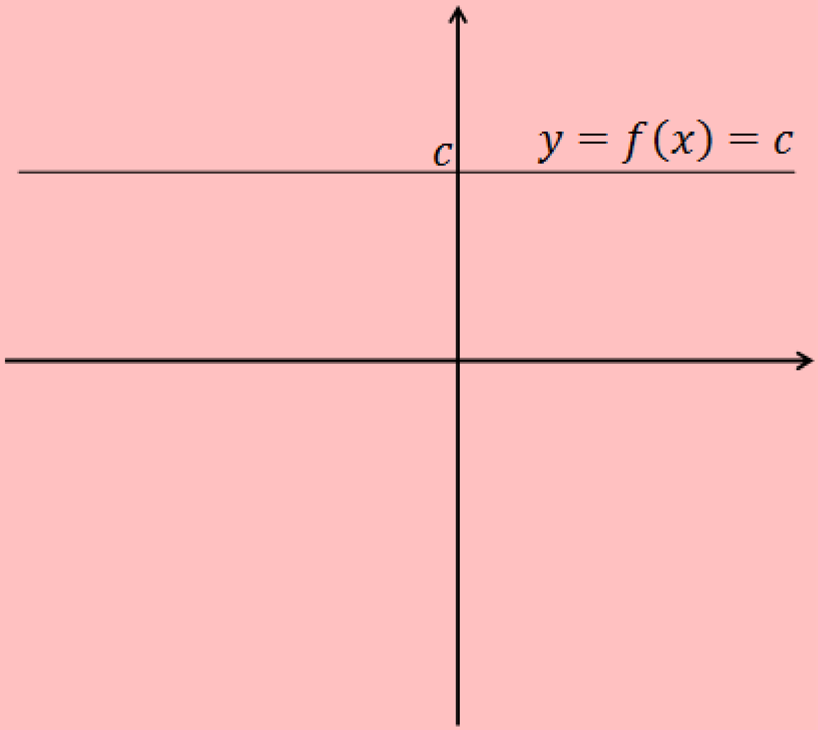
Constant functions;

Constant functions are the simplest types of functions. They are functions that contain only one element which is the constant. For any value of the independent variable, the function remains the same and hence, it has the same value throughout any given interval.

The function is in the form:

$$y = c$$
$$f(x) = c$$

c in the above is a constant.



Sketch of a constant function

Linear functions;

Linear functions are types of functions that are straight lines and are in the form;

$$f(x) = mx + c$$
$$y = mx + c$$

Where m and c are constants;

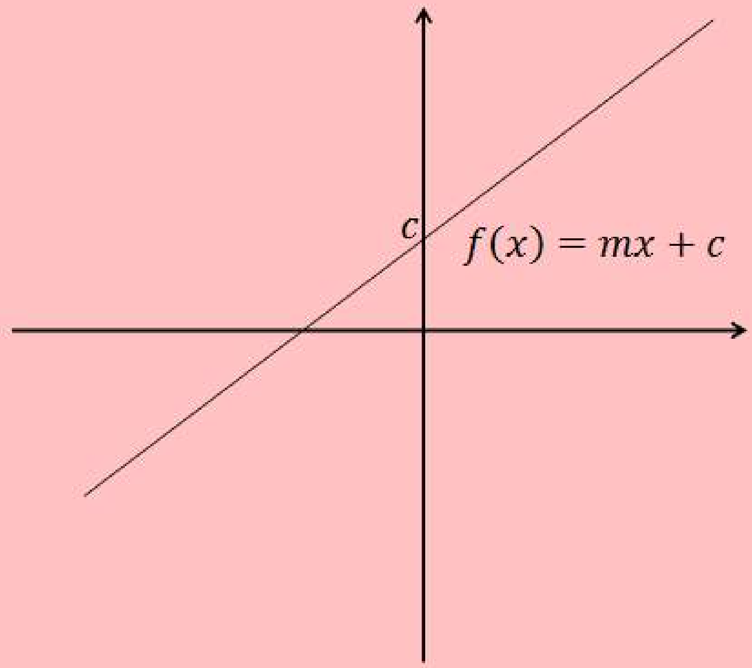
The power of x is always 1 and hence, any function with a power of x higher than or lower than 1 isn't a linear function. Linear functions contain only two possible terms, in term in x and the constant;

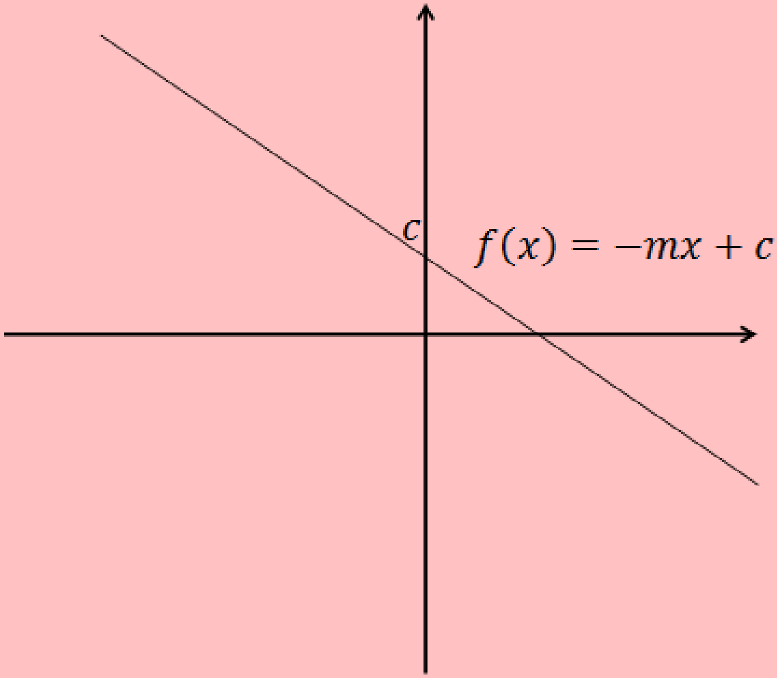
Examples of such functions are given below:

$$y = -5x$$

$$y = 3x - 8$$

$$f(x) = \frac{1}{4}x + 3$$





Sketches of linear functions

Quadratic functions;

They're functions of the form;

$$f(x) = ax^2 + bx + c$$
$$y = ax^2 + bx + c$$

where a, b and c are constants

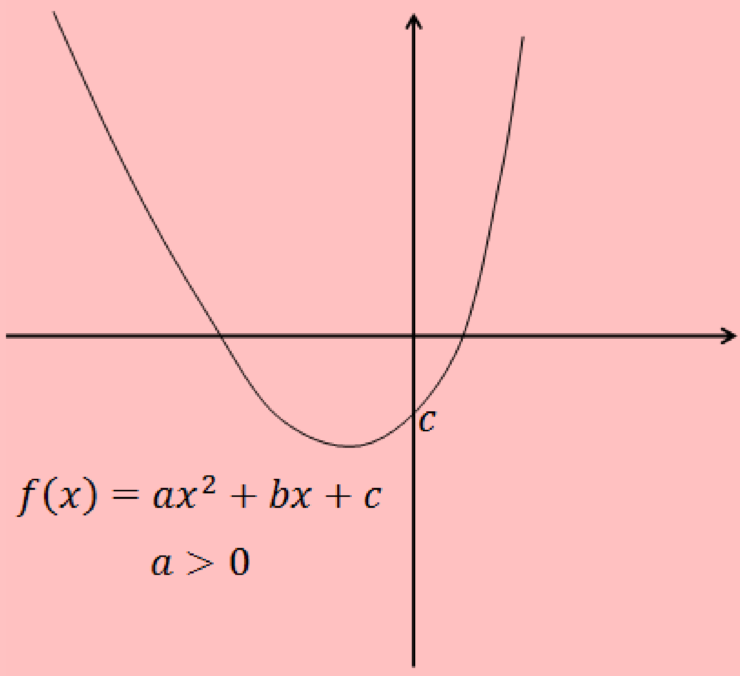
Their graphs are called **parabolas;** they're next to linear functions in simplicity; the nature of the coefficient of x^2 determines the nature of their graph, a positive value yields an open upward parabola while the negative value yields an open downward parabola. Here as well, the highest power of the independent variable is 2. It does not include negative powers too.

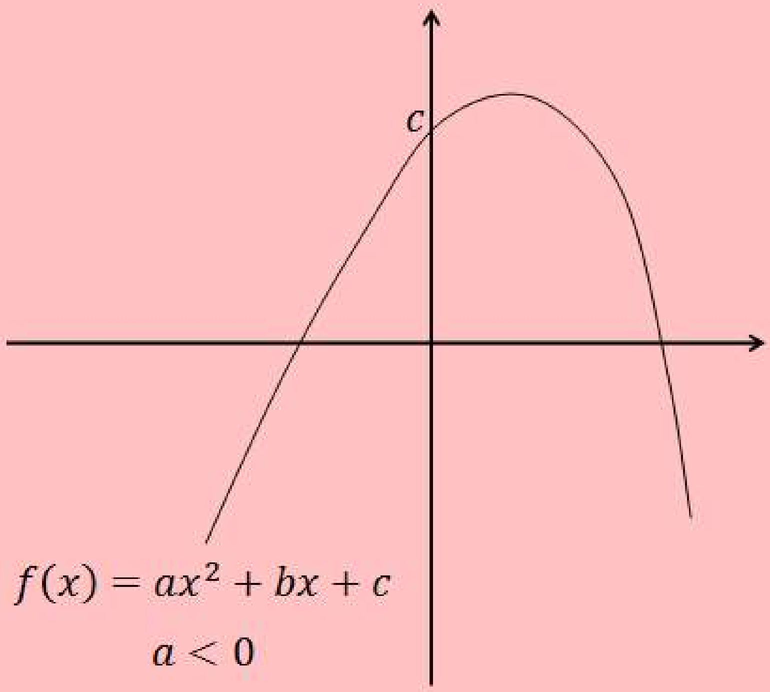
Examples of such functions are given below:

$$y = x^2 + 8x - 3$$

$$f(x) = x^2 + 9$$

$$f(x) = 4x - x^2$$





Sketches of quadratic functions

Cubic functions;

They're functions of the form;

$$f(x) = ax^3 + bx^x + cx + d$$

$$y = ax^3 + bx^x + cx + d$$

Where a, b, c and d are constants

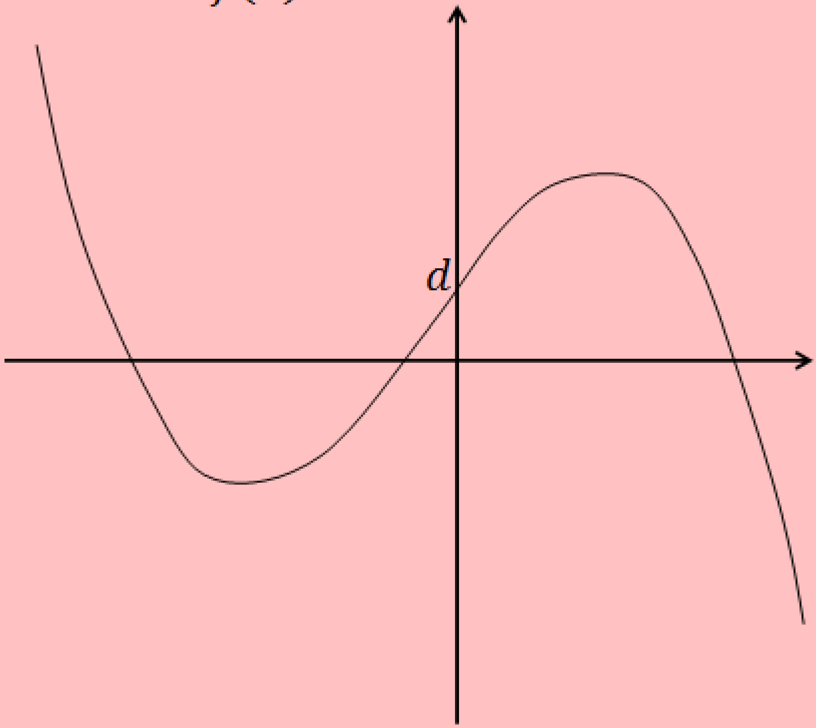
They may or may not contain the terms in x^2, x and the constant, however, the highest power of the independent variable is 3. It does not include negative powers.

Examples of such functions are given below:

$$y = 3x^3 - 2x^2 + 7x + 1$$

$$f(x) = x^3 + 3$$

$$f(x) = ax^3 + bx^2 + cx + d$$



Sketch of a cubic functions

Polynomial functions

All the immediate last three functions (linear, quadratic, cubic) we just considered are polynomial functions, i.e. functions in the form;

$$ax^n + bx^{n-1} + cx^{n-2} \ldots + dx^2 + ex + f$$

In the above, a, b, c, d, e, f are all constants.

This is strictly limited to whole number powers of the independent variable and the highest power of the independent variable is called **the degree of the polynomial.** Hence, a cubic function is a polynomial of the degree 3, the quadratic function, a polynomial of the degree 2 and a linear function, a polynomial of the degree 1.

The curves of polynomial functions are always smooth curves due to their uniform and continuous nature.

$$y = x^5 + 4x^4 - 3x^3 - x^2 - x + 9$$

$$y = \frac{1}{3}x^2 + x + 9$$

$$f(x) = x^3 + 7x^2 - 2x + 7$$

For sketches of a polynomial function, any sketch of a cubic, quadratic or linear function is an ideal example or sketch of a polynomial function.

Nonlinear functions

Any function whose graph is not a straight line is a nonlinear function. In essence, all other forms of functions expect the polynomial function of degree 1 (the linear functions) are not linear functions and hence, the examples of nonlinear functions are quite endless.

Power functions

They're functions of the form;

$$y = ax^b$$

Where a and b are constants; $a \neq 0$ and $b \neq 0$
A power function is different from a polynomial function because the power of x, the independent variable could be any form of number, negative,

positive, fractional and so on, hence, power functions generally give the functions where the independent variable is raised to a constant power.

Examples of such functions are given below:

$$f(x) = 3\sqrt{x} = 3x^{\frac{1}{2}}$$

$$f(x) = \frac{1}{x^2} = x^{-2}$$

Power functions have no peculiarity and hence, have no fixed sketch.

Rational functions;

A rational function is expressed in the form:

$$y = \frac{P(x)}{Q(x)}$$

Where both $P(x)$ and $Q(x)$ are polynomial functions.

A rational function is any function which can be defined by a rational fraction, i.e. an algebraic fraction where both the numerator and the denominator are polynomial functions.

Examples of such functions are given below:

$$y = \frac{x^3 - 2x}{2(x^2 - 5)}$$

$$y = \frac{x^2 + 1}{x}$$

$$y = \frac{5x - 1}{3x + 4}$$

Rational functions have no peculiarity and hence, have no fixed sketch.

Bilinear Rational functions;

This is a subclass of rational functions. A bilinear rational function is also expressed in the form:

$$y = \frac{P(x)}{Q(x)}$$

However, unlike in mere rational functions, we have that both $P(x)$ and $Q(x)$ are linear functions (which of course are also polynomial functions).

Hence, a bilinear rational function is any function which can be defined by a rational fraction where both numerator and denominator are linear functions.

Examples of such functions are given below:

$$y = \frac{x + 1}{x - 1}$$

$$y = \frac{5 - 3x}{4 + 7x}$$

$$y = \frac{5x - 1}{3x + 4}$$

Algebraic functions;

An algebraic function is a function that can be defined as the root of a polynomial equation.

Algebraic functions are algebraic expressions which have a finite number of terms, involving only algebraic operations, addition, subtraction, division, multiplication and such terms are raised to a fractional power.

Algebraic functions are direct opposites of transcendental functions, transcendental functions will be discussed in the latter pages.

Examples of such functions are given below:

$$y = \frac{1}{x}$$

$$y = \sqrt{x}$$

$$y = \frac{\sqrt{1 + x^3}}{x^{\frac{3}{7}} - 7x^{\frac{1}{3}}}$$

Algebraic functions have no peculiarity and hence, have no fixed sketch.

Exponential functions

They're functions of the form;

$$y = ab^x$$

Where a and b are constants; $a \neq 0$, $b > 0$

An exponential function is a type of function where the independent variable is an exponent.

It looks so much like the power function but they're extremely different functions. In power functions, x is raised to a constant power while in exponential functions, a constant is raised to the power of x, the independent variable.

In the exponential function (ab^x), if b is greater than one, the graph is increasing and if it is less than 1, the graph is decreasing.

It is noteworthy that b is not a negative number, hence, $b > 0$.

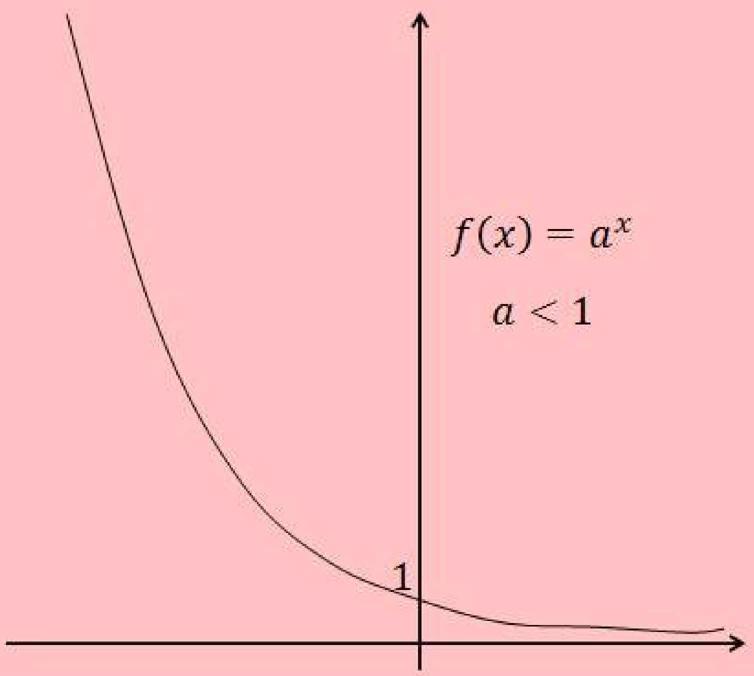
Examples of such functions are given below:

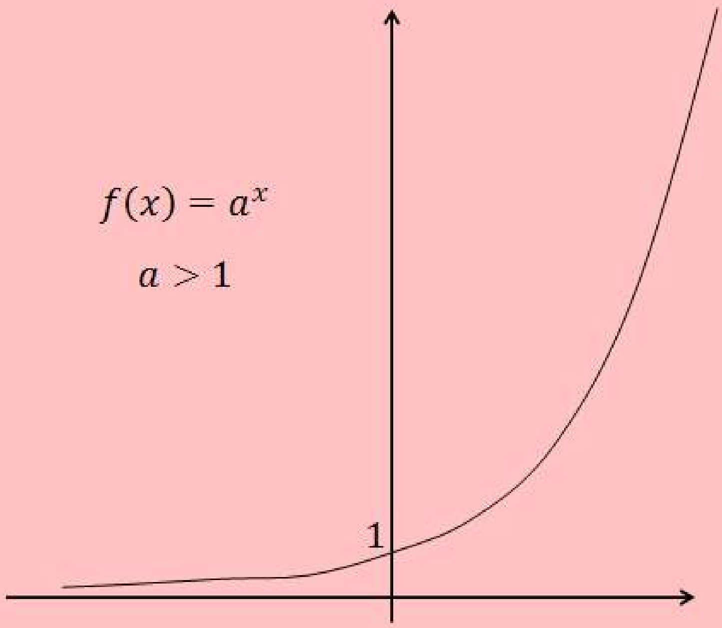
$$y = \left(\frac{1}{2}\right)^x$$

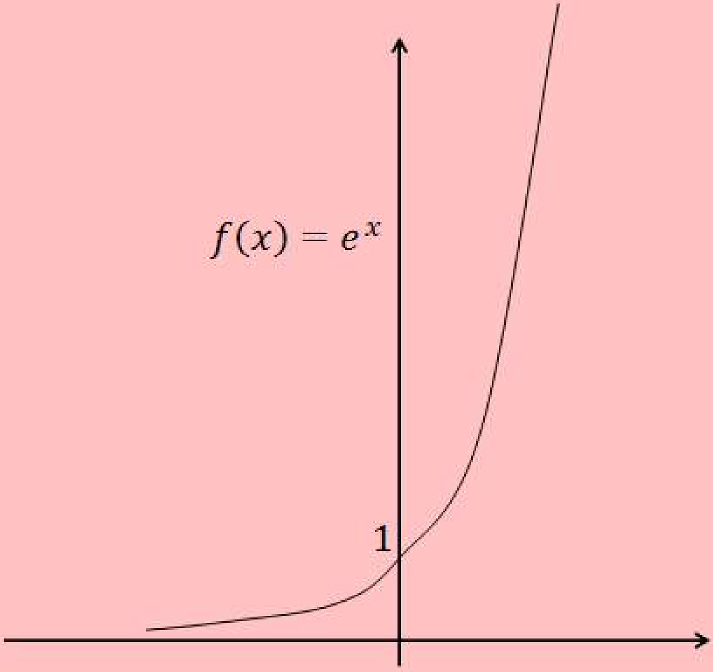
$$f(x) = 4^{3-x^2}$$

$$y = e^{2x-1}$$

e is the Euler's number, $e = 2.718$







Sketches of exponential functions

Logarithm functions

They're many ways to represent such functions but we'll take the forms;

$$y = a\log_b x + b$$

$$y = a\log_e x + b$$

Where a and b are constants; $a \neq 0$, $b \neq 0$

Here, a logarithm function is a function where the independent variable, x is expressed as the logarithm of a number to a certain base, b which is a constant. Logarithms functions do not exist for negative values of x, hence, in logarithm functions, the independent variable only takes a positive value; logarithm functions have a negative value within the range of 0 and 1 (0 and 1 not inclusive) while they are positive for any value greater than 1.

$$f(x) = 3\log_e(3x + 3) + 7$$

$$f(x) = 3\log_5(x^2 - 3)$$

Logarithm functions and exponential functions are very closely related.

In fact, any logarithm function can be written in its equivalent exponential function with the whole thing still meaning the same thing. For example, the function:

$$y = \log_e x$$

Can be written in its equivalent exponential form as:

$$x = e^y$$

Also; to any base b;

$$y = \log_b x$$

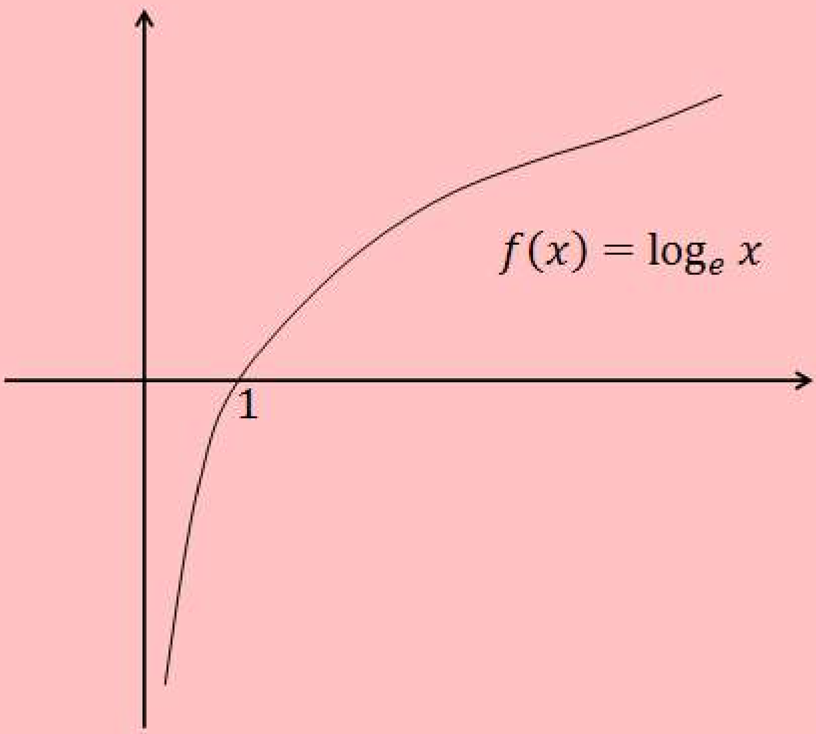
Can be written in its equivalent exponential form as:

$$x = b^y$$

The graphs of the logarithm functions and exponential functions are also very related as they are simply gotten from the other by reflecting the other about the axes.

An understanding of logarithms, which was still discussed in the chapter of the basic operations in mathematics will help understand the close relationship between logarithm and exponential functions.

Note also that the e *used as the base in this logarithm situation is the same as the Euler's number,* $e = 2.718$. *Which of course the form* $\log_e x$ *is the same as the natural logarithm,* $\ln x$.



Sketch of a logarithm functions

Trigonometric functions

They're many ways to represent such functions but we'll take the forms;

$$y = a\sin x + b$$

$$y = a\cos x + b$$

Where a and b are constants; $a \neq 0$

Trigonometric functions, also called angle functions are functions of an angle. They relate the angles of a triangle to the length of its side, hence, the ratio of two distances are expressed as the function of an angle. They're very important in the study of triangles and importantly in the study of periodic phenomena.

The basic trigonometric functions are six; the *sine, cosine, tangent, secant, cosecant* and the *cotangent*

It is noteworthy that:

The secant function is equal to the inverse of the cosine function;

$$\sec x = \frac{1}{\cos x}$$

The cosecant function is equal to the inverse of the sine function;

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

The cotangent function is equal to the inverse of the tangent function;

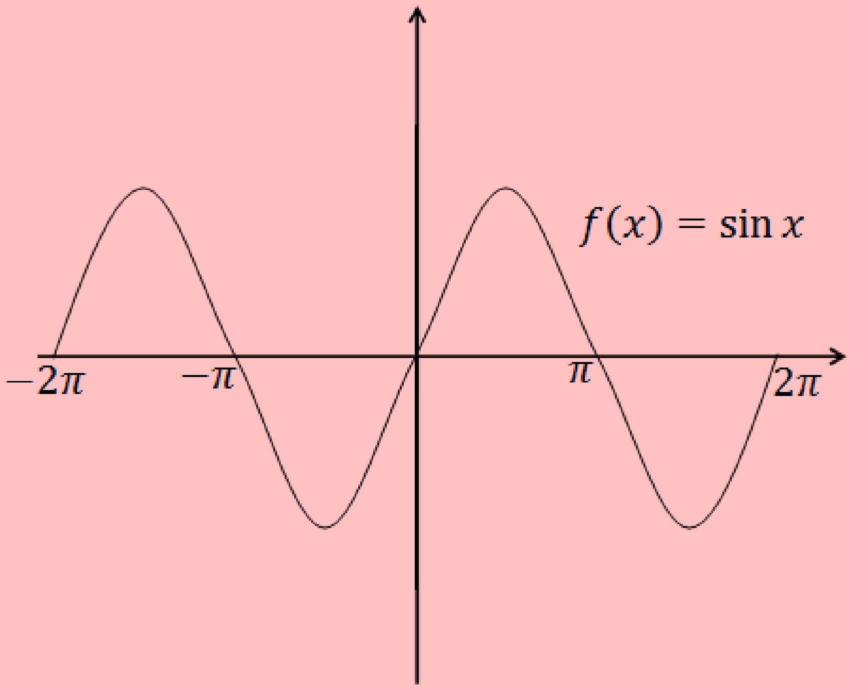
$$\cot x = \frac{1}{\tan x}$$

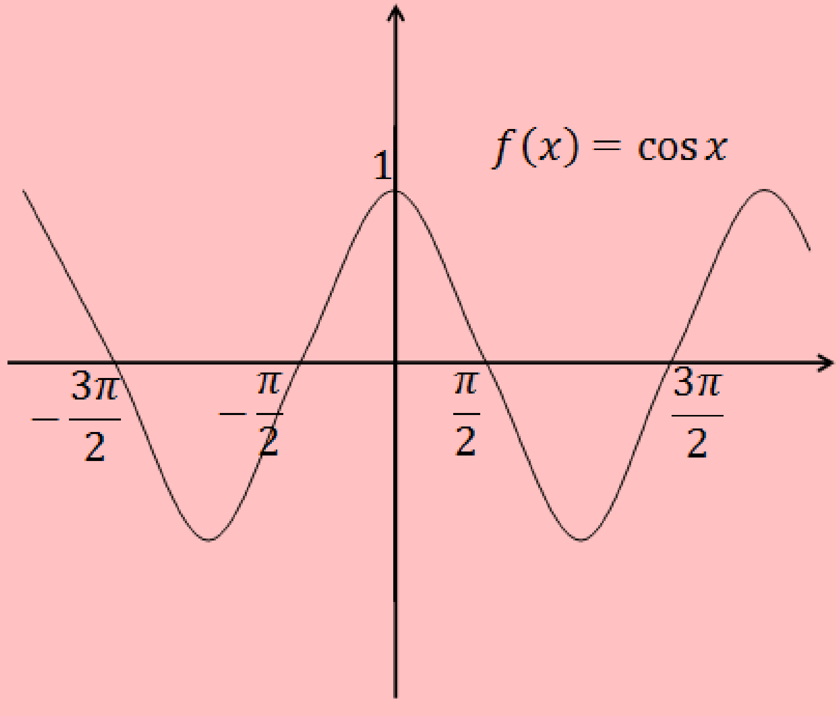
Examples of such functions are given below:

$$y = 5 \sin x + 3$$

$$y = \cos x$$

$$f(x) = 3 \cot(2x^3) - 8$$





Sketches of trigonometric functions

Transcendental functions

A transcendental function is an analytic function **that does not satisfy** a polynomial equation, in a sharp contrast to an algebraic function, it is in essence the opposite of an algebraic function. In other words, a transcendental function *"transcends"* algebra since it cannot be expressed in terms of **a finite sequence** of algebraic operations of addition, subtraction, division and multiplication.

The three main examples of transcendental functions have been discussed already, the exponential function, the logarithm function and the trigonometric function.

As an explanation, the value of sin 30 cannot be expressed as a finite sum of terms, hence, transcendental functions are non-algebraic.

Examples of transcendental functions are the examples of exponential, logarithm and trigonometric functions.

For sketches of transcendental functions, any sketch for exponential, logarithm or trigonometric functions is appropriate.

SOME OTHER TYPES OF FUNCTIONS ARE DISCUSSED BELOW

An additive function preserves the addition operation; thus; in an additive function;

$$f(x+y) = f(x) + f(y)$$

Examples of such functions are given below:

$$f(x) = 3x$$

To explain;
For:

$$f(x) = 3x$$

It means;

$$f(y) = 3y$$

And;

$$f(x+y) = 3(x+y)$$

Expanding;

$$f(x+y) = 3x + 3y$$

But since;

$$f(x) = 3x$$
$$f(y) = 3y$$

It means;

$$f(x) + f(y) = 3x + 3y$$

The above reveals that:

$$f(x + y) = f(x) + f(y)$$

A multiplicative function preserves the multiplication operation; in a multiplicative function;

$$f(x \times y) = f(x) \times f(y)$$

$$f(xy) = f(x)f(y)$$

Examples of such functions are:

$$f(x) = x^6$$

To explain; for:

$$f(x) = x^6$$

It means;

$$f(y) = y^6$$

And;

$$f(xy) = (xy)^6$$

Expanding by indices;

$$f(xy) = x^6y^6$$

But since;

$$f(x) = x^6$$
$$f(y) = y^6$$

It means;

$$f(x) \times f(y) = x^6 \times y^6$$

$$f(x)f(y) = x^6y^6$$

The above reveals that:

$$f(xy) = f(x)f(y)$$

An even function is a function that gives the same value for the negative value of the same independent variable. In an even function:

$$f(-x) = f(x)$$

Examples of even functions are;

$$f(x) = \cos x$$

$$y = x^2$$

$$y = |x|$$

- Show that the following are even functions.

  ✓ $f(x) = x^4 - x^2$
  ✓ $f(x) = \cos x$

<u>Solution</u>

✓ $f(x) = x^4 - x^2$

For an even function,

$$f(-x) = f(x)$$

Hence,
We need to show that this is true for the above function.

$$f(x) = x^4 - x^2$$

We know how to find $f(-x)$;

$$f(-x) = (-x)^4 - (-x)^2$$

We know that even powers of negative bases are end up positive.

$$f(-x) = x^4 - x^2$$

And hence, since;

$$f(x) = x^4 - x^2$$

Then;

$$f(-x) = f(x)$$

✓ $f(x) = \cos x$

For an even function,

$$f(-x) = f(x)$$

Hence,
We need to show that this is true for the above function.

$$f(x) = \cos x$$

We know how to find $f(-x)$;

$$f(-x) = \cos(-x)$$

By trigonometric rules;

$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$

Hence,

$$f(-x) = \cos x$$

And hence, since;

$$f(x) = \cos x$$

Then;

$$f(-x) = f(x)$$

The above trigonometric rules will reveal why $\sin x$ and $\tan x$ are odd functions, odd functions are discussed below.

An odd function is a function that gives the same negative equivalent value for the negative value of the same independent variable. In an even function:

$$f(-x) = -f(x)$$

Examples of odd functions are;

$$f(x) = \sin x$$

$$y = x^3$$

$$y = e^x - e^{-x}$$

$$f(x) = 2x^3 + 3x$$

- Show that the following are odd functions.

✓ $f(x) = 2x^3 + 3x$
✓ $f(x) = e^x - e^{-x}$

<u>Solution</u>

✓ $f(x) = 2x^3 + 3x$

For an odd function,

$$f(-x) = -f(x)$$

Hence,
We need to show that this is true for the above function.

$$f(x) = 2x^3 + 3x$$

We know how to find $f(-x)$;

$$f(-x) = 2(-x)^3 + 3(-x)$$

$$f(-x) = 2 \times -x^3 - 3x$$

$$f(-x) = -2x^3 - 3x$$

Looking at $f(-x)$, it can be factorized;

$$f(-x) = -(2x^3 + 3x)$$

And hence, since;

$$f(x) = 2x^3 + 3x$$

Then;

$$f(-x) = -f(x)$$

✓ $f(x) = e^x - e^{-x}$

For an odd function,

$$f(-x) = -f(x)$$

Hence,
We need to do the same as in the previous example.

$$f(x) = e^x - e^{-x}$$

We know how to find $f(-x)$;

$$f(-x) = e^{(-x)} - e^{-(-x)}$$

$$f(-x) = e^{-x} - e^{x}$$

Looking at $f(-x)$, it doesn't necessarily need to be factorized, however, factorizing -1, let's see what it'll yield;

$$f(-x) = -(-e^{-x} + e^{x})$$

$$f(-x) = -(e^{x} - e^{-x})$$

And hence, since;

$$f(x) = e^{x} - e^{-x}$$

Then;

$$f(-x) = -f(x)$$

Sinusoidal functions;

They're functions that are used for describing relationships whose graph forms are wave-like with respect to the independent variable; sinusoidal functions have a highest value which is called its **amplitude** and sinusoidal functions are also called periodic functions, sinusoidal functions repeat themselves in continuous process;

The graphs of *sine* and *cosine* functions are extremely good examples of sinusoidal functions.

Examples of sinusoidal functions are;

$$y = 5\cos(2x + 3)$$

$$f(x) = \sin(3x - 1)$$

Sinusoidal functions are mainly trigonometric functions and hence, for the sketches of sinusoidal functions, we'll be taking the same sketches for trigonometric functions.

MONOTONIC FUNCTIONS

They're strange types of functions which are either strictly increasing or decreasing, unlike other functions that are increasing and decreasing over separate intervals; the monotonic functions are either always increasing or always decreasing. **They're the classes of increasing or decreasing functions.**

We've seen a quite a lot of monotonic functions in the functions taken already.

However, firstly, it is expedient to take the definitions of increasing and decreasing functions.

Increasing functions are functions that increase in value as the value of the independent functions increase over an interval.

If $f(x)$ is increasing over the interval;

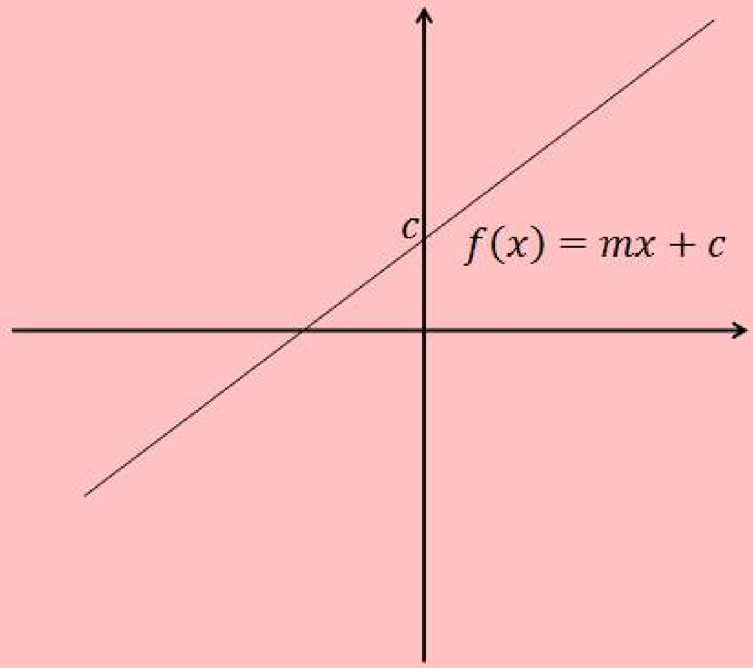
$$a \leq x \leq b$$

Then;

$$f(b) > f(a)$$

For all;

$$b > a \text{ in } a \leq x \leq b$$



Sketch of an increasing function

Decreasing functions are functions that decrease in value as the value of the independent functions increase over an interval.

If $f(x)$ is decreasing over the interval;

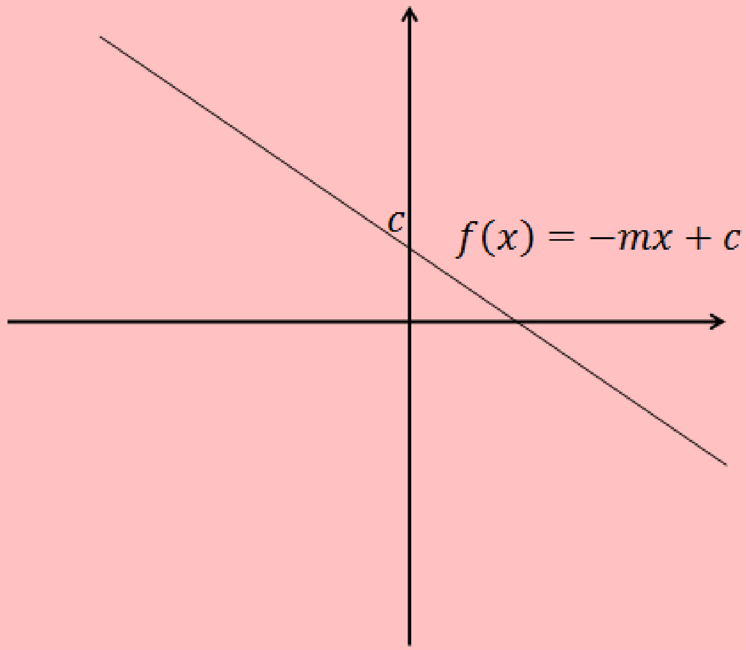
$$a \leq x \leq b$$

Then;

$$f(b) < f(a)$$

For all;

$$b > a \text{ in } a \leq x \leq b$$



Sketch of a decreasing function

Monotonic functions are hence, increasing or decreasing for all values of the function. And hence, not only within an interval, monotonic functions are either **strictly increasing** or **strictly decreasing** for all values of the independent variable.

Hence, for a **strictly monotonously increasing function;**

$$f(b) > f(a)$$

For all;

$$b > a$$

For a **strictly monotonously decreasing function;**

$$f(b) < f(a)$$

For all;

$$b > a$$

The linear functions are good examples of monotonic functions which are straight lines that are either all increasing or all decreasing.

Another very good example of monotonic functions are the exponential functions, they're all increasing functions when the constant being raised to the power of x is greater than 1 and all decreasing when the constant is less than 1.

Examples of monotonic functions are;

$$y = 3x + 7 \text{ (monotonic increasing)}$$

$$y = 8 - 3x \text{ (monotonic decreasing)}$$

$$y = e^{x} \text{ (monotonic increasing)}$$

$$y = \left(\frac{3}{5}\right)^{x} \text{ (monotonic decreasing)}$$

Real life examples of monotonic functions in the world of economics are the demand and supply function.

The supply functions are monotonic increasing since as the price of goods increase, supply increases on the part of the producer.

The demand functions are however monotonic decreasing since as the price of goods increase, demand decreases on the part of the consumers.

A good sketch of monotonic functions is sketches of the linear functions and exponential functions.

Other two types of functions are;

- The homogenous functions
- The harmonic functions

It makes no sense treating the two above functions now, they require knowledge of calculus, and they'll be treated in the chapter; partial differentiation, a chapter in this book, you'll find the definitions of the two in that chapter.

THIS IS THE FIRST MAIN CHAPTER IN THIS BOOK. YOU SHOULD SEE A TRAILER OF WHAT IS TO COME IN THE NEXT CHAPTERS. RELAX, CONCENTRATE AND STUDY AS YOU HAVE A SURE 'A' USING THIS ANDROID APPLICATION FOR YOUR SSC106 THIS SEMESTER.