PARTIAL DIFFERENTIATION

The concept of partial differentiation is something very simple; on a very serious note now, it's really simple.

I'm sure you can remember what multivariate (or multivariable) functions are: That's what we have here, we are not introducing any new rules of differentiation but we are just seeing ways of bringing in differentiation into multivariable functions as well, we have thoroughly dealt with differentiation of functions of just one variable (the independent variable) which are the single-variable functions.

Partial differentiation is the differentiation of multivariable functions which yields several partial derivatives (differential coefficients).

A partial derivative of a function of several variables (the multivariate functions) is its derivative with respect to one of the variables that constitute the function with other variables ***temporarily kept constant.***

Well, yes, I believe you must have seen this last statement one way or the other while studying differentiation in the previous section.

Unlike in derivatives of ordinary single-variable functions, **the swirly-*d* symbol ∂**, called *"tho"* or *"partial"* is used in partial derivatives of multivariable functions to distinguish between ordinary single-variable differential coefficients and partial multi-variable differential coefficients.

Now, before we continue, I'll like you to try this out:

Find $f'(x)$ if $f(x) = 2a^2(x^2 + 1)^3$

Did you get $f'(x) = 12xa^2(x + 1)^2$??? Then you should be on course to fully understanding the concept of partial derivatives;

Now, the function I have placed above is not a multivariate function but a single-variable function, it isn't a single-variable function because we can only see one variable on the right hand side but because we can clearly see that the function is defined as a single function, $f(x)$;

There is something important as well we'll need to understand from the above short differentiation of the single-variable function above, which you're supposed to know now anyway.

It's obvious that a^2 is unchanged in the derivative, now, the function, as a function of x, every other thing found in that function is as good as constant. We mentioned that severally when we treated differentiation. Hence, a is no different, a in this function is as good as a constant and hence, basically takes no effect in the derivative;

So……………$f(x) = 2a^2(x^2 + 1)^3$

$$\text{Put } u = x^2 + 1$$

$$\frac{du}{dx} = 2 \times x^{2-1} + 0 = 2x$$

Hence,

$$f(u) = 2a^2u^3$$

a is a constant;

$$f'(u) = 3 \times 2a^2 \times u^{3-1} = 6a^2u^2$$

$$f'(x) = \frac{df(u)}{du} \times \frac{du}{dx}$$

$$f'(x) = 6a^2u^2 \times 2x = 12xa^2u^2 \;;$$

Return what u is:

$$f'(x) = 12xa^2(x + 1)^2$$

Got the answer above? Then you got partial derivatives as well!

My major point is that a is as good as a constant.

Take that my major point in and let's move on;

Like I said earlier, we're not here to introduce new differentiation rules or something, it's basically the same thing, just few things to note, keep your hands tight and follow me gradually ☺☺☺

NOTATIONS IN PARTIAL DERIVATIVES

The notations in partial derivatives are given below:

Consider the multivariate functions below:

(i) $f(x, y) = 2x^2 - y^3 + 3xy$

(ii) $Z = 2x^3 - y^3 - x^2y$

(iii) $f(x_1, x_2, x_3, \dots, x_n) = \dots \dots \dots$

For the derivative of the function $f(x, y)$ with respect to x is given by: $\frac{\partial f}{\partial x}$ or f_x or $f_x(x, y)$ or f'_x or $\partial_x f$ or $D_x f$; in increasing order of generality;

Similarly, the derivative of the function $f(x, y)$ with respect to y is given by: $\frac{\partial f}{\partial y}$ or f_y or $f_y(x, y)$ or f_y' or $\partial_y f$ or $D_y f$; in increasing order of generality.

In the second example where the function is represented by a dependent variable Z, the derivative of the Z with respect to x is given by: $\frac{\partial z}{\partial x}$ or Z_x or Z_x' ; in increasing order of generality.

Similarly, the derivative of the Z with respect to y is given by: $\frac{\partial z}{\partial y}$ or Z_y or Z_y' ; in increasing order of generality.

The third example is no different, multivariable functions are not limited to two independent variables only, however, as it were, to find their partial derivatives, replace x and y in the notations above in (i) with x_1, x_2 and so on such as f_{x_1}, $\frac{\partial f}{\partial x_1}$ and following the same pattern for the notation. The above notations denote **the notations for the first order partial derivatives.**

I’ll assume you’ve been writing all these hosts of notations!

Let’s now explain this definition thoroughly; the function:

$$f(x) = 2a^2(x^2 + 1)^3$$

Could’ve as well being represented as:

$$f(x, a) = 2a^2(x^2 + 1)^3$$

The above should be understood from our knowledge of multivariate functions. The expression above would only mean that a is also a variable and not a constant, however, it is the case of $f(x)$.

$$f(x) = 2a^2(x^2 + 1)^3$$

It’s glaring a is nothing but an unknown constant!

Hence, in the function above of $f(x, a)$, it’ll be a situation of a multivariate function dependent on x and a. Since we have two variables the function is dependent on, none is better than the other, and hence, there has to be provision for the derivative of the function with respect to both variables;

Hence, for the function $f(x, a)$; the function can be differentiated both with respect to x and with respect to a; that is the meaning of partial differentiation. The differentiation is done partially, with respect to one variable at a time! When differentiating with respect to one variable, the other is assumed constant and vice-versa.

Hence, when differentiating $f(x, a)$ with respect to x, we'll be assuming that $f(x, a)$ is $f(x)$ and we'll evaluate $f'(x)$ which will now be the partial derivative of $f(x, a)$ with respect to x, which is actually, $f_x(x, a)$

In the same way, when differentiating $f(x, a)$ with respect to a, we'll be assuming that $f(x, a)$ is $f(a)$ and we'll evaluate $f'(a)$ which will now be the partial derivative of $f(x, a)$ with respect to a, which is actually, $f_a(x, a)$

This is just a thorough explanation of how partial differentiation pans out, not the assumption will be shown glaringly in real life solutions, let's complete the analysis of $f(x, a)$;

$$f(x, a) = 2a^2(x^2 + 1)^3$$

To find the partial derivative of $f(x, a)$ with respect to x; we assume $f(x, a)$ is $f(x)$; hence,

$$f(x) = 2a^2(x^2 + 1)^3$$

Then we find the derivative with respect to x;

It's been done above but I don't mind repeating it down here:

$$f(x) = 2a^2(x^2 + 1)^3$$

$$\text{Put } u = x^2 + 1$$

$$\frac{du}{dx} = 2 \times x^{2-1} + 0 = 2x$$

Hence,

$$f(u) = 2a^2u^3$$

a is a constant;

$$f'(u) = 3 \times 2a^2 \times u^{3-1} = 6a^2u^2$$

$$f'(x) = \frac{df(u)}{du} \times \frac{du}{dx}$$

$$f'(x) = 6a^2u^2 \times 2x = 12xa^2u^2 \;;$$

Return what u is:

$$f'(x) = 12xa^2(x+1)^2$$

Hence, we conclude that $f_x(x, a)$, the partial derivative with respect to x of $f(x, a)$ is;

$$f_x(x, a) = 12xa^2(x+1)^2$$

But then, when we wanna differentiate $f(x, a)$ partially with respect to a, we then assume the function is a single-variable function of a, $f(a)$; hence, we'll have:

$$f(a) = 2a^2(x^2+1)^3$$

In this case now, every other thing apart from a are rendered as constants, including the very big $(x^2 + 1)$ that is present, hence, we differentiate with respect to a;

$$f'(a) = 2 \times 2a^{2-1}(x^2+1)^3$$

$$f'(a) = 4a(x^2+1)^3$$

Hence, we conclude that $f_a(x, a)$, the partial derivative with respect to a of $f(x, a)$ is;

$$f_a(x, a) = 4a(x^2+1)^3$$

Thorough analysis there right? I believe you should've have understood this properly! That's it, let's take some examples before introducing any other order!

- Consider the function; $f(x, y) = x^2y^3$. Find the first order partials of the function.

The first order partials of the function which is a multivariate function; the function depending on two variables has two first order partial derivatives.

From our notations, the partials with respect to x and y respectively are f_x and f_y

Hence, we have;

$$f_x = \frac{\partial}{\partial x}(x^2y^3)$$

$$f_x = 2 \times y^3 \times x^{2-1} = 2xy^3$$

We differentiate this by regarding y as constant and hence, y^3 as a constant, y isn't regarded as a variable temporarily for the time being when we are differentiating with respect to x.

Notice that as I've said, we don't write out the assumption of taking the function as $f(x)$, however, you can always do that separately in case you need it to be easier for you to relate with.

Now, for the same function, to differentiate with respect to y, we'll now regard x as a constant and take y as the function we'll be differentiating with respect to now.
Hence,

$$f_y = \frac{\partial}{\partial y}(x^2 y^3)$$

$$f_y = 3 \times x^2 \times y^{3-1} = 3x^2 y^2$$

Seriously, partial differentiation is pretty easy; you should be done reading the entire topic in few hours. As usual with SSC106, put your mind into this. It's pretty easy!

Let's do some more partials, it's pretty interesting!

- Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in:

$$u = x^2 + xy + y^2$$

We'll be finding each derivative while taking the other variable as a constant.

Now, the derivative sums and differences still stands here; in short, all differentiation rules still hold, we just take a variable as a constant and treat it as we take a normal differentiation situation.

So,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2 + xy + y^2)$$

$$\frac{\partial u}{\partial x} = 2 \times x^{2-1} + 1 \times x^{1-1}y + 0$$

y^2 is a constant in this situation so the derivative is zero, now in xy, y is like a constant multiplying the function so it doesn't change; you should be coding that;

$$\frac{\partial u}{\partial x} = 2x + (1)(y) = 2x + y$$

Likewise,

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2 + xy + y^2)$$

$$\frac{\partial u}{\partial y} = 0 + 1 \times x \times y^{1-1} + 2 \times y^{2-1}$$

x^2 is a constant in this situation so the derivative is zero, also in xy, x is like a constant multiplying the function so it doesn't change; you should be coding that;

$$\frac{\partial u}{\partial x} = 0 + (1)(x) + 2y = x + 2y$$

That's it, let's solve some more! You love it right?

- Find the $Z_X + 2Z_Y$ given that:

$$Z = X^3 + Y^3 - 2X^2Y$$

Right, we know what to do, Z_X is the partial derivative of Z with respect to X;

We have;

$$Z_X = \frac{\partial}{\partial X}(X^3 + Y^3 - 2X^2Y)$$

$$Z_X = 3 \times X^{3-1} + 0 - 2 \times 2 \times X^{2-1} \times Y$$

Anything other than X is a constant, you should be getting used to that now;

$$Z_X = 3X^2 - 4XY$$

In the same way, we know that Z_Y is the partial derivative of Z with respect to Y;

We have;

$$Z_Y = \frac{\partial}{\partial Y}(X^3 + Y^3 - 2X^2Y)$$

$$Z_Y = 0 + 3 \times Y^{3-1} - 1 \times 2 \times X^2 \times Y^{1-1}$$

$$Z_Y = 3Y^2 - 2X^2$$

Now, going to what we have to evaluate, we get it thus;

$$Z_X + 2Z_Y$$

$$3X^2 - 4XY + 2(3Y^2 - 2X^2)$$

$$3X^2 - 4XY + 6Y^2 - 4X^2$$

$$6Y^2 - 4XY - X^2$$

Now, let's see product rule in partial differentiation.

- Find f_x' and f_y' given that:

$$f(x, y) = (4x - 2y)(3x + 5y)$$

It's just as normal as the most normal differentiation situation;

The product rule with respect to x will be given by:

$$f'_x = v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}$$

As normal as you can think of;

Just the change in the d sign for the differential coefficient symbol to the ∂ for partial differentiation;

Here; we assign our u and v normally;

$$u = (4x - 2y)$$
$$v = (3x + 5y)$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(4x - 2y)$$

$$\frac{\partial u}{\partial x} = 1 \times 4 \times x^{1-1} - 0 = 4$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(3x + 5y)$$

$$\frac{\partial v}{\partial x} = 1 \times 3 \times x^{1-1} + 0 = 3$$

Note that y is regarded as good as a constant in all the above situations, you should be very used to that now; we apply the product rule;

$$f_x' = v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}$$

$$f_x' = (3x + 5y)(4) + (4x - 2y)(3)$$

$$f_x' = 12x + 20y + 12x - 6y$$

$$f'_x = 24x + 14y$$

The product rule with respect to y will be given by:

$$f_y' = v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y}$$

Here; our assigned u and v still stands;

$$u = (4x - 2y)$$
$$v = (3x + 5y)$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(4x - 2y)$$

$$\frac{\partial u}{\partial y} = 0 - 1 \times 2 \times y^{1-1} = -2$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(3x + 5y)$$

$$\frac{\partial v}{\partial y} = 0 + 1 \times 5 \times y^{1-1} = 5$$

We know $x$has been regarded as good as a constant in all the above situations, you should be very used to that now; we apply the product rule;

$$f'_y = v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y}$$

$$f'_y = (3x + 5y)(-2) + (4x - 2y)(5)$$

$$f'_y = -6x - 10y + 20x - 10y$$

$$f'_y = 14x - 20y$$

Let's see quotient rule in partial differentiation.

- Find f'_x and f'_y given that

$$f(x, y) = \frac{5x + y}{x - 2y}$$

Same way, this isn't different from the normal quotient rule situation;

The quotient rule with respect to x will be given by:

$$f'_x = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$$

As normal as you can think of;

Just the change in the d sign for the differential coefficient symbol to the ∂ for partial differentiation;

Here; we assign our u and v normally;

$$u = 5x + y$$
$$v = x - 2y$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(5x + y)$$

$$\frac{\partial u}{\partial x} = 1 \times 5 \times x^{1-1} + 0 = 5$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(x - 2y)$$

$$\frac{\partial v}{\partial x} = 1 \times 1 \times x^{1-1} - 0 = 1$$

Note that y is regarded as good as a constant in all the above situations, you should be very used to that now; we apply the quotient rule;

$$f_x' = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$$

$$f_x' = \frac{(x-2y)(5) - (5x+y)(1)}{(x-2y)^2}$$

$$f_x' = \frac{5x - 10y - 5x - y}{(x-2y)^2}$$

$$f_x' = \frac{-11y}{(x-2y)^2}$$

The quotient rule with respect to y will be given by:

$$f_y' = \frac{v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y}}{v^2}$$

Here; our assigned u and v still stands;

$$u = 5x + y$$
$$v = x - 2y$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(5x + y)$$

$$\frac{\partial u}{\partial y} = 0 + 1 \times 1 \times y^{1-1} = 1$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(x - 2y)$$

$$\frac{\partial v}{\partial y} = 0 - 1 \times 2 \times y^{1-1} = -2$$

We know x has been regarded as good as a constant in all the above situations, you should be very used to that now; we apply the quotient rule;

$$f'_y = \frac{(x-2y)(1) - (5x+y)(-2)}{(x-2y)^2}$$

$$f'_y = \frac{x - 2y + 10x + 2y}{(x-2y)^2}$$

$$f'_y = \frac{11x}{(x-2y)^2}$$

Let's see how we roll in trigonometry situations;

- Find the first order partials of

$$u = \sin(3x + 2y)$$

Cool, this is a trigonometric function of function situation;

For the derivative with respect to x; we have;

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[\sin(3x + 2y)]$$

We need a substitution, just like the normal chain rule situation;

$$z = 3x + 2y$$

As a matter of fact, the chain rule applies here as well, just that we take it with respect to x when finding the partial derivative with respect to x and conversely, we take it with respect to y when finding the partial derivative with respect to y;

$$\frac{\partial z}{\partial x} = 1 \times 3 \times x^{1-1} + 0 = 3$$

Hence,

$$u = \sin z$$

$$\frac{\partial u}{\partial z} = \cos z$$

From chain rule;

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z} \times \frac{\partial z}{\partial x}$$

$$\frac{\partial u}{\partial x} = \cos z \times 3$$

$$\frac{\partial u}{\partial x} = 3 \cos z$$

Returning the value of z; we have

$$\frac{\partial u}{\partial x} = 3\cos(3x + 2y)$$

Using basically the same process but this time, with respect to y and with x taken as a constant, we have;

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[\sin(3x + 2y)]$$

We need the same substitution;

$$z = 3x + 2y$$

Chain rule applies here as well;

$$\frac{\partial z}{\partial y} = 0 + 1 \times 2 \times y^{1-1} = 2$$

Hence,

$$u = \sin z$$

$$\frac{\partial u}{\partial z} = \cos z$$

From chain rule;

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} \times \frac{\partial z}{\partial y}$$

$$\frac{\partial u}{\partial y} = \cos z \times 2$$

$$\frac{\partial u}{\partial y} = 2\cos z$$

Returning the value of z; we have

$$\frac{\partial u}{\partial y} = 2\cos(3x + 2y)$$

Let's see an exponential situation;

- Find Z_x and Z_y in $Z = e^{3x^2-4y}$

Basically the same whole thing!

This is an exponential function of function situation;

For the derivative with respect to x, we'll be taking y as a constant; we have;

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(e^{3x^2-4y}\right)$$

As usual in functions of functions (chain rule), we need a substitution;

$$u = 3x^2 - 4y$$

As a matter of fact, the chain rule applies here as well;

$$\frac{\partial u}{\partial x} = 2 \times 3 \times x^{2-1} + 0 = 6x$$

Hence,

$$Z = e^u$$

$$\frac{\partial Z}{\partial u} = e^u$$

From chain rule;

$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial u} \times \frac{\partial u}{\partial x}$$

$$\frac{\partial Z}{\partial x} = e^u \times 6x$$

$$\frac{\partial Z}{\partial x} = 6xe^u$$

Returning the value of u; we have

$$\frac{\partial Z}{\partial x} = 6xe^{3x^2-4y}$$

Using basically the same process but this time, with respect to y and with x taken as a constant, we have;

$$\frac{\partial Z}{\partial y} = \frac{\partial}{\partial y}\left(e^{3x^2-4y}\right)$$

We need the same substitution;

$$u = 3x^2 - 4y$$

As a matter of fact, the chain rule applies here as well;

$$\frac{\partial u}{\partial y} = 0 - 1 \times 4 \times y^{1-1} = -4$$

Hence,

$$Z = e^u$$

$$\frac{\partial Z}{\partial u} = e^u$$

From chain rule;

$$\frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial u} \times \frac{\partial u}{\partial y}$$

$$\frac{\partial Z}{\partial y} = e^u \times -4$$

$$\frac{\partial Z}{\partial y} = -4e^u$$

Returning the value of u; we have

$$\frac{\partial Z}{\partial y} = -4e^{3x^2-4y}$$

Let's move to a logarithm situation;

- Let $u = \log_2(2x^2 + y^2)$; find $u_x - u_y$.

Basically the same whole thing!

This is a logarithm function of function situation;

For the derivative with respect to x, we'll be taking y as a constant; we have;

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[\log_2(2x^2 + y^2)]$$

We need a substitution;

$$z = 2x^2 + y^2$$

The chain rule applies here as well;

$$\frac{\partial z}{\partial x} = 2 \times 2 \times x^{2-1} + 0 = 4x$$

Hence,

$$u = \log_2 z$$

$$\frac{\partial u}{\partial z} = \frac{1}{z \ln 2}$$

From chain rule;

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z} \times \frac{\partial z}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{1}{z \ln 2} \times 4x$$

$$\frac{\partial u}{\partial x} = \frac{4x}{z \ln 2}$$

Returning the value of z; we have

$$u_x = \frac{\partial u}{\partial x} = \frac{4x}{\ln 2 \, (2x^2 + y^2)}$$

Using basically the same process but this time, with respect to y and with x taken as a constant, we have;

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[\log_2(2x^2 + y^2)]$$

We need a substitution;

$$z = 2x^2 + y^2$$

The chain rule applies here as well;

$$\frac{\partial z}{\partial y} = 0 + 2 \times y^{2-1} = 2y$$

Hence,

$$u = \log_2 z$$

$$\frac{\partial u}{\partial z} = \frac{1}{z \ln 2}$$

From chain rule;

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} \times \frac{\partial z}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{1}{z \ln 2} \times 2y$$

$$\frac{\partial u}{\partial y} = \frac{2y}{z \ln 2}$$

Returning the value of z; we have

$$u_y = \frac{\partial u}{\partial y} = \frac{2y}{\ln 2\,(2x^2 + y^2)}$$

Hence, to evaluate what we are told to;

$$u_x - u_y$$

Substitute what we've solved already!

$$\frac{4x}{\ln 2\,(2x^2 + y^2)} - \frac{2y}{\ln 2\,(2x^2 + y^2)}$$

Same denominator so we can add straight!

$$\frac{4x - 2y}{\ln 2\,(2x^2 + y^2)} = \frac{2(2x - y)}{\ln 2\,(2x^2 + y^2)}$$

Hence,

$$u_x - u_y = \frac{2(2x - y)}{\ln 2\,(2x^2 + y^2)}$$

You know I'll always have tougher questions for us to solve together, that's why I explain it well.

- Given that $y = 4\sin 3x \cos 2t$; find the value of $3(y_x)^3(y_t)^2$

Cool, it's easy to be tempted that this is a product rule question but far from it, it isn't!

When we take the derivative with respect to x, we'll have that we'll be taking t as a constant, hence, $\cos 2t$ in essence is also a constant, hence, only $\sin 3x$ is regarded as a function;

$$y = 4\cos 2t \sin 3x$$

So put;

$$z = 3x$$

$$\frac{\partial z}{\partial x} = 1 \times 3 \times x^{1-1} = 3$$

Hence,

$$y = 4\cos 2t \sin z$$

Derivative of $\sin z$ is $\cos z$

$$\frac{\partial y}{\partial z} = 4\cos 2t \cos z$$

From chain rule;

$$y_x = \frac{dy}{dz} \times \frac{dz}{dx}$$

$$y_x = 4\cos 2t \cos z \times 3$$

$$y_x = 12\cos 2t \cos z$$

So return the substitution, $z = 3x$

$$y_x = 12\cos 2t \cos 3x$$

In the same way,

When we take the derivative with respect to t, we'll have that we'll be taking x as a constant, hence, $\sin 3x$ in essence is also a constant, hence, only $\cos 2t$ is regarded as a function;

$$y = 4\sin 3x \cos 2t$$

So put;

$$z = 2t$$

$$\frac{\partial z}{\partial t} = 1 \times 2 \times t^{1-1} = 2$$

Hence,

$$y = 4\sin 3x \cos z$$

Derivative of $\cos z$ is $-\sin z$

$$\frac{\partial y}{\partial z} = 4\sin 3x\,(-\sin z)$$

$$\frac{dy}{dz} = -4\sin 3x \sin z$$

From chain rule;

$$y_t = \frac{dy}{dz} \times \frac{dz}{dt}$$

$$y_t = -4 \sin 3x \sin z \times 2$$

$$y_t = -8 \sin 3x \sin z$$

So return the substitution, $z = 2t$

$$y_t = -8 \sin 3x \sin 2t$$

We are to evaluate this;

$$3(y_x)^3(y_t)^2$$

Hence; we have:

$$3(12 \cos 2t \cos 3x)^3(-8 \sin 3x \sin 2t)^2$$

Distribute the powers on the brackets;

$$3(12)^3(\cos 2t)^3(\cos 3x)^3(-8)^2(\sin 3x)^2(\sin 2t)^2$$

$$331776(\cos^3 2t \cos^3 3x \sin^2 3x \sin^2 2t)$$

SECOND ORDER PARTIAL DERIVATIVES

The concept of higher order partial derivatives is the next in the concept of partial derivatives; very similar to higher derivatives, it involves further differentiation of partial derivatives; hence, when a partial derivative is differentiated once again, we get a **higher order partial derivative**.

For the scope of this course, we'll limit this to the **second order partial derivative**. It's nothing different from what you know so far, since it is actually like the concept of higher derivatives in the single-variable function. It also involves continuous derivatives of functions.

Now, consider the function on x, y; $f(x, y)$

The first order derivatives are:

$$\frac{\partial}{\partial x}[f(x,y)] \quad \text{and} \quad \frac{\partial}{\partial y}[f(x,y)]$$

We express them in the form;

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}$$

$$f_x \quad \text{and} \quad f_y$$

And other notations which we've established already; so consider the following operations;

The partial derivatives above; they also constitute functions on x and y;

- Taking the partial derivative of $\frac{\partial f}{\partial x}$ with respect to x, we'll be having;

  $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$ which from what we've learnt in differentiation already will be:

$$\frac{\partial^2 f}{\partial x^2}$$

- Taking the partial derivative of $\frac{\partial f}{\partial x}$ with respect to y, we'll be having;

  $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ which from what we've learnt in differentiation already will be:

$$\frac{\partial^2 f}{\partial y \partial x}$$

  This is because the derivatives multiplying at the denominator are unlike;

- Taking the partial derivative of $\frac{\partial f}{\partial y}$ with respect to x, we'll be having;

  $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ which from what we've learnt in differentiation already will be:

$$\frac{\partial^2 f}{\partial x \partial y}$$

  This is because the derivatives multiplying at the denominator are unlike;

- Taking the partial derivative of $\frac{\partial f}{\partial y}$ with respect to y, we'll be having;

  $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$ which from what we've learnt in differentiation already will be:

$$\frac{\partial^2 f}{\partial y^2}$$

Notice these ***two second order partials*** shown below:

$$\frac{\partial^2 f}{\partial x^2} \text{ and } \frac{\partial^2 f}{\partial y^2}$$

They are called ***direct second order partial derivatives*** since they're differentiated partially with respect to the same variable twice;

Now, notice also these two:

$$\frac{\partial^2 f}{\partial y \partial x} \text{ and } \frac{\partial^2 f}{\partial x \partial y}$$

They're quite different from each other, in short very different, this is because the first involves the partial derivative with respect to x first then with respect to y; the second involves the partial derivative with respect to y first and then with respect to x; The above to derivatives are called ***mixed/crossed/indirect second order partial derivatives***.

Now, in the form of function notation;

f_{xx} denotes the direct second order partial derivative with respect to x; this is the same with the variable notation: $\frac{\partial^2 f}{\partial x^2}$

f_{yy} denotes the direct second order partial derivative with respect to y; this is the same with the variable notation: $\frac{\partial^2 f}{\partial y^2}$

f_{xy} denotes the indirect second order partial derivative, where it is differentiated with respect to x first, and then with respect to y; this is the same with the variable notation: $\frac{\partial^2 f}{\partial y \partial x}$

f_{yx} denotes the indirect second order partial derivative, where it is differentiated with respect to y first, and then with respect to x; this is the same with the variable notation: $\frac{\partial^2 f}{\partial x \partial y}$

The other notations will fall in place; hence, just like we know how to take it calmly in the first order partial derivatives, we can take the **mixed partials** calmly as well.

Let's take some examples;

- Find the first and second partial derivatives of the function;

$$z = 3x^2 + 4xy - 5y^2$$

We'll be finding each derivative while taking the other variable as a constant.

So,

$$z_x = \frac{\partial}{\partial x}(3x^2 + 4xy - 5y^2)$$

$$z_x = 2 \times 3 \times x^{2-1} + 1 \times 4 \times x^{1-1}y - 0$$

Everything concerning y is regarded constant, you should have coded that already;

$$z_x = 6x + 4y$$

Likewise,

$$z_y = \frac{\partial}{\partial y}(3x^2 + 4xy - 5y^2)$$

$$z_y = 0 + 4 \times 1 \times x \times y^{1-1} + 2 \times 5 \times y^{2-1}$$

Everything concerning x is regarded constant, you should have coded that already;

$$z_y = 4x - 10y$$

That's not all this time around, we'll proceed to the second order partials which will be done by finding the partial derivative of z_x and z_y each with respect to x and y respectively.

Taking partial derivative of z_x with respect to x

$$(z_x)_x = z_{xx} = \frac{\partial}{\partial x}(6x + 4y)$$

Anything other than x is regarded as a constant

$$z_{xx} = 1 \times 6 \times x^{1-1} + 0$$

$$z_{xx} = 6$$

Taking partial derivative of z_x with respect to y

$$(z_x)_y = z_{xy} = \frac{\partial}{\partial y}(6x + 4y)$$

Anything other than y is regarded as a constant

$$z_{xy} = 0 + 1 \times 4 \times y^{1-1}$$

$$z_{xy} = 4$$

Taking partial derivative of z_y with respect to x

$$(z_y)_x = z_{yx} = \frac{\partial}{\partial x}(4x - 10y)$$

Anything other than x is regarded as a constant

$$z_{yx} = 1 \times 4 \times x^{1-1} - 0$$

$$z_{yx} = 4$$

Taking partial derivative of z_y with respect to y

$$\left(z_y\right)_y = z_{yy} = \frac{\partial}{\partial y}(4x - 10y)$$

Anything other than y is regarded as a constant

$$z_{yy} = 0 - 1 \times 10 \times y^{1-1}$$

$$z_{yy} = -10$$

We can notice something between z_{xy} and z_{yx}; we can see that both are equal;

Let's not jump, let's see another example just over here!

- Find the first and second partial derivatives of the function;

$$u = 5x^3 + 3x^2y + 4y^3$$

Again, we'll be finding each derivative while taking the other variable as a constant.

So,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(5x^3 + 3x^2y + 4y^3)$$

$$\frac{\partial u}{\partial x} = 3 \times 5 \times x^{3-1} + 2 \times 3 \times x^{2-1}y - 0$$

Everything concerning y is regarded constant, you should have coded that already;

$$\frac{\partial u}{\partial x} = 15x^2 + 6xy$$

Likewise,

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(5x^3 + 3x^2y + 4y^3)$$

$$\frac{\partial u}{\partial y} = 0 + 1 \times 3 \times x^2 \times y^{1-1} + 3 \times 4 \times y^{3-1}$$

Everything concerning x is regarded constant, you should have coded that already;

$$\frac{\partial u}{\partial y} = 3x^2 + 12y^2$$

In same wise, we'll proceed to the second order partials which will be done by finding the partial derivative of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ each with respect to x and y respectively.

Taking partial derivative of $\frac{\partial u}{\partial x}$ with respect to x

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(15x^2 + 6xy)$$

Anything other than x is regarded as a constant

$$\frac{\partial^2 u}{\partial x^2} = 2 \times 15 \times x^{2-1} + 0$$

$$\frac{\partial^2 u}{\partial x^2} = 30x$$

Taking partial derivative of $\frac{\partial u}{\partial x}$ with respect to y

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}(15x^2 + 6xy)$$

Anything other than y is regarded as a constant

$$\frac{\partial^2 u}{\partial y \partial x} = 0 + 1 \times 6 \times x \times y^{1-1}$$

$$\frac{\partial^2 u}{\partial y \partial x} = 6x$$

Taking partial derivative of $\frac{\partial u}{\partial y}$ with respect to x

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}(3x^2 + 12y^2)$$

Anything other than x is regarded as a constant

$$\frac{\partial^2 u}{\partial x \partial y} = 2 \times 3 \times x^{2-1} + 0$$

$$\frac{\partial^2 u}{\partial x \partial y} = 6x$$

Taking partial derivative of $\frac{\partial u}{\partial y}$ with respect to y

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(3x^2 + 12y^2)$$

Anything other than y is regarded as a constant

$$\frac{\partial^2 u}{\partial y^2} = 0 + 2 \times 12 \times y^{2-1}$$

$$\frac{\partial^2 u}{\partial y^2} = 24y$$

Again, we can notice something between $\frac{\partial^2 u}{\partial y \partial x}$ and $\frac{\partial^2 u}{\partial x \partial y}$; we can see that both are equal; That is becoming more than just a coincidence, we can jump into a conclusion now:

That brings the concept of the ***equality of mixed partials*** known as **the Young's theorem**. This concept is also called the ***symmetry of second derivatives;***

It states that two complementary second order mixed partials of a continuous and twice differentiable function are equal; for a multivariate function dependent on x and y;

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

In function notation;

$$f_{xy} = f_{yx}$$

The Young's theorem isn't only constrained to functions dependent on two variables, generally; for every multivariate function;

$$f(x_1, x_2, x_3, \ldots, x_n);$$

$$f_{x_i x_j} = f_{x_j x_i}$$

Where i and j are 1, 2, 3 representing the variables that make up the function; this rule generally exists for **all multivariate functions** that can be differentiated <u>at least twice</u>; their indirect second order partials are always equal, there isn't any need to confirm with any examples under this;

Let's see other examples very quickly; we'll chip other things in subsequently, make sure you don't forget the Young's theorem we just discussed; it is a very important theorem in partial differentiation.

- Find the first and second partial derivatives of the function;

$$z = x \cos y - y \cos x$$

We'll be finding each derivative while taking the other variable as a constant.

So,

$$z_x = \frac{\partial}{\partial x}(x\cos y - y\cos x)$$

$$z_x = 1 \times x^{1-1}\cos y - y \times -\sin x$$

Everything concerning y is regarded constant, the major functions as far as these are concerned are x and $\cos x$ each multiplied by constants $\cos y$ and y respectively; Just needed to explain that but I believe you should have gotten that logic already; derivative of $\cos x$ is $-\sin x$

$$z_x = \cos y + y\sin x$$

Likewise,

$$z_y = \frac{\partial}{\partial y}(x\cos y - y\cos x)$$

$$z_y = x \times -\sin y - 1 \times y^{1-1}\cos x$$

Everything concerning x is regarded constant, the major functions as far as these are concerned are $\cos y$ and y each multiplied by constants x and $\cos x$ respectively; You should have gotten that logic already; derivative of $\cos y$ is $-\sin y$

$$z_y = -x\sin y - \cos x$$

We'll proceed to the second order partials which will be done by finding the partial derivative of z_x and z_y each with respect to x and y respectively.

Taking partial derivative of z_x with respect to x

$$(z_x)_x = z_{xx} = \frac{\partial}{\partial x}(\cos y + y \sin x)$$

Anything other than x is regarded as a constant

$$z_{xx} = 0 + y \times \cos x$$

Derivative of $\sin x$ is $\cos x$, $\cos y$ is a constant;

$$z_{xx} = y \cos x$$

Taking partial derivative of z_x with respect to y

$$(z_x)_y = z_{xy} = \frac{\partial}{\partial y}(\cos y + y \sin x)$$

Anything other than y is regarded as a constant

$$z_{xy} = -\sin y + 1 \times y^{1-1} \sin x$$

$$z_{xy} = -\sin y + \sin x$$

$$z_{xy} = \sin x - \sin y$$

Taking partial derivative of z_y with respect to x

$$\left(z_y\right)_x = z_{yx} = \frac{\partial}{\partial x}(-x\sin y - \cos x)$$

Anything other than x is regarded as a constant

$$z_{yx} = -1 \times x^{1-1}\sin y - (-\sin x)$$

$$z_{yx} = -\sin y + \sin x$$

$$z_{yx} = \sin x - \sin y$$

Taking partial derivative of z_y with respect to y

$$\left(z_y\right)_y = z_{yy} = \frac{\partial}{\partial y}(-x\sin y - \cos x)$$

Anything other than y is regarded as a constant

$$z_{yy} = -x \times (\cos y) - 0$$

$$z_{yy} = -x\cos y$$

$$z_{yy} = -x\cos y$$

Normally, we aren't surprised that $z_{xy} = z_{yx}$ since the function can be differentiated twice; Young's theorem holds for ***all multivariate functions***. It serves as a kind of check for the correctness of your workings in first and second order partial derivatives since it is true for all functions.

More,

- If $V = \ln(x^2 + y^2)$; find the first and second order partial derivatives of V: Also, find:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}$$

Basically the same whole thing we've been doing!

This is a logarithm function of function situation;

For the derivative with respect to x, we'll be taking y as a constant; we have;

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}[\ln(x^2 + y^2)]$$

We need a substitution, as per chain rule;

$$z = (x^2 + y^2)$$

The chain rule applies here as well;

$$\frac{\partial z}{\partial x} = 2 \times x^{2-1} + 0 = 2x$$

Hence,

$$V = \ln z$$

$$\frac{\partial V}{\partial z} = \frac{1}{z}$$

From chain rule;

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial z} \times \frac{\partial z}{\partial x}$$

$$\frac{\partial V}{\partial x} = \frac{1}{z} \times 2x$$

$$\frac{\partial u}{\partial x} = \frac{2x}{z}$$

Returning the value of z; we have

$$\frac{\partial V}{\partial x} = \frac{2x}{(x^2 + y^2)}$$

Using basically the same process but this time, with respect to y and with x taken as a constant, we have;

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}[\ln(x^2 + y^2)]$$

We need a substitution; same substitution still goes;

$$z = (x^2 + y^2)$$

The chain rule applies here as well;

$$\frac{\partial z}{\partial y} = 0 + 2 \times y^{2-1} = 2y$$

Hence,

$$V = \ln z$$

$$\frac{\partial V}{\partial z} = \frac{1}{z}$$

From chain rule;

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial z} \times \frac{\partial z}{\partial y}$$

$$\frac{\partial V}{\partial y} = \frac{2y}{z}$$

Returning the value of z; we have

$$\frac{\partial V}{\partial y} = \frac{2y}{(x^2 + y^2)}$$

We'll proceed to the second order partials which will be done by finding the partial derivative of $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$ each with respect to x and y respectively, this will be quite a tedious process with quotients involved here.

Taking partial derivative of $\frac{\partial V}{\partial x}$ with respect to x

$$\frac{\partial V}{\partial x} = \frac{2x}{(x^2 + y^2)}$$

This isn't different from the normal quotient rule situation;

The quotient rule with respect to x will be given by:

$$\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) = \frac{\partial^2 V}{\partial x^2} = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$$

As normal as you can think of;

Just the change in the d sign for the differential coefficient symbol to the ∂ symbol for partial differentiation;

Here; we assign our u and v normally;

$$u = 2x$$
$$v = x^2 + y^2$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(2x)$$

$$\frac{\partial u}{\partial x} = 1 \times 2 \times x^{1-1} + 0 = 2$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2)$$

$$\frac{\partial v}{\partial x} = 2 \times x^{2-1} + 0 = 2x$$

Note that y is regarded as good as a constant in all the above situations, you should be very used to that now; we apply the quotient rule;

$$\frac{\partial^2 V}{\partial x^2} = \frac{(x^2 + y^2)(2) - (2x)(2x)}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{2x^2 + 2y^2 - 4x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}$$

Taking partial derivative of $\frac{\partial V}{\partial x}$ with respect to y

$$\frac{\partial V}{\partial x} = \frac{2x}{(x^2 + y^2)}$$

This isn't different from the normal quotient rule situation;

The quotient rule with respect to y will be given by:

$$\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial x}\right) = \frac{v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y}}{v^2}$$

Here; we assign our u and v normally;

$$u = 2x$$
$$v = x^2 + y^2$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(2x)$$

$$\frac{\partial u}{\partial y} = 0$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2)$$

$$\frac{\partial v}{\partial y} = 0 + 2 \times y^{2-1} = 2y$$

Note that x is regarded as good as a constant in all the above situations, you should be very used to that now; we apply the quotient rule;

$$\frac{\partial^2 V}{\partial y \partial x} = \frac{(x^2 + y^2)(0) - (2x)(2y)}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 V}{\partial y \partial x} = \frac{-4xy}{(x^2 + y^2)^2}$$

Taking partial derivative of $\frac{\partial V}{\partial y}$ with respect to x

$$\frac{\partial V}{\partial y} = \frac{2y}{(x^2 + y^2)}$$

This isn't different from the normal quotient rule situation;

The quotient rule with respect to x will be given by:

$$\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y}\right) = \frac{v_1 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial v_1}{\partial x}}{v^2}$$

Here; we assign our u_1 and v_1 as u and v have been used in this same question for the quotient rule of $\frac{\partial V}{\partial x}$;

$$u_1 = 2y$$
$$v_1 = x^2 + y^2$$

$$\frac{\partial u_1}{\partial x} = \frac{\partial}{\partial x}(2y)$$

$$\frac{\partial u_1}{\partial x} = 0$$

$$\frac{\partial v_1}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2)$$

$$\frac{\partial v_1}{\partial x} = 2 \times x^{2-1} + 0 = 2x$$

Note that y is regarded as good as a constant in all the above situations, you should be very used to that now; we apply the quotient rule;

$$\frac{\partial^2 V}{\partial x \partial y} = \frac{(x^2 + y^2)(0) - (2y)(2x)}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 V}{\partial y \partial x} = \frac{-4xy}{(x^2 + y^2)^2}$$

Taking partial derivative of $\frac{\partial V}{\partial y}$ with respect to y

$$\frac{\partial V}{\partial y} = \frac{2y}{(x^2 + y^2)}$$

This isn't different from the normal quotient rule situation;

The quotient rule with respect to y will be given by:

$$\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) = \frac{v_1 \frac{\partial u_1}{\partial y} + u_1 \frac{\partial v_1}{\partial y}}{v^2}$$

Here; we assign our u_1 and v_1 as u and v have been used in this same question for the quotient rule of $\frac{\partial V}{\partial y}$;

$$u_1 = 2y$$
$$v_1 = x^2 + y^2$$

$$\frac{\partial u_1}{\partial y} = \frac{\partial}{\partial y}(2y)$$

$$\frac{\partial u_1}{\partial y} = 1 \times 2 \times y^{1-1} = 2$$

$$\frac{\partial v_1}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2)$$

$$\frac{\partial v_1}{\partial y} = 0 + 2 \times y^{2-1} = 2y$$

Note that x is regarded as good as a constant in all the above situations, you should be very used to that now; we apply the quotient rule;

$$\frac{\partial^2 V}{\partial y^2} = \frac{(x^2 + y^2)(2) - (2y)(2y)}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 V}{\partial y^2} = \frac{2x^2 + 2y^2 - 4y^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 V}{\partial y^2} = \frac{2x^2 - 2y^2}{(x^2 + y^2)^2}$$

To find:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}$$

$$\frac{2y^2 - 2x^2}{(x^2 + y^2)^2} + \frac{2x^2 - 2y^2}{(x^2 + y^2)^2}$$

$$\frac{2y^2 - 2x^2 + 2x^2 - 2y^2}{(x^2 + y^2)^2}$$

$$\frac{0}{(x^2 + y^2)^2} = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Asides what we've evaluated, it's nothing surprising that $\frac{\partial^2 V}{\partial y \partial x}$ is equal to $\frac{\partial^2 V}{\partial x \partial y}$. It's normal, that's the rule of **Young's theorem.** It serves as a kind of check for the correctness of your workings in first and second order partial derivatives since it is true for all functions.

This result however, isn't going to be taken casually; it's a special type of equation on its own.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

We're introducing the concept of the **Laplace's equation** and the **harmonic functions**. The Laplace equation is a second-order partial differentiation named after **Pierre-Simon Laplace** who first studied its properties.

It also exists over twice differentiable functions. The **Laplace equation** is the equation of the sum of all direct second order partial derivatives on special types of twice continuously differentiable functions which are generally called the **harmonic functions.**

The Laplace equation for $f(x_1, x_2, \dots, x_n)$ is:

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0$$

$$f_{x_1x_1} + f_{x_2x_2} + \cdots + f_{x_nx_n} = 0$$

For a three variable function; $f(x, y, z)$; the equation is:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

$$f_{xx} + f_{yy} + f_{zz} = 0$$

For a two variable function; $f(x, y)$; the equation is:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

$$f_{xx} + f_{yy} = 0$$

The Laplace equation is another unique equation just like the Young's theorem. However, the Laplace equation is basically for unique functions, it is **very far from true for all functions**.

So, let's take one more example of a function obeying the Laplace equation and conclude this; we take the Euler's theorem and the Jacobian theorem and we're good.

- Show that the function $z = e^x \sin y$ is a harmonic function.

To prove that it is a harmonic function; we have to prove that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

Basically the same whole thing we've been doing!

This looks like a product rule situation, however, it isn't as both terms are in different terms are in different variables, hence, when we are taking their partials, one becomes a constant;

For the derivative with respect to x, we'll be taking y as a constant; we have;

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[e^x \sin y]$$

$$\frac{\partial z}{\partial x} = e^x \times \sin y = e^x \sin y$$

e^x is a matter of a standard differential rule and $\sin y$ is as good as a constant;

Using basically the same process but this time, with respect to y and with x taken as a constant, we have;

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[e^x \sin y]$$

$$\frac{\partial z}{\partial y} = e^x \times \cos y = e^x \cos y$$

We'll proceed to the second order partials which in this case, we need only the direct second order partials to plug into the Laplace equation to confirm the harmonic nature of the given function.

We'll do this finding the partial derivative of $\frac{\partial z}{\partial x}$ each with respect to x and $\frac{\partial z}{\partial y}$ with respect to y.

Taking partial derivative of $\frac{\partial z}{\partial x}$ with respect to x

$$\frac{\partial z}{\partial x} = e^x \sin y$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(e^x \sin y)$$

The derivative of e^x remains e^x, $\sin y$ is a constant; everything not x is constant here;

$$\frac{\partial^2 z}{\partial x^2} = e^x \times \sin y = e^x \sin y$$

Taking partial derivative of $\frac{\partial z}{\partial y}$ with respect to y

$$\frac{\partial z}{\partial y} = e^x \cos y$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(e^x \cos y)$$

The derivative of $\cos y$ is $-\sin y$, e^x is a constant; everything not y is constant here;

$$\frac{\partial^2 z}{\partial y^2} = e^x \times -\sin y = -e^x \sin y$$

So, it’s time to check for this;

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

$$e^x \sin y + (-e^x \sin y)$$

$$e^x \sin y - e^x \sin y = 0$$

Hence, it is confirmed that the function is harmonic; since the sum of its direct second order partial derivatives is zero.

Basically, that's it about partial derivatives; we just need to see the two laws I promised above;

Before then, let's make do with this refreshment;

- Find the first and second order partials of

$$z = \ln(e^x + e^y)$$

Show that:

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$$

Cool,

Nothing new anyway! This is a logarithm function of function situation;

For the derivative with respect to x, we'll be taking y as a constant; we have;

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[\ln(e^x + e^y)]$$

We need a substitution;

$$u = e^x + e^y$$

The chain rule applies here as well;

$$\frac{\partial u}{\partial x} = e^x + 0 = e^x$$

Hence,

$$z = \ln u$$

$$\frac{\partial z}{\partial u} = \frac{1}{u}$$

From chain rule;

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \times \frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{1}{u} \times e^x$$

$$\frac{\partial z}{\partial x} = \frac{e^x}{u}$$

Returning the value of u; we have

$$\frac{\partial z}{\partial x} = \frac{e^x}{e^x + e^y}$$

Using basically the same process but this time, with respect to y and with x taken as a constant, we have;

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[\ln(e^x + e^y)]$$

We need a substitution; same substitution still goes;

$$u = e^x + e^y$$

The chain rule applies here as well;

$$\frac{\partial u}{\partial y} = 0 + e^y = e^y$$

Hence,

$$z = \ln u$$

$$\frac{\partial z}{\partial u} = \frac{1}{u}$$

From chain rule;

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \times \frac{\partial u}{\partial y}$$

$$\frac{\partial z}{\partial y} = \frac{1}{u} \times e^{y}$$

$$\frac{\partial z}{\partial y} = \frac{e^{y}}{u}$$

Returning the value of u; we have

$$\frac{\partial z}{\partial y} = \frac{e^{y}}{e^{x} + e^{y}}$$

Hence, we were told to prove this:

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$$

Let's prove the LHS:

$$\frac{e^{x}}{e^{x} + e^{y}} + \frac{e^{y}}{e^{x} + e^{y}}$$

$$\frac{e^{x} + e^{y}}{e^{x} + e^{y}} = 1$$

The whole thing cancels off into 1 and our proof is done!

So I think we can now check out for the remaining two remaining;

EULER'S THEOREM

The **Euler's theorem** deals with the concept of **homogenous functions** and is used to find the homogeneity of homogenous functions.

A homogenous function is one with multiplicative scaling behaviour, for a homogenous function, if all its arguments (the variables that make up the function) are multiplied by a factor; then the function is multiplied by some power of the factor;

Relax with all those English I'm using to compound your head, as usual, follow me. That's a reminder that you should've been writing since though. You also need the above for the sake of definition though.

For example; if a function; $f(x, y)$ is homogenous;

Let a multiply both x and y; then we have;

$$f(ax, ay) = a^k f(x, y)$$

Then the function is homogenous to the degree, k; the power of the factor multiplying the arguments is k and k is referred to as the **degree of homogeneity of the function**;

For example now;

- Find the degree of homogeneity of the function; $f(x, y) = x^2 + y^2$; let's see the fundamental rule of homogeneity;

Let a multiply all the argument of $f(x, y)$, then we have;

$$f(ax, ay) = (ax)^2 + (ay)^2$$

$$f(ax, ay) = a^2x^2 + a^2y^2$$

Factorize a^2;

$$f(ax, ay) = a^2(x^2 + y^2)$$

The power of a, the multiplying factor is 2 and hence, the homogeneity of the function is 2.

Definitely not all functions are homogenous; let's not go along taking too many examples; let's see what **Euler** has to say about homogenous functions because that is what you'll be really needing in proving homogenous functions in SSC106!

Euler has this to say; consider the condition for the homogeneity of $f(x, y)$;

$$f(ax, ay) = a^n f(x, y)$$

Where n is the homogeneity of the function, $f(x, y)$;

Then: given the first order partials of; $f(x, y)$

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$$

So, ***the sum of the product of the arguments and their first order partial derivatives of a function is equal to the degree of homogeneity of a function multiplying the function;*** that is the statement of **Euler's theorem**.

The rule is that simple; so let's apply this rule to find the homogeneity of some functions; let's see these;

Verify if the following functions are homogenous or not and find their homogeneity if they are homogenous;

- $U = x^2 + y^2$
- $U = \ln(x + y)$
- $f(x, y, z) = x^5y^2z^3$
- $U = x^5 + 2x^3y^2 + 9xy^4$
- $U = x + 5$
- $f(x, y) = \frac{x}{x^2+y^2}$
- $Z = \log_e\left(\frac{x}{y}\right)$

Cool,
First;

$$\boldsymbol{U = x^2 + y^2}$$

$$U_x = 2x^{1-1} + 0 = 2x$$

$$U_y = 0 + 2y^{1-1} = 2y$$

From the Euler's homogeneity theorem;

$$xU_x + yU_y = nU$$

Of course don't get it twisted; you should be fully aware of the differences in function notation and variable notations of partial derivatives; the relationship above is still the same as Euler's theorem;

Hence;

$$x(2x) + y(2y) = n(x^2 + y^2)$$

$$2x^2 + 2y^2 = n(x^2 + y^2)$$

Factorize on the left hand side;

$$2(x^2 + y^2) = n(x^2 + y^2)$$

Hence, here, $n = 2$; since it is factorable on the left to yield an expression in terms of the main function, it is homogenous and the factor multiplying it is the degree; hence, the function is **homogenous to the degree 2**.

NEXT!

$$\boldsymbol{U = \ln(x + y)}$$

$$U_x;$$

Put

$$z = x + y$$

$$\frac{\partial z}{\partial x} = x^{1-1} + 0 = 1$$

Hence, $U = \ln z$

$$\frac{\partial U}{\partial z} = \frac{1}{z}$$

Hence;

$$U_x = \frac{\partial U}{\partial z} \times \frac{\partial z}{\partial x} = \frac{1}{z} \times 1 = \frac{1}{z}$$

$$U_x = \frac{1}{x + y}$$

$$U_y;$$

Put $z = x + y$; same substitution can be used;

$$\frac{\partial z}{\partial y} = 0 + y^{1-1} = 1$$

Hence, $U = \ln z$

$$\frac{\partial U}{\partial z} = \frac{1}{z}$$

Hence;

$$U_y = \frac{\partial U}{\partial z} \times \frac{\partial z}{\partial y} = \frac{1}{z} \times 1 = \frac{1}{z}$$

$$U_y = \frac{1}{x+y}$$

From the Euler's homogeneity theorem;

$$xU_x + yU_y = nU$$

$$x\left(\frac{1}{x+y}\right) + y\left(\frac{1}{x+y}\right) = n(\ln(x+y))$$

$$\frac{x}{x+y} + \frac{y}{x+y} = n(\ln(x+y))$$

$$\frac{x+y}{x+y} = n(\ln(x+y))$$

$$1 = n(\ln(x+y))$$

There is obviously no value for n; hence, the function is not homogenous;

NEXT!

$$\boldsymbol{f(x, y, z) = x^5 y^2 z^3}$$

The homogenous function concept isn't limited to two-variable multivariate functions; let's take the first order partial derivatives of each argument;

$$f_x = 5x^{5-1}y^2z^3 = 5x^4y^2z^3$$

$$f_y = 2 \times x^5 y^{2-1} z^3 = 2x^5 y z^3$$

$$f_z = 3 \times x^5 y^2 z^{3-1} = 3x^5 y^2 z^2$$

From the Euler's homogeneity theorem;

$$x f_x + y f_y + z f_z = n f(x, y, z)$$

Hence, we have;

$$x(5x^4 y^2 z^3) + y(2x^5 y z^3) + z(3x^5 y^2 z^2) \\ = n(x^5 y^2 z^3)$$

Expand this and simplify gradually;

$$5x^5 y^2 z^3 + 2x^5 y^2 z^3 + 3x^5 y^2 z^3 = n(x^5 y^2 z^3)$$

$x^5 y^2 z^3$ can be factorized on the left hand side;

$$x^5 y^2 z^3 (5 + 2 + 3) = n(x^5 y^2 z^3)$$

$$x^5 y^2 z^3 (10) = n(x^5 y^2 z^3)$$

$$10x^5 y^2 z^3 = n(x^5 y^2 z^3)$$

Hence, here, by comparison, $n = 10$
The **degree of homogeneity of this function is 10.**

NEXT!

$$\boldsymbol{U = x^5 + 2x^3y^2 + 9xy^4}$$

Taking the first order partial derivatives;

$$U_x = 5x^{5-1} + 3 \times 2x^{3-1}y^2 + 1 \times 9x^{1-1}y^4$$

$$U_x = 5x^4 + 6x^2y^2 + 9y^4$$

$$U_y = 0 + 2 \times 2x^3y^{2-1} + 4 \times 9xy^{4-1}$$

$$U_y = 4x^3y + 36xy^3$$

From the Euler's homogeneity theorem;

$$xU_x + yU_y = nU$$

Hence, we have;

$$x(5x^4 + 6x^2y^2 + 9y^4) + y(4x^3y + 36xy^3) = n(x^5 + 2x^3y^2 + 9xy^4)$$

$$5x^5 + 6x^3y^2 + 9xy^4 + 4x^3y^2 + 36xy^4 = n(x^5 + 2x^3y^2 + 9xy^4)$$

$$5x^5 + 10x^3y^2 + 45xy^4 = n(x^5 + 2x^3y^2 + 9xy^4)$$

On the left hand side; 5 can be factorized;

$$5(x^5 + 2x^3y^2 + 9xy^4) = n(x^5 + 2x^3y^2 + 9xy^4)$$

Hence, we can see that;

$$\boldsymbol{n = 5}$$

Hence, **the degree of homogeneity of the function is 5.**

$$\boldsymbol{U = x + 5}$$

Let's see this one variable function, Euler's theorem also extends to single-variable functions;

$$U_x = x^{1-1} + 0 = 1$$

From the Euler's homogeneity theorem;

$$xU_x = nU$$

$$x(1) = n(x + 5)$$

$$x = nx + 5n$$

Very obviously, there is no comparison possible here and hence, this heads to nowhere; it is not a homogenous function;

NEXT!

$$\boldsymbol{f(x, y) = \frac{x}{x^2 + y^2}}$$

Taking the first order partial derivatives;

$$\frac{\partial f}{\partial x}$$

This requires quotient rule;
Put;

$$u = x$$

$$v = x^2 + y^2$$

I'm sure you still remember quotient rule in partial derivatives, we treated it here;

$$u = x$$

$$\frac{\partial u}{\partial x} = x^{1-1} = 1$$

$$v = x^2 + y^2$$

$$\frac{\partial v}{\partial x} = 2x^{2-1} + 0 = 2x$$

From our quotient rule;

$$\frac{\partial f}{\partial x} = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$$

$$\frac{\partial f}{\partial x} = \frac{(x^2 + y^2)(1) - (x)(2x)}{(x^2 + y^2)^2}$$

$$\frac{\partial f}{\partial x} = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial f}{\partial y}$$

This equally requires quotient rule;

Put same substitutions;

$$u = x$$

$$v = x^2 + y^2$$

$$u = x$$

$$\frac{\partial u}{\partial y} = 0$$

$$v = x^2 + y^2$$

$$\frac{\partial v}{\partial y} = 0 + 2y^{2-1} = 2y$$

From our quotient rule;

$$\frac{\partial f}{\partial y} = \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}$$

$$\frac{\partial f}{\partial y} = \frac{(x^2 + y^2)(0) - (x)(2y)}{(x^2 + y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{0 - 2xy}{(x^2 + y^2)^2} = \frac{-2xy}{(x^2 + y^2)^2}$$

From the Euler's homogeneity theorem;

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$$

Hence, we have;

$$x\left(\frac{y^2 - x^2}{(x^2 + y^2)^2}\right) + y\left(\frac{-2xy}{(x^2 + y^2)^2}\right) = n\left(\frac{x}{x^2 + y^2}\right)$$

$$\frac{x(y^2 - x^2)}{(x^2 + y^2)^2} - \frac{2xy^2}{(x^2 + y^2)^2} = n\left(\frac{x}{x^2 + y^2}\right)$$

$$\frac{xy^2 - x^3 - 2xy^2}{(x^2 + y^2)^2} = n\left(\frac{x}{x^2 + y^2}\right)$$

$$\frac{-x^3 - xy^2}{(x^2 + y^2)^2} = n\left(\frac{x}{x^2 + y^2}\right)$$

Factorize $-x$ in the numerator of the LHS;

$$-\frac{x(x^2+y^2)}{(x^2+y^2)^2} = n\left(\frac{x}{x^2+y^2}\right)$$

(x^2+y^2) cancels off in the LHS;

$$-\left(\frac{x}{x^2+y^2}\right) = n\left(\frac{x}{x^2+y^2}\right)$$

Hence, by comparison; $n = -1$
The **degree of homogeneity of this function is -1.**

LASTLY!

$$Z = \log_e\left(\frac{x}{y}\right)$$

This is a case of chain rule;

$$Z_x;$$

Put:

$$u = \frac{x}{y}$$

$$\frac{\partial u}{\partial x} = 1 \times x^{1-1} \times \frac{1}{y} = \frac{1}{y}$$

Hence, $Z = \log_e u$

$$\frac{\partial Z}{\partial u} = \frac{1}{u}$$

Hence;

$$Z_x = \frac{\partial Z}{\partial u} \times \frac{\partial u}{\partial x} = \frac{1}{u} \times \frac{1}{y} = \frac{1}{uy}$$

$$Z_x = \frac{1}{\left(\frac{x}{y}\right) y} = \frac{1}{x}$$

$$Z_y;$$

Put:

$$u = \frac{x}{y} = xy^{-1}$$

Same substitution can be used;

$$\frac{\partial u}{\partial y} = -1 \times x \times y^{-1-1} = -\frac{x}{y^2}$$

Hence, $Z = \log_e u$

$$\frac{\partial Z}{\partial u} = \frac{1}{u}$$

Hence;

$$Z_y = \frac{\partial Z}{\partial u} \times \frac{\partial u}{\partial y} = \frac{1}{z} \times -\frac{x}{y^2} = -\frac{x}{zy^2}$$

$$Z_y = -\frac{x}{\left(\frac{x}{y}\right) y^2} = -\frac{x}{xy} = -\frac{1}{y}$$

From the Euler's homogeneity theorem;

$$xZ_x + yZ_y = nZ$$

$$x\left(\frac{1}{x}\right) + y\left(-\frac{1}{y}\right) = n \log_e \left(\frac{x}{y}\right)$$

$$1 + (-1) = n \log_e \left(\frac{x}{y}\right)$$

$$1 - 1 = n \log_e \left(\frac{x}{y}\right)$$

$$0 = n \log_e \left(\frac{x}{y}\right)$$

Is there a value for n? Think about it, I'm waiting for you before you turn to the next page!

Thinking! Thinking! Thinking!

What did you come up with? Well,

$n = 0$ since 0 multiplying $\log_e\left(\frac{x}{y}\right)$ will make it zero.

Hence, The **degree of homogeneity of this function is 0.**

JACOBIAN THEOREM

So, let's move to **the Jacobian theorem**; that's the last theorem we'll be treating in partial derivatives; so far, we have treated the application of partial derivatives on static situations i.e. applying to singular situations such as the harmonic functions, the homogenous functions which deal with one function each.

Partial derivatives however also provide a means for testing whether there exists functional dependence (linear or nonlinear) among a set of n functions; this is related to the notion of **the Jacobian determinants (named after Jacobi)**;

Let's see what the Jacobian determinant is; we'll be limiting our Jacobian determinant to two functions only;

Consider two multivariate functions;

$$Z_1 = f(x, y)$$

$$Z_2 = g(x, y)$$

Jacobi, just like Euler, also has several theorems, which include the Jacobian inverse function theorem; however, we'll just be looking at the **functional dependence theorem**.

Each function has two partial derivatives; hence, the two functions have four first order partial derivatives which are;

$$\frac{\partial Z_1}{\partial x}, \quad \frac{\partial Z_1}{\partial y}, \quad \frac{\partial Z_2}{\partial x}, \quad \frac{\partial Z_2}{\partial y}$$

The Jacobian matrix is an array of the first order partial derivatives in a 2×2 matrix as shown below; the first order partial derivatives of the same function are arranged on the same row; thus we have; corresponding arguments (i.e. variables) are also placed on the same column;

$$J = \begin{bmatrix} \frac{\partial Z_1}{\partial x} & \frac{\partial Z_1}{\partial y} \\ \frac{\partial Z_2}{\partial x} & \frac{\partial Z_2}{\partial x} \end{bmatrix}$$

Above is the Jacobian matrix;

The Jacobian determinant is the determinant of the matrix above;

Let's see this example;

- Suppose we have two functions;

$$U = x + 2y$$

$$Z = 2x^2 - 4y^3$$

Then the Jacobian matrix is given thus:

We take the first order partials of the two functions;

$$\frac{\partial U}{\partial x} = x^{1-1} + 0 = 1$$

$$\frac{\partial U}{\partial y} = 0 + 1 \times 2y^{1-1} = 2$$

$$\frac{\partial Z}{\partial x} = 2 \times 2x^{2-1} - 0 = 4x$$

$$\frac{\partial Z}{\partial y} = 0 - 3 \times 4y^{3-1} = -12y^2$$

So, we'll arrange our Jacobian matrix thus;

$$J = \begin{pmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial Z}{\partial x} & \frac{\partial Z}{\partial y} \end{pmatrix}$$

The Jacobian matrix in this case hence is;

$$J = \begin{pmatrix} 1 & 2 \\ 4x & -12y^2 \end{pmatrix}$$

Then, the Jacobian determinant is given by;

$$|J| = \begin{vmatrix} 1 & 2 \\ 4x & -12y^2 \end{vmatrix}$$

$$|J| = 1(-12y^2) - (2)(4x) = -12y^2 - 8x$$

$$|J| = -4(2x + 3y^2)$$

The Jacobian test for the functional dependence among a set of functions is given by the following theorem which we can state as **the Jacobian theorem of functional dependence**:

The Jacobian determinant for a set of functions will be equal to zero for all values of the arguments of the functions if and only if the functions are (linearly or nonlinearly) dependent.

For the value of the Jacobian determinant to vanish (i.e. the determinant of the Jacobian matrix being equal to zero), then the two functions must be dependent.

So, let's see these examples; we'll get to know what it means when two functions are dependent here too!

- Femi asserts that $Z_1 = 3x - 6y$ and $Z_2 = 18y - 9x$ are related; is Femi right or wrong?

Right, let's tackle this; all we need to do is to solve for the Jacobian determinant and see its nature;

Taking the first order partials;

$$\frac{\partial Z_1}{\partial x} = 3x^{1-1} - 0 = 3$$

$$\frac{\partial Z_1}{\partial y} = 0 - 1 \times 6y^{1-1} = -6$$

$$\frac{\partial Z_2}{\partial x} = 0 - 1 \times 9x^{1-1} = -9$$

$$\frac{\partial Z_2}{\partial y} = 1 \times 18y^{1-1} - 0 = 18$$

Form the Jacobian matrix;

$$J = \begin{pmatrix} \frac{\partial Z_1}{\partial x} & \frac{\partial Z_1}{\partial y} \\ \frac{\partial Z_2}{\partial x} & \frac{\partial Z_2}{\partial y} \end{pmatrix}$$

$$J = \begin{pmatrix} 3 & -6 \\ -9 & 18 \end{pmatrix}$$

Evaluate the Jacobian determinant;

$$|J| = (3)(18) - (-6)(-9)$$

$$|J| = 54 - 54 = 0$$

Hence, the two functions are dependent or they are related as in the question; that means Femi isn't a liar!

***Now, what does it mean when two functions are related (dependent);

It means the formation of one of the functions is dependent on the other function; consider the just concluded example; let's multiply Z_1 by -3; we'll have;

$$-3(3x - 6y) = -9x + 18y = 18y - 9x$$

Hence, we can see that Z_2 was formed by multiplying Z_1 by a scalar, -3; that is called **linear dependence** because they're related by a scalar.

Let's see another example;

- Check if the following two functions are dependent;

$$Z_1 = x - 3y$$

$$Z_2 = x^2 - 6xy + 9y^2$$

All we need to do is to solve for the Jacobian determinant and see its nature;

Taking the first order partials;

$$\frac{\partial Z_1}{\partial x} = x^{1-1} - 0 = 1$$

$$\frac{\partial Z_1}{\partial y} = 0 - 1 \times 3y^{1-1} = -3$$

$$\frac{\partial Z_2}{\partial x} = 2 \times x^{2-1} - 1 \times 6x^{1-1}y + 0$$

$$\frac{\partial Z_2}{\partial x} = 2x - 6y$$

$$\frac{\partial Z_2}{\partial y} = 0 - 1 \times 6xy^{1-1} + 2 \times 9y^{2-1}$$

$$\frac{\partial Z_2}{\partial y} = -6x + 18y$$

Form the Jacobian matrix;

$$J = \begin{pmatrix} \frac{\partial Z_1}{\partial x} & \frac{\partial Z_1}{\partial y} \\ \frac{\partial Z_2}{\partial x} & \frac{\partial Z_2}{\partial y} \end{pmatrix}$$

$$J = \begin{pmatrix} 1 & -3 \\ 2x - 6y & -6x + 18y \end{pmatrix}$$

Evaluate the Jacobian determinant;

$$|J| = 1(-6x + 18y) - (-3)(2x - 6y)$$

$$|J| = -6x + 18y + 6x - 18y$$

$$|J| = 0$$

Hence, the two functions are dependent;;

Here, it also means the formation of one of the functions is dependent on the other function; consider the just concluded example; let's take the square of Z_1; we'll have;

$$(x - 3y)^2 = x^2 - 3xy - 3xy + 9y^2$$

$$(x - 3y)^2 = x^2 - 6xy + 9y^2$$

Hence, they're related as the second function is the square of the first function! This is a case of **nonlinear dependence** of two functions;

NOW! You're not to go ahead proofing how functions are related in exam situation, you just need to show it using the Jacobian determinant, once the Jacobian determinant is zero, the two functions are related (dependent) and if otherwise, the two functions are unrelated (or independent); I'm just showing you the essence of functional dependence and its meaning! Hence, once you have proved the Jacobian determinant to be equal to zero, you're up and running!

Let's see this case;

- Check if the following two functions are related;

$$Z_1 = x - 3y$$

$$Z_2 = 2x^3 - 5x^2y - 3xy^2$$

All we need to do is to solve for the Jacobian determinant and see its nature;

Taking the first order partials;

$$\frac{\partial Z_1}{\partial x} = x^{1-1} - 0 = 1$$

$$\frac{\partial Z_1}{\partial y} = 0 - 1 \times 3y^{1-1} = -3$$

$$\frac{\partial Z_2}{\partial x} = 3 \times 2x^{3-1} - 2 \times 5x^{2-1}y - 1 \times 3x^{1-1}y^2$$

$$\frac{\partial Z_2}{\partial x} = 6x^2 - 10xy - 3y^2$$

$$\frac{\partial Z_2}{\partial y} = 0 - 1 \times 5x^2y^{1-1} - 2 \times 3xy^{2-1}$$

$$\frac{\partial Z_2}{\partial y} = -5x^2 - 6xy$$

Form the Jacobian matrix;

$$J = \begin{pmatrix} \frac{\partial Z_1}{\partial x} & \frac{\partial Z_1}{\partial y} \\ \frac{\partial Z_2}{\partial x} & \frac{\partial Z_2}{\partial y} \end{pmatrix}$$

$$J = \begin{pmatrix} 1 & -3 \\ 6x^2 - 10xy - 3y^2 & -5x^2 - 6xy \end{pmatrix}$$

Evaluate the Jacobian determinant;

$$|J| = 1(-5x^2 - 6xy) \\ - (-3)(6x^2 - 10xy - 3y^2)$$

$$|J| = -5x^2 - 6xy + 3(6x^2 - 10xy - 3y^2)$$

$$|J| = -5x^2 - 6xy + 18x^2 - 30xy - 9y^2$$

$$|J| = 13x^2 - 36xy - 9y^2$$

Hence, the two functions are not in any way related; you can see the value of the Jacobian determinant is not equal to zero, hence, their value, the two functions are unrelated;

I guess we're good with the concept of partial differentiation **in the SSC106 way**.

It was quite a long walk though!