DEPARTMENT OF ECONOMICS
FACULTY OF SOCIAL SCIENCES
OBAFEMI AWOLOWO UNIVERSITY, ILE-IFE, NIGERIA
SSC106: MATHEMATICS FOR SOCIAL SCIENCES II
RAIN SEMESTER EXAMINATION (2016/2017 SESSION)

INSTRUCTIONS:

Attempt any three questions

Show all workings clearly

Time allowed: 2 hours

1. *(a)* With an appropriate example, distinguish between symmetric and orthogonal matrix.

   (b) Given the following system of simultaneous equations;

$$X_1 + X_2 - X_3 = 2$$
$$2X_1 - X_2 + X_3 = 3$$
$$3X_1 + X_2 + 2X_3 = 2$$

Find:

(i) the coefficient matrix of the system;

(ii) the value of the determinant;

(iii) the inverse of the coefficient matrix;

(iv) the solution values of the unknown.

OR

(a) If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

(i) Obtain A^1 and AA^1

(ii) What is the nature of matrix is AA^1

(b) Classify the following matrices by types;

(i) $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$

(iii) $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix}$

(iv) $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix}$

(v) $\begin{pmatrix} 4 & 6 \\ 0 & 5 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(i) Find A^{-1}

(ii) Show that: $AA^{-1} = I$

2. *(a)* Differentiate between the following pairs;

(i) Ordinary and Partial differential equations;

(ii) Order and degree of differential equations;

(b) Determine the order and degree of the following differential equations;

(i) $$\frac{dy}{dx} = 3y$$

(ii) $$\frac{d^2y}{dx^2} + (x^2 - 5)\frac{dy}{dx} + xy = 0$$

(iii) $$\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^5 + \frac{y}{x^2+1} = e^x$$

(iv) $$\frac{d^3y}{dx^3} + 6\sqrt{\left(\frac{dy}{dx}\right)^2 + y}$$

(c) Form differential equations from the following;

(i) $x^2 - e^y = a$

(ii) $y^2 - ax + a^2$

(iii) $x^2 + y^2 - 2ax + 1$

3. *(a)* Differentiate between definite and indefinite integrals.

(b) Solve the following integrals;

(i) $$\int \frac{4x-3}{x^2+4x+3}\,dx$$

(ii) $\displaystyle\int \frac{x+3}{x^2+5x+6}dx$

(iii) $\displaystyle\int \frac{2x+7}{x^2+7x-3}dx$

(iv) $\displaystyle\int 4x^3e^{x^4}dx$

(v) $\displaystyle\int e^{ax+b}dx$

(c) $\displaystyle\int_a^b (2x+3)dx$

(d) The rate of change of a quantity A is given by t^2+1. If $A=\frac{4}{3}$ when $t=1$, find A in terms of t.

4. Distinguish clearly between the following pairs of functions.

(a) Explicit and Implicit functions

(b) Increasing and decreasing functions

(c) Linear and Non-linear functions

(d) Even and Odd functions

(e) Exponential and Logarithm functions

5 *(a)* What do you understand by differential calculus.

(b) State with examples any five rules of differentiation

(c) State the Young's theorem

(d) Verify that:

$$Z = e^{x^2} + 8xy + 7y^2$$

satisfies Young's theorem.

(e) Determine the values of x and y that'll maximize the function: $z = xy + 2x$; subject to $4x + 2y = 60$

SOLUTION TO THE PAST QUESTIONS

All topics needed here have been thoroughly treated through and through, all you need to do is to sit down and answer all these questions like a joke, you are not expected to start reading from here but from the topics, you make a joke of yourself when you read the concept of any subject starting from the side of the past question; more so, this text isn't a past question but a complete guide to lead you to a straight A in your SSC106 exam.

Hence, keep calm and check the solutions to these past questions one-by-one;

GOOD LUCK AND GOD'S BEST!

SOLUTION TO THE SSC106 EXAMINATION 2016/2017 ACADEMIC SESSION

The instruction is you answer three questions. We'll be solving everything though, just sit tight as we solve.

Remember every singular solution here, you can get the full concept by reading this book and not the past question section exclusively;

SECTION A

Question 1

(a) With an appropriate example, distinguish between symmetric and orthogonal matrix.

(b) Given the following system of simultaneous equations;

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$$2X_1 - X_2 + X_3 = 3$$
$$3X_1 + X_2 + 2X_3 = 2$$

Find:

(i) the coefficient matrix of the system;
(ii) the value of the determinant;
(iii) the inverse of the coefficient matrix;
(iv) the solution values of the unknown.

OR

(a) If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

(i) Obtain A^1 and AA^1
(ii) What is the nature of matrix is AA^1

(b) Classify the following matrices by types;

(i) $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$

(iii) $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix}$

(iv) $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix}$

(v) $\begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(i) Find A^{-1}

(ii) Show that: $AA^{-1} = I$

You have a weird instruction here, to either answer the first two questions or the last three questions for the same marks. We'll solve the five of them though.

(a)

Orthogonal matrices and symmetric matrices; do they have a direct difference? I doubt, we'll just define them and put while between them.

A symmetric matrix is a square matrix that is equal to its transpose while an orthogonal matrix is a square matrix whose transpose is equal to its inverse;

For a symmetric matrix, A;

$$A = A^T$$

For an orthogonal matrix, A;

$$A^{-1} = A^T$$

(b)

$$\begin{gathered} X_1 + X_2 - X_3 = 2 \\ 2X_1 - X_2 + X_3 = 3 \\ 3X_1 + X_2 + 2X_3 = 2 \end{gathered}$$

Similar question in the note, just change of variable.

(i)

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

(ii)

$$\text{Let } A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

Hence; we have;

$$|A| = \begin{vmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

$$1\begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} - (1)\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix}$$

$$1[(-1)(2) - (1)(1)] - 1[(2)(2) - (1)(3)] - 1[(2)(1) - (-1)(3)]$$

$$|A| = 1(-3) - (1) - (5) = -9$$

(iii) Let's find the minors and cofactors of all the elements; as we know, it's minors first;

Let's find the minor elements;

$$min(7) = \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} \qquad min(-1) = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$$

$$min(-1) = \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} \qquad min(2) = \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}$$

$$min(-1) = \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} \qquad min(1) = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}$$

$$min(3) = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \qquad min(1) = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$min(2) = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}$$

Hence,

The matrix of minors is;

$$\text{minor}(A) = \begin{pmatrix} -3 & 1 & 5 \\ 3 & 5 & -2 \\ 0 & 3 & -3 \end{pmatrix}$$

From the cofactor matrix sign notation below:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Hence,

The matrix of cofactors is;

$$\text{cofactor}(A) = \begin{pmatrix} -3 & -1 & 5 \\ -3 & 5 & 2 \\ 0 & -3 & -3 \end{pmatrix}$$

(iv) To solve the equation, since we have the cofactor matrix already, we can straightforward have the adjoint matrix;

The adjoint which is the transpose of the cofactor matrices and hence inverse of the matrix and compute the remaining;

$$adj(A) = [\text{cofactor}(A)]^T$$

$$\text{adj}(A) = \begin{pmatrix} -3 & -1 & 5 \\ -3 & 5 & 2 \\ 0 & -3 & -3 \end{pmatrix}^T$$

$$\text{adj}(A) = \begin{pmatrix} -3 & -3 & 0 \\ -1 & 5 & -3 \\ 5 & 2 & -3 \end{pmatrix}$$

We have evaluated the determinant already and hence;

$$A^{-1} = -\frac{1}{9}\begin{pmatrix} -3 & -3 & 0 \\ -1 & 5 & -3 \\ 5 & 2 & -3 \end{pmatrix}$$

To solve the equation; pre-multiply both sides of this equation by A^{-1}

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

We have;

$$A^{-1}\begin{pmatrix}1 & 1 & -1\\ 2 & -1 & 1\\ 3 & 1 & 2\end{pmatrix}\begin{pmatrix}X_1\\ X_2\\ X_3\end{pmatrix} = -\frac{1}{9}\begin{pmatrix}-3 & -3 & 0\\ -1 & 5 & -3\\ 5 & 2 & -3\end{pmatrix}\begin{pmatrix}2\\ 3\\ 2\end{pmatrix}$$

The left hand side is reduces completely since A^{-1} multiplying A will yield the identity matrix; we start expanding the right hand side multiplication;

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}X_1\\ X_2\\ X_3\end{pmatrix} = -\frac{1}{9}\begin{pmatrix}(-3)(2)+(-3)(3)+(0)(2)\\ (-1)(2)+(5)(3)+(-3)(2)\\ (5)(2)+(2)(3)+(-3)(2)\end{pmatrix}$$

The left hand side simply yields the matrix of coefficients since it is multiplied by an identity matrix;

$$\begin{pmatrix}X_1\\ X_2\\ X_3\end{pmatrix} = -\frac{1}{9}\begin{pmatrix}-15\\ 7\\ 10\end{pmatrix}$$

Hence, expanding the matrices and applying the matrix equality rule;

$$X_1 = \frac{15}{9} = \frac{5}{3}$$

$$X_2 = -\frac{7}{9}$$

$$X_3 = -\frac{10}{9}$$

That's the solution!

(a)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

We're told to obtain A' which from our matrix studies, we know is the transpose of A;

$$A' = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

We're to also obtain AA'; hence, we have matrix multiplication; this will be quite tedious;

$$AA' = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

$$\begin{pmatrix} (1)(1)+(2)(2)+(3)(3) & (1)(4)+(2)(5)+(3)(6) & (1)(7)+(2)(8)+(3)(9) \\ (4)(1)+(5)(2)+(6)(3) & (4)(4)+(5)(5)+(6)(6) & (4)(7)+(5)(8)+(6)(9) \\ (7)(1)+(8)(2)+(9)(3) & (7)(4)+(8)(5)+(9)(6) & (7)(7)+(8)(8)+(9)(9) \end{pmatrix}$$

Sorry for the small font in the expansion, kindly zoom it;

$$AA' = \begin{pmatrix} 14 & 32 & 50 \\ 32 & 77 & 122 \\ 50 & 122 & 194 \end{pmatrix}$$

(ii)

Of course, we treated this in the book, the product of a matrix and its transpose is a **symmetric matrix;** that is, a matrix whose row elements are equal to its corresponding column elements, hence, such a matrix is equal to its transpose.

Hence, AA' is such that:

$$AA' = (AA')'$$

(b)

(i)

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

This wasn't mentioned in the textbook section; however;

A scalar matrix is a special type of diagonal matrix where every element on the main diagonal is the same scalar value. It is always equivalent to λI where λ is the same-valued element on the main diagonal and I is the identity matrix. Hence, in the above;

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} = 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 5I$$

Hence, (i) is a **scalar matrix.**

(ii)

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

Evaluating the transpose of the above matrix;

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

Hence, it is obvious the matrix above is equal to its transpose and is a symmetric matrix.

(iii)

$$\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix}$$

Evaluating the transpose of the above matrix;

$$\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{pmatrix}$$

Hence, it is obvious the matrix above is equal to the negative of its transpose and is a skew-symmetric matrix.

(iv)

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

For the matrix above, all elements not on its main diagonal is zero and hence, the matrix is a diagonal matrix.

(v)

$$\begin{pmatrix} 4 & 6 \\ 0 & 5 \end{pmatrix}$$

Above is an upper triangular matrix, it's not so easy to spot but see it clearly.

(c)

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We're required to find A^{-1}; will take quite some time, the normal time to find a matrix inverse.

Let's find the minor elements first;

$$min(1) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \qquad min(1) = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}$$

$$min(1) = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \qquad min(0) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$min(1) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \qquad min(0) = \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}$$

$$min(0) = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \qquad min(0) = \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}$$

$$min(1) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

Hence,
The matrix of minors is;

$$\text{minor}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

From the cofactor matrix sign notation below:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Hence,
The matrix of cofactors is;

$$\text{cofactor}(A) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

We can straightforward have the adjoint matrix;

$$adj(A) = [\text{cofactor}(A)]^T$$

$$\text{adj}(A) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}^T$$

$$\operatorname{adj}(A) = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We need the determinant for the inverse;

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$1[(1) - (0)] - 1[(0) - (0)] + 1[(0) - (0)]$$

$$|A| = 1$$

$$A^{-1} = \frac{\operatorname{adj}(A)}{|A|}$$

Hence,

$$A^{-1} = \frac{1}{1}\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence;

$$A^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We're told to show;

$$AA^{-1} = I$$

Hence; we multiply both!!

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Expand;

$$\begin{pmatrix} 1(1)+1(0)+1(0) & 1(-1)+1(1)+1(0) & 1(-1)+1(0)+1(1) \\ 0(1)+1(0)+0(0) & 0(-1)+1(1)+0(0) & 0(-1)+1(0)+0(1) \\ 0(1)+0(0)+1(0) & 0(-1)+0(1)+1(0) & 0(-1)+0(0)+1(1) \end{pmatrix}$$

Please zoom to view clearly if you need to!

Obviously;

$$AA^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The above is the identity matrix, I and hence, the prove is successful!

Question 2

(a) Differentiate between the following pairs;

(i) Ordinary and Partial differential equations;

(ii) Order and degree of differential equations;

(b) Determine the order and degree of the following differential equations;

(i) $\dfrac{dy}{dx} = 3y$

(ii) $\dfrac{d^2y}{dx^2} + (x^2 - 5)\dfrac{dy}{dx} + xy = 0$

(iii) $\left(\dfrac{d^3y}{dx^3}\right)^2 + \left(\dfrac{d^2y}{dx^2}\right)^5 + \dfrac{y}{x^2+1} = e^x$

(iv) $\dfrac{d^3y}{dx^3} + 6\sqrt{\left(\dfrac{dy}{dx}\right)^2} + y$

(c) Form differential equations from the following;

(i) $x^2 - e^y = a$

(ii) $y^2 - ax + a^2$

(iii) $x^2 + y^2 - 2ax + 1$

(a)

(i)

There isn't so much difference between the two, they both contain differential coefficients, but they contain different types of differential coefficients.

An ordinary differential equation is a differential equation involving one independent and one dependent variable while a partial differential equation involves partial derivatives. It is a differential equation involving a dependent variable and more than one independent variable.

(ii)

The order of a differential equation is the highest derivative involved in the differential equation while the degree of a differential equation is determined after the order of the differential equation has been determined

(b)

(i)

$$\frac{dy}{dx} = 3y$$

The highest derivative in the above is $\frac{dy}{dx}$, hence, the order is 1.

The highest derivative is raised to a power of 1 and hence, the degree is 1.

(ii)

$$\frac{d^2y}{dx^2} + (x^2 - 5)\frac{dy}{dx} + xy = 0$$

The highest derivative in the above is $\frac{d^2y}{dx^2}$, hence, the order is 2.

The highest derivative is raised to a power of 1 and hence, the degree is 1.

(iii)

$$\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^5 + \frac{y}{x^2+1} = e^x$$

The highest derivative in the above is $\frac{d^3y}{dx^3}$, hence, the order is 3.

The highest derivative is raised to a power of 2 and hence, the degree is 2.

(iv)

$$\frac{d^3y}{dx^3} + 6\sqrt{\left(\frac{dy}{dx}\right)^2 + y}$$

A differential equation must be simplified to an equation without roots before the degree is found, the order doesn't need simplification as it is straightforward in the highest derivative in the equation.

The highest derivative in the above is $\frac{d^3y}{dx^3}$, hence, the order is 1.

$$\frac{d^3y}{dx^3} + 6\sqrt{\left(\frac{dy}{dx}\right)^2} + y = 0$$

In simplifying the above, we have assumed the equation is equated to zero it is differential equations and not expression and hence, must have an equality sign!

We know we must remove all roots before finding the degree of the differential equation;

We have it thus:

$$\frac{d^3y}{dx^3} = -6\sqrt{\left(\frac{dy}{dx}\right)^2 + y}$$

Square both sides;

$$\left(\frac{d^3y}{dx^3}\right)^2 = \left(-6\sqrt{\left(\frac{dy}{dx}\right)^2 + y}\right)^2$$

$$\left(\frac{d^3y}{dx^3}\right)^2 = (-6)^2\left(\left(\frac{dy}{dx}\right)^2 + y\right)$$

$$\left(\frac{d^3y}{dx^3}\right)^2 = 36\left(\left(\frac{dy}{dx}\right)^2 + y\right)$$

$$\left(\frac{d^3y}{dx^3}\right)^2 = 36\left(\frac{dy}{dx}\right)^2 + 36y$$

Hence, after simplification, the highest derivative is raised to a power of 2 and hence, the degree is 2.

(c)

We learnt how to form differential equations, and hence, we'll do that now;

(i)

$$x^2 - e^y = a$$

Here; we have a case of an implicit function, I introduced it briefly at the end of differentiation and turns out, it's very much useful, we know the rule of implicit differentiation, when we differentiate y, we include $\frac{dy}{dx}$ and when we differentiate x, it is left as it is:

$$2x - e^y \frac{dy}{dx} = 0$$

Hence;

We have:

$$\frac{dy}{dx} = \frac{2x}{e^y}$$

Normally, from normal forming of differential equations, we would've have needed to go to the initial equation and express the arbitrary constant in terms of the variable and substitute into the differential equation, however, here, since the arbitrary constant isn't in the first derivative, the differential equation remains like that.

(ii)

$$y^2 - ax + a^2 = 0$$

Another case of implicit differentiation; we'll assume it is equated to zero since the question omitted the equality sign.

$$2 \times y^{2-1} \frac{dy}{dx} - 1 \times ax^{1-1} + 0 = 0$$

Hence;

$$2y\frac{dy}{dx} - a = 0$$

From here;

It is more convenient to make a the subject in the equation for $\frac{dy}{dx}$ than the initial equation, hence, we make a the subject here and substitute into the original equation. Hence, from here;

$$a = 2y\frac{dy}{dx}$$

We have:

Substitute this into the initial equation;

$$y^2 - ax + a^2 = 0$$

Hence;

$$y^2 - \left(2y\frac{dy}{dx}\right)x + \left(2y\frac{dy}{dx}\right)^2 = 0$$

Expanding;

$$y^2 - 2xy\frac{dy}{dx} + (2y)^2\left(\frac{dy}{dx}\right)^2 = 0$$

$$y^2 - 2xy\frac{dy}{dx} + 4y^2\left(\frac{dy}{dx}\right)^2 = 0$$

Above is the needed differential equation.

(iii)

$$x^2 + y^2 - 2ax + 1 = 0$$

Another case of implicit differentiation; we'll assume it is equated to zero since the question omitted the equality sign.

$$2 \times x^{2-1} + 2 \times y^{2-1}\frac{dy}{dx} - 2 \times ax^{1-1} + 0 = 0$$

Hence;

$$2x + 2y\frac{dy}{dx} - 2a = 0$$

From here;

It is more convenient to make a the subject in the equation for $\frac{dy}{dx}$ than the initial equation, hence, we make a the subject here and substitute into the original equation. Hence, from here;

$$2a = 2x + 2y\frac{dy}{dx}$$

We have:

Multiplying through by $\left(\frac{1}{2}\right)$;

$$\frac{1}{2}(2a) = \left(2x + 2y\frac{dy}{dx}\right) \times \frac{1}{2}$$

$$a = x + y\frac{dy}{dx}$$

Substitute into the initial equation;

$$x^2 + y^2 - 2ax + 1 = 0$$

Hence;

$$x^2 + y^2 - 2\left(x + y\frac{dy}{dx}\right)x + 1 = 0$$

Expanding;

$$x^2 + y^2 - 2\left(x^2 + xy\frac{dy}{dx}\right) + 1 = 0$$

$$x^2 + y^2 - 2x^2 - 2xy\frac{dy}{dx} + 1 = 0$$

Above is the needed differential equation!

Question 3

(a) Differentiate between definite and indefinite integrals.

(b) Solve the following integrals;

(i) $$\int \frac{4x-3}{x^2+4x+3}dx$$

(ii) $$\int \frac{x+3}{x^2+5x+6}dx$$

(iii) $$\int \frac{2x+7}{x^2+7x-3}dx$$

(iv) $$\int 4x^3e^{x^4}dx$$

(v) $$\int e^{ax+b}dx$$

(c) $$\int_a^b (2x+3)dx$$

(d) The rate of change of a quantity A is given by t^2+1. If $A=\frac{4}{3}$ when $t=1$, find A in terms of t.

(a)

Two simple differences;

- The indefinite integral gives the general form of the anti-derivative of a function; the definite integral gives the area under a curve between two given points and is the value gotten by evaluating the integral from the two limits.
- The indefinite integral also contains an arbitrary constant while a definite integral doesn't contain an arbitrary constant.

(b)

Integrals!

(i)

$$\int \frac{4x - 3}{x^2 + 4x + 3} dx$$

Checking it well, it isn't the integration by substitution as the derivative of the denominator and the numerator have no relationship; hence, it's by partial fraction this will be sorted out.

Partial fraction! Break it down;
Factorize the denominator;

$$x^2 + 4x + 3$$

$$x^2 + 3x + x + 3$$
$$x(x + 3) + 1(x + 3)$$
$$(x + 3)(x + 1)$$

Hence, we have;

$$\int \frac{4x - 3}{(x + 3)(x + 1)} dx \equiv \frac{A}{x + 3} + \frac{B}{x + 1}$$

Clear out every needed clearing, we're left with;

$$4x - 3 = A(x + 1) + B(x + 3)$$

Here,

$$\text{Let } x + 1 = 0$$

Here,

$$x = -1$$

Then, substitute in the main equation

$$4(-1) - 3 = A(-1 + 1) + B(-1 + 3)$$
$$-7 = A(0) + B(2)$$

Here,

$$B = -\frac{7}{2}$$

Again,

$$\text{Let } x + 3 = 0$$

Hence,

$$x = -3$$

Then, substitute in the main equation

$$4(-3) - 3 = A(-3+1) + B(-3+3)$$
$$-15 = A(-2) + B(0)$$

Here,

$$A = \frac{15}{2}$$

We have found our two values for A and B and we're good!

$$\frac{4x-3}{(x+3)(x+1)} \equiv \frac{15}{2(x+3)} - \frac{7}{2(x+1)}$$

Hence, the integral;

$$\int \frac{4x-3}{(x+3)(x+1)} dx$$
$$\equiv \int \left(\frac{15}{2(x+3)} - \frac{7}{2(x+1)}\right) dx$$

From the integrals of sums and differences; we'll have:

$$\frac{4x-3}{(x+3)(x+1)} \equiv \frac{15}{2(x+3)} - \frac{7}{2(x+1)}$$

$$\int \frac{4x-3}{(x+3)(x+1)}dx$$
$$\equiv \int \frac{15}{2(x+3)}dx - \int \frac{7}{2(x+1)}dx$$

Take the integrals separately;

$$\int \frac{15}{2(x+3)}dx$$

Put $u = x + 3$;

$$\frac{du}{dx} = 1$$

Hence,

$$dx = du$$

We have;

$$\int \frac{15}{2u}du = \frac{15}{2}\int \frac{1}{u}du = \frac{15}{2}(\ln u)$$

We have;

$$\frac{15}{2}\ln(x+3)$$

Next!

$$\frac{7}{2(x+1)}$$

Put $z = x + 1$;

$$\frac{dz}{dx} = 1$$

Hence,

$$dx = dz$$

We have;

$$\int \frac{7}{2z} dz = \frac{7}{2}\int \frac{1}{z} dz = \frac{7}{2}(\ln z)$$

We have;

$$\frac{7}{2}\ln(x+1)$$

Finally, let's combine everything;

$$\int \frac{15}{2(x+3)} dx - \int \frac{7}{2(x+1)} dx$$

$$\frac{15}{2}\ln(x+3) - \frac{7}{2}\ln(x+1) + C$$

Of course, our arbitrary constant is added; these two separate integrals are again nothing special, just integral of linear functions.

(ii)

$$\int \frac{x+3}{x^2+5x+6}\,dx$$

Again, checking it well, it isn't the integration by substitution as the derivative of the denominator and the numerator have no relationship; hence, it's by partial fraction this will be sorted out.

Factorize the denominator;

$$x^2+5x+6$$
$$x^2+3x+2x+6$$
$$x(x+3)+2(x+3)$$
$$(x+3)(x+2)$$

Hence, we have;

$$\int \frac{x+3}{(x+3)(x+2)}\,dx \equiv \frac{A}{x+3}+\frac{B}{x+2}$$

Seeing the above, I have made it very plain, we needn't go ahead for partial fractions even though if we used partial fractions, it won't go ahead wrong if you don't make any mistake; see this;

$$\frac{(x+3)}{(x+3)(x+2)}$$

$(x+3)$ cancels off; we're left with;

$$\frac{1}{(x+2)}$$

So, if you'd still gone ahead with the partial fractions, you'd have gotten the value of A as zero and B as 1. Which will yield the fraction above;

Hence, the integral;

$$\int \frac{x+3}{(x+3)(x+2)}\,dx \equiv \int \frac{1}{x+2}\,dx$$

Linear substitution;

$$\int \frac{1}{x+2}\,dx$$

Put $u = x + 2$;

$$\frac{du}{dx} = 1$$

Hence,

$$dx = du$$

We have;

$$\int \frac{1}{u} du = (\ln u)$$

We have;

$$\ln(x + 2) + C$$

Of course, our arbitrary constant is added, nothing special!

(iii)

$$\int \frac{2x + 7}{x^2 + 7x - 3} dx$$

This time around, we have a case of substitution, a case of:

$$\frac{f'(x)}{f(x)}$$

Hence;

$$u = x^2 + 7x - 3$$

$$\frac{du}{dx} = 2x + 7$$

Hence;

$$dx = \frac{du}{2x + 7}$$

We now have:

$$\int \frac{2x+7}{u} \times \frac{du}{2x+7}$$

$2x + 7$ cancels off; we have:

$$\int \frac{1}{u} du$$

Standard!

$$\ln u$$

Return u;

$$\ln(x^2 + 7x - 3) + C$$

Very *very* simple! Our arbitrary constant must be added!

(iv)

$$\int 4x^3 e^{x^4} dx$$

Another case of substitution! We have a case of substitution, a case of:

$$f'(x)g[f(x)]$$

Hence;

$$u = x^4$$

$$\frac{du}{dx} = 4x^3$$

Hence;

$$dx = \frac{du}{4x^3}$$

We now have:

$$\int 4x^3 e^u \frac{du}{4x^3}$$

$4x^3$ cancels off; we have:

$$\int e^u du$$

Standard integral;

$$e^u$$

Return u;

$$e^{x^4} + C$$

Very *very* simple! Our arbitrary constant must be added!

(v)

$$\int e^{ax+b} dx$$

Linear substitution!

$$u = ax + b$$

Hence,

$$\frac{du}{dx} = 1 \times ax^{1-1} + 0$$

$$\frac{du}{dx} = a$$

And;

$$dx = \frac{du}{a}$$

We have;

$$\int e^u \frac{du}{a}$$

a is a constant, bring it out!

$$\frac{1}{a} \int e^u \, du$$

Standard integral!

$$\frac{1}{a} e^u$$

Return u;

$$\frac{1}{a} e^{ax+b} + C$$

Very *very* simple! Our arbitrary constant must be added!

(c)

$$\int_a^b (2x+3)dx$$

Definite integral; we need the indefinite integral first!

$$\int (2x+3)dx$$

Integral of sums;

$$\int 2xdx + \int 3dx$$

Power rule;

$$2\left[\frac{x^{1+1}}{1+1}\right] + 3\left[\frac{x^{0+1}}{0+1}\right]$$

$$2\left(\frac{x^2}{2}\right) + 3(x)$$

$$x^2 + 3x$$

We don’t need the arbitrary constant since we are seeking a definite integral; hence;

$$\int_a^b (2x+3)dx = [x^2+3x]_a^b$$

$$(b)^2 + 3(b) - [(a)^2 + 3(a)]$$

We have;

$$b^2 + 3b - a^2 - 3a$$

We can simplify this in several ways

$$b^2 - a^2 = (b-a)(b+a)$$

difference of two squares

$$3b - 3a = 3(b-a)$$

Hence; we have;

$$b^2 - a^2 + 3b - 3a$$

$$(b-a)(b+a) + 3(b-a)$$

Factorize $(b-a)$;

$$(b-a)(b+a+3)$$

(d)

Integration upon integration!

The rate of change of quantity A is

$$t^2 + 1$$

Hence;

$$\frac{dA}{dt} = t^2 + 1$$

We have;

$$dA = (t^2 + 1)dt$$

Integrate both sides;

$$\int dA = \int (t^2 + 1)dt$$

The left hand side readily removes d; and we now sort out the right by integration of sums;

$$A = \int t^2 dt + \int 1dt$$

$$A = \left[\frac{t^{2+1}}{2+1}\right] + \left[\frac{t^{0+1}}{0+1}\right]$$

$$A = \frac{t^3}{3} + t + C$$

The arbitrary constant is added; however, we are given information on how to find the true value of the arbitrary constant.

$$A = \frac{4}{3} \text{ when } t = 1$$

Hence;

$$\frac{4}{3} = \frac{(1)^3}{3} + (1) + C$$

We have;

$$\frac{4}{3} = \frac{1}{3} + 1 + C$$

Adding fractions;

$$\frac{4}{3} = \frac{4}{3} + C$$

Hence;

$$C = 0$$

Finally; we have;

$$A = \frac{t^3}{3} + t$$

Question 4

Distinguish clearly between the following pairs of functions.

(a) Explicit and Implicit functions
(b) Increasing and decreasing functions
(c) Linear and Non-linear functions
(d) Even and Odd functions
(e) Exponential and Logarithm functions

All these stories questions, it's very boring solving them.

(a)

An explicit function is stated solely in terms of the independent variable(s) and the direction of the function is certain while the implicit function isn't expressed solely in terms of any variable and the direction of the function is not certain.

(b)

Increasing functions are functions which are increase in value as the value of the independent functions increase over an interval.

If $f(x)$ is increasing over the interval;

$$a \leq x \leq b$$

Then;

$$f(b) > f(a)$$

For all;

$$b > a \text{ in } a \leq x \leq b$$

WHILE

Decreasing functions are functions which are increase in value as the value of the independent functions increase over an interval.

If $f(x)$ is decreasing over the interval;

$$a \leq x \leq b$$

Then;

$$f(b) < f(a)$$

For all;

$$b > a \text{ in } a \leq x \leq b$$

(c)

Linear functions are types of functions that are straight lines and are in the form;

$$f(x) = mx + c$$

Where m and c are constants; The power of x is always 1

WHILE

Non-linear functions are types of functions whose graphs are not straight lines; they have no fixed form as every functions apart from functions of the polynomial of degree 1 are all non-linear functions.

(d)

An odd function is a function that gives the same negative equivalent value for the negative value of the same independent variable. In an even function:

$$f(-x) = -f(x)$$

WHILE

An even function is a function that gives the same value for the negative value of the same independent variable. In an even function:

$$f(-x) = f(x)$$

(e)

Exponential functions are of the form;

$$y = ab^x$$

Where a and b are constants;

An exponential function is a type of function where the independent variable is an exponent.

WHILE

Logarithm functions are of the form;

$$y = a \log x + b$$

$$y = a \ln x + b$$

Where a and b are constants;

In logarithm functions, the independent variable, x is expressed as the logarithm of a number, a positive number to be precise.

Question 5

(a) What do you understand by differential calculus.

(b) State with examples any five rules of differentiation.

(c) State the Young's theorem

(d) Verify that:

$$Z = e^{x^2} + 8xy + 7y^2$$

satisfies Young's theorem.

(e) Determine the values of x and y that'll maximize the function: $z = xy + 2x$; subject to $4x + 2y = 60$

(a)

Differential calculus (also known as differentiation) basically means the process of finding the derivative or differential coefficient of a function.

Differentiation can also be defined as the branch of calculus that deals with the rate of change in one quantity with respect to another.

(b)

We're told to state with examples, five rules of differentiation; let's list some of them:

(i)

The power rule states that:
If:

$$y = x^n$$

Then:

$$\frac{dy}{dx} = nx^{n-1}$$

For example:

$$y = x^2$$

$$\frac{dy}{dx} = 2x^{2-1} = 2x$$

(ii)

The exponential rule states that:
If:

$$y = e^x$$

Then;

$$\frac{dy}{dx} = e^x$$

And if:

$$y = e^{f(x)}$$

Then:

$$\frac{dy}{dx} = f'(x)e^{f(x)}$$

And if:

$$y = a^x$$

Then;

$$\frac{dy}{dx} = a^x \ln a$$

And if:

$$y = a^{f(x)}$$

Then:

$$\frac{dy}{dx} = \ln a\, f'(x)a^{f(x)}$$

For example:

$$y = e^{3x}$$

$$\frac{dy}{dx} = 3 \times e^{3x} = 3e^{3x}$$

(iii)

The logarithm rule:

If:

$$y = \log_e x$$

Then:

$$\frac{dy}{dx} = \frac{1}{x}$$

And if:

$$y = \log_e f(x)$$

Then:

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

If:

$$y = \log_a x$$

Then:

$$\frac{dy}{dx} = \frac{1}{x \ln a}$$

And if:

$$y = \log_a f(x)$$

Then:

$$\frac{dy}{dx} = \frac{f'(x)}{f(x) \ln a}$$

For example:

$$y = \log_e 4x$$

$$\frac{dy}{dx} = \frac{4}{4x} = \frac{1}{x}$$

(iv)

Chain rule states that:

If:

$$u = f(x)$$

And:

$$y = f(u)$$

Then:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

For example:

$$u = 4x^2 + 3x + 2$$

$$y = e^{4x^2+3x+2}$$

We have:

$$\frac{du}{dx} = 8x + 3$$

And:

$$y = e^u$$

And:

$$\frac{dy}{du} = e^u$$

Then:

$$\frac{dy}{dx} = e^u \times (8x + 3) = (8x + 3)e^{4x^2+3x+2}$$

(v)

Product rule states that:

If:

$$y = u \times v$$

Where:

u and v are functions of x;

Then:

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

For example:

$$y = (2x + 3)(x^2 - 6)$$

$$u = 2x + 3$$

$$\frac{du}{dx} = 2$$

$$v = x^2 - 6$$

$$\frac{dv}{dx} = 2x$$

Then:

$$\frac{dy}{dx} = (x^2 - 6)(2) + (2x + 3)(2x)$$

$$\frac{dy}{dx} = 2x^2 - 12 + 4x + 6x = 2x^2 + 10x - 12$$

(c)

Young's theorem states that two complementary second order mixed partials of a continuous and twice differentiable function are equal; for a multivariate function dependent on x and y;

(d)

$$Z = e^{x^2} + 8xy + 7y^2$$

It'll be quite long to verify Young's theorem here;

$$Z_x = \frac{\partial}{\partial x}\left(e^{x^2}\right) + \frac{\partial}{\partial x}(8xy) + \frac{\partial}{\partial x}(7y^2)$$

For:

$$\frac{\partial}{\partial x}\left(e^{x^2}\right)$$

From chain rule formula; it'll be:

$$\frac{\partial}{\partial x}\left(e^{x^2}\right) = 2x \times e^{x^2} = 2xe^{x^2}$$

Hence;

$$Z_x = 2xe^{x^2} + 1 \times 8x^{1-1}y + 0$$

$$Z_x = 2xe^{x^2} + 8y$$

$$Z_y = \frac{\partial}{\partial y}\left(e^{x^2} + 8xy + 7y^2\right)$$

$$Z_y = 0 + 1 \times 8xy^{1-1} + 2 \times 7y^{2-1}$$

$$Z_y = 8x + 14y$$

Going further for the indirect partial derivatives;

$$Z_{xy} = \frac{\partial}{\partial y}(Z_x) = \frac{\partial}{\partial y}\left(2xe^{x^2} + 8y\right)$$

$$Z_{xy} = 0 + 1 \times 8y^{1-1} = 8$$

$$Z_{yx} = \frac{\partial}{\partial x}\left(Z_y\right) = \frac{\partial}{\partial x}(8x + 14y)$$

$$Z_{yx} = 1 \times 8x^{1-1} + 0 = 8$$

Hence;
Proved:

$$Z_{xy} = Z_{yx} = 8$$

(e)

To maximize $z = xy + 2x$
Subject to $4x + 2y = 60$

This question has probably been set a million times in SSC106 exams!

Here, the objective function is:

$$xy + 2x$$

The constraint function is:

$$4x + 2y - 60 = 0$$

Note that it has been equated to zero;

Hence, using the method of Lagrangean multiplier; we have;

$$\mathcal{L}(x, y, \lambda) = xy + 2x - \lambda(4x + 2y - 60)$$

Take the first order partials;

$$\mathcal{L}_x = 1 \times x^{1-1}y + 1 \times 2x^{1-1} - \lambda(1 \times 4x^{1-1} + 0 - 0)$$

$$\mathcal{L}_x = y + 2 - 4\lambda$$

$$\mathcal{L}_y = 1 \times xy^{1-1} + 0 - \lambda(0 + 1 \times 2y^{1-1} - 0)$$

$$\mathcal{L}_y = x - 2\lambda$$

$$\mathcal{L}_\lambda = 0 + 0 - 1 \times \lambda^{1-1}(4x + 2y - 60)$$

$$\mathcal{L}_\lambda = -4x - 2y + 60$$

Equate each to zero:

$$y + 2 - 4\lambda = 0 \dots \dots \dots (1)$$
$$x - 2\lambda = 0 \dots \dots \dots (2)$$

$$-4x - 2y + 60 = 0$$

Clear;

$$-2x - y = -30 \dots \dots \dots (3)$$

(1) and (2) can easily be sorted out:

From (1);

$$4\lambda = y + 2$$

From (2)

$$2\lambda = x$$

Multiply through by 2;

$$4\lambda = 2x$$

Hence;

It infers;

$$2x = y + 2$$

Since both are equal to 4λ;

Hence;

$$2x - y = 2 \ldots \ldots \ldots (4)$$

Solving (4) simultaneously with (3);

$$2x - y = 2 \ldots \ldots \ldots (4)$$
$$-2x - y = -30 \ldots \ldots \ldots (3)$$

Add both;

$$-2y = -28$$
$$y = 14$$

From (4);

$$2x - (14) = 2$$
$$2x = 16$$
$$x = 8$$

From (2);

$$x = 2\lambda$$
$$8 = 2\lambda$$
$$\lambda = 4$$

Hence, to maximize Z, find its value at the optimum point;

$$Z = xy + 2x$$

Hence;

Maximum Z;

$$Z = (8)(14) + 2(8)$$

$$Z = 128$$

THAT'S IT! TWO YEARS AGO! THIS WAS WRITTEN!