DIFFERENTIATION

Differentiation, also known as differential calculus is the first aspect of calculus. Differentiation, as it were, is basically wide, as already explained, calculus basically is about differentiation and integration; and hence, differentiation constitutes half of the very celebrated word ***"calculus."***

As a student of this course, SSC106, a piece of advice is to approach differentiation ***calmly,*** trust me, it's nothing extremely difficult.

Differentiation as a concept may seem difficult to introduce, especially for your colleagues in MTH courses (science and technology students) where differentiation is taken from the basis of the limit of a function. However, introducing differentiation in SSC106 over here is much simpler.

You might be a little familiar with seeing expressions like $\frac{dy}{dx}$ and the likes, well; the concept of that expression is differentiation, also known as differential calculus.

Differentiation basically means the process of finding the derivative or differential coefficient of a function.

Now, that's just the basic meaning.

However, the derivative (or differential coefficient) of a function really does have a wide range of meaning.

Most basically and commonly, **the derivative of a function is either the gradient of that curve (in form of a graph) OR the rate of change of a value with respect to another.**

Hence, **differentiation can also be defined as the branch of calculus that deals with the rate of change in one quantity with respect to another**, mostly to the nearest approximation.

Now, let's stop the grammar and move to plain English.

Differential coefficients are always expressed as the rate of change of a variable with respect to another; now, from our studies of functions, we usually have a dependent variable and independent variable(s). Therefore, derivatives are the rate of change of the dependent variable with respect to the independent variable; and derivatives can also be expressed in function notation.

Now, let's peep at the notations in differential calculus.

The derivative with respect to a variable, say x, for any given function is denoted as

$$\frac{d}{dx}(\text{the given function})$$

The derivative of y with respect to x is denoted as:

$$\frac{dy}{dx}, \text{ this is read as } \textit{'dee y dee x'}$$

In the same way, the derivative of u with respect to z is denoted as:

$$\frac{du}{dz}, \text{ this is read as } \textit{'dee u dee z'}$$

Now, $\frac{dy}{dx}$ is actually the same as $\frac{d}{dx}(y)$ which is read as *dee dee x of y*. This is because y will be standing in place of the given function.

Now, in function notation (the $f(x)$ notation), the derivative of that function with respect to x is denoted as:

$f'(x)$, this is read as *'f-prime of x'*

where that is denoted with ***an apostrophe sign;*** however, some texts and questions denote the derivative of $f(x)$ with respect to x as $f^1(x)$ where the number 1 is used in place of the apostrophe sign; both mean the same thing, however, in the context of this book, we'll be using the notation of the apostrophe sign which is in the form, $f'(x)$.

So, from this, it is obvious that:

$$\frac{d}{dx}[f(x)] = f'(x)$$

meaning the derivative with respect to x of $f(x)$ is $f'(x)$.

Now, if y is a function of x, which implies

$$y = f(x)$$

$$\text{then} \quad \frac{dy}{dx} = f'(x)$$

Both $\frac{dy}{dx}$ and $f'(x)$ will be used interchangeably based on the question in context.

So, with this we have a firm knowledge of what differentiation is now, at least, for what will need.

Having asserted the fact that derivatives basically refer to the rate of change of variables with respect to another, it means that $\frac{dy}{dx}$ refers to the rate of change of y with respect to x.

Now, we'll move to the rules of differentiation, basically, that is what it's all about basically for this course;

THE DERIVATIVE OF A CONSTANT

Firstly, the most basic rule of differentiation is **the derivative of a constant.**

Now, when a function is a constant, it means ***it doesn't change;*** hence, the rate of change of that function is zero.

If $y = c$, where c is a constant

$$\frac{dy}{dx} = 0$$

So, that's it, when functions are constant functions, their rate of change is zero.

In the function above too; i.e. $y = c$

$$\frac{dy}{du} = 0$$

<u>**With respect to any variable,**</u> the derivative of a constant is zero. You'll get to understand more of this statement as we go on.

Now, that is basically all about derivatives of constants.

THE ALGEBRA POWER RULE

Now, here is a very important rule in differential calculus; the derivative of a function with a **constant** power. It is normally regarded to as **the derivative of $\boldsymbol{y = x^n}$ or $\boldsymbol{f(x) = x^n}$**

Now, the rule is like this;

If:

$$y = x^n$$

Then:

$$\frac{dy}{dx} = nx^{n-1}$$

In essence, let's say this in words, drop the power down and reduce it by 1. Drop the power down implies that you multiply the whole function by the initial power; the second step is to subtract 1 from the initial power, hence, the multiplication must be done before the subtraction of one. Afterwards, you'd have successfully found the derivative of that function.

Let's have about ten examples or thereabouts now:

KICKOFF!

- Find $\frac{dy}{dx}$ if $y = x^2$.

Now, this is basically the simplest form of question you can have. With this, you can understand the lengthy grammar I have written above.

To find $\frac{dy}{dx}$, the power of x, the independent variable, which is want we want to differentiate with respect to, is 2.

Hence, to find $\frac{dy}{dx}$, the independent variable needs to be identified first of all, which is x, since the denominator is dx and the dependent variable is dy since it is the numerator of the differential coefficient. Functions must be expressed explicitly, with the dependent variable on the left and the independent variable on the right. Implicit functions can be differentiated too, but we'll not be taking that until the end of this chapter. For now, we'll be taking explicit differentiation (differentiation of explicit functions) where the dependent variable is expressed separately on the left of the equation.

This story above is of extreme importance; the independent variable most importantly must be identified first before differentiation is done.

Now in $\frac{dy}{dx}$, the independent variable is x and hence, the function is to be differentiated with respect to x. Now, in this rule of powers of the independent variable, it is important ***that the power is solely a constant,*** in finding $\frac{dy}{dx}$, the power we're talking about here **must not** contain anything x or y.

Hence, the rule can also be stated as the algebra **constant power rule.** The power must be firstly identified as a constant before you continue with the algebra power rule. Jog back to the topic **of functions**, this rule is meant for power functions and **not exponential functions**.

I hope you quite get these stories?

Here:

$$y = x^2$$

From our notations:

$$\frac{dy}{dx} = \frac{d}{dx}(x^2)$$

To find $\frac{dy}{dx}$, the power here is 2; so what will be done is to multiply the whole function by 2, and then reduce the power of 2 by 1; hence,

$$\frac{dy}{dx} = 2 \times x^{2-1}$$

Therefore, $\boldsymbol{\frac{dy}{dx} = 2 \times x^{1} = 2 \times x = 2x}$

Now, we'll still see several examples, so let's move fast;

- Differentiate with respect to x the function;

$$f(x) = x^{\frac{3}{2}}$$

Easy! To differentiate $f(x)$ with respect to x, we'll be getting $f'(x)$

We'll identify the independent variable as x

Here,

$$f'(x) = \frac{d}{dx}\left(x^{\frac{3}{2}}\right)$$

Next, the power of x is $\frac{3}{2}$ and hence, we'll multiply the function by $\frac{3}{2}$ and subtract 1 from it.

$$f'(x) = \frac{3}{2} \times x^{\frac{3}{2}-1}$$

Hence,

$$f'(x) = \frac{3}{2} \times x^{\frac{1}{2}} = \frac{3}{2}x^{\frac{1}{2}}$$

Now, $x^{\frac{1}{2}}$ is same as the square root of $x \rightarrow \sqrt{x}$

Therefore, we have:

$$\boldsymbol{f'(x) = \frac{3}{2}\sqrt{x}}$$

More…

- If $u = \frac{1}{z}$, find $\frac{du}{dz}$

Now, we can identify the independent variable as z which in essence, we're differentiating with respect to z, hence, we need identify the power of z. Here:

$$\frac{du}{dz} = \frac{d}{dz}\left(\frac{1}{z}\right)$$

Now, we have $\frac{1}{z}$, hence, from the negative power law of indices,

We can express this as $= \frac{1}{z^1}$, since the power of z is 1.

Afterwards, to make this linear i.e., removing the fractional part of this, we'll be having it that the power in the denominator is negative when transferred to the numerator.

$$u = z^{-1}$$

Now, we have seen the power of z, therefore, without much issues, we multiply the function by the power of -1 and then, subtract 1 from the power.

$$\frac{du}{dz} = -1 \times z^{-1-1}$$

Hence,

$$\frac{du}{dz} = -1 \times z^{-2}$$

Now, we can make z^{-2} look better by reapplying the negative power law and bringing it to the denominator to eliminate the negative power; hence:

$$z^{-2} = \frac{1}{z^2}$$

Back to our derivative;

$$\boldsymbol{\frac{du}{dz} = -1 \times \frac{1}{z^2} = -\frac{1}{z^2}}$$

Before we move to the next examples under constant powers of the independent variable, let's see this quick rule.

Now, when we have something like this:

$$y = af(x)$$

A situation when a function of the independent variable is multiplied by a constant, a constant in the case of differentiation often refers to every other variable or constants other than the independent and dependent variable, **for example, when solving for $\frac{dy}{dx}$, values like $1, 7, a, b, z$ and a host of all others are all regarded to as constants since they are neither x nor y.**

The long and short of this rule is that when a function is multiplied by a constant, the constant doesn't affect the function; hence,

When,

$$y = af(x)$$

$$\frac{dy}{dx} = a \times f'(x) = af'(x)$$

Hence, the present of a constant doesn't affect us; we'll simply work on the function and return the constant in place;

!!! Keep all these rules to heart; they're of extreme importance in all our studies of differentiation. Trust me they'll show up at every point.

Now, I think we can continue with our examples, you will actually understand with our further examples; I think we're moving to the fourth one now:

- If $f(a) = 3a^4$, find $f'(a)$

Simple, we have the independent variable in this case as a and hence we'll be differentiating with respect to a;

Here,

$$f'(a) = \frac{d}{da}[f(a)]$$

Now, in the new rule we introduced, we'll simply ignore the constant 3 and differentiate a^4 with respect to a.

Therefore;

$$f'(a) = 3 \times \frac{d}{da}(a^4)$$

We are ignoring the constant and not taking it as zero as if we are differentiating a constant, it is when differentiating a constant [alone] that it is equal to zero, as you can see above, 3 is still there.

Now, the power of a is 4 and hence, we'll multiply the whole thing by 4 and reduce the power by 1, i.e., subtracting 1 from 4.

$$f'(a) = 3 \times 4 \times a^{4-1}$$

Hence,

$$\boldsymbol{f'(a) = 12 \times a^3 = 12a^3}$$

Next!

- Find $\dfrac{dy}{dx}$ if $y = 7\sqrt[3]{x^2}$

Alright…

We have the independent variable as x, full stop.

Also, 7 is multiplying the function of x and hence, it can be ignored.

Hence, here;

$$\frac{dy}{dx} = 7 \times \frac{d}{dx}\left(\sqrt[3]{x^2}\right)$$

Now, we have to properly manipulate $\sqrt[3]{x^2}$, here, the third root means it is raised to a power of $\frac{1}{3}$, hence, $\sqrt[3]{x^2}$ is same as $(x^2)^{\frac{1}{3}}$, next, we'll expand the power of x which is $2 \times \frac{1}{3} = \frac{2}{3}$, hence, the story of $\sqrt[3]{x^2}$ is the same as $x^{\frac{2}{3}}$…

Therefore, the differential coefficient is same as:

$$\frac{dy}{dx} = 7 \times \frac{d}{dx}\left(x^{\frac{2}{3}}\right)$$

Now, the power of x is $\frac{2}{3}$, hence, we'll multiply the whole thing by $\frac{2}{3}$ and subtract 1 from the power of x.

Hence, we have that:

$$\frac{dy}{dx} = 7 \times \frac{2}{3} \times x^{\frac{2}{3}-1}$$

Hence,

$$\frac{dy}{dx} = \frac{14}{3} \times x^{-\frac{1}{3}}$$

We can modify the negative power from the negative law rule to convert:

$$x^{-\frac{1}{3}} \to \frac{1}{x^{\frac{1}{3}}}$$

Hence, we have:

Hence,

$$\frac{dy}{dx} = \frac{14}{3} \times \frac{1}{x^{\frac{1}{3}}} = \frac{14}{3} \times \frac{1}{\sqrt[3]{x}} = \frac{14}{3\sqrt[3]{x}}$$

Sure you noticed the transition of $x^{\frac{1}{3}}$ to $\sqrt[3]{x}$. It's all about indices.

Next!

- Differentiate $12x^{3a-2}$ with respect to x;

Now, since we are to differentiate with respect to x, the independent variable is x and hence, we can express this as a function of x.

$$f(x) = 12x^{3a-2}$$

Here, we have the power of x as $(3a - 2)$, we're still safe to use this as a power since it is not containing any of our independent or dependent variables constituting the function, that confirms it as a constant.

Here, 12 is multiplying the function, you know the normal rule since it is also a mere constant!

$$f'(x) = 12 \times \frac{d}{dx}(x^{3a-2})$$

Multiply the whole expression by the power, $(3a - 2)$ and subtract 1 from the power.

$f'(x) = 12 \times (3a - 2) \times x^{3a-2-1}$;

Notice the use of brackets in $(3a - 2)$ when it was multiplying the whole expression, as the power must be taken as one entity; hence, it is multiplied as a whole;

Hence, $f'(x) = (36a - 24) \times x^{3a-3}$

Therefore,

$$f'(x) = (36a - 24)x^{3a-3}$$

Now, let's see a host of other examples; I don't think you need quite an excessive explanation on this again;

- Find $\frac{dv}{da}$ if:

$$v = -\frac{2}{3}a^{\frac{3}{4}}$$

Now, this is nothing different from where we've been so far;

Independent variable is a;

$$\frac{dv}{dx} = \frac{d}{da}\left(-\frac{2}{3}a^{\frac{3}{4}}\right)$$

$-\frac{2}{3}$ is a constant, hence:

$$\frac{dv}{dx} = -\frac{2}{3} \times \frac{d}{da}\left(a^{\frac{3}{4}}\right)$$

$$\frac{dv}{dx} = -\frac{2}{3} \times \frac{3}{4} \times a^{\frac{3}{4}-1}$$

$$\frac{dv}{dx} = -\frac{1}{2} \times a^{-\frac{1}{4}}$$

Here, we know $a^{-\frac{1}{4}}$ is also $\frac{1}{a^{\frac{1}{4}}}$;

Hence,

$$\frac{dv}{dx} = -\frac{1}{2} \times \frac{1}{a^{\frac{1}{4}}} = -\frac{1}{2} \times \frac{1}{\sqrt[4]{a}}$$

$$\frac{dv}{dx} = -\frac{1}{2\sqrt[4]{a}}$$

Just two more!

- Differentiate with respect to x, $\frac{a}{\sqrt{x}}$

We're differentiating with respect to x, x is the independent variable, make this a function of x;

$$\text{Here, } f(x) = \frac{a}{\sqrt{x}}$$

Here, a (essentially a constant since we're differentiating with respect to x) is multiplying a function of x; we have;

$$f(x) = a \times \frac{1}{\sqrt{x}}$$

Hence,

$$f'(x) = a \times \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right)$$

Simplify $\frac{1}{\sqrt{x}}$ thus;

$$\frac{1}{\sqrt{x}} = \frac{1}{x^{\frac{1}{2}}} = x^{-\frac{1}{2}}$$

Hence,

$$f'(x) = a \times \frac{d}{dx}\left(x^{-\frac{1}{2}}\right)$$

Multiply and subtract one from the power; you know that rule already;

$$f'(x) = a \times -\frac{1}{2} \times x^{-\frac{1}{2}-1}$$

Hence,

$$f'(x) = a \times -\frac{1}{2} \times x^{-\frac{3}{2}} = -\frac{a}{2} \times \frac{1}{x^{\frac{3}{2}}} = -\frac{a}{2x^{\frac{3}{2}}}$$

Hence,

$$\boldsymbol{f'(x) = -\frac{a}{2\sqrt{x^3}}}$$

If all those final simplification of the indices are looking strange, go back to the basic chapter in this text, and refresh yourself on indices.

And the last one for this section!

- Find $\frac{dy}{dx}$ in:

$$y = \frac{q}{\sqrt{x^n}}$$

Cool, now the independent variable is x, and the dependent variable is y, hence, q and n are constants;

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{q}{\sqrt{x^n}}\right)$$

q is constant;

$$\frac{dy}{dx} = q \times \frac{d}{dx}\left(\frac{1}{\sqrt{x^n}}\right);$$

$\frac{1}{\sqrt{x^n}}$ is simplified thus;

$$\frac{1}{(x^n)^{\frac{1}{2}}} = \frac{1}{x^{\frac{n}{2}}} = x^{-\frac{n}{2}}$$

Sure you understood that simplification;

Thus, we have;

$$\frac{dy}{dx} = q \times \frac{d}{dx}\left(x^{-\frac{n}{2}}\right)$$

Hence,

$$\frac{dy}{dx} = q \times -\frac{n}{2} \times x^{-\frac{n}{2}-1}$$

$$\frac{dy}{dx} = -\frac{qn}{2} \times x^{\frac{-n-2}{2}} = -\frac{qn}{2} x^{-\frac{(n+2)}{2}}$$

Never mind that the answer isn't looking attractive, we have it like that; note that in $-n-2$; -1 is factorized to give $-(n+2)$; by negative power rule;

$$\frac{dy}{dx} = -\frac{qn}{2} \times \frac{1}{x^{\frac{n+2}{2}}}$$

And by fractional power rule;

$$\frac{dy}{dx} = -\frac{qn}{2} \times \frac{1}{\sqrt{x^{n+2}}}$$

Hence;

$$\frac{dy}{dx} = -\frac{qn}{2\sqrt{x^{n+2}}}$$

Relax! It's just indices!

Now, this lengthy time we used in building all these isn't wasted at all, it is the very bedrock to understanding differentiation (especially in explicit cases); All these careful explanations are what you need in building carefully every other differential rule; the aspect of identifying constants from the independent variable is key in really smashing the study of differentiation, follow me into something very small that comes next, **the derivative of sums and differences;**

THE DERIVATIVE OF SUMS AND DIFFERENCES

This basically is just a continuation of where we stopped; however, it brings us into the realities of the commonest functions we see; it's of no strange fact most functions are not only expressed as singular terms but as sum of terms, hence, we'll have this rule thus:

When a function is expressed as the sum of different functions of the same independent variable, then the derivative of that function is the sum of the individual derivatives of the functions that make up the function.

This story can be summarized thus:

If:

$$y = g(x) + h(x) + u(x) + v(x)$$

Then:

$$\frac{dy}{dx} = g'(x) + h'(x) + u'(x) + v'(x)$$

Here, if we have different terms summed up in a function, the derivative of the function is the sum of the derivatives of each of those functions; when you see sum, differences aren't exempted, differences and sums are basically the same type of operation, though directly opposite.

Thus:

If:

$$y = g(x) - h(x) - u(x) - v(x)$$

Then:

$$\frac{dy}{dx} = g'(x) - h'(x) - u'(x) - v'(x)$$

Let's see these:

- Find $f'(x)$ if $f(x) = 2x^3 - 5x^2 + 2$

Right! This is very simple and we have smashed this already in the previous section;

Which amongst $2x^3$ or $5x^2$ or 2 is a problem for you to differentiate? None, I guess;

So the whole thing is that as:

$f(x) = 2x^3 - 5x^2 + 2$

$$f'(x) = \frac{d}{dx}(2x^3) - \frac{d}{dx}(5x^2) + \frac{d}{dx}(2)$$

Now, these functions are differentiated separately and added and subtracted appropriately based on the sign in the original function;

$$f'(x) = 3 \times 2 \times x^{3-2} - 2 \times 5 \times x^{2-1} + 0$$

You sure know the derivative of 2 (a constant) is zero!

Hence,

$$f'(x) = 6x^2 - 10x$$

Time has been comprehensively taken to break the derivative of these functions properly into the special steps of their derivative; here during sum and differences, you should have gotten the trick well now; so let's move on fast, we won't bother explaining in details how these functions are differentiated, we've learnt them in detail;

Next!

- Find $\frac{dy}{dx}$ if:

$$y = 3x^2 + \frac{1}{x}$$

This is straightforward; the independent variable is x; convert $\frac{1}{x}$ to x^{-1} and complete the rest;

$$y = 3x^2 + x^{-1}$$

$$\frac{dy}{dx} = \frac{d}{dx}(3x^2) + \frac{d}{dx}(x^{-1})$$

Hence, $\frac{dy}{dx} = 2 \times 3 \times x^{2-1} + (-1 \times x^{-1-1})$

$$\frac{dy}{dx} = 6x - x^{-2}$$

$$\frac{dy}{dx} = 6x - \frac{1}{x^2}$$

Here is another;

- Find $\frac{dV}{dx}$ if:

$$V = x^a - 2x^b + x^{-c}$$

where a, b and c are constants.

Fine, this isn't in any way a problem;

$$\frac{dV}{dx} = \frac{d}{dx}(x^a) - \frac{d}{dx}(2x^b) + \frac{d}{dx}(x^{-c})$$

Now, we don't need to be told a, b and c are constants because they're automatically constants since they're neither V nor x which are the dependent and independent variables;

Since a, b and c are constants, they'll be treated with the same rule, of course you understand that;

$$\frac{dV}{dx} = a \times x^{a-1} - b \times 2 \times x^{b-1} + (-c \times x^{-c-1})$$

$$\frac{dV}{dx} = ax^{a-1} - 2bx^{b-1} - cx^{-(c+1)}$$

I'm sure the factorization of -1 in $-c - 1$ is not strange, it becomes $-(c + 1)$, we can apply negative power law to $x^{-(c+1)}$ to make it $\frac{1}{x^{c+1}}$,

Hence, our answer is

$$\frac{dV}{dx} = ax^{a-1} - 2bx^{b-1} - \frac{c}{x^{(c+1)}}$$

As cool as you like, I guess that answer looks attractive!

Next…

- Differentiate with respect to x,

$$\sqrt{x} + \frac{1}{\sqrt{x}} - 3$$

Now, take it one by one, break the square roots into fractional powers and solve!

$$\sqrt{x} \rightarrow x^{\frac{1}{2}}$$

Hence;

$$\frac{1}{\sqrt{x}} \rightarrow \frac{1}{x^{\frac{1}{2}}} = x^{-\frac{1}{2}}$$

Hence, since the independent variable is x (we're differentiating with respect to x), we take it as a function of x;

$$f(x) = x^{\frac{1}{2}} + x^{-\frac{1}{2}} - 3$$

$$f'(x) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) + \frac{d}{dx}\left(x^{-\frac{1}{2}}\right) - \frac{d}{dx}(3)$$

We'll apply the constant power rule and differentiate;

$$f'(x) = \frac{1}{2} \times x^{\frac{1}{2}-1} + \left(-\frac{1}{2} \times x^{-\frac{1}{2}-1}\right) - 0$$

$$f'(x) = \frac{1}{2} \times x^{-\frac{1}{2}} - \frac{1}{2} \times x^{-\frac{1}{2}-1}$$

$$f'(x) = \frac{1}{2} \times x^{-\frac{1}{2}} - \frac{1}{2} \times x^{-\frac{3}{2}}$$

Convert by indices;

$$x^{-\frac{1}{2}} \to \frac{1}{x^{\frac{1}{2}}} \to \frac{1}{\sqrt{x}}$$

And

$$x^{-\frac{3}{2}} \to \frac{1}{x^{\frac{3}{2}}} \to \frac{1}{\sqrt{x^3}}$$

Hence;

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$$

And this, another example for the derivative of sum and difference;

- If $f(u) = \frac{u^3+2u^2+1}{u}$, find $f'(1)$.

Cool, here, we'll need to simplify this a bit before we continue, don't get puzzled that you're seeing something too strange over here;

So right,

$$f(u) = \frac{u^3 + 2u^2 + 1}{u}$$

This can be simplified by dividing the numerators separately instead of jointly; it's a common mathematics rule that:

$$\frac{a+b+c}{d} \text{ is equivalent to } \frac{a}{d} + \frac{b}{d} + \frac{c}{d}$$

That is, if the denominator is just one, it can be distributed on the numerators.

Hence, here,

$$f(u) = \frac{u^3}{u} + \frac{2u^2}{u} + \frac{1}{u}$$

Simplifying via indicial rule;

$$f(u) = u^{3-1} + 2u^{2-1} + 1 \times u^{-1};$$

$$f(u) = u^2 + 2u + u^{-1}$$

These are powers as constants so it's a simple rule; Here;

$$f'(u) = 2 \times u^{2-1} + 2 \times u^{1-1} + (-1 \times u^{-1-1})$$

Hence,

$$f'(u) = 2u + 2 \times u^{0} - 1 \times u^{-2}$$

Anything raised to power 0 is 1; $u^0 = 1$ and u^{-2} is $\frac{1}{u^2}$

Hence;

$$f'(u) = 2u + 2 - \frac{1}{u^2}$$

But we were told to evaluate $f'(1)$, I'm sure you understand that, substitute 1 for u in $f'(u)$;

Since;

$$f'(u) = 2u + 2 - \frac{1}{u^2}$$

We have:

$$f'(1) = 2(1) + 2 - \frac{1}{1^2}$$

$$f'(1) = 3$$

- The distance covered by a particle after time, t is given by $s = ut + \frac{1}{2}at^2$, find the rate at which the particle is moving with respect to its time, t.

Wow, looks so much like a science question; however, it's not more of it though; it's simply a question on the derivative of sum and differences. Now, we are taking the independent variable as t since we're told to find the rate of change with respect to t, hence, all other terms are taking as constants. Only t and s are taking as variables; Now, we have:

$$s = ut + \frac{1}{2}at^2$$

We require $\frac{ds}{dt}$ since we need the rate of change of s with respect to t;

$$\frac{ds}{dt} = \frac{d}{dt}(ut) + \frac{d}{dt}\left(\frac{1}{2}at^2\right)$$

Apply the rule of constant powers to the independent variable, t

$$\frac{ds}{dt} = 1 \times u \times t^{1-1} + 2 \times \frac{1}{2} \times a \times t^{2-1}$$

$$\frac{ds}{dt} = u \times t^0 + 1 \times a \times t$$

We have:

$$\frac{ds}{dt} = u + at$$

It's that simple.

Now, we approach the most important part of differential calculus; once this is understood, you can stroll over the entirety of differentiation without stress. It is called **the function of a function.**

THE CHAIN RULE: THE FUNCTION OF A FUNCTION

Now, looking at the functions we've been looking at so far, you can notice it very obviously that the functions we have been working on have been in the form of the simplest expression of a function like x, u, z and etc been raised to constant powers or divided and so on and so forth.

Examples of the functions new worked on included functions like $x^{\frac{1}{2}}$, $x^{-\frac{1}{2}}$, t^2, x^{-c}, $x^{-1}\left(\frac{1}{x}\right)$,

u^2, u^3 and a lots of other functions we've worked upon.

Now, notice it that all these functions are expressed **as basic functions** where the constant powers or the basic functions represented are in the form of **the basic variables**, x, u, t and so on. We simply applied the constant power law directly on those functions normally.

Now consider this, what do you think will be the case when we have something like this: $(2x - 3)^5$; do you feel we'll simply drop the power of 5 down and subtract 1 from it? No, actually ... ... That brings the differential rule known as **the function of a function.**

Now, it's like this, functions are expressed in the form $f(x), f(u)$ and so on; hence, x, u in this case are the most basic ways the independent variables of such functions can be expressed. Hence, when differentiating with rules applying to differentiation, such rules **apply "directly" only when the terms in the function are expressed in the most basic way.**

In other cases, as well, the rules are perfectly applied, however, this is done with a constraint of the concept of the function of a function.

Now, when applying differential rules to functions not expressed in their most basic form, it must be expressed in the most basic form by certain substitutions; it is done like this:

Let's take $y = (2x - 3)^5$ as our specimen.

Note that anything other than x expressed alone, we'll be taking this process below:

The function which is enough to be called a function of its own is taken out! Now, $(2x - 3)$ is not a basic way a function of x is expressed; hence, $(2x - 3)$ is in itself a function of x;

Here, y can be seen as a function of $(2x - 3)$;

Now, let's do some manipulations;

$$\text{Let's put } u = 2x - 3$$

Now, we now have that u is a function of x;

Now, we can now have neatly that:

$$y = u^5 \text{ since } u = 2x - 3$$

We now have that y is a function of u

Now, $y = u^5$ has now been expressed in form of the most basic way a function of u can be now be differentiated since u itself is what is raised to a constant power.

Now, we have that:

$$y = u^5$$

$\frac{dy}{du}$ will be $5u^4$, I'm sure that's not a challenge at all.

But remember that u is just an intruder, more so, we need $\frac{dy}{dx}$ and not $\frac{dy}{du}$ right? So this is where the rule of function of function is;

From:

$$u = 2x - 3$$

We see that $\frac{du}{dx} = 2$, Now we have two differential coefficients here;

We now have both $\frac{dy}{du}$ and $\frac{du}{dx}$ ... ... So what's next?

Consider this product;

$$\frac{dy}{du} \times \frac{du}{dx}$$

Now try to reduce this product without actually cancelling the terms physically; you can see that du will cancel out to leave you with $\frac{dy}{dx}$ which is actually what we need.

Hence, that brings the rule of the function of a function:

If y is a function of u, i.e. $y = f(u)$ and that u itself is a function of x, i.e. $u = g(x)$

Then,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

The statement above in read is the statement definition of the chain rule;

u, in chain rule, is serving as a kind of middle man!

We can see from our specimen, that:

$$\frac{dy}{dx} = 2 \times 5u^4 = 10u^4$$

u however was an intruder,

$$u = 2x - 3$$

And it has to be returned to its original value; therefore;

$$\frac{dy}{dx} = 10(2x - 3)^4$$

Now, the function of function rule isn't in any way limited to one substitution, as many substitutions are made based on the complexity of the function, the function of function rule stands when functions can be expressed as functions of other functions;

Now, consider this product;

$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{du} \times \frac{du}{dx}$$

This occurs when $y = f(z)$ and $z = g(u)$ and $u = h(x)$;

The above product exists when substitutions are made twice; in cases of complex functions; you should see an example that looks like that in the course of this.

The function of function rule is generally known as the chain rule.

I guess the whole basic form thing is understood anyway? You'd understand more and more as we consider other rules of differentiation.

Also, I think other examples will lighten you up the more:

Let's move a bit faster;

Find the derivative of y with respect to x in the following;

- $y = 4(3x^2 - 2)^3$
- $y = \sqrt{1 - 2x^3}$

- $y = \frac{5}{(6-x^2)^3}$
- $y = \frac{3}{\sqrt{1+x^2}}$
- $y = \frac{a}{\sqrt{bx^2+cx+d}}$

Wow, how about five for a start? Let's kick start this ASAP!

- $y = 4(3x^2 - 2)^3$

Now, $3x^2 - 2$ is a function on its own and hence, we cannot just blankly apply the constant power rule here, we'll make a substitution for $3x^2 - 2$;

Now, put:

$$u = 3x^2 - 2$$

It'll make things easy now;

y will hence become;

$$y = 4u^3$$

This is now basic and can be differentiated using the constant power rule.

Now,

$u = 3x^2 - 2$	$y = 4u^3$
$\frac{du}{dx} = 2 \times 3 \times x^{2-1} - 0$	$\frac{dy}{du} = 3 \times 4 \times u^{3-1}$
$\frac{du}{dx} = 6x$	$\frac{dy}{du} = 12u^2$

Now, the chain rule tells us that;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Hence;

$$\frac{dy}{dx} = 12u^2 \times 6x$$

I'm sure this isn't strange but let me remind you *sha*, in this product, since no addition or subtraction sign is involved, the numbers can be multiplied and the unlike terms left as they are; hence,

$$\frac{dy}{dx} = 12 \times 6 \times u^2 \times x = 72u^2x$$

The intruder can now excuse us,

$$u = 3x^2 - 2$$

Fix that back!

$$\frac{dy}{dx} = 72(3x^2 - 2)^2 x$$

For the proper orderliness;

$$\frac{dy}{dx} = 72x(3x^2 - 2)^2$$

Right, we'll move on:::

- $y = \sqrt{1 - 2x^3}$

This is nothing difficult, put:

$$u = 1 - 2x^3$$

It will make it:

$$y = \sqrt{u}$$

Hence;

$u = 1 - 2x^3$	$y = \sqrt{u}$
$\frac{du}{dx} = 0 - 3 \times 2 \times x^{3-1}$	$y = u^{\frac{1}{2}}$

$$\frac{du}{dx} = -6x^2 \qquad \frac{dy}{du} = \frac{1}{2} \times u^{\frac{1}{2}-1}$$

By proper simplification;

$$\frac{du}{dx} = -6x^2 \qquad \frac{dy}{du} = \frac{1}{2\sqrt{u}}$$

Now, all these basic simplification processes have been trashed in the previous parts of this topic, you shouldn't find them difficult at all again.

Now, the chain rule tells us that;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{u}} \times -6x^2 = -\frac{6x^2}{2\sqrt{u}}$$

Put the true value of u in place;

$$u = 1 - 2x^3$$

We have;

$$\frac{dy}{dx} = -\frac{6x^2}{2\sqrt{1 - 2x^3}}$$

2 cancels off 6;

$$\frac{dy}{dx} = -\frac{3x^2}{\sqrt{1-2x^3}}$$

Next!

- $y = \frac{5}{(6-x^2)^3}$

$6 - x^2$ isn't basic, hence, put:

$$u = 6 - x^2$$

That simplifies y to:

$$y = \frac{5}{u^3} = 5u^{-3}$$

$u = 6 - x^2$ $\qquad$ $y = 5u^{-3}$

$\frac{du}{dx} = 0 - 2 \times x^{2-1}$ $\qquad$ $\frac{dy}{dx} = -3 \times 5 \times u^{-3-1}$

$\frac{du}{dx} = -2x$ $\qquad$ $\frac{dy}{dx} = -15u^{-4}$

The chain rule tells us that;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = -15u^{-4} \times -2x = -15 \times\ -2 \times x \times u^{-4}$$

$$\frac{dy}{dx} = 30xu^{-4} = \frac{30x}{u^4}$$

Sure you understood that negative power transition!

Bring u back to what it is:

$$u = 6 - x^2$$

Hence,

$$\frac{dy}{dx} = \frac{30x}{(6 - x^2)^4}$$

- $y = \frac{3}{\sqrt{1+x^2}}$

Here; we put $u = 1 + x^2$; since $1 + x^2$ is a function on its own too!

Hence, the function y becomes:

$$y = \frac{3}{\sqrt{u}} = \frac{3}{u^{\frac{1}{2}}} = 3u^{-\frac{1}{2}}$$

$u = 1 + x^2$	$y = 3u^{-\frac{1}{2}}$

$$\frac{du}{dx} = 0 + 2 \times x^{2-1}$$

$$\frac{du}{dx} = 2x$$

$$\frac{dy}{du} = -\frac{1}{2} \times 3 \times u^{-\frac{1}{2}-1}$$

$$\frac{dy}{du} = -\frac{3}{2} u^{-\frac{3}{2}}$$

Simplifying $\frac{dy}{du}$ by indices, we have:

$$\frac{dy}{du} = -\frac{3}{2} \times \frac{1}{u^{\frac{3}{2}}} = -\frac{3}{2\sqrt{u^3}}$$

The chain rule tells us that;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = -\frac{3}{2\sqrt{u^3}} \times 2x = -\frac{3x}{\sqrt{u^3}}$$

Return u back to position; $u = 1 + x^2$

$$\frac{dy}{dx} = -\frac{3x}{\sqrt{(1 + x^2)^3}}$$

- $y = \frac{a}{\sqrt{bx^2+cx+d}}$

Here, we know we are regarding a, b, c and d as constants as they're neither of the dependent variable nor the independent variable.

Put:

$$u = bx^2 + cx + d$$

Here;

$$\frac{du}{dx} = 2 \times b \times x^{2-1} + 1 \times c \times x^{1-1} + 0$$

$$\frac{du}{dx} = 2bx + c$$

Since,

$$u = bx^2 + cx + d$$

$$y = \frac{a}{\sqrt{u}} = \frac{a}{u^{\frac{1}{2}}} = au^{-\frac{1}{2}}$$

Here, a is a constant so you remember how its done, we simply use power rule:

$$\frac{dy}{du} = -\frac{1}{2} \times a \times u^{-\frac{1}{2}-1}$$

$$\frac{dy}{du} = -\frac{a}{2} \times u^{-\frac{3}{2}} = -\frac{a}{2} \times \frac{1}{u^{\frac{3}{2}}}$$

$$\frac{dy}{du} = -\frac{a}{2} \times \frac{1}{\sqrt{u^3}} = -\frac{a}{2\sqrt{u^3}}$$

The chain rule tells us that;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Hence;

$$\frac{dy}{dx} = -\frac{a}{2\sqrt{u^3}} \times (2bx + c)$$

$$\frac{dy}{dx} = -\frac{a(2bx + c)}{2\sqrt{u^3}}$$

Put u in place; $u = bx^2 + cx + d$

$$\frac{dy}{dx} = -\frac{a(2bx + c)}{2\sqrt{(bx^2 + cx + d)^3}}$$

I think that's it about the chain rule and basically, that's the main thing to know about the whole idea of explicit differentiation.

Lol, someone is feeling like: "what rubbish is this guy saying? There are still over 100 pages to go and you're saying we've known the main thing, what's the correlation?" Well, don't worry, you'll see it yourself that chain rule is the backbone of every other subsequent rule of differentiation!

I'll want to skip the derivative of products and quotients for the time being and take us through the journeys of the trigonometric, the logarithm and the exponential functions. We'll get back to the product and quotient rule in a short while.

And so, like I said, there isn't any challenge again with anything about explicit differentiation if and only if you have mastered the rule of the function of a function.

Now, we move into the **derivative of trigonometric functions.**

DERIVATIVE OF TRIGONOMETRIC FUNCTIONS

The derivative of trigonometric functions is basically some set of rules you need to master, after that apply the rule of the function of a function and that's it.

Here are the rules;

If: $\boldsymbol{y = \sin x}$

$$\boldsymbol{\frac{dy}{dx} = \cos x}$$

If: $\boldsymbol{y = \cos x}$

$$\boldsymbol{\frac{dy}{dx} = -\sin x}$$

If: $\boldsymbol{y = \tan x}$

$$\boldsymbol{\frac{dy}{dx} = \sec^2 x}$$

By derivation from these above functions with the following facts;

$$\sec x = \frac{1}{\cos x}$$

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

$$\cot x = \frac{1}{\tan x}$$

If: $\boldsymbol{y = \sec x}$

$$\boldsymbol{\frac{dy}{dx} = \sec x \tan x}$$

If: $\boldsymbol{y = \operatorname{cosec} x}$

$$\boldsymbol{\frac{dy}{dx} = -\cot x \operatorname{cosec} x}$$

If: $\boldsymbol{y = \cot x}$

$$\boldsymbol{\frac{dy}{dx} = -\operatorname{cosec}^2 x}$$

The rules above are just rules you need to commit to memory, as for the scope of this course; that's all you need; after committing them to memory, you apply the function of function and you're in.

Let's see this; before we continue though; here are some basic rules in trigonometry;

$\sin^2 x$ means the square of $\sin x = (\sin x)^2$

Same;

$\cos^2 x$ means the square of $\cos x = (\sin x)^2$

$\tan^2 x$ means the square of $\tan x = (\tan x)^2$

$\sec^2 x$ means the square of $\sec x = (\sec x)^2$

$\operatorname{cosec}^2 x$ means the square of $\operatorname{cosec} x = (\operatorname{cosec} x)^2$

$\cot^2 x$ means the square of $\cot x = (\cot x)^2$

Also, this is key;

$$\tan x = \frac{\sin x}{\cos x}$$

Which in the same way, since:

$$\cot x = \frac{1}{\tan x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

Now, while you are memorizing all those; let's see these quick examples;

Chain rule in trigonometric differentiation rule;

The rules given above are in the most basic forms of the trigonometric functions, however, most often, it isn't the most basic forms of trigonometric functions that are differentiated, mostly the trigonometric forms are expressed in terms of other functions. Hence, let's see how it comes into play here;

Consider;

$$y = \sin f(x)$$

Let;

$$u = f(x)$$

It implies;

$$y = \sin u$$

Hence, y has now been expressed in the most basic sine expression.

$$\frac{du}{dx} = f'(x)$$

And by the rules we've just listed out (just above in page 52);

$$\frac{dy}{du} = \cos u$$

From chain rule;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Hence;

$$\frac{dy}{dx} = \cos u \times f'(x)$$

But;

$$u = f(x)$$

Hence;

$$\frac{dy}{dx} = f'(x)\cos f(x)$$

Hence;

We have that:

If: $\boldsymbol{y = \sin f(x)}$

$$\boldsymbol{\frac{dy}{dx} = f'(x)\cos f(x)}$$

Similarly; by the same chain rule application;

If: $\boldsymbol{y = \cos f(x)}$

$$\boldsymbol{\frac{dy}{dx} = -f'(x)\sin f(x)}$$

If: $\boldsymbol{y = \tan f(x)}$

$$\boldsymbol{\frac{dy}{dx} = f'(x)\sec^2 f(x)}$$

If: $\boldsymbol{y = \sec f(x)}$

$$\boldsymbol{\frac{dy}{dx} = f'(x) \sec f(x) \tan f(x)}$$

If: $\boldsymbol{y = \operatorname{cosec} f(x)}$

$$\boldsymbol{\frac{dy}{dx} = -f'(x) \cot f(x) \operatorname{cosec} f(x)}$$

If: $\boldsymbol{y = \cot f(x)}$

$$\boldsymbol{\frac{dy}{dx} = -f'(x) \operatorname{cosec}^2 f(x)}$$

The above help us to differentiate functions not expressed in the most basic forms but still very light very quickly;

For instance;
The derivative of $\sin(-x)$ is simply gotten as $-\cos(-x)$ since the derivative of $-x$ (in this case serving as $f(x)$ is -1).

Hence; let's see an instance, the chain rule trigonometry occurs in this form with the formula;

$$y = \sin\ 2x$$

This is like $\sin f(x)$ where in this case, $f(x)$ is $2x$;

For:

$$f(x) = 2x$$
$$f'(x) = 2$$

And hence, from the formula;

If: $\boldsymbol{y = \sin f(x)}$

$$\boldsymbol{\frac{dy}{dx} = f'(x) \cos f(x)}$$

We'd have the derivative of $\sin 2x$ as:

$$\frac{dy}{dx} = 2 \cos 2x$$

Similarly; for:

$$\cos(-3x)$$

We have:

$$f(x) = -3x$$
$$f'(x) = -3$$

And hence; from the formula;

If: $\boldsymbol{y = \cos f(x)}$

$$\boldsymbol{\frac{dy}{dx} = -f'(x)\sin f(x)}$$

We'd have:

$$\frac{dy}{dx} = -(-3)\sin(-3x) = 3\sin(-3x)$$

For now, though, we will still substitute properly for chain rule. It is always safe to substitute properly to avoid mistakes. **For cases of simple functions of functions, we can use the above method, the safest method though is to substitute appropriately.**

So in the examples below, we'd be substituting properly by chain rule, if you can cope with the direct use of the formula though, you can keep up with it.

Find $\frac{dy}{dx}$ in each of the following:

- $y = \cos 2x$
- $y = \sin \frac{1}{3} x$
- $y = \cos^2 x$
- $y = \tan^3 x^7$
- $y = \sec 6x$
- $y = -2 \operatorname{cosec} \left(-\frac{3}{4} x^3\right)$

All these are quite soft to take;

- $y = \cos 2x$

So you've known the derivative of $\cos x$ from the trigonometric rules; however, we have $2x$ here and not x, hence the function of function is here:

We'll now have; put:

$$u = 2x$$

$$\frac{du}{dx} = 1 \times 2 \times x^{1-1} = 2 \times x^0 = 2 \times 1 = 2$$

Hence, we now have this after the substitution:

$$y = \cos u$$

Here we can now apply the rule softly;

$$y = \cos u$$

Hence,

$$\frac{dy}{du} = -\sin u$$

That's as simple as that; the rules you were given that:

If: $\boldsymbol{y = \cos x}$

$$\boldsymbol{\frac{dy}{dx} = -\sin x}$$

Here,

From the chain rule;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Hence,

$$\frac{dy}{dx} = -\sin u \times 2 = -2\sin u$$

Return what u means in place;

$$\frac{dy}{dx} = -2\sin 2x$$

You can see obviously that it's nothing difficult in any way. You only need to properly understand the function of function rule; the rest are just a matter of a little memorizing.

- $y = \sin\frac{1}{3}x$

Simple, $\frac{1}{3}x$ isn't a basic value; hence, we do the regular;

$$u = \frac{1}{3}x$$

$$\frac{du}{dx} = 1 \times \frac{1}{3} \times x^{1-1} = \frac{1}{3} \times x^{0} = \frac{1}{3}$$

Hence, from our substitution;

$$y = \sin u$$

From the rule that;

If: $\boldsymbol{y = \sin x}$

$$\boldsymbol{\frac{dy}{dx} = \cos x}$$

Hence, with $y = \sin u$

$$\frac{dy}{du} = \cos u$$

From the chain rule;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Hence,

$$\frac{dy}{dx} = \cos u \times \frac{1}{3} = \frac{1}{3} \cos u$$

Return what u means in place;

$$\frac{dy}{dx} = \frac{1}{3} \cos \frac{1}{3} x$$

We need to move faster; this is too simple;

- $y = \cos^2 x$

Now, from the brief rules of trigonometry we saw, we can see that;

$y = \cos^2 x$ implies $y = (\cos x)^2$

Now, seeing this critically;

We see this as a constant power situation, but just as the previous cases, the base raised to the constant power is not the basic value of x…

Hence, what do we do here?

$$y = (\cos x)^2$$

Hence, this is now, in this case, a constant power rule situation with the function within which we'll be substituting for, the cosine trigonometric function.

Normal thing as we've been doing above;

Put:

$$u = \cos x$$

Hence; from the matter of our derivative of trigonometric functions rule;

$$\frac{du}{dx} = -\sin x$$

From our substitution, we're left with;

$$y = u^2$$

Here:

$$\frac{dy}{du} = 2 \times u^{2-1} = 2 \times u = 2u$$

As usual, from the chain rule;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Hence,

$$\frac{dy}{dx} = 2u \times -\sin x = -2u\sin x$$

Return what u means in place; $u = \cos x$

$$\frac{dy}{dx} = -2\cos x \sin x$$

That looks a bit tricky; it's still all about the function of a function. You don't mind reading it all over, do you?

Let's get to the next question;

- $y = \tan^3 x^7$

Now, from the trigonometric rule we saw, we can see that;

$$\tan^3 x^7 = (\tan x^7)^3$$

Sure that wasn't too strange, it's not limited to only squares, when a power is written immediately above the trigonometric function, it is a power of the whole function.

So… Here is also a constant power situation where we have the base not in the basic form but as a function on its own;

No stories; put:

$$u = \tan x^7$$

This is getting interesting; we can't find $\frac{du}{dx}$ directly since the basic rule is for $\tan x$;

Hence, another substitution loading… … …

Put:

$$z = x^7$$

Here, we can simply have;

$$\frac{dz}{dx} = 7 \times x^{7-1} = 7x^6$$

Now, it means $u = \tan z$; right???

Check it well, it is so, since $z = x^7$

$$u = \tan z$$

From this law:

If: $\boldsymbol{y = \tan x}$

$$\boldsymbol{\frac{dy}{dx} = \sec^2 x}$$

Therefore; here,

$$\frac{du}{dz} = \sec^2 z$$

But since $u = \tan x^7$

$$y = u^3$$

That is because, $y = (\tan x^7)^3$

Hence,

$$\frac{dy}{du} = 3 \times u^{3-1} = 3u^2$$

Now, we have three different differential coefficients;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dz} \times \frac{dz}{dx}$$

That's the case of having three differential coefficients which I already promised you we'll be seeing a case as such; hence,

$$\frac{dy}{dx} = 3u^2 \times \sec^2 z \times 7x^6 = 21x^6u^2 \sec^2 z$$

We simply multiplied everything above together;
Now, return the base values;

$$z = x^7$$

$$u = \tan z = \tan x^7$$

Hence;

$$\frac{dy}{dx} = 21x^6(\tan x^7)^2 \sec^2 x^7$$

$$\frac{dy}{dx} = 21x^6 . \tan^2 x^7 . \sec^2 x^7$$

Looks quite some complex answer; three terms multiplied together but that's just how it is – not difficult at all, just take it **very gradually** as it is, ***it really is very very basic***;

- $y = \sec 6x$

Here, quite easier, put:

$$u = 6x$$

$$\frac{du}{dx} = 1 \times 6 \times x^{1-1} = 6x^0 = 6$$

Quite an easy substitution;

$$y = \sec u$$

From this:

If: $\boldsymbol{y = \sec x}$

$$\boldsymbol{\frac{dy}{dx} = \sec x \tan x}$$

Here,

$$\frac{dy}{du} = \sec u \tan u$$

From the chain rule;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \sec u \tan u \times 6 = 6 \sec u \tan u$$

Return the value of $u = 6x$

$$\frac{dy}{dx} = 6 \sec 6x \tan 6x$$

Wawu, the last one;

- $y = -2 \operatorname{cosec}\left(-\frac{3}{4}x^3\right)$

Cool, this is quite straightforward, make the straightforward substitution;

$$u = -\frac{3}{4}x^3$$

$$\frac{du}{dx} = 3 \times -\frac{3}{4} \times x^{3-1} = -\frac{9}{4}x^2$$

Obviously, here;

$$y = -2 \operatorname{cosec} u;$$

From this law:

If: $\boldsymbol{y = \operatorname{cosec} x}$

$$\boldsymbol{\frac{dy}{dx} = -\cot x \operatorname{cosec} x}$$

Hence, here:

$$\frac{dy}{du} = -2 \times -\cot u \operatorname{cosec} u$$

Note from the above that -2 in y is a mere constant multiplying the function and hence, it has nothing to affect the derivative since it'll just multiply the derivative.

Hence, $-2 \times -$ cancels off to become a positive value;

$$\frac{dy}{du} = 2 \cot u \operatorname{cosec} u$$

From the chain rule;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = 2 \cot u \operatorname{cosec} u \times \left(-\frac{9}{4}x^2\right)$$

$$\frac{dy}{dx} = -\frac{9}{2}x^2 \cot u \operatorname{cosec} u$$

Notice the cancelling 2 and 4 and the coming in of the negative sign from there!

Return the value of:

$$u = -\frac{3}{4}x^3$$

$$\frac{dy}{dx} = -\frac{9}{2}x^2 \cot\left(-\frac{3}{4}x^3\right) \operatorname{cosec}\left(-\frac{3}{4}x^3\right)$$

Don't mind the complexity of all these functions please; just take it gradually, that's the whole essence of constant substitution to avoid the complexity along the way but only in the final answer;

Let's see this one; just two more!

- Evaluate $f'(\theta)$ if:

$$f(\theta) = \sin\sqrt{4\theta^2 - a^2}$$

Now, trust you know any variable can be the independent variable, you know we usually note the independent variable before anything else; Here, the independent variable is θ; hence, a is a constant;

So;

$$f(\theta) = \sin\sqrt{4\theta^2 - a^2}$$

We'll make our substitutions to find $f'(\theta)$ since this is a trigonometric function but it isn't in its most basic form;
put:

$$u = \sqrt{4\theta^2 - a^2}$$

Convert the square root to power $\frac{1}{2}$

$$u = (4\theta^2 - a^2)^{\frac{1}{2}}$$

We can't handle this constant power situation on its own since it isn't basic; hence, we make another substitution;

$$z = 4\theta^2 - a^2$$

We should be able to handle this now, remember vividly that we are differentiating with respect to θ and not with respect to a:

$$\frac{dz}{d\theta} = 2 \times 4 \times \theta^{2-1} - 0$$

Obviously, a is a constant and has a derivative of zero;

$$\frac{dz}{d\theta} = 8\theta$$

Up next!

$$u = z^{\frac{1}{2}}$$

Since $z = 4\theta^2 - a^2$

Hence, by power rule,

$$\frac{du}{dz} = \frac{1}{2} \times z^{\frac{1}{2}-1} = \frac{1}{2} z^{-\frac{1}{2}} = \frac{1}{2\sqrt{z}}$$

I'm sure you can handle all the simplification of $z^{-\frac{1}{2}}$ saga;

Since this is a chain rule situation; we'll let;

$y = f(\theta)$ so that $f'(\theta)$ can be denoted as $\frac{dy}{dx}$

Next; since:

$$f(\theta) = \sin\sqrt{4\theta^2 - a^2}$$

$$y = \sin\sqrt{4\theta^2 - a^2} = \sin u$$

Since $u = (4\theta^2 - a^2)^{\frac{1}{2}}$

From the rule:

If: $\boldsymbol{y = \sin x}$

$$\boldsymbol{\frac{dy}{dx} = \cos x}$$

Hence,

$$\frac{dy}{du} = \cos u$$

Now, we have the triple chain rule;

$$\frac{dy}{d\theta} = \frac{dy}{du} \times \frac{du}{dz} \times \frac{dz}{d\theta}$$

$$\frac{dy}{d\theta} = \cos u \times \frac{1}{2\sqrt{z}} \times 8\theta$$

Return the base values;

$$z = 4\theta^2 - a^2$$

$$u = z^{\frac{1}{2}} = (4\theta^2 - a^2)^{\frac{1}{2}} = \sqrt{4\theta^2 - a^2}$$

Hence,

$$\frac{dy}{d\theta} = \cos\sqrt{4\theta^2 - a^2} \times \frac{1}{2\sqrt{4\theta^2 - a^2}} \times 8\theta$$

Multiply and blend everything together;

$$\frac{dy}{d\theta} = \frac{8\theta \cos\sqrt{4\theta^2 - a^2}}{2\sqrt{4\theta^2 - a^2}}$$

Now, since $y = f(\theta)$ as we introduced it for the sake of the chain rule expression; we have that;

$$f'(\theta) = \frac{8\theta \cos\left(\sqrt{4\theta^2 - a^2}\right)}{2\sqrt{4\theta^2 - a^2}}$$

Wow, such an ambiguous answer, but I guess it only got ambiguous at the end though; hence, the need for substitution of the function of a function. So... ... ... Let's have a short visit at your past question!

- If $x = r\sin\theta$ and $y = r\cos\theta$, show that:

(i) $$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2$$

(ii) $$\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2 = 1$$

So, some question this is:
Here, x is serving as a dependent variable, yeah, everything is possible;

So we have two functions here:

$$x = r\sin\theta$$
$$y = r\cos\theta$$

In (i); we're told to evaluate both $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ and work out some things on them.

$$x = r\sin\theta$$

To find $\frac{dx}{d\theta}$, we'll know that r is a constant since we're differentiating with respect to θ; hence,

$$\frac{dx}{d\theta} = r \times \frac{d}{d\theta}(\sin\theta)$$

Of course from our trigonometric rules, we know the derivative of $\sin\theta$ is $\cos\theta$ just like for $\sin x$ is $\cos x$, just a change of variable;

$$\frac{dx}{d\theta} = r \times \cos\theta = r\cos\theta$$

$$y = r\cos\theta$$

To find $\frac{dy}{d\theta}$, same way, we'll know that r is a constant since we're differentiating with respect to θ; hence,

$$\frac{dy}{d\theta} = r \times \frac{d}{d\theta}(\cos\theta)$$

Of course from our trigonometric rules, we know the derivative of $\cos\theta$ is $-\sin\theta$ just like for $\cos x$ is $-\sin x$, just a change of variable;

$$\frac{dy}{d\theta} = r \times -\sin\theta = -r\sin\theta$$

Now, we're told to prove this:

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2$$

To prove, we show that the left hand side is equal to the right hand side of the equation;

To do this, we take the squares of our derivatives just like it is in the question;

$$\left(\frac{dx}{d\theta}\right)^2 = (r\cos\theta)^2 = r^2(\cos\theta)^2 = r^2(\cos^2\theta)$$

$$\left(\frac{dy}{d\theta}\right)^2 = (-r\sin\theta)^2 = r^2(\sin\theta)^2 = r^2(\sin^2\theta)$$

Taking their sum, we have:

$$r^2(\cos^2\theta) + r^2(\sin^2\theta)$$

r^2 is common between both, factorize it:

$$r^2[\cos^2\theta + \sin^2\theta]$$

Now, this is a solid trigonometric identity:

$$\boldsymbol{\sin^2\theta + \cos^2\theta = 1}$$

Hence, we have that this reduces to:

$$r^2(1) = r^2$$

Both $(\cos^2\theta + \sin^2\theta)$ and $(\sin^2\theta + \cos^2\theta)$ are same things

Hence, the first one has been proved since the sum of the squares of both derivatives yield r^2!

Next (ii), we're to prove;

$$\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2 = 1$$

Here, we'll need to evaluate $\frac{dx}{dr}$ and $\frac{dy}{dr}$

$$x = r \sin\theta$$

To find $\frac{dx}{dr}$, we'll know that θ and hence, $\sin\theta$ is a constant, since we're differentiating with respect to r; hence,

$$\frac{dx}{dr} = \frac{d}{dr}(r) \times \sin\theta$$

$$\frac{dx}{dr} = 1 \times r^{1-1} \times \sin\theta$$

$$\frac{dx}{dr} = \sin\theta \times 1 = \sin\theta$$

$$y = r \cos\theta$$

To find $\frac{dy}{dr}$, we'll know that θ and hence, $\cos\theta$ is a constant, since we're differentiating with respect to r; hence,

$$\frac{dy}{dr} = \frac{d}{dr}(r) \times \cos\theta$$

$$\frac{dy}{dr} = 1 \times r^{1-1} \times \cos\theta$$

$$\frac{dy}{dr} = \cos\theta \times 1 = \cos\theta$$

To prove:

$$\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2 = 1$$

Evaluate the squares of the derivatives we've found which will show the left hand side equal to the right hand side;

$$\left(\frac{dx}{dr}\right)^2 = (\sin\theta)^2 = \sin^2\theta$$

$$\left(\frac{dy}{dr}\right)^2 = (\cos\theta)^2 = \cos^2\theta$$

Taking their sum, we have:

$$\sin^2\theta + \cos^2\theta$$

You were just told now, just recently;

$$\sin^2\theta + \cos^2\theta = 1$$

Hence, what we have is simply equal to 1.

We have proved the second one too since the sum of the squares of both derivatives yield 1!

Up next is the **derivative of logarithm function;**

THE DERIVATIVE OF LOGARITHM FUNCTIONS

Again; this is a simple concept that involves you just having to memorize a simple law, it's that simple:

Now, it's like this;

If: $y = \log_a x$

$$\frac{dy}{dx} = \frac{1}{x \ln a}$$

If: $y = \log_e x$

$$\frac{dy}{dx} = \frac{1}{x}$$

Now, let's make clear some things here;

$\log_e x$ is called the natural log of a number which is the logarithm of a number to a base e.

The natural logarithm $\log_e x$ is also represented as $\ln x$. And hence:

$$\ln a = \log_e a$$

Now, looking at the derivative of $\log_a x$ which has its derivative as:

$$\frac{1}{x \ln a}$$

We can connote that; if:

$$y = \log_a x$$

$$\frac{dy}{dx} = \frac{1}{x \log_e a}$$

Now, in the case of the derivative of $\log_e x$;

We can replace a with e in the derivative of $\log_a x$ since it's only the logarithm bases that differ by a and e

Hence, since if $y = \log_a x$

$$\frac{dy}{dx} = \frac{1}{x \log_e a}$$

Then,

If $y = \log_e x$

$$\frac{dy}{dx} = \frac{1}{x \log_e e}$$

Now, I guess you should still remember that in SSC105 that the logarithm of same base and log number is 1; hence, we have that;

$$\frac{dy}{dx} = \frac{1}{x \times 1} = \frac{1}{x}$$

Hence, that's the reason the derivative of the natural log of an independent variable, x is $\frac{1}{x}$

The difference basically between the derivative of a natural log of a variable x and the derivative of the log to any base, a of the same variable x is just the constant $\ln a$ as $\ln a$ $(\log_e a)$ is a constant in this case since neither a nor e are the independent nor the dependent variable.

Hence, these are the logarithm rules once again;

If: $\boldsymbol{y = \log_a x}$

$$\frac{dy}{dx} = \frac{1}{x \ln a}$$

If: $\boldsymbol{y = \log_e x}$

$$\frac{dy}{dx} = \frac{1}{x}$$

So, that's how it is; like in the case of trigonometric rules, you need to put the law of the function of a function firmly to mind over here!

But before seeing examples, let's see this:

Chain rule in the logarithm differentiation rule;

The two rules given above are in the most basic forms of the logarithm functions, however, most often, it isn't the most basic forms of logarithm functions that are differentiated, mostly the logarithm forms are expressed in terms of other functions. Hence, let's see how it comes into play here;

Consider;

$$y = \log_a f(x)$$

Let;

$$u = f(x)$$

It implies;

$$y = \log_a u$$

Hence, y has now been expressed in the most basic log expression.

$$\frac{du}{dx} = f'(x)$$

And by the rules we've just listed out (just above in page 83 – previous page);

$$\frac{dy}{du} = \frac{1}{u \ln a}$$

From chain rule;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Hence;

$$\frac{dy}{dx} = \frac{1}{u \ln a} \times f'(x)$$

But;

$$u = f(x)$$

Hence;

$$\frac{dy}{dx} = f'(x)\frac{1}{f(x)\ln a} = \frac{f'(x)}{\ln a\, f(x)}$$

Hence;

We have that:

If: $\boldsymbol{y = \log_a f(x)}$

$$\boldsymbol{\frac{dy}{dx} = \frac{f'(x)}{\ln a\, f(x)}}$$

Similarly; by the same chain rule application;

If: $\boldsymbol{y = \log_e f(x)}$

$$\boldsymbol{\frac{dy}{dx} = \frac{f'(x)}{f(x)}}$$

The above given standard definitions of the logarithm rules of differentiation and can help to evaluate, straight, the derivatives of some logarithm functions not expressed in the most basic form. Let's see an example below:

Consider;

$$y = \log_e(-x)$$

In this case;

$$f(x) = -x$$
$$f'(x) = -1$$

Hence;

From the formula;

If: $\boldsymbol{y = \log_e f(x)}$

$$\boldsymbol{\frac{dy}{dx} = \frac{f'(x)}{f(x)}}$$

Hence; in this case; it'll be:

$$\frac{dy}{dx} = \frac{-1}{-x} = \frac{1}{x}$$

Similarly; for:

$$y = \log_2(4x)$$

$$f(x) = 4x$$
$$f'(x) = 4$$

Hence;

From the formula;

If: $\boldsymbol{y = \log_a f(x)}$

$$\boldsymbol{\frac{dy}{dx} = \frac{f'(x)}{\ln a \ f(x)}}$$

$$\frac{dy}{dx} = \frac{4}{\ln 2\,(4x)} = \frac{1}{x \ln 2}$$

Hence, the formula can help in straightforwardly using the chain rule in light logarithm situations; the safest way still is to substitute properly though, and that's what we'd be using below, as you move further, you can easily use the formula; ensure you don't make errors, **though!**

Also; before we continue still;

The logarithm rules of differentiation can be really confused many times. Especially for the difference in the natural log and log to a certain base. As already established;

$$\frac{d}{dx}(\log_e x) = \frac{1}{x}$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

However, many times, we see instances like this:

$$\frac{d}{dx}(\log x)$$

We know that $\log x$ is same as $\log_{10} x$ since log without base is the base 10; hence;

$$\frac{d}{dx}(\log_{10} x) = \frac{1}{x \ln 10}$$

So, let's start!

Find the derivative with respect to x of the following functions;

- $y = \log(3x^2 + 4)$
- $y = \log_e(\sin x - \cos x)$
- $y = \log_2 \sqrt{\tan^3 x}$
- $y = \log_e(ax^2 - bx + c)^4$
- $y = \log(\log x)$

Cool, let the solving begin;

- $y = \log(3x^2 + 4)$

When a log is expressed without a base, it is to the base 10.

$$y = \log_{10}(3x^2 + 4)$$

So, this is in the form of $\log_a x$, here, $a = 10$;

But, before we do that, we can see that $(3x^2 + 4)$ is another function, so we have to apply the function of a function here;
Put:

$$u = 3x^2 + 4$$

$$\frac{du}{dx} = 2 \times 3 \times x^{2-1} = 6x$$

Hence,

$$y = \log_{10} u$$

From this rule;

If: $\boldsymbol{y = \log_a x}$

$$\boldsymbol{\frac{dy}{dx} = \frac{1}{x \ln a}}$$

$$\frac{dy}{du} = \frac{1}{u \ln 10}$$

From our normal, very normal chain rule;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{u \ln 10} \times 6x = \frac{6x}{u \ln 10}$$

Return $u = 3x^2 + 4$ into the derivative;

$$\frac{dy}{dx} = \frac{6x}{(3x^2 + 4) \ln 10} = \frac{6x}{\ln 10\, (3x^2 + 4)}$$

- $y = \log_e(\sin x - \cos x)$

Right, we have a logarithm function here, however, as we have known several times that the logarithm function must be made basic to apply the rule, in the function of function way.

Hence, put:

$$u = \sin x - \cos x$$

This chain rule should be very basic already right?

$$\frac{du}{dx} = \frac{d}{dx}(\sin x) - \frac{d}{dx}(\cos x)$$

Straight from trigonometric rules;

$$\frac{du}{dx} = \cos x - (-\sin x) = \cos x + \sin x$$

Hence, from the substitution package;

$$y = \log_e u$$

From logarithm derivative rule; this is the natural logarithm;

If: $\boldsymbol{y = \log_{e} x}$

$$\boldsymbol{\frac{dy}{dx} = \frac{1}{x}}$$

Hence;

$$\frac{dy}{du} = \frac{1}{u}$$

From our normal, very normal chain rule;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{u} \times (\cos x + \sin x) = \frac{(\cos x + \sin x)}{u}$$

Return $u = \sin x - \cos x$

$$\frac{dy}{dx} = \frac{\cos x + \sin x}{\sin x - \cos x}$$

As much as the above looks like a relation that can be simplified; it cannot be; no factorization can simplify this; oh yeah, none!

Next!

- $y = \log_2 \sqrt{\tan^3 x}$

Now, of course we'll make our substitution,

Put:

$$u = \sqrt{\tan^3 x}$$

However, before rushing into making a thoughtless extra substitution that'll lead to another needless substitution; calm down and do this first!

$\tan^3 x$ implies $(\tan x)^3$

Hence, we have:

$$\sqrt{(\tan x)^3} = [(\tan x)^3]^{\frac{1}{2}}$$

after converting the square root to power of $\frac{1}{2}$
Hence, expanding the double powers, we have:

$$u = (\tan x)^{\frac{3}{2}}$$

So, this is a constant power situation; however, we can't apply it as $\tan x$ is another function on its own, we need another substitution

$$z = \tan x$$

This is basic and hence, from trigonometric rule;

$$\frac{dz}{dx} = \sec^2 x$$

Therefore,

$$u = z^{\frac{3}{2}}$$

Power rule;

$$\frac{du}{dz} = \frac{3}{2} \times z^{\frac{3}{2}-1} = \frac{3}{2} \times z^{\frac{1}{2}} = \frac{3\sqrt{z}}{2}$$

Hence,

$$y = \log_2 u; \text{since } u = (\tan x)^{\frac{3}{2}}$$

Hence, from:

If: $\boldsymbol{y = \log_a x}$

$$\boldsymbol{\frac{dy}{dx} = \frac{1}{x \ln a}}$$

$$\frac{dy}{du} = \frac{1}{u \ln 2}$$

Now, the triple chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dz} \times \frac{dz}{dx}$$

$$\frac{dy}{dx} = \frac{1}{u \ln 2} \times \frac{3\sqrt{z}}{2} \times \sec^2 x$$

Return the base values; $z = \tan x$

$$u = z^{\frac{3}{2}} = (\tan x)^{\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{1}{(\tan x)^{\frac{3}{2}} \ln 2} \times \frac{3\sqrt{\tan x}}{2} \times \sec^2 x$$

$$\frac{dy}{dx} = \frac{3(\tan x)^{\frac{1}{2}} \sec^2 x}{\ln 2\,(\tan x)^{\frac{3}{2}}}$$

Let's simplify the powers of $\tan x$ between the numerator and the denominator; from indices;

$$\frac{dy}{dx} = \frac{3(\tan x)^{\frac{1}{2}-\frac{3}{2}} \sec^2 x}{\ln 2} = \frac{3(\tan x)^{-1} \sec^2 x}{\ln 2}$$

$(\tan x)^{-1}$ is converted to $\frac{1}{(\tan x)^1}$

$$\frac{dy}{dx} = \frac{3 \sec^2 x}{\ln 2 \tan x}$$

Wow, getting more interesting;

- $y = \log_e(ax^2 - bx + c)^4$

Put:

$$u = (ax^2 - bx + c)^4$$

without wasting time;

This cannot be differentiated directly by power rule, introduce another substitution;

$$z = ax^2 - bx + c$$

Now this can be differentiated;

$$\frac{dz}{dx} = 2 \times a \times x^{2-1} - 1 \times b \times x^{1-1} + 0$$

$$\frac{dz}{dx} = 2ax - b$$

I don't think I need to mention that obviously, a, b and c will be constants in this situation.
Hence, we have:

$$u = z^4$$

Easily,

$$\frac{du}{dz} = 4 \times z^{4-1} = 4z^3$$

In the same way, you can see the easy function of function substitution for every question we've been facing:

We end up in:

$$y = \log_e u$$

Since $u = z^4 = (ax^2 - bx + c)^4$

$$\frac{dy}{du} = \frac{1}{u}$$ as simple as you like

Now, the triple chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dz} \times \frac{dz}{dx}$$

$$\frac{dy}{dx} = \frac{1}{u} \times 4z^3 \times (2ax - b)$$

$$\frac{dy}{dx} = \frac{4z^3(2ax - b)}{u}$$

Return the real values;

$$z = ax^2 - bx + c$$
$$u = z^4 = (ax^2 - bx + c)^4$$

$$\frac{dy}{dx} = \frac{4(ax^2 - bx + c)^3(2ax - b)}{(ax^2 - bx + c)^4}$$

Clear $(ax^2 - bx + c)$ in powers of 3 and 4 (indices)

$$\frac{dy}{dx} = \frac{4(ax^2 - bx + c)^{3-4}(2ax - b)}{1}$$

$$\frac{dy}{dx} = \frac{4(ax^2 - bx + c)^{-1}(2ax - b)}{1}$$

Converting the negative power;

$$\frac{dy}{dx} = \frac{4(2ax - b)}{(ax^2 - bx + c)}$$

- $y = \log(\log x)$

Yeah, this is directly from your past question; it's looking weird but be calm, it's probably the shortest we'll be solving in the logarithm rules problems we've been facing;

Now,

$$y = \log(\log x)$$

$\log x$ is another function on its own; hence, we need to make a substitution thus:

$$u = \log x$$

No base implies;

$$u = \log_{10} x$$

Hence,

$$\frac{du}{dx} = \frac{1}{x \ln 10}$$

Since $u = \log x$

$$y = \log u$$

Hence,

$$y = \log_{10} u$$

$$\frac{dy}{du} = \frac{1}{u \ln 10}$$

Hence,

From chain rule;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{u \ln 10} \times \frac{1}{x \ln 10} = \frac{1}{xu\ (\ln 10)^2}$$

Return the value of:

$$u = \log x$$

Notice that ln 10 multiplied twice produced $(\ln 10)^2$

$$\frac{dy}{dx} = \frac{1}{x \log x\,(\ln 10)^2}$$

So, that's it about that; we'll see more examples after the whole thing.

We move to the **derivative of exponential functions.**

DERIVATIVE OF EXPONENTIAL FUNCTIONS

OKAY, so this is a very interesting part; Throwback to the aspect of the part I kept insisting on constant power, several times I mentioned that word constant to reinstate the fact that the power mentioned must be a constant in the sense that it contains neither the independent nor the dependent variable. **That law is simply called the power function rule**; however, the word constant was meant to reinstate that.

So, think about differentiating this function;

$$y = 3^x$$

to find $\frac{dy}{dx}$, I guess we will do this right?

Yes! We drop down the power and subtract 1 from it right?

$$\frac{dy}{dx} = x \times 3^{x-1} = x(3^{x-1})$$

Lol! Thanks for the rubbish solving; let's gently cancel the nonsense first;

$$\cancel{\frac{dy}{dx} = x \times 3^{x-1} = x(3^{x-1})}$$

So, that was on a lighter note, let's now say something serious;

When in the case where power of any base raised to a certain power is the independent variable or sometimes the dependent variable, then, it is called an exponential function. Go back to the topic of functions; you'll see the definition of an exponential function. The power is mostly the independent variable and hence, in this case, the normal [algebra] power rule is totally inapplicable here.

Now, finding the derivatives of these functions come basically in two forms;

For a general constant, a

If: $\boldsymbol{y = a^x}$

$$\boldsymbol{\frac{dy}{dx} = a^x \ln a}$$

If: $\boldsymbol{y = e^x}$

$$\boldsymbol{\frac{dy}{dx} = e^x}$$

Now, I didn't make any mistake there;

If $y = e^x$,

$$\frac{dy}{dx} \text{ is also } \boldsymbol{e^x}$$

I guess that's the easiest to remember. For a^x too, not so different,

$$\frac{dy}{dx} \text{ is also } \boldsymbol{a^x \ln a}$$

$\ln a$ is a constant so it isn't really different from the major function. Remember you also need to commit these two rules to memory.

Before examples again in this case, let's see this;

Chain rule in the exponential differentiation rule;

The two rules given above are in the most basic forms of the exponential functions, however, most often, it isn't the most basic forms of exponential functions that are differentiated, mostly the exponential forms are expressed in terms of other functions. Hence, let's see how it comes into play here;

Consider;

$$y = a^{f(x)}$$

Let;

$$u = f(x)$$

It implies;

$$y = a^{u}$$

Hence, y has now been expressed in the most basic exponent expression.

$$\frac{du}{dx} = f'(x)$$

And by the rules we've just listed out (just above in page 100 – two pages back);

$$\frac{dy}{du} = a^u \ln a$$

From chain rule;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Hence;

$$\frac{dy}{dx} = a^u \ln a \times f'(x)$$

But;

$$u = f(x)$$

Hence;

$$\frac{dy}{dx} = f'(x)a^{f(x)} \ln a = \ln a\, f'(x)a^{f(x)}$$

Hence;

We have that:

If: $\boldsymbol{y = a^{f(x)}}$

$$\boldsymbol{\frac{dy}{dx} = \ln a\, f'(x)a^{f(x)}}$$

Similarly; by the same chain rule application;

If: $\boldsymbol{y = e^{f(x)}}$

$$\boldsymbol{\frac{dy}{dx} = f'(x)e^{f(x)}}$$

The above given standard definitions of the exponential rules of differentiation and can help to evaluate, straight, the derivatives of some exponential functions not given in the most basic form. Let's see an example below:

Consider;

$$y = e^{-2x}$$

In this case;

$$f(x) = -2x$$

$$f'(x) = -2$$

Hence;

From the formula;

If: $\boldsymbol{y = e^{f(x)}}$

$$\boldsymbol{\frac{dy}{dx} = f'(x)e^{f(x)}}$$

Hence; in this case; it'll be:

$$\frac{dy}{dx} = -2e^{-2x}$$

Similarly; for:

$$y = 3^{4x}$$

$$f(x) = 4x$$
$$f'(x) = 4$$

Hence;
From the formula;

If: $\boldsymbol{y = a^{f(x)}}$

$$\boldsymbol{\frac{dy}{dx} = \ln a\, f'(x) a^{f(x)}}$$

$$\frac{dy}{dx} = \ln 3\,(4)a^{4x} = 4\ln 3\, a^{4x}$$

Hence, the formula can help in straightforwardly using the chain rule in light exponential situations; the safest way still is to substitute properly though, and that's what we'd be using below, as you move further, you can easily use the formula; ensure you don't make errors, **though!**

Let's see some examples; remember the function of function rule is always needed.

Let's see these;

Find the derivative of the following with respect to x.

- $y = e^{\sin x}$
- $y = e^{\log x + 2x^3}$
- $y = e^{\log \cos x}$
- $y = 3^{-3x^3 + \frac{1}{x}}$

- $y = n^{\sqrt[4]{x^3+2x}}$

Alright, we have our differentiation rules already; up next is to put the function of function at the back of our minds and we solve;

- $y = e^{\sin x}$

Here, the rule is meant for e^x, the most basic form, hence, we need to make substitution for $\sin x$

Put:

$$u = \sin x$$

From trig rules;

$$\frac{du}{dx} = \cos x$$

Now, we can now easily have:

$$y = e^u$$

From our rules;

If: $\boldsymbol{y = e^x}$

$$\boldsymbol{\frac{dy}{dx} = e^x}$$

$$\frac{dy}{du} = e^u$$

From chain rule;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = e^u \times \cos x = e^u \cos x$$

Fix the value of:

$$u = \sin x$$

We have:

$$\frac{dy}{dx} = e^{\sin x} \cos x$$

Easy! Right? Of course it should, the function of function seems like the most important thing in the whole of differentiation.

- $y = e^{\log x + 2x^3}$

Here, we need to make the substitution, just normally.

Put:

$$u = \log x + 2x^3$$

$u = \log_{10} x + 2x^3$ (no base implies base 10)

Differentiate the summed functions separately (derivatives of sums and differences)

$$\frac{du}{dx} = \frac{1}{x \ln 10} + 3 \times 2 \times x^{3-1}$$

$$\frac{du}{dx} = \frac{1}{x \ln 10} + 6x^2$$

Hence, after the substitution;

$$y = e^u$$

From our rules;

$$\frac{dy}{du} = e^u$$

From chain rule;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = e^u \times \left(\frac{1}{x \ln 10} + 6x^2\right)$$

Note that the bracket is important since the sum of two terms is involved in the product and not only the first term multiplying e^u;

Fix back the value of $u = \log x + 2x^3$

$$\frac{dy}{dx} = e^{\log x + 2x^3}\left(\frac{1}{x \ln 10} + 6x^2\right)$$

That's it about that;

- $y = e^{\log \cos x}$

You should understand this fully now; we make our substitution;

$$u = \log \cos x$$

We can't solve directly for the differential coefficient of u with respect to x as the logarithm function is not in the basic form; hence, another substitution:

$$z = \cos x$$

This can go straight now, from trigonometric rules:

$$\frac{dz}{dx} = -\sin x$$

Now,

$$u = \log z$$

Since $z = \cos x$

Here,

$$u = \log_{10} z \text{ (no base implies base 10)}$$

$$\frac{du}{dz} = \frac{1}{z \ln 10}$$

Hence, with our substitutions;

$$y = e^u$$

$$\frac{dy}{du} = e^u$$

We now have a triple chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dz} \times \frac{dz}{dx}$$

$$\frac{dy}{dx} = e^u \times \frac{1}{z \ln 10} \times -\sin x$$

Fix the values of:

$$z = \cos x$$

$$u = \log z = \log \cos x$$

$$\frac{dy}{dx} = e^{\log \cos x} \times \frac{1}{\cos x \ln 10} \times -\sin x$$

$$\frac{dy}{dx} = \frac{-\sin x\, e^{\log \cos x}}{\cos x \ln 10}$$

But;

$$\frac{\sin x}{\cos x} = \tan x$$

Hence;

$$\frac{dy}{dx} = \frac{-\tan x\, e^{\log \cos x}}{\ln 10}$$

Cool, next!

- $y = 3^{-3x^3 + \frac{1}{x}}$

Without stories, quite a little story anyway; this is 3 raised to a power of a function of x; this is in the form a^x even though not in the basic form, 3 here is taking the place of a.

So, nonetheless, we have to make a substitution, in the form;

$$u = -3x^3 + \frac{1}{x}$$

Indices;

$$u = -3x^3 + x^{-1}$$

$$\frac{du}{dx} = (3 \times -3 \times x^{3-1}) + (-1 \times x^{-1-1})$$

$$\frac{du}{dx} = -9x^2 - x^{-2} = -9x^2 - \frac{1}{x^2}$$

Factorizing;

$$\frac{du}{dx} = -9x^2 - x^{-2} = -\left(9x^2 + \frac{1}{x^2}\right)$$

With: $u = -3x^3 + \frac{1}{x}$

$$y = 3^u$$

$$\frac{dy}{du} = 3^u \ln 3$$

You may want to glance at this again:

If: $\boldsymbol{y = a^x}$

$$\boldsymbol{\frac{dy}{dx} = a^x \ln a}$$

If: $\boldsymbol{y = e^x}$

$$\boldsymbol{\frac{dy}{dx} = e^x}$$

So back to where we were;
From chain rule;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = 3^u \ln 3 \times -\left(9x^2 + \frac{1}{x^2}\right)$$

Fix:

$$u = -3x^3 + \frac{1}{x}$$

$$\frac{dy}{dx} = 3^{-3x^3+\frac{1}{x}} \ln 3 \times -\left(9x^2 + \frac{1}{x^2}\right)$$

$$\frac{dy}{dx} = \ln 3 \left(3^{-3x^3+\frac{1}{x}}\right) \times -\left(9x^2 + \frac{1}{x^2}\right)$$

The minus sign goes to the front and;

$$\frac{dy}{dx} = -\ln 3 \left(3^{-3x^3+\frac{1}{x}}\right)\left(9x^2 + \frac{1}{x^2}\right)$$

I'd want us to leave the answer this way!

- $y = n^{\sqrt[4]{x^3+2x}}$

All these are just meant for ambiguity sake. Since n is a constant (we are differentiating with respect to x), this is in the form of a^x, as usual, we need a substitution.

$$u = \sqrt[4]{x^3 + 2x}$$

Let's turn the root to a fractional power. This is the fourth root, hence, a power of $\frac{1}{4}$

We have:

$$u = (x^3 + 2x)^{\frac{1}{4}}$$

Another substitution is needed;

$$z = x^3 + 2x$$

This can be differentiated now;

$$\frac{dz}{dx} = 3 \times x^{3-1} + 1 \times 2 \times x^{1-1}$$

$$\frac{dz}{dx} = 3x^2 + 2$$

Hence,

$$u = z^{\frac{1}{4}}$$

Since $z = x^3 + 2x$

$$\frac{du}{dz} = \frac{1}{4} \times z^{\frac{1}{4}-1} = \frac{1}{4} \times z^{-\frac{3}{4}}$$

$$\frac{du}{dz} = \frac{1}{4z^{\frac{3}{4}}}$$

Here,

$$y = n^u$$

Since $u = z^{\frac{1}{4}} = (x^3 + 2x)^{\frac{1}{4}}$

$$\frac{dy}{du} = n^u \ln n$$

We can use the $\log_e$ expression here for clarity since $\ln n$ looks like a double-n situation;

The above is the derivative of the form a^x since n is a constant;

Let's apply chain rule to dissolve everything;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dz} \times \frac{dz}{dx}$$

$$\frac{dy}{dx} = n^{u} \log_e n \times \frac{1}{4z^{\frac{3}{4}}} \times 3x^2 + 2$$

Fix the original values back in this;

$$z = x^3 + 2x$$

$$u = z^{\frac{1}{4}} = (x^3 + 2x)^{\frac{1}{4}}$$

$$\frac{dy}{dx} = n^{(x^3+2x)^{\frac{1}{4}}} \log_e n \times \frac{1}{4(x^3 + 2x)^{\frac{3}{4}}} \times 3x^2 + 2$$

Let's simplify this as much as possible:

$$\frac{dy}{dx} = \log_e n \left(n^{\sqrt[4]{x^3+2x}}\right)\left(\frac{3x^2 + 2}{4(x^3 + 2x)^{\frac{3}{4}}}\right)$$

Right there! We have treated the major rules in differentiation, at least, based on **the SSC106 way.**

There are still some few things of extreme importance though;

- The product rule
- The quotient rule

These two rules are as important as the whole knowledge you've acquired above, however, the knowledge everything above is all you need to know the two rules, interesting? That's for you to find out.

Cool…

THE PRODUCT RULE

Unlike in sums and differences of products where individual derivatives are just added; the same isn't so when functions are multiplied.

The rule when two functions that basically cannot be expanded are multiplied is given by this;

Let u and v be two functions of x

And:

$$\boldsymbol{y = u \times v}$$

Then

$$\boldsymbol{\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}}$$

So the logic is, take the second, multiply it by the derivative of the first and take the first, multiply it by the derivative of the second, add both together.

That's it about it; examples will tell us more!

Let's find some derivatives using this product rule:

Find $\frac{dy}{dx}$ in each of the following:

- $y = (1 - 2x + 3x^2)(4 - 5x^2)$
- $y = \sqrt{x}(1 + 2x)^2$
- $y = x^3(3 - 2x + 4x^2)^{\frac{1}{2}}$
- $y = \sin x \cos x$
- $y = e^{x^2} \cos(2x + 3)$
- $y = 4x^2 \sin x - 3x^2 \cos x$
- $y = \sin^3 x \tan 2x$

Quite some questions on our hands; let's begin:

- $y = (1 - 2x + 3x^2)(4 - 5x^2)$

This is a product of two functions;

Let’s classify our functions into u and v; since it’s a product of two things, we must classify one as u and the other as v;

$$u = 1 - 2x + 3x^2$$

And hence,

$$v = 4 - 5x^2$$

$$\frac{du}{dx} = 0 - (1 \times 2 \times x^{1-1}) + 2 \times 3 \times x^{2-1}$$

$$\frac{du}{dx} = -2 + 6x$$

$$\frac{dv}{dx} = 0 - 2 \times 5 \times x^{2-1}$$

$$\frac{dv}{dx} = -10x$$

From our product formula:

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

Hence, substitute appropriately;

$$\frac{dy}{dx} = (4 - 5x^2)(-2 + 6x) + (1 - 2x + 3x^2)(-10x)$$

Expand and add appropriately:

$$\frac{dy}{dx} = -8 + 24x + 10x^2 - 30x^3 + (-10x + 20x^2 - 30x^3)$$

$$\frac{dy}{dx} = -60x^3 + 30x^2 + 14x - 8$$

- $y = \sqrt{x}(1 + 2x)^2$

This is a product of two functions;

Let's classify our functions into u and v

$$u = \sqrt{x}$$

And hence,

$$v = (1 + 2x)^2$$

$$u = x^{\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{1}{2} \times x^{\frac{1}{2}-1} = \frac{1}{2} \times x^{-\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

Now before we solve for $\frac{dv}{dx}$, let me note something; the second term is taken **as a whole** and hence you mustn't be tempted to take v as $(1 + 2x)$ and be confusing yourself into extra substitutions and get hooked.

The **whole of the second function** is taken as v and when solving for $\frac{dv}{dx}$, any other substitution can be made if need be. Same thing for u, the whole of it is taken.

$$v = (1 + 2x)^2$$

$(1 + 2x)$ isn't basic, hence, by chain rule, we will have:

$$\frac{dv}{dx} = 2(1 + 2x) \times 2 = 4(1 + 2x)$$

If that seemed somehow, you can do it gradually, the above is basic chain rule;

From our product formula:

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

Hence, substitute appropriately;

$$\frac{dy}{dx} = (1 + 2x)^2\left(\frac{1}{2\sqrt{x}}\right) + \sqrt{x}[4(1 + 2x)]$$

$$\frac{dy}{dx} = \frac{(1 + 2x)^2}{2\sqrt{x}} + \sqrt{x}(4 + 8x)$$

Quite some manipulation can be done in this case; add both fractions with $2\sqrt{x}$ as the common denominator;

$$\frac{dy}{dx} = \frac{(1 + 2x)^2 \times 1 \ + \ \sqrt{x}(4 + 8x) \times 2\sqrt{x}}{2\sqrt{x}}$$

$\sqrt{x} \times 2\sqrt{x} = 2x$ as $\sqrt{x}$ twice cancels the root sign from the fact that $x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x$

$$\frac{dy}{dx} = \frac{(1 + 2x)^2 \ + \ 2x(4 + 8x)}{2\sqrt{x}}$$

Expand well now;

$$\frac{dy}{dx} = \frac{1 + 4x + 4x^2 + 8x + 16x^2}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{20x^2 + 12x + 1}{2\sqrt{x}}$$

So the whole thing looks really simpler now;

- $y = x^3(3 - 2x + 4x^2)^{\frac{1}{2}}$

Here, our product is taken thus;

$$u = x^3$$

And hence,

$$v = (3 - 2x + 4x^2)^{\frac{1}{2}}$$

You know we're taking the whole of v

$$u = x^3$$

$$\frac{du}{dx} = 3 \times x^{3-1} = 3x^2$$

$$v = (3 - 2x + 4x^2)^{\frac{1}{2}}$$

v will be differentiated using chain rule;

Put:

$$z = 3 - 2x + 4x^2$$

$$\frac{dz}{dx} = 0 - (1 \times 2 \times x^{1-1}) + 2 \times 4 \times x^{2-1}$$

$$\frac{dz}{dx} = -2 + 8x$$

Therefore;

$$v = z^{\frac{1}{2}}$$

$$\frac{dv}{dz} = \frac{1}{2} \times z^{\frac{1}{2}-1} = \frac{1}{2} \times z^{-\frac{1}{2}}$$

$$\frac{dv}{dz} = \frac{1}{2\sqrt{z}}$$

Chain rule;

$$\frac{dv}{dx} = \frac{dv}{dz} \times \frac{dz}{dx}$$

$$\frac{dv}{dx} = \frac{1}{2\sqrt{z}} \times (-2 + 8x)$$

Put: $z = 3 - 2x + 4x^2$;

$$\frac{dv}{dx} = \frac{1}{2\sqrt{3 - 2x + 4x^2}} \times (-2 + 8x)$$
$$\frac{dv}{dx} = \frac{8x - 2}{2\sqrt{3 - 2x + 4x^2}}$$

From our product formula:

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

Hence, substitute appropriately;

$$\frac{dy}{dx} = (3 - 2x + 4x^2)^{\frac{1}{2}}(3x^2) + (x^3)\left(\frac{8x - 2}{2\sqrt{3 - 2x + 4x^2}}\right)$$

Add both fractions with $2\sqrt{3 - 2x + 4x^2}$ as the common denominator;

$$\frac{dy}{dx} = \frac{(3 - 2x + 4x^2)^{\frac{1}{2}}(3x^2) \times 2\sqrt{3 - 2x + 4x^2} + x^3(8x - 2)}{2\sqrt{3 - 2x + 4x^2}}$$

$(3-2x+4x^2)^{\frac{1}{2}}$ occurring twice becomes $(3-2x+4x^2)$;

$$\frac{dy}{dx} = \frac{6x^2(3-2x+4x^2)+8x^4-2x^3}{2\sqrt{3-2x+4x^2}}$$

Expand;

$$\frac{dy}{dx} = \frac{18x^2-12x^3+24x^4+8x^4-2x^3}{2\sqrt{3-2x+4x^2}}$$

$$\frac{dy}{dx} = \frac{32x^4-14x^3+18x^2}{2\sqrt{3-2x+4x^2}}$$

- $y = \sin x \cos x$

Alright, we have a trigonometric something here:

However, this is a product of two separate functions, this is not the **sin of cos $\boldsymbol{x}$** but the product of **sin $\boldsymbol{x}$** and **cos $\boldsymbol{x}$**; the **sin of cos $\boldsymbol{x}$** will be in the form: **sin cos $\boldsymbol{x}$** where you'll simply be applying the function of function by substituting for **cos $\boldsymbol{x}$**.

So, here,

$$u = \sin x$$

And hence

$$v = \cos x$$

$$\frac{du}{dx} = \cos x \text{, which is straight from trig rules}$$

$$\frac{dv}{dx} = -\sin x \text{, this is also straight from trig rules}$$

From our product formula:

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

Hence, substitute appropriately;

$$\frac{dy}{dx} = (\cos x)(\cos x) + (\sin x)(-\sin x)$$

$$\frac{dy}{dx} = \cos^2 x - \sin^2 x$$

Short and simple!

- $y = e^{x^2}\cos(2x + 3)$

This is a product of two functions, one exponential and one trigonometric;

We'll take it thus:

$$u = e^{x^2}$$

And hence

$$v = \cos(2x + 3)$$

$$u = e^{x^2}$$

Here, e^{x^2} isn't the basic exponential function so we have to make the substitution;

Put:

$$z = x^2$$

$$\frac{dz}{dx} = 2 \times x^{2-1} = 2x$$

Hence,

$$u = e^z$$

Hence, this has been made straight;

$$\frac{du}{dz} = e^z$$

Hence,

$$\frac{du}{dx} = \frac{du}{dz} \times \frac{dz}{dx}$$

$$\frac{du}{dx} = e^z \times (2x) = 2xe^z$$

Put the value of z in place; $z = x^2$

$$\frac{du}{dx} = 2xe^{x^2}$$

That's about $\frac{du}{dx}$;

$$v = \cos(2x + 3)$$

$\frac{dv}{dx}$ cannot be evaluated straight too;

Put:

$$w = 2x + 3$$

$$\frac{dw}{dx} = 1 \times 2 \times x^{1-1} + 0$$

$$\frac{dw}{dx} = 2$$

Hence,

$$v = \cos w$$

Since: $w = 2x + 3$

Hence, from trig rules;

$$\frac{dv}{dw} = -\sin w$$

Chain rule is:

$$\frac{dv}{dx} = \frac{dv}{dw} \times \frac{dw}{dx}$$

$$\frac{dv}{dx} = -\sin w \times 2 = -2\sin w$$

Put back in place, $w = 2x + 3$

$$\frac{dv}{dx} = -2\sin(2x + 3)$$

From our product formula:

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

Hence, substitute appropriately;

$$\frac{dy}{dx} = \cos(2x + 3) \times 2xe^{x^2} + \left(e^{x^2}\right)(-2\sin(2x + 3))$$

$$\frac{dy}{dx} = 2xe^{x^2}.\cos(2x+3) - 2e^{x^2}\sin(2x+3)$$

We could factorize but let's just have this as that; On second thought, let's factorize it to make it neater;

$$\frac{dy}{dx} = 2e^{x^2}[x\cos(2x+3) - \sin(2x+3)]$$

- $y = 4x^2 \sin x - 3x^2 \cos x$

Wawu; this is the derivative of sums;

Now:

$$\frac{dy}{dx} = \frac{d}{dx}(4x^2 \sin x) - \frac{d}{dx}(3x^2 \cos x)$$

Now, these individual derivatives are products on their own; so we'll evaluate their products separately:

$$\frac{d}{dx}(4x^2 \sin x)$$

$$u = 4x^2$$

And hence

$$v = \sin x$$

$$u = 4x^2$$

$$\frac{du}{dx} = 2 \times 4 \times x^{2-1} = 8x$$

$$v = \sin x$$

$$\frac{dv}{dx} = \cos x \text{ , trigonometric rules;}$$

From our product formula:

$$\frac{d}{dx}(4x^2 \sin x) = v\frac{du}{dx} + u\frac{dv}{dx}$$

Hence, substitute appropriately;

$$\frac{d}{dx}\left(4x^2 \sin x\right) = (\sin x)(8x) + \left(4x^2\right)(\cos x)$$

$$\frac{d}{dx}(4x^2 \sin x) = 8x \sin x + 4x^2 \cos x$$

Now, up next is:

$$\frac{d}{dx}(3x^2 \cos x)$$

$$u = 3x^2$$

And hence:

$$v = \cos x$$

$$u = 3x^2$$

$$\frac{du}{dx} = 2 \times 3 \times x^{2-1} = 6x$$

$$v = \cos x$$

$$\frac{dv}{dx} = -\sin x \text{ , trigonometric rules;}$$

From our product formula:

$$\frac{d}{dx}(3x^2 \cos x) = v\frac{du}{dx} + u\frac{dv}{dx}$$

Hence, substitute appropriately;

$$\frac{d}{dx}\left(3x^2 \cos x\right) = (\cos x)(6x) + \left(3x^2\right)(-\sin x)$$

$$\frac{d}{dx}(3x^2 \cos x) = 6x \cos x - 3x^2 \sin x$$

Therefore, finally:

$$\frac{dy}{dx} = \frac{d}{dx}(4x^2 \sin x) - \frac{d}{dx}(3x^2 \cos x)$$

$$\frac{dy}{dx} = 8x \sin x + 4x^2 \cos x - (6x \cos x - 3x^2 \sin x)$$

$$\frac{dy}{dx} = 8x \sin x + 4x^2 \cos x - 6x \cos x + 3x^2 \sin x$$

Factorize $\cos x$ and $\sin x$ respectively;

$$\frac{dy}{dx} = \cos x\,(4x^2 - 6x) + \sin x\,(8x + 3x^2)$$

- $y = \sin^3 x \tan 2x$

This is a product of two functions, two trigonometric functions;

We'll take it thus:

$$u = \sin^3 x$$

And hence

$$v = \tan 2x$$

$$u = \sin^3 x$$

Here, it means:

$$u = (\sin x)^3$$

We need to perform a substitution here;

$$z = \sin x$$

Hence,

$$\frac{dz}{dx} = \cos x$$

We thus have;

$$u = z^3$$

Since $z = \sin x$

Here,

$$\frac{du}{dz} = 3 \times z^{3-1} = 3z^2$$

Our chain rule tells us that;

$$\frac{du}{dx} = \frac{du}{dz} \times \frac{dz}{dx}$$

$$\frac{du}{dx} = 3z^2 \times \cos x = 3z^2 \cos x$$

Put back: $z = \sin x$;

$$\frac{du}{dx} = 3(\sin x)^2 \cos x = 3 \sin^2 x \cos x$$

$$v = \tan 2x$$

From chain rule, we differentiate this simple case as:

$$\frac{dv}{dx} = 2 \times \sec^2 2x = 2\sec^2 2x$$

From our product formula:

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

Hence, substitute appropriately;

$$\frac{dy}{dx} = (\tan 2x)(3\sin^2 x \cos x) + (\sin^3 x)(2\sec^2 2x)$$

$$\frac{dy}{dx} = 3\sin^2 x \cos x \tan 2x + 2\sin^3 x \sec^2 2x$$

Cool, quite some questions have really been trashed out though: Sure you're loving it;;;

Fine; product and division goes together; hence, there are also rules for differentiating functions that are two separate functions with one dividing the other; it is called the **derivative of a quotient.**

THE QUOTIENT RULE

Just like in product rule, derivatives are not just blankly divided;

The rule when two functions are expressed as quotients is given by this;

Let u and v be two functions of x

And:

$$y = \frac{u}{v}$$

Then:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

So the logic is, take the denominator, multiply it by the derivative of the numerator, subtract it from taking the numerator and multiplying it by the derivative of the denominator, divide the whole thing by the square of the denominator.

Note always that it's the denominator multiplied by the numerator first, you mustn't forget as it is subtraction.

That's it about it; examples will tell us more!

Let's find some derivatives using this quotient rule:

Evaluate the derivative of the following:

- $y = \frac{1+x^2}{1-x^2}$
- $y = \frac{3+2x+x^2}{\sqrt{1+x}}$
- $y = \frac{\sqrt[3]{\left(1+3x^2\right)^2}}{x}$
- $y = \ln\left(\frac{1-3x^2}{1+3x^2}\right)^{\frac{1}{2}}$
- $y = \sqrt{\frac{\cos 2x}{1+\sin 2x}}$
- $y = \frac{\sec x - \tan x}{\sec x + \tan x}$

- $y = \frac{x^2 \sin x}{(x+1)(x^2-1)}$

Right, let's kick-start this:

- $y = \frac{1+x^2}{1-x^2}$

Here, compare this with the form $\frac{u}{v}$, we have:

$$u = 1 + x^2$$

And hence,

$$v = 1 - x^2$$

$$u = 1 + x^2$$

$$\frac{du}{dx} = 0 + 2 \times x^{2-1} = 2x$$

$$v = 1 - x^2$$

$$\frac{dv}{dx} = 0 - (2 \times x^{2-1}) = -2x$$

Here is the quotient rule;

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Hence;

$$\frac{dy}{dx} = \frac{[(1-x^2)(2x) - (1+x^2)(-2x)]}{(1-x^2)^2}$$

$$\frac{dy}{dx} = \frac{[2x - 2x^3 - (-2x - 2x^3)]}{(1-x^2)^2}$$

$$\frac{dy}{dx} = \frac{4x}{(1-x^2)^2}$$

Simple as that!

- $y = \frac{3+2x+x^2}{\sqrt{1+x}}$

Here;

$$u = 3 + 2x + x^2$$

And hence;

$$v = \sqrt{1+x}$$

$$\frac{du}{dx} = 0 + (1 \times 2 \times x^{1-1}) + (2 \times x^{2-1})$$

$$\frac{du}{dx} = 2 + 2x$$

$$v = \sqrt{1 + x}$$

Here, chain rule, a substitution is needed,

$$z = 1 + x$$

$$\frac{dz}{dx} = 0 + (1 \times x^{1-1}) = 1$$

Hence,

$$v = \sqrt{z} = z^{\frac{1}{2}}$$

$$\frac{dv}{dz} = \frac{1}{2} \times z^{\frac{1}{2}-1} = \frac{1}{2} \times z^{-\frac{1}{2}} = \frac{1}{2\sqrt{z}}$$

Hence,

$$\frac{dv}{dx} = \frac{dv}{dz} \times \frac{dz}{dx}$$

$$\frac{dv}{dx} = \frac{1}{2\sqrt{z}} \times 1$$

Put $z = 1 + x$

$$\frac{dv}{dx} = \frac{1}{2\sqrt{1 + x}} \times 1 = \frac{1}{2\sqrt{1 + x}}$$

Here, we'll apply the quotient rule now:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{\sqrt{1+x}(2+2x) - (3+2x+x^2)\left(\frac{1}{2\sqrt{1+x}}\right)}{\left(\sqrt{1+x}\right)^2}$$

Now, we need to manipulate the numerator first; then with the denominator;

Now,

$$\frac{a}{b} = a \times \frac{1}{b}$$

Hence, here we can do this as, converting the denominator to multiplication while it is inverted:

$$\left[\sqrt{1+x}(2+2x) - \frac{(3+2x+x^2)}{2\sqrt{1+x}}\right] \times \frac{1}{\left(\sqrt{1+x}\right)^2}$$

Adding within the numerator, we have; note also that; $\left(\sqrt{1+x}\right)^2 = (1+x)$

$$\left[\frac{2\sqrt{1+x}\sqrt{1+x}(2+2x) - (3+2x+x^2) \times 1}{2\sqrt{1+x}}\right]\frac{1}{(1+x)}$$

Note; $\sqrt{1+x}\sqrt{1+x} = (1+x)$

$$\frac{dy}{dx} = \frac{2(1+x)(2+2x) - (3+2x+x^2)}{2\sqrt{1+x} \times (1+x)}$$

$$\frac{dy}{dx} = \frac{(2+2x)(2+2x) - (3+2x+x^2)}{2\sqrt{1+x} \times (1+x)}$$

$$\frac{dy}{dx} = \frac{4+8x+4x^2-3-2x-x^2}{2(1+x)^{\frac{3}{2}}}$$

$$\frac{dy}{dx} = \frac{3x^2+6x+1}{2(1+x)^{\frac{3}{2}}}$$

- $y = \frac{\sqrt[3]{\left(1+3x^2\right)^2}}{x}$

Straightforward, Here

$$u = \sqrt[3]{(1+3x^2)^2}$$

And

$$v = x$$

$$u = \sqrt[3]{(1+3x^2)^2}$$

Convert the cube root to power $\left(\frac{1}{3}\right)$

$$u = ((1 + 3x^2)^2)^{\frac{1}{3}}$$

Expand the powers through indices;

$$u = (1 + 3x^2)^{\frac{2}{3}}$$

We need a substitution,

$$z = 1 + 3x^2$$

Here,

$$\frac{dz}{dx} = 0 + (2 \times 3 \times x^{2-1}) = 6x$$

Hence,

$$u = z^{\frac{2}{3}}$$

Since; $z = 1 + 3x^2$

$$\frac{du}{dz} = \frac{2}{3} \times z^{\frac{2}{3}-1} = \frac{2}{3} \times z^{-\frac{1}{3}}$$

$$\frac{du}{dz} = \frac{2}{3z^{\frac{1}{3}}}$$

Chain rule!

$$\frac{du}{dx} = \frac{du}{dz} \times \frac{dz}{dx}$$

$$\frac{du}{dx} = \frac{2}{3z^{\frac{1}{3}}} \times 6x = \frac{12x}{3z^{\frac{1}{3}}}$$

Put: $z = 1 + 3x^2$

$$\frac{du}{dx} = \frac{12x}{3(1+3x^2)^{\frac{1}{3}}} = \frac{4x}{(1+3x^2)^{\frac{1}{3}}}$$

$$v = x$$

Soft!

$$\frac{dv}{dx} = 1$$

Quotient rule!

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(x)\left(\frac{4x}{(1+3x^2)^{\frac{1}{3}}}\right) - \left((1+3x^2)^{\frac{2}{3}}\right)(1)}{(x)^2}$$

$$\frac{dy}{dx} = \frac{\frac{4x^2}{(1+3x^2)^{\frac{1}{3}}} - (1+3x^2)^{\frac{2}{3}}}{x^2}$$

Add the numerator; and convert the denominator to $\frac{1}{x^2}$

$$\frac{dy}{dx} = \left[\frac{4x^2 - (1+3x^2)^{\frac{2}{3}}(1+3x^2)^{\frac{1}{3}}}{(1+3x^2)^{\frac{1}{3}}}\right] \times \frac{1}{x^2}$$

$$\frac{dy}{dx} = \left[\frac{4x^2 - (1+3x^2)^{\frac{2}{3}+\frac{1}{3}}}{(1+3x^2)^{\frac{1}{3}}}\right] \times \frac{1}{x^2}$$

$$\frac{dy}{dx} = \left[\frac{4x^2 - (1+3x^2)}{(1+3x^2)^{\frac{1}{3}}}\right] \times \frac{1}{x^2}$$

$$\frac{dy}{dx} = \left[\frac{4x^2 - 1 - 3x^2}{(1+3x^2)^{\frac{1}{3}}}\right] \times \frac{1}{x^2}$$

$$\frac{dy}{dx} = \frac{x^2 - 1}{x^2(1+3x^2)^{\frac{1}{3}}}$$

- $y = \ln\left(\frac{1-3x^2}{1+3x^2}\right)^{\frac{1}{2}}$

Aiit, the quotient situation is inside the natural log;

So, here;

$$y = \ln\left(\frac{1 - 3x^2}{1 + 3x^2}\right)^{\frac{1}{2}}$$

So, we'll make the substitution;

$$a = \left(\frac{1 - 3x^2}{1 + 3x^2}\right)^{\frac{1}{2}}$$

Again, we need another substitution since we cannot apply the constant power law directly over here;

$$b = \frac{1 - 3x^2}{1 + 3x^2}$$

So, now we need to evaluate $\frac{db}{dx}$ using quotient rule:

$$u = 1 - 3x^2$$

And

$$v = 1 + 3x^2$$

$$\frac{du}{dx} = 0 - (2 \times 3 \times x^{2-1}) = -6x$$

$$\frac{dv}{dx} = 0 + (2 \times 3 \times x^{2-1}) = 6x$$

From quotient rule;

$$\frac{db}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{db}{dx} = \frac{(1 + 3x^2)(-6x) - (1 - 3x^2)(6x)}{(1 + 3x^2)^2}$$

$$\frac{db}{dx} = \frac{-6x - 18x^3 - (6x - 18x^3)}{(1 + 3x^2)^2}$$

$$\frac{db}{dx} = \frac{-12x}{(1 + 3x^2)^2}$$

Hence,

$$a = b^{\frac{1}{2}}$$

Since

$$b = \frac{1 - 3x^2}{1 + 3x^2}$$

$$\frac{da}{db} = \frac{1}{2} \times b^{\frac{1}{2}-1} = \frac{1}{2} \times \frac{1}{b^{\frac{1}{2}}}$$

$$\frac{da}{db} = \frac{1}{2b^{\frac{1}{2}}}$$

We now have, as we progress;

$$y = \ln a$$

And hence, from logarithm rules;

$$\frac{dy}{da} = \frac{1}{a}$$

We can now apply chain rule;

$$\frac{dy}{dx} = \frac{dy}{da} \times \frac{da}{db} \times \frac{db}{dx}$$

Notice the tactical avoidance of u and v in the substitution processes since they'll still be used in the quotient rule, we made use of substitutions of

a and b, just like in some other examples we used letters like z and w;

$$\frac{dy}{dx} = \frac{1}{a} \times \frac{1}{2b^{\frac{1}{2}}} \times \frac{-12x}{(1+3x^2)^2}$$

Return the values;

$$b = \frac{1-3x^2}{1+3x^2}$$

$$a = b^{\frac{1}{2}} = \left(\frac{1-3x^2}{1+3x^2}\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{\left(\frac{1-3x^2}{1+3x^2}\right)^{\frac{1}{2}}} \times \frac{1}{2\left(\frac{1-3x^2}{1+3x^2}\right)^{\frac{1}{2}}} \times \frac{-12x}{(1+3x^2)^2}$$

The power of a base can be split on a quotients or product; hence, we have:

$$\frac{dy}{dx} = \frac{1}{\frac{(1-3x^2)^{\frac{1}{2}}}{(1+3x^2)^{\frac{1}{2}}}} \times \frac{1}{2 \times \frac{(1-3x^2)^{\frac{1}{2}}}{(1+3x^2)^{\frac{1}{2}}}} \times \frac{-6x}{(1+3x^2)^2}$$

Mathematically;

$$\frac{1}{\frac{a}{b}} = \frac{b}{a}$$

Hence, we have:

$$\frac{dy}{dx} = \frac{(1+3x^2)^{\frac{1}{2}}}{(1-3x^2)^{\frac{1}{2}}} \times \frac{(1+3x^2)^{\frac{1}{2}}}{2(1-3x^2)^{\frac{1}{2}}} \times \frac{-6x}{(1+3x^2)^2}$$

Multiply everything together since they are connected by multiplication;

$$\frac{dy}{dx} = \frac{(1+3x^2)^{\frac{1}{2}} \times (1+3x^2)^{\frac{1}{2}} \times -6x}{(1-3x^2)^{\frac{1}{2}} \times 2 \times (1-3x^2)^{\frac{1}{2}} \times (1+3x^2)^2}$$

Simplify the like bases, using indices;

$$\frac{dy}{dx} = \frac{(1+3x^2)^{\frac{1}{2}+\frac{1}{2}}}{(1-3x^2)^{\frac{1}{2}+\frac{1}{2}}} \times \frac{-6x}{(1+3x^2)^2}$$

$$\frac{dy}{dx} = \frac{(1+3x^2)}{(1-3x^2)} \times \frac{-6x}{(1+3x^2)^2}$$

Finally! $(1+3x^2)$ cancels out

$$\frac{dy}{dx} = \frac{-6x}{(1-3x^2)(1+3x^2)}$$

$$\frac{dy}{dx} = -\frac{6x}{1-9x^4}$$

- $y = \sqrt{\frac{\cos 2x}{1+\sin 2x}}$

Now, we convert the square root to power $\frac{1}{2}$

$$y = \left(\frac{\cos 2x}{1+\sin 2x}\right)^{\frac{1}{2}}$$

We need to make a substitution here;

$$w = \frac{\cos 2x}{1+\sin 2x}$$

Now, we need to find $\frac{dw}{dx}$ using quotient rule;

Here;

$$u = \cos 2x$$

And

$$v = 1+\sin 2x$$

$$u = \cos 2x$$

By chain rule; we will differentiate u

$$\frac{du}{dx} = -\sin 2x \times 2 = -2\sin 2x$$

For v now;

$$v = 1 + \sin 2x$$

$$\frac{dv}{dx} = 0 + \frac{d}{dx}(\sin 2x)$$

For the derivative of $\sin 2x$, it is gotten from chain rule as:

$$\cos 2x \times 2$$

Hence;

$$\frac{dv}{dx} = 2\cos 2x$$

We'll now apply the quotient rule here:

$$\frac{dw}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dw}{dx} = \frac{(1 + \sin 2x)(-2 \sin 2x) - (\cos 2x)(2 \cos 2x)}{(1 + \sin 2x)^2}$$

$$\frac{dw}{dx} = \frac{-2 \sin 2x - 2(\sin 2x)^2 - 2(\cos 2x)^2}{(1 + \sin 2x)^2}$$

Factorize -2

$$\frac{dw}{dx} = \frac{-2(\sin 2x + \sin^2 2x + \cos^2 2x)}{(1 + \sin 2x)^2}$$

$\sin^2 2x + \cos^2 2x = 1$ since $\sin^2 x + \cos^2 x = 1$; same way $\sin^2(99x) + \cos^2(99x)$ will still be 1.

Here we have:

$$\frac{dw}{dx} = \frac{-2(\sin 2x + 1)}{(1 + \sin 2x)^2}$$

$\sin 2x + 1$ and $1 + \sin 2x$ are the same thing; hence, we have this below and they can cancel off by indices;

$$\frac{dw}{dx} = \frac{-2(1 + \sin 2x)^{1-2}}{1} = -\frac{2}{(1 + \sin 2x)}$$

Now, but since w we just differentiated using quotient rule is:

$$w = \frac{\cos 2x}{1 + \sin 2x}$$

$$y = w^{\frac{1}{2}}$$

$$\frac{dy}{dw} = \frac{1}{2} \times w^{\frac{1}{2}-1} = \frac{1}{2} \times w^{-\frac{1}{2}}$$

$$\frac{dy}{dw} = \frac{1}{2w^{\frac{1}{2}}}$$

Fix back:

$$w = \frac{\cos 2x}{1 + \sin 2x}$$

$$\frac{dy}{dw} = \frac{1}{2\left(\frac{\cos 2x}{1+\sin 2x}\right)^{\frac{1}{2}}} = \frac{1}{\frac{2(\cos 2x)^{\frac{1}{2}}}{(1+\sin 2x)^{\frac{1}{2}}}}$$

$$\frac{dy}{dw} = \frac{(1 + \sin 2x)^{\frac{1}{2}}}{2(\cos 2x)^{\frac{1}{2}}}$$

Hence, from chain rule:

$$\frac{dy}{dx} = \frac{dy}{dw} \times \frac{dw}{dx}$$

$$\frac{dy}{dx} = \frac{(1 + \sin 2x)^{\frac{1}{2}}}{2(\cos 2x)^{\frac{1}{2}}} \times -\frac{2}{(1 + \sin 2x)}$$

2 cancels out, then, sorting $(1 + \sin 2x)$ out by indices;

$$\frac{dy}{dx} = \frac{(1 + \sin 2x)^{\frac{1}{2}-1}}{(\cos 2x)^{\frac{1}{2}}} \times -1$$

Sort out the negative powers and all… Finally!

$$\frac{dy}{dx} = -\frac{1}{(1 + \sin 2x)^{\frac{1}{2}}(\cos 2x)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = -\frac{1}{[(1 + \sin 2x)(\cos 2x)]^{\frac{1}{2}}}$$

The above is because $a^{\frac{1}{2}}b^{\frac{1}{2}} = (ab)^{\frac{1}{2}}$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{(1+\sin 2x)(\cos 2x)}}$$

Next!

- $y = \frac{\sec x - \tan x}{\sec x + \tan x}$

$$u = \sec x - \tan x$$
$$v = \sec x + \tan x$$
$$\frac{du}{dx} = \sec x \tan x - \sec^2 x$$

These are direct derivative laws; you may want to be refreshed on this:

If: $\boldsymbol{y = \sin x}$

$$\boldsymbol{\frac{dy}{dx} = \cos x}$$

If: $\boldsymbol{y = \cos x}$

$$\boldsymbol{\frac{dy}{dx} = -\sin x}$$

If: $\boldsymbol{y = \tan x}$

$$\frac{dy}{dx} = \sec^2 x$$

If: $y = \sec x$

$$\frac{dy}{dx} = \sec x \tan x$$

If: $y = \operatorname{cosec} x$

$$\frac{dy}{dx} = -\cot x \operatorname{cosec} x$$

If: $y = \cot x$

$$\frac{dy}{dx} = -\operatorname{cosec}^2 x$$

So….. Let's get back to where we were;

$$\frac{du}{dx} = \sec x \tan x - \sec^2 x$$

$$v = \sec x + \tan x$$

$$\frac{dv}{dx} = \sec x \tan x + \sec^2 x$$

Direct laws again!

So, we'll apply the quotient rule here:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(\sec x + \tan x)(\sec x \tan x - \sec^2 x) - (\sec x - \tan x)(\sec x \tan x + \sec^2 x)}{(\sec x + \tan x)^2}$$

Expanding extensively!

$$\frac{dy}{dx} = \frac{\sec^2 x \tan x - \sec^3 x + \sec x \tan^2 x - \sec^2 x \tan x - (\sec^2 x \tan x + \sec^3 x - \sec x \tan^2 x - \sec^2 x \tan x)}{(\sec x + \tan x)^2}$$

Several terms cancel out and we're left with:

$$\frac{dy}{dx} = \frac{2\sec x \tan^2 x - 2\sec^3 x}{(\sec x + \tan x)^2}$$

$$\frac{dy}{dx} = \frac{2\sec x\,(\tan^2 x - \sec^2 x)}{(\sec x + \tan x)^2}$$

$\tan^2 x - \sec^2 x = (\tan x + \sec x)(\tan x - \sec x)$
This is by the difference of two squares;

$$\frac{dy}{dx} = \frac{2\sec x\,(\tan x + \sec x)(\tan x - \sec x)}{(\sec x + \tan x)^2}$$

$(\tan x + \sec x)$ cancels out and have:

$$\frac{dy}{dx} = \frac{2\sec x\,(\tan x - \sec x)}{(\tan x + \sec x)}$$

Fine, that answer seems very compact as in really. Up to the last example on quotient rule;

- $y = \frac{x^2 \sin x}{(x+1)(x^2-1)}$

$$u = x^2 \sin x$$
$$v = (x + 1)(x^2 - 1)$$

Wawu,

$$u = x^2 \sin x$$

These individual ones need product rule on their own before their derivatives are evaluated;

$$u = x^2 \sin x$$

However, we’ll need separate variables for the product situation;

$$u = a \times b$$

Here,

$$\frac{du}{dx} = b\frac{da}{dx} + a\frac{db}{dx}$$

It's nothing fixed on u and v, it's about the derivative of the second multiplied by the first function added to the derivative of the first multiplied by the second function. So,

$$a = x^2$$

$$\frac{da}{dx} = 1 \times 2 \times x^{2-1} = 2x$$

$$b = \sin x$$

$$\frac{db}{dx} = \cos x$$

Hence,

$$\frac{du}{dx} = b\frac{da}{dx} + a\frac{db}{dx}$$

$$\frac{du}{dx} = (\sin x)(2x) + (x^2)(\cos x)$$

$$\frac{du}{dx} = 2x\sin x + x^2\cos x$$

$$v = (x+1)(x^2-1)$$

Another product situation, we have to introduce new variables again;

$$v = c \times e$$

$$\frac{dv}{dx} = \ e\frac{dc}{dx} + c\frac{de}{dx}$$

$$c = x + 1$$

$$\frac{dc}{dx} = 1 \times x^{1-1} = 1$$

$$e \ = (x^2 - 1)$$

$$\frac{de}{dx} = 2 \times x^{2-1} - 0 = 2x$$

$$\frac{dv}{dx} = e\frac{dc}{dx} + c\frac{de}{dx}$$

$$\frac{dv}{dx} = (x^2-1)(1) + (x+1)(2x)$$

$$\frac{dv}{dx} = x^2 - 1 + 2x^2 + 2x$$

$$\frac{dv}{dx} = 3x^2 + 2x - 1$$

Now, we've now successfully gotten $\frac{du}{dx}$ and $\frac{dv}{dx}$

Let's now bundle them to the quotient rule;

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Hence,

$$\frac{dy}{dx} = \frac{(x+1)(x^2-1)(2x\sin x + x^2\cos x) - (x^2\sin x)(3x^2+2x-1)}{[(x+1)(x^2-1)]^2}$$

$$\frac{dy}{dx} = \frac{(x^3+x^2-x-1)(2x\sin x + x^2\cos x) - (x^2\sin x)(3x^2+2x-1)}{(x+1)^2(x^2-1)^2}$$

You can zoom it a bit to make the fraction more visible. This is the answer, this is how the final answer looks like after proper expansion:

$$\frac{dy}{dx} = \frac{\cos x\,(x^5+x^4-x^3-x^2) - \sin x\,(x^4+x^2+2x)}{(x+1)^2(x^2-1)^2}$$

FINALLY, We're almost through with explicit differentiation;

Now, we have just one more concept remaining, the concept **higher derivatives!**

It's as good as concluded actually but this is also very important; higher derivatives are no big deal; let's come quickly to dissolve it quickly; you still have integration to read.

HIGHER DERIVATIVES

So, *owkai*... ... ... The concept of higher derivatives goes thus:

Consider a function of x, $f(x)$ with:

$$y = f(x)$$

The derivative with respect to x is given by:

$$\frac{dy}{dx} \quad or \quad f'(x)$$

We're fully aware of this, so consider a situation where we want to further differentiate this; as we know, in this explicit differentiation situation, we can see it clearly that $\frac{dy}{dx}$ or $f'(x)$ also gives some other function of x;

So, think of performing this operation;

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

That is, differentiating $\frac{dy}{dx}$ with respect to x
We have a differential coefficient in the form;

$\frac{d^2y}{dx^2}$ pronounced *dee two y dee x two*

Hence, we have;

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

In the same way;
We can decide not to stop and still differentiate further, we'll be having;

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

It can continue,

$$\frac{d}{dx}\left(\frac{d^3y}{dx^3}\right) = \frac{d^4y}{dx^4}$$

In function notation; all these aren't left out;

Further differentiating $f'(x)$; we have;

$$\frac{d}{dx}\big(f(x)\big) = f'(x)$$

In the same way;
We can decide not to stop and still differentiate further, we'll be having;

$$\frac{d}{dx}\big(f'(x)\big) = f''(x)$$

It can continue,

$$\frac{d}{dx}\big(f''(x)\big) = f'''(x)$$

And so on;

Like you know, some books may represent all these as $f^1(x)$, $f^{11}(x)$, $f^{111}(x)$ and so on;

The concept of further differentiating differential coefficients is called the concept of **higher derivatives.**

From the concept of higher derivatives;

$\frac{dy}{dx}$ is called the **first derivative of *y*** with respect to x;

$\frac{d^2y}{dx^2}$ is called the **second derivative of *y*** with respect to x;

$\frac{d^3y}{dx^3}$ is called the **third derivative of *y*** with respect to x;

And so on;

So, quickly, let's see some examples;

Find the first, second and third derivatives in:

- $y = (3x - 5)^5$

- $y = \log_e(2x-9)^2$

Cool, let's begin;

- $y = (3x-5)^5$

$$\text{To find } \frac{dy}{dx}, \text{ put } u = 3x - 5$$

$$u = 3x - 5$$

$$\frac{du}{dx} = 3$$

$$y = u^5$$

$$\frac{dy}{du} = 5 \times u^{5-1} = 5u^4$$

From chain rule;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = 5u^4 \times 3 = 15u^4$$

Return: $u = 3x - 5$

$$\frac{dy}{dx} = 15(3x-5)^4$$

So, going further for the higher derivatives;

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}[15(3x-5)^4]$$

Again, the same substitution can be used;

$$u = 3x - 5$$

$$\frac{du}{dx} = 3$$

$$\left(\frac{dy}{dx}\right) = 15u^4$$

The derivative with respect to u is:

$$\frac{d}{du}\left(\frac{dy}{dx}\right) = 4 \times 15u^{4-1} = 60u^3$$

From chain rule;

$$\frac{d^2y}{dx^2} = \frac{d}{du}\left(\frac{dy}{dx}\right) \times \frac{du}{dx}$$

$$\frac{d^2y}{dx^2} = 60u^3 \times 3 = 180u^3$$

Return: $u = 3x - 5$

$$\frac{d^2y}{dx^2} = 180(3x - 5)^3$$

For the third derivative;

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^3}\right) = \frac{d}{dx}[180(3x - 5)^3]$$

Again, the same substitution can be used;

$$u = 3x - 5$$

$$\frac{du}{dx} = 3$$

$$\left(\frac{d^2y}{dx^2}\right) = 180u^3$$

The derivative with respect to u is:

$$\frac{d}{du}\left(\frac{d^2y}{dx^2}\right) = 3 \times 180u^{3-1} = 540u^2$$

From chain rule;

$$\frac{d^3y}{dx^3} = \frac{d}{du}\left(\frac{d^2y}{dx^2}\right) \times \frac{du}{dx}$$

$$\frac{d^3y}{dx^3} = 540u^2 \times 3 = 1620u^2$$

Return $u = 3x - 5$

$$\frac{d^3y}{dx^3} = 1620(3x - 5)^2$$

- $y = \log_e(2x - 9)^2$

To find $\frac{dy}{dx}$, put

$$u = (2x - 9)^2$$

Another substitution is needed;

$$z = 2x - 9$$

$$\frac{dz}{dx} = 2x^{1-1} - 0 = 2$$

$$u = z^2$$

$$\frac{du}{dz} = 2z^{2-1} = 2z$$

$$\frac{du}{dx} = \frac{dz}{dx} \times \frac{du}{dz}$$

$$\frac{du}{dx} = 2 \times 2z = 4z$$

Hence,

$$\frac{du}{dx} = 4(2x - 9)$$

$$y = \log_e u$$

$$\frac{dy}{du} = \frac{1}{u}$$

From chain rule;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{u} \times 4(2x - 9) = \frac{4(2x - 9)}{u}$$

Return $u = (2x - 9)^2$

$$\frac{dy}{dx} = \frac{4(2x - 9)}{(2x - 9)^2}$$

$(2x - 9)$ is reduced by one power;

$$\frac{dy}{dx} = \frac{4}{(2x-9)}$$

So, going further for the higher derivatives;

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[\frac{4}{2x-9}\right]$$

Now, the substitution can be used;

$$z = 2x - 9$$

And $\frac{dz}{dx} = 2$ as used above already

$$\left(\frac{dy}{dx}\right) = \frac{4}{z} = 4z^{-1}$$

The derivative with respect to z is:

$$\frac{d}{dz}\left(\frac{dy}{dx}\right) = -1 \times 4u^{-1-1} = -4z^{-2}$$

$$\frac{d}{dz}\left(\frac{dy}{dx}\right) = -\frac{4}{z^2}$$

From chain rule;

$$\frac{d^2y}{dx^2} = \frac{d}{dz}\left(\frac{dy}{dx}\right) \times \frac{dz}{dx}$$

$$\frac{d^2y}{dx^2} = -\frac{4}{z^2} \times 2 = -\frac{8}{z^2}$$

Return $z = 2x - 9$

$$\frac{d^2y}{dx^2} = -\frac{8}{(2x-9)^2}$$

For the third derivative;

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^3}\right) = \frac{d}{dx}\left[-\frac{8}{(2x-9)^2}\right]$$

Again, the same substitution can be used;

$$z = 2x - 9$$

And $\frac{dz}{dx} = 2$ as used above already

$$\left(\frac{d^2y}{dx^2}\right) = -\frac{8}{z^2} = -8z^{-2}$$

The derivative with respect to z is:

$$\frac{d}{dz}\left(\frac{d^2y}{dx^2}\right) = -2 \times -8u^{-2-1} = 16z^{-3}$$

$$\frac{d}{dz}\left(\frac{d^2y}{dx^2}\right) = \frac{16}{z^3}$$

From chain rule;

$$\frac{d^3y}{dx^3} = \frac{d}{dz}\left(\frac{d^2y}{dx^2}\right) \times \frac{dz}{dx}$$

$$\frac{d^3y}{dx^3} = \frac{16}{z^3} \times 2 = \frac{32}{z^3}$$

Return $u = 2x - 9$

$$\frac{d^3y}{dx^3} = \frac{32}{(2x-9)^3}$$

Hope you didn't get the substitutions here twisted, keep calm and go over it again.

Let's see these further two:

Find the first and second derivatives of;

- $y = \sin(x^2 + 3)$

- $y = e^{x^4}$

FIRST!

- $y = \sin(x^2 + 3)$

$$\text{To find } \frac{dy}{dx}, \text{ put } u = x^2 + 3$$

$$u = x^2 + 3$$

$$\frac{du}{dx} = 2x^{2-1} + 0 = 2x$$

$$y = \sin u$$

$$\frac{dy}{du} = \cos u$$

From chain rule;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \cos u \times 2x = 2x \cos u$$

Return $u = x^2 + 3$

$$\frac{dy}{dx} = 2x\cos(x^2 + 3)$$

So, going further for the higher derivatives;

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}[2x\cos(x^2 + 3)]$$

This isn't as straightforward as you think, this is a product, hence;

$$\left(\frac{dy}{dx}\right) = 2x \times \cos(x^2 + 3)$$

Let

$$\left(\frac{dy}{dx}\right) = a \times b$$

$$a = 2x$$

$$\frac{da}{dx} = 2x^{1-1} = 2$$

$$b = \cos(x^2 + 3)$$

With the substitution; $u = x^2 + 3$

$$\frac{du}{dx} = 2x$$

We have;

$$b = \cos u$$

$$\frac{db}{du} = -\sin u$$

From chain rule;

$$\frac{db}{dx} = \frac{db}{du} \times \frac{du}{dx}$$

$$\frac{db}{dx} = -\sin u \times 2x = -2x \sin u$$

$$\frac{db}{dx} = -\sin u \times 2x = -2x \sin(x^2 + 3)$$

Product rule;

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = b\frac{da}{dx} + a\frac{db}{dx}$$

$$\frac{d^2y}{dx^2} = [\cos(x^2 + 3)](2) + (2x)[-2x\sin(x^2 + 3)]$$

$$\frac{d^2y}{dx^2} = 2\cos(x^2 + 3) - 4x^2\sin(x^2 + 3)$$

- $y = e^{x^4}$

Here;

$$\text{To find } \frac{dy}{dx}\text{, put } u = x^4$$

$$u = x^4$$

$$\frac{du}{dx} = 4x^{4-1} = 4x^3$$

$$y = e^u$$

$$\frac{dy}{du} = e^u$$

From chain rule;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = e^u \times 4x^3 = 4x^3 e^u$$

Return $u = x^4$

$$\frac{dy}{dx} = 4x^3 e^{x^4}$$

So, going further for the higher derivatives;

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[4x^3e^{x^4}\right]$$

This also isn't as straightforward as you think; this is a product, hence;

$$\left(\frac{dy}{dx}\right) = 4x^3 \times e^{x^4}$$

Let:

$$\left(\frac{dy}{dx}\right) = a \times b$$

$$a = 4x^3$$

$$\frac{da}{dx} = 3 \times 4x^{3-1} = 12x^2$$

$$b = e^{x^4}$$

Now,

$$\text{since } b = e^{x^4} \text{ and } y = e^{x^4}$$

$$\frac{db}{dx} = \frac{dy}{dx} = 4x^3e^{x^4}$$

Note that we have solved for $\frac{dy}{dx}$ above:

Product rule;

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = b\frac{da}{dx} + a\frac{db}{dx}$$

$$\frac{d^2y}{dx^2} = \left[e^{x^4}\right](12x^2) + \left(4x^3\right)\left[4x^3e^{x^4}\right]$$

$$\frac{d^2y}{dx^2} = 12x^2e^{x^4} + 16x^6e^{x^4}$$

Let's take some further examples;

- If $f(\theta) = \frac{\sin\theta}{1+\cos\theta}$; find $f''(\theta)$

To find $f'(\theta)$ first, this is a clear case of quotient rule;

$$f(\theta) = \frac{u}{v}$$

$$u = \sin\theta$$

$$\frac{du}{d\theta} = \cos\theta$$

$$v = 1 + \cos\theta$$

$$\frac{dv}{d\theta} = 0 + (-\sin\theta) = -\sin\theta$$

All these are matters of standard differential coefficients;

Here;

$$\frac{df(\theta)}{d\theta} = f'(\theta) = \frac{v\frac{du}{d\theta} - u\frac{dv}{d\theta}}{v^2}$$

$$f'(\theta) = \frac{(1+\cos\theta)(\cos\theta) - (\sin\theta)(-\sin\theta)}{(1+\cos\theta)^2}$$

$$f'(\theta) = \frac{(\cos\theta + \cos^2\theta + \sin^2\theta)}{(1+\cos\theta)^2}$$

From basic trigonometry identities;

$$\cos^2\theta + \sin^2\theta = 1$$

Hence;

$$f'(\theta) = \frac{(\cos\theta + 1)}{(1+\cos\theta)^2} = \frac{(1+\cos\theta)}{(1+\cos\theta)^2}$$

$1 + \cos\theta$ cancels out;

$$f'(\theta) = \frac{1}{(1 + \cos\theta)}$$

Going further for the second derivative;
This is a simpler case;

$$f''(\theta) = \frac{d}{d\theta}\left(f'(\theta)\right) = \frac{d}{d\theta}\left(\frac{1}{(1 + \cos\theta)}\right)$$

Using the substitution above;

$$v = 1 + \cos\theta$$

$$\frac{dv}{d\theta} = -\sin\theta$$

Hence;

$$\frac{df(\theta)}{d\theta} = f'(\theta) = \frac{1}{v} = v^{-1}$$

The derivative with respect to v is:

$$\frac{d}{dv}\left(f'(\theta)\right) = -1 \times v^{-1-1} = -\frac{1}{v^2}$$

Then, by chain rule;

$$\frac{d^2 f(\theta)}{d\theta^2} = f''(\theta) = \frac{d}{dv}\left(f'(\theta)\right) \times \frac{dv}{d\theta}$$

$$f''(\theta) = -\frac{1}{v^2} \times -\sin\theta = \frac{\sin\theta}{v^2}$$

Return $v = 1 + \cos\theta$

$$f''(\theta) = \frac{\sin\theta}{(1 + \cos\theta)^2}$$

- Find $\frac{d^2 y}{dx^2}$ in:

$$y = e^x - e^{-x}$$

Hence show that;

$$\frac{d^2 y}{dx^2} - y = 0$$

$$y = e^x - e^{-x}$$

From the normal derivative rule of sums and differences;

$$\frac{dy}{dx} = \frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x})$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(e^{-x})$$

By basic chain rule;

$$\frac{d}{dx}(e^{-x}) = e^{-x} \times -1 = -e^{-x}$$

$$\frac{dy}{dx} = \frac{d}{dx}(e^{x}) - \frac{d}{dx}(e^{-x})$$

$$\frac{dy}{dx} = e^{x} - (-e^{-x})$$

$$\frac{dy}{dx} = e^{x} + e^{-x}$$

Going further;

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(e^{x} + e^{-x})$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(e^{x}) + \frac{d}{dx}(e^{-x})$$

$$\frac{d}{dx}(e^{x}) = e^{x}$$

$$\text{As seen above; } \frac{d}{dx}(e^{-x}) = -e^{-x}$$

Hence,

$$\frac{d^2y}{dx^2} = e^x + (-e^{-x})$$

$$\frac{d^2y}{dx^2} = e^x - e^{-x}$$

Hence, what we're told to show;

$$\frac{d^2y}{dx^2} - y = 0$$

$$\frac{d^2y}{dx^2} = e^x - e^{-x}$$

$$y = e^x - e^{-x}$$

Hence, let's prove:

$$e^x - e^{-x} - (e^x - e^{-x})$$

$$e^x - e^{-x} - e^x + e^{-x}$$

The whole thing cancels off to zero and the prove is done;

Never mind if the last three examples seem somehow, they are things you know and are simple concepts from chain rule and all. Understanding the other ones is very important!

So, that draws the curtain on differentiation (differential calculus); sure you enjoyed every bit of it; don't hesitate to re-read any concept you aren't clear with.

The notes are self-explanatory; it's been nice sitting to teach yourself the first branch calculus.

UP NEXT is partial differentiation – the next chapter! However, it is expedient we learn a bit of implicit differentiation, it is very important you know it as it is quite important in real life functions.

IMPLICIT DIFFERENTIATION

Implicit functions as we learnt in our studies of functions are functions that its direction isn't certain.

Such functions occur many a times in real life functions and as a matter of fact, though the first 185 pages of this topic focuses on explicit differentiation (the differentiation of explicit functions), implicit functions as well do have derivatives.

Hence, implicit differentiation is the process of differentiating implicit functions.

How then are implicit functions differentiated? Let's see this!

$$x + y + xy^2 = 5$$

The above is a very good example of an implicit function. Implicit differentiation makes use of the rules of explicit differentiation and hence, nothing new is being learnt here;

In dealing with implicit differentiation, each term is taken as an entity and differentiated on its own.

We then make a choice of a variable to make the dependent variable and the independent variable.

Mostly in the case of x and y variables, the variable y is taken as the dependent variable and x as the independent variable. **Note y is not the dependent variable neither is x the independent variable, they are taken as that for the sake of the implicit differentiation.**

(i) Now, as each term is taken as an entity, **terms that contain both variables** are differentiated twice. Terms that contain only one of the two variables are differentiated once.

(ii) When any term is differentiated is with respect to the **assumed dependent variable**, the differential coefficient with respect to the **assumed independent variable** is included in the derivative.

Now, let's use our primitive example to explain all the following.

$$x + y + xy^2 = 5$$

We will be taking our assumed dependent variable as y and our independent variable as x;

Our first entity is x;

x contains only variable and hence according to rule (i), it is differentiated only once.

The derivative of x according to power rule is:

$$1 \times x^{1-1} = 1$$

According to the rule (ii), the derivative is not with respect to the assumed dependent variable (the dependent variable is assumed to be y), hence, we have here that nothing is included in the derivative, hence, in the implicit differentiation of our example, the first term in the differentiation process from our first entity is:

$$1$$

The second entity is y;

y contains only variable and hence according to rule (i), it is differentiated only once.

The derivative of y according to power rule is:

$$1 \times y^{1-1} = 1$$

According to the rule (ii), the derivative is with respect to the assumed dependent variable (the dependent variable is assumed to be y), hence, we have here that we have to include the differential coefficient with respect to the **assumed independent variable** to its derivative, hence, in the implicit differentiation of our example, the second term in the differentiation process from our second entity is:

$$1 \times \frac{dy}{dx} = \frac{dy}{dx}$$

We are multiplying the derivative of y by the differential coefficient since y is the assumed dependent variable.

The third entity is xy^2;

xy^2 contains the two variables and hence according to rule (i), it is differentiated twice.

Twice means that first, with respect to the assumed independent variable and secondly with respect to the assumed dependent variable.

Hence; we have:

$$xy^2$$

To differentiate with respect to x, we differentiate the x terms and take every other thing constant;

Hence; the derivative with respect to x is:

$$1 \times x^{1-1} \times y^2 = y^2$$

Since it is with respect to x we have differentiated now, rule (ii) makes us understand that no differential coefficient is included.

Now, the same term is differentiated with respect to y and every other thing is taken as constant;

Here; we have:

$$2 \times xy^{2-1} = 2xy$$

Since it is with respect to y we have differentiated now, rule (ii) makes us understand that the differential coefficient must be included.

$$2xy \times \frac{dy}{dx} = 2xy\frac{dy}{dx}$$

Now, in implicit differentiation, we replace every term with their corresponding implicit derivatives while maintaining the signs between them, in the case of terms differentiated twice, the sum of the two derivatives are used to replace the term differentiated twice with brackets properly used.

$$x + y + xy^2 = 5$$

Replacing;

$$1 + \frac{dy}{dx} + \left(y^2 + 2xy\frac{dy}{dx}\right) = 0$$

Hence, to find the derivative now, we make the differential coefficient the subject of the relation;

Firstly, take all terms not containing $\frac{dy}{dx}$ to the other side of the equality;

$$\frac{dy}{dx} + 2xy\frac{dy}{dx} = -y^2 - 1$$

Factorize $\frac{dy}{dx}$ on the left; factorize -1 on the right;

$$\frac{dy}{dx}(1+2xy) = -(y^2+1)$$

Divide both sides by $(1+2xy)$;

$$\frac{dy}{dx} = -\frac{y^2+1}{1+2xy}$$

Hence, in implicit derivatives, the differential coefficient mostly contains both terms and unlike explicit cases where the derivative is always only in terms of the independent variable;

Let's take three to five more examples and we are done! We will take x as the independent variable.

Differentiate the following functions implicitly.

- $xy^3 - 2xe^y = a$
- $\log_e x - 2x^3 \sin y + y^3 = 0$
- $2x^3 - 4x^2y + \tan y = 0$

- $\frac{x}{y} - y^2 + e^x \ln y = 0$

- $y^3 + \ln x - \cos y - x^2 = 7$

We'll be done soon! Let's take it one by one!

- $xy^3 - 2xe^y = a$

We differentiate each entity, I'll still break it down gradually, as you proceed however, you can do it without doing it entity by entity.

$$xy^3$$

With respect to x; we have:

$$1 \times x^{1-1}y^3 = y^3$$

With respect to y; we have;

$$3 \times xy^{3-1} = 3xy^2 \times \frac{dy}{dx}$$

$$3xy^2 \frac{dy}{dx}$$

Of course we know the differential coefficient is included when differentiating with respect to y;

Next term;

$$2xe^{y}$$

Differentiating with respect to x; we have;

$$1 \times 2x^{1-1}e^{y} = 2e^{y}$$

Differentiating with respect to y; we have;

$$2x(e^{y}) = 2xe^{y} \times \frac{dy}{dx}$$

$$2xe^{y}\frac{dy}{dx}$$

You know the derivative of e^{y} is still e^{y} and we will include the differential coefficient;

Lastly;

$$a$$

a is but a constant since it is neither x nor y, hence, the derivative is zero!

Hence, we'll now replace each term by its implicit derivative and maintain the signs within, we'll employ brackets now that there is a minus sign in the midst of the terms this time.

$$xy^3 - 2xe^y = a$$

Replacing;

$$y^3 + 3xy^2\frac{dy}{dx} - \left(2e^y + 2xe^y\frac{dy}{dx}\right) = 0$$

$$y^3 + 3xy^2\frac{dy}{dx} - 2e^y - 2xe^y\frac{dy}{dx} = 0$$

Do the needful, make the subject of the relation and complete the work.

$$3xy^2\frac{dy}{dx} - 2xe^y\frac{dy}{dx} = 2e^y - y^3$$

Notice the change of signs as we cross the equality, factorize $\frac{dy}{dx}$;

$$\frac{dy}{dx}(3xy^2 - 2xe^y) = 2e^y - y^3$$

Dividing through;

$$\frac{dy}{dx} = \frac{2e^y - y^3}{3xy^2 - 2xe^y}$$

NEXT!

- $\log_e x - 2x^3 \sin y + y^3 = 0$

We differentiate each entity, I'll still break it down gradually, as you proceed however, you can do it without doing it entity by entity. We will do so as we proceed;

$$\log_e x$$

This only contains one variable and hence, we're differentiating only with respect to x; we have a standard derivative;

$$\frac{1}{x}$$

Next term is;

$$2x^3 \sin y$$

Differentiating with respect to x; we have;

$$3 \times 2x^{3-1} \sin y = 6x^2 \sin y$$

Differentiating with respect to y; we have;

$$2x^3(\cos y) \times \frac{dy}{dx} = 2x^3 \cos y \frac{dy}{dx}$$

We know the derivative of $\sin y$ is $\cos y$ and we are including the differential coefficient;

Lastly;

$$y^3$$

This only contains one variable and hence, we're differentiating only with respect to y; we have the power rule;

$$3 \times y^{3-1} \times \frac{dy}{dx} = 3y^2 \frac{dy}{dx}$$

We know we are including the differential coefficient since we are differentiating with respect to y;

Hence, we'll replace each term by its implicit derivative and maintain the signs within, we'll employ brackets when there is a minus sign in the midst of the terms.

$$\log_e x - 2x^3 \sin y + y^3 = 0$$

Replacing;

$$\frac{1}{x} - \left(6x^2 \sin y + 2x^3 \cos y \frac{dy}{dx}\right) + 3y^2 \frac{dy}{dx} = 0$$

Expanding;

$$\frac{1}{x} - 6x^2 \sin y - 2x^3 \cos y \frac{dy}{dx} + 3y^2 \frac{dy}{dx} = 0$$

Do the needful, make $\frac{dy}{dx}$ the subject of the relation and complete the work.

$$-2x^3 \cos y \frac{dy}{dx} + 3y^2 \frac{dy}{dx} = 6x^2 \sin y - \frac{1}{x}$$

Notice the change of signs as we cross the equality, factorize $\frac{dy}{dx}$ on the left; add the fractions on the right with the common denominator as x!

$$\frac{dy}{dx}(3y^2 - 2x^3 \cos y) = \frac{6x^2 \sin y\,(x) - 1}{x}$$

$$\frac{dy}{dx}(3y^2 - 2x^3 \cos y) = \frac{6x^3 \sin y - 1}{x}$$

The above is since: $6x^2 \sin y\,(x) = 6x^3 \sin y$

Dividing through;

$$\frac{dy}{dx} = \frac{6x^3 \sin y - 1}{x} \div (3y^2 - 2x^3 \cos y)$$

$$\frac{dy}{dx} = \frac{6x^3 \sin y - 1}{x} \times \frac{1}{(3y^2 - 2x^3 \cos y)}$$

$$\frac{dy}{dx} = \frac{6x^3 \sin y - 1}{x(3y^2 - 2x^3 \cos y)}$$

$$\frac{dy}{dx} = \frac{6x^3 \sin y - 1}{3xy^2 - 2x^4 \cos y}$$

NEXT;

$$2x^3 - 4x^2y + \tan y = 0$$

We differentiate each entity;

$$2x^3$$

With respect to x;

$$3 \times 2x^{3-1} = 6x^2$$

Next;

$$4x^2y$$

First, with respect to x;

$$2 \times 4x^{2-1}y = 8xy$$

Second, with respect to y;

$$4x^2y^{1-1} \times \frac{dy}{dx} = 4x^2\frac{dy}{dx}$$

We know we are including the differential coefficient since we are differentiating with respect to y;

Lastly;

$$\tan y$$

Standard derivative;

$$\sec^2 y \times \frac{dy}{dx} = \sec^2 y\frac{dy}{dx}$$

We know we are including the differential coefficient since we are differentiating with respect to y;

The last term after the equality is 0 which is nothing but a constant with a derivative of zero.

Hence, we'll replace each term by its implicit derivative and maintain the signs within, we'll employ brackets when there is a minus sign in the midst of the terms.

$$2x^3 - 4x^2y + \tan y = 0$$

Replacing;

$$6x^2 - \left(8xy + 4x^2\frac{dy}{dx}\right) + \sec^2 y\frac{dy}{dx} = 0$$

$$6x^2 - 8xy - 4x^2\frac{dy}{dx} + \sec^2 y\frac{dy}{dx} = 0$$

Sort out and factorize;

$$-4x^2\frac{dy}{dx} + \sec^2 y\frac{dy}{dx} = 8xy - 6x^2$$

$$\frac{dy}{dx}(\sec^2 y - 4x^2) = 8xy - 6x^2$$

Hence;

$$\frac{dy}{dx} = \frac{8xy - 6x^2}{\sec^2 y - 4x^2}$$

NEXT;

- $\frac{x}{y} - y^2 + e^x \ln y = 0$

We differentiate each entity;

$$\frac{x}{y}$$

The above contains two variables and can be expressed as shown below for ease of differentiation;

$$x\left(\frac{1}{y}\right) = x(y^{-1})$$

Hence, differentiating with respect to x, we have;

$$1 \times x^{1-1}\left(\frac{1}{y}\right) = \frac{1}{y}$$

Differentiating with respect to y, we have;

$$x(-1 \times y^{-1-1}) \times \frac{dy}{dx} = -xy^{-2}\frac{dy}{dx}$$

$$-\frac{x}{y^2}\left(\frac{dy}{dx}\right)$$

Next;

$$y^2$$

With respect to y;

$$2 \times y^{2-1} \times \frac{dy}{dx} = 2y\frac{dy}{dx}$$

Next;

$$e^x \ln y$$

First, with respect to x;

$$(e^x) \ln y$$

We know the derivative of e^x remains e^x

Second, with respect to y;

$$e^x \left(\frac{1}{y}\right) \times \frac{dy}{dx} = \frac{e^x}{y}\left(\frac{dy}{dx}\right)$$

The last term after the equality is 0 which is nothing but a constant with a derivative of zero.

Hence, we'll replace each term by its implicit derivative and maintain the signs within, we'll employ brackets when there is a minus sign in the midst of the terms.

$$\frac{x}{y} - y^2 + e^x \ln y = 0$$

Replacing;

$$\frac{1}{y} + \left(-\frac{x}{y^2}\left(\frac{dy}{dx}\right)\right) - \left(2y\frac{dy}{dx}\right) + \left(e^x \ln y + \frac{e^x}{y}\left(\frac{dy}{dx}\right)\right) = 0$$

Expanding;

$$\frac{1}{y} - \frac{x}{y^2}\left(\frac{dy}{dx}\right) - 2y\frac{dy}{dx} + e^x \ln y + \frac{e^x}{y}\left(\frac{dy}{dx}\right) = 0$$

Sorting out and factorizing; signs will change as we cross the equality signs;

$$-\frac{x}{y^2}\left(\frac{dy}{dx}\right) - 2y\frac{dy}{dx} + \frac{e^x}{y}\left(\frac{dy}{dx}\right) = -\frac{1}{y} - e^x \ln y$$

$$\frac{dy}{dx}\left(\frac{e^x}{y} - \frac{x}{y^2} - 2y\right) = -\left(\frac{1}{y} + e^x \ln y\right)$$

Add terms within both brackets;

$$\frac{dy}{dx}\left(\frac{e^x(y) - x(1) - 2y(y^2)}{y^2}\right) = -\left(\frac{1 + e^x \ln y\,(y)}{y}\right)$$

$$\frac{dy}{dx}\left(\frac{e^x y - x - 2y^3}{y^2}\right) = -\left(\frac{1 + e^x y \ln y}{y}\right)$$

To isolate $\frac{dy}{dx}$; we will divide through by what is multiplying it;

$$\frac{dy}{dx} = -\left(\frac{1 + e^x y \ln y}{y}\right) \div \left(\frac{e^x y - x - 2y^3}{y^2}\right)$$

Division turns to multiplication with reciprocal following suit;

$$\frac{dy}{dx} = -\left(\frac{1 + e^x y \ln y}{y}\right) \times \frac{y^2}{e^x y - x - 2y^3}$$

y^2 and y cancel off to reduce;

$$\frac{dy}{dx} = -\left(\frac{1 + e^x y \ln y}{1}\right) \times \frac{y}{e^x y - x - 2y^3}$$

We have;

$$\frac{dy}{dx} = -\frac{y(1 + e^x y \ln y)}{e^x y - x - 2y^3}$$

Expanding;

$$\frac{dy}{dx} = -\frac{y + e^x y^2 \ln y}{e^x y - x - 2y^3}$$

LAST EXAMPLE!

$$y^3 + \ln x - \cos y - x^2 = 7$$

We'll differentiate each entity; each contains one variable!

$$y^3$$

With respect to y;

$$3 \times y^{3-1} \times \frac{dy}{dx} = 3y^2 \frac{dy}{dx}$$

Next;

$$\ln x$$

Standard derivative with respect to x;

$$\frac{1}{x}$$

Next;

$$\cos y$$

Standard derivative with respect to y;

$$(-\sin y) \times \frac{dy}{dx} = -\sin y\left(\frac{dy}{dx}\right)$$

Next;

$$x^2$$

With respect to x;

$$2 \times x^{2-1} = 2x$$

The last term is 7, which is a constant, its derivative either way is zero!

Hence, we'll replace each term by its implicit derivative and maintain the signs within, we'll employ brackets when there is a minus sign in the midst of the terms.

$$y^3 + \ln x - \cos y - x^2 = 7$$

Replacing;

$$3y^2\frac{dy}{dx}+\frac{1}{x}-\left(-\sin y\frac{dy}{dx}\right)-(2x)=0$$

Expand;

$$3y^2\frac{dy}{dx}+\frac{1}{x}+\sin y\frac{dy}{dx}-2x=0$$

Sort out and factorize;

$$3y^2\frac{dy}{dx}+\sin y\frac{dy}{dx}=2x-\frac{1}{x}$$

Add on the right;

$$\frac{dy}{dx}(3y^2+\sin y)=\frac{2x^2-1}{x}$$

Divide through;

$$\frac{dy}{dx}=\frac{2x^2-1}{x}\div(3y^2+\sin y)$$

$$\frac{dy}{dx}=\frac{2x^2-1}{x}\times\frac{1}{3y^2+\sin y}$$

$$\frac{dy}{dx} = \frac{2x^2 - 1}{x(3y^2 + \sin y\,)}$$

$$\frac{dy}{dx} = \frac{2x^2 - 1}{3xy^2 + x\sin y}$$

That wraps up differentiation as a whole! UP NEXT is partial differentiation, it also this with two variables but in this case, it is the explicit form where one variable depends on two or more variables, hence, it isn't implicit.

DO NOT CONFLICT IMPLICIT DIFFERENTIATION WITH PARTIAL DIFFERENTIATION