DEPARTMENT OF ECONOMICS
FACULTY OF SOCIAL SCIENCES
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SSC106: MATHEMATICS FOR SOCIAL SCIENCES II

RAIN SEMESTER EXAMINATION (2017/2018 SESSION)

INSTRUCTIONS:

- Attempt all questions in **Section A;**
- Answer any two questions from **Section B.**
- Show all workings clearly

Time allowed: 120 minutes

SECTION A (20 marks)

(i) Find y as an explicit function of x when $y = 2t^2 + 16t - 4$ (where $t = x^2 + 4$)

(ii) Establish whether this function is homogenous and determine the degree of homogeneity.

$$Y = 3x^2$$

(iii) If $f(x) = 1 + x^2 + x^4$, show that $f(-a) = f(a)$ for all values of a.

(iv) Find $\dfrac{dy}{dx}$ if $y = \dfrac{\sin x}{\cos x}$

(v) If $f(x) = x^3 + 4x^2 - 3x - 15$, find the value(s) of x for which $f'(x) = 0$

(vi) If $y = \log(e^x - e^{-x})$, find $\dfrac{dy}{dx}$

(vii) Find $\displaystyle\int \sqrt[3]{x^2}\, dx$

(viii) Find $\displaystyle\int e^{4x+3}\, dx$

(ix) Any matrix which its determinant equal to zero is called ……………….

(x) Given that matrix A is $m \times n$ and matrix B is $n \times p$, then the solution of the matrix multiplication AB is …………….

(xi) For any matrix A and B with a given scalar k, verify that $k(A + B) = kA + kB$; where;

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \\ 0 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -1 \\ 2 & 7 \\ 1 & 6 \end{bmatrix} \text{ and } k = 2$$

(xii) Ascertain whether the function:

$$y = f(x) = x^3 - 6x^2 + 12$$

has a point of inflection.

SECTION B

1. *(a)* State the necessary and sufficient conditions for relative optima of a function: $y = f(x)$

(b) Given that $Y = \ln x_1 + 2\ln x_2$ subject to $2x_1 + 3x_2 = 18$. Obtain the value of x_1 and x_2 that maximizes the function given that the second order condition is satisfied.

(c) Find the relative optimum values of the function:

$$Z = f(x, y) = x^2 + y^2 + 4x - 8y$$

2. *(a)* Explain the basic difference between Differential calculus and integral calculus.

(b) Explain the following:

(i) $\int_{9}^{1} X\sqrt{1+2X}\,dX$

(ii) $\int_{5}^{5} \frac{X^4}{e^X+9}\,dX$

(iii) $\int 3X^2 \sin(X^3+1)\,dX$

(iv) $\int_{1}^{9} 3\sqrt{Z-1}\,dZ$

(v) $\int 2Xe^{X^2}\,dX$

(c) Find:

$$\int \frac{x+1}{x^2+2x+6}\,dx$$

3. *(a)* Distinguish between ordinary and partial differential equations.

(b) (i) State Euler's theorem.

(ii) Use Euler's theorem to determine the degree of homogeneity of the function:

$$z = x^2 + 6xy + y^2$$

(c) (i) What is harmonic function.

(ii) Is the function $z = \sin x \cos y$ harmonic?

4. *(a)* Differentiate among diagonal matrix, scalar matrix and identity matrix.

(b) What is orthogonal matrix? Identify the salient features of orthogonal matrix.

(c) Define the term Trace of a matrix. If A and B are square matrices of the same order, show that:

$$tr(A + B) = trA + trB$$

(d) If the linear market model of demand and supply for one commodity is of the form:

$Qd = a + bP$ (demand relation)

$Qs = c + dP$ (supply relation)

$Q_d = Q_s = Q$ (equilibrium relation)

Determine the equilibrium price and quality variables using Crammer's rule.

SOLUTION TO THE PAST QUESTIONS

All topics needed here have been thoroughly treated through and through, all you need to do is to sit down and answer all these questions like a joke, you are not expected to start reading from here but from the topics, you make a joke of yourself when you read the concept of any subject starting from the side of the past question; more so, this text isn't a past question but a complete guide to lead you to a straight A in your SSC106 exam.

Hence, keep calm and check the solutions to these past questions one-by-one;

GOOD LUCK AND GOD'S BEST!

SOLUTION TO THE SSC106 EXAMINATION 2017/2018 ACADEMIC SESSION

The instruction is you answer all questions in the **Section A** and only two questions from Section B, we'll be answering everything in both sections in this book though; keep calm and move with me;

Remember every singular solution here, you can get the full concept by reading this book and not the past question section exclusively;

SECTION A

Question (i)

Find y as an explicit function of x when $y = 2t^2 + 16t - 4$ (where $t = x^2 + 4$)

Here, we have a basic question on the basics of a function. We have:

$$y = 2t^2 + 16t - 4$$

To convert y to an explicit function of x, we simply substitute for t anywhere we see t in the function of y, knowing that $t = x^2 + 4$

Hence;

$$y = 2t^2 + 16t - 4$$

Substitute; $t = x^2 + 4$

$$y = 2(x^2 + 4)^2 + 16(x^2 + 4) - 4$$

We'd be doing some gradual expansion

$$y = 2(x^4 + 8x^4 + 16) + 16x^2 + 64 - 4$$

$$y = 2x^4 + 16x^2 + 32 + 16x^2 + 60$$

$$y = 2x^4 + 32x^2 + 92$$

Question (ii)

Establish whether this function is homogenous and determine the degree of homogeneity.

$$Y = 3x^2$$

We'd be using the Euler's theorem; as we attempt to find the degree of homogeneity, we'd know whether or not it is homogenous, we saw a case of one variable in the textbook section;

$$Y = 3x^2$$

We have:

$$Y_x = 2 \times 3x^{2-1} = 6x$$

Y_x of course is the derivative of Y with respect to x;

From Euler's theorem;

The degree of homogeneity, n, for one variable is:

$$xY_x = nY$$

Hence;

$$x(6x) = n(3x^2)$$

Here;

$$6x^2 = n(3x^2)$$

Dividing through by $3x^2$;

$$n = 2$$

Hence, the function is homogenous! And it is homogenous to the degree of 2.

Question (iii)

If $f(x) = 1 + x^2 + x^4$, show that $f(-a) = f(a)$ for all values of a.

Another basic question on functions, let's show what we are told to show;

$$f(x) = 1 + x^2 + x^4$$

Hence;

Basic things we know how to do under functions;

$$f(-a) = 1 + (-a)^2 + (-a)^4$$

Even powers of negative bases end up positive;

$$f(-a) = 1 + a^2 + a^4$$

Also;

$$f(a) = 1 + (a)^2 + (a)^4$$

Hence;

$$f(a) = 1 + a^2 + a^4$$

Clearly, it is shown that:

$$f(-a) = f(a)$$

You can impress them to tell them that **you know** that such a function is called **an even function.**

Question (iv)

Find $\frac{dy}{dx}$ if $y = \frac{\sin x}{\cos x}$

Here; simple differentiation;

Quotient rule;

Ops, no, no, no;

From basic trigonometry!

$$\frac{\sin x}{\cos x} = \tan x$$

Hence;

$$y = \tan x$$

And as a basic formula,

$$\frac{dy}{dx} = \sec^2 x$$

Looked too easy right? Well, if you'd not figured it out and you went through quotient rule, you'd have still gotten the same answer, **actually;**

Let's try! Quotient rule;

$$u = \sin x$$
$$v = \cos x$$

Basic;

$$\frac{du}{dx} = \cos x$$

And;

$$\frac{dv}{dx} = -\sin x$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Hence;

$$\frac{dy}{dx} = \frac{\cos x\,(\cos x) - \sin x\,(-\sin x)}{(\cos x)^2}$$

$$\frac{dy}{dx} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

But; from basic trigonometry;

$$\cos^2 x + \sin^2 x = 1$$

Hence;

$$\frac{dy}{dx} = \frac{1}{\cos^2 x}$$

But;

$$\frac{1}{\cos x} = \sec x$$

Hence;

$$\frac{dy}{dx} = \frac{1}{\cos^2 x} = \sec^2 x$$

SIMPLE! Same answer but longer process; open your eyes to do it in the shortest way!

Question (v)

If $f(x) = x^3 + 4x^2 - 3x - 15$, find the value(s) of x for which $f'(x) = 0$

Here;

$$f(x) = x^3 + 4x^2 - 3x - 15$$

Hence;
Basic derivative;

$$f'(x) = 3 \times x^{3-1} + 2 \times 4x^{2-1} - 1 \times 3x^{1-1} - 0$$

$$f'(x) = 3x^2 + 8x - 3$$

We're told to find the values of x for $f'(x) = 0$
Hence; we solve;

$$f'(x) = 3x^2 + 8x - 3 = 0$$

This is a quadratic equation which we'll solve by factorization, if you have difficulty factorizing, you can safely make use of the quadratic formula;

$$3x^2 + 8x - 3 = 0$$

$$3x^2 + 9x - x - 3 = 0$$

$$3x(x + 3) - 1(x + 3) = 0$$

$$(x + 3)(3x - 1) = 0$$

Hence;

$$x + 3 = 0$$

$$x = -3$$

Or

$$3x - 1 = 0$$

$$3x = 1$$

$$x = \frac{1}{3}$$

Hence, the two values for which $f'(x) = 0$ are:

$$x = -3 \quad \text{and} \quad x = \frac{1}{3}$$

Question (vi)

If $y = \log(e^x - e^{-x})$, find $\frac{dy}{dx}$

Here;
Substitution;

$$u = e^x - e^{-x}$$

$$\frac{du}{dx} = e^x - (-e^{-x})$$

The derivative of e^x is e^x from straight formula, and derivative of e^{-x} from basic chain rule exponential formula is $-e^{-x}$

$$\frac{du}{dx} = e^x + e^{-x}$$

Hence;

$$y = \log u = \log_{10} u$$

$$\frac{dy}{du} = \frac{1}{u \ln 10}$$

Chain rule;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Hence;

$$\frac{dy}{dx} = \frac{1}{u \ln 10} \times (e^x + e^{-x})$$

$$\frac{dy}{dx} = \frac{e^x + e^{-x}}{u \ln 10}$$

Return u;

$$\frac{dy}{dx} = \frac{e^x + e^{-x}}{\ln 10\,(e^x - e^{-x})}$$

Question (vii)

Find $\int \sqrt[3]{x^2}\, dx$

From indices;

$$\sqrt[3]{x^2} = (x^2)^{\frac{1}{3}} = x^{2\times\frac{1}{3}} = x^{\frac{2}{3}}$$

Hence;

$$\int \sqrt[3]{x^2}\, dx = \int x^{\frac{2}{3}}\, dx$$

From integration power rule; the integral is:

$$\left[\frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1}\right] + C = \frac{x^{\frac{5}{3}}}{\frac{5}{3}} + C$$

Finally;

$$\frac{3}{5}x^{\frac{5}{3}} + C$$

The arbitrary constant is never forgotten!

Question (viii)

Find $\int e^{4x+3}\,dx$

Here; we have linear substitution;

$$u = 4x + 3$$

$$\frac{du}{dx} = 4$$

Hence;

$$dx = \frac{du}{4}$$

The integral becomes;

$$\int e^{u}\frac{du}{4}$$

Constant out;

$$\frac{1}{4}\int e^{u}du$$

Straight integral;

$$\frac{1}{4}[e^u] + C$$

Return u;

$$\frac{1}{4}e^{4x+3} + C$$

Question (ix)

Any matrix which its determinant equal to zero is called ………………

It is simply called a **singular matrix.**

Question (x)

Given that matrix A is $m \times n$ and matrix B is $n \times p$, then the solution of the matrix multiplication AB is ……………

Directly from the textbook section;
As a matter of rule, if A is of the order $m \times n$ and B is of the order $n \times p$;
Then: AB will be of order $m \times p$.

Question (xi)

For any matrix A and B with a given scalar k, verify that $k(A+B)=kA+kB$; where;

$$A=\begin{bmatrix}1 & -2\\3 & 5\\0 & 4\end{bmatrix} \text{ and } B=\begin{bmatrix}0 & -1\\2 & 7\\1 & 6\end{bmatrix} \text{ and } k=2$$

Here; $k=2$

$$(A+B)=\begin{bmatrix}1 & -2\\3 & 5\\0 & 4\end{bmatrix}+\begin{bmatrix}0 & -1\\2 & 7\\1 & 6\end{bmatrix}$$

$$(A+B)=\begin{bmatrix}1+0 & -2+(-1)\\3+2 & 5+7\\0+1 & 4+6\end{bmatrix}=\begin{bmatrix}1 & -3\\5 & 12\\1 & 10\end{bmatrix}$$

Hence;

$$k(A+B)=2\begin{bmatrix}1 & -3\\5 & 12\\1 & 10\end{bmatrix}=\begin{bmatrix}2 & -6\\10 & 24\\2 & 20\end{bmatrix}$$

Also;

$$kA=2\begin{bmatrix}1 & -2\\3 & 5\\0 & 4\end{bmatrix}=\begin{bmatrix}2 & -4\\6 & 10\\0 & 8\end{bmatrix}$$

And;

$$kB = 2\begin{bmatrix} 0 & -1 \\ 2 & 7 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 4 & 14 \\ 2 & 12 \end{bmatrix}$$

Hence;

$$kA + kB = \begin{bmatrix} 2 & -4 \\ 6 & 10 \\ 0 & 8 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 4 & 14 \\ 2 & 12 \end{bmatrix}$$

$$kA + kB = \begin{bmatrix} 2+0 & -4+(-2) \\ 6+4 & 10+14 \\ 0+2 & 8+12 \end{bmatrix} = \begin{bmatrix} 2 & -6 \\ 10 & 24 \\ 2 & 20 \end{bmatrix}$$

Evidently;

$$k(A+B) = kA + kB$$

PROVED!

Question (xii)

Ascertain whether the function:

$$y = f(x) = x^3 - 6x^2 + 12$$

has a point of inflection.

Since we are looking for a point of inflection (inflexion), we'd have to differentiate this and check it till **the second order conditions;**

Hence;

$$y = f(x) = x^3 - 6x^2 + 12$$

$$\frac{dy}{dx} = 3 \times x^{3-1} - 2 \times 6x^{2-1} + 0$$

$$\frac{dy}{dx} = 3x^2 - 12x$$

Going further to the second order derivative;

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(3x^2 - 12x)$$

$$\frac{d^2y}{dx^2} = 2 \times 3x^{2-1} - 1 \times 12x^{1-1}$$

$$\frac{d^2y}{dx^2} = 6x - 12$$

Now, for a point of inflection;

$$\frac{d^2y}{dx^2} = 0$$

Hence;
For this function, we'd solve and check if there is a possible value of x for which this function's second derivative is equal to zero; hence, solving;

$$\frac{d^2y}{dx^2} = 6x - 12 = 0$$

$$6x = 12$$
$$x = 2$$

Hence, the function has a point of inflection and it exists at $x = 2$.

N.B.: You may be wondering why we didn't bother to solve for the first derivative equal to zero at first, reason is this, the solution of the first derivative equal to zero gives the turning points. However, in a curve, the point of inflection may be a turning point **but isn't necessarily always a turning point.** The minimum and maximum points are **always turning points**, and that is why to find them, we solve for the first derivative equal to zero and check for its nature from the second derivative. However, since the question is asking directly for the point of inflection, we can go directly to solve it for the second derivative equal to zero. Sometimes, we solve for the turning points and check for the nature and discover that it is a point of inflection, in that case, it is both a turning point and a point of inflection, **however, it doesn't have to be a turning point for it to be a point of inflection, once the second derivative is equal to zero, it is a point of inflection.**

I sincerely hope that is clear???

SECTION B

Larger questions over here, let's solve them all in a bit.

Question 1

(a) State the necessary and sufficient conditions for relative optima of a function: $y = f(x)$

(b) Given that $Y = \ln x_1 + 2\ln x_2$ subject to $2x_1 + 3x_2 = 18$. Obtain the value of x_1 and x_2 that maximizes the function given that the second order condition is satisfied.

(c) Find the relative optimum values of the function:

$$Z = f(x, y) = x^2 + y^2 + 4x - 8y$$

(a)

The conditions are in your notes; they are the allied calculus conditions for relative optima.

For stationary points on a given function,

$$y = f(x)$$

where:

$$\frac{dy}{dx} = 0 \quad \text{or} \quad f'(x) = 0$$

If;

$$\frac{d^2y}{dx^2} > 0, f''(x) > 0$$

Then the stationary point is **a minimum point.**

If;

$$\frac{d^2y}{dx^2} < 0, f''(x) < 0$$

Then the stationary point is **a maximum point.**

If;

$$\frac{d^2y}{dx^2} = 0, f''(x) = 0$$

Then the stationary point is **a point of inflexion**.

(b)

The question is just like:

Maximize $Y = \ln x_1 + 2\ln x_2$
subject to $2x_1 + 3x_2 = 18$

Before we continue, we have a very critical statement here;

given that the second order condition is satisfied.

Here, it means that we don't have to go ahead proving that it is a maximum case of optimization using the second order conditions, hence, we'd just optimize it and conclude. It means we are safe to use the method of Lagragean multipliers since we won't have to check for the second order conditions. Understood?

Let's form our Lagragean equation;

Here, the objective function is:

$$\ln x_1 + 2\ln x_2$$

The constraint function is:

$$2x_1 + 3x_2 - 18 = 0$$

Note that it has been equated to zero;

Hence, the Lagrangean equation is;

$$\mathcal{L}(x_1, x_2, \lambda) \\ = \ln x_1 + 2\ln x_2 - \lambda(2x_1 + 3x_2 - 18)$$

Take the first order partials;

$$\mathcal{L}_{x_1} = \frac{1}{x_1} + 0 - \lambda(1 \times 2x_1^{(1-1)} + 0 - 0)$$

I'm sure you should know that the derivative of $\ln x_1$ is $\frac{1}{x_1}$ and derivative of $2x_1$ is 2, all partially with respect to x_1

$$\mathcal{L}_{x_1} = \frac{1}{x_1} - 2\lambda$$

$$\mathcal{L}_{x_2} = 0 + 2\left(\frac{1}{x_2}\right) - \lambda(0 + 1 \times 3x_2^{(1-1)} - 0)$$

$$\mathcal{L}_{x_2} = \frac{2}{x_2} - 3\lambda$$

$$\mathcal{L}_{\lambda} = 0 + 0 - 1 \times \lambda^{1-1}(2x_1 + 3x_2 - 18)$$

$$\mathcal{L}_{\lambda} = -2x_1 - 3x_2 + 18$$

Equate each to zero;

$$\frac{1}{x_1} - 2\lambda = 0 \ldots \ldots \ldots (1)$$

$$\frac{2}{x_2} - 3\lambda = 0 \ldots \ldots \ldots (2)$$

$$-2x_1 - 3x_2 + 18 = 0$$
$$-2x_1 - 3x_2 = -18$$

Divide through by -1

$$2x_1 + 3x_2 = 18 \ldots \ldots \ldots (3)$$

(1) and (2) can be easily sorted out; you have to calm down and understand this this time around since you can't use crammer's rule for this situation here;

From (1);

$$\frac{1}{x_1} = 2\lambda$$

From (2);

$$\frac{2}{x_2} = 3\lambda$$

$$\mathbf{3} \times (\mathbf{1}): \frac{3}{x_1} = 6\lambda$$

And;

$$\mathbf{2} \times (\mathbf{2}): \frac{4}{x_2} = 6\lambda$$

Hence;
We can conclude that:

$$\frac{3}{x_1} = \frac{4}{x_2}$$

Since both are equal to 6λ;
Cross multiplying;

$$3x_2 = 4x_1$$

Hence; dividing through;

$$x_2 = \frac{4x_1}{3} \dots \dots \dots (4)$$

We can now substitute (4) into (3);

$$2x_1 + 3\left(\frac{4x_1}{3}\right) = 18$$

Clearing fractions;

$$2x_1 + 4x_1 = 18$$

Hence;

$$6x_1 = 18$$
$$x_1 = 3$$

From (4);

$$x_2 = \frac{4x_1}{3} = \frac{4(3)}{3}$$

$$x_2 = 4$$

Let's solve for λ for full marks;
From (1);

$$\frac{1}{x_1} = 2\lambda$$

Hence;

$$2\lambda = \frac{1}{3}$$

Solving;

$$\lambda = \frac{1}{6}$$

Hence, from the need of our question, we were told to find the values of x_1 and x_2 that maximizes Y;

The values are;

$$x_1 = 3$$
$$x_2 = 4$$

We were not told to find that maximum value hence we needn't substitute into Y.

(c)

$$Z = f(x, y) = x^2 + y^2 + 4x - 8y$$

For a multivariate function, we can also find the relative optima. We touched it under differentiation applications – Pg. 32.

We'll evaluate the first order partials;

$$Z_x = 2 \times x^{2-1} + 0 + 1 \times 4x^{1-1} - 0$$

$$Z_x = 2x + 4$$

$$Z_y = 0 + 2 \times y^{2-1} + 0 - 1 \times 8y^{1-1}$$

$$Z = 2y - 8$$

We find the stationary points for x and y separately;

$$Z_x = 2x + 4 = 0$$
$$2x = -4$$
$$x = -2$$

$$Z_y = 2y - 8$$
$$2y = 8$$
$$y = 4$$

We'd test for their natures separately as well using the second order conditions;

Test for x, take the second derivative of Z_x with respect to x;

$$Z_{xx} = \frac{d}{dx}(Z_x)$$

$$Z_{xx} = \frac{d}{dx}(2x + 4)$$

$$Z_{xx} = 1 \times 2x^{1-1} + 0 = 2$$

Now, since the second derivative gives a straight constant positive value, we needn't do any substitution as for any value, it'll be the same constant value, it shows that the **singular stationary point ($x = -2$) is a minimum point.**

Test for y, take the second derivative of Z_y with respect to y;

$$Z_{yy} = \frac{d}{dy}(Z_y)$$

$$Z_{yy} = \frac{d}{dy}(2y - 8)$$

$$Z_{yy} = 1 \times 2y^{1-1} - 0 = 2$$

Again, the second derivative gives a straight constant positive value, we needn't do any substitution as for any value, it'll be the same constant value, it shows that the **singular stationary point ($y = 4$) is a minimum point.**

Since we need the optimum values; we have a minimum for x at $x = -2$ and we also have to take the minimum point for y at $y = 4$, since both are ***correspondingly minimum***; we can have a minimum point for the function as a whole;

To get the minimum value of the function, we'll take our minimum points to the **main function** which are corresponding minimums, hence **the minimum point is $\boldsymbol{x = -2}$ and $\boldsymbol{y = 4}$**

$$Z = f(x, y) = x^2 + y^2 + 4x - 8y$$

$$Z = f(-2,4) = (-2)^2 + (4)^2 + 4(-2) - 8(4)$$

$$Z = f(-2,4) = 4 + 16 - 8 - 32$$

$$Z = f(-2,4) = -20$$

Hence, the minimum value of the function is -20;

Question 2

(a) Explain the basic difference between Differential calculus and integral calculus.

(b) Explain the following:

(i) $\int_{9}^{1} X\sqrt{1+2X}\, dX$

(ii) $\int_{5}^{5} \frac{X^4}{e^X+9}\, dX$

(iii) $\int 3X^2 \sin(X^3+1)\ dX$

(iv) $\int_{1}^{9} 3\sqrt{Z-1}\, dZ$

(v) $\int 2Xe^{X^2}\ dX$

(c) Find:

$$\int \frac{x+1}{x^2+2x+6}\, dx$$

These questions are a little bit thick;

(a)

Differential calculus is the process of finding the derivative of a function while **integral**

calculus is the process of finding a function from its derivative.

Differentiation deals with the rate of change of a quantity with respect to another, while **integration** deals with the area under the curve of a function. The two processes are reverse processes with respect to each other.

(b)

Integrals;

(i)

$$\int_{9}^{1} X\sqrt{1+2X}\, dX$$

Looking at this, we can't resolve this one except by integration it **by parts;**

By normal wisdom, we'd make X to be u so that it can just be differentiated straight into 1, a constant;

$$u = X$$

And hence,

$$\frac{dv}{dX} = \sqrt{1+2X}$$

So, we'll deal with each separately, we have to differentiate u with respect to X and to integrate $\frac{dv}{dX}$ with respect to x as well.

$$u = X$$

Hence,

$$\frac{du}{dX} = 1 \times X^{1-1} = 1$$

And:

$$\frac{dv}{dX} = \sqrt{1 + 2X}$$

Hence,

$$dv = \sqrt{1 + 2X}dX$$

Integrate!

$$\int dv = \int (1 + 2X)^{\frac{1}{2}} dX$$

We require a linear substitution;

$$z = 1 + 2X$$
$$\frac{dz}{dX} = 2$$

Hence;

$$dX = \frac{dz}{2}$$

Hence, the integral becomes;

$$\int dv = \int (z)^{\frac{1}{2}} \frac{dz}{2}$$

$$v = \frac{1}{2} \int z^{\frac{1}{2}} \, dz$$

Power rule;

$$v = \frac{1}{2} \left[\frac{z^{\frac{1}{2}+1}}{\frac{1}{2} + 1} \right] = \frac{1}{2} \left[\frac{z^{\frac{3}{2}}}{\frac{3}{2}} \right]$$

$$v = \frac{1}{2} \times \frac{2}{3} \times z^{\frac{3}{2}} = \frac{1}{3} z^{\frac{3}{2}}$$

Return z;

$$v = \frac{1}{3}(1 + 2X)^{\frac{3}{2}}$$

Hence, we have all we need; rush to the integration by parts formula making the appropriate substitutions for **all terms**:

$$\int u \frac{dv}{dX} dX = uv - \int v \frac{du}{dX} dX$$

$$\int X\sqrt{1+2X}\,dX$$
$$= X\left(\frac{1}{3}(1+2X)^{\frac{3}{2}}\right) - \int \frac{1}{3}(1+2X)^{\frac{3}{2}}(1)\,dX$$

$$\int X\sqrt{1+2X}\,dX = \frac{X}{3}(1+2X)^{\frac{3}{2}} - \int \frac{1}{3}(1+2X)^{\frac{3}{2}}\,dX$$

Hence, we need the integral;

$$\int \frac{1}{3}(1+2X)^{\frac{3}{2}}\,dX$$

To complete the solution;

Similar linear substitution;

$$z = 1 + 2X$$
$$\frac{dz}{dX} = 2$$

Hence;

$$dX = \frac{dz}{2}$$

Hence, the integral becomes;

$$\int \frac{1}{3}(z)^{\frac{3}{2}}\frac{dz}{2}$$

Bring all constants OUT!

$$\frac{1}{3} \times \frac{1}{2} \int z^{\frac{3}{2}}\, dz$$

Power rule;

$$\frac{1}{6}\left[\frac{z^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right] = \frac{1}{6}\left[\frac{z^{\frac{5}{2}}}{\frac{5}{2}}\right]$$

$$\frac{1}{6} \times \frac{2}{5} \times z^{\frac{5}{2}} = \frac{1}{15} z^{\frac{5}{2}}$$

Return z;

$$\int \frac{1}{3}(1+2X)^{\frac{3}{2}}\, dX = \frac{1}{15}(1+2X)^{\frac{5}{2}}$$

Tracing back everything;

$$\int X\sqrt{1+2X}\, dX = \frac{X}{3}(1+2X)^{\frac{3}{2}} - \int \frac{1}{3}(1+2X)^{\frac{3}{2}}\, dX$$

Input the last integral;

$$\int X\sqrt{1+2X}\, dX = \frac{X}{3}(1+2X)^{\frac{3}{2}} - \left[\frac{1}{15}(1+2X)^{\frac{5}{2}}\right]$$

Hence;

$$\int X\sqrt{1+2X}\,dX = \frac{X(1+2X)^{\frac{3}{2}}}{3} - \frac{(1+2X)^{\frac{5}{2}}}{15}$$

Let’s now input the limits; we’ll have:

$$\int_{9}^{1} X\sqrt{1+2X}\,dX = \left[\frac{X(1+2X)^{\frac{3}{2}}}{3} - \frac{(1+2X)^{\frac{5}{2}}}{15}\right]_{9}^{1}$$

We have:

$$\int_{9}^{1} X\sqrt{1+2X}\,dX$$

$$= \left(\frac{1\big(1+2(1)\big)^{\frac{3}{2}}}{3} - \frac{\big(1+2(1)\big)^{\frac{5}{2}}}{15}\right) - \left(\frac{9\big(1+2(9)\big)^{\frac{3}{2}}}{3} - \frac{\big(1+2(9)\big)^{\frac{5}{2}}}{15}\right)$$

$$\int_{9}^{1} X\sqrt{1+2X}\,dX$$

$$= \left(\frac{(3)^{\frac{3}{2}}}{3} - \frac{(3)^{\frac{5}{2}}}{15}\right) - \left(\frac{9(19)^{\frac{3}{2}}}{3} - \frac{(19)^{\frac{5}{2}}}{15}\right)$$

CALCULATOR!!!!!

$$\int_9^1 X\sqrt{1+2X}\,dX$$
$$= \left(\frac{5.196}{3} - \frac{15.588}{15}\right) - \left(\frac{9(82.819)}{3} - \frac{(1573.562)}{15}\right)$$

Finish up!

$$\int_9^1 X\sqrt{1+2X}\,dX$$
$$= (1.732 - 1.039) - (248.457 - 104.904)$$

$$\int_9^1 X\sqrt{1+2X}\,dX = 1.732 - 1.039 - 248.457 + 104.904$$

$$\int_9^1 X\sqrt{1+2X}\,dX = -142.86$$

WOW!

(ii)

$$\int_5^5 \frac{X^3}{e^X + 9}\,dX$$

WOW! This is a question I love so much – a question a million people will adjudge to have no solution.

Rightly as we established in the textbook section of this app, not all functions have an integral that can be expressed as their anti-derivative (i.e. not all functions can be integrated using integral rules), and as a matter of fact, the function here is just an example of such function – it actually cannot be integrated.

However, looking at this, we see a light into how to solve the **definite integral problem**, since the situation is an integral with **equal limits.**

Hence, we'd do this, we will assume thus:
Let:

$$\int \frac{X^3}{e^X + 9}\, dX = h(X)$$

Since we know we cannot arrive at the integral, we assume $h(X)$ is the final result of integrating the function, however, we know that we cannot find out what $h(X)$ really is; but then, **with the assumption above,** we'd have:

Applying the limits;

$$\int_{5}^{5} \frac{X^3}{e^X + 9}\, dX = [h(X)]_5^5$$

From the rule of limits; we'd have:

$$\int_{5}^{5} \frac{X^3}{e^X + 9}\, dX = h(5) - h(5)$$

$$\int_{5}^{5} \frac{X^3}{e^X + 9}\, dX = 0$$

Hence, the final undisputable answer is **zero!**

As a matter of rule actually, when a definite integral is done with both the upper and lower limit equal, the value **is always zero!**

(iii)

$$\int 3X^2 \sin(X^3 + 1)\, dX$$

This is a case of integration by substitution, from our textbook, a **CASE 3** situation;

Substitute;

$$u = X^3 + 1$$

$$\frac{du}{dX} = 3 \times X^{3-1} + 0 = 3X^2$$

Hence;

$$dX = \frac{du}{3X^2}$$

The integral becomes;

$$\int 3X^2 \sin(u) \frac{du}{3X^2}$$

$3X^2$ cancels off; we have:

$$\int \sin(u)\, du$$

Standard integral;

$$-\cos u$$

Return u plus fix arbitrary constant;

$$-\cos(X^3 + 1) + C$$

(iv)

$$\int_{1}^{9} 3\sqrt{Z-1}\,dZ$$

Linear substitution;

$$u = Z - 1$$

$$\frac{du}{dZ} = 1 \times Z^{1-1} - 0 = 1$$

And;

$$dZ = du$$

Hence;
The integral becomes;

$$\int 3\sqrt{u}\,du$$

Constant OUT!

$$3\int \sqrt{u}\,du$$

Power rule;

$$3\left[\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right] = 3\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]$$

$$3 \times \frac{2}{3} u^{\frac{3}{2}} = 2u^{\frac{3}{2}}$$

Return u;

$$2(Z-1)^{\frac{3}{2}}$$

No need of arbitrary constant, so let's input the limits;

$$\int_1^9 3\sqrt{Z-1}\,dZ = \left[2(Z-1)^{\frac{3}{2}}\right]_1^9$$

$$\int_1^9 3\sqrt{Z-1}\,dZ = \left(2(9-1)^{\frac{3}{2}}\right) - \left(2(1-1)^{\frac{3}{2}}\right)$$

$$\int_1^9 3\sqrt{Z-1}\,dZ = \left(2(8)^{\frac{3}{2}}\right) - \left(2(0)^{\frac{3}{2}}\right)$$

$$\int_1^9 3\sqrt{Z-1}\,dZ = \left(2(2^3)^{\frac{3}{2}}\right) - 0$$

$$\int_1^9 3\sqrt{Z-1}\,dZ = \left(2 \times 2^{\frac{9}{2}}\right)$$

Calculator!

$$\int_1^9 3\sqrt{Z-1}\,dZ = (2 \times 22.627) = 45.25$$

(v)

$$\int 2Xe^{X^2}\,dX$$

Another case of substitution, a **CASE 3** from our textbook section.

Substitute;

$$u = X^2$$

$$\frac{du}{dX} = 2 \times X^{2-1} = 2X$$

Hence;

$$dX = \frac{du}{2X}$$

The integral becomes;

$$\int 2Xe^{u}\frac{du}{2X}$$

$2X$ cancels off; we have:

$$\int e^u du$$

Standard integral;

$$e^u$$

Return u plus fix arbitrary constant;

$$e^{X^2} + C$$

(c)

$$\int \frac{x+1}{x^2+2x+6} dx$$

Very simple **(CASE ONE)** substitution integral;

This is because the derivative of the denominator is $2x + 2$ which can be factorized into $2(x + 1)$ and will turn out to be in terms of the numerator; hence, we substitute appropriately;

$$u = x^2 + 2x + 6$$

$$\frac{du}{dx} = 2 \times x^{2-1} + 1 \times 2x^{1-1} + 0$$

$$\frac{du}{dx} = 2x + 2 = 2(x + 1)$$

Hence;

$$dx = \frac{du}{2(x + 1)}$$

Hence, the integral becomes;

$$\int \frac{x + 1}{u} \frac{du}{2(x + 1)}$$

$(x + 1)$ cancels off. We have:

$$\int \frac{1}{u} \frac{du}{2}$$

Constant OUT!

$$\frac{1}{2} \int \frac{1}{u}\, du$$

Standard integral;

$$\frac{1}{2} \ln u$$

Return u;

$$\frac{1}{2} \ln(x^2 + 2x + 6) + C$$

Arbitrary constant is never forgotten!

Question 3

(a) Distinguish between ordinary and partial differential equations.

(b) (i) State Euler's theorem.

(ii) Use Euler's theorem to determine the degree of homogeneity of the function:

$$z = x^2 + 6xy + y^2$$

(c) (i) What is harmonic function.

(ii) Is the function $z = \sin x \cos y$ harmonic?

(a)

An ordinary differential equation is a differential equation involving one independent and one dependent variable. For example, an ordinary equation in y and x (with y the dependent variable and x the independent variable) will involve only differential coefficients of $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ and so on. A partial differential equation **however** involves partial derivatives. It is a differential equation involving a dependent variable and more than one independent variable. For example, partial differential equations will include differential coefficients of $\frac{\partial y}{\partial x}, \frac{\partial y}{\partial t}$ in a single equation.

(b)

(i)

Euler's theorem states that the sum of the product of the arguments and their first order partial derivatives of a function is equal to the degree of homogeneity of a function multiplying the function itself.

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$$

where n is the degree of homogeneity;

(ii)

$$z = x^2 + 6xy + y^2$$

$$\frac{\partial z}{\partial x} = 2x^{2-1} + 1 \times 6x^{1-1}y + 0$$

$$\frac{\partial z}{\partial x} = 2x + 6y$$

$$\frac{\partial z}{\partial y} = 0 + 1 \times 6xy^{1-1} + 2y^{2-1}$$

$$\frac{\partial z}{\partial y} = 6x + 2y$$

Hence,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

$$x(2x + 6y) + y(6x + 2y) = n(x^2 + 6xy + y^2)$$

Expand!

$$2x^2 + 6xy + 6xy + 2y^2 = n(x^2 + 6xy + y^2)$$

Simplify;

$$2x^2 + 12xy + 2y^2 = n(x^2 + 6xy + y^2)$$

Factorization can be done on the left hand side;

$$2(x^2 + 6xy + y^2) = n(x^2 + 6xy + y^2)$$

Hence, by comparison;

$$n = 2$$

(c)

(i)

A harmonic function is a continuous twice differentiable function whose sum of all its direct second order derivatives is equal to zero; the harmonic function doesn't hold for all twice differentiable functions, hence, only certain types of functions are harmonic.

For a harmonic function; $f(x, y)$;

$$f_{xx} + f_{yy} = 0$$

(ii)

$$z = \sin x \cos y$$

For a harmonic function:

$$(z)_{xx} + (z)_{yy} = 0$$

Hence, we'll be finding the direct second order partial derivatives;

$$z = \sin x \cos y$$

$$z_x = (\cos x)\cos y$$

Please ensure you've read partial derivatives before coming here; You'll look totally lost if you've not read partial derivatives;

$$z_y = \sin x\,(-\sin y)$$

$$z_y = -\sin x \sin y$$

As a reminder, when differentiating with respect to x, y and in essence, $\cos y$ is nothing but a constant, same in differentiating with respect to y, x and in essence, $\sin x$ is nothing but a constant.

Going further;

$$(z)_{xx} = \frac{\partial}{\partial x}(z_x) = \frac{\partial}{\partial x}\left((\cos x)\cos y\right)$$

$$z_{xx} = (-\sin x)\cos y$$

$$z_{xx} = -\sin x \cos y$$

$$z_{yy} = \frac{\partial}{\partial y}\left(z_y\right) = \frac{\partial}{\partial y}(-\sin x \sin y)$$

$$z_{yy} = -\sin x\,(\cos y)$$

$$z_{yy} = -\sin x \cos y$$

Hence;

$$z_{xx} + z_{yy}$$

$$-\sin x \cos y + (-\sin x \cos y)$$

$$z_{xx} + z_{yy} = -2\sin x \cos y$$

Hence, the function is not harmonic!

Question 4

(a) Differentiate among diagonal matrix, scalar matrix and identity matrix.

(b) What is orthogonal matrix? Identify the salient features of orthogonal matrix.

(c) Define the term Trace of a matrix. If *A* and *B* are square matrices of the same order, show that:

$$tr(A + B) = trA + trB$$

(d) If the linear market model of demand and supply for one commodity is of the form:

$Qd = a + bP$ (demand relation)

$Qs = c + dP$ (supply relation)

$Q_d = Q_s = Q$ (equilibrium relation)

Determine the equilibrium price and quality variables using Crammer's rule.

(a)

The best way to distinguish between the three is to define the three of them separately.

A diagonal matrix is a square matrix that is such that every element that is not on its main diagonal is equal to zero. Example below:

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

As for the scalar matrix, we didn't mention this in the notes section, however, I'd explain it shortly here.

A scalar matrix is a special type of diagonal matrix where all the elements not on the main diagonal is zero and every element on the main diagonal is the same scalar value. It is always equivalent to λI where λ is the same-valued element on the main diagonal and I is the identity matrix. Examples below:

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3I$$

$$\begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{pmatrix} = -5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -5I$$

An identity matrix is a special type of diagonal matrix where all the elements not on the main diagonal is zero and every element on the main diagonal is unity (1). An identity matrix is shown below:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, summarily, the three types of matrix are similar. In the three (i.e. diagonal, scalar and identity) matrices, elements not on the main diagonal are equal to zero. In the diagonal matrix, the main diagonal contains any type of scalar, in the scalar, the main diagonal contains the same scalar value and in the identity matrix, the main diagonal contains the value 1.

(b)

An orthogonal matrix is a square matrix whose transpose is equal to its inverse; it is a special type of matrix related to both the transpose of a matrix and the inverse of a matrix.

For an orthogonal matrix;

$$A^T = A^{-1}$$

Hence;

$$AA^T = I$$

The orthogonal matrix has two major (salient) features;

- The first is that the transpose of the matrix is equal to its inverse;

- And of course, the determinant of the matrix is -1 or 1.

These two above properties were mentioned in the textbook section.

(c)

The trace of a matrix is a unique operation for square matrices; the **trace of a matrix** is the sum of all elements on the main diagonal (the diagonal from the upper left to the lower right).

For the second part;

To show this, the only way is to represent primitive (sample) matrices;
Let;

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \text{ and } B = \begin{pmatrix} m & n & o \\ p & q & r \\ s & t & u \end{pmatrix}$$

Then $A + B$ is;

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} + \begin{pmatrix} m & n & o \\ p & q & r \\ s & t & u \end{pmatrix}$$

$$A + B = \begin{pmatrix} a+m & b+n & c+o \\ d+p & e+q & f+r \\ g+s & h+t & i+u \end{pmatrix}$$

Then; $\mathrm{tr}(A + B)$ is (the sum of the terms on the diagonal):

$$\mathrm{tr}(A + B) = (a + m) + (e + q) + (i + u)$$
$$\mathrm{tr}(A + B) = a + m + e + q + i + u$$

Then, let's find the separate traces and add;

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \text{ and } B = \begin{pmatrix} m & n & o \\ p & q & r \\ s & t & u \end{pmatrix}$$

$$\mathrm{tr}(A) = a + e + i$$

$$\mathrm{tr}(B) = m + q + u$$

Then,

$$\mathrm{tr}(A) + \mathrm{tr}(B) = a + e + i + m + q + u$$

If you rearrange the terms as you like (added terms can be arranged anyhow)

$$\mathrm{tr}(A) + \mathrm{tr}(B) = a + m + e + q + i + u$$

Hence, it is proved that: $\mathrm{tr}(A + B) = \mathrm{tr}A + \mathrm{tr}B$

(d)

$Qd = a + bP$ (demand relation)

$Qs = c + dP$ (supply relation)

$Q_d = Q_s = Q$ (equilibrium relation)

Now let's take this **gradually;**
The last equation (equilibrium relation) is an equation you should know already, from our studies, the market is at equilibrium when **the demand and supply have a common value** (i.e. when demand is equal to supply), hence, that is what we have there;

$Qd = a + bP$ (demand relation)

$Qs = c + dP$ (supply relation)

Hence, at equilibrium;

$$Q_d = Q_s = Q$$

Hence;

The demand and supply equations at equilibrium quantity will hence, be given by:

$Q = a + bP$ (demand relation)

$Q = c + dP$ (supply relation)

Now, we are told to attack the above equation using Crammer's rule; we have the statement in the question:

Determine the equilibrium price and quality variables using Crammer's rule.

I believe quality there meant "quantity."

Hence, we need to solve for the value of Q, the equilibrium quantity and value of P, the equilibrium price.

Hence, we have to rearrange our equation to put the needed unknown variables $(Q$ and $P)$on the left and others on the right;

$Q - bP = a$ (demand relation)

$Q - dP = c$ (supply relation)

Solving both simultaneously to find Q and P **using the required Crammer's rule,** we have:

$$\Delta = \begin{vmatrix} 1 & -b \\ 1 & -d \end{vmatrix}$$

I hope the above is clear, coefficients of the needed variables are what we need in Crammer's rule.

$$\Delta = (1 \times -d) - (-b \times 1) = -d + b$$
$$\Delta = b - d$$

To find Δ_Q, replace the column of Q with the column of solutions;

$$\Delta_Q = \begin{vmatrix} a & -b \\ c & -d \end{vmatrix} = (a \times -d) - (-b \times c)$$

$$\Delta_Q = bc - ad$$

To find Δ_P, replace the column of P with the column of solutions;

$$\Delta_P = \begin{vmatrix} 1 & a \\ 1 & c \end{vmatrix} = (1 \times c) - (a \times 1)$$

$$\Delta_P = c - a$$

Hence;
Equilibrium price, P;

$$P = \frac{\Delta_P}{\Delta} = \frac{c - a}{b - d}$$

Equilibrium quantity, Q;

$$Q = \frac{\Delta_Q}{\Delta} = \frac{bc - ad}{b - d}$$

THAT WRAPS UP THE MOST RECENT PAST QUESTIONS WE HAVE!