DEPARTMENT OF ECONOMICS
FACULTY OF SOCIAL SCIENCES
OBAFEMI AWOLOWO UNIVERSITY, ILE-IFE, NIGERIA
SSC106: MATHEMATICS FOR SOCIAL SCIENCES II
RAIN SEMESTER EXAMINATION (2007/2008 SESSION)

INSTRUCTIONS:

- Attempt all questions in **Section A** and any two questions from **Section B.**
- Show all workings clearly

Time allowed: 2 hours

SECTION A

1. Enumerate three techniques of evaluating determinants of matrices in increasing order of generality.
2. Distinguish between a vector and a scalar.
3. State the conformability condition for matrix multiplication.
4. Differentiate between explicit and implicit functions.

5. Toyin asserts that the following functions are unrelated:

$$Z_1 = 3x - y$$

and

$$Z_2 = 9x^2 - 6xy + y^2$$

Is Toyin's assertion correct?

6. Outline the allied Calculus conditions for relative optima.

7. State Young's Theorem.

8. If a firm's average production function is

$$AP = \frac{200}{L} + 5L - \frac{L^2}{4}$$

   (i) Derive the firm's production function;
   (ii) Compute its marginal product when $L = 10$.

9. Obtain the sum of the direct second-order partial derivatives of the function: $Z = ax^2 + bxy + cy^2 + dx + ey + f$.

10. Integrate with respect to x the expression: $(3ax^2 + 2bx + c)e^{ax^3+bx^2+cx+d}$

11. State the Young's theorem.

12. If $Z = e^{\log x \log y}$, evaluate $\frac{\partial^2 Z}{\partial x \partial y}$ and $\frac{\partial^2 Z}{\partial y \partial x}$

SECTION B

1. *(a)* Distinguish between orthogonal and idempotent matrices.

   (b) If A and B are two orthogonal square matrices of the same order, prove that their product, either way, is also orthogonal.

   (c) Solve the following linear equation system by Crammer's rule.

   $3x_1 + 2x_2 = c_1$
   $4x_1 + x_2 = c_2$, where $c_1 = 1$ and $c_2 = 3$

2. *(a)* Differentiate with respect to x:

   (i) $\log_e(\log_e x)$

   (ii) $\log_e(x^2 + 3x)$

   (iii) $e^{\sin x}$

   (iv) $(x^2 - 2x)e^x$

(b) If $x = r \sin\theta$ and $y = r\cos\theta$, show that:

(i) $$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2$$

(ii) $$\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2 = 1$$

3. *(a)* Use Lagrangean method to find the stationary value of $Z = xy + 2x$ subject to $4x + 2y = 60$.

(b) Integrate the following expression:

(i) $$\frac{x^n}{1+n}$$

(ii) $$\frac{\cos x}{1+\sin x}$$

(iii) $$x^4 - x^3 - x^2 + \frac{1}{x} - 1$$

SOLUTION TO THE PAST QUESTIONS

All topics needed here have been thoroughly treated through and through, all you need to do is to sit down and answer all these questions like a joke, you are not expected to start reading from here but from the topics, you make a joke of yourself when you read the concept of any subject starting from the side of the past question; more so, this text isn't a past question but a complete guide to lead you to a straight A in your SSC106 exam.

Hence, keep calm and check the solutions to these past questions one-by-one;

GOOD LUCK AND GOD'S BEST!

SOLUTION TO THE SSC106 EXAMINATION 2007/2008 ACADEMIC SESSION

The instruction is you answer all questions in the **Section A** and only two questions from Section B, we'll be answering everything in both sections in this book though; keep calm and move with me;

Remember every singular solution here, you can get the full concept by reading this book and not the past question section exclusively;

SECTION A

Question 1

Enumerate three techniques of evaluating determinants of matrices in increasing order of generality.

Well, this is a simple question, you know all the method but this one you're been asked in increasing order of generality, that may prove tricky. So, this is it; the open scissors technique is used at virtually all times since it is even used when finding the determinants of 3×3 matrices and higher, hence, it is the most general, the Laplace expansion technique is next since it is used for 3×3 matrices and higher, the Sarrus method is the least common, it is only limited to 3×3 matrices and as a matter of fact, is mostly learnt for the sake of formalities. Hence; we have the list in increasing order of generality as:

- Open scissors technique
- The Laplace expansion method
- The Sarrus method.

Question 2

Distinguish between a vector and a scalar.

I mentioned this in the textbook in the studies of scalar multiplication in matrices;

A **vector** has magnitude and direction while a **scalar** has magnitude only. **Full stop!**

Question 3

State the conformability condition for matrix multiplication.

The conformability condition for matrix multiplication states that for two matrices to be multiplied; the number of columns in the pre-multiplier must be equal to the number of rows in the post-multiplier.

Question 4

Differentiate between explicit and implicit functions.

An **explicit function** is stated solely in terms of the independent variable(s) and the direction of the function is certain while the **implicit function** isn't expressed solely in terms of any variable and the direction of the function is not certain.

Question 5

Toyin asserts that the following functions are unrelated:

$$Z_1 = 3x - y$$

and

$$Z_2 = 9x^2 - 6xy + y^2$$

Is Toyin's assertion correct?

The Toyin story is just formalities, you're just asked to check if the functions are related (or dependent). If you find out that they're related, you conclude Toyin's assertion is incorrect and if they aren't related, you conclude it is a correct assertion.

All we need to do is to solve for the Jacobian determinant and see its nature;

Taking the first order partials;

$$\frac{\partial Z_1}{\partial x} = 3x^{1-1} - 0 = 3$$

$$\frac{\partial Z_1}{\partial y} = 0 - 1 \times 1y^{1-1} = -1$$

$$\frac{\partial Z_2}{\partial x} = 2 \times 9x^{2-1} - 1 \times 6x^{1-1}y + 0$$

$$\frac{\partial Z_2}{\partial x} = 18x - 6y$$

$$\frac{\partial Z_2}{\partial y} = 0 - 1 \times 6xy^{1-1} + 2 \times y^{2-1}$$

$$\frac{\partial Z_2}{\partial y} = -6x + 2y$$

Form the Jacobian matrix;

$$J = \begin{pmatrix} \frac{\partial Z_1}{\partial x} & \frac{\partial Z_1}{\partial y} \\ \frac{\partial Z_2}{\partial x} & \frac{\partial Z_2}{\partial y} \end{pmatrix}$$

$$J = \begin{pmatrix} 3 & -1 \\ 18x - 6y & -6x + 2y \end{pmatrix}$$

Evaluate the Jacobian determinant;

$$|J| = 3(-6x + 2y) - (-1)(18x - 6y)$$

$$|J| = -18x + 6y + 18x - 6y$$

$$|J| = 0$$

Hence, the two functions are dependent (related);

And then Toyin's assertion is incorrect!

Question 6

Outline the allied Calculus conditions for relative optima.

The conditions are in your notes;

For stationary points on a given function, $y = f(x)$; where:

$$\frac{dy}{dx} = 0 \quad \text{or} \quad f'(x) = 0$$

If;

$$\frac{d^2y}{dx^2} > 0, f''(x) > 0$$

Then the stationary point is **a minimum point.**
If;

$$\frac{d^2y}{dx^2} < 0, f''(x) < 0$$

Then the stationary point is **a maximum point.**

If;

$$\frac{d^2y}{dx^2} = 0, f''(x) = 0$$

Then the stationary point is **a point of inflexion**.

Question 7

State Young's Theorem.

It states that two complementary second order mixed partials of a continuous and twice differentiable function are equal; for a multivariate function dependent on x and y;

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

In function notation;

$$f_{xy} = f_{yx}$$

Question 8

If a firm's average production function is

$$AP = \frac{200}{L} + 5L - \frac{L^2}{4}$$

(i) Derive the firm's production function;

(ii) Compute its marginal product when $L = 10$.

We know that the average function for any function is gotten by dividing the function by the unit it depends on, hence, the average production function is given by;

$$AP = \frac{P(L)}{L}$$

Hence;

$$P(L) = AP \times L$$

$$P(L) = \left(\frac{200}{L} + 5L - \frac{L^2}{4}\right) \times L$$

Expanding;

$$P(L) = 200 + 5L^2 - \frac{L^3}{4}$$

Hence, we have gotten the firm's production form, $P(L)$;

To compute its marginal product when $L = 10$; we need the marginal production function first!

$$P'(L) = \frac{d}{dL}\big(P(L)\big) = \frac{d}{dL}\left(200 + 5L^2 - \frac{L^3}{4}\right)$$

$$P'(L) = 0 + 2 \times 5L^{2-1} - 3 \times \frac{L^{3-1}}{4}$$

$$P'(L) = 10L - \frac{3}{4}L^2$$

Hence;
At $L = 10$, the marginal product is:

$$P'(10) = 10(10) - \frac{3}{4}(10)^2$$

$$P'(10) = 100 - 75$$

$$P'(10) = 25$$

Question 9

Obtain the sum of the direct second-order partial derivatives of the function:

$Z = ax^2 + bxy + cy^2 + dx + ey + f.$

The direct second order partial derivatives are:

$$\frac{\partial^2 Z}{\partial x^2} \text{ and } \frac{\partial^2 Z}{\partial y^2}$$

Let's find the above partial derivatives;

$$Z = ax^2 + bxy + cy^2 + dx + ey + f$$

Of course you know everything about partial differentiation, when we're considering x, y is constant and vice versa.

$$Z = ax^2 + bxy + cy^2 + dx + ey + f$$

$$\frac{\partial Z}{\partial x} = 2 \times ax^{2-1} + 1 \times bx^{1-1}y + 0 + 1 \times x^{1-1} + 0 + 0$$

$$\frac{\partial Z}{\partial x} = 2ax + by + d$$

$$\frac{\partial Z}{\partial y} = 0 + 1 \times bxy^{1-1} + 2 \times cy^{2-1} + 0 + 1 \times ey^{1-1} + 0$$

$$\frac{\partial Z}{\partial y} = bx + 2cy + e$$

Going for the second-order direct partials;

$$\frac{\partial^2 Z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial Z}{\partial x}\right) = \frac{\partial}{\partial x}(2ax + by + d)$$

$$\frac{\partial^2 Z}{\partial x^2} = 1 \times 2ax^{1-1} + 0 + 0$$

$$\frac{\partial^2 Z}{\partial x^2} = 2a$$

$$\frac{\partial^2 Z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial Z}{\partial y}\right) = \frac{\partial}{\partial x}(bx + 2cy + e)$$

$$\frac{\partial^2 Z}{\partial y^2} = 0 + 1 \times 2cy^{1-1} + 0$$

$$\frac{\partial^2 Z}{\partial x^2} = 2c$$

Their sum will therefore be:

$$\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2} = 2a + 2c = 2(a + c)$$

Question 10

Integrate with respect to x the expression:
$(3ax^2 + 2bx + c)e^{ax^3+bx^2+cx+d}$

This is an integration substitution of:

$$f'(x)g[f(x)]$$

$$\int (3ax^2 + 2bx + c)e^{ax^3+bx^2+cx+d}\, dx$$

Substitution!

$$u = ax^3 + bx^2 + cx + d$$

$$\frac{du}{dx} = 3 \times ax^{3-1} + 2 \times bx^{2-1} + 1 \times cx^{1-1} + 0$$

$$\frac{du}{dx} = 3ax^2 + 2bx + c$$

Hence;

$$dx = \frac{du}{3ax^2 + 2bx + c}$$

Hence, the integral becomes after substituting u and dx;

$$\int (3ax^2 + 2bx + c)e^u \frac{du}{3ax^2 + 2bx + c}$$

$3ax^2 + 2bx + c$ cancels off;

$$\int e^u \, du$$

Standard integral;

$$e^u + C$$

Arbitrary constant has been added, we now return the true value of u;

$$e^{ax^3+bx^2+cx+d} + C$$

Question 11

This is a repetition of question 7, scroll up and check.

Question 12

If: $Z = e^{\log x \log y}$, evaluate:

$$\frac{\partial^2 Z}{\partial x \partial y} \text{ and } \frac{\partial^2 Z}{\partial y \partial x}$$

This is an exponential function of function situation;

For the derivative with respect to x, we'll be taking y as a constant; we have;

$$\frac{\partial Z}{\partial x} = \frac{\partial}{\partial x}\left(e^{\log x \log y}\right)$$

As usual in functions of functions (chain rule), we need a substitution;

$$u = \log x \log y$$

$\log y$ is a constant in this case, hence, we differentiate $\log x$;

$$\frac{\partial u}{\partial x} = \frac{1}{x \ln 10} \log y = \frac{\log y}{x \ln 10}$$

Hence,

$$Z = e^u$$

$$\frac{\partial Z}{\partial u} = e^u$$

From chain rule;

$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial u} \times \frac{\partial u}{\partial x}$$

$$\frac{\partial Z}{\partial x} = e^u \times \frac{\log y}{x \ln 10}$$

$$\frac{\partial Z}{\partial x} = \frac{e^u \log y}{x \ln 10}$$

Returning the value of u; we have

$$\frac{\partial Z}{\partial x} = \frac{e^{\log x \log y} \log y}{x \ln 10}$$

Using basically the same process but this time, with respect to y and with x taken as a constant, we have;

$$\frac{\partial Z}{\partial y} = \frac{\partial}{\partial y}\left(e^{\log x \log y}\right)$$

We need the same substitution;

$$u = \log x \log y$$

$\log x$ is a constant in this case, hence, we differentiate $\log y$;

$$\frac{\partial u}{\partial y} = \log x \frac{1}{y \ln 10} = \frac{\log x}{y \ln 10}$$

Hence,

$$Z = e^u$$

$$\frac{\partial Z}{\partial u} = e^u$$

From chain rule;

$$\frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial u} \times \frac{\partial u}{\partial y}$$

$$\frac{\partial Z}{\partial y} = e^{u} \times \frac{\log x}{y \ln 10}$$

$$\frac{\partial Z}{\partial y} = \frac{e^{u} \log x}{y \ln 10}$$

Returning the value of u; we have

$$\frac{\partial Z}{\partial y} = \frac{e^{\log x \log y} \log x}{y \ln 10}$$

It'll be quite tedious to find:

$$\frac{\partial^2 x}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 x}{\partial y \partial x}$$

However,
Nothing is skipped in **THE SSC106 WAY;**
hence, we'll do it.

$$\frac{\partial^2 Z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial Z}{\partial y}\right)$$

Hence;

$$\frac{\partial^2 Z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{e^{\log x \log y} \log x}{y \ln 10}\right)$$

This is a product, and not a quotient actually, when we are differentiating with respect to x, we'll have the following (outside the bracket) as constants;

$$\frac{1}{y \ln 10}\left(e^{\log x \log y} \log x\right)$$

Hence, only the terms in the bracket are functions which form a product; we have:

$$u = e^{\log x \log y}$$
$$v = \log x$$

Hence,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(e^{\log x \log y}\right)$$

The derivative above is the same derivative for $\frac{\partial Z}{\partial x}$

Hence;

$$\frac{\partial u_1}{\partial x} = \frac{e^{\log x \log y} \log y}{x \ln 10}$$

$$v = \log x$$

Straight!

$$\frac{\partial v}{\partial x} = \frac{1}{x \ln 10}$$

Hence;

Product rule;

$$\frac{\partial^2 Z}{\partial x \partial y} = \frac{1}{y \ln 10}\left(v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}\right)$$

Notice that we have brought the constants along with it to multiply the derivative of the product; hence;

$$\frac{\partial^2 Z}{\partial x \partial y} = \frac{1}{y \ln 10}\left(\log x \frac{e^{\log x \log y} \log y}{x \ln 10} + e^{\log x \log y}\left(\frac{1}{x \ln 10}\right)\right)$$

Simplifying!

$$\frac{\partial^2 Z}{\partial x \partial y} = \frac{1}{y \ln 10}\left(\frac{(\log x \log y)e^{\log x \log y}}{x \ln 10} + \frac{e^{\log x \log y}}{x \ln 10}\right)$$

Factorize $e^{\log x \log y}$;

$$\frac{\partial^2 Z}{\partial x \partial y} = \frac{e^{\log x \log y}}{y \ln 10}\left(\frac{(\log x \log y)}{x \ln 10} + \frac{1}{x \ln 10}\right)$$

Add the terms in bracket, they have the same denominator;

$$\frac{\partial^2 Z}{\partial x \partial y} = \frac{e^{\log x \log y}}{y \ln 10}\left(\frac{\log x \log y + 1}{x \ln 10}\right)$$

It's as usual no big deal; just try your possible best to follow it one by one!

Next one;

$$\frac{\partial^2 Z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial Z}{\partial x}\right)$$

Hence;

$$\frac{\partial^2 Z}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{e^{\log x \log y} \log y}{x \ln 10}\right)$$

This is a product, and not a quotient actually, when we are differentiating with respect to y, we'll have the following as constants;

$$\frac{1}{x \ln 10}\left(e^{\log x \log y} \log y\right)$$

Hence, only the terms in the bracket are functions which form a product; we have:

$$u = e^{\log x \log y}$$
$$v = \log y$$

Hence,

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(e^{\log x \log y}\right)$$

The derivative above is the same derivative for $\frac{\partial Z}{\partial y}$

Hence;

$$\frac{\partial u_1}{\partial y} = \frac{e^{\log x \log y} \log x}{y \ln 10}$$

$$v = \log y$$

Straight!

$$\frac{\partial v}{\partial y} = \frac{1}{y \ln 10}$$

Hence;

Product rule;

$$\frac{\partial^2 Z}{\partial y \partial x} = \frac{1}{x \ln 10}\left(v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}\right)$$

Notice that we have brought the constants along with it to multiply the derivative of the product; hence;

$$\frac{\partial^2 Z}{\partial y \partial x} = \frac{1}{x \ln 10}\left(\log y\left(\frac{e^{\log x \log y} \log x}{y \ln 10}\right) + e^{\log x \log y}\left(\frac{1}{y \ln 10}\right)\right)$$

Simplifying!

$$\frac{\partial^2 Z}{\partial y \partial x} = \frac{1}{x \ln 10}\left(\frac{(\log x \log y) e^{\log x \log y}}{y \ln 10} + \frac{e^{\log x \log y}}{y \ln 10}\right)$$

Factorize $e^{\log x \log y}$;

$$\frac{\partial^2 Z}{\partial x \partial y} = \frac{e^{\log x \log y}}{x \ln 10}\left(\frac{(\log x \log y)}{y \ln 10} + \frac{1}{y \ln 10}\right)$$

Add the terms in bracket, they have the same denominator;

$$\frac{\partial^2 Z}{\partial x \partial y} = \frac{e^{\log x \log y}}{x \ln 10}\left(\frac{\log x \log y + 1}{y \ln 10}\right)$$

Quite some question! It shouldn't be in section A, perhaps it was put there mistakenly *sha*. It has successfully added to your knowledge though.

SECTION B

I hope you were not thinking we were through already; work has just started.

Question 1

(a) Distinguish between orthogonal and idempotent matrices.

(b) If A and B are two orthogonal square matrices of the same order, prove that their product, either way, is also orthogonal.

(c) Solve the following linear equation system by Crammer's rule.

$3x_1 + 2x_2 = c_1$

$4x_1 + x_2 = c_2$, where $c_1 = 1$ and $c_2 = 3$

(a)

There isn't a direct difference as it were between idempotent and orthogonal matrices; so I guess we'll just define them both.

An idempotent matrix is a matrix which when multiplied by itself is still equal to itself while **an orthogonal matrix** is a square matrix whose transpose is equal to its inverse;

For an idempotent matrix;

$$A^2 = A$$

For an orthogonal matrix, A;

$$A^T = A^{-1}$$

(b)

This has been solved in the notes already;

For an orthogonal matrix, A;

$$A^T = A^{-1}$$

To test for their product, either way! We are to prove that the matrices AB and BA are both orthogonal.

Now, if A and B are orthogonal, it follows that:

$$A^T = A^{-1}$$
$$B^T = B^{-1}$$

To prove that AB and BA are orthogonal, we must test for the value of the transposes of AB and BA;

Hence, for AB; we test for:

$$(AB)^T$$

Now, from transpose rules;

$$(AB)^T = B^T A^T$$

Hence,
We have:

$$(AB)^T = B^T A^T$$

From the fundamental information we have for A and B that they are orthogonal, we know that:

$$A^T = A^{-1}$$
$$B^T = B^{-1}$$

Hence, by substitution for B^T and A^T, we have;

$$(AB)^T = B^{-1} A^{-1}$$

Also, from inverse rules;

$$(AB)^{-1} = B^{-1} A^{-1}$$

Hence, we can hence substitute for $B^{-1} A^{-1}$ in our equation to arrive that:

$$(AB)^T = (AB)^{-1}$$

Hence, it is proved that the transpose of AB is equal to its inverse which is the condition for it to be orthogonal.

Going to the next part, proving for BA;

For BA; we test for:

$$(BA)^T$$

Now, from transpose rules;

$$(AB)^T = B^T A^T$$

Hence,
We have:

$$(BA)^T = A^T B^T$$

From the fundamental information we have for A and B that they are orthogonal, we know that:

$$A^T = A^{-1}$$
$$B^T = B^{-1}$$

Hence, by substitution for A^T and B^T, we have;

$$(BA)^T = A^{-1} B^{-1}$$

Also, from inverse rules; we know that the inverse of a product is the reverse product of their individual inverses:

$$(AB)^{-1} = B^{-1}A^{-1}$$

It follows that:

$$(BA)^{-1} = A^{-1}B^{-1}$$

Hence, we can hence substitute for $A^{-1}B^{-1}$ in our equation to arrive that:

$$(BA)^T = (BA)^{-1}$$

Hence, it is proved that the transpose of BA is equal to its inverse which is the condition for it to be orthogonal.

(c)

$$3x_1 + 2x_2 = c_1$$
$$4x_1 + x_2 = c_2$$

Where $c_1 = 1$ and $c_2 = 3$
Hence, the equations can be restated as:

$$3x_1 + 2x_2 = 1$$
$$4x_1 + x_2 = 3$$

Writing our crammer's rule determinants since this one has been fully arranged as it is the best way it can be arranged;

We can see the coefficients of the variable which we need clearly here; the variables that determine these equations are x_1 and x_2 and hence, we'll be solving for them, for the first determinant, we have it thus;

$$\Delta = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$$

$$\Delta = (3 \times 1) - (4 \times 2) = 3 - 8 = -5$$

We proceed to find the determinant for the first variable; the variable, x_1; replacing the column that contain the coefficients of x_1 with the solutions of the equations;

$$\Delta_{x_1} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

$$\Delta_{x_1} = (1 \times 1) - (3 \times 2) = 1 - 6 = -5$$

To find, Δ_{x_2}, we'll be replacing the column that contain the coefficients of x_2 with the solutions of the equations;

$$\Delta_{x_2} = \begin{vmatrix} 3 & 1 \\ 4 & 3 \end{vmatrix}$$

$$\Delta_{x_2} = (3 \times 3) - (4 \times 1) = 9 - 4 = 5$$

So, here; we have;

$$x_1 = \frac{\Delta_{x_1}}{\Delta} = \frac{-5}{-5} = 1$$

$$x_2 = \frac{\Delta_{x_2}}{\Delta} = -\frac{5}{5} = -1$$

Hence, the solution of the equation is $x_1 = 1$ and $x_2 = -1$;

Question 2

(a) Differentiate with respect to x:

(i) $\log_e(\log_e x)$

(ii) $\log_e(x^2 + 3x)$

(iii) $e^{\sin x}$

(iv) $(x^2 - 2x)e^x$

(b) If $x = r\sin\theta$ and $y = r\cos\theta$, show that:

(i) $$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2$$

(ii) $$\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2 = 1$$

(a)

Some questions these are; we'll be applying rules of differentiation;

(i)

$$\log_e(\log_e x)$$

Let;

$$y = \log_e(\log_e x)$$

Hence;

This is a function of function substitution case;

$$u = \log_e x$$

Straight!

$$\frac{du}{dx} = \frac{1}{x}$$

Hence;

$$y = \log_e u$$

Straight;

$$\frac{dy}{du} = \frac{1}{u}$$

From chain rule;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Hence;

$$\frac{dy}{dx} = \frac{1}{u} \times \frac{1}{x}$$

Return u;

$$\frac{dy}{dx} = \frac{1}{(\log_e x) \times x}$$

$$\frac{dy}{dx} = \frac{1}{x \log_e x}$$

(ii)

$$\log_e(x^2 + 3x)$$

Let;

$$y = \log_e(x^2 + 3x)$$

Hence;

This is a function of function substitution case;

$$u = x^2 + 3x$$

Power rule;

$$\frac{du}{dx} = 2x + 3$$

Hence;

$$y = \log_e u$$

Straight;

$$\frac{dy}{du} = \frac{1}{u}$$

From chain rule;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Hence;

$$\frac{dy}{dx} = \frac{1}{u} \times (2x + 3)$$

Return u;

$$\frac{dy}{dx} = \frac{2x + 3}{x^2 + 3x}$$

(iii)

$$e^{\sin x}$$

Let;

$$y = e^{\sin x}$$

Hence;

This is a function of function substitution case;

$$u = \sin x$$

Straight!

$$\frac{du}{dx} = \cos x$$

Hence;

$$y = e^{u}$$

Straight;

$$\frac{dy}{du} = e^{u}$$

From chain rule;

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Hence;

$$\frac{dy}{dx} = e^{u} \times (\cos x)$$

Return u;

$$\frac{dy}{dx} = e^{\sin x} \cos x$$

(iv)

$$(x^2 - 2x)e^{x}$$

Let;

$$y = (x^2 - 2x)e^{x}$$

Short product rule;

$$u = x^2 - 2x$$

$$v = e^x$$

Here;

$$\frac{du}{dx} = 2x - 2$$

$$\frac{dv}{dx} = e^x$$

Hence;

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

Hence;

$$\frac{dy}{dx} = (e^x)(2x - 2) + (x^2 - 2x)(e^x)$$

Factorize e^x;

$$\frac{dy}{dx} = e^x\left(2x - 2 + (x^2 - 2x)\right)$$

$$\frac{dy}{dx} = e^x(x^2 - 2)$$

(b)

The question has been sorted out in the notes already!

So we have two functions here:

$$x = r\sin\theta$$
$$y = r\cos\theta$$

In (i);

We're told to evaluate both $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ and work out some things on them.

$$x = r\sin\theta$$

$$\frac{dx}{d\theta} = r \times \frac{d}{d\theta}(\sin\theta)$$

$$\frac{dx}{d\theta} = r \times \cos\theta = r\cos\theta$$

$$y = r\cos\theta$$

$$\frac{dy}{d\theta} = r \times \frac{d}{d\theta}(\cos\theta)$$

$$\frac{dy}{d\theta} = r \times -\sin\theta = -r\sin\theta$$

We're told to prove this:

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2$$

To prove, we show that the left hand side is equal to the right hand side of the equation;

To do this, we take the squares of our derivatives just like it is in the question;

$$\left(\frac{dx}{d\theta}\right)^2 = (r\cos\theta)^2 = r^2(\cos\theta)^2 = r^2(\cos^2\theta)$$

$$\left(\frac{dy}{d\theta}\right)^2 = (-r\sin\theta)^2 = r^2(\sin\theta)^2 = r^2(\sin^2\theta)$$

Taking their sum, we have:

$$r^2(\cos^2\theta) + r^2(\sin^2\theta)$$

r^2 is common between both, factorize it:

$$r^2[\cos^2\theta + \sin^2\theta]$$

From trigonometry;

$$\sin^2\theta + \cos^2\theta = 1$$

Hence, we have that this reduces to:

$$r^2(1) = r^2$$

Hence, the first one has been proved since the sum of the squares of both derivatives yield r^2!

Next (ii),

we're to prove;

$$\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2 = 1$$

Here, we'll need to evaluate $\frac{dx}{dr}$ and $\frac{dy}{dr}$

$$x = r\sin\theta$$

$$\frac{dx}{dr} = \frac{d}{dr}(r) \times \sin\theta$$

$$\frac{dx}{dr} = 1 \times r^{1-1} \times \sin\theta$$

$$\frac{dx}{dr} = \sin\theta \times 1 = \sin\theta$$

$$y = r\cos\theta$$

$$\frac{dy}{dr} = \frac{d}{dr}(r) \times \cos\theta$$

$$\frac{dy}{dr} = 1 \times r^{1-1} \times \cos\theta$$

$$\frac{dy}{dr} = \cos\theta \times 1 = \cos\theta$$

To prove:

$$\left(\frac{dx}{dr}\right)^2 + \left(\frac{dy}{dr}\right)^2 = 1$$

Evaluate the squares of the derivatives we've found which will show the left hand side equal to the right hand side;

$$\left(\frac{dx}{dr}\right)^2 = (\sin\theta)^2 = \sin^2\theta$$

$$\left(\frac{dy}{dr}\right)^2 = (\cos\theta)^2 = \cos^2\theta$$

Taking their sum, we have:

$$\sin^2\theta + \cos^2\theta$$

From trigonometry;

$$\sin^2\theta + \cos^2\theta = 1$$

Hence, what we have is simply equal to 1.

We have proved the second one too since the sum of the squares of both derivatives yield 1!

Question 3

(a) Use Lagrangean method to find the stationary value of $Z = xy + 2x$ subject to $4x + 2y = 60$.

(b) Integrate the following expression:

(i) $\frac{x^n}{1+n}$

(ii) $\frac{\cos x}{1+\sin x}$

(iii) $x^4 - x^3 - x^2 + \frac{1}{x} - 1$

(a)

This can be solved by both direct substitution and Lagrangean equation, however, we are told to use Lagrangean method; hence, we will do so;

So here; the objective function is: $Z = xy + 2x$

The constraint function is $4x + 2y = 60$

We'll express the constraint function equated to zero;

Hence,

$$4x + 2y - 60 = 0$$

So, let's write the Lagrangean expression here, introducing the Lagrangean multiplier:

Following the rule below:

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda \times g(x, y)$$

We have:

$$\mathcal{L}(x, y, \lambda) = xy + 2x - \lambda(4x + 2y - 60)$$

$$\mathcal{L}(x, y, \lambda) = xy + 2x - 4x\lambda - 2y\lambda + 60\lambda$$

Take the first order partials of $\mathcal{L}(x, y, \lambda)$ with respect to x, y and λ. As you are differentiating partially for any variable in the function, the rest are taken as constants, as usual.

$$\mathcal{L}_x = 1 \times x^{1-1} \times y + 1 \times 2 \times x^{1-1} - 1 \times 4 \times x^{1-1} \times \lambda - 0 + 0$$

$$\mathcal{L}_x = y + 2 - 4\lambda$$

$$\mathcal{L}_y = 1 \times x \times y^{1-1} + 0 - 0 - 1 \times 2 \times y^{1-1} \times \lambda + 0$$

$$\mathcal{L}_y = x - 2\lambda$$

$$\mathcal{L}_\lambda = 0 + 0 - 1 \times 4 \times x \times \lambda^{1-1} - 1 \times 2 \times y \times \lambda^{1-1} + 1 \times 60 \times \lambda^{1-1}$$

$$\mathcal{L}_\lambda = -4x - 2y + 60$$

From the first order partials, equate everything to zero, I mean each of the first partials; that's similar to the first order conditions;

$\mathcal{L}_x = y + 2 - 4\lambda = 0 \ldots \ldots \ldots(1)$

$\mathcal{L}_y = x - 2\lambda = 0 \ldots \ldots \ldots(2)$

$\mathcal{L}_\lambda = -4x - 2y + 60 = 0 \ldots \ldots \ldots(3)$

Solving simultaneously;

From (2);

$$x = 2\lambda \ldots \ldots \ldots(4)$$

Put (4) in (3);

$$-4x - 2y + 60 = 0$$

$$-4(2\lambda) - 2y + 60 = 0$$

$$-8\lambda - 2y = -60 \ldots \ldots \ldots(5)$$

Combine (5) with (1) since both are in y and λ

$-8\lambda - 2y = -60 \ldots \ldots \ldots (5)$

$y + 2 - 4\lambda = 0 \ldots \ldots \ldots (1)$

$\mathbf{2 \times (1)}: 2y - 8\lambda = -4 \ldots \ldots \ldots (6)$

Subtract (6) from (5)

$-8\lambda - 2y = -60 \ldots\ldots\ldots(5)$

$2y - 8\lambda = -4 \ldots\ldots\ldots(6)$

$-4y = -56$

$$\text{Here, } y = 14$$

From (3)

$-4x - 2y + 60 = 0$

Hence,

$$-4x - 2(14) + 60 = 0$$
$$-4x - 28 + 60 = 0$$
$$4x = 32$$
$$x = 8$$

The value of λ is of no necessity of no use to the solution, but let's put it for full marks.

From (5);

$$-8\lambda - 2y = -60$$

$y = 14;$

$$-8\lambda - 2(14) = -60$$
$$-8\lambda = -32$$
$$\lambda = 4$$

USE CRAMMER'S RULE IF THE ABOVE METHOD SEEMS CREEPY!

$$Z = xy + 2x$$

For the stationary value, we evaluate the value of Z at the optimum points.

$$Z = (8)(14) + 2(8) = 128$$

(b)

Integration questions;

(i)

$$\frac{x^n}{1+n}$$

We have:

$$\int \frac{x^n}{1+n} dx$$

We are to integrate with respect to x as we'll just assume since we were not told;

It's extremely simple but it looks confusing as the expression itself even looks like the power rule of integration, but as it were, it isn't, just treat it like a normal fraction; the first thing is to bring out the constant since anything that is not x is a constant.

$$\frac{1}{1+n}\int x^n dx$$

Apply the integral;

$$\frac{1}{1+n}\left[\frac{x^{n+1}}{n+1}\right]+C$$

We have;

$$\frac{1}{n+1}\times\frac{x^{n+1}}{n+1}+C$$

$$\frac{x^{n+1}}{(n+1)^2}+C$$

(ii)

$$\frac{\cos x}{1+\sin x}$$

We have:

$$\int\frac{\cos x}{1+\sin x}dx$$

This is a case of substation, the case of:

$$\frac{f'(x)}{f(x)}$$

This has even been solved in the notes;
The numerator is the derivative of the function the second function depends on, hence, our substitution is going on smoothly!

$$u = 1 + \sin x$$

$$\frac{du}{dx} = \cos x$$

Hence,

$$dx = \frac{du}{\cos x}$$

Substitute back into the integral, substitute for $(1 + \sin x)$ and for dx;

$$\int \frac{\cos x}{u} \times \frac{du}{\cos x}$$

$\cos x$ cancels out! Leaving:

$$\int \frac{1}{u} du$$

The integral, a standard integral;

$$\ln u + C$$

We won't forget our arbitrary constant, return u, the integral is:

$$\ln(1 + \sin x) + C$$

(iii)

$$x^4 - x^3 - x^2 + \frac{1}{x} - 1$$

We have:

$$\int \left(x^4 - x^3 - x^2 + \frac{1}{x} - 1\right) dx$$

Integral of sums; we have straight power rules;

$$\int x^4\,dx - \int x^3\,dx - \int x^2\,dx + \int \frac{1}{x}\,dx - \int 1\,dx$$

Integrate using the power rule, with the exception of $\frac{1}{x}$ which is a standard exception integral;

$$\left[\frac{x^{4+1}}{4+1}\right] - \left[\frac{x^{3+1}}{3+1}\right] - \left[\frac{x^{2+1}}{2+1}\right] + \ln x - \left[\frac{x^{0+1}}{0+1}\right]$$

$$\frac{x^5}{5} - \frac{x^4}{4} - \frac{x^3}{3} - x + \ln x$$

DONE!