APPLICATIONS OF DIFFERENTIATION

Much time has been spent to comprehensively destroy the concept of differentiation. I believe the topic (differential calculus) is far from anything you're afraid of now; so we want to learn some basic applications of the differentiation you learnt in the very previous chapter; both the ordinary and partial differentiation principles;

Though differentiation has a crazy wild range of applications; I mean a really really vast and wild range of applications; we'll be limiting our studies here to a very few some which are in **the SSC106 way,** that's the scope of this book, to comprehensively give you all you need **within the scope of this course**.

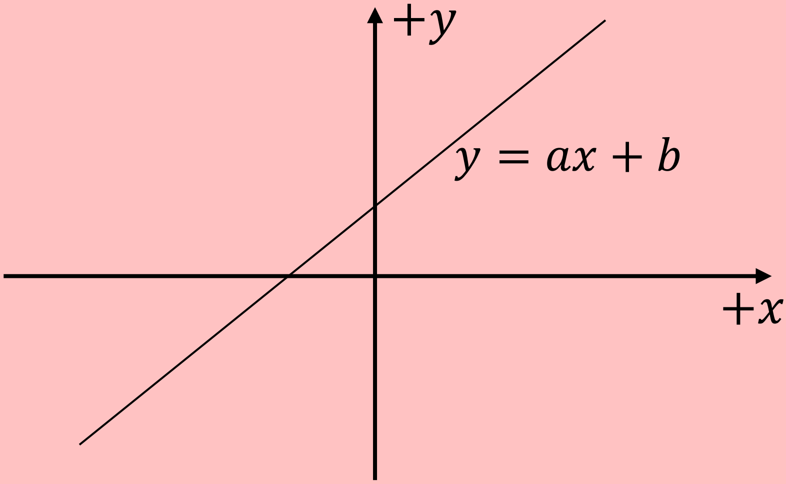
So, some applications of differential calculus include the gradient of the tangent to a curve, the gradient of the normal to a curve, the concept of increasing and decreasing functions, the concept of approximation, rate of change, rectilinear motion, optimization of functions, curve sketching, and to mention a few.

In the scope of this course; we'll be considering two very major concepts; the concepts of **increasing and decreasing functions** and the concept of **optimization of functions**.

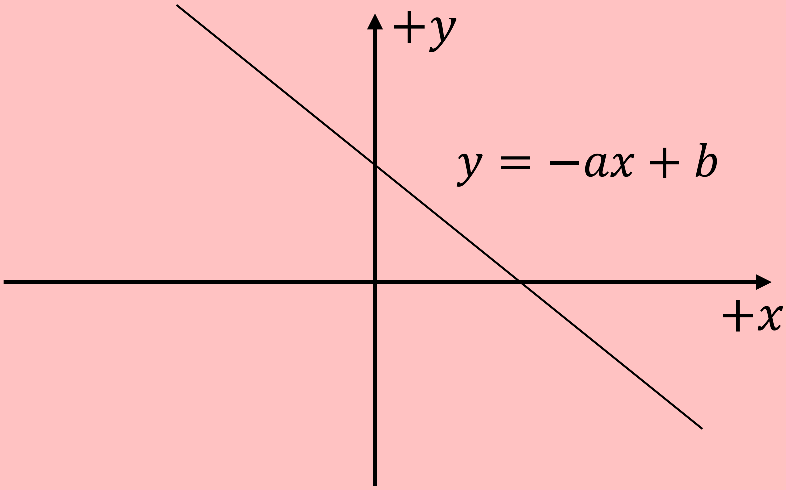
So, straight, we'll move to the concept of increasing and decreasing functions. We'll explain this with the two most basic functions which are the linear functions.

The functions; $\boldsymbol{y = ax + b}$ and $\boldsymbol{y = -ax + b}$

Sketches of the two above functions are given below, let's carefully consider the two graphs below:



The graph of $y = ax + b$



The graph of $y = -ax + b$

By just mere sight, we notice the first graph looks to be increasing (going up) and the second graph looks to be decreasing (going down).

Hence, let's do something strategic; let's evaluate the derivatives with respect to x of the both functions.

$$y = ax + b$$

$$\frac{dy}{dx} = 1 \times a \times x^{1-1} + 0$$

$$\frac{dy}{dx} = a$$

$$y = -ax + b$$

$$\frac{dy}{dx} = 1 \times -a \times x^{1-1} + 0$$

$$\frac{dy}{dx} = -a$$

What do we notice about the two values, the first is positive whilst the other is negative; hence, we can assume that a function is increasing when its first derivative is positive and it is decreasing when the first derivative is negative;

However, it is beyond an assumption, it is reality and in fact, a law in differentiation applications that **a function is increasing if its derivative is greater than zero (positive)** and that **a function is decreasing if its derivative is less than zero (negative).**

That is the law, take it in; let's not go deriving too many things, let's just take the laws this way!

So, let's see some examples;

Find the range of values for which the following functions are:

- increasing;
- decreasing.

(a) $x^2 - 3x + 6$

(b) $\frac{x^2}{3} + \frac{x}{6} + \frac{1}{3}$

Alright, for the first function, we have the situation given this way:

(a) Let $y = x^2 - 3x + 6$

Hence,

$$\frac{dy}{dx} = 2 \times x^{2-1} - 1 \times 3 \times x^{1-1} + 0$$

$$\frac{dy}{dx} = 2x - 3$$

Now, from our law;

for the function to be increasing;

$$\frac{dy}{dx} \text{ is positive, hence, } \frac{dy}{dx} > 0$$

Now, from our law;

for the function to be decreasing;

$$\frac{dy}{dx} \text{ is negative, hence, } \frac{dy}{dx} < 0$$

A positive number is greater than zero, and a negative number is less than zero; I guess that's fine?

So for this question;

For the range of values for which it is increasing;

$$2x - 3 > 0$$

$$2x > 3$$

$$x > \frac{3}{2}$$

For the range of values for which it is increasing;

$$2x - 3 < 0$$

$$2x < 3$$

$$x < \frac{3}{2}$$

Above is a simple case of inequality which you should be familiar with;

The second one;

(b) $\frac{x^2}{3} + \frac{x}{6} + \frac{1}{3}$

Normally, we need to find the first derivative at first before doing any other thing,

$$\frac{dy}{dx} = 2 \times \frac{1}{3} \times x^{2-1} + 1 \times \frac{1}{6} \times x^{1-1} + 0$$

$$\frac{dy}{dx} = \frac{2}{3}x + \frac{1}{6}$$

Now, from our law;

for the function to be increasing;

$$\frac{dy}{dx} > 0$$

for the function to be decreasing;

$$\frac{dy}{dx} < 0$$

So for this question;

For the range of values for which it is increasing;

$$\frac{2}{3}x + \frac{1}{6} > 0$$

$$\frac{2}{3}x > -\frac{1}{6}$$

$$x > -\frac{1}{6} \times \frac{3}{2}$$

$$x > -\frac{1}{4}$$

For the range of values for which it is increasing;

$$\frac{2}{3}x + \frac{1}{6} < 0$$

$$\frac{2}{3}x < -\frac{1}{6}$$

$$x < -\frac{1}{6} \times \frac{3}{2}$$

$$x < -\frac{1}{4}$$

Let's see another type of question;

- Verify if the following function is increasing or decreasing:

$$y = x^3 - 5x^2 + 2x - 3$$

at:

(i) $x = -4$
(ii) $x = 2$
(iii) $x = 0$
(iv) $x = -3$
(v) $x = 1$

Right;
Since it's an increasing and decreasing situation, we have the first condition of testing for the nature of the first derivative; let's find that first derivative:

$$y = x^3 - 5x^2 + 2x - 3$$

$$\frac{dy}{dx} = 3 \times x^{3-1} - 2 \times 5 \times x^{2-1} + 1 \times 2 \times x^{1-1} - 0$$

$$\frac{dy}{dx} = 3x^2 - 10x + 2$$

Hence, since we're not asked to find ranges, we're told to find the nature of the function (whether increasing or decreasing) at specific values; so let's fix the values we need one by one and test if they'll be positive or negative;

At $x = -4$;

$$\frac{dy}{dx} = 3(-4)^2 - 10(-4) + 2$$

$$\frac{dy}{dx} = 48 + 40 + 2 = 90$$

90 is greater than zero, hence, it is increasing at $x = -4$.

At $x = 2$

$$\frac{dy}{dx} = 3(2)^2 - 10(2) + 2$$

$$\frac{dy}{dx} = 12 - 20 + 2 = -6$$

-6 is less than zero, hence, it is decreasing at $x = 2$.

At $x = 0$

$$\frac{dy}{dx} = 3(0)^2 - 10(0) + 2$$

$$\frac{dy}{dx} = 0 + 0 + 2 = 2$$

2 is greater than zero, hence, it is increasing at $x = 0$.

At $x = -3$

$$\frac{dy}{dx} = 3(-3)^2 - 10(-3) + 2$$

$$\frac{dy}{dx} = 27 + 30 + 2 = 59$$

59 is greater than zero, hence, it is increasing at $x = -4$.

At $x = 1$

$$\frac{dy}{dx} = 3(1)^2 - 10(1) + 2$$

$$\frac{dy}{dx} = 3 - 10 + 2 = -5$$

-5 is less than zero, hence, it is increasing at $x = 1$.

Do this for me;

Find the nature (if increasing or decreasing) of the following function at the following points;

$$f(u) = \frac{x^4}{4} - \frac{x^3}{4} - 6x^2 + \frac{x}{2} - 7$$

At:

(i) $u = -1$

(ii) $u = 1$

(iii) $u = -3$

(iv) $u = 2$

(v) $u = 0$

And here are the answers;

(i) increasing;
(ii) decreasing;
(iii) decreasing;
(iv) decreasing;
(v) increasing.

That's all about the whole thing, it's nothing out of the world, and it's pretty easy.

So, let's move to the next part of the applications of differentiation.

OPTIMIZATION OF FUNCTIONS

Yeah, that's the real deal in the applications of differential calculus; optimization is crazily used in the fields of economic analysis, as in like crazily, it is used vastly. It isn't worth it if you have attended SSC106 fully without properly understanding the concept of function optimization. So, without wasting time, let's see what we mean when we say we are optimizing a function.

Generally, in life, in mathematics and other areas of operation researches, optimization is the selection of the best elements (mostly with respect to a criterion) from a set of available alternatives;

In simplest case which is the case of a mathematical or economic function, a situation of optimizing functions is called an **optimization problem**.

In the simple case of a mathematical or economic function, ***an optimization problem consists of maximizing or minimizing a real function by the use of input values within an allowed set to compute the value of the function.***

A function to be optimized is called an **objective function** with a variety of types of objective functions.

Optimization problems;

Consider the following problem;

Given a function: $A = f(x)$;

For an element, x_1 in A such that; $f(x_1)$ yields the ***maximum possible*** value of A;

For an element, x_2 in A such that; $f(x_2)$ yields the ***minimum possible*** value of A;

Then such a formulation (or problem) is called an optimization problem. It is widely involved in real-world and theoretical problems in the field of sciences and in the field of economic analysis, it indeed is an indispensable concept in the field of administration and social sciences.

Now, it's like this, **objective functions** are of varying categories;

- a **loss function** or a **cost function** which generally are to be minimized;
- a **utility function** or **fitness function** which are generally meant to be maximized; the **energy functions** in the field of sciences are also desirable to be maximized.

Optimization of functions is basically of two types; **the constrained** and the **unconstrained optimization problems.** And in the scope of this, we'll be taking the both; the unconstrained and the constrained mathematical optimization.

So let's take a little focus on **the unconstrained optimization problems**.

As we know for now, we all know that optimization at this level is the process of finding the minimum and maximum values;

One of Fermat's theorem states that the optima of unconstrained problems are found at **the stationary points** of the objective function that brought about the optimization problem.

Hence, in this point we are, we have given an extremely strong point to knowing what to do in unconstrained optimizations.

Stationary points occur from what we have learnt so far, from the concept of the increasing and decreasing functions, we know that their derivatives are greater than zero and less than zero respectively.

Now, consider a point that is **neither increasing nor decreasing**; such point is called **a stationary point**. You should be thinking of what that'll imply now, and I guess you should be right, since it's somewhere between increasing and decreasing, then the **derivative of a function at its stationary value(s) is equal to zero.**

Hence, we have the law that:

$$y = f(x)$$

At stationary value;

$$\frac{dy}{dx} = 0 \quad \text{or} \quad f'(x) = 0$$

That's the basic rules of finding the stationary points on a function;

The stationary point rule above is called:
'first-order condition' – FOC, for relative optima.

Keep taking notes of definitions of terms and writing them down, SSC106 tests your knowledge on definitions very well and that is why time is taken to explicitly express them for you to see them clearly.

I believe it's that simple, as simple as that, this also includes the aspect of partial derivatives we can find the stationary points of the separate independent variables in multivariate functions by equating their separate first order partials to zero.

We'll see examples in the of both situations as we continue;

The stationary points of a function are not sufficient proves for the complete optimization of a function, i.e. the first order conditions for relative optima are not enough to completely optimize a function. The stationary points only give us point where the function is neither increasing nor decreasing, however, it doesn't give the nature of the stationary point. The idea of optimization of functions is finding ***the maximum and/or minimum points (and values)*** of a function.

These cases can be distinguished by checking the second derivative of the objective functions; the second derivative test constitutes the:

'second-order conditions' – FOC, for relative optima, they are called condition**(s)** owing to the fact that they are three, the second order conditions distinguish stationary points into the maximum, the minimum and the point of inflexion;

The second order conditions for relative optima are given below:

For stationary points on a given function, $y = f(x)$; where:

$$\frac{dy}{dx} = 0 \quad \text{or} \quad f'(x) = 0$$

If;

$$\frac{d^2y}{dx^2} > 0, f''(x) > 0$$

Then the stationary point is **a minimum point**.

If;

$$\frac{d^2y}{dx^2} < 0, f''(x) < 0$$

Then the stationary point is **a maximum point**.

If;

$$\frac{d^2y}{dx^2} = 0, f''(x) = 0$$

Then the stationary point is **a point of inflexion**.

Those above three are called the second-order conditions for relative optima;

The combination of the first order and second order conditions for relative optima are jointly called **the allied calculus conditions for relative optima**.

For unconstrained functions, the conditions above (the first and second order) give the conditions for the **complete optimization** of such functions.

Let's see some quick examples;

- Find the stationary points of the following functions and determine their nature;
  (a) $y = x^2(x-9)$
  (b) $y = x(x^2-12)$
  (c) $y = x^3 - 9x^2 + 27x - 27$

Kk, let's begin:

(a) $y = x^2(x-9)$

The first thing to do when dealing with stationary points, we'll find their first derivatives;

$$y = x^2(x-9)$$

No need of thinking of product rule; we can easily expand this;

$$y = x^3 - 9x^2$$

$$\frac{dy}{dx} = 3 \times x^{3-1} - 2 \times 9 \times x^{2-1}$$

$$\frac{dy}{dx} = 3x^2 - 18x$$

So, at the stationary point; $\frac{dy}{dx} = 0$

Hence, here:

$$3x^2 - 18x = 0$$

We factorize $3x$ in this;

$$3x(x - 6) = 0$$

Hence,

$$3x = 0 \text{ or } (x - 6) = 0$$

Break this down;

$$3x = 0$$

Hence,

$$\boldsymbol{x = 0}$$

Also,

$$x - 6 = 0$$

$$\boldsymbol{x = 6}$$

Hence, we have two stationary values ($x = 0$ and $x = 6$); hence, we need to test for their natures;

$$\frac{dy}{dx} = 3x^2 - 18x$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(3x^2 - 18x)$$

$$\frac{d^2y}{dx^2} = 2 \times 3 \times x^{2-1} - 1 \times 18 \times x^{1-1}$$

$$\frac{d^2y}{dx^2} = 6x - 18$$

Now, this is how the nature of the stationary points are gotten, we have two stationary points;

$$x = 0$$
$$x = 6$$

We test for each of the stationary points in the second derivative of the function;

$$\frac{d^2y}{dx^2} = 6x - 18$$

At $x = 0$;

$$\frac{d^2y}{dx^2} = 6(0) - 18 = -18$$

-18 is less than zero, hence, from our second order conditions, $\boldsymbol{x = 0}$ **is a maximum point.**

Now, don't get it twisted, our needed value is still the $x = 0$; the second order test is as it is called, is a mere test to confirm the nature of the stationary point and not in any way another value; we are just using it to test if in the second derivative, the point is greater than, less than or equal to zero for an appropriate test of the nature of the stationary point.

So, similarly,

At $x = 6$;

$$\frac{d^2y}{dx^2} = 6(6) - 18 = 36 - 18 = 18$$

18 is greater than zero, hence, from our second order conditions, **$x = 6$ is a minimum point.**

So, to the second question;

(b) $y = x(x^2 - 12)$

The first thing to do when dealing with stationary points, we'll find their first derivatives;

$$y = x(x^2 - 12)$$

Again, we needn't think of the product rule; we can easily expand this;

$$y = x^3 - 12x$$

$$\frac{dy}{dx} = 3 \times x^{3-1} - 1 \times 12 \times x^{1-1}$$

$$\frac{dy}{dx} = 3x^2 - 12$$

So, at the stationary point;

$$\frac{dy}{dx} = 0$$

Hence, here:

$$3x^2 - 12 = 0$$

Hence,

$$3x^2 = 12$$

$$x^2 = \frac{12}{3} = 4$$

Taking square roots of both sides;

$$x = \pm\sqrt{4} = \pm 2$$

I'm sure you aren't strange with the concept of the fact that square roots have plus and minus values, since the square root of a number is the positive and negative of the square root value.

We’ll need to break this dual value down as well to have the two stationary values that we have here;

$$x = -2 \text{ or } x = 2$$

Hence, we have two stationary values; hence, we need to test for their natures;

$$\frac{dy}{dx} = 3x^2 - 12$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(3x^2 - 12)$$

$$\frac{d^2y}{dx^2} = 2 \times 3 \times x^{2-1} - 0$$

$$\frac{d^2y}{dx^2} = 6x$$

Now, we already know how the natures of the stationary points are gotten, we have two stationary points;

$$x = -2$$
$$x = 2$$

We test for each of the stationary points in the second derivative of the function;

$$\frac{d^2y}{dx^2} = 6x$$

At $x = -2$;

$$\frac{d^2y}{dx^2} = 6(-2) = -12$$

-12 is less than zero, hence, from our second order conditions, $\boldsymbol{x = -2}$ **is a maximum point.**

So, similarly,

At $x = 2$;

$$\frac{d^2y}{dx^2} = 6(2) = 12$$

12 is greater than zero, hence, from our second order conditions, $\boldsymbol{x = 6}$ **is a minimum point.**

To the third question;

(c) $y = x^3 - 9x^2 + 27x - 27$

We know the first thing to do when dealing with stationary points; we'll find their first derivatives;

$$y = x^3 - 9x^2 + 27x - 27$$

$$\frac{dy}{dx} = 3 \times x^{3-1} - 2 \times 9 \times x^{2-1} + 1 \times 27 \times x^{1-1} - 0$$

$$\frac{dy}{dx} = 3x^2 - 18x + 27$$

So, at the stationary point;

$$\frac{dy}{dx} = 0$$

Hence, here:

$$3x^2 - 18x + 27 = 0$$

Hence, this is a quadratic equation; we'll take this quadratic equation by factorization; use the quadratic formula if you can't factorize;

$$3x^2 - 9x - 9x + 27 = 0$$

$$3x(x-3) - 9(x-3)$$

$$(3x-9)(x-3) = 0$$

Hence,

$$(3x-9) = 0 \text{ or } (x-3) = 0$$

Break this down;

$$3x - 9 = 0$$

$$3x = 9$$

$$x = 3$$

Also,

$$x - 3 = 0$$

$$x = 3$$

Hence, we seem to have two stationary values; however, both values are the same, hence, we have just one stationary point at $x = 3$, we need to test for its nature;

$$\frac{dy}{dx} = 3x^2 - 18x + 27$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(3x^2 - 18x + 27)$$

$$\frac{d^2y}{dx^2} = 2 \times 3 \times x^{2-1} - 1 \times 18 \times x^{1-1} + 0$$

$$\frac{d^2y}{dx^2} = 6x - 18$$

Now, we have our stationary point;

$$x = 3$$

We test for the stationary point in the second derivative of the function;

$$\frac{d^2y}{dx^2} = 6x - 18$$

At $x = 3$;

$$\frac{d^2y}{dx^2} = 6(3) - 18 = 18 - 18 = 0$$

0 is zero in itself, hence, from our second order conditions, $\boldsymbol{x = 3}$ **is a point of inflexion.**

That's basically how it's done, we don't need bulk of examples; you already know how to differentiate all diverse forms of explicit functions; let's see how this is applied to partial derivatives; Oops, let's see one more example to introduce the concept of maximum and minimum values; it's nothing new, just keep calm!

So, we'll see this question dealing with the minimum and maximum values, an extension of the minimum and maximum concept;

So,

- Find the relative optima, if they exist, for the function;

$$f(x) = x^3 - 6x^2 + 9x - 4$$

To find the relative optima, we find the optimal values, whether maxima, minima or inflexion.

We know the first thing to do when dealing with stationary points; we'll find their first derivatives;

$$f(x) = x^3 - 6x^2 + 9x - 4$$

$$f' = 3 \times x^{3-1} - 2 \times 6 \times x^{2-1} + 1 \times 9 \times x^{1-1} - 0$$

$$f'(x) = 3x^2 - 12x + 9$$

So, at the stationary point; $f'(x) = 0$

Hence, here:

$$3x^2 - 12x + 9 = 0$$

Hence, this is a quadratic equation; we'll take this quadratic equation by factorization; use the quadratic formula if you can't factorize;

$$3x^2 - 3x - 9x + 9 = 0$$

$$3x(x-1) - 9(x-1) = 0$$

$$(3x-9)(x-1) = 0$$

Hence,

$$(3x-9) = 0 \text{ or } (x-1) = 0$$

Break this down;

$$3x - 9 = 0$$

$$3x = 9$$

$$x = 3$$

Also,

$$x - 1 = 0$$

$$x = 1$$

Hence, we have two stationary values; hence, we need to test for their natures;

$$f'(x) = 3x^2 - 12x + 9$$

$$f''(x) = \frac{d}{dx}\big(f'(x)\big) = \frac{d}{dx}(3x^2 - 12x + 9)$$

$$f''(x) = 2 \times 3 \times x^{2-1} - 1 \times 12 \times x^{1-1} + 0$$

$$f''(x) = 6x - 12$$

Now, we know how the natures of the stationary points are gotten, we have two stationary points;

$$x = 3$$
$$x = 1$$

We test for each of the stationary points in the second derivative of the function;

$$f''(x) = 6x - 12$$

At $x = 3$;

$$f'' = 6(3) - 12 = 18 - 12 = 6$$

6 is greater than zero, hence, from our second order conditions, $x = 3$ is a minimum point.

And, similarly,

At $x = 1$;

$$f'' = 6(1) - 12 = 6 - 12 = -6$$

-6 is less than zero, hence, from our second order conditions, $x = 1$ is a maximum point.

I guess all that were formalities; we know all those, so, let's evaluate now, the minimum and maximum values (the relative optima), finding maximum values goes beyond just finding the points;

Now, we have the minimum point; $x = 3$;

Now, to get the minimum value; we'll be evaluating the value of that function at $x = 3$; now, we have evaluated the value of x (the independent variable) for which the function itself is minimum.

Hence, we be evaluate $f(3)$ to find our minimum value since the corresponding minimum point is $x = 3$;

$$f(x) = x^3 - 6x^2 + 9x - 4$$

$$f(3) = (3)^3 - 6(3)^2 + 9(3) - 4$$

$$f(3) = 27 - 54 + 27 - 4$$

$$f(3) = -4$$

Hence, the **minimum value possible in this function is -4.**

Now, we also have the maximum point; $x = 1$;

Now, to get the maximum value; we'll be evaluating the value of that function at $x = 1$; now, we have evaluated the value of x (the independent variable) for which the function itself is maximum.

Hence, we be evaluate $f(1)$ to find our maximum value since the corresponding maximum point is $x = 1$;

$$f(x) = x^3 - 6x^2 + 9x - 4$$

$$f(1) = (1)^3 - 6(1)^2 + 9(1) - 4$$

$$f(1) = 1 - 6 + 9 - 4$$

$$f(1) = 0$$

Hence, the **maximum value possible in this function is 0.**

So, before we move to the constrained problem situation; let's see how unconstrained problems are dealt with in the case of multivariate functions;

- Find the minimum value of the function;

$$C(x, y) = 3x^2 - 12x + y^3 - 12y$$

So, we'll evaluate the first order partials of the function with respect to both variables;

$$C_x = 2 \times 3 \times x^{2-1} - 1 \times 12 \times x^{1-1} + 0 - 0$$

$$C_x = 6x - 12$$

$$C_y = 0 - 0 + 3 \times y^{3-1} - 1 \times 12 \times y^{1-1}$$

$$C_y = 3y^2 - 12$$

So, regularly; we'll find their separate stationary points with the first order conditions, only difference in this case is that we have to first order derivatives which are the first order partials, to find the stationary points for x and y separately.

At stationary point;

$$C_x = 0$$

$$6x - 12 = 0$$

$$6x = 12$$

$$x = 2$$

Hence, we have only one stationary point from C_x

At stationary point for y;

$$C_y = 0$$

$$3y^2 - 12 = 0$$

$$3y^2 = 12$$

$$y^2 = 4$$

Take square roots;

$$y = \pm 2$$

Break them down;

$$y = -2 \quad \text{or} \quad y = 2$$

Hence, we have only two stationary points from C_y

So we have the separate stationary points, let's now test for their nature with the second order conditions, separately for each variable.

Test for x, take the second derivative of C_x with respect to x;

$$C_{xx} = \frac{d}{dx}(C_x)$$

$$C_{xx} = \frac{d}{dx}(6x - 12)$$

$$C_{xx} = 1 \times 6 \times x^{1-1} - 0 = 6$$

Now, since the second derivative gives a straight constant positive value, we needn't do any substitution as for any value, it'll be the same constant value, it shows that the **singular stationary point ($x = 2$) is a minimum point.**

Test for y, take the second derivative of C_y with respect to y;

$$C_{yy} = \frac{d}{dy}(C_y)$$

$$C_{yy} = \frac{d}{dy}(3y^2 - 12)$$

$$C_{yy} = 2 \times 3 \times y^{2-1} - 0 = 6y$$

So, this time around, we have two stationary points from C_y, hence, we have to substitute them in C_{yy} for their nature. Let's test for our two y stationary values here;

$$C_{yy} = 6y$$

At $y = -2$;

$$C_{yy} = 6(-2) = -12$$

Hence, $\boldsymbol{y = -2}$ **is a maximum point** since its second derivative value is less than zero (-12)

At $y = 2$

$$C_{yy} = 6(2) = 12$$

Hence, $\boldsymbol{y = 2}$ **is a minimum point** since its second derivative value is less than zero (12)

Since we need the optimum minimum values; we have a minimum for x at $x = 2$ and we also have to take the minimum point for y, since both are ***correspondingly minimum***, and hence, we'll take the point $y = 2$;

To get the minimum value of the function, we'll take our minimum points to the **main function** which are corresponding minimums, hence **the minimum point is** $\boldsymbol{x = 2}$ **and** $\boldsymbol{y = 2}$

$$C(x, y) = 3x^2 - 12x + y^3 - 12y$$

$$C(2,2) = 3(2)^2 - 12(2) + (2)^3 - 12(2)$$

$$C(2,2) = 12 - 24 + 8 - 24$$

$$C(2,2) = -24$$

Hence, the minimum value of the function is -24;

Now, the function doesn't have a maximum value as only y has a maximum value, x only has a minimum value and hence, we only have a minimum (or maximum) value for the whole function, when substituting, the two natures of the stationary points for both variable must be the same (either either maximum or both minimum).

And of course you know the notation to be used when the multivariate function is expressed as a dependent variable form, you'll be using notations such as $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ since we're dealing with partial derivatives;

UP! To the constrained optimization problems!

CONSTRAINED OPTIMIZATION PROBLEMS

We have seen how we freely optimize functions applying the first and second order conditions to them; however, in extremely practical cases, optimization is done subject to constraints; constraints means we'll find optimal values for a function while at the same time, still satisfying a condition. Examples of such cases include a desire to minimize cost while still producing a certain amount of products, maximizing the area of a field while we have a fixed perimeter among other of such cases; there are basically two methods of solving constrained optimization problems:

- The method of direct substitution;
- The Lagrangean method also known as the method of Lagrange multipliers.

In the cases of constrained problems, we have at least two different functions, **the objective function** and **the constraint function**. The objective function, same as the case of the unconstrained problems, is the function to be optimized (i.e. to be maximized or minimized). The constraint function is the condition that must be met, while optimizing the function;

In constrained optimization, the constrained functions could be more than one, however, for the scope of SSC106, we'll be limited to problems involving just one objective function and one constraint, we'll also limit our work to the use of two variables;

Also; we won't be dwelling much on the method of direct substitution, but on the method of Lagrange multipliers;

However, we won't be abandoning it completely, hence, let's quickly see the method of direct substitution for optimizing constrained problems.

The method of direct substitution;

The method of direct substitution is very short and simple; especially now that we're dealing with one objective function and one constraint function, subject to two variables; the process is simple; one variable is made the subject of the relation in the constraint function; it is substituted into the objective function, which becomes a one-variable function; the objective function is then optimized using the allied calculus conditions for relative optima.

Let's use an example to explain this, as usual ☺☺☺

- Maximize the objective function;

$$Z = xy + 2x$$

x, y subject to the constraint $4x + 2y = 60$

Cool, in direct substitution, the basic process is simple, make either x or y the subject in the constraint function;

The constraint function here is:

$$4x + 2y = 60;$$

I think I like y, let's make y the subject;

$$2y = 60 - 4x$$

$$y = \frac{60}{2} - \frac{4x}{2}$$

$$y = 30 - 2x$$

Now, take this into the objective function!

The objective function is;

$$Z = xy + 2x$$

Substitute for y; from our constraint function

$$Z = x(30 - 2x) + 2x$$

Expand and make it a one-variable function.

$$Z = 30x - 2x^2 + 2x$$

$$Z = 32x - 2x^2$$

Then, to optimize this (for a maximization in this process), we know allied calculus conditions here, we'll simply treat this as a situation to maximize Z with respect to x only, but relax, we aren't going to abandon y, we'll come back later;

$$\frac{dZ}{dx} = 1 \times 32 \times x^{1-1} - 2 \times 2 \times x^{2-1}$$

$$\frac{dZ}{dx} = 32 - 4x$$

At stationary point;

$$\frac{dZ}{dx} = 0$$

$$32 - 4x = 0$$

$$4x = 32$$

$$x = 8$$

We have just one stationary point, let's test for the second derivative; it should be a maximum point though; since that's what the question requires;

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}\left(\frac{dZ}{dx}\right) = \frac{d}{dx}(32 - 4x)$$

$$\frac{d^2Z}{dx^2} = 0 - 1 \times 4 \times x^{1-1} = -4$$

So as expected, the second derivative is a constant negative and hence a maximum value; So, now, we have our $x = 8$ as our value for the maximizing of the objective function whilst still obeying the constraint;

Hence, let's bring back our y;

$$y = 30 - 2x$$

$$y = 30 - 2(8) = 30 - 16 = 14$$

Hence, **the maximum values occur when $x = 8$ and $y = 14$;** while the constraint is still fully obeyed.

Let's switch to practical cases;

- Find the greatest product of two numbers whose sum is 12.

Now, we'll extract our objective and constraint functions from here;

Let the two numbers be x and y

The sum must be 12; hence, our constraint is;

$$x + y = 12$$

We want to find the greatest product; it in essence means we want to maximize the product of the two numbers subject to the constraint:

$$x + y = 12$$

Hence, the question can be summarized as:

Maximize $P = xy$
Subject to $x + y = 12$

Here, P is the product and we want to maximize it, the product of x and y is xy and hence, we want to maximize P, the product;

Hence, let's go through the short process;

The constraint function here is:

$$x + y = 12;$$

Let's still use y, let's make y the subject;

$$y = 12 - x$$

Now, take this into the objective function!

The objective function is;

$$P = xy$$

Substitute for y; from our constraint function

$$P = x(12 - x)$$

Expand and make it a one-variable function.

$$P = 12x - x^2$$

Then, to optimize this (for a maximization in this process), we know allied calculus conditions here, we'll simply treat this as a situation to maximize P with respect to x only.

$$\frac{dP}{dx} = 1 \times 12 \times x^{1-1} - 2 \times x^{2-1}$$

$$\frac{dP}{dx} = 12 - 2x$$

At stationary point;

$$\frac{dP}{dx} = 0$$

$$12 - 2x = 0$$

$$2x = 12$$

$$x = 6$$

We have just one stationary point, let's test for the second derivative; it should be a maximum point though; since that's what the question requires;

$$\frac{d^2P}{dx^2} = \frac{d}{dx}\left(\frac{dP}{dx}\right) = \frac{d}{dx}(12 - 2x)$$

$$\frac{d^2P}{dx^2} = 0 - 1 \times 2 \times x^{1-1} = -2$$

So as expected, the second derivative is a constant negative and hence a maximum value; So, now, we have our $x = 6$ as our value for the maximizing of the objective function whilst still obeying the constraint;

Hence, let's bring back our y;

$$y = 12 - x$$

$$y = 12 - 6 = 6$$

Hence, **the maximum values occur when $x = 6$ and $y = 6$;** while the constraint that their sum is 12 is still; the two numbers for the optimum situation is 6 and 6.

Hence, to determine the greatest product, we'll have;

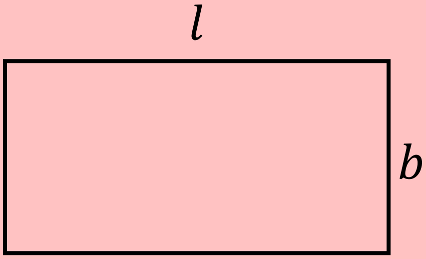
$$P = xy$$

$$P = 6 \times 6 = 36$$

for the maximum value subject to the constraint that their sum is 12

- 100m of wire is available for fencing a rectangular piece of land. Find the dimensions of the land which maximizes the area. Hence, determine the maximum area of the fence.

A rectangle is shown below:



Hence, the perimeter is the sum round the rectangle; the perimeter formula is shown below, that's just it.

$$P = 2(l + b)$$

So, let's successfully analyze our constrained problem here:

If $100m$ is used to fence the rectangular land through, that means the perimeter is $100m$

Hence, we can create our constraint function that:

$$2(l + b) = 100$$

Expand:

$$2l + 2b = 100$$

We have our constraint function;

What then is our objective function? The desire in the question is to maximize the area of the land, hence, what is the objective function?

The area of a rectangle is its length times its breadth. Hence, our objective function is:

$$A = lb$$

We want to maximize the area, subject to the constraint we've already seen.

Hence, the question can be summarized as:

Maximize $A = lb$
Subject to $2l + 2b = 100$

Hence, let's go through the short process;

The constraint function here is:

$$2l + 2b = 100;$$

Making use of b, let's make b the subject;

$$2b = 100 - 2l$$

$$b = \frac{100}{2} - \frac{2l}{2}$$

$$b = 50 - l$$

Now, take this into the objective function!

The objective function is;

$$A = lb$$

Substitute for b; from our constraint function

$$A = l(50 - l)$$

Expand and make it a one-variable function.

$$P = 50l - l^2$$

Then, to optimize this (for a maximization in this process), we know allied calculus conditions here, we'll simply treat this as a situation to maximize A with respect to l only.

$$\frac{dA}{dl} = 1 \times 50 \times l^{1-1} - 2 \times l^{2-1}$$

$$\frac{dP}{dx} = 50 - 2l$$

At stationary point;

$$\frac{dA}{dl} = 0$$

$$50 - 2l = 0$$

$$2l = 50$$

$$l = 25$$

We have just one stationary point, let's test for the second derivative; it should be a maximum point though; since that's what the question requires;

$$\frac{d^2A}{dl^2} = \frac{d}{dl}\left(\frac{dA}{dl}\right) = \frac{d}{dl}(50 - 2l)$$

$$\frac{d^2A}{dl^2} = 0 - 1 \times 2 \times x^{1-1} = -2$$

So as expected, the second derivative is a constant negative and hence a maximum value;

So, now, we have our $l = 25$ as our value for the maximizing of the objective function whilst still obeying the constraint;

Hence, let's bring back our b;

$$b = 50 - l$$

$$b = 50 - 25 = 25$$

Hence, **the maximum values occur when $\boldsymbol{l} = \mathbf{25}$ and $\boldsymbol{b} = \mathbf{25}$;** while the constraint that their perimeter is 100 is still obeyed; the two dimensions for the optimum situation is $l = 25$ and $b = 25$.

Hence, to determine the greatest area, we'll have;

$$A = lb$$

$$P = 25 \times 25 = 625m^2$$

For the maximum value subject to the constraint that their perimeter is $100m$.

Note that we used $625m^2$ since the perimeter was expressed in m (metres).

Alright, so we can now move to the main stuff we're here to do, the solving of constrained problems using the method of Lagrange multipliers.

The Lagrangean method;

The idea of the Lagrangean method (also known as the method of Lagrange multipliers) was named after **Joseph-Louis Lagrange.** It is a strategy for optimizing functions which are subject to equality constraints.

Several constrained problems can be easily transformed into unconstrained problems with the help of the Lagrange multipliers (mostly represented by the Greek, ***lambda*** alphabet **(λ).** The **Lagrangean method** is also spelt as **Lagrangian method** which is a key way of solving constrained problems. Multiple constraints can easily be solved by the method of Lagrange multipliers; however, just like we've said earlier, this will be limited to one objective function and one constraint functions in two variables.

Consider, an objective function; $f(x, y)$

And a constraint function, $g(x, y) = 0$

The Lagrangean equation is given by:

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda \times g(x, y)$$

For instance;

If the constraint is; $4x + 2y = 60$

It'll expressed as $4x + 2y - 60 = 0$

If the constraint is $y = x$

It'll be expressed as $y - x = 0$

The newly expressed form takes the place of $g(x, y)$, the whole requirement is that everything in the constraint function is equated to zero.

The Lagrangean expression is formed thus; with the Lagrangean multiplier introduced;

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda \times g(x, y)$$

Notice the Lagrangean function is now in form of a multivariate function on three variables because of the introduction of the Lagrange multiplier; hence, the whole thing becomes a singular function after successful expansion;

So, the Lagrangean multiplier is in inform of the Laplace transform symbol, $\mathcal{L}$. The symbol is shown magnified below;

$$\mathcal{L}$$

The Lagrange multiplier method of optimization is then concluded thus;

Take the first order partials of $\mathcal{L}$ with respect to x, y and with λ; three equations in three unknowns are gotten; then the equations are solved simultaneously. Could appear as a tedious work in the aspect of simultaneous equations but it isn't tedious as it were; you can decide to use the Crammer's rule if you get stuck; you sure have been thought Crammer's rule very well in the course of this book.

For the second order conditions, the natures of the direct second order partial derivatives are gotten and the normal allied conditions are checked;

For maximum; the direct second order derivatives are less than zero;

For maximum; the direct second order derivatives are greater than zero;

The second order conditions however are quite tricky in the case of Lagrangean method of constrained function optimization because at times, the direct second order partial derivatives give a zero value, tempting you to think they're points of inflexion but that's not the case; when such happens, further analysis is needed to determine the nature, which is beyond the scope of this course and hence, we do not mostly check for the nature of the optimized values unless when asked to do so specifically, such cases when you have been told to do so, the functions will be in ways that the direct second order partial derivatives exist;

So, let's begin;

- Max $Z = xy + 2x$;
  x, y subject to the constraint $4x + 2y = 60$.

So, this is the first question we treated using the direct substitution method; let's see this:

So here; the objective function is: $Z = xy + 2x$

The constraint function is $4x + 2y = 60$

We'll express the constraint function equated to zero, note that it can be expressed either way, either all terms are taken to the right hand of the equality sign or all terms are taken to the left hand of the equality sign:

Hence, either:

$$4x + 2y - 60 = 0$$

Or

$$60 - 4x - 2y = 0$$

Can be used!

So, let's write the Lagrangean expression here, introducing the Lagrangean multiplier:

Following the rule below:

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda \times g(x, y)$$

We have:

$$\mathcal{L}(x, y, \lambda) = xy + 2x - \lambda(4x + 2y - 60)$$

I hope this is gotten; it isn't difficult at all, let's expand this function.

$$\mathcal{L}(x, y, \lambda) = xy + 2x - 4x\lambda - 2y\lambda + 60\lambda$$

Take the first order partials of $\mathcal{L}(x, y, \lambda)$ with respect to x, y and λ. I'm sure you remember the partial derivative we just discussed in the just previous chapter. As you are differentiating partially for any variable in the function, the rest are taken as constants, as usual.

$$\mathcal{L}_x = 1 \times x^{1-1} \times y + 1 \times 2 \times x^{1-1} - 1 \times 4 \times x^{1-1} \times \lambda - 0 + 0$$

$$\mathcal{L}_x = y + 2 - 4\lambda$$

$$\mathcal{L}_y = 1 \times x \times y^{1-1} + 0 - 0 - 1 \times 2 \times y^{1-1} \times \lambda + 0$$

$$\mathcal{L}_y = x - 2\lambda$$

$$\mathcal{L}_\lambda = 0 + 0 - 1 \times 4 \times x \times \lambda^{1-1} - 1 \times 2 \times y \times \lambda^{1-1} + 1 \times 60 \times \lambda^{1-1}$$

$$\mathcal{L}_\lambda = -4x - 2y + 60$$

From the first order partials, equate everything to zero, I mean each of the first partials; that's similar to the first order conditions;

$$\mathcal{L}_x = y + 2 - 4\lambda = 0 \ldots\ldots\ldots(1)$$

$$\mathcal{L}_y = x - 2\lambda = 0 \ldots\ldots\ldots(2)$$

$$\mathcal{L}_\lambda = -4x - 2y + 60 = 0 \ldots\ldots\ldots(3)$$

Solving simultaneously;

From (2);

$$x = 2\lambda \ldots \ldots \ldots (4)$$

Put (4) in (3);

$$-4x - 2y + 60 = 0$$

$$-4(2\lambda) - 2y + 60 = 0$$

$$-8\lambda - 2y = -60 \ldots \ldots \ldots (5)$$

Combine (5) with (1) since both are in y and λ

$-8\lambda - 2y = -60 \ldots \ldots \ldots (5)$

$y + 2 - 4\lambda = 0 \ldots \ldots \ldots (1)$

$\mathbf{2} \times (\mathbf{1})$: $2y - 8\lambda = -4 \ldots \ldots \ldots (6)$

Subtract (6) from (5)

$-8\lambda - 2y = -60 \ldots \ldots \ldots (5)$

$2y - 8\lambda = -4 \ldots \ldots \ldots (6)$

$-4y = -56$

Here, $y = 14$

From (3)

$-4x - 2y + 60 = 0$

Hence,

$$-4x - 2(14) + 60 = 0$$
$$-4x - 28 + 60 = 0$$
$$4x = 32$$
$$x = 8$$

The value of λ is of no necessity of no use to the solution, though it has a significance I won't like to bother you with, it can still be gotten though, as it is part of the simultaneous equation.

From (5);

$$-8\lambda - 2y = -60$$

$y = 14$;

$$-8\lambda - 2(14) = -60$$

$$-8\lambda = -32$$

$$\lambda = 4$$

Run back to our method solving using the direct substitution method, you can see it clearly that the answers are the same, hence, as it were, they aren't different at all, both methods are the same, also, the direct substitution method can be easier; true. However, the Lagrangean method has the huge edge over the direct substitution, it can work easily with double constraints and of course, it can still solve an optimization problem even when in the constraint function, a variable cannot be expressed in terms of the other. Hence, the Lagrangean method still has a huge edge, more so, it could be required specifically in your SSC106 exam and of course you know this is in **the SSC106 way**.

Let's see the second order conditions for the optimization of constrained functions using the Lagrangean method even though we're not asked to do so;

In the above example; let's bring;

$$\mathcal{L}_x = y + 2 - 4\lambda$$
$$\mathcal{L}_y = x - 2\lambda$$

$$\mathcal{L}_{xx} = 0 + 0 - 0 = 0$$
$$\mathcal{L}_{yy} = 0 - 0 = 0$$

Hence, you can see that they give zero values here which cannot help decide the nature of the optimal points; however, it doesn't mean they can't be determined, determining it is beyond the scope of this book though;

Let's see some few more examples and see where the second order conditions can be seen in the Lagrangean method;

- Using the Lagrangean method, optimize $8x^2 - xy + 12y^2$ subject to the constraint: $x = 42 - y$;
- Check for the nature of the optimal points.

Cool,
Here, our objective function is:

$$8x^2 - xy + 12y^2$$

Our constraint function is:

$$x = 42 - y$$

Express it equated to zero;

$$x + y - 42 = 0$$

Let's form our Lagrangean equation;

$$\mathcal{L}(x, y, \lambda) = 8x^2 - xy + 12y^2 - \lambda(x + y - 42)$$

Take the first order partials;

$$\mathcal{L}_x = 2 \times 8x^{2-1} - 1 \times x^{1-1} \times y + 0 - \lambda(1 \times x^{1-1} + 0 - 0)$$

$$\mathcal{L}_x = 16x - y - \lambda$$

$$\mathcal{L}_y = 0 - 1 \times x \times y^{1-1} + 2 \times 12y^{2-1}$$
$$- \lambda(0 + 1 \times y^{1-1} - 0)$$

$$\mathcal{L}_y = -x + 24y - \lambda$$

$$\mathcal{L}_\lambda = 0 - 0 + 0 - 1 \times \lambda^{1-1}(x + y - 42)$$

$$\mathcal{L}_\lambda = -x - y + 42$$

Notice we have not expanded λ in this situation, we're differentiating with the brackets in place; it makes our work simpler since each terms in the brackets are different; we can differentiate partially as we need; that's just a needless explanation anyway; you can expand the brackets before the partial differentiation process if you're confused about how we differentiated without expanding the bracket with λ; let's continue;

For the first order conditions, equate each to zero;

$$16x - y - \lambda = 0 \ldots \ldots \ldots (1)$$

$$-x + 24y - \lambda = 0 \ldots \ldots \ldots (2)$$

$$-x - y + 42 = 0$$

$$-x - y = -42 \ldots \ldots \ldots (3)$$

Let's solve these equations using Crammer's rule; we can use anyone we like;

Here, we have three equations;

$$16x - y - \lambda = 0 \ldots\ldots\ldots(1)$$

$$-x + 24y - \lambda = 0 \ldots\ldots\ldots(2)$$

$$-x - y = -42 \ldots\ldots\ldots(3)$$

The equivalent matrix determinant needed here is :

$$\Delta = \begin{vmatrix} 16 & -1 & -1 \\ -1 & 24 & -1 \\ -1 & -1 & 0 \end{vmatrix}$$

Evaluating this determinant;

$$\Delta = 16[(24)(0) - (-1)(-1)] - (-1)[(-1)(0) - (-1)(-1)] + (-1)[(-1)(-1) - (24)(-1)]$$

$$\Delta = 16(-1) + 1(-1) - 1(25)$$
$$\Delta = -16 - 1 - 25 = -42$$

To evaluate Δ_x, we replace the column of x with the column matrix of answers;

$$\Delta_x = \begin{vmatrix} 0 & -1 & -1 \\ 0 & 24 & -1 \\ -42 & -1 & 0 \end{vmatrix}$$

Evaluating this determinant;

$$\Delta_x = 0[(24)(0) - (-1)(-1)] \\ - (-1)[(0)(0) - (-1)(-42)] \\ + (-1)[(0)(-1) - (24)(-42)]$$

$$\Delta_x = 0(-1) + 1(-42) - 1(1008)$$
$$\Delta_x = 0 - 42 - 1008 = -1050$$

To evaluate Δ_y, we replace the column of y with the column matrix of answers;

$$\Delta_y = \begin{vmatrix} 16 & 0 & -1 \\ -1 & 0 & -1 \\ -1 & -42 & 0 \end{vmatrix}$$

Evaluating this determinant;

$$\Delta_y = 16[(0)(0) - (-1)(-42)] \\ - 0[(-1)(0) - (-1)(-1)] \\ + (-1)[(-1)(-42) - (0)(-1)]$$

$$\Delta_y = 16(-42) + 0(-1) - 1(42)$$
$$\Delta_y = -714$$

To evaluate Δ_λ, we replace the column of λ with the column matrix of answers;

$$\Delta_\lambda = \begin{vmatrix} 16 & -1 & 0 \\ -1 & 24 & 0 \\ -1 & -1 & -42 \end{vmatrix}$$

Evaluating this determinant;

$$\begin{aligned}\Delta_\lambda = 16[(24)(-42) - (0)(-1)] \\ - (-1)[(-1)(-42) - (0)(-1)] \\ + 0[(-1)(-1) - (-1)(24)]\end{aligned}$$

$$\Delta_\lambda = 16(-1008) + 1(42) - 0(25)$$

$$\Delta_\lambda = -16086$$

Hence, we have our x, y and λ values as;

$$x = \frac{\Delta_x}{\Delta} = \frac{-1050}{-42} = 25$$

$$y = \frac{\Delta_y}{\Delta} = \frac{-714}{-42} = 17$$

$$\lambda = \frac{\Delta_\lambda}{\Delta} = \frac{-16086}{-42} = 383$$

Hence, we have our optimal values; let's see the second part, we're asked to find the nature of the optimal values;

Head back to the first order partials; we need that of the partials with respect to x and y;

$$\mathcal{L}_x = 16x - y - \lambda$$

$$\mathcal{L}_y = -x + 24y - \lambda$$

$$\mathcal{L}_{xx} = \frac{\partial}{\partial x}(16x - y - \lambda)$$

$$\mathcal{L}_{xx} = 1 \times 16x^{1-1} - 0 - 0 = 16$$

$$\mathcal{L}_{yy} = \frac{\partial}{\partial y}(-x + 24y - \lambda)$$

$$\mathcal{L}_{yy} = 0 + 1 \times 24y^{1-1} - 0 = 24$$

Hence, we can see that the both second order partials are positive values; hence, from our second order conditions, they're minimum values and hence, our answer is concluded;

$\boldsymbol{x = 25}$ **and** $\boldsymbol{y = 17}$ are the optimum points (minimum points in this case) of the function: $8x^2 - xy + 12y^2$ subject to the constraint: $x = 42 - y$;

To get the optimum (minimum) value subject to the constraint, substitute the optimum points in the main objective function!

$$f(x, y) = 8x^2 - xy + 12y^2$$

$$f(25,17) = 8(25)^2 - (25)(17) + 12(17)^2$$

$$f(25,17) = 8043$$

Now, there's no point repeating all the examples in the method of direct substitution, it's all about extracting the objective and the constraint function and then finishing up in the Lagrangean equation!

One last example;

- Using the Lagrangean multiplier method, find a rectangular consumption basket which has the largest area for a given perimeter.

Here, we are not given any values but we know formulas; so who cares?

$$A = l \times b = lb$$

This is the area of the rectangular basket; we need the maximum area and hence the above is the objective function;

The perimeter is;

$$P = 2(l + b)$$

Expanding;

$$P = 2l + 2b$$

Hence, since it is a given perimeter, it means the perimeter is what the desire is meant to be fixed, hence, the constraint function is given above;

Regularly in Lagrangean functions, we'll express the constraint equaled to zero;

$$2l + 2b - P = 0$$

Hence, we'll write our Lagrangean function now with our Lagrangean multiplier involved;

$$\mathcal{L}(l, b, \lambda) = lb - \lambda(2l + 2b - P)$$

Here, our first order partials are:

$$\mathcal{L}_l = 1 \times l^{1-1}b - \lambda(1 \times 2l^{1-1} + 0 - 0)$$

$$\mathcal{L}_l = b - 2\lambda$$

$$\mathcal{L}_b = 1 \times lb^{1-1} - \lambda(0 + 1 \times 2b^{1-1} - 0)$$

$$\mathcal{L}_b = l - 2\lambda$$

$$\mathcal{L}_\lambda = 0 - 1 \times \lambda^{1-1}(2l + 2b - P)$$

$$\mathcal{L}_\lambda = -2l - 2b + P$$

Solve for the first order conditions;

$$\mathcal{L}_l = b - 2\lambda = 0 \ldots \ldots \ldots (1)$$

$$\mathcal{L}_b = l - 2\lambda = 0 \ldots \ldots \ldots (2)$$

$$\mathcal{L}_\lambda = -2l - 2b + P = 0 \ldots \ldots \ldots (3)$$

From (1);

$$b - 2\lambda = 0$$
$$b = 2\lambda \ldots \ldots \ldots (4)$$

$$l - 2\lambda = 0$$
$$l = 2\lambda \ldots \ldots \ldots (5)$$

From (4) and (5), it follows that;

$$l = b \ldots \ldots \ldots (6)$$

This is since both are equal to 2λ

Put (6) into (3);

$$-2l - 2b + P = 0$$

$$-2l - 2(l) + P = 0$$

Hence,

$$-P = -4l$$

Hence,

$$l = \frac{P}{4}$$

Since $l = b$

$$b = \frac{P}{4}$$

From (1);

$$b - 2\lambda = 0$$

$$\frac{P}{4} - 2\lambda = 0$$

$$\lambda = \frac{P}{8}$$

Hence, the optimal values for the largest area is when the length,

$$l = \frac{P}{4}$$

and the breadth,

$$b = \frac{P}{4}$$

And so, optimization is a very important concept generally in virtually all fields of the world such as mechanics, economics, engineering, research, geophysics and several other fields.

In economics and finances, the optimization problem has several uses such as the studies of utility maximization and expenditure minimization, profit maximization and also a whole range of economic equilibrium owning to interdependent optimizing of different factors affecting the economy such as labour, consumers, investors, government and etc.

Optimization problems are indispensable concepts generally in the field of economics, finances and administration.

You’ll be seeing a glimpse of the thorough use of both constrained and unconstrained optimization in real life economics and social sciences in the topic, in this book, that deals with the application of calculus to economic and social sciences.