DIFFERENTIAL EQUATIONS

Now that we have thrashed the two concepts of calculus which are differentiation and integration *(mind you at least for the SSC106 way)*, let's now move to a very interesting concept → **differential equations.** Promise it's a very simple stuff, you can have my word ☺☺☺ as far as it is within the scope of SSC106 and more importantly, as far as you understood the concepts of differentiation and integration. However, ***never try to dabble into this if you haven't studied both differentiation and integration*** in the previous section of this book.

So without further ado, let's start.

A differential equation is simply a mathematical relationship that describes the relationship between functions and their various derivatives (their various differential coefficients). In simple terms, a differential equation consists of derivative(s), could be just one derivative or more, identifying a differential equation is very easy; an equation where derivatives are shown (with mathematical operations such as adding, subtraction and etc) is simply a differential equation.

Let's move a bit quicker; the types of differential equations:

Differential equations can be classified into types but based on different criteria such as based on types, based on linearity, based on homogeneity and so on, the criteria of classification is actually far from exhaustive but for the scope of SSC106, we'll see it's classification based on its type only, I'm sure you prefer it that way!

CLASSIFYING DIFFERENTIAL EQUATIONS BASED ON TYPES

- **An ordinary differential equation** is a differential equation involving one independent and one dependent variable. For example, an ordinary equation in y and x(with y the dependent variable and x the independent variable) will involve only differential coefficients of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and so on. Examples of ordinary differential equations are given below:

$$\frac{dy}{dx} + 5x = 5y$$

$$\frac{d^2y}{dx^2} = 3^{x+y}$$

$$\frac{d^2u}{dx^2} - x\frac{du}{dx} + u = 0$$

- **A partial differential equation** involves partial derivatives. It is a differential equation involving a dependent variable and more than one independent variable. For example, partial differential equations will include differential coefficients of $\frac{\partial y}{\partial x}, \frac{\partial y}{\partial t}$ in a single equation. Examples of partial differential equations are given below:

$$\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} = x^3 - t^3$$

$$\frac{\partial^2 y}{\partial x^2} - c^2\frac{\partial^2 y}{dt^2} = x$$

In summary, an ordinary differential equation is a differential equation involving one independent variable while a partial differential equation is a differential equation involving more than one independent variable.

THE ORDER AND DEGREE OF DIFFERENTIAL EQUATIONS

The order and degree of differential equations doesn't have an exhaustive list as it were; let's see what the order and degree of a differential equation talks about:

The order of a differential equation is the highest derivative involved in the differential equation. You recall the concept of higher derivatives right where we have derivatives in the order of $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$ and so on. The order of a differential equation is the highest order of derivative present in that equation, I'm sure that is easy to understand, let's see from these illustrations.

$$x\frac{dy}{dx} - y^2 = 0$$

is a differential equation of the first order since the highest derivative in the equation is $\frac{dy}{dx}$ which is a first derivative.

$$xy\frac{d^2y}{dx^2} - y^2 \sin x = 5$$

is a differential equation of the second order since the highest derivative in the equation is $\frac{d^2y}{dx^2}$ which is a second derivative.

$$\frac{d^3y}{dx^3} - y\frac{dy}{dx} + e^{4x} = 0$$

is a differential equation of the third order since the highest derivative in the equation is $\frac{d^3y}{dx^3}$ which is a third derivative. Note that $\frac{dy}{dx}$ is also present but the highest derivative present is $\frac{d^3y}{dx^3}$ which is a third derivative, hence it is a third order differential equation.

Partial derivatives also have their order and the rule is no different at all, the highest derivative present wins the contest for the order of the differential equation. Here:

$$\frac{\partial u}{\partial t} = 6u\frac{\partial u}{dx} - \frac{\partial^3 u}{\partial x^3}$$

is a differential equation of the third order as obviously the highest derivative in the equation is $\frac{\partial^3 u}{\partial x^3}$ which is a third partial derivative.

Now, **the degree of a differential equation** is determined after the order of the differential equation has been determined. The degree of a differential equation is the power the highest derivative has been raised to. Hence, that derivative that won the contest of the order of a differential equation, the power it is raised is called the degree of the differential equation; from the illustration below:::

$$\frac{dy}{dx} - \cos x = 0$$

has a degree 1 since the highest derivative, $\frac{dy}{dx}$ is raised to a power of 1.

Confused? You shouldn't be, let's see this:

$$\frac{\partial u}{\partial t} = 6u\frac{\partial u}{dx} - \frac{\partial^3 u}{\partial x^3}$$

has a degree 1 since the highest derivative, $\frac{\partial^3 u}{\partial x^3}$ is raised to a power of 1.

$$\left(\frac{d^2y}{dx^2}\right)^4 - xy\frac{dy}{dx} = 2x - y$$

has a degree of the 4 since the highest derivative, $\frac{d^2y}{dx^2}$ is raised to a power of 4.

Check out this example:

EXAMPLE I:

If a and b are the order and degree, respectively of the following differential equation, find the value of $b - a$.

$$\left(\frac{d^4y}{dx^4}\right)^8 - \left(\frac{dy}{dx}\right)^{25} + 15 = 0;$$

Now, as you can obviously see, the highest derivative in this equation is $\frac{d^4y}{dx^4}$ which outlines the fact that the order of this differential equation is 4.

Now, don't be rabble-roused by the large figure of 25 to take it as the degree of the differential equation, the degree is 8 please as the highest derivative, $\frac{d^4y}{dx^4}$, is raised to a power of 8.

We're told to find $b - a$ where a and b are the order and degree of the differential equation respectively, hence, a, the order, is 4;
b, the degree, is 8.

Therefore, $b - a = 8 - 4 = 4$. As simple as that!

EXAMPLE II:

Find the order and degree of the differential equation given below:

$$\frac{d^3y}{dx^3} + 12\left(\frac{d^2y}{dx^2}\right)^3 = x^3\frac{d^2y}{dx^2};$$

This is not too different from what we've done; however, I'm drawn to the little fact that the appearance of $\frac{d^2y}{dx^2}$ twice could mislead you to taking the order as 2 and consequently the degree as 3; however, whether it appears a million time isn't your concern, the most important thing is to find the highest derivative and forget every other thing: Therefore, the highest derivative is $\frac{d^3y}{dx^3}$ and consequently, the order of this differential equation is 3. Let's leave it that simple.

The degree of the differential equation is therefore 1 since $\frac{d^3y}{dx^3}$ is raised to a power of 1. **Full stop.**

Err, wait, before the **full stop**, watch out for $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}$ and other higher derivatives as they can easily deceive you to you thinking them as the degree of differential equations. The degree remains the power of the differential coefficient and not the order of the differential coefficients. So long as these three listed and other higher appear as shown above, their degrees are 1. If they appear like this $\left(\frac{d^2y}{dx^2}\right)^2, \left(\frac{d^3y}{dx^3}\right)^2, \left(\frac{d^4y}{dx^4}\right)^2$, their degrees are 2 and so on.

One last example;

EXAMPLE III:

Find the order and degree of the differential equation given below:

$$4\frac{d^3y}{dx^3} + \sqrt{12\left(\frac{d^2y}{dx^2}\right)^3 + 4y^3} = x^2y\frac{d^4y}{dx^4}$$

$$4\frac{d^3y}{dx^3} + \sqrt{12\left(\frac{d^2y}{dx^2}\right)^3 + 4y^3} = x^2y\frac{d^4y}{dx^4}$$

In a differential equation, finding the order is as good as seeing the highest derivative in the differential equation. Hence, the order of the differential equation is 4 since the highest derivative is $\frac{d^4y}{dx^4}$

However, for the degree, all roots in a differential equation must be cleared before anything can be done to the differential equation. Hence, we must work a bit on this differential equation before we can get the degree of the differential equation. Hence;

$$4\frac{d^3y}{dx^3} + \sqrt{12\left(\frac{d^2y}{dx^2}\right)^3 + 4y^3} = x^2y\frac{d^4y}{dx^4}$$

Isolate the root; we have:

$$\sqrt{12\left(\frac{d^2y}{dx^2}\right)^3 + 4y^3} = x^2y\frac{d^4y}{dx^4} + 4\frac{d^3y}{dx^3}$$

Square both sides; we have;

$$\left(\sqrt{12\left(\frac{d^2y}{dx^2}\right)^3+4y^3}\right)^2=\left(x^2y\frac{d^4y}{dx^4}+4\frac{d^3y}{dx^3}\right)^2$$

The root cancels off on the left, we'll expand the right;

$$12\left(\frac{d^2y}{dx^2}\right)^3+4y^3 = \left(x^2y\frac{d^4y}{dx^4}+4\frac{d^3y}{dx^3}\right)\left(x^2y\frac{d^4y}{dx^4}+4\frac{d^3y}{dx^3}\right)$$

$$12\left(\frac{d^2y}{dx^2}\right)^3+4y^3 = \left(x^2y\frac{d^4y}{dx^4}\right)\left(x^2y\frac{d^4y}{dx^4}\right)+\left(x^2y\frac{d^4y}{dx^4}\right)\left(4\frac{d^3y}{dx^3}\right)+\left(4\frac{d^3y}{dx^3}\right)\left(x^2y\frac{d^4y}{dx^4}\right)+\left(4\frac{d^3y}{dx^3}\right)\left(4\frac{d^3y}{dx^3}\right)$$

We have;

$$12\left(\frac{d^2y}{dx^2}\right)^3+4y^3 = (x^2y)^2\left(\frac{d^4y}{dx^4}\right)^2+4x^2y\left(\frac{d^4y}{dx^4}\right)\left(\frac{d^3y}{dx^3}\right)+4x^2y\left(\frac{d^4y}{dx^4}\right)\left(\frac{d^3y}{dx^3}\right)+16\left(\frac{d^3y}{dx^3}\right)^2$$

$$12\left(\frac{d^2y}{dx^2}\right)^3 + 4y^3$$
$$= x^4y^2\left(\frac{d^4y}{dx^4}\right)^2 + 8x^2y\left(\frac{d^4y}{dx^4}\right)\left(\frac{d^3y}{dx^3}\right)$$
$$+ 16\left(\frac{d^3y}{dx^3}\right)^2$$

Hence, we have the simplified differential equation as shown above with the roots removed; hence, obviously here;

The power of $\frac{d^4y}{dx^4}$ is 2 after the simplification and hence the degree of the differential equation is 2.

Expansion shouldn't be too difficult for you, we treated the concept of expansion in the basic operations in mathematics section.

EXAMPLE IV: Find the order and degree of the differential equation given below:

$$\frac{d^3y}{dx^3} + 6\sqrt{\left(\frac{dy}{dx}\right)^2} + y = 0$$

A differential equation must be simplified to an equation without roots before the degree is found, the order doesn't need simplification as it is straightforward in the highest derivative in the equation.

The highest derivative in the above is $\frac{d^3y}{dx^3}$, hence, the order is 1.

$$\frac{d^3y}{dx^3} + 6\sqrt{\left(\frac{dy}{dx}\right)^2} + y = 0$$

As we know, we must simplify and remove all the roots before we find the degree. The simplification in this case has no much troubles though, within the square root, we have;

$$\sqrt{\left(\frac{dy}{dx}\right)^2} = \left(\left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}} = \left(\frac{dy}{dx}\right)^{2\times\frac{1}{2}} = \left(\frac{dy}{dx}\right)$$

We have:

$$\frac{d^3y}{dx^3} + 6\left(\frac{dy}{dx}\right) + y = 0$$

Hence, after simplification, the highest derivative is raised to a power of 1 and hence, the degree is 1.

SOLVING DIFFERENTIAL EQUATIONS

This is where the real work is now; however, as I have assured you in the beginning of this chapter, for the scope of SSC106, you have just quite a few worries. Basically, we're going to be limited to solving first order differential equations, also, you just need to settle down and follow step-by-step as we dissect it together just in a moment.

Now, we have seen several differential equations above and seeing the nature of what differential equations are. Now, firstly, let's see how differential equations are formed; forming differential equations and solving differential equations are two directly opposite processes.

Now, let's get the ball rolling... ... ...

Formation of differential equations:

Forming differential equations is quite a very pretty easy stuff to do, solving it is quite complex a bit though. For the time being, let's focus on the forming we're here to do.

Alright! **Example 1:**

EXAMPLE 1:

Derive a differential equation from the function:

$$y = x + \frac{A}{x}$$

where A is constant.

Okay. So, let me give you the first hint, the master hint to formation of differential equations.

So here is it: When you have one arbitrary constant, the differential equation to be formed will be of the first order.

When you have two arbitrary constants, the differential equation to be formed will be of the second order.

You've gotten that in no time right??? Why not? That's simple enough:

When you have three arbitrary constants, the differential equation to be formed will be of the third order. And so on.

So… … … How many arbitrary constants are in this question? One, which is A, hence, the differential equation formed will be of the first order, that implies that the highest derivative in the differential equation will be $\frac{dy}{dx}$ since it is a function in which y depends on x.

That means all we need to find is $\frac{dy}{dx}$, manipulate it and we're done.

So… … …

$$y = x + \frac{A}{x}$$

can be expressed as:

$$y = x + Ax^{-1}$$

$$\frac{dy}{dx} = 1x^{1-1} + (-1 \times A \times x^{-1-1})$$

Sure differentiation of functions is no issue for you.

$$\frac{dy}{dx} = x^0 - Ax^{-2}$$

$$\frac{dy}{dx} = 1 - \frac{A}{x^2}$$

So, we're almost through. Forming differential equations is that straightforward. Now, how do we go about this?

Forming differential equations involve having equations in term of their derivative with the arbitrary constant eliminated. Hence, it can be done in two ways, either substitution is made from the derivative equation to the original equation or from the original equation to the derivative equation. The choice of what to substitute in terms of the other is checked by convenience.

Go back to the initial equation and express A in terms of x and y (this is because it is easy to make A the subject in the initial equation).

Hence, from the equation, $y = x + \frac{A}{x}$

$$\frac{A}{x} = y - x$$

$$A = x(y - x)$$

So, carry this A that is expressed in terms of x and y into the $\frac{dy}{dx}$ own ☺;

$$\frac{dy}{dx} = 1 - \frac{A}{x^2}$$

Put $A = x(y - x)$

$$\frac{dy}{dx} = 1 - \frac{x(y - x)}{x^2}$$

x is reduced by indices in $\frac{x(y-x)}{x^2}$

We have the differential equation as

$$\frac{dy}{dx} = 1 - \frac{(y - x)}{x}$$

Clear the equation with x;

$$x\frac{dy}{dx} = x - (y - x)$$

$$x\frac{dy}{dx} = x - y + x$$

$$x\frac{dy}{dx} = 2x - y$$

Above is the needed differential equation.

That's easy enough or what do you think? ☺

Let's see this one:

EXAMPLE 2: Form a differential equation from $u = Ax^3 + 2x^2$

Cool, we have just one arbitrary constant here; which is A, 2 isn't an arbitrary constant, sure you understood that from our studies of integration. Hence, it'll lead us to a first order differential equation, we need only $\frac{du}{dx}$; yeah, this one is a function where u depends on x.

$$u = Ax^3 + 2x^2$$

$$\frac{du}{dx} = 3 \times Ax^{3-2} + 2 \times 2x^{2-1}$$

$$\frac{du}{dx} = 3Ax^2 + 4x$$

Now, we have $\frac{du}{dx}$, what next?

Express A in terms of u and x from the first equation and substitute in $\frac{du}{dx}$; again, it is very easy to make A the subject in the initial equation and hence, we'll do so; simple…

$$u = Ax^3 + 2x^2$$

$$Ax^3 = u - 2x^2$$

$$A = \frac{u - 2x^2}{x^3}$$

Take this to meet $\frac{du}{dx}$

$$\frac{du}{dx} = 3Ax^2 + 4x$$

Substitute; $A = \frac{u-2x^2}{x^3}$

$$\frac{du}{dx} = 3\left(\frac{u - 2x^2}{x^3}\right)x^2 + 4x$$

Clear out x^2;

$$\frac{du}{dx} = 3\left(\frac{u - 2x^2}{x}\right) + 4x$$

Multiply through by x

$$x\frac{du}{dx} = 3(u - 2x^2) + 4x^2$$

I guess that's it! We can leave it this way!

Some more to go, let's see where two arbitrary constants are in the original equation. You should know that we'll reach the second derivative in that case. Right? Example 3!

EXAMPLE 3: Form a differential equation from $y = Ax^2 + Bx$; where A and B are constants.

Cool, here, we have two arbitrary constants!

I don't need excessive explanations here; we'll be reaching $\frac{d^2y}{dx^2}$ here since two arbitrary constants are involved, the resulting differential equation will be of the second order. Straight up!

$$y = Ax^2 + Bx$$

$$\frac{dy}{dx} = 2 \times Ax^{2-1} + 1 \times Bx^{1-1}$$

$$\frac{dy}{dx} = 2Ax + B$$

Let's get $\frac{d^2y}{dx^2}$ from here, by differentiating $\frac{dy}{dx}$:

$$\frac{d^2y}{dx^2} = 1 \times 2Ax^{1-1} + 0$$

$$\frac{d^2y}{dx^2} = 2A$$

So settle down and see how we'll manipulate it to eliminate A and B from our differential equation which will be of the second order.

It is very easy to make A the subject of the relation in $\frac{d^2y}{dx^2}$ and hence, we'll start from here;

Here, from:

$$\frac{d^2y}{dx^2} = 2A$$

$$A = \frac{1}{2}\frac{d^2y}{dx^2}$$

We’ll then substitute it into this:

$$\frac{dy}{dx} = 2Ax + B$$

we have;

$$\frac{dy}{dx} = 2\left(\frac{1}{2}\frac{d^2y}{dx^2}\right)x + B$$

Here, we can now get an expression for B by making B the subject of the relation;

$$\frac{dy}{dx} = x\frac{d^2y}{dx^2} + B$$

$$B = \frac{dy}{dx} - x\frac{d^2y}{dx^2}$$

Now that we have an expression for A and B in terms of their derivatives, we can substitute them in the initial equation:

$$y = Ax^2 + Bx$$

We’ll come out with our differential equation.

$$y = Ax^2 + Bx$$

Substitute:

$$A = \frac{1}{2}\frac{d^2y}{dx^2} \quad \text{and} \quad B = \frac{dy}{dx} - x\frac{d^2y}{dx^2}$$

$$y = \left(\frac{1}{2}\frac{d^2y}{dx^2}\right)x^2 + \left(\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right)x$$

We have:

$$y = \frac{x^2}{2}\frac{d^2y}{dx^2} + x\frac{dy}{dx} - x^2\frac{d^2y}{dx^2}$$

There are two terms in $\frac{d^2y}{dx^2}$ which we can add up;

$$y = x\frac{dy}{dx} + \frac{x^2}{2}\frac{d^2y}{dx^2} - x^2\frac{d^2y}{dx^2}$$

$$y = x\frac{dy}{dx} + \left(\frac{x^2}{2} - x^2\right)\frac{d^2y}{dx^2}$$

Finally, adding up the terms in the bracket;

$$y = x\frac{dy}{dx} - \frac{x^2}{2}\frac{d^2y}{dx^2}$$

That's our differential equation.

***Don't get it twisted, in the initial one including one arbitrary constant, we expressed for A from the first equation and substituted in the first derivative;

However, here, we started from the second derivative, found an expression for the first arbitrary constant (A), moved to the first derivative and found an expression for the second arbitrary constant (B) and finally substituted all into the main function.
I've explained it already, we substitute by what it is easy to make the subject of the relation, and hence, anywhere it is convenient to make an arbitrary constant the subject of the relation, it is made there and then substituting into the next equation.

We could've also started the initial one including one arbitrary constant, from the first derivative too, however, we have used the most convenient. Just follow it calmly!

Let's see some cases with implicit functions;

EXAMPLE 4: Form a differential equation:

$$x^2 - e^y = a$$

where a is a constant.

Cool, here, we have one arbitrary constant since the only arbitrary constant is a and hence, we need just $\frac{dy}{dx}$!

Here; we have a case of an implicit function, it was introduced briefly at the end of differentiation and turns out, it's very much useful, we know the rule of implicit differentiation, when we differentiate y, we include $\frac{dy}{dx}$ and when we differentiate x, it is left as it is: this is actually a simple case, we won't be breaking it down like we did there, we'll do it right though, if you have not studied implicit differentiation, kindly head to page 174 of the chapter differentiation, you're not expected to be skipping topics.

$$2x - e^y \frac{dy}{dx} = 0$$

Hence;

We have:

$$\frac{dy}{dx} = \frac{2x}{e^y}$$

Normally, from normal forming of differential equations, as we have seen in the last three, we would've have needed to go to the initial equation and express the arbitrary constant in terms of the variable and substitute into the differential equation (or vice-versa), however, here, since the arbitrary constant isn't in the first derivative, the differential equation remains like that and needs no substitution, the goal of having an equation in terms of derivatives without the arbitrary constant has been achieved already.

EXAMPLE 5: Form a differential equation:

$$y^2 - ax + a^2 = 0$$

where a is a constant.

This is another case of implicit differentiation;

Another simple case, we know the rule of implicit differentiation, the implicit differentiation process is shown below;

$$\left(2 \times y^{2-1}\frac{dy}{dx}\right) - (1 \times ax^{1-1}) + 0 = 0$$

Note that a^2 is a constant and its derivative is zero!

Hence;

$$2y\frac{dy}{dx} - a = 0$$

From here;

It is more convenient to make a the subject in the equation for $\frac{dy}{dx}$ than the initial equation, hence, we make a the subject here and substitute into the original equation. Hence, from here;

$$a = 2y\frac{dy}{dx}$$

We have:

Substitute this into the initial equation;

$$y^2 - ax + a^2 = 0$$

Hence;

$$y^2 - \left(2y\frac{dy}{dx}\right)x + \left(2y\frac{dy}{dx}\right)^2 = 0$$

Expanding;

$$y^2 - 2xy\frac{dy}{dx} + (2y)^2\left(\frac{dy}{dx}\right)^2 = 0$$

$$y^2 - 2xy\frac{dy}{dx} + 4y^2\left(\frac{dy}{dx}\right)^2 = 0$$

Above is the needed differential equation.

Last example!

EXAMPLE 6: Form a differential equation:

$$x^2y - 3\sin x = 3a^2$$

where a is a constant.

This is another case of implicit differentiation;

Another simple case, we know the rule of implicit differentiation, the implicit differentiation process is shown below;

$$\left(2 \times x^{2-1}y + 1 \times y^{1-1}\frac{dy}{dx}\right) - \left(3(\cos x)\right) = 0$$

Note that $3a^2$ is a constant and its derivative is zero! Derivative of $\sin x = \cos x$

Hence;

$$2xy + \frac{dy}{dx} - 3\cos x = 0$$

From here; the equation containing the derivative has no arbitrary constant and hence, requires no substitution, the equation in itself is the required differential equation. Once the derivative contains no arbitrary constant, it is the needed differential equation.

Solving differential equations:

So okay! Solving differential equations involves a more discrete process; we cannot go ahead to attack all the differential equations we've formed in the previous section and decide to jump into solving them like that, *lol*, it's not done that way. Differential equations are solved according to their types and orders; as for SSC106 first of all, we won't be solving the differential equations of the second order at all. We'll be limited to first order equations too! And even then, we'll also not exhaust the types of possible situations in the first order differential equations.

Solving first order differential equations:

Solving first order differential equations is quite simple, at least, for a start, however, could get trickier as it moves on: however, for the limit of SSC106, it's quite pretty easy, excited, isn't it? ☺

There are quite a number of methods of solving first order differential equations which is constrained to the type of differential equation you have to solve:

Methods include:
- By direct integration
- By separating variables
- Substitution in homogenous equations
- The use of integrating factors in linear equations
- The Bernoulli's equation

We'll be considering the first two only! And hence, as you can see, you won't know much about differential equations after all.

Cool... ... ..., let's go!

The method of direct integration: This method of solving first order differential equation is used when asides the differential coefficients (the derivative), every other term are functions of just one variable which is either the dependent variable or the independent variable. Such equations are in the form:

$$\frac{dy}{dx} = f(x); \qquad \frac{dy}{dx} = f(y);$$

Here, it's just a simple case solved by direct integration; Example!

EXAMPLE 7: Solve the differential equation:

$$\frac{dy}{dx} = 3x^2 - 6x + 5$$

Now, this is just one very simple question; it's not different from our studies of integration:

Here: this is in the form:

$$\frac{dy}{dx} = f(x)$$

$$\frac{dy}{dx} = 3x^2 - 6x + 5$$

Multiply both sides by dx;

$$dy = (3x^2 - 6x + 5)\, dx$$

We'll simply integrate both sides; with respect to their respective variables;

$$\int dy = \int (3x^2 - 6x + 5)\, dx$$

Of course you know the integral of dy is simply y as it is like the integral, $\int 1\, dy$ which is $\frac{y^{0+1}}{0+1}$ which gives y.

The right hand side is the integral

$$\int (3x^2 - 6x + 5)\, dx$$

which can be expressed in terms of its separate integrals;

Here;

$$y = \int 3x^2\,dx - \int 6x\,dx + \int 5\,dx$$

$$y = \frac{3x^{2+1}}{2+1} - \frac{6x^{1+1}}{1+1} + \frac{5x^{0+1}}{0+1} + C$$

And of course your arbitrary constant is added;

$$y = \frac{3x^3}{3} - \frac{6x^2}{2} + \frac{5x}{1} + C$$

$$y = x^3 - 3x^2 + 5x + C$$

Done! It's that easy. NEXT.

EXAMPLE 8: Solve the differential equation

$$x\frac{dy}{dx} = 5x^3 + 4$$

Okay! This looks as if it's not what we just did; however, it's still the same thing; it's still a situation of just one variable which will yield direct substitution, just check it out and see.

$$x\frac{dy}{dx} = 5x^3 + 4$$

Now divide through by x;

$$x\frac{dy}{dx} \times \frac{1}{x} = \frac{5x^3}{x} + \frac{4}{x}$$

Recall dividing through and multiplying by the reciprocal are same processes;

$$\frac{dy}{dx} = 5x^2 + 4x^{-1}$$

Now, we're back in the form: $\dfrac{dy}{dx} = f(x)$

Solve like the immediate previous example; multiplying both sides by dx;

$$dy = (5x^2 + 4x^{-1})\, dx$$

Taking integrals

$$\int dy = \int (5x^2 + 4x^{-1})\, dx$$

Integrating separating;

$$y = \int 5x^2\, dx + \int 4x^{-1}\, dx$$

Now, recall the integral of a power of -1 is the natural logarithm;

$$y = \frac{5x^{2+1}}{2+1} + 4\ln x + C$$

Of course your arbitrary constant is added.

$$y = \frac{5x^3}{3} + 4\ln x + C$$

I guess that's it; easy right? Yes, it is. Two more;

EXAMPLE 9: Solve the differential equation;

$$\frac{dy}{dx} = (y-1)(y-2)$$

Fine, here we have the equation in the form

$$\frac{dy}{dx} = f(y)$$

See how it's done!

$$\frac{dy}{dx} = (y-1)(y-2)$$

Take reciprocals of both sides;

$$\frac{dx}{dy} = \frac{1}{(y-1)(y-2)}$$

We'll multiply through by dy

$$dx = \frac{dy}{(y-1)(y-2)}$$

Now this right hand side is a simple equation in one variable that can be integrated with respect to y; the left hand side is nothing special.

$$\int dx = \int \frac{dy}{(y-1)(y-2)}$$

We'll break $\frac{1}{(y-1)(y-2)}$ into partial fractions; we've learnt that thoroughly during the studies of integration.

Let's evaluate this integral:

$$\int \frac{dy}{(y-1)(y-2)}$$

The left hand side is readily broken down into:

$$\frac{1}{(y-1)(y-2)} \equiv \frac{A}{y-1} + \frac{B}{y-2}$$

Multiplying through by $(y-1)(y-2)$; we have:

$1 = A(y-2) + B(y-1)$; put $y = 1$

$1 = A(1-2) + B(1-1);$

$1 = A(-1)$

$$A = -1$$

$1 = A(y-2) + B(y-1)$; put $y = 2$

$1 = A(2-2) + B(2-1);$

$1 = B(1);$

$$B = 1$$

$$\therefore \frac{1}{(y-1)(y-2)} \equiv \frac{-1}{y-1} + \frac{1}{y-2}$$

$$\int \frac{dy}{(y-1)(y-2)} = \int \frac{-1}{y-1} dy + \int \frac{1}{y-2}\, dy$$

$$\int \frac{dy}{(y-1)(y-2)} = -\frac{\ln(y-1)}{1} + \frac{\ln(y-2)}{1}$$

Rearrange:

$$\int \frac{dy}{(y-1)(y-2)} = \ln(y-2) - \ln(y-1)$$

$$\int \frac{dy}{(y-1)(y-2)} = \ln\frac{(y-2)}{(y-1)}$$

Therefore, back to our differential equation:

$$\int dx = \int \frac{dy}{(y-1)(y-2)}$$

$$x = \ln\frac{(y-2)}{(y-1)} + C$$

And that's it! Of course the arbitrary constant is always added in integration.

There's nothing more to explain per say, however, there's something slight to introduce over here;

EXAMPLE 10: Solve the equation:

$$e^{x^2}\frac{dy}{dx} = 4x$$

given that $y = 3$ when $x = 0$;

Now, this is nothing different from what we've been doing actually. However, we can see a little bit unfamiliar stuff at the end of that question.

Now,

$$e^{x^2}\frac{dy}{dx} = 4x$$

Divide through by e^{x^2} to have a one variable function on our right hand side;

$$\frac{dy}{dx} = \frac{4x}{e^{x^2}}$$

Multiply both sides by dx;

$$dy = \frac{4x}{e^{x^2}}\,dx$$

Integrate the right hand side and the left hand side is integrated straight-off to y.

$$\int dy = \int \frac{4x}{e^{x^2}}\,dx$$

Now, let's integrate the right hand side;

$$\int \frac{4x}{e^{x^2}}\,dx$$

Put $u = x^2$

This is a case of the integral of:

$$\int f'(x)\,g[f(x)]dx$$

$$\frac{du}{dx} = \frac{d}{dx}(x^2)$$

$$\frac{du}{dx} = 2x$$

Here:

$$dx = \frac{du}{2x}$$

Since $u = x^2$;

$$\int \frac{4x}{e^u}\frac{du}{2x}$$

Cancel out x; we'll now have:

$$\int \frac{2}{e^u}\, du$$

We have:

$$2\int \frac{1}{e^u}\, du$$

Subsequently we know: $\frac{1}{e^u} = e^{-u}$; hence;

$$2\int e^{-u}\, du$$

Again, Put $z = -u$

$$\frac{dz}{du} = -1$$

Hence,

$$du = -dz$$

We have,

$$2\int e^z \times -dz$$

We have:

$$-2\int e^z\, dz$$

We can now integrate since the integral of e^z is still e^z; Hence, we have:

$$-2(e^z)$$

As the integral;

Substitute;

$$-2e^z \text{ and } z = -u$$

and $u = x^2$; we have:

$$-2e^{-x^2}$$

By indices law, we have

$$-\frac{2}{e^{x^2}}$$

We can now go back to our integral;

$$-2 \times -\frac{2}{e^{x^2}} = \frac{4}{e^{x^2}}$$

Hence;

$$\int dy = \int \frac{4x}{e^{x^2}}\, dx$$

$$y = -\frac{2}{e^{x^2}} + C$$

Of course the arbitrary constant is added. Now, after such a long journey; we have our answer.

However, there is something called a general solution and a particular solution. The general solution involves the arbitrary constant since no information is given as to how we'll get the true value of the arbitrary constant. However, a particular solution can be gotten when information is given concerning corresponding values of the variables involved in the differential equation.

Here, we are told $y = 3$ when $x = 0$; So, what do we do? Simply substitute y and x into this equation you're given and solve for your constant C.

$$y = -\frac{2}{e^{x^2}} + C$$

$y = 3$ when $x = 0$

$$3 = -\frac{2}{e^{0^2}} + C$$

$$3 = -\frac{2}{e^0} + C$$

$e^0 = 1$; hence;

$$3 = -\frac{2}{1} + C$$

$$C = 5$$

Therefore, the particular solution is given by:

$$y = -\frac{2}{e^{x^2}} + 5$$

The above is gotten by putting the actual value of C, in that way, problems are solved specifically and not with an arbitrary constant.

So, I believe that is fully understood; you should understand perfectly so long you understood differentiation and integration **especially the**

aspect of integration. As you can see for yourself, integration is very key in understanding of differential equations.

So… … … Without further ado, it's time to move to the method of separating variables.

The method of separating variables: This method of solving first order differential equations is used when asides the differential coefficients (i.e. the dy and the dx part), separate functions of the two variables involved (the independent and dependent variables) can be expressed as products (two terms multiplied together) or quotients (two terms divided by each other). Sums and differences between functions of the two variables aren't allowed in this method and separate functions multiplied and divided alone. The method is solved by separating the separate functions into their corresponding separate differential coefficients to their various functions. For example, the differential coefficients for variables y and x in the form, $\frac{dy}{dx}$, the functions of y are taken to meet dy and the functions of x are taken to meet with dx, that's the simple logic.

Now, this story, understand it as you may, is useless without we applying this into an example!

Let's have this example!

EXAMPLE 11: Solve the differential equation:

$$\frac{dy}{dx} = \frac{3y}{2x}$$

Now, the first thing to do over here is to separate dy and dx into their appropriate locations in the left hand side and right hand side respectively such that both of them are at the numerators, this can be done by taking dx to the right hand side since the dy is already on the numerator on the left hand side.

$$dy = \frac{3y}{2x}\, dx$$

Now, separate $3y$ which is a function of y to meet dy and $2x$ which is a function of x to meet dx. Now, note that this isn't done anyhow but with valid mathematical process, hence, here, $3y$ is crossed into the left hand side, while $2x$ since is already with dx is left there.

Now, we have:

$$\frac{dy}{3y} = \frac{dx}{2x}$$

The next thing to do is to integrate separately with respect to their respective variables;

$$\int \frac{dy}{3y} = \int \frac{dx}{2x}$$

Let's integrate the left hand side and right hand side separately; it's better slow than wrong;

LHS:

$$\int \frac{dy}{3y}$$

Put $u = 3y$;

$$\frac{du}{dy} = 3$$

Here,

$$dy = \frac{du}{3}$$

Yielding;

$$\int \frac{1}{u}\frac{du}{3}$$

We now have:

$$\frac{1}{3}\int \frac{du}{u}$$

Integrate straight where this is a direct integral rule.

$$\frac{1}{3}\int \frac{du}{u} = \frac{1}{3}(\ln u) = \frac{1}{3}(\ln 3y)$$

RHS:

$$\int \frac{dx}{2x}$$

Put $z = 2x$;

$$\frac{dz}{dx} = 2$$

Here,

$$dx = \frac{dz}{2}$$

Yielding;

$$\int \frac{1}{z}\frac{dz}{2} = \frac{1}{2}\int \frac{dz}{z}$$

We now have:

$$\frac{1}{2}\int \frac{dz}{z}$$

Integrate straight where this is also a direct integral rule.

$$\frac{1}{2}\int\frac{dz}{z}=\frac{1}{2}(\ln z)=\frac{1}{2}(\ln 2x)$$

Hence, $LHS = RHS$, our arbitrary constant is added just once since the sum of arbitrary constants is still an arbitrary constant; hence, we add one arbitrary constant;

$$\frac{1}{3}(\ln 3y)=\frac{1}{2}(\ln 2x)+C$$

This is not just all, differential equations are simplified to the maximum level; multiply through by 6

$$2(\ln 3y)=3(\ln 2x)+6C$$

Since;

$$a\log b=\log b^{a}$$

$$2(\ln 3y)=\ln(3y)^{2}$$

$$3(\ln 2x)=\ln(2x)^{3}$$

Hence; we have:

$$\ln(3y)^2 = \ln(2x)^3 + 6C$$

We have: $6C$ is still one random constant; put D

Let me explain in detail;

For any arbitrary constant like 3, if multiplied by 6, we'll have $6 \times 3 = 18$. 18 is another arbitrary constant and hence, rather than going around with $6(3)$, 18 makes it shorter.

$$\ln 9y^2 = \ln 8x^3 + D$$

D is a natural log of any random constant; put $\ln A$; don't get it twisted, we only need it for the sake of simplification; since two logs added can be simplified; and constants lead to other constants; the log of a constant is another constant and so on.

Let me explain in detail;

The value of $\log 2 = 0.3010$.

2 is a constant, and 0.3010 is a constant as well, hence, if 0.3010 had been D, then, it means it could have has well been represented as $\log A$ where A is now 2. We make use of such

simplifications for the sake of having something uniform to simplify equations together. A logarithm cannot be added to a number but can be added to a fellow logarithm. Hence, it is better to express 0.3010 as $\log 2$ to allow for simplification. Who cares as far as we are dealing with arbitrary constants, they can be manipulated anyhow. Hence, we have the situation shown below:

$$\ln 9y^2 = \ln 8x^3 + \ln A$$

Of course you know log rules;

$$\log a + \log b = \log(ab)$$

$$\ln 9y^2 = \ln(8x^3 \times A)$$

$$\ln 9y^2 = \ln 8Ax^3$$

Hence, the log numbers can be equated;

$$9y^2 = 8Ax^3$$

Where A is a constant. That's the solution of the differential equation;

EXAMPLE 12: Solve the differential equation:

$$\frac{dy}{dx} = \frac{2x}{y+1}$$

Separate the variables to the appropriate locations, however, dx must be made a numerator in the RHS first.

$$dy = \frac{2x\ dx}{y+1}$$

We have:

$$(y+1)dy = 2x\ dx$$

Integrate both sides with respect to their respective variables; we have successfully separated the variables as $(y + 1)$ is a function of y and $2x$ is a function of x.

$$\int (y+1)dy = \int 2x\ dx$$

Let's solve for the LHS;

$$\int (y+1)dy = \int y dy + \int 1 dy$$

$$\left[\frac{y^{1+1}}{1+1}\right] + \left[\frac{y^{0+1}}{0+1}\right] = \frac{y^2}{2} + y$$

RHS:

$$\int 2x\,dx = 2\int x\,dx$$

$$2\left[\frac{x^{1+1}}{1+1}\right] = 2 \times \frac{x^2}{2} = x^2$$

Hence, $LHS = RHS$, our arbitrary constant is added just once since the sum of arbitrary constants is still an arbitrary constant; hence, we add one arbitrary constant;

$$\frac{y^2}{2} + y = x^2 + C$$

Multiply through by 2

$$y^2 + 2y = 2x^2 + 2C$$

$$y^2 + 2y = 2x^2 + 2C$$

$$y^2 + 2y = 2x^2 + 2$$

$2C$ is also some constant; put $2C = A$

Therefore;

$$y^2 + 2y = 2x^2 + A$$

That's the solution of the differential equation;

Moving further;

EXAMPLE 13: Solve:

$$\frac{dy}{dx} = (1 + x)(1 + y)$$

Now, this looks like direct integration but it is definitely not.

Firstly, transfer dx as a numerator in the right hand side;

$$dy = dx(1 + x)(1 + y)$$

Firstly, I'm sure we know $1 + x$ and $1 + y$ are functions of x and y respectively and they're multiplied, so the variables can still be separated.

Separating variables… … …

$$\frac{dy}{1 + y} = dx(1 + x)$$

Integrate both sides with respect to their respective variables;

$$\int \frac{dy}{1+y} = \int (1+x)dx$$

LHS:

$$\int \frac{dy}{1+y}$$

Put:

$$u = 1 + y$$

$$\frac{du}{dy} = 0 + 1 = 1; \text{ here; } dy = du$$

$$\int \frac{du}{u} = \ln u \,(\text{this is an integral rule})$$

Hence; since: $u = 1 + y$

$$\int \frac{dy}{1+y} = \ln(1+y)$$

RHS:

$$\int (1+x)dx = \int 1dx + \int xdx$$

$$\left[\frac{x^{0+1}}{0+1}\right]+\left[\frac{x^{1+1}}{1+1}\right]=x+\frac{x^2}{2}$$

Hence, $LHS = RHS$, our arbitrary constant is added just once since the sum of arbitrary constants is still an arbitrary constant; hence, we add one arbitrary constant;

$$\ln(1+y)=x+\frac{x^2}{2}+C$$

Multiply through by 2

$$2\ln(1+y)=2x+x^2+2C$$

From log rules; $2\ln(1+y)=\ln(1+y)^2$

$$\ln(1+y)^2=x^2+2x+2C$$

$2C$ is also some constant, put $2C=A$

$$\ln(1+y)^2=x^2+2x+A$$

No further simplification is possible here, neither x^2 nor $2x$ can be expressed in form of a log since they are not arbitrary constants; hence, we have to stop where we are above.

EXAMPLE 14: Solve

$$\frac{dy}{dx} = \frac{y^2 + xy^2}{x^2y - x^2}$$

Lol… … … … … … … …

We seem to be crossing our boundary now, how can these variables be separated; but don't you think those terms can be factorized; both at the numerator and the denominator;

$$\frac{dy}{dx} = \frac{y^2(1 + x)}{x^2(y - 1)}$$

Now, you can see separate functions of x and y showing up now, can you see them; in brackets and alone; so let's do the usual; transfer dx to the RHS as a numerator;

$$dy = \frac{y^2(1 + x)dx}{x^2(y - 1)}$$

Separate the variables now;

$$\frac{(y - 1)dy}{y^2} = \frac{(1 + x)dx}{x^2}$$

Integrate both sides with respect to their respective variables;

$$\int \frac{(y-1)}{y^2} dy = \int \frac{(1+x)}{x^2} dx$$

LHS:

$$\int \frac{(y-1)}{y^2} dy$$

Break the fraction down since it has one denominator;

$$\int \left(\frac{y}{y^2} - \frac{1}{y^2}\right) dy = \int \frac{y}{y^2} dy - \int \frac{1}{y^2} dy$$

$$\int \frac{1}{y} dy - \int y^{-2} dy$$

Integrating each as they require;

$$\ln y - \left(\frac{y^{-2+1}}{-2+1}\right) = \ln y - \left(\frac{y^{-1}}{-1}\right)$$

$$\ln y + y^{-1} = \ln y + \frac{1}{y}$$

RHS:

$$\int \frac{(1+x)}{x^2} dx$$

$$\int \left(\frac{1}{x^2} + \frac{x}{x^2}\right) dx = \int \frac{1}{x^2} dx + \int \frac{1}{x} dx$$

$$\int x^{-2} dx + \int \frac{1}{x} dx$$

$$\frac{x^{-2+1}}{-2+1} + \ln x = \frac{x^{-1}}{-1} + \ln x$$

$$-x^{-1} + \ln x =$$

$$-\frac{1}{x} + \ln x = \ln x - \frac{1}{x}$$

Hence, $LHS = RHS$, our arbitrary constant is added just once since the sum of arbitrary constants is still an arbitrary constant; hence, we add one arbitrary constant;

$$\ln y + \frac{1}{y} = \ln x - \frac{1}{x} + C$$

Which gives the solution to the differential equation; further simplification makes no practical meaning actually. Check it a bit very well.

EXAMPLE 15: Solve:

$$xy\frac{dy}{dx} = \frac{x^2+1}{y+1}$$

Lots of rearrangement needs to be done;

$$xy\frac{dy}{dx} = \frac{x^2+1}{y+1}$$

Take dx where it is supposed to be and transfer the x functions to dx, the y functions to dy. We are separating the variables, do it carefully as you can see dx and x moving to right, $(y+1)$ moving to the left and the rest remaining;

$$y(y+1)dy = \frac{x^2+1}{x}dx$$

Integrate both sides with respect to their respective variables;

$$\int y(y+1)dy = \int \frac{x^2+1}{x}dx$$

LHS:

$$\int y(y+1)dy = \int (y^2+y)dy$$

Expansion and then Integral of sums;

$$\int y^2\,dy + \int y\,dy = \frac{y^{2+1}}{2+1} + \frac{y^{1+1}}{1+1}$$

$$\frac{y^3}{3} + \frac{y^2}{2}$$

RHS:

Break down the fractions since only one denominator is there;

$$\int \frac{x^2+1}{x}dx = \int \left(\frac{x^2}{x} + \frac{1}{x}\right)dx$$

$$\int \left(x + \frac{1}{x}\right)dx = \int x\,dx + \int \frac{1}{x}dx$$

$$\frac{x^{1+1}}{1+1} + \ln x = \frac{x^2}{2} + \ln x$$

Hence, $LHS = RHS$, our arbitrary constant is added just once since the sum of arbitrary constants is still an arbitrary constant; hence, we add one arbitrary constant;

$$\frac{y^3}{3} + \frac{y^2}{2} = \frac{x^2}{2} + \ln x + C$$

Clear by multiplying through by 6;

$$2y^3 + 3y^2 = 3x^2 + 6\ln x + 6C$$

$6C$ is some constant, put $6C = A$
Also; from log rules;
$6\ln x = \ln x^6$

$$2y^3 + 3y^2 = 3x^2 + \ln x^6 + A$$

EXAMPLE 16: Solve

$$x\frac{dy}{dx} = y + xy$$

Here is the addition sign threatening to spoil the rule again, but you should know factorization is likely to be involved; factorize the RHS;

$$x\frac{dy}{dx} = y(1 + x)$$

Now we have distinct functions of x and y so we can successfully separate the variables; take dx where it belongs and do the job.

$$\frac{dy}{y} = \frac{1+x}{x} dx$$

Integrate both sides with respect to their respective variables;

$$\int \frac{dy}{y} = \int \frac{1+x}{x} dx$$

LHS:

$$\int \frac{dy}{y} = \ln y$$

This is softly done from a direct rule of integration;
RHS:

$$\int \frac{1+x}{x} dx = \int \left(\frac{1}{x} + \frac{x}{x}\right) dx$$

$$\int \left(\frac{1}{x} + 1\right) dx = \int \frac{1}{x} dx + \int 1 dx$$

$$\ln x + \frac{x^{0+1}}{0+1} = \ln x + x$$

Hence, $LHS = RHS$, our arbitrary constant is added just once since the sum of arbitrary constants is still an arbitrary constant; hence, we add one arbitrary constant;

$$\ln y = \ln x + x + c$$

Let's do some manipulations;

$$\ln y - \ln x = x + c$$

Since $\log a - \log b = \log\left(\frac{a}{b}\right)$

$$\ln\left(\frac{y}{x}\right) = x + c$$

$\ln\left(\frac{y}{x}\right)$ implies the natural log, $\log_e\left(\frac{y}{x}\right)$

From log rules;

$$\log_b a = c, \text{ then } a = b^c$$

It follows that;

$$\frac{y}{x} = e^{x+c}$$

Break these indices;

$$a^{m+n} = a^m \times a^n$$

Similarly;

$$\frac{y}{x} = e^x \times e^c$$

e^c is some constant, regard the constant as A

$$\frac{y}{x} = e^x \times A = Ae^x$$

Cross multiply;

$$y = Ae^x \times x = Axe^x$$

$$y = Axe^x$$

EXAMPLE 17: Solve:

$$y \tan x \frac{dy}{dx} = (4 + y^2) \sec^2 x$$

The x and y are on their own so separate the variables; take $(4 + y^2)$ to where it is meant on the left, and port dx and its friend, $\tan x$ to the right;

$$\frac{y}{4 + y^2} dy = \frac{\sec^2 x}{\tan x} dx$$

Integrate both sides with respect to their respective variables;

$$\int \frac{y}{4+y^2}\,dy = \int \frac{\sec^2 x}{\tan x}\,dx$$

LHS:

$$\int \frac{y}{4+y^2}\,dy$$

Obviously; the derivative denominator will be in terms of the numerator;

Put: $u = 4 + y^2$

$$\frac{du}{dy} = 0 + 2 \times y^{2-1} = 2y$$

Hence,

$$dy = \frac{du}{2y}$$

Hence, the integral; y will cancel out to leave a function that can be integrated and the substitution is replaced;

$$\int \frac{y}{u}\frac{du}{2y} = \int \frac{1}{u}\frac{du}{2} = \frac{1}{2}\int \frac{du}{u}$$

$$\frac{1}{2}\ln u = \frac{1}{2}\ln(4+y^2)$$

RHS:

$$\int \frac{\sec^2 x}{\tan x} dx$$

Obviously; the derivative denominator will be in terms of the numerator;

Put:

$$z = \tan x$$

$$\frac{dz}{dx} = \sec^2 x \,;$$

Hence,

$$dx = \frac{dz}{\sec^2 x}$$

Hence, the integral becomes; $\sec^2 x$ will cancel out to leave a function that can be integrated and the substitution is replaced;

$$\int \frac{\sec^2 x}{z} \frac{dz}{\sec^2 x} = \int \frac{dz}{z}$$

$$\ln z = \ln \tan x$$

Hence, $LHS = RHS$, our arbitrary constant is added just once since the sum of arbitrary constants is still an arbitrary constant; hence, we add one arbitrary constant;

$$\frac{1}{2}\ln(4 + y^2) = \ln\tan x + C$$

Multiply through by 2;

$$\ln(4 + y^2) = 2\ln\tan x + 2C$$

$$\ln(4 + y^2) = \ln(\tan x)^2 + 2C$$

$$\ln(4 + y^2) - \ln(\tan x)^2 = 2C$$

From;

$$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$$

$2C$ is a some constant, put that as D

$$\ln\frac{(4 + y^2)}{\tan^2 x} = D$$

From log rules, we have that;

$$e^{D} = \frac{(4+y^2)}{\tan^2 x}$$

e^D is still some constant, so let's regard that as A

$$A = \frac{(4+y^2)}{\tan^2 x}$$

Hence,

$$(4+y^2) = A\tan^2 x$$

$$y^2 = A\tan^2 x - 4$$

Above is the solution to the differential equation.

So let's see this last example;

EXAMPLE 18: Solve:

$$\frac{dy}{dx} = e^{2x+3y}$$

Kk, yeah, this is extremely easy even though it looks a hard bone to crack;

Aiit, from indicial rule;

$$a^{m+n} = a^m \times a^n$$

Hence, here we have:

$$\frac{dy}{dx} = e^{2x} \times e^{3y}$$

So relatively easy, we can separate our variables and solve;

$$\frac{dy}{e^{3y}} = e^{2x} dx$$

$$e^{-3y} dy = e^{2x} dx$$

Integrate both sides with respect to their respective variables;

$$\int e^{-3y} dy = \int e^{2x} dx$$

LHS:

$$\int e^{-3y} dy$$

Substitution is needed; put $z = -3y$

$$\frac{dz}{dy} = 1 \times -3 \times y^{1-1} = -3;$$

Hence;

$$dy = \frac{dz}{-3}$$

$$\int e^{z}\frac{dz}{-3} = -\frac{1}{3}\int e^{z}$$

Finishing off and returning the true value of *z*

$$-\frac{1}{3}(e^{z}) = -\frac{1}{3}e^{-3y}$$

RHS:

$$\int e^{2x}dx$$

Substitution is needed; put $u = 2x$

$$\frac{du}{dx} = 1 \times 2 \times x^{1-1} = 2$$

Hence,

$$dx = \frac{du}{2}$$

$$\int e^{u}\frac{du}{2} = \frac{1}{2}\int e^{u}$$

$$\frac{1}{2}(e^{u}) = \frac{1}{2}e^{2x}$$

Hence, $LHS = RHS$, our arbitrary constant is added just once since the sum of arbitrary constants is still an arbitrary constant; hence, we add one arbitrary constant;

$$-\frac{1}{3}e^{-3y} = \frac{1}{2}e^{2x} + C$$

Clear by multiplying through by 6 and afterwards doing some soft rearrangement;

$$-2e^{-3y} = 3e^{2x} + 6C$$

$$3e^{2x} + 2e^{-3y} = -6C$$

$-6C$ is some constant, regard that as A

$$3e^{2x} + 2e^{-3y} = A$$

So, that's it about differential equations;

So, have these questions as your food! ☺☺☺☺☺

Remember all the rules please, place the differential coefficients where they are supposed to be; and separate the variables carefully; and exhaust the integral rules you know to integrate any needed integral. Remember also that we have several *several* manipulations before reaching our final answers especially manipulations in expressing operations under constants as constants. Put each and every one of them at the back of your mind. Remember we add the arbitrary constant once, and as much as we add it once, it is of extreme importance if at all you'll be getting the final answer of the solution of differential equations.

i. Solve: $x\dfrac{dy}{dx} = x^2 + 2x - 3$

ii. Solve: $\dfrac{dy}{dx} = \dfrac{y}{x}$

iii. Solve: $\dfrac{dy}{dx} = (y+2)(x+1)$

iv. Solve: $\dfrac{dy}{dx} = xy - y$

v. Solve: $\dfrac{\sin x}{1 + y}\dfrac{dy}{dx} = \cos x$

vi. Solve: $\dfrac{dy}{dx} = e^{x-2y}$

vii. Solve: $(x^2 - 1)\dfrac{dy}{dx} = x - 2xy$

HINT: e^c is still a constant

viii. Solve the equation given the initial condition that $x = 1$ when $y = 2$

$$(1 + x^3)\frac{dy}{dx} = x^2 y$$

ix. Solve the equation: $x(y - 3)\dfrac{dy}{dx} = 4y$

x. Solve: $x^3 + (y + 1)^2\dfrac{dy}{dx} = 0$

HINT: Transfer x^3 to the RHS

xi. Solve the differential equation:

$$x^2(y+1) + y^2(x-1)\frac{dy}{dx} = 0$$

HINT: Transfer $x^2(y+1)$ to the RHS

xii. $(1+e^{-x})\sin y\dfrac{dy}{dx} = -\cos y\,;$

Given that $y = 45°$ when $x = 0$

And below are the answers! Solve and confirm your answers; ensure you solve everything.

i. $y = \dfrac{x^2}{2} + 2x - 3\ln x + C$

ii. $y = Ax$

iii. $\ln(y+2) = \dfrac{x^2}{2} + x + C$

iv. $\ln y = \dfrac{x^2}{2} - x + C$

v. $y = A\sin x - 1$

vi. $e^{2y} = 2e^{x} + A$

vii. $(1-2y)(x^2-1) = A$

viii. Before substitution of real values;

$$y^3 = A(1 + x^3)$$

After substituting the values, $x = 1, y = 2$

$$y^3 = 4(1 + x^3)$$

ix. The last phase of the solution;

$$y + C = \ln x^4 y^3$$

Hence,

$$e^{y+C} = e^{\ln x^4 y^3}$$

Exponent and natural log cancels out;

$$e^y e^C = x^4 y^3$$

e^C is some constant, A

$$Ae^y = x^4 y^3$$

x. $4(y + 1)^3 + 3x^4 = A$

xi. Use partial fraction integration (which are improper fractions that'll involve polynomial division first) in:

$$\int \frac{y^2}{y+1} dy \quad and \quad -\int \frac{x^2}{x-1} dx$$

$$x^2 + y^2 + 2x - 2y + 2\ln(y+1) + 2\ln(x-1) = A$$

xii. Use polynomial division in:

$$\int \frac{dx}{1+e^{-x}}$$

The last phase of the solution;

$$\ln \cos y = x + \ln(1+e^{-x}) + C$$

$$\ln(1+e^{-x}) - \ln \cos y = -x - C$$

$$\ln\left(\frac{1+e^{-x}}{\cos y}\right) = -x - C$$

Take exponent of both sides, the natural log cancels out

$$\frac{1+e^{-x}}{\cos y} = e^{-x-C}$$

$$\frac{1+e^{-x}}{\cos y} = e^{-x} \times e^{-C}$$

e^{-C} is some constant, put $e^{-C} = A$

e^{-x} as a negative power can be converted to a positive power at the denominator.

$$\frac{1+e^{-x}}{\cos y} = \frac{A}{e^{x}}$$

$$\frac{e^{x}(1+e^{-x})}{\cos y} = A$$

$$\frac{e^{x}+e^{x}\times e^{-x}}{\cos y} = A$$

$$\frac{e^{x}+e^{x-x}}{\cos y} = A$$

$$\frac{e^{x}+e^{0}}{\cos y} = A$$

$$\frac{(e^{x}+1)}{\cos y} = A$$

Fixing the real values; $y = 45°$ when $x = 0$

$$\frac{(e^{0}+1)}{\cos 45} = A$$

Solving for A;

$$\frac{2}{\frac{1}{\sqrt{2}}} = A; \quad A = 2\sqrt{2}$$

Therefore, the solution of the equation is:

$$\frac{(e^x + 1)}{\cos y} = 2\sqrt{2}$$

$$(e^x + 1) = 2\sqrt{2}\cos y$$