PACM Preliminary Exam

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1 Harmonic Analysis: Charles Fefferman

1. Suppose the Laplacian of a function u on $\mathbb{R}^3$ belongs to L^p what can you say about the second derivatives of u? What if $p = 1$ or ∞?
2. Define the truncated Hilbert transform by the formula

$$T(f, \delta)(x) = \int_{|y| \geqslant \delta} f(x - y) \frac{dy}{y}.$$

   Prove that the limit of $T(f, \delta)$ as $\delta \to 0$ exists pointwise everywhere if f is C^∞ of compact support, and almost everywhere if $f \in L^1$.

The first question went smoothly. I stated the identity involving Riesz transforms and then sketched the proof of the Calderón-Zygmund theorem. Fefferman was mostly interested in the weak-$(1, 1)$ bound, but he didn't make me do any nitty gritty details. After that, he was satisfied when I said the magic word "duality." During all of this, Fefferman said very little and mostly just nodded his head enthusiastically whenever I said anything correct.

I wasn't expecting the second question. I proved the case where $f \in C_c^\infty$, but then handwaved the generalization to $f \in L^1$ by saying that since there is a maximal inequality here, it is just a standard density argument. Fefferman pressed me on how to prove the maximal inequality, and I said that it is analogous to the proof that I just did of Calderón-Zygmund. He agreed, but said that there is one crucial point for L^2 boundedness that I need to mention. I thought he was talking about the two tricks involving complex exponentials that Stein uses to prove that truncated Calderón-Zymund kernels have bounded Fourier transform, but fumbled in saying this. He didn't seem convinced but said he was happy with harmonic analysis and suggested that we should move on and could come back to this later.

Sarnak and Shapiro were very quiet during this topic. Sarnak did make a comment about how this stuff takes him back and that he has a handwritten copy of Stein's singular integrals book consisting of notes taken by someone whose name I didn't catch.

2 Ergodic Theory: Peter Sarnak

1. Discuss and outline proofs of various versions of the L^2 and pointwise ergodic theorems.
2. Define the Kolmogoroff-Sinai entropy of a dynamical system and give some examples and applications.
3. Show that the geodesic flow on a negativley curved surface is ergodic— what can you say about its entropy?
4. Define what is meant by a uniquely ergodic topological dynamical system (X, T). Give some examples and show that the horocyle motion on the unit tangent space to a compact hyperbolic surface is uniquely ergodic.

This was the smoothest part of the exam. I stated the L^2 ergodic theorem and sketched an elementary proof, but Sarnak stopped me halfway through. He then asked me about the relationship between this theorem and the spectral theorem, and thankfully Anshul Adve had taught me that a week prior! I sketched the proof of the pointwise ergodic theorem that I learned in Einsiedler-Ward; he was fine with me saying that most everything is analogous to the proof of the Lebesgue differentiation theorem. I started giving the details of the last step, but he stopped me halfway through again.

We then moved to entropy, where I outlined the definition and said that it allows us to show that Bernoulli schemes are nonisomorphic. Sarnak asked me about intuition for entropy, and proceeded to explain that there is a theorem that guarantees that the definition of entropy is correct. Namely, a system has zero entropy if and only if with probability 1, "you can predict the future from the past." He also educated me about topological entropy.

Lastly, he said that he didn't realize that the exam is only 1.5 hours, so instead of making me do questions three and four, he asked me to prove that an irrational rotation of the circle is uniquely ergodic. I used Fourier series to prove that it is ergodic, and then he prompted me on how to conclude unique ergodicity from that.

Fefferman and Shapiro were very quiet during this part. Sarnak was amused by my use of the acronym "WTS" for "want to show." Fefferman said that all of the MAT 215 kids use it.

3 Functional Analysis: Jacob Shapiro

1. Prove that Schatten implies compact.
2. Prove that if U is unitary and $U - I$ is compact then $\sigma(U)$ cannot be the whole unit circle.
3. Prove that if two orthogonal projections have infinite range and kernel, then there is a norm-continuous map (within the space of such orthogonal projections) that interpolates between them.
4. Let A be the discrete Laplacian on $\ell^2(\mathbb{Z})$ normalized so that its spectrum is $[0, 4]$. Estimate the decay rate of the matrix elements of the spectral projection $P = \chi_{[-10,2]}(A)$, i.e. calculate how P_{xy} decays as a function of $|x - y|$. Now do the same for the Greens function $G(x, y; z) = [(A - zI)^{-1}](x, y)$ where $z = 2 + i\varepsilon$, ε is small.
5. Prove that if A is a bounded self-adjoint operator on $\ell^2(\mathbb{Z})$ that has only ac spectrum then $[\exp(itA)X^2 \exp(-itA)](0, 0)$ grows to infinity at least like t^2 as $t \to \infty$.
6. Prove that if A is a bounded self-adjoint operator on $\ell^2(\mathbb{Z})$ then $\dim \ker A$ is upper semicontinuous. Find an example which is not lower semicontinuous.
7. Find a bounded linear operator A on a separable Hilbert space such that $\dim \ker A$, $\dim \ker A^*$ are both finite, but $\operatorname{im} A$ is not closed.
8. Prove that if a bounded linear operator A on a separable Hilbert space has $\dim \operatorname{coker} A$ finite, then $\operatorname{im} A$ is closed, which moreover implies that $|A|^2$ is strictly lower bounded outside of its kernel.
9. Show that the Fourier transform as a map from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ is unitary and make an eigendecomposition of it.
10. Prove Atkinson's theorem.

This part of the exam was rocky. During prep time, I wrote out solutions to questions 1, 7, and 9, but Shapiro said that he wanted to go in order! Sarnak made a comment about how the junior faculty all come into these exams with their research questions.

I gave two proofs that Schatten implies compact. The first was really the proof that trace-class implies compact in Reed and Simon, and the second proceeded by considering the essential spectrum as in Shapiro's notes. I was then able to do question 2 fairly quickly with some small hints from Shapiro and Fefferman.

Question three went poorly. I had no idea what to do at first. Fefferman gave me lots of hints and Shapiro chimed in often too. By making me draw pictures in two dimensions, they helped me realize that the idea is unitary evolution between the kernels and images. However, when it came to writing down the path explicitly, my brain completely turned off. All three professors were trying to help me, but I was very nervous and simply not thinking. Eventually Shapiro let it go and we moved on to question four, which was even worse. After some time of them trying to help me, but me clearly being unable to think, they let it go again and asked me to step out of the room.

4 Conclusion

After a couple minutes Shapiro brought me back inside and they congratulated me! I was upset because of my performance at the end, but they insisted that I did well and that they understand that I got worn out. The exam lasted about an hour and 45 minutes. We chatted about the books I studied, and then I stayed back with Shapiro and he explained question 4 to me.