Probability theory — Boris Hanin

Suppose

$$S_n = \sum_{i=1}^{n} X_i$$

is a sum of mean zero random variables $X_i \in \mathbb{R}$.

Question 1.

- State the central limit theorem
- What are some methods for proving the central limit theorem? Implement one of them.
- What is the rate of convergence in the central limit theorem? Can this rate be improved?

Comments. I stated the CLT and gave the proof via characteristic functions and Lévy's continuity theorem. This was okay, but I tried giving all details and misremembered some. In the course of the proof I wrote down a bound that contains a term like $\mathbb{E}\left[\min\left\{|X_1|^2, |X_1|^3\right\}\right]$, and Boris took issue with this since I made no third moment assumption. I knew this worked, but couldn't give a precise explanation of it. Boris was satisfied and moved on. Michael interjected and asked about CLT-type results where the limit is not normal. I mentioned Poisson approximation and Tracy-Widom distribution, and that was enough. (In hindsight, I think he was asking for a comment on stable laws.) For rates, I had no clue — we ended up discussing Berry-Esseen briefly, but I had very little concrete to say about it.

Question 2.

- Suppose X_i are a.s. bounded. What can you say about

  $$\mathbb{P}\left(|S_n| > t\right) \qquad t \in \mathbb{R}_+$$

  when $n \gg 1$? What if X_i are not a.s. bounded?

- Write

$$\rho_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i}.$$

  If $\mathbb{P}_X = \mathrm{Law}(X_i)$, in what sense does ρ_n converge to $\mathbb{P}_X$?

Comments. I discussed Hoeffding's inequality and gave the proof up to bounding a moment generating function, which I forgot how to do, but mumbled some things. I then gave the sub-gaussian version. Boris asked for an example of a sub-gaussian distribution and I said Gaussian — true, but he wants a better one and the best I could come up with is Bernoulli. Non-sub gaussian? I said something heavy-tailed, like Lévy. I had written the sub-gaussian norm on the board, so Boris asked me what the sub-gaussian norm of $\mathcal{N}(0,1)$ was. I didn't know so he asked me to calculate the Gaussian integral. I tried and failed, and Michael empathized and said: "I don't like to differentiate in public." Boris: "That doesn't rule out integration!" For the second part, I discussed weak convergence of measures and Boris wanted to know some things about the relation of this to convergence w.r.t a metric on a space of probability measures. I got myself into a pickle by mentioning relative entropy, gave the definition, and discussed some things about it. In hindsight, I should have mentioned Wasserstein spaces since I know far more about that than the relative entropy distance. Idk what I was thinking.

Question 3. Suppose

$$A = (a_{ij})_{i \le i, j \le n} \qquad a_{ij} \sim \mathcal{N}(0,1)$$

- What is the expected Frobenius norm $\mathbb{E}\left[\|A\|_F^2\right]$ of A? What does this tell us about the average squared singular value of A?
- What is the order in n of the largest singular value of A? How does it concentrate?
- Consider the empirical distribution of singular values $s_1 \ge s_2 \ge \cdots \ge s_n$ of A,

$$\mu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{s_i}.$$

  What can you say about μ_n as $n \to \infty$?

Comments. When I finished discussing question 2, Boris asked how long he has, and Peter said about 30 minutes. More than 30 minutes had passed, so we decided to skip after I said I didn't get to this one during prep. (I didn't study much random matrix theory for this, so I was very happy about this.)

Partial differential equations — Peter Constantin

Question 1. Consider the system

$$\begin{cases} u_t + Au_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = u_0 & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}$$

where $u(x, t) \in \mathbb{R}^n$, $x \in \mathbb{R}$, and A is a real, constant (in time and space) matrix. Suppose u_0 is compactly supported. Can you solve the system? What happens to the L^∞ norm of solutions? What about the L^1 norm? Can you give examples of different behaviors?

Comments. I blanked at first and did the only thing that came to mind: diagonalizing A. I found a system of n scalar transport equations I knew how to solve, thank god. Peter: "Solve." Solved. I said L^∞ norm is constant, and gave a brief argument. I said something different might happen to the L^1 norm, and Peter said no and I realized why. Then we talked about different behaviors based on the spectrum of A, which I sort of knew how to discuss (different speeds of components), but Peter helped with some details.

Question 2. Let $\Omega \subseteq \mathbb{R}^3$ be open, bounded with smooth boundary. Consider

$$-\Delta u + u^5 = f$$

with boundary condition $u|_{\partial\Omega} = 0$. Assume $f \in L^2(\Omega)$. Discuss existence, uniqueness, and regularity of solutions.

Comments. I barely read anything from Chapter 8 of Evans and was more prepared to talk elliptic and parabolic, so I was winging this one. I knew how to write down the energy functional and mentioned coercivity and lower semi-continuity. Peter asked how to show coercivity and I wrote some things down and mentioned Poincaré inequality and he was okay with that. I hemmed and hawed about convexity and lower semi-continuity. Then Peter held my hand through a mostly one-sided conversation about regularity, and I messed up some exponents and wrote many stupid things on the board.

Statistical mechanics and spin glasses — Michael Aizenman

Describe/outline the Ruelle probability cascade:

- Define it
- Discuss its dynamical quasi-stationarity property
- Role in a heuristic explanation of the relevance of the Parisi variational ansatz for the free energy of the SK spin glass model.
- Outline a variational computation of the free energy assuming the RPC does indeed capture the SK Gibbs states' structure.

Comments. This was the least smooth part of the exam. I spent most of my prep time here attempting to jot down the details of the construction. I managed to do this, and sketched some drawings and loose arguments about representing the Parisi functional with RPC. I really didn't know what he meant by quasi-stationarity property — I thought he meant Bolthausen-Sznitman invariance, or Ghirlanda-Guerra identities, so I wrote some things down about them. When I presented, I was stopped before I finished the construction. Michael wanted me to think about a simpler case, and ended up trying to teach me his intuition for these things. I learned a thing or two, but in the end, we talked very little about what I had prepared. I explained how the distribution of overlaps is determined by the functional order parameter, and Michael asked me to connect this intuitively to some things I discussed at the beginning with Boris. He was happy with these answers, then we chatted how to interpret Ghirlanda-Guerra.

I was asked to step out and waited in the hall. I expected this to be a short time, but it went on for what might have been like 10 minutes. I became very scared I might not have passed given how long it was taking. Then Michael opened the door with a smile and Boris said, "I have good news, and I have bad news. The good news is you passed, but the bad news is you're stuck with me for a few years."

I ended up walking towards ORFE with Boris, and he shared some great wisdom. He also said that they took so long because Michael and Peter were chatting about the preservation of the L^1 norm in the PDE problem and the wait had nothing to do with me.