1 Mathematical Physics - Michael Aizenman

Anderson Localization: The random Schrodinger operator on ℓ^2 over the d-dimensional lattice $\mathbb{Z}^d$, with i.i.d. random potential is an operator of the form: $H(\omega) = -\Delta + \lambda V(\omega)$ with $-\Delta$ the discrete second difference operator, and $\{V(x)\}$ distributed as i.i.d. random variables, and λ a control parameter. Focusing on the case that $V(x)$ are uniformly distributed in $[0,1]$:

1. What is the (typical) spectrum of H?
2. Prove, or explain the key steps in the proof, that at λ large enough the operator has a full set of proper (localized) eigenvalues.
3. In that case, in what way does the unitary evolution under $\exp(-itH)$ differ from that under the potential-free operator $\exp(it\Delta)$?

Comments: This was mostly OK. For (1) I presented the proof of Pastur and Kunz-Souillard which were fairly standard. Jacob then asked me to prove that the supports of the different spectral types are deterministic. I kind of froze for a bit, but he hinted that I should consider resolvents, I was able to give a somewhat sketchy argument. He seemed underwhelmed, but we didn't go further with it. (2) was a bit of a mess. I started with the fractional moment bound for the eigenfunction correlator, but Michael wanted me to give an intuitive explanation to why it holds. I told him that I probably wouldn't be able to do that, and tried to present an intuitive argument as to why square summability of the eigenfunction correlators implies dynamic localization instead. He seemed OK with that. Proving the single-step bound got very technical so we just skipped forward. For (3), I just listed a bunch of stuff, and by then everyone just wanted to move on. Michael then told me to summarize everything in 2 words, which was not enough words.

2 Differential Geometry - Mihalis Dafermos

1. Discuss the definition of smooth manifold and explain how to show that the standard unit n-sphere $\mathbb{S}^n$ is a smooth manifold. Can this be done more easily than by explicitly exhibiting charts? Now consider the group $O(n)$ consisting of real $n \times n$ matrices A such that $\langle Av, Aw\rangle = \langle v, w\rangle$ for all vectors $v, w \in \mathbb{R}^n$. Show that one may naturally define on $O(n)$ the structure of a smooth manifold.
2. We say that a Riemannian manifold $(\mathcal{M}, g)$ is geodesically complete if all geodesics through all points can be extended for arbitrary time (or equivalently, if for all $p \in \mathcal{M}$, the exponential map $\exp_p$ may be defined globally as a map $\exp_p : T_p\mathcal{M} \to \mathcal{M}$). Give an example of a geodesically complete manifold and describe its geodesics. Show that if $(\mathcal{M}, g)$ is connected and $\mathcal{U} \subset \mathcal{M}$ is a proper open subset then $(\mathcal{U}, g|_{\mathcal{U}})$ taken as a Riemannian manifold necessarily fails to be geodesically complete.
3. Do all geodesically incomplete Riemannian manifolds arise as in 2. above? That is to say, if $(\mathcal{M}, g)$ fails to be geodesically complete, is there always a geodesically complete $(\bar{\mathcal{M}}, \bar{g})$ of the same dimension and a map $\phi\colon \mathcal{M} \to \bar{\mathcal{M}}$ such that $\phi(\mathcal{M}) \subset \mathcal{M}$ is open, $\phi\colon \mathcal{M} \to \phi(\mathcal{M})$ is a diffeomorphism and $\phi^*\bar{g} = g$?

Comments: (1) and (2) are standard and we went through it very quickly. (3) was kind of unfortunate because I didn't actually study any Riemannian geometry (I was mostly doing differential topology). Michael told me to think of the Greeks, which didn't help at all. I struggled through this but eventually got to an answer after many many hints and a long time. In hindsight, this is quite easy if you have the right idea (cone without point).

3 Functional Analysis - Jacob Shapiro

1. State the Atiyah-Janich theorem and sketch a proof for the case that the base space X is a point.
2. Provide Kuiper's theorem.
3. Prove the existence of a unique square root for a positive element in a C-star algebra.

4. Define the projection-valued spectral measure of a bounded self-adjoint operator H via the Herglotz-Pick function. Provide an example of H where the Hilbert space is isomorphic to $L^2(\mathbb{R}, \mu)$ where μ is the spectral measure associated with one vector, and an example when this cannot be the case.
5. Let X be a Banach space. Provide an example where $X^{**} \cong X$ yet X is not reflexive.
6. Let X be a Banach space. Find a linear functional on X^* which is continuous w.r.t. the weak-star topology on X^* but not continuous w.r.t. the weak topology on X^*.
7. Is the polar decomposition of a bounded linear operator on a separable Hilbert space unique?
8. Prove that a self-adjoint bounded linear operator which has essential spectrum at zero is compact and provide a counter-example if you drop self-adjointness.
9. Prove that Schatten implies compact.
10. Prove the Fedosov formula.

Comments: This was probably the longest part of the exam. I only managed to do the first 4 questions within the prep time, and I presented solutions to (2) and (3), before Jacob started asking random questions. He started with asking me to show (9), and I got stuck doing in on the spot (which he seemed a bit annoyed by) but he decided to move on. There were quite a few questions afterwards... I don't really remember them all but they include (a) is there a compact operator that is not trace class (b) when does an operator have a symbol, what is the spectral type of the bilateral shift (c) if P_1 and P_2 are two different projections which has infinite dimensional kernel and image, then there exists a paths within such projections connecting the two (d) are there cube roots in a general C^*-algebra (I think Michael asked this one actually), (e) give an example of an operator with only singularly continuous spectrum.