Linear Algebra

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Preliminary Assessment

1. **(a)** Is the set of polynomials of degree $n \in \mathbb{Z}_{\geq 0}$ over a field $\mathbb{F}$ a vector space? If it is, prove it. If it is not, state which axioms are not satisfied and provide counterexamples.

   (b) Prove that the set of polynomials of degree at most n is a vector space. You may assume that the set of continuous functions $\mathscr{C}(\mathbb{F})$ in $\mathbb{F}$ is a vector space. Why is this helpful?

   In what follows, let $P_2(\mathbb{R})$ be the vector space of polynomials of degree at most 2 and $\mathbb{V}$ denote the subset of $P_2(\mathbb{R})$ such that:

   $$\int_0^1 p(t)\,\mathrm{d}t = 0.$$

   (c) Prove that $\mathbb{V}$ is a subspace of $P_2(\mathbb{R})$.

   (d) Construct a basis for $\mathbb{V}$ and prove it is indeed a basis. What is the dimension of $\mathbb{V}$?

2. Let $P_n(\mathbb{F})$ be the vector space of polynomials of degree at most n with coefficients in a field $\mathbb{F}$. Define the transformation $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ by

   $$T(p(t)) = 2p'(t) - 3p''(t).$$

   (a) Prove that T is a linear transformation.

   (b) Find a basis for the nullspace of T.

   (c) Given the bases $\beta := \{1, t, t^2, t^3\}$ and $\gamma := \{1, t-1, t^2-1\}$ for $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$ respectively, determine the matrix $[T]_\beta^\gamma$ which represents T. That is, if v is a vector in $P_3(\mathbb{R})$, $w = T(v)$, and $[v]_\beta$ represents the coordinates of v in the β-basis, find the matrix $[T]_\beta^\gamma$ for which $[w]_\gamma = [T]_\beta^\gamma [v]_\beta$.

3. Let $\beta := \{(1,1), (1,-1)\}$ and $\beta' := \{(2,4), (3,1)\}$.

   (a) Verify that β and β' are bases for $\mathbb{R}^2$.

   (b) Construct the matrix Q which changes β'-coordinates to β-coordinates. That is, if $v \in \mathbb{R}^2$ and $[v]_\beta$ represents the coordinates of v in the β-basis, determine the matrix Q such that $[v]_\beta = Q[v]_{\beta'}$.

4. Let $M_n(\mathbb{F})$ denote the vector space of $n \times n$ matrices over a field $\mathbb{F}$ and define $T : P_2(\mathbb{R}) \to M_2(\mathbb{R})$ by

   $$T(f(x)) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix}.$$

   Find a basis for the range $R(T)$ of T and a basis for the nullspace $N(T)$ of T. Verify that $\dim R(T) + \dim N(T) = 3 = \dim P_2(\mathbb{R})$.

5. Define the following matrix.

   $$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

   Determine the eigenvalues of A. Use these eigenvalues to write $A = PDP^{-1}$ where D is a diagonal matrix and P is nonsingular.

6. Suppose $\{v_1, \ldots, v_n\}$ is an orthogonal set of vectors. Let $\|\cdot\|$ denote the norm generated by the inner product $\langle \cdot, \cdot \rangle$. Prove

   $$\left\| \sum_{i=1}^n a_i v_i \right\|^2 = \sum_{i=1}^n |a_i|^2 \|v_i\|^2,$$

   where $a_1, \ldots, a_n$ are scalars.

7. Let $\mathbb{V}$ be a finite-dimensional inner product space over $\mathbb{C}$ and suppose $T : \mathbb{V} \to \mathbb{V}$ is linear. Then there exists a unique linear transformation $T^* : \mathbb{V} \to \mathbb{V}$ (called the *adjoint* of T) such that $\langle T(v), w \rangle = \langle v, T^*(w) \rangle$ for all v, w in $\mathbb{V}$. Using only the definition above and the axioms of an inner product, show that if $T = T^*$, then the eigenvalues of T are real.

8. Consider the vector space $\mathbb{V} := \{p(t) = a + bt^2 \mid a, b \in \mathbb{R}\}$. Let ω_1 and ω_2 be linear functionals on $\mathbb{V}$ such that $\omega_1\{p(t)\} = p(1)$ and $\omega_2\{p(t)\} = p(2)$. Find the basis for $\mathbb{V}$ for which $\{\omega_1, \omega_2\}$ is the dual basis.