Homework # 05

Math 140
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Chapter 04 | Problem 1.1

Let $\mathscr{U} := \{(u^1, u^2) \in \mathbb{R}^2 : -\pi < u^1 < \pi, -\pi < u^2 < \pi\}$ and define

$$\boldsymbol{x}(u^1, u^2) = ((2 + \cos u^1)\cos u^2, (2 + \cos u^1)\sin u^2, \sin u^1).$$

(a) Prove that $\boldsymbol{x}$ is a simple surface. ($\boldsymbol{x}(\mathscr{U})$ looks like the surface of a donut or innertube.)

(b) Compute $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{n}$ as functions of u^1 and u^2.

Solution

(a) Recall the following definition.

Definition 1. *A C^k coordinate patch (or simple surface) is a one-to-one C^k function $\boldsymbol{x} : \mathscr{U} \to \mathbb{R}^3$ for some $k \geq 1$, where $\mathscr{U}$ is an open subset of $\mathbb{R}^2$ with coordinates u^1 and u^2 and $\partial\boldsymbol{x}/\partial u^1 \times \partial\boldsymbol{x}/\partial u^2 \neq \mathbf{0}$ on $\mathscr{U}$.*

In this case, $\mathscr{U} = (-\pi, \pi) \times (-\pi, \pi)$, so $\mathscr{U}$ is indeed an open subset of $\mathbb{R}^2$. Further, $\boldsymbol{x}$ is C^∞ since its components are products of C^∞ functions in $\mathbb{R}$. Additionally, $\boldsymbol{x}$ is one-to-one, since its components are products of one-to-one functions in $(-\pi, \pi) \subset \mathbb{R}$. All that remains to be shown is that the cross product of the tangent vectors never vanishes on $\mathscr{U}$. We have

$$\frac{\partial\boldsymbol{x}}{\partial u^1} = (-\sin u^1 \cos u^2, -\sin u^1 \sin u^2, \cos u^1),$$
$$\frac{\partial\boldsymbol{x}}{\partial u^2} = (-(2 + \cos u^1)\sin u^2, (2 + \cos u^1)\cos u^2, 0).$$

Let $\boldsymbol{e}_i$ denote the i'th standard basis vector in $\mathbb{R}^3$. Then, we can write:

$$\begin{aligned}
\frac{\partial\boldsymbol{x}}{\partial u^1} \times \frac{\partial\boldsymbol{x}}{\partial u^2} &= \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ -\sin u^1 \cos u^2 & -\sin u^1 \sin u^2 & \cos u^1 \\ -(2+\cos u^1)\sin u^2 & (2+\cos u^1)\cos u^2 & 0 \end{vmatrix} \\
&= \begin{vmatrix} -\sin u^1 \sin u^2 & \cos u^1 \\ (2+\cos u^1)\cos u^2 & 0 \end{vmatrix} \boldsymbol{e}_1 - \begin{vmatrix} -\sin u^1 \cos u^2 & \cos u^1 \\ -(2+\cos u^1)\sin u^2 & 0 \end{vmatrix} \boldsymbol{e}_2 + \begin{vmatrix} -\sin u^1 \cos u^2 & -\sin u^1 \sin u^2 \\ -(2+\cos u^1)\sin u^2 & (2+\cos u^1)\cos u^2 \end{vmatrix} \boldsymbol{e}_3 \\
&= -[2 + \cos u^1]\big(\cos u^1 \cos u^2, \cos u^1 \sin u^2, \sin u^1\big).
\end{aligned}$$

This vector never vanishes, since $\cos u^1 \neq -2$ for any $u_1 \in (-\pi, \pi)$, and $\cos u^i = 0 \iff \sin u^i \neq 0$, and similarly, $\sin u^i = 0 \iff \cos u^i \neq 0$. Hence, we see that $\boldsymbol{x}$ satisfies the definition of a simple surface, which completes the problem.

(b) Most of the work here was completed in (a). By definition, $\boldsymbol{x}_i := \partial\boldsymbol{x}/\partial u^i$ for $i = 1, 2$. Hence:

$$\boxed{\begin{aligned} \boldsymbol{x}_1 &= \big(-\sin u^1 \cos u^2, -\sin u^1 \sin u^2, \cos u^1\big), \\ \boldsymbol{x}_2 &= \big(-(2 + \cos u^1)\sin u^2, (2 + \cos u^1)\cos u^2, 0\big). \end{aligned}}$$

Additionally, a vector normal to the two is their cross product. Hence,

$$\boxed{\boldsymbol{n} = \frac{\boldsymbol{x}_1 \times \boldsymbol{x}_2}{\|\boldsymbol{x}_1 \times \boldsymbol{x}_2\|} = -\big(\cos u^1 \cos u^2, \cos u^1 \sin u^2, \sin u^1\big)}$$

since $\|\boldsymbol{x}_1 \times \boldsymbol{x}_2\| = 2 + \cos u^1$.

Chapter 04 | Problem 1.2

Consider a curve in the (r, z) plane given by $r = r(t) > 0, z = z(t)$. If this curve is rotated about the z axis, we obtain a *surface of revolution.* We may parameterize this surface in the following manner. It is useful to use coordinates t and θ instead of u^1 and u^2, where t measures position on the curve and θ measures how far the curve has been rotated. The surface is given by

$$\boldsymbol{x}(t, \theta) = (r(t)\cos\theta, r(t)\sin\theta, z(t)).$$

(a) Prove that $\boldsymbol{x}$ is a simple surface if the original curve $\boldsymbol{\alpha}(r(t), z(t))$ was regular and one-to-one and if $-\pi < \theta < \pi$ by computing $\boldsymbol{x}_1 = \partial\boldsymbol{x}/\partial t$, $\boldsymbol{x}_2 = \partial\boldsymbol{x}/\partial\theta$.

(b) Show that Problem 1.1 is a surface of revolution.

(c) Show that Example 1.9 is a surface of revolution.

The t-curves are called *meridians* and the θ-curves are called *circles of latitude.* The z-axis is called the *axis of revolution.*

(d) What are the meridians and circles of latitude of Problem 1.1 and Example 1.9?

Solution

(a) Since $\boldsymbol{\alpha}(r(t), z(t))$ is (assumed to be) regular, it is defined on an open subset of $\mathbb{R}$, say (a, b). In this case, $\mathscr{U} = (a, b) \times (-\pi, \pi) \subset \mathbb{R}^2$ is open since it is a Cartesian product of open sets. Further $\boldsymbol{x}(t, \theta) = (r(t)\cos\theta, r(t)\sin\theta, z(t))$ is injective. Indeed, suppose $t = t_0 \in (a, b)$ is fixed. Then $\boldsymbol{x}(t_0, \theta)$ is injective since $(\sin\theta, \cos\theta, 0)$ is injective on $(-\pi, \pi)$. If instead $\theta = \theta_0 \in (-\pi, \pi)$ is fixed, then $\boldsymbol{x}(t, \theta_0)$ is injective since $r(t)$ is injective. Finally, if r is C^k and z is C^ℓ for some $k, \ell \geq 1$, then $\boldsymbol{x}$ is $C^{\min\{k,\ell\}}$, so it is class C^m for some $m \geq 1$. Hence, by Definition 1, we need only check that the cross product of the partial derivatives of $\boldsymbol{x}$ never vanish on $\mathscr{U}$. Indeed

$$\begin{aligned}
\boldsymbol{x}_1 &= \frac{\partial\boldsymbol{x}}{\partial t} = (\quad \dot{r}(t)\cos\theta, \dot{r}(t)\sin\theta, \dot{z}(t)),\\
\boldsymbol{x}_2 &= \frac{\partial\boldsymbol{x}}{\partial\theta} = (-r(t)\sin\theta, r(t)\cos\theta, 0).
\end{aligned}$$

Let $\{\boldsymbol{e}_i\}_{i=1}^3$ denote the standard basis for $\mathbb{R}^3$. Then the cross product reads:

$$\boldsymbol{x}_1 \times \boldsymbol{x}_2 = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \dot{r}(t)\cos\theta & \dot{r}(t)\sin\theta & \dot{z}(t) \\ -r(t)\sin\theta & r(t)\cos\theta & 0 \end{vmatrix} = (-r(t)\dot{z}(t)\cos\theta)\boldsymbol{e}_1 - (r(t)\dot{z}(t)\sin\theta)\boldsymbol{e}_2 + (r(t)\dot{r}(t))\boldsymbol{e}_3$$

Or in components,

$$\boldsymbol{x}_1 \times \boldsymbol{x}_2 = r(t)\big(-\dot{z}(t)\cos\theta, -\dot{z}(t)\sin\theta, \dot{r}(t)\big).$$

Note the magnitude:

$$\|\boldsymbol{x}_1 \times \boldsymbol{x}_2\| = r(t)\sqrt{\dot{z}^2(t) + \dot{r}^2(t)}.$$

By assumption, $r(t) > 0$ for all $t \in (a, b)$ and as such never vanishes. Since $\boldsymbol{\alpha} = \boldsymbol{\alpha}(r(t), z(t))$ is regular, we have $\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t} \neq 0$ for all $t \in (a, b)$. As such,

$$\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t} = \frac{\partial\boldsymbol{\alpha}}{\partial r}\dot{r}(t) + \frac{\partial\boldsymbol{\alpha}}{\partial z}\dot{z}(t) \neq 0 \quad \forall t \in (a, b).$$

Evidently, $\dot{r}(t)$ and $\dot{z}(t)$ cannot both vanish simultaneously, for if they did $\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t}$ would vanish, contradicting the above. It follows that $\sqrt{\dot{z}^2(t) + \dot{r}^2(t)}$ is never zero, and so $\|\boldsymbol{x}_1 \times \boldsymbol{x}_2\| \neq 0$ on $\mathscr{U}$. Hence, $\boldsymbol{x}$ is simple surface.

(b) For the coordinate patch in Problem 1.1, write:

$$\boldsymbol{x}(u^1, u^2) = \big(r(u^1)\cos u^2, r(u^1)\sin u^2, z(u^1)\big)$$

where $r(u^1) := 2+\cos u^1$ and $z(u^1) := \sin u^1$. This appears to be a surface of revolution with $t = u^1$ and $\theta = u^2$. Indeed, $\mathscr{U} = (-\pi, \pi) \times (-\pi, \pi)$, which is consistent with the domain of a surface of revolution, and $r(u^2) = 2 + \cos u^1 \geq 1 > 0$. Hence, $\boldsymbol{x}$ is manifestly a surface of revolution.

(c) The coordinate patch from Example 1.9 in Millman and Parker is as follows. For $\mathscr{W} = \{(w^1, w^2) \in \mathbb{R}^2 : -\pi/2 < w^1 < \pi/2, -\pi < w^2 < \pi\}$, let

$$\boldsymbol{z}(w^1, w^2) = \big(\cos w^1 \cos w^2, \cos w^1 \sin w^2, \sin w^1\big).$$

We will take for granted that $\boldsymbol{z}$ is a coordinate patch. Again, write

$$\boldsymbol{z}(w^1, w^2) = \big(r(w^1)\cos w^2, r(w^1)\sin w^2, z(w^1)\big)$$

for $r(w^1) := \cos w^1$ and $z(w^1) := \sin w^1$. This also appears to be a surface of revolution with $t = w^1$ and $\theta = w^2$, and the domain $\mathscr{W}$ is again consistent with that of a surface of revolution (namely, $\theta \equiv w^2 \in (-\pi, \pi)$). Further, for $w^1 \in (-\pi/2, \pi/2)$, $r(w^1) = \cos w^1 > 0$. Hence, $\boldsymbol{z}$ is manifestly a surface of revolution.

(d) We have the following definition.

Definition 2. *Let $\boldsymbol{x} : \mathscr{U} \to \mathbb{R}^3$ be a simple surface. The u^1-curve, $\boldsymbol{\alpha}$, through $P = \boldsymbol{x}(a, b)$ is given by $\boldsymbol{\alpha}(u^1) = \boldsymbol{x}(u^1, b)$. The u^1-curve, $\boldsymbol{\beta}$, through P is given by $\boldsymbol{\beta}(u^2) = \boldsymbol{x}(a, u^2)$.*

For Example 1.9, the meridians and circles of latitude are obvious, as they coincide with the usual geographical notions of meridians and latitudes, since $\boldsymbol{z}$ is S^2 (minus the $180°$ (cartographical) meridian).

For Problem 1.1, the meridians are the cross sections of the torus where the planes of intersection contain the z-axis. The circles of latitude are the cross sections of the torus where the planes of intersection are perpendicular to the z-axis. For example, one such pair of meridians and circles of latitude are

$$\boldsymbol{\alpha}(u^1) \equiv \boldsymbol{x}(u^1, 0) = (2 + \cos u^1, 0, \sin u^1),$$
$$\boldsymbol{\beta}(u^2) \equiv \boldsymbol{x}(0, u^2) = (3\cos u^2, 3\sin u^2, 0).$$

Here $\boldsymbol{\alpha}$ is a meridian whose image is a circle in the r, z plane displaced two units to the right of the origin, and $\boldsymbol{\beta}$ is a circle of latitude whose image is a circle about the axis of revolution obtained by rotating the point $(1, 0, 0)$ around the axis of revolution.

I am not sure if I have sufficiently answered this question. It is rather open-ended.

Chapter 04 | Problem 1.14

Let $\boldsymbol{\alpha} : (a,b) \to \mathbb{R}^3$ be a unit speed curve with $\kappa \neq 0$. Let

$$\mathscr{U} = \{(s,t) \in \mathbb{R}^2 : a < s < b, t \neq 0\}.$$

Define $\boldsymbol{x} : \mathscr{U} \to \mathbb{R}^3$ by $\boldsymbol{x}(s,t) = \boldsymbol{\alpha}(s) + t\boldsymbol{\alpha}'(s)$. Prove that $\boldsymbol{x}$ is a simple surface, provided $\boldsymbol{x}$ is one-to-one. $\boldsymbol{x}$ is called the *tangent developable surface* of $\boldsymbol{\alpha}$.

Solution

Suppose $\boldsymbol{\alpha}$ is a C^k unit speed curve. Then $\boldsymbol{x}(s,t) = \boldsymbol{\alpha}(s) + t\boldsymbol{\alpha}'(s)$ is C^{k-1}. Indeed, in the standard basis for $\mathbb{R}^3$, we can write:

$$\boldsymbol{x}(s,t) = \left(\alpha^1(s) + t\frac{\mathrm{d}\alpha^1}{\mathrm{d}s}, \alpha^2(s) + t\frac{\mathrm{d}\alpha^2}{\mathrm{d}s}, \alpha^3(s) + t\frac{\mathrm{d}\alpha^3}{\mathrm{d}s}\right).$$

If we fix $s = s_0 \in \mathbb{R}$, the components of $\boldsymbol{x}(s_0, t)$ are seen to be polynomials in t and are therefore class C^∞. If instead we fix $t = t_0 \in \mathbb{R}$, the components of $\boldsymbol{x}(s, t_0)$ depend on the components of $\boldsymbol{\alpha}$ and $\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}s}$, which are class C^k and C^{k-1} respectively. Hence, $\boldsymbol{x}$ is class C^{k-1}. Further, $\boldsymbol{x}$ is injective since $\boldsymbol{\alpha}$ is (assumed to be) injective.

Additionally, I claim that our domain $\mathscr{U} = (a,b) \times \mathbb{R} \setminus \{0\} \subset \mathbb{R}^2$ is open. Note that (a,b) is open, and that $(\mathbb{R} \setminus \{0\})^c = \{0\}$ is closed, so $\mathbb{R} \setminus \{0\}$ is open. Since the Cartesian product of open sets is open, $\mathscr{U}$ is indeed open.

Hence, by Definition 1, we need only check that the cross product of the partial derivatives never vanishes. We have

$$\boldsymbol{x}_1 = \frac{\partial \boldsymbol{x}}{\partial s} = \boldsymbol{\alpha}'(s) + t\boldsymbol{\alpha}''(s),$$
$$\boldsymbol{x}_2 = \frac{\partial \boldsymbol{x}}{\partial t} = \boldsymbol{\alpha}'(s).$$

So their cross product reads

$$\boldsymbol{x}_1 \times \boldsymbol{x}_2 = (\boldsymbol{\alpha}'(s) + t\boldsymbol{\alpha}''(s)) \times \boldsymbol{\alpha}'(s) = t\boldsymbol{\alpha}'(s) \times \boldsymbol{\alpha}''(s)$$

since $\boldsymbol{\alpha}' \times \boldsymbol{\alpha}' \equiv 0$. I claim that this never vanishes.

Note that $t \neq 0$ in $\mathscr{U}$. Also, we have

$$\boldsymbol{\alpha}'(s) = \boldsymbol{T}(s)$$
$$\boldsymbol{\alpha}''(s) = \boldsymbol{T}'(s) = \kappa \boldsymbol{N}$$

from Frenet-Serret theory. As such $\boldsymbol{\alpha}' \times \boldsymbol{\alpha}'' = \boldsymbol{T} \times \kappa\boldsymbol{N} = \kappa(\boldsymbol{T} \times \boldsymbol{N}) = \kappa$ since $\boldsymbol{T}$ and $\boldsymbol{N}$ are everywhere orthonormal. But, $\kappa \neq 0$ by assumption, and so we conclude $\boldsymbol{\alpha}''(s) \times \boldsymbol{\alpha}'(s) \neq \boldsymbol{0}$. Hence $\boldsymbol{x}_1 \times \boldsymbol{x}_2$ is everywhere nonzero on $\mathscr{U}$, and so $\boldsymbol{x}$ is a smooth surface.

Chapter 04 | Problem 2.1

The coordinate patch of Problem 1.2 does not cover the entire surface of revolution — it omits points that would correspond to $\theta = \pm\pi$. Define a second coordinate patch with $0 < \varphi < 2\pi$, check the overlap condition, and thus show that the surface of revolution is a surface as in Example 2.3.

Solution

We could use the same coordinate patch as used in Problem 1.2, changing only the angle of definition. We have the following two coordinate patches.

$$\begin{aligned} \mathscr{U} &= (a,b) \times (-\pi,\pi) &\quad : \quad& \boldsymbol{x}(t,\theta) = (r(t)\cos\theta, r(t)\sin\theta, z(t)) \\ \mathscr{V} &= (a,b) \times (0,2\pi) &\quad : \quad& \boldsymbol{y}(s,\theta) = (r(s)\cos\varphi, r(s)\sin\varphi, z(s)), \end{aligned}$$

I claim that these two coordinate patches cover the surface of revolution S. Note that both $\boldsymbol{x}$ and $\boldsymbol{y}$ are proper and that every point on S is either in $\boldsymbol{x}(\mathscr{U})$ or in $\boldsymbol{y}(\mathscr{V})$. Hence to prove S is a surface, we need only find a C^k coordinate transform

$$\boldsymbol{y}^{-1} \circ \boldsymbol{x} \equiv \boldsymbol{\psi} : \left(\boldsymbol{x}^{-1}\Big(\boldsymbol{x}(\mathscr{U}) \cap \boldsymbol{y}(\mathscr{V})\Big)\right) \to \left(\boldsymbol{y}^{-1}\Big(\boldsymbol{x}(\mathscr{U}) \cap \boldsymbol{y}(\mathscr{V})\Big)\right),$$

i.e., a C^k bijection between the preimages of the overlapping regions in S.

We can take $s = t$ since the domains (a,b) are the same. From here, the choice of bijection is clear, since the map $(t,\theta) \to (t,\varphi)$ is isomorphic to a horizontal translation of the region $(a,b) \times (-\pi,\pi)$ to $(a,b) \times (0,2\pi)$. Hence, we choose

$$(s,\varphi) = \boldsymbol{\psi}(t,\theta) \equiv (t, \theta + \pi)$$

for all points $t \in (a,b)$ and $\theta \in (-\pi,0) \cup (0,\pi)$. Clearly the map is C^∞, since its components are polynomials, and its inverse $\boldsymbol{\psi}^{-1}(s,\varphi) = (s, \varphi - \pi)$ is C^∞ for identical reasons. Hence $\boldsymbol{\psi}$ is a valid coordinate transform, and S is indeed a surface covered by the coordinate patches $\boldsymbol{x}$ and $\boldsymbol{y}$.

Chapter 04 | Problem 3.1

Show that the metric coefficients for the surface of revolution (Problem 1.2) are given by the matrix

$$\begin{pmatrix} \dot{r}^2 + \dot{z}^2 & 0 \\ 0 & r^2 \end{pmatrix}.$$

Solution

Recall the metric coefficients are defined by $\mathsf{g}_{ij} := \langle \boldsymbol{x}_i, \boldsymbol{x}_j \rangle$ where $\boldsymbol{x}_i := \partial \boldsymbol{x} / \partial u^i$ for $i \in \{t, \theta\}$. We have

$$\boldsymbol{x}(t, \theta) = (r(t)\cos\theta, r(t)\sin\theta, z(t))$$

and so

$$\begin{aligned} \boldsymbol{x}_1 &= \frac{\partial \boldsymbol{x}}{\partial t} = (\quad \dot{r}(t)\cos\theta, \dot{r}(t)\sin\theta, \dot{z}(t)), \\ \boldsymbol{x}_2 &= \frac{\partial \boldsymbol{x}}{\partial t} = (-r(t)\sin\theta, r(t)\cos\theta, 0). \end{aligned}$$

So the metric components read

$$\begin{aligned} \mathsf{g}_{11} &= \langle \boldsymbol{x}_1, \boldsymbol{x}_1 \rangle = \dot{r}^2\cos^2\theta + \dot{r}^2\sin^2\theta + \dot{z}^2 = \dot{r}^2 + \dot{z}^2, \\ \mathsf{g}_{12} = \mathsf{g}_{21} &= \langle \boldsymbol{x}_2, \boldsymbol{x}_1 \rangle = r\dot{r}\sin\theta\cos\theta - r\dot{r}\sin\theta\cos\theta = 0, \\ \mathsf{g}_{22} &= r^2\sin^2\theta + r^2\cos^2\theta = r^2, \end{aligned}$$

where I have suppressed the t-dependence for brevity. Hence, the metric takes the form

$$\boxed{(\mathsf{g}_{ij}) = \begin{pmatrix} \dot{r}^2 + \dot{z}^2 & 0 \\ 0 & r^2 \end{pmatrix}}$$

as expected.

Chapter 04 | Problem 3.5

For a coordinate patch $\boldsymbol{x} : \mathscr{U} \to \mathbb{R}^3$ show that u^1 is arc length on the u^1-curves if and only if $\mathsf{g}_{11} \equiv 1$.

Solution

Recall that if $\boldsymbol{\alpha} : (a, b) \to \mathbb{R}^3$ is unit-speed, then $\|\boldsymbol{\alpha}'(s)\| = 1$ on (a, b). From Definition 2, the appropriate generalization is that $\boldsymbol{x} : \mathscr{U} \to \mathbb{R}^3$ is unit-speed along the u^1-curves if and only if $\left\|\boldsymbol{x}_1(u^1, u^2)\right\| = 1$ on $\mathscr{U}$ (for a fixed u^2). But

$$\mathsf{g}_{11} \equiv 1 \iff \langle \boldsymbol{x}_1, \boldsymbol{x}_1 \rangle \equiv 1 \iff \|\boldsymbol{x}_1\|^2 \equiv 1 \iff \|\boldsymbol{x}_1\| \equiv 1.$$

Hence u^1 is arc length on the u^1-curves if and only if $\mathsf{g}_{11} \equiv 1$.
