Homework # 03

Math 140
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Chapter 02

Problem 5.2

Find a unit speed curve $\boldsymbol{\alpha}(s)$ with $\kappa(s) = 1/(1+s^2)$ and $\tau \equiv 0$.

Solution

I will proceed as in Chapter 02, Example 5.3 of Millman and Parker. Let us make the following change of variables:

$$t(s) = \int_0^s \kappa(\sigma)\,\mathrm{d}\sigma = \int_0^s \frac{1}{1+\sigma^2}\,\mathrm{d}\sigma = \arctan(s). \tag{1}$$

Such a change of variables is admissible since $t'(s) = \kappa > 0$, which implies t is one-to-one and both t, κ are smooth. In what follows, let $u' := \mathrm{d}u/\mathrm{d}s$ and $\dot{u} = \mathrm{d}u/\mathrm{d}t$. Recall the Frenet-Serret Theorem:[1]

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix} \tag{2}$$

Hence, in the s basis, we have

$$\mathbf{T}' = \kappa\mathbf{N} \quad ; \quad \mathbf{N}' = -\kappa\mathbf{T}.$$

In the t basis, we have $u'(t(s)) = t'(s)\dot{u}(t) = \kappa\dot{u}$ for all smooth functions u. Hence, the above equations take the form

$$\dot{\mathbf{T}} = \mathbf{N} \quad ; \quad \dot{\mathbf{N}} = -\mathbf{T}.$$

So the utility of the substitution (1) is that it decomposes Eq. (2) into a set of coupled constant-coefficient first order linear differential equations. Differentiating the first and using the second gives us an uncoupled second order ODE:

$$\ddot{\mathbf{T}} = \dot{\mathbf{N}} = -\mathbf{T}.$$

The most general solution to the above is

$$\mathbf{T}(t) = \mathbf{a}\cos t + \mathbf{b}\sin t$$

where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ are constant. We find $\boldsymbol{\alpha}$ by integrating over s:

$$\boldsymbol{\alpha}(s) = \mathbf{a}\int_0^s \cos t(\sigma)\,\mathrm{d}\sigma + \mathbf{b}\int_0^s \sin t(\sigma)\,\mathrm{d}\sigma + \mathbf{c} \tag{3}$$

where $\mathbf{c} \in \mathbb{R}^3$ is another constant, In fact, $\mathbf{c} = \boldsymbol{\alpha}(0)$, since both integrals vanish at $s = 0$. Since we are looking for an arbitrary curve with the given curvature, not the most general one, we might as well set $\mathbf{c} = \mathbf{0}$. Next, we examine the integrals in Eq. (3). Both integrands can be expressed in terms of algebraic functions. Consider $\cos(\arctan(\sigma))$. Let $\theta = \arctan(\sigma)$ so that $\sigma = \tan\theta$. Then:

$$\sec^2\theta = 1 + \tan^2\theta = 1 + \sigma^2 \quad \Longrightarrow \quad \cos(\theta) = \frac{1}{\sqrt{1+\sigma^2}}.$$

For $\sin(\arctan(\sigma))$, we have:

$$\sigma = \tan\theta = \frac{\sin\theta}{\cos\theta} \quad \Longrightarrow \quad \sin\theta = \frac{\sigma}{\sqrt{1+\sigma^2}}.$$

Hence, the first of the integrals is

$$\int_0^s \cos t(\sigma)\,\mathrm{d}\sigma = \int_0^s \frac{1}{\sqrt{1+\sigma^2}}\,\mathrm{d}\sigma = \operatorname{arcsinh}(s).$$

[1]Theorem 4.2 of Chapter 2 of Millman and Parker (page 30).

The second:

$$\int_0^s \sin t(\sigma)\,\mathrm{d}\sigma = \frac{1}{2}\int_0^s \frac{2\sigma}{\sqrt{1+\sigma^2}}\,\mathrm{d}\sigma = \sqrt{1+s^2}-1.$$

Hence, our curve is

$$\boldsymbol{\alpha}(s) = \mathbf{a}(\operatorname{arcsinh}(s)) + \mathbf{b}(\sqrt{1+s^2}-1).$$

The vectors $\mathbf{a}$ and $\mathbf{b}$ are not arbitrary, however. Note:

$$\mathbf{N} = \dot{\mathbf{T}} = -\mathbf{a}\sin t + \mathbf{b}\cos t$$
$$\frac{\mathrm{d}\mathbf{N}}{\mathrm{d}t} = -\mathbf{a}\cos t - \mathbf{b}\sin t.$$

Then,

$$0 = \left\langle \mathbf{N}, \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}t}\right\rangle = \tfrac{1}{2}(|\mathbf{a}|^2 - |\mathbf{b}|^2)\sin 2t + \langle \mathbf{a}, \mathbf{b}\rangle(\cos^2 t - \sin^2 t).$$

At $t = 0$, we get $0 = \langle \mathbf{a}, \mathbf{b}\rangle$, so the vectors are orthogonal. Then,

$$0 = (|\mathbf{a}|^2 - |\mathbf{b}|^2)\sin 2t.$$

For this to be true for all t, we require $|\mathbf{a}|^2 = |\mathbf{b}|^2$. Finally,

$$1 = |\mathbf{N}|^2 = |\mathbf{a}|^2\sin^2 t + |\mathbf{b}|^2\cos^2 t = |\mathbf{a}|^2 = |\mathbf{b}|^2.$$

Hence, $|\mathbf{a}| = |\mathbf{b}| = 1$. That is, the vectors are orthonormal. One possible choice is $\mathbf{a} = \mathbf{x}$ and $\mathbf{b} = \mathbf{y}$. Hence, one such unit speed curve with the given curvature and torsion is

$$\boxed{\boldsymbol{\alpha}(s) = \left(\operatorname{arcsinh}(s), \sqrt{1+s^2}-1, 0\right).}$$

Problem 5.3

Prove that the only plane or spherical unit speed curves of constant curvature are circles.

Solution

Suppose $\boldsymbol{\alpha}$ is a planar unit speed curve. Then, we are free to choose a basis so that

$$\boldsymbol{\alpha}(s) = (x(s), y(s), 0)$$

where $x, y : (a,b) \to \mathbb{R}$ for some $(a,b) \subseteq \mathbb{R}$. We have the following lemma.

Lemma. *Let $\boldsymbol{\alpha}$ be a planar unit speed curve. Then the curvature of $\boldsymbol{\alpha}$ is constant if and only if the image of $\boldsymbol{\alpha}$ is a circle.*

Proof. ($\Longleftarrow$) Suppose the image of $\boldsymbol{\alpha}$ is a circle in the plane. Then, up to orientation, we have[2]

$$\boldsymbol{\alpha}(s) = \left(r\cos(\tfrac{s}{r}), r\sin(\tfrac{s}{r}), 0\right)$$

where $r > 0$ is the radius of the circle. Then,

$$\mathbf{T} = \boldsymbol{\alpha}' = \left(-\sin(\tfrac{s}{r}), \cos(\tfrac{s}{r}), 0\right)$$
$$\mathbf{T}' = \left(-\tfrac{1}{r}\cos(\tfrac{s}{r}), -\tfrac{1}{r}\sin(\tfrac{s}{r}), 0\right).$$

Hence,

$$\kappa(s) = \left|\mathbf{T}'(s)\right| = \sqrt{\left(-\tfrac{1}{r}\cos(\tfrac{s}{r})\right)^2 + \left(-\tfrac{1}{r}\sin(\tfrac{s}{r})\right)^2} = \frac{1}{r}$$

which is constant.

[2] It was proved in lecture that this parameterization of a circle is unit speed, so I will not prove it again here.

($\Longrightarrow$) Suppose κ is a constant. Since $\boldsymbol{\alpha}$ is planar, $\tau \equiv 0$ for all s. Then, by the Frenet-Serret Theorem (2), we have

$$\mathbf{T}' = \kappa \mathbf{N} \quad ; \quad \mathbf{N}' = -\kappa \mathbf{T}.$$

Differentiating the former and using the latter, we find

$$\mathbf{T}'' = \kappa \mathbf{N}' = -\kappa^2 \mathbf{T}.$$

The most general solution to the above is

$$\mathbf{T}(s) = \mathbf{a} \cos ks + \mathbf{b} \sin ks$$

where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. We can now find $\boldsymbol{\alpha}$ by direct integration:

$$\boldsymbol{\alpha}(s) = \mathbf{a}\tfrac{1}{\kappa} \sin ks - \mathbf{b}\tfrac{1}{\kappa} \cos ks + \mathbf{c}$$

where $\mathbf{c} \in \mathbb{R}^3$ is arbitrary. We know from Problem 5.2 that $\mathbf{a}$ and $\mathbf{b}$ form an orthonormal basis of a two-dimensional subspace of $\mathbb{R}^3$. Hence, in the $\{\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}\}$ basis, we find

$$\boldsymbol{\alpha}(s) = \mathbf{c} + \left(\tfrac{1}{\kappa} \sin ks, -\tfrac{1}{\kappa} \cos ks, 0\right).$$

Hence, the image of $\boldsymbol{\alpha}$ is a circle of radius $r = \frac{1}{\kappa}$ and center $\mathbf{c}$. This completes the proof of the lemma. $\square$

Next, we consider the case where $\boldsymbol{\alpha}$ lies on the surface of a sphere. We have the following lemma.

Lemma. *Let $\boldsymbol{\alpha}$ be a spherical unit speed curve with constant curvature. Then the image of $\boldsymbol{\alpha}$ is a circle.*

Proof. Suppose $\boldsymbol{\alpha}$ is a spherical curve with constant curvature κ. Since $\boldsymbol{\alpha}$ is spherical, $\exists r \in \mathbb{R} \setminus \{0\}, \mathbf{m} \in \mathbb{R}^3$ such that $\langle \boldsymbol{\alpha} - \mathbf{m}, \boldsymbol{\alpha} - \mathbf{m} \rangle = r^2$. By the fundamental theorem of curves,[3] we can take $\mathbf{m} = \mathbf{0}$ without loss of generality. Differentiating this equation:

$$0 = \frac{\mathrm{d}}{\mathrm{d}s} \langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle = 2 \langle \boldsymbol{\alpha}, \mathbf{T} \rangle . \tag{4}$$

In particular, this implies

$$\boldsymbol{\alpha}(s) = \langle \boldsymbol{\alpha}, \mathbf{N} \rangle \mathbf{N} + \langle \boldsymbol{\alpha}, \mathbf{B} \rangle \mathbf{B}. \tag{5}$$

Differentiating Eq. (4) again, we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}s} \langle \boldsymbol{\alpha}, \mathbf{T} \rangle = \langle \mathbf{T}, \mathbf{T} \rangle + \langle \boldsymbol{\alpha}, \kappa \mathbf{N} \rangle .$$

In particular,

$$-\frac{1}{\kappa} = \langle \boldsymbol{\alpha}, \mathbf{N} \rangle . \tag{6}$$

Since κ is constant, we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}s} \left\{ -\frac{1}{\kappa} \right\} = \frac{\mathrm{d}}{\mathrm{d}s} \langle \boldsymbol{\alpha}, \mathbf{N} \rangle = \langle \mathbf{T}, \mathbf{N} \rangle + \langle \boldsymbol{\alpha}, -\kappa \mathbf{T} + \tau \mathbf{B} \rangle .$$

In particular, since $\langle \mathbf{T}, \mathbf{N} \rangle = 0$ by orthonormality and $\langle \boldsymbol{\alpha}, \mathbf{T} \rangle = 0$ by Eq. (4), we find

$$0 = \tau \langle \boldsymbol{\alpha}, \mathbf{B} \rangle . \tag{7}$$

Suppose by way of contradiction that $\tau \neq 0$ so that $\langle \boldsymbol{\alpha}, \mathbf{B} \rangle = 0$. Then from Eq.'s (5) and (6), we have

$$\boldsymbol{\alpha}(s) = -\frac{1}{\kappa} \mathbf{N}.$$

If we differentiate once more, we get

$$\mathbf{T} = -\frac{1}{\kappa} \Big[-\kappa \mathbf{T} + \tau \mathbf{B} \Big] = \mathbf{T} - \frac{\tau}{\kappa} \mathbf{B},$$

[3] See Problem 5.8 (page 5)

i.e., that $\tau = 0$ (since $\mathbf{B} \neq \mathbf{0}$), which is a contradiction. Hence, we are forced to conclude $\langle \boldsymbol{\alpha}, \mathbf{B} \rangle \neq 0$, and so we require

$$\tau = 0$$

to satisfy Eq. (7). This implies $\boldsymbol{\alpha}$ is planar, but the only planar curves that lie on a sphere are circles.[4] Hence, $\boldsymbol{\alpha}$ is a circle.

□

Problem 5.7

Let $\boldsymbol{\alpha}(s)$ be a unit speed curve with $\kappa > 0$ and $\tau > 0$. Let

$$\boldsymbol{\beta}(t) = \int_0^s \mathbf{B}(\sigma)\,\mathrm{d}\sigma\,.$$

(a) Prove that $\boldsymbol{\beta}$ is unit speed.

(b) Show that the Frenet-Serret apparatus $\{\bar{\kappa}, \bar{\tau}, \bar{\mathbf{T}}, \bar{\mathbf{N}}, \bar{\mathbf{B}}\}$ of $\boldsymbol{\beta}$ satisfies $\bar{\kappa} = \tau, \bar{\tau} = \kappa, \bar{\mathbf{T}} = \mathbf{B}, \bar{\mathbf{N}} = -\mathbf{N}$, and $\bar{\mathbf{B}} = \mathbf{T}$.

Solution

(a) A curve $\boldsymbol{\gamma}$ is unit speed if and only if $|\mathrm{d}\boldsymbol{\gamma}/\mathrm{d}s| = 1$ for all s. But

$$\frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\int_0^s \mathbf{B}(\sigma)\,\mathrm{d}\sigma = \mathbf{B}(s)$$

by the fundamental theorem of calculus. Since $\mathbf{B}(s)$ is a unit vector for all s, we have

$$\left|\frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}s}\right| = |\mathbf{B}(s)| \equiv 1$$

for all s. Hence, $\boldsymbol{\beta}$ is indeed unit speed.

(b) Recall that the Frenet-Serret apparatus is defined for a curve $\boldsymbol{\gamma}$ as follows:

$$\begin{cases} \mathbf{T}(s) & = & \frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}s} \\ \mathbf{N}(s) & = & \frac{1}{\kappa}\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} \\ \mathbf{B}(s) & = & \mathbf{T} \times \mathbf{N} \\ \kappa(s) & = & \left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right| \\ \tau(s) & = & -\left\langle \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}, \mathbf{N}\right\rangle \end{cases}$$

For $\boldsymbol{\beta}$, we have

$$\bar{\mathbf{T}} = \frac{\mathrm{d}\boldsymbol{\beta}}{\mathrm{d}s} = \mathbf{B}.$$

Hence $\bar{\mathbf{T}} = \mathbf{B}$, which we found in (a). Next,

$$\frac{\mathrm{d}\bar{\mathbf{T}}}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s} - \tau\mathbf{N} \qquad \Longrightarrow \qquad \bar{\kappa} = |-\tau\mathbf{N}| = \tau$$

where in the last step on the left I used Eq. (2) to write $\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}$ in $\boldsymbol{\alpha}$'s Frenet-Serret basis, and on th right I used $|\mathbf{N}| = 1$. Hence $\bar{\kappa} = \tau$. Next,

$$\bar{\mathbf{N}} = \frac{1}{\bar{\kappa}}\frac{\mathrm{d}\bar{\mathbf{T}}}{\mathrm{d}s} = \frac{1}{\tau}(-\tau\mathbf{N}) = -\mathbf{N},$$

[4]This is an elementary result from analytic geometry, but it is a mess to prove, so I hope it was not required!

so $\bar{\mathbf{N}} = -\mathbf{N}$. The bi-normal vector is

$$\bar{\mathbf{B}} = \bar{\mathbf{T}} \times \bar{\mathbf{N}} = \mathbf{B} \times (-\mathbf{N}) = \mathbf{N} \times \mathbf{B} = \mathbf{T}$$

since $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is an oriented orthonormal basis. Hence $\bar{\mathbf{B}} = \mathbf{T}$. Differentiating the above and using Eq. (2), we get

$$\frac{\mathrm{d}\bar{\mathbf{B}}}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \kappa\mathbf{N}.$$

Finally, this implies

$$\bar{\tau} = -\left\langle \frac{\mathrm{d}\bar{\mathbf{B}}}{\mathrm{d}s}, \bar{\mathbf{N}} \right\rangle = -\left\langle \kappa\mathbf{N}, -\mathbf{N} \right\rangle = \kappa,$$

or $\bar{\tau} = \kappa$. Thus, we recover the Frenet-Serret apparatus from the prompt.

Problem 5.8

Let $\boldsymbol{\alpha}(s)$ be a helix with $\kappa = \tau > 0$. If $\boldsymbol{\beta}$ is defined as in Problem 5.7, show that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are congruent; that is, they are the same curve up to position in space.

Solution

If $\kappa, \tau > 0$, then the curve $\boldsymbol{\beta}$ is well-defined. Since $\kappa = \tau$, the Frenet-Serret relations proved in the previous question imply

$$\bar{\kappa} = \kappa,$$
$$\bar{\tau} = \tau.$$

We have the following theorem from lecture:[5]

Theorem (The Fundamental Theorem of Curves)**.** *Given an interval (a, b) containing 0, two smooth functions $\bar{\kappa}, \bar{\tau} : (a, b) \to \mathbb{R}$ with $\bar{\kappa} > 0$, a point $\mathbf{x}_0 \in \mathbb{R}^3$, and an oriented orthonormal basis $\{\mathbf{D}, \mathbf{E}, \mathbf{F}\} \subset \mathbb{R}^3$, there is a unique curve $\boldsymbol{\alpha}$ with unit speed so that $\boldsymbol{\alpha}(0) = \mathbf{x}_0$, $\mathbf{T}(0) = \mathbf{D}, \mathbf{N}(0) = \mathbf{E}, \mathbf{B}(0) = \mathbf{F}$ and $\kappa(s) = \bar{\kappa}(s)$ and $\tau(s) = \bar{\tau}(s)$.*

That is, the curvature and torsion specify a curve up to a translation and rotation in $\mathbb{R}^3$. By the foregoing theorem, we can choose a basis in $\mathbb{R}^3$ so that $\boldsymbol{\alpha}$ takes the form:[6]

$$\boldsymbol{\alpha}(s) = (r \cos \omega s, r \sin \omega s, h\omega s)$$

where $r, h \in (0, \infty)$ and $\omega := 1/\sqrt{r^2 + h^2}$. In this case,

$$\kappa = \left| \frac{\mathrm{d}^2\boldsymbol{\alpha}}{\mathrm{d}s^2} \right| = \omega^2 r > 0$$

which is constant in s. This implies that $\kappa : \mathbb{R} \to \mathbb{R}$ is smooth in s. This further implies that $\bar{\kappa}, \bar{\tau} : \mathbb{R} \to \mathbb{R}$ are smooth in s, and that $\bar{\kappa} > 0$. In particular, since $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ have the same curvature and the same torsion, then by the fundamental theorem of curves, the two curves $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are identical up to position in space.

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[5] Theorem 5.2 of Chapter 2 of Millman and Parker (page 42), i.e., the fundamental theorem of curves.

[6] This form of the helix was derived in lecture and proved to be unit speed, so I will omit the calculation.

Problem 6.4

Find the curvature and torsion of $\boldsymbol{\beta}(t) = (e^t \cos t, e^t \sin t, e^t)$.

Solution

Note that $\boldsymbol{\beta}$ is not unit-speed, so the canonical method of calculating curvature and torsion is not valid unless we reparameterize. However, the relationship between the arc lengh s and the parameter t is nonlinear, so it is better to avoid reparameterization in this case. We proved in lecture that[7] if $\boldsymbol{\beta}(t)$ is a regular curve that is not unit speed, then:

$$\begin{cases} \mathbf{T}(t) &= \dot{\boldsymbol{\beta}} \,/\, |\dot{\boldsymbol{\beta}}| \\ \mathbf{B}(t) &= \dot{\boldsymbol{\beta}} \times \ddot{\boldsymbol{\beta}} \,/\, |\dot{\boldsymbol{\beta}} \times \ddot{\boldsymbol{\beta}}| \\ \mathbf{N}(t) &= \mathbf{B} \times \mathbf{T} \\ \kappa(t) &= |\dot{\boldsymbol{\beta}} \times \ddot{\boldsymbol{\beta}}| \,/\, |\dot{\boldsymbol{\beta}}|^3 \\ \tau(t) &= [\dot{\boldsymbol{\beta}}, \ddot{\boldsymbol{\beta}}, \dddot{\boldsymbol{\beta}}] \,/\, |\dot{\boldsymbol{\beta}} \times \ddot{\boldsymbol{\beta}}|^2 \end{cases}$$

In the last line, I introduced the following notation:

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] := \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}),$$

i.e., $[\mathbf{u}, \mathbf{v}, \mathbf{w}]$ is the scalar triple product of $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. In any case, we need the derivatives of $\boldsymbol{\beta}$. We have

$$\begin{aligned} \boldsymbol{\beta} &= e^t(\cos t, \sin t, 1), \\ \dot{\boldsymbol{\beta}} &= e^t(\cos t - \sin t, \cos t + \sin t, 1), \\ \ddot{\boldsymbol{\beta}} &= e^t(-2\sin t, 2\cos t, 1), \\ \dddot{\boldsymbol{\beta}} &= e^t(-2\sin t - 2\cos t, -2\sin t + 2\cos t, 1). \end{aligned}$$

Then

$$\dot{\boldsymbol{\beta}} \times \ddot{\boldsymbol{\beta}} = e^{2t} \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \cos t - \sin t & \cos t + \sin t & 1 \\ -2\sin t & 2\cos t & 1 \end{vmatrix} = e^{2t}(\sin t - \cos t, -\sin t - \cos t, 2)$$

and so $|\dot{\boldsymbol{\beta}} \times \ddot{\boldsymbol{\beta}}| = \sqrt{6} e^{2t}$. Next,

$$[\dot{\boldsymbol{\beta}}, \ddot{\boldsymbol{\beta}}, \dddot{\boldsymbol{\beta}}] = e^{3t} \begin{vmatrix} \cos t - \sin t & \cos t + \sin t & 1 \\ -2\sin t & 2\cos t & 1 \\ -2\sin t - 2\cos t & -2\sin t + 2\cos t & 1 \end{vmatrix} = 2e^{3t}.$$

Finally,

$$|\dot{\boldsymbol{\beta}}| = e^t \sqrt{(\cos t - \sin t)^2 + (\cos t + \sin t)^2 + 1} = \sqrt{3} e^t.$$

Hence, the curvature and torsion read

$$\boxed{\begin{aligned} &\kappa(t) = |\dot{\boldsymbol{\beta}} \times \ddot{\boldsymbol{\beta}}| \,/\, |\dot{\boldsymbol{\beta}}|^3 = \tfrac{\sqrt{2}}{3} e^{-t}, \\ &\tau(t) = [\dot{\boldsymbol{\beta}}, \ddot{\boldsymbol{\beta}}, \dddot{\boldsymbol{\beta}}] \,/\, |\dot{\boldsymbol{\beta}} \times \ddot{\boldsymbol{\beta}}|^2 = \tfrac{1}{3} e^{-t}. \end{aligned}}$$

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[7] Proposition 6.1 of Chapter 2 of Millman and Parker (page 46).