Linear Algebra

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We make the following definitions.

Definition 1 (Sum)**.** If S_1 and S_2 are nonempty subsets of a vector space V, then the sum of S_1 and S_2 is $S_! + S_2 := \{x + y \mid x \in S_1, y \in S_2\}$.

Definition 2 (Direct sum)**.** A vector space V is called the direct sum of W_1 and W_2 if W_1 and W_2 are subspaces of V with $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$. We denote the V is the direct sum of W_1 and W_2 by writing $V = W_1 \oplus W_2$.

Problem Set 1: Vector Spaces

1. Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cap W_2$ is a subspace of V. (*Note that* $\{0\}$ *is a subspace of every vector space.*)

2. Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

3. Let W_1 and W_2 be subspaces of a vector space V.

   (a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

   (b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

   (c) Prove that $V = W_1 \oplus W_2$ if and only if each vector in V can be *uniquely* written as $x_1 + x_2$ where $x_1 \in W_1$ and $x_2 \in W_2$.

4. Suppose V is a finite dimensional vector space and U_1, U_2, …, U_m are subspaces of V such that $V = U_1 \oplus U_2 \oplus \cdots \oplus U_m$. Prove $\dim V = \dim U_1 + \cdots + \dim U_m$.

5. Suppose U_1, U_2, and U_3 are subspaces of a vector space V.

   (a) Prove
   $$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$
   This may convince you that the law of inclusion-exclusion holds for vector spaces.

   (b) Show that it is *not necessarily* true that
   $$\begin{aligned}\dim(U_1 + U_2 + U_3) = {} & \dim U_1 + \dim U_2 + \dim U_3 \\ & - \dim(U_1 \cap U_2) - \dim(U_2 \cap U_3) - \dim(U_1 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3)\end{aligned}$$
   [*Hint: Consider* $V = \mathbb{R}^2$.] This shows that that the law of inclusion-exclusion *does not* hold for vector spaces in general.

6. The techniques we learn about in linear algebra work in any field, but there are pathological properties associated with finite fields that may lead to confusion. Consider the set $S = \{(1,1,0)^\top, (1,0,1)^\top, (0,1,1)^\top\}$ as a subset of the vector space $\mathbb{F}^3$.

   (a) If $\mathbb{F} = \mathbb{R}$, show that S is a basis for $\mathbb{F}^3$.

   (b) If $\mathbb{F} = \mathbb{F}_2$ (the field of integers modulo 2), then S is not linearly independent and hence is not a basis for $\mathbb{F}^3$.

7. Suppose $S = \{v_1, \ldots, v_m\}$ is a linearly independent subset of a finite dimensional vector space V of dimension $n > m$. Show that S can be extended to a basis for V; that is, construct a basis for V of the form $\{v_1, \ldots, v_m, v_{m+1}, \ldots v_n\} = S \cup \{v_{m+1}, \ldots v_n\}$.

8. Consider the set $V = \{p \in P_3(\mathbb{R}) \mid p'(1) = 0\}$. Prove that V is a subspace of $P_3(\mathbb{R})$ and construct a basis for V. What is its dimension?