Homework # 04

Math 140
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Chapter 03

Problem 2.1

Prove Lemma 2.2.

Lemma (2.2)**.** *For a unit speed plane curve* $\boldsymbol{\alpha}$ *we have that*

$$\mathbf{t}(s) = \mathbf{T}(s) \quad ; \quad \mathbf{n}(s) = \pm\mathbf{N}(s)$$

at all points for which $\mathbf{N}(s)$ *is defined,* $\kappa(s) = |k(s)|$*,* $\mathbf{n}(s)$ *is differentiable, and* $\mathbf{n}'(s) = -k(s)\mathbf{t}(s)$*.*

Solution

The first equality is obvious, since the definitions are the same, i.e.,

$$\mathbf{t}(s) \equiv \frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}s} \equiv \mathbf{T}(s).$$

For the remainder of the proof, we extend to $\mathbb{R}^3$ by choosing a basis so that $\boldsymbol{\alpha}(s) = (x(s), y(s), 0)$. Then:

$$\mathbf{n}(s) = (-y'(s), x'(s), 0),$$
$$\mathbf{N}(s) = (x''(s)/\kappa, y''(s)/\kappa, 0).$$

In any case, we see that both $\mathbf{n}$ and $\mathbf{N}$ live in the plane in which $\boldsymbol{\alpha}$ is contained. And:

$$\langle \boldsymbol{\alpha}, \mathbf{n} \rangle \equiv 0$$
$$\text{and } \langle \boldsymbol{\alpha}, \mathbf{N} \rangle \equiv 0$$

by construction, so $\mathbf{n} \parallel \mathbf{N}$. Since both are unit vectors, this implies that they are equal up to orientation, i.e., $\mathbf{n} = \pm\mathbf{N}$, as expected.

Next, we show $\kappa = |k|$. Indeed,

$$\kappa = \left|\mathbf{T}'(s)\right| = \sqrt{(x'')^2 + (y'')^2};$$
$$k = \langle \mathbf{t}', \mathbf{n} \rangle = \langle (x'', y'', 0), (-y', x', 0) \rangle = x'y'' - x''y'.$$

But as we proved above,

$$\mathbf{n} = \pm\mathbf{N} \quad \Longleftrightarrow \quad -y' = \pm x''/\kappa \text{ , } x' = \pm y''/\kappa$$

Hence

$$|k| = \left|x'y'' - x''y'\right| = \frac{\left|\pm(x'')^2 \pm (y'')^2\right|}{|\kappa|} = \frac{\left|(x'')^2 + (y'')^2\right|}{\kappa},$$

where in the last step I used the fact that κ is non-negative. Since for any u, $|u| = \sqrt{u^2}$, we can write

$$|k| = \frac{\sqrt{[(x'')^2 + (y'')^2]^2}}{\sqrt{(x'')^2 + (y'')^2}} = \sqrt{(x'')^2 + (y'')^2} \equiv \kappa.$$

Hence $\kappa = |k|$.

Since $\mathbf{N} = \pm\mathbf{n}$ and $\mathbf{N}$ is differentiable, $\mathbf{n}$ is also differentiable. Since $\{\mathbf{t}, \mathbf{s}\}$ forms a basis for the subspace of $\mathbb{R}^2$ on which $\boldsymbol{\alpha}$ lives, we have

$$\mathbf{n}'(s) = \langle \mathbf{t}(s), \mathbf{n}'(s) \rangle \, \mathbf{t}(s) + \langle \mathbf{n}(s), \mathbf{n}'(s) \rangle \, \mathbf{n}(s).$$

But $|\mathbf{n}(s)| \equiv 1$ is constant, so $\langle \mathbf{n}(s), \mathbf{n}'(s) \rangle \equiv 0$. Also, since $\mathbf{t}$ and $\mathbf{n}$ are everywhere orthogonal, we have

$$0 = \langle \mathbf{t}(s), \mathbf{n}(s) \rangle \quad \Longrightarrow \quad 0 = \langle \mathbf{t}(s), \mathbf{n}(s) \rangle' = \langle \mathbf{t}'(s), \mathbf{n}(s) \rangle + \langle \mathbf{t}(s), \mathbf{n}'(s) \rangle .$$

From Lemma 2.1 in Millman and Parker, $\mathbf{t}'(s) = k(s)\mathbf{n}(s)$. Hence,

$$\langle \mathbf{t}(s), \mathbf{n}'(s) \rangle = -\langle \mathbf{t}'(s), \mathbf{n}(s) \rangle = -\langle k(s)\mathbf{n}(s), \mathbf{n}(s) \rangle = -k(s)$$

since $\mathbf{n}$ is a unit vector. Hence,

$$\mathbf{n}'(s) = -k(s)\mathbf{t}(s) + 0 \cdot \mathbf{n}(s) = -k(s)\mathbf{t}(s)$$

as expected.

Problem 5.4

Prove that the Four-Vertex Theorem is the hypothesis "closed" is omitted by considering the parabola $\boldsymbol{\alpha}(t) = (t, t^2)$.

Solution

The Four-Vertex Theorem states that a strictly convex simple closed curve has at least four distinct vertices, i.e., points for which $k(s)$ is a maximum or a minimum. Clearly the parabola is strictly convex, since the tangent line to the parabola does not intersect the curve at any other point than the point of tangency. It is also simple, because $t \mapsto (t, t^2)$ is injective for $t \in \mathbb{R}$. However, $\boldsymbol{\alpha}$ is not closed, so it fails to meet the requirements for the theorem.

Note that the parabola is not parameterized by arc length. The relationship between t and s is nonlinear, so it would be unwise to try to reparameterize. Instead, note the following.

Definition (Stationary points)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Then if $\exists x \in \mathbb{R}$ for which $f'(x) = 0$, we say that x is a stationary point point of f.*

Lemma. *If $f : \mathbb{R} \to \mathbb{R}$ is everywhere differentiable, non-constant, changes sign finitely many times, and has finitely many zeros, then the number of stationary points of $|f|$ is equal to the number of stationary points of f.*

Proof. If f has a constant sign on $\mathbb{R}$, then $|f(x)| = \pm f(x)$ and the stationary points of $|f|$ are the same as the stationary points of f. Otherwise, suppose the zeros of f occur at the ordered sequence of points $\{x_0, \ldots, x_n\}$. Then we can decompose $\mathbb{R}$ into a union of open intervals and their corresponding endpoints:

$$\mathbb{R} = (-\infty, x_0) \cup \{x_0\} \cup \bigcup_{i=0}^{n-1} [(x_i, x_{i+1}) \cup \{x_{i+1}\}] \cup (x_n, \infty)$$

Then on each interval $I_i := (x_i, x_{i+1})$ for $i \in \mathbb{Z}_n$, we have $|f(x)| = \pm f(x)$ and the stationary points of $|f|\big|_{I_i}$ are the same as the stationary points of $f\big|_{I_i}$. Similarly, the number of stationary points does not change on the intervals $(-\infty, x_0)$ and (x_n, ∞). Next we consider the behavior of f in a neighborhood $B_\varepsilon(x_i) := (x_i - \varepsilon, x_i + \varepsilon)$ where $0 < \varepsilon < \min\{(x_{j+1} - x_j)/2 : j \in \mathbb{Z}_n\}$ of each of the endpoints x_i. If f has a constant sign on $B_\varepsilon(x_i) \setminus \{x_i\}$, then again $|f(x)| = \pm f(x)$ and the number of stationary points does not change. If f changes sign on either side of x_i, then $f'(x_i)$ does not exist (since f is continuous and everywhere differentiable). In particular, x_i is not a stationary point, and the number of stationary points does not change. This completes the proof. $\square$

We can use Proposition 6.1 from Chapter 02 of Millman and Parker to calculate the curvature in terms of the parameter t. Indeed,

$$\kappa(t) = |\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}| \;/\; |\dot{\boldsymbol{\alpha}}|^3$$

where we are treating $\boldsymbol{\alpha} \in \mathbb{R}^3$. Suppose for simplicity that $\boldsymbol{\alpha}(t) = (t, t^2, 0)$. Then:

$$\dot{\boldsymbol{\alpha}} = (1, 2t, 0)$$
$$\ddot{\boldsymbol{\alpha}} = (0, 2, 0).$$

Hence, $|\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}| = (0, 0, 2)$ and $|\dot{\boldsymbol{\alpha}}|^3 = (1 + 4t^2)^{3/2}$. Hence,

$$\kappa(t) = \frac{2}{(1 + 4t^2)^{3/2}}.$$

We can find the locations of the maxima and minima of κ by finding the critical points of κ, i.e., the points for which $\kappa'(s)$ vanishes. We have

$$\dot{\kappa}(t) = -\frac{24t}{(1 + 4t^2)^{5/2}}.$$

We see that $\dot{\kappa}(t) = 0$ if and only if $t = 0$ (and $\dot{\kappa}(t)$ is everywhere defined). And, since

$$\kappa(t) = \kappa(s(t)) \implies \dot{\kappa}(t) = \kappa'(s)\dot{s}(t)$$

where $\dot{s}(t) \equiv |\dot{\boldsymbol{\alpha}}(t)| = \sqrt{1 + 4t^2} \neq 0$ for all t, we see that $\kappa'(s) = 0$ if and only if $\dot{\kappa}(t) = 0$. Hence, $\kappa(s)$ has only one critical point, and it is a stationary point. Since $k(s) = |\kappa(s)|$, we see by the lemma that $\boldsymbol{\alpha}$ has exactly one vertex. Thus, the number of vertices of $\boldsymbol{\alpha}$ is strictly less than four, and as such the condition that $\boldsymbol{\alpha}$ is closed is necessary for the Four-Vertex Theorem.

Chapter 05

Problem 1.1

Prove Corollary 1.2.

Corollary (1.2)**.** *The tangent spherical image of a regular closed curve does not lie in any closed hemisphere unless it lies in the great circle that bounds the hemisphere.*

Solution

This is a trivial consequence of Lemma 1.1:

Lemma. *If $\boldsymbol{\alpha}$ is a regular closed curve, then its tangent spherical image does not lie in any open hemisphere.*

Since the tangent spherical image of $\boldsymbol{\alpha}$ is confined to S^2 and it does not lie in any open hemisphere $H \subset S^2$, then some part of it lies in $S^2 \setminus H$. In particular, it either lies on the boundary of the hemisphere ∂H or not in any hemisphere at all.

Problem 1.2

Let $\boldsymbol{\alpha}(s)$ be a closed space curve. Suppose $0 \leq \kappa \leq 1/R$ for some real number $R > 0$. Prove that the length of $\boldsymbol{\alpha}$ is at least $2\pi R$.

Solution

If $\kappa \leq 1/R$, then the total curvature is bounded. Indeed, if the length of $\boldsymbol{\alpha}$ is L, then

$$\int_0^L \kappa \, \mathrm{d}s \leq \int_0^L \frac{1}{R} \, \mathrm{d}s = \frac{1}{R} \int_0^L \mathrm{d}s = \frac{L}{R}.$$

Since $\boldsymbol{\alpha}$ is a closed space curve, Fenchel's theorem implies that the total curvature is at least 2π. Hence

$$2\pi \leq \int_0^L \kappa \, \mathrm{d}s \leq \frac{L}{R}.$$

Multiplying both sides by R gives the desired result:

$$\boxed{L \geq 2\pi R.}$$
