Homework # 02

Math 140
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Chapter 02

Problem 2.5

Reparameterize the curve $\boldsymbol{\alpha}(t) = (\cosh t, \sinh t, t)$ by arc length.

Solution

Recall that arc length s is defined by

$$s := \int_0^t \left|\boldsymbol{\alpha}'(u)\right| \mathrm{d}u$$

where $\boldsymbol{\alpha}'(u) = \mathrm{d}\boldsymbol{\alpha}/\mathrm{d}u$. In this case:

$$\boldsymbol{\alpha}'(u) = \frac{\mathrm{d}}{\mathrm{d}u}(\cosh u, \sinh u, u) = (\sinh u, \cosh u, 1).$$

Then, the norm reads

$$\left|\boldsymbol{\alpha}'(u)\right| = \sqrt{\sinh^2 u + \cosh^2 u + 1}.$$

Since $\cosh^2 u - \sinh^2 u = 1$, we can write $1 + \sinh^2 u = \cosh^2 u$. Hence,

$$\left|\boldsymbol{\alpha}'(u)\right| = \sqrt{2}|\cosh u| = \sqrt{2}\cosh u$$

since $\cosh u > 0$ for all u. So,

$$s = \int_0^t \left|\boldsymbol{\alpha}'(u)\right| \mathrm{d}u = \sqrt{2}\sinh(t) \implies t = \sinh^{-1}\left(\frac{s}{\sqrt{2}}\right).$$

So by definition, we have

$$\sinh(t) = \sinh\left(\sinh^{-1}\left(\frac{s}{\sqrt{2}}\right)\right) = \frac{s}{\sqrt{2}}.$$

But $\cosh(t)$ is not as simple. The inverse hyperbolic sine function is well-known, so it will be stated here without proof. Indeed,

$$t = \sinh^{-1}\left(\frac{s}{\sqrt{2}}\right) = \ln\left(\frac{s}{\sqrt{2}} + \sqrt{\frac{s^2}{2} + 1}\right).$$

Let $x = s/\sqrt{2}$. Then, we have[1]

$$\begin{aligned}
\cosh(\sinh^{-1}(x)) &= \frac{e^{\sinh^{-1}(x)} + e^{-\sinh^{-1}(x)}}{2} \\
&= \frac{1}{2}\left(\left[\sqrt{x^2+1} + x\right] + \left[\frac{1}{\sqrt{x^2+1} + x}\right]\right) \\
&= \frac{1}{2}\left(\frac{(\sqrt{x^2+1} + x)^2 + 1}{\sqrt{x^2+1} + x}\right) \\
&= \frac{1}{2}\left(\frac{2(x^2+1) + 2x\sqrt{x^2+1}}{\sqrt{x^2+1} + x}\right) \\
&= \frac{x(x + \sqrt{x^2+1}) + 1}{\sqrt{x^2+1} + x} \\
&= x + \frac{1}{\sqrt{x^2+1} + x} \cdot \frac{\sqrt{x^2+1} - x}{\sqrt{x^2+1} - x}
\end{aligned}$$

[1] I have never been sure of how much algebra I should show on a homework assignment like this. Should I show every step, or can I skip the process entirely and just state $\cosh(\sinh^{-1}(x)) = \sqrt{x^2+1}$?

$$\therefore \cosh(\sinh^{-1}(x)) = \sqrt{x^2+1}.$$

Hence, we find

$$\cosh\left(\sinh^{-1}\left(\frac{s}{\sqrt{2}}\right)\right) = \sqrt{\frac{s^2}{2}+1}.$$

Finally, this gives us the arc-length parameterization

$$\boxed{\boldsymbol{\alpha}(s) = \left(\sqrt{\frac{s^2}{2}+1}\ ,\ \frac{s}{\sqrt{2}}\ ,\ \ln\left[\frac{s}{\sqrt{2}} + \sqrt{\frac{s^2}{2}+1}\right]\right).}$$

Problem 3.2

Show that

$$\boldsymbol{\alpha}(s) = \left(\frac{(1+s)^{3/2}}{3}, \frac{(1-s)^{3/2}}{3}, \frac{s}{\sqrt{2}}\right)$$

is a unit speed curve and compute its Frenet-Serret apparatus.

Solution

First we will show that $\boldsymbol{\alpha}(s)$ is a unit-speed. We have

$$\boldsymbol{\alpha}'(s) = \left(\frac{(1+s)^{1/2}}{2}, -\frac{(1-s)^{1/2}}{2}, \frac{1}{\sqrt{2}}\right)$$

so that

$$\left|\boldsymbol{\alpha}'(s)\right| = \sqrt{\left[\frac{(1+s)^{1/2}}{2}\right]^2 + \left[-\frac{(1-s)^{1/2}}{2}\right]^2 + \left[\frac{1}{\sqrt{2}}\right]^2} = \sqrt{\frac{1+s}{4} + \frac{1-s}{4} + \frac{1}{2}} = 1.$$

So, $\boldsymbol{\alpha}(s)$ is indeed a unit-speed curve.

The Frenet-Serret apparatus of $\boldsymbol{\alpha}(s)$ is the ordered set $\{\kappa(s), \tau(s), \mathbf{T}, \mathbf{N}, \mathbf{B}\}$, i.e., the associated curvature, torsion function, tangent vector, normal vector, and bi-normal vector. We already fount the tangent, since $\boldsymbol{\alpha}$ is a unit-speed curve:

$$\mathbf{T} = \boldsymbol{\alpha}'(s) = \left(\frac{(1+s)^{1/2}}{2}, -\frac{(1-s)^{1/2}}{2}, \frac{1}{\sqrt{2}}\right).$$

Next, the curvature is

$$\kappa(s) = \left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right|.$$

So,

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \left(\frac{(1+s)^{-1/2}}{4}, \frac{(1-s)^{-1/2}}{4}, 0\right)$$

$$\therefore \kappa(s) = \sqrt{\left[\frac{(1+s)^{-1/2}}{4}\right]^2 + \left[\frac{(1-s)^{-1/2}}{4}\right]^2} = \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{1-s^2}}.$$

Thus, the normal vector $\mathbf{N} = \mathbf{T}'/\kappa$ is

$$\mathbf{N} = \left(\sqrt{\frac{1-s}{2}}\ ,\ \sqrt{\frac{1+s}{2}}\ ,\ 0\right).$$

This gives a bi-normal vector $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ of
$$\mathbf{B} = \left(-\frac{\sqrt{1+s}}{2} \,,\, \frac{\sqrt{1-s}}{2} \,,\, \frac{1}{\sqrt{2}} \right).$$
Finally, we need the torsion $\tau = -\langle \mathbf{B}', \mathbf{N} \rangle$. Differentiating $\mathbf{B}$ gives
$$\mathbf{B}' = \left(-\frac{1}{4\sqrt{1+s}} \,,\, -\frac{1}{4\sqrt{1-s}} \,,\, 0 \right).$$
Thus, the torsion reads
$$\tau = -\left[-\frac{1}{4\sqrt{1+s}} \cdot \sqrt{\frac{1-s}{2}} - \frac{1}{4\sqrt{1-s}} \cdot \sqrt{\frac{1+s}{2}} \right] = \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{1-s^2}}.$$
Hence, the Frenet-Serret apparatus is the following set of functions:
$$\begin{cases} \kappa(s) &= \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{1-s^2}} \\ \tau(s) &= \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{1-s^2}} \\ \mathbf{T}(s) &= \left(\frac{\sqrt{1+s}}{2}, -\frac{\sqrt{1-s}}{2}, \frac{1}{\sqrt{2}} \right) \\ \mathbf{N}(s) &= \left(\sqrt{\frac{1-s}{2}} \,,\, \sqrt{\frac{1+s}{2}} \,,\, 0 \right) \\ \mathbf{B}(s) &= \left(-\frac{\sqrt{1+s}}{2} \,,\, \frac{\sqrt{1-s}}{2} \,,\, \frac{1}{\sqrt{2}} \right) \end{cases}$$

Problem 4.1

Prove that $\kappa\tau = -\langle \mathbf{T}', \mathbf{B}' \rangle$.

Solution

By definition, we have $\kappa = |\mathbf{T}'|$ and $\tau = -\langle \mathbf{B}', \mathbf{N} \rangle$. So,
$$\kappa\tau = -\left|\mathbf{T}'\right| \langle \mathbf{B}', \mathbf{N} \rangle = -\left\langle \mathbf{B}', \left|\mathbf{T}'\right|^{-1} \mathbf{N} \right\rangle.$$
The second equality follows from the fact that the inner product is linear in each component and that for each s, $\kappa \in \mathbb{R}$. The definition of the normal vector is such that $\mathbf{N} = \frac{1}{\kappa}\mathbf{T}'$, so $\mathbf{T}' = \kappa\mathbf{N} = |\mathbf{T}'|\mathbf{N}$. This is precisely the second component of the inner product. Thus,
$$\kappa\tau = -\langle \mathbf{B}', \mathbf{T}' \rangle = -\langle \mathbf{T}', \mathbf{N}' \rangle.$$
The second equality here follows from the symmetry of the inner product, i.e., that $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$. Thus proves the conjecture in the problem statement.

Problem 4.6

Find the equation of the normal plane to $\boldsymbol{\alpha}(t) = (e^t, \cos t, 3t^2)$ at $t = 1$. (*Note: t is not arc length!*)

Solution

We could begin by reparameterizing the curve by arc length, but since the elements of $\boldsymbol{\alpha}$ are mostly nonlinear, doing so would be impossible in closed form. Instead, we quote a result from the lecture, that if $\boldsymbol{\beta}(t)$ is a regular curve, then
$$\mathbf{T} = \frac{\dot{\boldsymbol{\beta}}}{|\dot{\boldsymbol{\beta}}|}$$

where $[\mathbf{u}, \mathbf{v}, \mathbf{w}] := \langle \mathbf{u}, (\mathbf{v} \times \mathbf{w}) \rangle$ is the triple scalar product. Note that the normal plane to a curve $\boldsymbol{\alpha}$ is the plane whose normal vector is $\mathbf{T}$. Let $\mathbf{v} = (x, y, z)$ be on the normal plane. Then the equation of the normal plane to $\boldsymbol{\alpha}$ at $t = 1$ is given by:

$$0 = \langle \mathbf{v} - \boldsymbol{\alpha}(1), \mathbf{T}(1) \rangle .$$

First, we have

$$\boldsymbol{\alpha}(1) = (e^1, \cos 1, 3 \cdot 1^2) = (e, \cos 1, 3).$$

Next, let's calculate the derivatives of $\boldsymbol{\alpha}$:

$$\dot{\boldsymbol{\alpha}} = (e^t, -\sin t, 6t)$$
$$\therefore |\dot{\boldsymbol{\alpha}}| = \sqrt{e^2 + \sin^2 t + 36t^2}$$

Hence at $t = 1$, the tangent vector is

$$\mathbf{T}(1) = \left(\frac{e}{e^2 + \sin^2(1) + 36}, -\frac{\sin(1)}{e^2 + \sin^2(1) + 36}, \frac{6}{e^2 + \sin^2(1) + 36} \right).$$

Hence, the equation of the normal plane is

$$0 = \left\langle (x - e, y - \cos(1), z - 3), \left(\frac{e}{\sqrt{e^2 + \sin^2(1) + 36}}, -\frac{\sin(1)}{\sqrt{e^2 + \sin^2(1) + 36}}, \frac{6}{\sqrt{e^2 + \sin^2(1) + 36}} \right) \right\rangle$$
$$= \frac{e(x - e) - \sin(1)(y - \cos(1)) + 6(z - 3)}{\sqrt{e^2 + \sin^2(1) + 36}}$$

Clearing the fractions and writing the equation in standard form, we get

$$\boxed{ex - \sin(1)y + 6z = e^2 - \tfrac{1}{2}\sin(2) + 18}$$

Problem 4.24

Let $\boldsymbol{\alpha}(s)$ be a unit speed curve with $\kappa \neq 0, \tau \neq 0$ and $\rho = 1/\kappa, \sigma = 1/\tau$. Assume $\rho^2 + (\rho'\sigma)^2 =$ constant $= a^2$ where $a > 0$. Prove that the image of $\boldsymbol{\alpha}$ lies on a sphere of radius a. (*Hint:* Show $\boldsymbol{\alpha} + \rho\mathbf{N} + \rho'\sigma\mathbf{B}$ is constant. Call this $\mathbf{m}$. It should be the center of the sphere. This is motivated by Proposition 4.10.)

Solution

Let $\mathbf{m} := \boldsymbol{\alpha} + \rho\mathbf{N} + \rho'\sigma\mathbf{B}$. I claim that $\mathbf{m}$ is constant. We have

$$\frac{\mathrm{d}\mathbf{m}}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\left\{ \boldsymbol{\alpha} + \rho\mathbf{N} + \rho'\sigma\mathbf{B} \right\} = \mathbf{T} + \rho'\mathbf{N} + \rho\mathbf{N}' + (\rho'\sigma)'\mathbf{B} + (\rho'\sigma)\mathbf{B}'. \tag{1}$$

Recall the Frenet-Serret theorem:[2]

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}$$

Replacing the derivatives of $\mathbf{N}$ and $\mathbf{B}$ in Eq. (1) with their Frenet-Serret decomposition gives:

$$\frac{\mathrm{d}\mathbf{m}}{\mathrm{d}s} = \underbrace{(1 - \rho\kappa)}_{m_T'}\mathbf{T} + \underbrace{(\rho' - \rho'\sigma\tau)}_{m_N'}\mathbf{N} + \underbrace{(\rho\tau + [\rho'\sigma]')}_{m_B'}\mathbf{B}.$$

[2]Theorem 4.2 of Chapter 2 of Millman and Parker (page 30).

We have $m_T' = 1-\rho\kappa = 1-(\frac{1}{\kappa})\kappa = 1-1 = 0$. Similarly, $m_N' = \rho'[1-\sigma\tau] = \rho'[1-(\frac{1}{\tau})\tau] = \rho'[1-1] = 0$. Note that since $\rho^2 + (\rho'\sigma)^2 = a^2 \in \mathbb{R}$, differentiating gives

$$2\rho\rho' + 2(\rho'\sigma)(\rho'\sigma)' = 0 \implies (\rho'\sigma)' = -\frac{\rho}{\sigma} = -\rho\tau.$$

Hence, $m_B' = \rho\tau + [\rho'\sigma]' = \rho\tau - \rho\tau = 0$. Hence, $\mathrm{d}\mathbf{m}/\mathrm{d}s = 0$ which implies $\mathbf{m}$ is constant. Now, I claim that the image of $\boldsymbol{\alpha}$ lies on a sphere of radius a and center $\mathbf{m}$. To prove this, write

$$\begin{aligned}\langle \boldsymbol{\alpha} - \mathbf{m}, \boldsymbol{\alpha} - \mathbf{m}\rangle &= \langle \boldsymbol{\alpha} - (\boldsymbol{\alpha} + \rho\mathbf{N} + \rho'\sigma\mathbf{B})\ ,\ \boldsymbol{\alpha} - (\boldsymbol{\alpha} + \rho\mathbf{N} + \rho'\sigma\mathbf{B})\rangle \\ &= (-1)^2 \langle \rho\mathbf{N} + \rho'\sigma\mathbf{B}\ ,\ \rho\mathbf{N} + \rho'\sigma\mathbf{B}\rangle \\ &= \left[\rho^2 \langle \mathbf{N}, \mathbf{N}\rangle + 2\rho\rho'\sigma \langle \mathbf{N}, \mathbf{B}\rangle + (\rho'\sigma)^2 \langle \mathbf{B}, \mathbf{B}\rangle\right].\end{aligned}$$

Since $\beta := \{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is an orthonormal basis for $\mathbb{R}^3$, we have $\langle \mathbf{e}_i, \mathbf{e}_j\rangle = \delta_{ij}$ for all $\mathbf{e}_i \in \beta$. As such,

$$\langle \boldsymbol{\alpha} - \mathbf{m}, \boldsymbol{\alpha} - \mathbf{m}\rangle = \left[\rho^2 \langle \mathbf{N}, \mathbf{N}\rangle + 2\rho\rho'\sigma \langle \mathbf{N}, \mathbf{B}\rangle + (\rho'\sigma)^2 \langle \mathbf{B}, \mathbf{B}\rangle\right] = \rho^2 + (\rho'\sigma)^2 \equiv a^2,$$

i.e., $\langle \boldsymbol{\alpha} - \mathbf{m}, \boldsymbol{\alpha} - \mathbf{m}\rangle = a^2 \iff |\boldsymbol{\alpha} - \mathbf{m}| = a$, so the image of $\boldsymbol{\alpha}$ lies on a sphere of radius a and center $\mathbf{m}$, which proves the conjecture.
