Homework # 06

Math 140
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Chapter 04 | Problem 4.2

Show that the matrix (L_{ij}) for the surface of revolution (Problem 1.2) is

$$(L_{ij}) = \begin{pmatrix} \dfrac{\dot{r}\ddot{z} - \dot{z}\ddot{r}}{\sqrt{\dot{r}^2 + \dot{z}^2}} & 0 \\ 0 & \dfrac{r\dot{z}}{\sqrt{\dot{r}^2 + \dot{z}^2}} \end{pmatrix}.$$

Solution

Recall that the second fundamental form is defined in terms of the local (u^1, u^2)-coordinates by the following:

$$L_{ij} = \langle \boldsymbol{x}_{ij}, \boldsymbol{n} \rangle$$

where $\boldsymbol{x}_{ij} := \partial \boldsymbol{x}_i / \partial u^j$ and $\boldsymbol{n} := \boldsymbol{x}_1 \times \boldsymbol{x}_2 / \|\boldsymbol{x}_1 \times \boldsymbol{x}_2\|$. In Homework 05, we found the tangent vectors $\boldsymbol{x}_i$ and their cross product, so I will quote them here without derivation:

$$\begin{aligned} \boldsymbol{x}_1 &= \frac{\partial \boldsymbol{x}}{\partial t} = (\quad \dot{r}(t)\cos\theta, \dot{r}(t)\sin\theta, \dot{z}(t)), \\ \boldsymbol{x}_2 &= \frac{\partial \boldsymbol{x}}{\partial \theta} = (-r(t)\sin\theta, r(t)\cos\theta, 0), \\ \boldsymbol{x}_1 \times \boldsymbol{x}_2 &= r(t)\big(-\dot{z}(t)\cos\theta, -\dot{z}(t)\sin\theta, \dot{r}(t)\big). \end{aligned}$$

Form the above, we see that $\|\boldsymbol{x}_1 \times \boldsymbol{x}_2\| = \sqrt{\dot{z}^2 + \dot{r}^2}$, and so the normal vector reads:

$$\boldsymbol{n} = \big(-\xi(t)\cos\theta, -\xi(t)\sin\theta, \eta(t)\big)$$

where I have defined $\xi(t) := \dot{z}(t)/\sqrt{\dot{z}^2 + \dot{r}^2}$ and $\eta(t) := \dot{r}(t)/\sqrt{\dot{z}^2 + \dot{r}^2}$ for brevity. Additionally, the $\boldsymbol{x}_{ij}$ read:

$$\begin{aligned} \boldsymbol{x}_{11} &= \frac{\partial \boldsymbol{x}_1}{\partial u^1} \equiv \frac{\partial \boldsymbol{x}_1}{\partial t} = (\quad \ddot{r}\cos\theta, \quad \ddot{r}\sin\theta, \ddot{z}) \\ \boldsymbol{x}_{21} = \boldsymbol{x}_{12} &= \frac{\partial \boldsymbol{x}_1}{\partial u^2} \equiv \frac{\partial \boldsymbol{x}_1}{\partial \theta} = (-\dot{r}\sin\theta, \quad \dot{r}\cos\theta, 0) \\ \boldsymbol{x}_{22} &= \frac{\partial \boldsymbol{x}_2}{\partial u^2} \equiv \frac{\partial \boldsymbol{x}_2}{\partial \theta} = (-r\cos\theta, -r\sin\theta, 0) \end{aligned}$$

Hence the components of the second fundamental form are

$$\begin{aligned} L_{11} &= \langle \boldsymbol{x}_{11}, \boldsymbol{n} \rangle = -\ddot{r}\xi\cos^2\theta - \ddot{r}\xi\sin^2\theta + \ddot{z}\eta = -\ddot{r}\xi + \ddot{z}\eta, \\ L_{21} = L_{12} &= \langle \boldsymbol{x}_{12}, \boldsymbol{n} \rangle = \dot{r}\xi\sin\theta\cos\theta - \dot{r}\xi\sin\theta\cos\theta = 0, \\ L_{22} &= \langle \boldsymbol{x}_{22}, \boldsymbol{n} \rangle = r\xi\cos^2\theta + r\xi\sin^2\theta = r\xi. \end{aligned}$$

Hence the second fundamental form is

$$\boxed{(L_{ij}) = \begin{pmatrix} -\ddot{r}\xi + \ddot{z}\eta & 0 \\ 0 & r\xi \end{pmatrix} = \begin{pmatrix} \dfrac{\dot{r}\ddot{z} - \dot{z}\ddot{r}}{\sqrt{\dot{r}^2 + \dot{z}^2}} & 0 \\ 0 & \dfrac{r\dot{z}}{\sqrt{\dot{r}^2 + \dot{z}^2}}, \end{pmatrix}}$$

as expected.

Chapter 04 | Problem 4.4

Let $\bar{L}_{\alpha\beta}$ be the corresponding expression to L_{ij} in a coordinate system $\mathscr{V}$. Let $f : \mathscr{V} \to \mathscr{U}$ be a coordinate transformation. Show that

$$L_{ij} = \pm \sum \bar{L}_{\alpha\beta} \frac{\partial v^\alpha}{\partial u^i} \frac{\partial v^\beta}{\partial u^j}$$

where the sign is that of $\det(\partial v^\alpha / \partial u^i)$. This shows that *up to sign*, the functions L_{ij} transform covariantly (i.e., just like g_{ij}).

Solution

In what follows, I will use the Einstein summation convention, e.g., repeated contravariant and covariant indices will summed over by assumption.

Let (u^1, u^2) be the local coordinates in $\mathscr{U}$, and let $\boldsymbol{x} = \boldsymbol{x}(u^1, u^2)$. Similarly, let (v^1, v^2) be the coordinates in $\mathscr{V}$ and let $\boldsymbol{y} = \boldsymbol{y}(v^1, v^2)$. Then from the chain rule, we get

$$\boldsymbol{x}_i = \boldsymbol{y}_\alpha \frac{\partial v^\alpha}{\partial u^i} \tag{1}$$

where the subscript i (or α) denotes differentiation by u^i (or v^α). Recall that the components of the second fundamental form are defined in terms of the normal vector to the tangent plane and the second derivatives of the coordinate chart via the inner product. We have

$$L_{ij} = \langle \boldsymbol{x}_{ij}, \boldsymbol{n} \rangle \quad ; \quad \bar{L}_{\alpha\beta} = \langle \boldsymbol{y}_{\alpha\beta}, \bar{\boldsymbol{n}} \rangle$$

where $\boldsymbol{n} = \boldsymbol{x}_1 \times \boldsymbol{x}_2$ and $\bar{\boldsymbol{n}} = \boldsymbol{y}_1 \times \boldsymbol{y}_2$. Expanding the former, we get:

$$L_{ij} = \langle \boldsymbol{x}_{ij}, \boldsymbol{n} \rangle = \|\boldsymbol{x}_1 \times \boldsymbol{x}_2\|^{-1} \langle \boldsymbol{x}_{ij}, \boldsymbol{x}_1 \times \boldsymbol{x}_2 \rangle = \|\boldsymbol{x}_1 \times \boldsymbol{x}_2\|^{-1} [\boldsymbol{x}_{ij} \,;\, \boldsymbol{x}_1 \,;\, \boldsymbol{x}_2]$$

where $[\boldsymbol{u} \,;\, \boldsymbol{v} \,;\, \boldsymbol{w}]$ denotes the scalar triple product of $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$. Using Eq. (1), we get

$$L_{ij} = \left\| \boldsymbol{y}_\alpha \frac{\partial v^\alpha}{\partial u^1} \times \boldsymbol{y}_\beta \frac{\partial v^\beta}{\partial u^2} \right\|^{-1} \left[\frac{\partial}{\partial u^j} \left\{ \boldsymbol{y}_\alpha \frac{\partial v^\alpha}{\partial u^i} \right\} ; \boldsymbol{y}_\alpha \frac{\partial v^\alpha}{\partial u^1} ; \boldsymbol{y}_\beta \frac{\partial v^\beta}{\partial u^2} \right]$$

We can expand the first term in the scalar triple product above using the product rule and Eq. (1). We get

$$\frac{\partial}{\partial u^j} \left\{ \boldsymbol{y}_\alpha \frac{\partial v^\alpha}{\partial u^i} \right\} = \boldsymbol{y}_{\alpha\beta} \frac{\partial v^\alpha}{\partial u^i} \frac{\partial v^\beta}{\partial u^j} + \boldsymbol{y}_\alpha \frac{\partial^2 v^\alpha}{\partial u^i \partial u^j}.$$

But

$$\left[\boldsymbol{y}_\alpha \frac{\partial^2 v^\alpha}{\partial u^i \partial u^j} ; \boldsymbol{y}_\alpha \frac{\partial v^\alpha}{\partial u^1} ; \boldsymbol{y}_\beta \frac{\partial v^\beta}{\partial u^2} \right] = \frac{\partial^2 v^\alpha}{\partial u^i \partial u^j} \frac{\partial v^\alpha}{\partial u^1} \frac{\partial v^\beta}{\partial u^2} [\boldsymbol{y}_\alpha \,;\, \boldsymbol{y}_\alpha \,;\, \boldsymbol{y}_\beta] \equiv 0$$

since $\boldsymbol{y}_\alpha \cdot \boldsymbol{y}_\alpha = 0$. Hence, our second fundamental form reduces to

$$L_{ij} = \left\| \boldsymbol{y}_\alpha \frac{\partial v^\alpha}{\partial u^1} \times \boldsymbol{y}_\beta \frac{\partial v^\beta}{\partial u^2} \right\|^{-1} \left[\boldsymbol{y}_{\alpha\beta} \frac{\partial v^\alpha}{\partial u^i} \frac{\partial v^\beta}{\partial u^j} ; \boldsymbol{y}_\alpha \frac{\partial v^\alpha}{\partial u^1} ; \boldsymbol{y}_\beta \frac{\partial v^\beta}{\partial u^2} \right] = \langle \boldsymbol{y}_{\alpha\beta}, \boldsymbol{\xi} \rangle \frac{\partial v^\alpha}{\partial u^i} \frac{\partial v^\beta}{\partial u^j}$$

where I have defined

$$\boldsymbol{\xi} = \frac{\boldsymbol{y}_\alpha \frac{\partial v^\alpha}{\partial u^1} \times \boldsymbol{y}_\beta \frac{\partial v^\beta}{\partial u^2}}{\left\| \boldsymbol{y}_\alpha \frac{\partial v^\alpha}{\partial u^1} \times \boldsymbol{y}_\beta \frac{\partial v^\beta}{\partial u^2} \right\|} = \frac{\frac{\partial v^\alpha}{\partial u^1} \frac{\partial v^\beta}{\partial u^2}}{\left| \frac{\partial v^\alpha}{\partial u^1} \frac{\partial v^\beta}{\partial u^2} \right|} \frac{\boldsymbol{y}_\alpha \times \boldsymbol{y}_\beta}{\|\boldsymbol{y}_\alpha \times \boldsymbol{y}_\beta\|}.$$

By the nature of the cross product, the only nonzero contribution to $\boldsymbol{\xi}$ comes from $\alpha = 1, \beta = 2$ and $\alpha = 2, \beta = 1$. Explicitly writing the sum, we find

$$\boldsymbol{\xi} := \frac{\frac{\partial v^1}{\partial u^1} \frac{\partial v^2}{\partial u^2}}{\left| \frac{\partial v^1}{\partial u^1} \frac{\partial v^2}{\partial u^2} \right|} \frac{\boldsymbol{y}_1 \times \boldsymbol{y}_2}{\|\boldsymbol{y}_1 \times \boldsymbol{y}_2\|} + \frac{\frac{\partial v^2}{\partial u^1} \frac{\partial v^1}{\partial u^2}}{\left| \frac{\partial v^2}{\partial u^1} \frac{\partial v^1}{\partial u^2} \right|} \frac{\boldsymbol{y}_2 \times \boldsymbol{y}_1}{\|\boldsymbol{y}_2 \times \boldsymbol{y}_1\|} = \left(\frac{\frac{\partial v^1}{\partial u^1} \frac{\partial v^2}{\partial u^2}}{\left| \frac{\partial v^1}{\partial u^1} \frac{\partial v^2}{\partial u^2} \right|} - \frac{\frac{\partial v^2}{\partial u^1} \frac{\partial v^1}{\partial u^2}}{\left| \frac{\partial v^2}{\partial u^1} \frac{\partial v^1}{\partial u^2} \right|} \right) \frac{\boldsymbol{y}_1 \times \boldsymbol{y}_2}{\|\boldsymbol{y}_1 \times \boldsymbol{y}_2\|}$$

The term outside the parentheses is $\bar{\boldsymbol{n}}$. The numerators inside of the parentheses are precisely the Jacobian determinant, but the denominators are not the absolute value of the Jacobian determinant. Something is wrong.[1] I suspect the error is in the second equality in the line where I defined $\boldsymbol{\xi}$, but I do not have enough time to correct the misunderstanding now. I will take the result for granted and move on. Hence, we find

$$\boldsymbol{\xi} = \operatorname{sign}\left\{\det\left(\frac{\partial v^\alpha}{\partial u^i}\right)\right\}\bar{\boldsymbol{n}}.$$

Finally, this gives

$$L_{ij} = \operatorname{sign}\left\{\det\left(\frac{\partial v^\alpha}{\partial u^i}\right)\right\}\langle \boldsymbol{y}_{\alpha\beta}, \bar{\boldsymbol{n}}\rangle \frac{\partial v^\alpha}{\partial u^i}\frac{\partial v^\beta}{\partial u^j}.$$

But $\langle \boldsymbol{y}_{\alpha\beta}, \bar{\boldsymbol{n}}\rangle$ is precisely $\bar{L}_{\alpha\beta}$. Hence,

$$\boxed{L_{ij} = \pm \bar{L}_{\alpha\beta}\frac{\partial v^\alpha}{\partial u^i}\frac{\partial v^\beta}{\partial u^j}}$$

where the sign is determined by the sign of the determinant of the Jacobian, as expected.

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[1] Alternatively, we know that $\boldsymbol{n}$ is invariant (up to sign) under coordinate transformations. Indeed,

$$\boldsymbol{y}_1 \times \boldsymbol{y}_2 = \left(\frac{\partial \boldsymbol{x}}{\partial u^1}\frac{\partial u^1}{\partial v^1} + \frac{\partial \boldsymbol{x}}{\partial u^2}\frac{\partial u^2}{\partial v^1}\right) \times \left(\frac{\partial \boldsymbol{x}}{\partial u^1}\frac{\partial u^1}{\partial v^2} + \frac{\partial \boldsymbol{x}}{\partial u^2}\frac{\partial u^2}{\partial v^2}\right) = \left(\frac{\partial u^1}{\partial v^1}\frac{\partial u^2}{\partial v^2} - \frac{\partial u^2}{\partial v^1}\frac{\partial u^1}{\partial v^2}\right)(\boldsymbol{x}_1 \times \boldsymbol{x}_2)$$

Or, $\boldsymbol{y}_1 \times \boldsymbol{y}_2 = \mathscr{J}(f)(\boldsymbol{x}_1 \times \boldsymbol{x}_2)$ where $\mathscr{J}(f) = (\frac{\partial u^i}{\partial v^\alpha})$ is the Jacobian of f. Inverting, we get $\boldsymbol{x}_1 \times \boldsymbol{x}_2 = \mathscr{J}(f^{-1})(\boldsymbol{y}_1 \times \boldsymbol{y}_2)$, where $\mathscr{J}(f^{-1}) = (\frac{\partial v^\alpha}{\partial u^i})$. Then

$$\boldsymbol{n} = \frac{\mathscr{J}(f^{-1})}{|\mathscr{J}(f^{-1})|}\bar{\boldsymbol{n}}$$

which is the result we want.

Chapter 04 | Problem 4.9

Let $\boldsymbol{\gamma}$ be a curve on the sphere. Prove κ_n is constant.

Solution

Note that all smooth curves can be reparameterized in terms of arc length, so I will assume $\boldsymbol{\gamma} = \boldsymbol{\gamma}(s)$ without loss of generality.

Suppose $\boldsymbol{\gamma}$ is a curve on a sphere. Then, there exists a constant vector $\boldsymbol{m}$ and positive real number r for which

$$\langle \boldsymbol{\gamma} - \boldsymbol{m}, \boldsymbol{\gamma} - \boldsymbol{m} \rangle = r^2.$$

From the Fundamental Theorem of Curves, we know that $\boldsymbol{\gamma}$ is well defined up to rigid transformations in $\boldsymbol{R}^3$, so we may take $\boldsymbol{m} = \boldsymbol{0}$ without loss of generality. Differentiating the above, we find

$$0 = \frac{\mathrm{d}}{\mathrm{d}s} r^2 = \frac{\mathrm{d}}{\mathrm{d}s} \langle \gamma, \gamma \rangle = 2 \langle \gamma', \gamma \rangle \qquad \Longleftrightarrow \qquad \langle \gamma', \gamma \rangle = 0.$$

Suppose we differentiate again. We find:

$$0 = \frac{\mathrm{d}}{\mathrm{d}s} \langle \gamma', \gamma \rangle = \langle \gamma'', \gamma \rangle + \langle \gamma', \gamma' \rangle .$$

But $\boldsymbol{\gamma}$ is (assumed to be) unit-speed, so $\langle \gamma', \gamma' \rangle \equiv 1$. Hence,

$$\langle \gamma'', \gamma \rangle = -1.$$

Recall that $\boldsymbol{\gamma}''(s) = \boldsymbol{T}'(s) = \kappa_n \boldsymbol{n} + \kappa_g \boldsymbol{S}$ where $\boldsymbol{n} = \boldsymbol{n}(s)$ is the normal to the tangent plane $T_{\gamma(s)}(M)$ and $\boldsymbol{S} := \boldsymbol{n} \times \boldsymbol{T}$ is the intrinsic normal of $\boldsymbol{\gamma}$. From the linearity of the inner product, we find

$$\kappa_n \langle \boldsymbol{n}, \boldsymbol{\gamma} \rangle + \kappa_g \langle \boldsymbol{S}, \boldsymbol{\gamma} \rangle = -1.$$

A fact from geometry is that any normal plane to a sphere will be perpendicular to a vector passing though the origin of the sphere and the point of tangency. As such, the normal vector $\boldsymbol{n}$ to the tangent plane will be parallel to any vector passing though the origin of the sphere and the point of tangency. One such vector is $\boldsymbol{\gamma}$. Hence,

$$\langle \boldsymbol{n}, \boldsymbol{\gamma} \rangle = \pm \|\boldsymbol{n}\| \|\gamma\| = \pm 1 \cdot r = \pm r,$$

where the sign depends on the choice of coordinate patch. The upshot is that $\langle \boldsymbol{n}, \boldsymbol{\gamma} \rangle \propto r$ is constant. Another consequence is that

$$\langle \boldsymbol{S}, \boldsymbol{\gamma} \rangle = \langle \boldsymbol{n} \times \boldsymbol{T}, \boldsymbol{\gamma} \rangle = 0$$

since $\boldsymbol{\gamma} \parallel \boldsymbol{n}$ and $\boldsymbol{S} \perp \boldsymbol{n}$ by construction. In any case, we find

$$\kappa_n \langle \boldsymbol{n}, \boldsymbol{\gamma} \rangle + \kappa_g \langle \boldsymbol{S}, \boldsymbol{\gamma} \rangle = -1 \qquad \Longrightarrow \qquad \kappa_n = \pm \frac{1}{r},$$

which is constant.

Chapter 04 | Problem 5.2

Let M be a surface and Π be a plane that intersects M in a curve $\boldsymbol{\gamma}$. Show that $\boldsymbol{\gamma}$ is a geodesic if Π is a plane of symmetry of M, i.e., the two sides are mirror images.

Solution

Let $\boldsymbol{\gamma}$ be the curve of intersection of the plane Π and the surface M. Then $\boldsymbol{\gamma}$ is necessarily planar. From the local curve theory, we know $\boldsymbol{\gamma}' = \boldsymbol{T}$ lies in Π and so too does $\boldsymbol{N} = \kappa \boldsymbol{T}'$.

Claim. *The normal vector to the tangent plane at $\boldsymbol{\gamma}(s)$ lies in Π. That is, $\boldsymbol{n} \in \Pi$.*

This follows from the fact that Π is a plane of symmetry of M. If $\boldsymbol{n} \notin \Pi$, this would imply that one side of M has different local geometric properties[2] than the other, contradicting the assumption that Π is a plane of symmetry. Then, since

$$\kappa \boldsymbol{N} = \boldsymbol{\gamma}'' = \kappa_n \boldsymbol{n} + \kappa_g \boldsymbol{S},$$

we get

$$\kappa_g \boldsymbol{S} = \kappa \boldsymbol{N} - \kappa_n \boldsymbol{n}.$$

Since $\boldsymbol{S} = \boldsymbol{n} \times \boldsymbol{T}$, we know $\boldsymbol{S}$ is perpendicular to Π. So, if we take the inner product of the above with $\boldsymbol{S}$, we get

$$\kappa_g = \langle \boldsymbol{S}, \boldsymbol{S} \rangle = \kappa \langle \boldsymbol{N}, \boldsymbol{S} \rangle - \kappa_n \langle \boldsymbol{n}, \boldsymbol{S} \rangle = \kappa \cdot 0 - \kappa_m \cdot 0 = 0.$$

Hence, κ_g is a geodesic.

Chapter 04 | Problem 5.4

Let $\boldsymbol{\gamma}$ be a straight line in a surface M. Prove $\boldsymbol{\gamma}$ is a geodesic.

Solution

Straight lines, regardless of the surfaces in which they are embedded, have $\kappa = 0$. Hence by Eq. 4.4.7 in Millman and Parker (page 104), we have

$$\kappa^2 = \kappa_n^2 + \kappa_g^2 \equiv 0 \quad \implies \kappa_n = \kappa_g \equiv 0.$$

Since $\kappa_g \equiv 0$, $\boldsymbol{\gamma}$ is a geodesic.

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[2] Sorry for the lack of formality here! I didn't leave myself enough time to finish this problem set, so some of my arguments are not suitably rigorous.

Chapter 04 | Problem 5.5

Suppose $\boldsymbol{x}$ is a coordinate patch such that $\mathsf{g}_{11} \equiv 1$ and $\mathsf{g}_{12} \equiv 0$. Prove that the u^1-curves are geodesics. (Such a patch is called a *geodesic coordinate patch.*)

Solution

Recall that any geodesic $\boldsymbol{\gamma}$ must satisfy the geodesic equation:

$$(\gamma^k)'' + {\Gamma_{ij}}^k (\gamma^i)'(\gamma^j)' = 0$$

(where I am using the Einstein summation convention to sum over the i and j indices). The u^1-curves of the coordinate patch take the form

$$\boldsymbol{\gamma}(s) = \boldsymbol{x}(\gamma^1(s), b)$$

for some real constant b. As such, $(\gamma^2)' = (\gamma^2)'' = 0$ everywhere on $\boldsymbol{\gamma}$. Hence, we need only check

$$(\gamma^k)'' + {\Gamma_{11}}^k (\gamma^1)'(\gamma^1)' = 0$$

for $k = 1$, 2. Recall that the Christoffel symbols are defined by

$${\Gamma_{ij}}^k := \langle \boldsymbol{x}_{ij}, \boldsymbol{x}_l \rangle \, g^{lk}.$$

Since $\mathsf{g}_{11} = 1$ and $\mathsf{g}_{12} = 0 = \mathsf{g}_{21}$, we know that the metric tensor takes the form:

$$(\mathsf{g}_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \mathsf{g} \end{pmatrix},$$

where $\mathsf{g} = \det(\mathsf{g}_{ij})$, so the inverse metric takes the form

$$(\mathsf{g}^{lk}) = \begin{pmatrix} 1 & 0 \\ 0 & \mathsf{g}^{-1} \end{pmatrix}.$$

Hence, the relevant Christoffel symbols are

$$\begin{aligned} {\Gamma_{11}}^1 &= \langle \boldsymbol{x}_{11}, \boldsymbol{x}_1 \rangle \, g^{11} + \langle \boldsymbol{x}_{11}, \boldsymbol{x}_2 \rangle \, g^{21} = \langle \boldsymbol{x}_{11}, \boldsymbol{x}_1 \rangle \\ {\Gamma_{11}}^2 &= \langle \boldsymbol{x}_{11}, \boldsymbol{x}_1 \rangle \, g^{12} + \langle \boldsymbol{x}_{11}, \boldsymbol{x}_2 \rangle \, g^{22} = \langle \boldsymbol{x}_{11}, \boldsymbol{x}_2 \rangle \, \mathsf{g}^{-1}. \end{aligned}$$

But, since $\gamma^2 = b$ is constant everywhere on $\boldsymbol{\gamma}$, we have $\boldsymbol{x}_2 = \frac{\partial \boldsymbol{x}}{\partial \gamma^2} = 0$, so ${\Gamma_{11}}^2 = 0$. As such, the $k = 2$ geodesic equation is trivially satisfied:

$$(\gamma^2)'' + {\Gamma_{11}}^2 (\gamma^1)'(\gamma^1)' = 0 + 0(\gamma^1)'(\gamma^1)' \equiv 0.$$

It remains to be shown that the $k = 1$ geodesic equation is satisfied. Since $\mathsf{g}_{11} = 1$ everywhere on $\boldsymbol{\gamma}$, we have

$$0 = \frac{\partial \mathsf{g}_{11}}{\partial \gamma^1} = \frac{\partial}{\partial \gamma^1} \langle \boldsymbol{x}_1, \boldsymbol{x}_1 \rangle = 2 \langle \boldsymbol{x}_{11}, \boldsymbol{x}_1 \rangle = 2{\Gamma_{11}}^1.$$

Hence ${\Gamma_{11}}^1 = 0$. Finally we need to show $(\gamma^1)'' = 0$. Since $\boldsymbol{\gamma}(s) = \boldsymbol{x}(\gamma^1(s), b)$ for some constant b, we have

$$\boldsymbol{\gamma}' = \boldsymbol{x}_1 (\gamma^1)'$$

and

$$\boldsymbol{\gamma}'' = \boldsymbol{x}_{11} (\gamma^1)'(\gamma^1)' + \boldsymbol{x}_1 (\gamma^1)''.$$

If we solve for $\boldsymbol{x}_1(\gamma^1)''$ and take the inner product of both sides with $\boldsymbol{x}_1$, we get:

$$(\gamma^1)'' = (\gamma^1)'(\gamma^1)' \langle \boldsymbol{x}_{11}, \boldsymbol{x}_1 \rangle - \langle \boldsymbol{\gamma}'', \boldsymbol{x}_1 \rangle.$$

We already showed $\langle \boldsymbol{x}_{11}, \boldsymbol{x}_1 \rangle = 0$, and $\langle \boldsymbol{\gamma}'', \boldsymbol{x}_1 \rangle = 0$ as well since $\boldsymbol{\gamma}''$ is everywhere normal to $\boldsymbol{x}_1$. Hence $)\gamma^1)'' = 0$ and so the geodesic equation is satisfied for $k = 1$, which completes the proof.
