How do the Christoffel Symbols Transform?

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In what follows, I will use the Einstein summation convention, e.g., repeated contravariant and covariant indices will summed over by assumption.

Lemma 1. *Let $\mathscr{U}, \mathscr{V}$ be coordinate systems in $\mathbb{R}^2$, and let $\boldsymbol{x}(x^1, x^2)$, $\bar{\boldsymbol{x}}(\bar{x}^1, \bar{x}^2)$ be coordinate charts for $\mathscr{U}, \mathscr{V}$ respectively. Let L_{ij} denote the coefficients of the second fundamental form in $\mathscr{U}$ and $\bar{L}_{\mu\nu}$ be the coefficients in $\mathscr{V}$. If there exists a coordinate transformation from $\mathscr{V}$ to $\mathscr{U}$, then*

$$L_{ij} = \pm \bar{L}_{\mu\nu} \frac{\partial \bar{x}^\mu}{\partial x^i} \frac{\partial \bar{x}^\nu}{\partial x^j}$$

where the sign is that of $\det\left(\frac{\partial \bar{x}^\mu}{\partial x^\nu}\right)$.

Proof. Let (x^1, x^2) be the local coordinates in $\mathscr{U}$, and let $\boldsymbol{x} = \boldsymbol{x}(x^1, x^2)$. Similarly, let $(\bar{x}^1, \bar{x}^2)$ be the coordinates in $\mathscr{V}$ and let $\bar{\boldsymbol{x}} = \bar{\boldsymbol{x}}(\bar{x}^1, \bar{x}^2)$. Then from the chain rule, we get

$$\boldsymbol{x}_i = \bar{\boldsymbol{x}}_\mu \frac{\partial \bar{x}^\mu}{\partial x^i} \tag{1}$$

where the subscript i (or μ) denotes differentiation by x^i (or $\bar{x}^\mu$). Recall that the components of the second fundamental form are defined in terms of the normal vector to the tangent plane and the second derivatives of the coordinate chart via the inner product. We have

$$L_{ij} = \langle \boldsymbol{x}_{ij}, \boldsymbol{n} \rangle \quad ; \quad \bar{L}_{\mu\nu} = \langle \bar{\boldsymbol{x}}_{\mu\nu}, \bar{\boldsymbol{n}} \rangle$$

where $\boldsymbol{n} = \boldsymbol{x}_1 \times \boldsymbol{x}_2$ and $\bar{\boldsymbol{n}} = \bar{\boldsymbol{x}}_1 \times \bar{\boldsymbol{x}}_2$. Expanding the former, we get:

$$L_{ij} = \langle \boldsymbol{x}_{ij}, \boldsymbol{n} \rangle = \|\boldsymbol{x}_1 \times \boldsymbol{x}_2\|^{-1} \langle \boldsymbol{x}_{ij}, \boldsymbol{x}_1 \times \boldsymbol{x}_2 \rangle = \|\boldsymbol{x}_1 \times \boldsymbol{x}_2\|^{-1} [\boldsymbol{x}_{ij} \,;\, \boldsymbol{x}_1 \,;\, \boldsymbol{x}_2]$$

where $[\boldsymbol{u} \,;\, \boldsymbol{v} \,;\, \boldsymbol{w}]$ denotes the scalar triple product of $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$. Using Eq. (1), we get

$$L_{ij} = \left\| \bar{\boldsymbol{x}}_\mu \frac{\partial \bar{x}^\mu}{\partial x^1} \times \bar{\boldsymbol{x}}_\nu \frac{\partial \bar{x}^\nu}{\partial x^2} \right\|^{-1} \left[\frac{\partial}{\partial x^j} \left\{ \bar{\boldsymbol{x}}_\mu \frac{\partial \bar{x}^\mu}{\partial x^i} \right\} ; \bar{\boldsymbol{x}}_\mu \frac{\partial \bar{x}^\mu}{\partial x^1} ; \bar{\boldsymbol{x}}_\nu \frac{\partial \bar{x}^\nu}{\partial x^2} \right]$$

We can expand the first term in the scalar triple product above using the product rule and Eq. (1). We get

$$\frac{\partial}{\partial x^j} \left\{ \bar{\boldsymbol{x}}_\mu \frac{\partial \bar{x}^\mu}{\partial x^i} \right\} = \bar{\boldsymbol{x}}_{\mu\nu} \frac{\partial \bar{x}^\mu}{\partial x^i} \frac{\partial \bar{x}^\nu}{\partial x^j} + \bar{\boldsymbol{x}}_\mu \frac{\partial^2 \bar{x}^\mu}{\partial x^i \partial x^j}.$$

But

$$\left[\bar{\boldsymbol{x}}_\mu \frac{\partial^2 \bar{x}^\mu}{\partial x^i \partial x^j} ; \bar{\boldsymbol{x}}_\mu \frac{\partial \bar{x}^\mu}{\partial x^1} ; \bar{\boldsymbol{x}}_\nu \frac{\partial \bar{x}^\nu}{\partial x^2} \right] = \frac{\partial^2 \bar{x}^\mu}{\partial x^i \partial x^j} \frac{\partial \bar{x}^\mu}{\partial x^1} \frac{\partial \bar{x}^\nu}{\partial x^2} [\bar{\boldsymbol{x}}_\mu \,;\, \bar{\boldsymbol{x}}_\mu \,;\, \bar{\boldsymbol{x}}_\nu] \equiv 0$$

since $\bar{\boldsymbol{x}}_\mu \cdot \bar{\boldsymbol{x}}_\mu = 0$. Hence, our second fundamental form reduces to

$$L_{ij} = \left\| \bar{\boldsymbol{x}}_\mu \frac{\partial \bar{x}^\mu}{\partial x^1} \times \bar{\boldsymbol{x}}_\nu \frac{\partial \bar{x}^\nu}{\partial x^2} \right\|^{-1} \left[\bar{\boldsymbol{x}}_{\mu\nu} \frac{\partial \bar{x}^\mu}{\partial x^i} \frac{\partial \bar{x}^\nu}{\partial x^j} \, ; \, \bar{\boldsymbol{x}}_\mu \frac{\partial \bar{x}^\mu}{\partial x^1} \, ; \, \bar{\boldsymbol{x}}_\nu \frac{\partial \bar{x}^\nu}{\partial x^2} \right] = \langle \bar{\boldsymbol{x}}_{\mu\nu}, \boldsymbol{\xi} \rangle \frac{\partial \bar{x}^\mu}{\partial x^i} \frac{\partial \bar{x}^\nu}{\partial x^j}$$

Additionally (***this still needs to be proved***),

$$\boldsymbol{\xi} = \operatorname{sign}\left\{ \det \left(\frac{\partial \bar{x}^\mu}{\partial x^i} \right) \right\} \bar{\boldsymbol{n}}.$$

Finally, this gives

$$L_{ij} = \operatorname{sign}\left\{ \det \left(\frac{\partial \bar{x}^\mu}{\partial x^i} \right) \right\} \langle \bar{\boldsymbol{x}}_{\mu\nu}, \bar{\boldsymbol{n}} \rangle \frac{\partial \bar{x}^\mu}{\partial x^i} \frac{\partial \bar{x}^\nu}{\partial x^j}.$$

But $\langle \bar{\boldsymbol{x}}_{\mu\nu}, \bar{\boldsymbol{n}} \rangle$ is precisely $\bar{L}_{\mu\nu}$. Hence,

$$\boxed{L_{ij} = \pm \bar{L}_{\mu\nu} \frac{\partial \bar{x}^\mu}{\partial x^i} \frac{\partial \bar{x}^\nu}{\partial x^j}}$$

where the sign is determined by the sign of the determinant of the Jacobian, as expected. □

Theorem 1. *Let* ${\Gamma_{ij}}^k$ *denote the components of the Chrisoffel symbols in a coordinate system* $\mathscr{U}$ *and let* ${\overline{\Gamma}_{\mu\nu}}^\gamma$ *be the components of the Chrisoffel symbols in* $\mathscr{V}$*. If there exists a coordinate transformation between* $\mathscr{V}$ *and* $\mathscr{U}$*, then*

$${\overline{\Gamma}_{\mu\nu}}^\gamma = \left({\Gamma_{ij}}^k \frac{\partial x^i}{\partial \bar{x}^\mu} \frac{\partial x^j}{\partial \bar{x}^\nu} + \frac{\partial^2 x^k}{\partial \bar{x}^\mu \partial \bar{x}^\nu} \right) \frac{\partial \bar{x}^\gamma}{\partial x^k}$$

Proof. In this proof, we will write Gauss's equations (2) in a coordinate coordinate system $\mathscr{V}$ and use prior results about the transformation properties of the second fundamental form to replace all quantities *except* the Christoffel symbols with their expressions in a coordinate system $\mathscr{U}$. We will then compare coefficients to determine what the Christoffel should be in $\mathscr{U}$. Recall Gauss' formulas:

$$\boldsymbol{x}_{ij} = L_{ij} \boldsymbol{n} + {\Gamma_{ij}}^k \boldsymbol{x}_k \tag{2}$$

where the $\{L_{ij}\}$ are the components of the second fundamental form and the $\{{\Gamma_{ij}}^k\}$ are the components of the Christoffel symbols. Again let (x^1, x^2) be the local coordinates in $\mathscr{U}$, and let $\boldsymbol{x} = \boldsymbol{x}(x^1, x^2)$. Similarly, let $(\bar{x}^1, \bar{x}^2)$ be the coordinates in $\mathscr{V}$ and let $\bar{\boldsymbol{x}} = \bar{\boldsymbol{x}}(\bar{x}^1, \bar{x}^2)$. This time, we will use the chain rule in the reverse direction:

$$\bar{\boldsymbol{x}}_\mu = \boldsymbol{x}_i \frac{\partial x^i}{\partial \bar{x}^\mu} \tag{3}$$

We compute the second derivatives using the product rule:

$$\bar{\boldsymbol{x}}_{\mu\nu} = \frac{\partial}{\partial \bar{x}^\nu} \left\{ \boldsymbol{x}_i \frac{\partial x^i}{\partial \bar{x}^\mu} \right\} = \boldsymbol{x}_{ij} \frac{\partial x^i}{\partial \bar{x}^\mu} \frac{\partial x^j}{\partial \bar{x}^\nu} + \boldsymbol{x}_i \frac{\partial^2 x^i}{\partial \bar{x}^\mu \partial \bar{x}^\nu}. \tag{4}$$

We proved in Lemma 1 that the components of the second fundamental form L_{ij} transform doubly covariantly up to the sign of the determinant of the Jacobian, and so the product transforms as

$$\bar{L}_{\mu\nu} \bar{\boldsymbol{n}} = \frac{\partial x^i}{\partial \bar{x}^\mu} \frac{\partial x^j}{\partial \bar{x}^\nu} L_{ij} \boldsymbol{n} \tag{5}$$

Using Eq.'s (3), (4), and (5), we find that in the coordinate system $\mathscr{U}$, Gauss's equations (2) take the form

$$\bar{\boldsymbol{x}}_{\mu\nu} = \bar{L}_{\mu\nu}\bar{\boldsymbol{n}} + \overline{\Gamma}_{\mu\nu}{}^{\lambda}\bar{\boldsymbol{x}}_{\lambda} \quad \Longrightarrow \quad \boldsymbol{x}_{ij}\frac{\partial x^i}{\partial \bar{x}^\mu}\frac{\partial x^j}{\partial \bar{x}^\nu} + \boldsymbol{x}_i \frac{\partial^2 x^i}{\partial \bar{x}^\mu \partial \bar{x}^\nu} + \frac{\partial x^i}{\partial \bar{x}^\mu}\frac{\partial x^j}{\partial \bar{x}^\nu} L_{ij}\boldsymbol{n} + \overline{\Gamma}_{\mu\nu}{}^{\lambda}\frac{\partial x^i}{\partial \bar{x}^\lambda}\boldsymbol{x}_i.$$

What we need to do now is compare this equation to the simplified form stated in Eq. (2). First, collect terms involving $\boldsymbol{x}_i$:

$$\boldsymbol{x}_{ij}\frac{\partial x^i}{\partial \bar{x}^\mu}\frac{\partial x^j}{\partial \bar{x}^\nu} = \frac{\partial x^i}{\partial \bar{x}^\mu}\frac{\partial x^j}{\partial \bar{x}^\nu} L_{ij}\boldsymbol{n} + \left(\overline{\Gamma}_{\mu\nu}{}^{\lambda}\frac{\partial x^i}{\partial \bar{x}^\lambda} - \frac{\partial^2 x^i}{\partial \bar{x}^\mu \partial \bar{x}^\nu}\right)\boldsymbol{x}_i.$$

Next, recall[1] that $\frac{\partial x^i}{\partial \bar{x}^\mu}\frac{\partial \bar{x}^\mu}{\partial x^j} = \delta^i{}_j$. If we contract both sides with two factors of $\frac{\partial \bar{x}^\mu}{\partial x^i}$, we will recover the $\boldsymbol{x}_{ij}$'s on the left hand side, which is what we want in accordance with Eq. (2). So,

$$\begin{aligned}
\boldsymbol{x}_{ij}\frac{\partial x^i}{\partial \bar{x}^\mu}\frac{\partial \bar{x}^\mu}{\partial x^k}\frac{\partial x^j}{\partial \bar{x}^\nu}\frac{\partial \bar{x}^\nu}{\partial x^\ell} &= L_{ij}\boldsymbol{n}\frac{\partial x^i}{\partial \bar{x}^\mu}\frac{\partial \bar{x}^\mu}{\partial x^k}\frac{\partial x^j}{\partial \bar{x}^\nu}\frac{\partial \bar{x}^\nu}{\partial x^\ell} + \left(\overline{\Gamma}_{\mu\nu}{}^{\lambda}\frac{\partial x^i}{\partial \bar{x}^\lambda}\frac{\partial \bar{x}^\mu}{\partial x^k}\frac{\partial \bar{x}^\nu}{\partial x^\ell} - \frac{\partial^2 x^i}{\partial \bar{x}^\mu \partial \bar{x}^\nu}\frac{\partial \bar{x}^\mu}{\partial x^k}\frac{\partial \bar{x}^\nu}{\partial x^\ell}\right)\boldsymbol{x}_i \\
\boldsymbol{x}_{ij}\delta^i{}_k\delta^j{}_\ell &= L_{ij}\boldsymbol{n}\delta^i{}_k\delta^j{}_\ell + \left(\overline{\Gamma}_{\mu\nu}{}^{\lambda}\frac{\partial x^i}{\partial \bar{x}^\lambda}\frac{\partial \bar{x}^\mu}{\partial x^k}\frac{\partial \bar{x}^\nu}{\partial x^\ell} - \frac{\partial^2 x^i}{\partial \bar{x}^\mu \partial \bar{x}^\nu}\frac{\partial \bar{x}^\mu}{\partial x^k}\frac{\partial \bar{x}^\nu}{\partial x^\ell}\right)\boldsymbol{x}_i \\
\boldsymbol{x}_{k\ell} &= L_{k\ell}\boldsymbol{n} + \Gamma_{k\ell}{}^{i}\boldsymbol{x}_i.
\end{aligned}$$

Evidently, we must have

$$\Gamma_{k\ell}{}^{i} = \overline{\Gamma}_{\mu\nu}{}^{\lambda}\frac{\partial x^i}{\partial \bar{x}^\lambda}\frac{\partial \bar{x}^\mu}{\partial x^k}\frac{\partial \bar{x}^\nu}{\partial x^\ell} - \frac{\partial^2 x^i}{\partial \bar{x}^\mu \partial \bar{x}^\nu}\frac{\partial \bar{x}^\mu}{\partial x^k}\frac{\partial \bar{x}^\nu}{\partial x^\ell}$$

if we want to satisfy Eq. (2). We need to solve the above for the $\overline{\Gamma}_{\mu\nu}{}^{\lambda}$'s, which we can do by contracting both sides with two factors of $\frac{\partial x^i}{\partial \bar{x}^\mu}$ and one factor of $\frac{\partial \bar{x}^\mu}{\partial x^i}$. So,

$$\begin{aligned}
\overline{\Gamma}_{\mu\nu}{}^{\lambda}\frac{\partial x^i}{\partial \bar{x}^\lambda}\frac{\partial \bar{x}^\mu}{\partial x^k}\frac{\partial \bar{x}^\nu}{\partial x^\ell} &= \Gamma_{k\ell}{}^{i} + \frac{\partial^2 x^i}{\partial \bar{x}^\mu \partial \bar{x}^\nu}\frac{\partial \bar{x}^\mu}{\partial x^k}\frac{\partial \bar{x}^\nu}{\partial x^\ell} \\
\overline{\Gamma}_{\mu\nu}{}^{\lambda}\frac{\partial x^i}{\partial \bar{x}^\lambda}\frac{\partial \bar{x}^\rho}{\partial x^i}\frac{\partial \bar{x}^\mu}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^\alpha}\frac{\partial \bar{x}^\nu}{\partial x^\ell}\frac{\partial x^\ell}{\partial \bar{x}^\beta} &= \Gamma_{k\ell}{}^{i}\frac{\partial \bar{x}^\rho}{\partial x^i}\frac{\partial x^k}{\partial \bar{x}^\alpha}\frac{\partial x^\ell}{\partial \bar{x}^\beta} + \frac{\partial^2 x^i}{\partial \bar{x}^\mu \partial \bar{x}^\nu}\frac{\partial \bar{x}^\mu}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^\alpha}\frac{\partial \bar{x}^\nu}{\partial x^\ell}\frac{\partial x^\ell}{\partial \bar{x}^\beta}\frac{\partial \bar{x}^\rho}{\partial x^i} \\
\overline{\Gamma}_{\mu\nu}{}^{\lambda}\delta^\rho{}_\lambda\delta^\mu{}_\alpha\delta^\nu{}_\beta &= \Gamma_{k\ell}{}^{i}\frac{\partial \bar{x}^\rho}{\partial x^i}\frac{\partial x^k}{\partial \bar{x}^\alpha}\frac{\partial x^\ell}{\partial \bar{x}^\beta} + \frac{\partial^2 x^i}{\partial \bar{x}^\mu \partial \bar{x}^\nu}\delta^\mu{}_\alpha\delta^\nu{}_\beta\frac{\partial \bar{x}^\rho}{\partial x^i} \\
\overline{\Gamma}_{\alpha\beta}{}^{\rho} &= \left(\Gamma_{k\ell}{}^{i}\frac{\partial x^k}{\partial \bar{x}^\alpha}\frac{\partial x^\ell}{\partial \bar{x}^\beta} + \frac{\partial^2 x^i}{\partial \bar{x}^\alpha \partial \bar{x}^\beta}\right)\frac{\partial \bar{x}^\rho}{\partial x^i},
\end{aligned}$$

which is exactly the transformation law we wanted to show. $\square$

[1] This is just the chain rule! Write $\delta^i{}_j = \frac{\partial x^i}{\partial x^j}$, then assert $x^i = x^i(\bar{x}^\mu)$ and use the chain rule.