Linear Algebra

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January 2022

Definition 1 (Row-equivalence)**.** Let A and B be $n \times n$ matrices. Then A is said to be row-equivalent to B if B can be derived from A by applying a finite sequence of elementary row operations to A.

Definition 2 (Elementary matrices)**.** An elementary matrix is an $n \times n$ matrix which differs from the $n \times n$ identity matrix by one and only one elementary row operation.

Remark. Since there are three different "types" of elementary row operations (i.e., swapping two rows, multiplying a row by a nonzero constant, and adding a multiple of one row to another), there are three different "types" of elementary matrices. It is important to consider all three when conducting the following proofs.

Problem Set 3: Matrix Theory

1. Let
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 7 \end{pmatrix}.$$

   (a) Compute $\det(A)$ by performing elementary row operations on A to obtain its reduced row echelon form. Make sure to keep track of how to determinant changes upon performing a given elementary row operation.

   (b) Determine both a basis for the row space of A and a basis for the column space of A. [*Hint: use the reduced row echelon form from part (a).*]

2. Suppose we have the following linear system.
$$\begin{aligned} x + 2y + 3z &= 7 \\ x + 3y + 5z &= 11 \\ x + 5y + 7z &= 13. \end{aligned}$$

   Solve this system by computing the inverse of the coefficient matrix.

3. This question deals with properties of elementary matrices. They are not of too much interest on their own, but they will be used in the next problem in a substantial way, so we motivate them here to lessen the burden.

(a) Show that performing an elementary row operation to a matrix A is the same as left-multiplying A by an elementary matrix *of the same type.*[1] Feel free to just prove this result for 3×3 matrices instead of dealing with the general $n \times n$ case.

(b) Show that the inverse of an elementary matrix is an elementary matrix. Again, feel free to restrict to 3×3 matrices if you want.

(c) Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Write A as a product of elementary matrices by computing the reduced row echelon form of A and keeping track of the elementary row operations you used.

4. In this exercise, you will prove that the determinant of a product is the product of the determinants. First, we need some theory.

(a) Let A be an $n \times n$ matrix and let E be an $n \times n$ elementary matrix. Show that $\det(EA) = \det(E)\det(A)$.

(b) Let A be an $n \times n$ matrix. Prove that the following are equivalent.

i. A is invertible.
ii. $Ax = b$ has a unique solution for every $b \in \mathbb{F}^n$.
iii. $Ax = 0$ has only the trivial solution.
iv. A is row-equivalent to the identity $\mathbf{I}$.
v. A can be written as the product of elementary matrices.

(Not all of these facts will be relevant for this proof, but they are important in general.)

(c) Let A and B be $n \times n$ matrices. Prove

$$\det(AB) = \det(A)\det(B)$$

by treating the cases in which A is invertible and A is not inverible separately. You can assume without loss of generality that B is invertible.

[*Hint: If A is inverible, use part (b). If it is not, then note that A is row-equivalent to a matrix C with at least one row (or at least one column) composed entirely of zeros (why?).*]

5. Prove that if A is invertible if and only if $\det(A) \neq 0$.

[*Hint: use the result of the previous problem for the forward implication, and write A as a product of elementary matrices and its reduced row echelon form for the reverse implication. What is A's reduced row echelon form if* $\det(A) \neq 0$*?*]

6. In this problem we will deduce Cramer's rule for solving a system of linear equations.

(a) Suppose we have the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2, \end{aligned}$$

[1] In particular, consider each possible type of elementary row operation you could perform on A. For example, if B is obtained from A by swapping rows i and j, show $B = E_1 A$ where E_1 is an elementary matrix formed by swapping rows i and j. Repeat for each type of elementary row operation.

with $a_{11}a_{22} - a_{21}a_{12} \neq 0$. Use elimination to solve the system and show that

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} =: \frac{\det A_1}{\det A} \quad ; \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} =: \frac{\det A_2}{\det A}$$

where I have used A_i to denote the matrix whose i'th column is replaced by the vector b.

(b) Deduce a generalization of Cramer's rule for a system of n equations and n unknowns. (You do not have to prove this result.)

(c) Use Cramer's rule to solve the system

$$\begin{aligned} -x + 2y - 3z &= 1, \\ 2x \qquad + \ \ z &= 0, \\ 3x - 4y + 4z &= 2 \end{aligned}$$

7. Cramer's rule gives an elegant solution to a linear system of equations which is occasionally useful when the matrices are sparse and/or small, but in general it is totally useless for actually solving linear systems. This is because for a system of n equations in n unknowns, one has to calculate $n+1$ determinants, each of which has $\mathcal{O}(n!)$ time complexity when computing determinants naïvely, given a net time complexity of $\mathcal{O}((n+1)!)$, which is substantially slower than just solving the system using Gaussian elimination ($\mathcal{O}(n^3)$)[2]

   One of the most efficient algorithms (typically quoted as $\mathcal{O}(n^2)$ for computing RREF) for solving linear systems involves the so-called *LU* factorization of the coefficient matrix. That is, we would like to compute a lower-triangular matrix L and an upper triangular matrix U such that $A = LU$. Suppose

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 7 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}}_{U}.$$

   We can deduce a candidate for U by setting U equal to the row echelon form of A (*not the reduced row echelon form*). Determine what L must be by examining the elements of the product.[3]

   [*Hint: Equate the* 1×1 *elements to get* $1 = \ell_{11}u_{11}$. *This determines* ℓ_{11} *in terms of the now-known* u_{11}. *Repeat for the other diagonal elements. The remaining* ℓ_{ij} *can be found by solving a few simple linear equations.*]

8. Let A_n be an $n \times n$ matrix of the form

$$A_n = \begin{pmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 0 & a_{23} & \cdots & a_{2n} \\ 0 & 0 & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{(n-1)n} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

[2] See this Wikipedia page for more data on time complexity.

[3] As far as I can tell, setting U equal to the row echelon form of A is part of the standard procedure. You can also determine L by keeping track of the row operations you perform on A to determine U and constructing a corresponding product of elementary matrices which relate A and U. This product is evidently guaranteed to be lower triangular, and you can prove that if you want. I don't care enough to do it myself or to ask you to do it.

(a) Compute A_3^2 and A_3^3 (that is, write out an explicit 3×3 matrix of this form and square and cube it).

(b) Make a conjecture about how $n \times n$ matrices of this form behave when you raise them to powers.

(c) Prove your conjecture.