Homework # 01

Math 140
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Chapter 01

Problem 3.5

Let $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ be an orthonormal basis for $\mathbb{R}^3$ with $\mathbf{a} \times \mathbf{b} = \mathbf{c}$. Prove $\mathbf{b} \times \mathbf{c} = \mathbf{a}$ and $\mathbf{c} \times \mathbf{a} = \mathbf{b}$.

Solution

Recall that the components of a cross product can be decomposed into a sum involving the Levi-Civita symbol, i.e.,

$$(\mathbf{u} \times \mathbf{v})_k = \sum_i \sum_j \varepsilon_{ijk} u^i v^j$$

We can use this to deduce a useful vector identity. Consider the following:

$$\begin{aligned}(\mathbf{u} \times \mathbf{v} \times \mathbf{w})_k &= \sum_i \sum_j \varepsilon_{ijk} u_i (\mathbf{v} \times \mathbf{w})_j \\ &= \sum_i \sum_j \varepsilon_{ijk} u_i \sum_l \sum_m \varepsilon_{lmj} v_l w_m \\ &= \sum_i \sum_j \sum_l \sum_m \varepsilon_{ijk} \varepsilon_{lmj} u_i v_l w_m.\end{aligned}$$

By the cyclic nature of the Levi-Civita symbol, we can write $\varepsilon_{ijk} = \varepsilon_{kij}$. Recall the contracted epsilon identity:

$$\sum_j \varepsilon_{kij} \varepsilon_{lmj} = \delta_{kl}\delta_{im} - \delta_{km}\delta_{il}.$$

Hence, we get

$$\begin{aligned}(\mathbf{u} \times \mathbf{v} \times \mathbf{w})_k &= \sum_i \sum_j \sum_l \sum_m \varepsilon_{ijk} \varepsilon_{lmj} u_i v_l w_m \\ &= \sum_i \sum_l \sum_m (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il}) u_i v_l w_m \\ &= \sum_i \sum_l \sum_m \delta_{kl}\delta_{im} u_i v_l w_m - \sum_i \sum_l \sum_m \delta_{km}\delta_{il} u_i v_l w_m \\ &= \sum_m \Big(\sum_l \delta_{kl} v_l\Big)\Big(\sum_i \delta_{im} u_i\Big) w_m - \sum_l \Big(\sum_m \delta_{km} w_m\Big)\Big(\sum_i \delta_{il} u_i\Big) v_l \\ &= \sum_m v_k u_m w_m - \sum_l w_k u_l v_l \\ &= \Big(\sum_m u_m w_m\Big) v_k - \Big(\sum_l u_l v_l\Big) w_k \\ &= [(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}]_k\end{aligned}$$

Hence, we have

$$\mathbf{u} \times \mathbf{v} \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}. \tag{1}$$

This will be useful in proving the claim in the problem statement.

Now suppose $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is an orthonormal basis for $\mathbf{R}^3$ with $\mathbf{a} \times \mathbf{b} = \mathbf{c}$. Taking the cross product on the right of both sides with $\mathbf{a}$, we get

$$\mathbf{c} \times \mathbf{a} = \mathbf{a} \times \mathbf{b} \times \mathbf{a} \overset{(1)}{=} (\mathbf{a} \cdot \mathbf{a})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{a} = \mathbf{b}.$$

The final equality follows from the orthonormality of the basis $\beta := \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, i.e., that $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ for $\mathbf{e}_i \in \beta$. Thus

$$\boxed{(\mathbf{a} \times \mathbf{b} = \mathbf{c}) \implies (\mathbf{c} \times \mathbf{a} = \mathbf{b}).}$$

To prove $\mathbf{b} \times \mathbf{c} = \mathbf{a}$, we repeat a similar argument and cross the above equation on the right with $\mathbf{c}$:

$$\mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a} \times \mathbf{c} \overset{(1)}{=} (\mathbf{c} \cdot \mathbf{c})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{c} = \mathbf{a}.$$

Hence

$$\boxed{(\mathbf{c} \times \mathbf{a} = \mathbf{b}) \implies (\mathbf{b} \times \mathbf{c} = \mathbf{a})}$$

which completes the proof.

Problem 4.1

Let $\mathbf{a} = (2, 1, -3)$, $\mathbf{b} = (1, 0, 1)$, $\mathbf{c} = (0, -1, 3)$. What is the equation of

(a) the line through $\mathbf{a}$ parallel to $\mathbf{b}$;
(b) the line through $\mathbf{b}$ and $\mathbf{c}$;
(c) the plane through $\mathbf{b}$ perpendicular to $\mathbf{a}$;
(d) the plane through $\mathbf{c}$ parallel to $\mathbf{a}$ and $\mathbf{b}$;
(e) the sphere with center $\mathbf{a}$ and radius 2?

Solution

Throughout this problem, I will assume that we are adopting the Euclidean inner product on $\mathbb{R}^3$.

(a) A parameterization of the line through $\mathbf{a}$ parallel to $\mathbf{b}$ is manifestly $\boldsymbol{\ell}_a(t) = \mathbf{a} + t\mathbf{b}$. We see that the line passes through $\mathbf{a}$ at $t = 0$ and travels parallel to $\mathbf{b}$ as t increases. In this case,

$$\boxed{\boldsymbol{\ell}_a(t) = (2 + t, 1, -3 + t).}$$

(b) A parameterization of the line through both $\mathbf{b}$ and $\mathbf{c}$ is $\boldsymbol{\ell}_b(t) = \mathbf{b} + t(\mathbf{c} - \mathbf{b})$. Here, the line passes through the tip of $\mathbf{b}$ at $t = 0$ and passes through the tip of $\mathbf{c}$ at $t = 1$. In the present case,

$$\boxed{\boldsymbol{\ell}_b(t) = (1 - t, -t, 1 + 2t).}$$

(c) The plane which passes through $\mathbf{b}$ and is perpendicular to $\mathbf{a}$ is the set of all vectors $\mathbf{v}$ such that $\langle \mathbf{v} - \mathbf{b}, \mathbf{a} \rangle = 0$. Let $\mathbf{v} = (x, y, z)$. Then

$$\begin{aligned} 0 &= \langle \mathbf{v} - \mathbf{b}, \mathbf{a} \rangle \\ &= \langle (x - 1, y, z - 1), (2, 1, -3) \rangle \\ &= 2(x - 1) + y - 3(z - 1). \end{aligned}$$

Alternatively, in standard form, the equation is

$$\boxed{2x + y - 3z = -1.}$$

(d) Note that a plane parallel to both $\mathbf{a}$ and $\mathbf{b}$ must necessarily be perpendicular to $\mathbf{a} \times \mathbf{b}$. Hence, the plane which passes through $\mathbf{c}$ and is parallel to both $\mathbf{a}$ and $\mathbf{b}$ is the same as the plane which passes through $\mathbf{c}$ and is perpendicular to $\mathbf{a} \times \mathbf{b}$. Here,

$$\mathbf{a} \times \mathbf{b} = (1, -[2 + 3], -1) = (1, -5, -1).$$

Then, repeating the same procedure as in (c) with $\mathbf{v} = (x, y, z)$, we find that the equation of the plane is

$$\begin{aligned} 0 &= \langle \mathbf{v} - \mathbf{c}, \mathbf{a} \times \mathbf{b} \rangle \\ &= \langle (x, y+1, z-3), (1, -5, -1) \rangle \\ &= x - 5(y+1) - (z-3). \end{aligned}$$

Or, in standard form,

$$\boxed{x - 5y - z = 2.}$$

(e) Let $\mathbf{v} = (x, y, z)$ lie on the surface of the sphere in question. Then $\mathbf{v} - \mathbf{a}$ points from the center of the sphere to the surface. The length of this vector is $|\mathbf{v} - \mathbf{a}|$ and is also the radius 2. Hence, the equation of the sphere is $|\mathbf{v} - \mathbf{a}| = 2$, or equivalently $\langle \mathbf{v} - \mathbf{a}, \mathbf{v} - \mathbf{a} \rangle = 4$. Hence,

$$\begin{aligned} 4 &= \langle \mathbf{v} - \mathbf{a}, \mathbf{v} - \mathbf{a} \rangle \\ &= \langle (x-2, y-1, z+3), (x-2, y-1, z+3) \rangle \\ &= (x-2)^2 + (y-1)^2 + (z+3)^2. \end{aligned}$$

So the equation of the sphere of radius 2 and center $\mathbf{a}$ is

$$\boxed{(x-2)^2 + (y-1)^2 + (z+3)^2 = 2.}$$

Problem 5.1

Prove Lemma 5.1.

Lemma (5.1)**.** *Let* $\mathbf{f}, \mathbf{g} : \mathbb{R} \to V$ *and suppose that* V *has an inner product* $\langle\ ,\ \rangle$*. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \mathbf{f}, \mathbf{g} \rangle = \left\langle \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t}, \mathbf{g} \right\rangle + \left\langle \mathbf{f}, \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t} \right\rangle.$$

In particular, if $|\mathbf{f}|$ *is constant then* $\mathrm{d}\mathbf{f}/\mathrm{d}t$ *is perpendicular to* $\mathbf{f}$*.*

Solution

Fix a basis $\{\mathbf{v}_i\}$ for V. Then, there exists sets of real functions $\{f^i(t)\}$ and $\{g^j(t)\}$ for which

$$\begin{aligned} \mathbf{f}(t) &= \sum_i f^i(t) \mathbf{v}_i, \\ \mathbf{g}(t) &= \sum_j g^j(t) \mathbf{v}_j. \end{aligned}$$

Then, using the linearity of the inner product, we find

$$\langle \mathbf{f}(t), \mathbf{g}(t) \rangle = \sum_i \sum_j f^i(t) g^j(t) \langle \mathbf{v}_i, \mathbf{v}_j \rangle. \tag{2}$$

Since the t-dependence lies entirely with the functions $\{f^i(t)\}$ and $\{g^j(t)\}$, differentiation follows trivially from the product rule. Indeed,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \langle \mathbf{f}(t), \mathbf{g}(t) \rangle &= \sum_i \sum_j \langle \mathbf{v}_i, \mathbf{v}_j \rangle \frac{\mathrm{d}}{\mathrm{d}t} \{ f^i(t) g^j(t) \} \\ &= \sum_i \sum_j \langle \mathbf{v}_i, \mathbf{v}_j \rangle \left(\frac{\mathrm{d}f^i}{\mathrm{d}t} g^j + f^i \frac{\mathrm{d}g^j}{\mathrm{d}t} \right) \end{aligned}$$

$$= \sum_i \sum_j \frac{\mathrm{d}f^i}{\mathrm{d}t} g^j \langle \mathbf{v}_i, \mathbf{v}_j \rangle + \sum_i \sum_j f^i \frac{\mathrm{d}g^j}{\mathrm{d}t} \langle \mathbf{v}_i, \mathbf{v}_j \rangle.$$

Comparing the above to Eq. (2), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{f}, \mathbf{g} \rangle = \left\langle \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t}, \mathbf{g} \right\rangle + \left\langle \mathbf{f}, \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t} \right\rangle$$

which is precisely the result of Lemma 5.1. □

Now suppose that $|\mathbf{f}|$ is constant. Then, $|\mathbf{f}|^2$ is also constant. From Lemma 5.1:

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{f}|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{f}, \mathbf{f} \rangle = \left\langle \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t}, \mathbf{f} \right\rangle + \left\langle \mathbf{f}, \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t} \right\rangle.$$

By the symmetry of the inner product, we have $\langle \mathrm{d}\mathbf{f}/\mathrm{d}t\,, \mathbf{f} \rangle = \langle \mathbf{f}, \mathrm{d}\mathbf{f}/\mathrm{d}t \rangle$, so we find $2\langle \mathrm{d}\mathbf{f}/\mathrm{d}t\,, \mathbf{f} \rangle = 0$, or that $\langle \mathrm{d}\mathbf{f}/\mathrm{d}t\,, \mathbf{f} \rangle = 0$, i.e., that $\mathrm{d}\mathbf{f}/\mathrm{d}t$ is perpendicular to $\mathbf{f}$. □

Note that all of the steps in the proof above are logically consistent in either direction. Hence, this is sufficient to claim that $|\mathbf{f}|$ *is constant* $\iff$ $\mathrm{d}\mathbf{f}/\mathrm{d}t$ *is perpendicular to* $\mathbf{f}$. This will be useful in Problem 1.7 of Chapter 02.

Problem 5.3

Prove that for any $t \in \mathbb{R}$, $\cosh^2(t) - \sinh^2(t) = 1$. (Since $x^2 - y^2 = 1$ is a hyperbola, this gives the origin of the term "hyperbolic function." Also, you should think about the analogy between this equation and $\cos^2\theta + \sin^2\theta = 1$.)

Solution

This can be proved via direct calculation. Let $t \in \mathbb{R}$. Then,

$$\begin{aligned}
\cosh^2(t) - \sinh^2(t) &= \left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2 \\
&= \frac{e^{2t} + 2 + e^{-2t}}{4} - \frac{e^{2t} - 2 + e^{-2t}}{4} \\
&= \tfrac{1}{4}\left(e^{2t} + 2 + e^{-2t} - e^{2t} + 2 - e^{-2t}\right) \\
&= 1
\end{aligned}$$

Hence $\cosh^2(t) - \sinh^2(t) = 1$ for all $t \in \mathbb{R}$.

Problem 5.4

Prove that $\cosh'(t) = \sinh(t)$, and $\sinh'(t) = \cosh(t)$ for all $t \in \mathbb{R}$.

Solution

This can also be proved via direct calculation. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\cosh(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{e^t + e^{-t}}{2}\right)$$

$$\begin{aligned}
&= \frac{\frac{\mathrm{d}}{\mathrm{d}t}e^t + \frac{\mathrm{d}}{\mathrm{d}t}e^{-t}}{2} \\
&= \frac{e^t - e^{-t}}{2} \\
&= \sinh(t).
\end{aligned}$$

Similarly, we find

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\sinh(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{e^t - e^{-t}}{2}\right) \\
&= \frac{\frac{\mathrm{d}}{\mathrm{d}t}e^t - \frac{\mathrm{d}}{\mathrm{d}t}e^{-t}}{2} \\
&= \frac{e^t + e^{-t}}{2} \\
&= \cosh(t).
\end{aligned}$$

Chapter 02

Problem 1.1

(a) Show that $\boldsymbol{\alpha}(t) = (\sin 3t \cos t, \sin 3t \sin t, 0)$ is a regular curve.
(b) Find the equation of the tangent line to $\boldsymbol{\alpha}$ at $t = \pi/3$.

Solution

(a) Clearly $\boldsymbol{\alpha}$ is a parameterized curve, since $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^3$ is smooth (since each component of $\boldsymbol{\alpha}$ is a smooth function in $\mathbb{R}$). For $\boldsymbol{\alpha}$ to be a regular curve, we need to show that $\mathrm{d}\boldsymbol{\alpha}/\mathrm{d}t \neq \mathbf{0}$. We have

$$\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t} = 3(\cos 3t \cos t - \sin 3t \sin t, \cos 3t \sin t + \sin 3t \cos t, 0).$$

We recognize the first and second components of $\mathrm{d}\boldsymbol{\alpha}/\mathrm{d}t$ as the expansion of $\cos(3t+t)$ and $\sin(3t+t)$, respectively, which gives

$$\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t} = 3(\cos(4t), \sin(4t), 0).$$

If $\boldsymbol{\alpha}$ was not a regular curve, then there would exist at least one t for which $\sin(4t) = \cos(4t) = 0$. But, $\sin(4t) = 0 \iff t = n\pi/4$ for $n \in \mathbb{Z}$ and $\cos(4t) = 0 \iff t = (2m+1)\pi/8$ for $m \in \mathbb{Z}$. Such a t would then require

$$\frac{n\pi}{4} = \frac{(2m+1)\pi}{8} \iff n = m + \frac{1}{2}.$$

Hence, n and m cannot both be integers, so there cannot be such a t. Thus, $\mathrm{d}\boldsymbol{\alpha}/\mathrm{d}t$ is never zero and $\boldsymbol{\alpha}$ is a regular curve.

(b) A parameterization of the tangent line to $\boldsymbol{\alpha}$ at $t = t_0$ is

$$\boldsymbol{\ell}(t_0; \mu) = \boldsymbol{\alpha}(t_0) + \mu \left.\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t}\right|_{t=t_0}$$

where $\mu \in \mathbb{R}$. So in general, we get

$$\boldsymbol{\ell}(t; \mu) = (\sin 3t \cos t + 3\mu \cos 4t, \sin 3t \sin t + 3\mu \sin 4t, 0).$$

Plugging in $t = \pi/3$ gives

$$\boxed{\boldsymbol{\ell}(\tfrac{\pi}{3}; \mu) = -\tfrac{\mu}{2}(3, \sqrt{3}, 0).}$$

Problem 1.6

Let $\boldsymbol{\alpha}(\theta) = (e^\theta \cos\theta, e^\theta \sin\theta, 0)$. Prove that the angle between $\boldsymbol{\alpha}$ and $\mathbf{T}$ is constant. (A curve with this property is called a logarithmic spiral.)

Solution

Modulo π, the angle between $\boldsymbol{\alpha}(\theta)$ and $\mathbf{T}$ is

$$\varphi = \arccos\left(\frac{\langle \boldsymbol{\alpha}, \mathbf{T}\rangle}{|\boldsymbol{\alpha}||\mathbf{T}|}\right)$$

assuming the Euclidean metric. First, we find $\mathbf{T} = \left|\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}\theta}\right|^{-1} \frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}\theta}$. We have

$$\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}\theta} = (e^\theta[\cos\theta - \sin\theta], e^\theta[\cos\theta + \sin\theta], 0).$$

The magnitude is therefore

$$\left|\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}\theta}\right| = \sqrt{\left\langle \frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}\theta}, \frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}\theta} \right\rangle} = \sqrt{2}e^{\theta}.$$

Thus, the tangent vector is

$$\mathbf{T} = \left|\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}\theta}\right|^{-1} \frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}\theta} = \left(\frac{\cos\theta - \sin\theta}{\sqrt{2}}, \frac{\cos\theta + \sin\theta}{\sqrt{2}}, 0 \right).$$

Then, the inner product reads

$$\begin{aligned}
\langle \boldsymbol{\alpha}, \mathbf{T} \rangle &= \left\langle (e^{\theta}\cos\theta, e^{\theta}\sin\theta, 0), \left(\frac{\cos\theta - \sin\theta}{\sqrt{2}}, \frac{\cos\theta + \sin\theta}{\sqrt{2}}, 0 \right) \right\rangle \\
&= \frac{e^{\theta}}{\sqrt{2}}[\cos\theta(\cos\theta - \sin\theta) + \sin\theta(\cos\theta + \sin\theta)] \\
&= \frac{e^{\theta}}{\sqrt{2}}.
\end{aligned}$$

Furthermore, the magnitudes of the two vectors also read

$$\begin{aligned}
|\boldsymbol{\alpha}| &= \sqrt{\langle (e^{\theta}\cos\theta, e^{\theta}\sin\theta, 0), (e^{\theta}\cos\theta, e^{\theta}\sin\theta, 0) \rangle} = e^{\theta} \\
|\mathbf{T}| &= 1 \quad \text{(by construction)}
\end{aligned}$$

Hence, we find

$$\varphi = \arccos\left(\frac{\langle \boldsymbol{\alpha}, \mathbf{T} \rangle}{|\boldsymbol{\alpha}||\mathbf{T}|} \right) = \arccos\left(\frac{e^{\theta}/\sqrt{2}}{e^{\theta}} \right) = \arccos\left(\frac{1}{\sqrt{2}} \right) = \frac{\pi}{4},$$

which is indeed constant. Hence, $\boldsymbol{\alpha}$ is a logarithmic spiral.

Problem 1.7

Let $\boldsymbol{\alpha}(t)$ be a regular curve. Suppose there is a point $\mathbf{a} \in \mathbb{R}^3$ such that $\boldsymbol{\alpha}(t) - \mathbf{a}$ is orthogonal to $\mathbf{T}(t)$ for all t. Prove that $\boldsymbol{\alpha}(t)$ lies on a sphere. (*Hint:* What should be the center of the sphere?)

Solution

If there is a point $\mathbf{a} \in \mathbb{R}^3$ such that $\boldsymbol{\alpha}(t) - \mathbf{a}$ is orthogonal to $\mathbf{T}(t)$ for all t, then necessarily

$$\langle \boldsymbol{\alpha}(t) - \mathbf{a}, \mathbf{T}(t) \rangle = 0.$$

Then, since $\mathbf{T} := \left|\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t}\right|^{-1} \frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t}$, by the linearity of the inner product, we can multiply through by $\left|\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t}\right|$ and get:

$$\left\langle \boldsymbol{\alpha}(t) - \mathbf{a}, \frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}t} \right\rangle = 0.$$

Additionally, since $\mathbf{a}$ is constant in t, $\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} = 0$ and we can equivalently write

$$\left\langle \boldsymbol{\alpha}(t) - \mathbf{a}, \frac{\mathrm{d}}{\mathrm{d}t}\{\boldsymbol{\alpha}(t) - \mathbf{a}\} \right\rangle = 0.$$

In the proof of Lemma 5.1[1], I proved that, for a function $\mathbf{f} : \mathbb{R} \to \mathbb{R}^3$, $|\mathbf{f}|$ is constant $\iff$ $\mathrm{d}\mathbf{f}/\mathrm{d}t$ is perpendicular to $\mathbf{f}$. Letting $\mathbf{f}(t) := \boldsymbol{\alpha}(t) - \mathbf{a}$ (which is indeed a map $\mathbb{R} \to \mathbb{R}^3$), we find

$$\left\langle \boldsymbol{\alpha}(t) - \mathbf{a}, \frac{\mathrm{d}}{\mathrm{d}t}\{\boldsymbol{\alpha}(t) - \mathbf{a}\} \right\rangle = 0 \implies |\boldsymbol{\alpha}(t) - \mathbf{a}| = r$$

[1] See Problem 5.1 of Chapter 01

where $r \in \mathbb{R}^+ \cup \{0\}$ (since magnitudes are axiomatically non-negative). Indeed, this proves that $\boldsymbol{\alpha}(t)$ lies on a sphere of radius r with center $\mathbf{a}$ for all $t \in \mathrm{dom}\{\boldsymbol{\alpha}\}$.