Linear Algebra

Cody Vig

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Problem Set 5: Inner Product Spaces

1. In lecture, we showed that $\{f_n\} = \{e^{int}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for a subset of the space $\mathscr{H}$ of continuous complex-valued functions $f : [0, 2\pi] \to \mathbb{C}$ with inner product

$$\langle f, g\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\,\mathrm{d}t.$$

Then, for any $f \in \mathscr{H}$ we can write

$$f(t) = \sum_{n=-\infty}^{\infty} \langle f, f_n\rangle f_n(t) \equiv \sum_{n=-\infty}^{\infty} c_n e^{int}.$$

This is the complex Fourier series for f (c.f. the Taylor series for f). We can construct the *real* Fourier series of f (so long as f is real-valued rather than complex) by instead considering the orthonormal basis

$$\beta = \{1\} \cup \{\sin(nt)\}_{n\in\mathbb{N}} \cup \{\cos(nt)\}_{n\in\mathbb{N}}$$

(where $\mathbb{N} = \{1, 2, \ldots\}$). It is customary to write $f \in \mathscr{H}$ as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt))$$

for unknown constants $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$

(a) Show that $\beta \setminus \{1\}$ is orthonormal and that β is orthogonal under the given inner product $\langle f, g\rangle := \frac{1}{\pi}\int_0^{2\pi} f(t)g(t)\,\mathrm{d}t$.[1]

(b) Find the Fourier coefficients a_n and b_n (in the same way we found them for $\{e^{int}\}$).

(c) Compute the real Fourier series for $f(t) = t$ on the interval $[0, 2\pi]$.

2. Given that $\beta = \{1, x, x^2\}$ is a basis for $P_2(\mathbb{R})$, use the Gram-Schmidt orthogonalization process to find an associated orthonormal basis for $P_2(\mathbb{R})$ under the inner product

$$\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x.$$

[1] This basis can be made orthonormal by replacing 1 with $1/\sqrt{2}$, but this is not the typical convention for writing Fourier series. This is why we have the coefficient $1/2$ on the a_0 term. The reason for this convention is that the Fourier coefficients $\{a_n\}$ agree with a_0 at $n = 0$.

Then find the Fourier coefficients of $f(x) = a + bx + cx^2$ relative to this new basis and use them to write f in terms of this basis.

The basis you end up with is closely related to the *Legendre polynomials*. Indeed, the sequence of functions $\{P_n(x)\}$ you get from applying the Gram Schmidt process to the basis $\{1, x, x^2, \ldots\}$ for $P(\mathbb{R})$, and then "normalizing" by requiring $P_n(1) = 1$, are the Legendre polynomials. These appear *everywhere* in physics; most notably, probably, in quantum mechanics, where derivatives of these functions describe the spherical geometry of Hydrogen orbitals (c.f., the spherical harmonics).

3. In this problem, you will (almost) solve the problem that made Euler famous.

   (a) Prove that for any orthonormal basis $\beta := \{e_1, \ldots, e_n\}$ for $\mathbf{V}$ and any $x \in \mathbf{V}$, we have

   $$\|x\|^2 = \sum_{i=1}^{n} |\langle x, e_i \rangle|^2.$$

   (b) Compute the Fourier coefficients $\langle f, f_n \rangle$ for $f(t) = t \in \mathscr{H}$ with the basis $\{e^{int}\}_{n\in\mathbb{Z}}$ (i.e., the inner product space studied in Problem 1).

   (c) Use your result from (b) to write

   $$\|f\|^2 = \sum_{n=-\infty}^{-1} |\langle f, f_n \rangle|^2 + |\langle f, 1 \rangle|^2 + \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2.$$

   You should be able to simplify the two series to a single series. Solve the equation for this sum.

   There are a few issues with this proof. The first of which is that we have not proved $\{e^{int}\}$ is actually a basis for this space, I have only told you it is. The second flaw is that part (b) is only clearly true if $\mathbf{V}$ is finite dimensional, whereas $\mathscr{H}$ is infinite dimensional. These details can be made rigorous though! Also note that we can replace $f(t) = t$ with other functions to deduce other interesting series identities.

4. The goal of this question it to introduce a few topics from a branch of math called *functional analysis*, which is essentially infinite-dimensional linear algebra at the graduate level. First, we introduce the notion of a *normed space* and recall the definition of a metric space.

   Definition 1. Let $\mathbf{V}$ be a vector space over a field $\mathbb{F}$ (where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$). Regardless of whether or not $\mathbf{V}$ is an inner product space, we may still define a norm $\|\cdot\|_{\mathbf{V}}$ as a real-valued function on $\mathbf{V}$ satisfying the following three conditions for all x and y in $\mathbf{V}$ and c in $\mathbb{F}$:

   i $\|x\|_{\mathbf{V}} \geq 0$, and $\|x\|_{\mathbf{V}} = 0$ if and only if $x = 0$;
   ii $\|ax\|_{\mathbf{V}} = |a|\|x\|_{\mathbf{V}}$;
   iii $\|x + y\|_{\mathbf{V}} \leq \|x\|_{\mathbf{V}} + \|y\|_{\mathbf{V}}$.

   If $\mathbf{V}$ has such a norm, we call it a *normed space*.

   Definition 2. A set X is said to be a *metric space* if with with any two points p and q in X there is associated a real number $d(p, q)$, called the *distance from p to q*, such that

   i $d(p, q) > 0$ if $p \neq q$ and $d(p, p) = 0$;

ii $d(p,q) = d(q,p)$;
iii $d(p,q) \le d(p,r) + d(r,q)$ for any r in X.

(a) Show that every inner product space is a normed space, but not every normed space is an inner product space.

[*Hint: You may want to complete the next two problems before attempting the reverse implication (Use Problem 6 and the contrapositive of Problem 7).*]

(b) Prove that every normed space is metric space.

(c) Prove that every inner product space is a metric space.

This proves {inner product spaces} $\subset$ {normed spaces} $\subset$ {metric spaces}. The problem is over now, but what follows is some exposition on why this matters. This realization gives rise to the notion of a *topological vector space*, in which all common notions of point-set topology are applied to vector spaces via their metric space structure, including notions of openness, compactness, and density. One of the most desirable conditions a metric space could have is completeness; i.e., that every Cauchy sequence in the space converges in the space. There are associated notions of completeness for normed spaces and inner product spaces.

Definition 3. A *Banach space* is a normed vector space which is complete with respect to the norm. A *Hilbert space* is an inner product space which is complete with respect to its inner product.

There are even notions of topological vector spaces which inherit their topological properties from something other than a metric space structure. Examples include the set of infinitely differentiable functions, or the set of holomorphic functions on a open domain.

5. In the previous problem, we discussed the notion of a normed space. In this problem, we will explore an example of a family of norms on $\mathbb{F}^n$ (some of which are *not* induced by an inner product).

Definition 4. Let $x = (x_1, x_2, \ldots, x_n)$ be a vector in $\mathbb{F}^n$ (where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$). The ℓ_p norm is then defined to be

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

where $p \ge 1$ is a real number.

(a) Show that the ℓ_p norm is indeed a norm on $\mathbb{F}^n$.

(b) Define $\|x\|_\infty := \lim_{p\to\infty} \|x\|_p$. Find a closed form expression for $\|x\|_\infty$ and prove it is also a norm.

6. If a norm is induced by an inner product, but we do not know what that inner product is, we can reconstruct it from the norm in a clever way. In this problem, we will derive the *polarization identity.* In what follows, let $\mathbf{V}$ be an inner product space over $\mathbb{C}$ and suppose $x, y \in \mathbf{V}$.

(a) Show that $\operatorname{Im}\{\langle x, y\rangle\} = \operatorname{Re}\{\langle x, iy\rangle\}$.

(b) Prove that $\|x \pm y\|^2 = \|x\|^2 + \|y\|^2 \pm 2\operatorname{Re}\{\langle x, y\rangle\}$.

(c) Write $\langle x, y\rangle = \operatorname{Re}\{\langle x, y\rangle\} + i\operatorname{Im}\{\langle x, y\rangle\}$ in terms of the $\|x \pm y\|^2$.

(d) [*Bonus*] From part (b), derive the *parallelogram law*: $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$. What does this tell you about parallelograms in $\mathbb{R}^2$ with the standard inner product, for example?

7. You may have wondered if there is a notion of an *outer product* related to the inner product, and indeed there is! To motivate the definition, recall what the inner product is. Essentially, it is a mapping $\langle \cdot\,, \cdot \rangle : \mathbf{V} \times \mathbf{V} \to \mathbb{F}$; that is, a map which reduces the "rank" by 1. An outer product essentially does the opposite. It is a map from $\mathbf{V} \times \mathbf{V} \to \mathscr{L}(\mathbf{V}) \cong M_{n\times n}(\mathbb{F})$, where $n = \dim(\mathbf{V})$. In $\mathbb{F}^n$, if $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, then the outer product is defined to be

$$x \otimes y = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \otimes \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & & \vdots \\ x_ny_1 & x_ny_2 & \cdots & x_ny_n \end{pmatrix}. \tag{1}$$

Or, in index notation, $(x \otimes y)_{ij} = x_i y_j$. If $\mathbb{F} = \mathbb{C}$, we may write $x \otimes y = xy^\dagger$, where $\dagger$ denotes the conjugate transpose, $(y^\dagger)_i = (\bar{y}^\top)_i$. If $\mathbb{F} = \mathbb{R}$, $y^\dagger = y^\top$. (Compare to the standard inner product $\langle x, y \rangle = y^\dagger x$). The outer product is a special case of the *tensor product.*

The outer product can also be abstractly defined as the transformation $\mathbf{V} \times \mathbf{V} \to \mathbb{F}$ given by[2] $(x \otimes y)(v) := \langle y, v \rangle x$. Show that this definition gives the same result as the matrix definition (1) when $\mathbf{V}$ is $\mathbb{F}^n$.

[2] This is yet another reason why I prefer Dirac notation, $\langle x|y \rangle := \langle y, x \rangle$, where we interpret $|y\rangle$ as a vector and $\langle x| = (|x\rangle)^\dagger$ as a dual vector. Then we can write $x \otimes y = |x\rangle\langle y|$ and the definition becomes $(|x\rangle\langle y|)\,|v\rangle =$ "$|x\rangle\langle y|v\rangle$" $= |x\rangle\,(\langle y|v\rangle) = \langle y|v\rangle\,|x\rangle$, which has the appearance of commutation between inner and outer products. The quotes around the term $|x\rangle\langle y|v\rangle$ symbolize that this notation is technically ambiguous. It could be referring to $(|x\rangle\langle y|)\,|v\rangle$, or it could be referring to $|x\rangle\,(\langle y|v\rangle)$. The elegance of Dirac notation is that both of these terms are equal, so there is no ambiguity in writing $|x\rangle\langle y|v\rangle$.