Homework # 10

Math 140
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By

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Chapter 06 | Problem 2.3

Let $\boldsymbol{x}$ be a geodesic coordinate patch with

$$(\mathsf{g}_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & h^2 \end{pmatrix}$$

and $h > 0$. Show that ${\Gamma_{12}}^2 = {\Gamma_{21}}^2 = h_1/h$, ${\Gamma_{22}}^1 = -h_1 h$, ${\Gamma_{22}}^2 = h_2/h$, and all other ${\Gamma_{ij}}^k$ are zero, where $h_i = \partial h / \partial u^i$. Show that $K = -h_{11}/h$.

Solution

We can use Gauss' formula to determine the Christoffel symbols. Recall:

$${\Gamma_{ij}}^k = \tfrac{1}{2}\mathsf{g}^{k\ell}\left(\frac{\partial \mathsf{g}_{j\ell}}{\partial u^i} + \frac{\partial \mathsf{g}_{\ell i}}{\partial u^j} - \frac{\partial \mathsf{g}_{ij}}{\partial u^\ell}\right) \tag{1}$$

(where the Einstein summation convention is imposed). Since $(\mathsf{g}_{ij}) = \operatorname{diag}(1, h^2)$, we see that all terms with fewer than two indices containing a 2 are zero (since only $\partial \mathsf{g}_{22} / \partial u^i \neq 0$). Further, we know by symmetry that ${\Gamma_{ij}}^k = {\Gamma_{ji}}^k$. The remaining nonzero Christoffel symbols are

$$\begin{aligned}
{\Gamma_{21}}^2 = {\Gamma_{12}}^2 &= \tfrac{1}{2}\mathsf{g}^{2\ell}\left(\frac{\partial \mathsf{g}_{2\ell}}{\partial u^1} + \frac{\partial \mathsf{g}_{\ell 1}}{\partial u^2} - \frac{\partial \mathsf{g}_{12}}{\partial u^\ell}\right) = \tfrac{1}{2}\mathsf{g}^{22}\frac{\partial \mathsf{g}_{22}}{\partial u^1} \\
{\Gamma_{22}}^1 &= \tfrac{1}{2}\mathsf{g}^{1\ell}\left(\frac{\partial \mathsf{g}_{2\ell}}{\partial u^2} + \frac{\partial \mathsf{g}_{\ell 2}}{\partial u^2} - \frac{\partial \mathsf{g}_{22}}{\partial u^\ell}\right) = -\tfrac{1}{2}\mathsf{g}^{11}\frac{\partial \mathsf{g}_{22}}{\partial u^1} \\
{\Gamma_{22}}^2 &= \tfrac{1}{2}\mathsf{g}^{2\ell}\left(\frac{\partial \mathsf{g}_{2\ell}}{\partial u^2} + \frac{\partial \mathsf{g}_{\ell 2}}{\partial u^2} - \frac{\partial \mathsf{g}_{22}}{\partial u^\ell}\right) = \tfrac{1}{2}\mathsf{g}^{22}\frac{\partial \mathsf{g}_{22}}{\partial u^2}.
\end{aligned}$$

In the above, I have taken advantage of the fact that

$$(\mathsf{g}^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1/h^2 \end{pmatrix}$$

and so $\mathsf{g}^{ij} = 0$ if $i \neq j$. In any case, $\frac{\partial \mathsf{g}_{22}}{\partial u^i} = 2hh_i$ and $\mathsf{g}^{22} = 1/h^2$, and so the Chirstoffel symbols are indeed as given in the prompt. From Problem 8.7 of Chapter 04, we know that if $\mathsf{g}_{11} = 1$ and $\mathsf{g}_{12} = 0$, then

$$\frac{\partial^2 \sqrt{\mathsf{g}_{22}}}{\partial (u^2)^2} + K\sqrt{\mathsf{g}_{22}} = 0.$$

In this case, $\mathsf{g}_{22} = h^2$ and so $\frac{\partial^2 \sqrt{\mathsf{g}_{22}}}{\partial (u^1)^2} = \frac{\partial}{\partial u^1}\{h_1\} = h_{11}$. Hence

$$K = -\frac{1}{\sqrt{\mathsf{g}_{22}}}\frac{\partial^2 \sqrt{\mathsf{g}_{22}}}{\partial (u^2)^2} = -\frac{h_{11}}{h},$$

as expected.

Chapter 06 | Problem 2.4

Suppose that there is a simple surface $\boldsymbol{x} : \mathscr{U} \to \mathbb{R}^3$ with

$$\mathscr{U} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \quad \text{and} \quad (\mathsf{g}_{ij}) = \begin{pmatrix} y^{-2} & 0 \\ 0 & y^{-2} \end{pmatrix}.$$

Show that ${\Gamma_{11}}^2 = -{\Gamma_{12}}^1 = -{\Gamma_{21}}^1 = -{\Gamma_{22}}^2 = y^{-1}$ and all other ${\Gamma_{ij}}^k$ are zero. Compute K.

Solution

Here $u^1 = x$ and $u^2 = y$, but I will continue using index notation for the benefit of the summation convention. This question will be answered in much the same way as the previous. There are in general $2^3 = 8$ Christoffel symbols to find, but since ${\Gamma_{ij}}^k = {\Gamma_{ji}}^k$, the number of Christoffel symbols we need to compute decreases to 6. Using Eq. (1), we find

$$\begin{aligned}
{\Gamma_{11}}^1 &= \tfrac{1}{2}\mathsf{g}^{1\ell}\left(\frac{\partial \mathsf{g}_{1\ell}}{\partial u^1} + \frac{\partial \mathsf{g}_{\ell 1}}{\partial u^1} - \frac{\partial \mathsf{g}_{11}}{\partial u^\ell}\right) &&= \quad 0 \\
{\Gamma_{11}}^2 &= \tfrac{1}{2}\mathsf{g}^{2\ell}\left(\frac{\partial \mathsf{g}_{1\ell}}{\partial u^1} + \frac{\partial \mathsf{g}_{\ell 1}}{\partial u^1} - \frac{\partial \mathsf{g}_{11}}{\partial u^\ell}\right) &&= -\tfrac{1}{2}\mathsf{g}^{22}\frac{\partial \mathsf{g}_{11}}{\partial u^2} \\
{\Gamma_{21}}^1 = {\Gamma_{12}}^1 &= \tfrac{1}{2}\mathsf{g}^{1\ell}\left(\frac{\partial \mathsf{g}_{2\ell}}{\partial u^1} + \frac{\partial \mathsf{g}_{\ell 1}}{\partial u^2} - \frac{\partial \mathsf{g}_{12}}{\partial u^\ell}\right) &&= \quad \tfrac{1}{2}\mathsf{g}^{11}\frac{\partial \mathsf{g}_{11}}{\partial u^2} \\
{\Gamma_{21}}^2 = {\Gamma_{12}}^2 &= \tfrac{1}{2}\mathsf{g}^{2\ell}\left(\frac{\partial \mathsf{g}_{2\ell}}{\partial u^1} + \frac{\partial \mathsf{g}_{\ell 1}}{\partial u^2} - \frac{\partial \mathsf{g}_{12}}{\partial u^\ell}\right) &&= \quad 0 \\
{\Gamma_{22}}^1 &= \tfrac{1}{2}\mathsf{g}^{1\ell}\left(\frac{\partial \mathsf{g}_{2\ell}}{\partial u^2} + \frac{\partial \mathsf{g}_{\ell 2}}{\partial u^2} - \frac{\partial \mathsf{g}_{22}}{\partial u^\ell}\right) &&= \quad 0 \\
{\Gamma_{22}}^2 &= \tfrac{1}{2}\mathsf{g}^{2\ell}\left(\frac{\partial \mathsf{g}_{2\ell}}{\partial u^2} + \frac{\partial \mathsf{g}_{\ell 2}}{\partial u^2} - \frac{\partial \mathsf{g}_{22}}{\partial u^\ell}\right) &&= \quad \tfrac{1}{2}\mathsf{g}^{22}\frac{\partial \mathsf{g}_{22}}{\partial u^2}
\end{aligned}$$

In the above, I have taken advantage of the fact that (g_{ij}) is diagonal and so $\frac{\partial \mathsf{g}_{ij}}{\partial u^k} = 0$ whenever $i \neq j$. Further, $\mathsf{g}_{ij} = \delta_{ij} y^{-2}$, and so $\mathsf{g}^{ij} = \delta^{ij} y^2$. Since $\frac{\partial \mathsf{g}_{11}}{\partial u^i} = \frac{\partial \mathsf{g}_{22}}{\partial u^i} = -2y^{-3}$, we find

$$\begin{aligned}
{\Gamma_{11}}^2 &= -\tfrac{1}{2}\mathsf{g}^{22}\frac{\partial \mathsf{g}_{11}}{\partial u^2} = \frac{1}{y}, \\
{\Gamma_{21}}^1 &= \tfrac{1}{2}\mathsf{g}^{11}\frac{\partial \mathsf{g}_{11}}{\partial u^2} = -\frac{1}{y}, \\
{\Gamma_{12}}^1 &= \tfrac{1}{2}\mathsf{g}^{11}\frac{\partial \mathsf{g}_{11}}{\partial u^2} = -\frac{1}{y}, \\
{\Gamma_{22}}^2 &= \tfrac{1}{2}\mathsf{g}^{22}\frac{\partial \mathsf{g}_{22}}{\partial u^2} = -\frac{1}{y}.
\end{aligned}$$

Hence the Chrisoffel symbols are indeed as given in the question. From Problem 8.6 in Chapter 04, we know that if (g_{ij}) is diagonal, then

$$K = -\frac{1}{2\sqrt{\mathsf{g}}}\left(\frac{\partial}{\partial u^2}\left\{\frac{1}{\sqrt{\mathsf{g}}}\frac{\partial \mathsf{g}_{11}}{\partial u^2}\right\} + \frac{\partial}{\partial u^1}\left\{\frac{1}{\sqrt{\mathsf{g}}}\frac{\partial \mathsf{g}_{22}}{\partial u^1}\right\}\right)$$

Here $\mathsf{g} = y^{-4}$ and so $\frac{\partial}{\partial u^1}\left\{\frac{1}{\sqrt{\mathsf{g}}}\frac{\partial \mathsf{g}_{22}}{\partial u^1}\right\} = 0$ since $u^1 = x$ and the expression is independent of x. Hence

$$K = -\frac{y^2}{2}\frac{\partial}{\partial y}\left\{\frac{-2}{y}\right\} = \frac{2y^2}{2y^2} = -1.$$

Chapter 06 | Problem 3.1

Prove that if $\boldsymbol{\gamma}$ bounds a region $\mathscr{R}$ that is entirely contained in a single coordinate patch, then a field of unit vectors exists in $\mathscr{R}$.

Solution

We can prove this claim by constructing one such field of unit vectors. Since $\mathscr{R}$ is a region contained entirely inside of a single coordinate patch, the vector $\boldsymbol{x}_1$ exists and is well-defined, continuous, and nonzero everywhere inside $\mathscr{R}$. Hence the field of unit vectors $\boldsymbol{x}_1/\|\boldsymbol{x}_1\|$ exists in $\mathbb{R}$.[1]

Chapter 06 | Problem 3.2

Prove that there is a field of unit vectors defined on all of the sphere except the south pole. (*Hint:* Problem 1.10 of Chapter 4.)

Solution

For convenience, define $\Omega := S^2 \setminus \{\text{the north pole}\}$. If we use the stereographic projection as defined in Problem 1.10 of Chapter 04, then we can define a field of unit vectors on the sphere except the *north* pole. Up to rigid motion, this answers the question given here. Recall that the stereographic projections assigns points on $\mathbb{R}^2$ to points on S^2 by embedding $\mathbb{R}^2$ into $\mathbb{R}^3$ and further embedding S^2 in such a way that $S^2 \cap \mathbb{R}^2$ is the unit circle in $\mathbb{R}^2$ centered at the origin. We then take a line segment which on one end connects to the north pole of the sphere and on the other connects to some point $(x, y, 0)$ in the plane. The intersection of this line with the sphere (less the north pole) defines a map

$$\varphi : \Omega \to \mathbb{R}^2$$

which is called the *stereographic projection* of the sphere onto $\mathbb{R}^2$. If for each $\boldsymbol{p} \in \Omega$ we construct a unit vector $\boldsymbol{v_p}$ parallel to the line segment used to define this projection and directed towards $\mathbb{R}^2$, then take the projection of this vector onto $T_{\boldsymbol{p}}(\Omega)$, then we have a field of unit vectors defined on Ω. Simply rotate this sphere to map the north pole to the south and we are done.[2]

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[1] This is either incorrect or not in the spirit of the question since my response does not mention the boundary curve $\boldsymbol{\gamma}$, but I do not have enough time this week to find a better solution!

[2] Sorry for the lack of formality! I am out of time.