Linear Algebra

Cody Vig

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Preliminary Assessment

1. **(a)** Is the set of polynomials of degree $n \in \mathbb{Z}_{\geq 0}$ over a field $\mathbb{F}$ a vector space? If it is, prove it. If it is not, state which axioms are not satisfied and provide counterexamples.

   Solution. No, the set of polynomials of degree $n \in \mathbb{Z}$ over a field $\mathbb{F}$ is not a vector space. The set is not closed. For example, x^2 and $-x^2$ are in $P_2(\mathbb{R})$ but $x^2 + (-x^2) = 0$ is not.

   (b) Prove that the set of polynomials of degree at most n is a vector space. You may assume that the set of continuous functions $\mathscr{C}(\mathbb{F})$ in $\mathbb{F}$ is a vector space. Why is this helpful?

   Solution. Since $P_n(\mathbb{R}) \subseteq \mathscr{C}(\mathbb{F})$, it is sufficient to prove that $P_n(\mathbb{R})$ is nonempty and closed under addition and scalar multiplication. Since the identity function $\mathrm{id}_{\mathscr{C}(\mathbb{F})}$ can be regarded as a polynomial of degree 0, $\mathrm{id}_{\mathscr{C}(\mathbb{F})} \in P_n(\mathbb{F})$ for any n and so $\mathrm{id}_{\mathscr{C}(\mathbb{F})}$ is nonempty. Let p and q be in $P_n(\mathbb{F})$ and c be in $\mathbb{F}$. Then we can write

$$p(t) := \sum_{i=0}^{n} a_i t^i \quad ; \quad q(t) := \sum_{i=0}^{n} b_i t^i$$

   where the a_i and b_i are elements of $\mathbb{F}$. then

$$[p+q](t) = \sum_{i=0}^{n} (a_i + b_i) t^i \quad ; \quad [cp](t) = \sum_{i=0}^{n} (ca_i) t^i$$

   Hence $p + q$ and cp are also elements of $P_n(\mathbb{F})$. Hence $P_n(\mathbb{F})$ is closed under addition and scalar multiplication and so $P_n(\mathbb{F})$ is indeed a vector space.

   In what follows, let $P_2(\mathbb{R})$ be the vector space of polynomials of degree at most 2 and $\mathbb{V}$ denote the subset of $P_2(\mathbb{R})$ such that:

$$\int_0^1 p(t)\,\mathrm{d}t = 0.$$

   (c) Prove that $\mathbb{V}$ is a subspace of $P_2(\mathbb{R})$.

   Solution. Since $0 \in P_2(\mathbb{R})$ and $\int_0^1 0\,\mathrm{d}t = 0$, we have $0 \in \mathbb{V}$ and so $\mathbb{V}$ is nonempty. Suppose p and q are elements of bV and c is real. Then $\int_0^1 p(t)\,\mathrm{d}t = 0 \int_0^1 q(t)\,\mathrm{d}t$, and so

$$\int_0^1 (p(t) + q(t))\,\mathrm{d}t = \int_0^1 p(t)\,\mathrm{d}t + \int_0^1 q(t)\,\mathrm{d}t = 0 + 0 = 0$$

   and

$$\int_0^1 cp(t)\,\mathrm{d}t = c \int_0^1 p(t)\,\mathrm{d}t = c \cdot 0 = 0.$$

   Hence $p + q$ and cp are also elements of $\mathbb{V}$. This proves that $\mathbb{V}$ is closed under addition and multiplication, and so $\mathbb{V}$ is indeed a vector space.

(d) Construct a basis for $\mathbb{V}$ and prove it is indeed a basis. What is the dimension of $\mathbb{V}$?

Solution. Any element p of $P_2(\mathbb{R})$ can be written as $p(t) = a_0 + a_1 t + a_2 t^2$. If p is in $\mathbb{V}$, then

$$0 = \int_0^1 p(t)\,\mathrm{d}t = \int_0^1 \left(a_0 + a_1 t + a_2 t^2\right)\mathrm{d}t = a_0 + \tfrac{1}{2}a_1 + \tfrac{1}{3}a_2.$$

Hence if p is in $\mathbb{V}$, we may write $a_0 = -\frac{1}{2}a_1 - \frac{1}{3}a_2$, and so

$$p(t) = a_0 + a_1 t + a_2 t^2 + \left(-\tfrac{1}{2}a_1 - \tfrac{1}{3}a_2\right) + a_1 t + a_2 t^2 = \tfrac{a_1}{2}(2t-1) + \tfrac{a_2}{3}\left(3t^2-1\right).$$

Recall that a basis for a vector space is any set which is linearly independent and which spans the vector space. By construction, $\beta := \{2t-1, 3t^2-1\}$ spans $\mathbb{V}$, so we need only verify that β is linearly independent. But since $(2t-1) \neq c(3t^2-1)$ for any real c, the vectors are indeed linearly independent. Hence β is a basis for $\mathbb{V}$, and so $\dim \mathbb{V} = |\beta| = 2$.

2. Let $P_n(\mathbb{F})$ be the vector space of polynomials of degree at most n with coefficients in a field $\mathbb{F}$. Define the transformation $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ by

$$T(p(t)) = 2p'(t) - 3p''(t).$$

(a) Prove that T is a linear transformation.

Solution. It is sufficient to show that for any p and q in $P_3(\mathbb{R})$ and a in $\mathbb{R}$ that

$$T(ap+q) = aT(p) + T(q).$$

We have

$$\begin{aligned} T(ap+q) = 2(ap(t)+q(t))' - 3(ap(t)+q(t))'' &= 2ap'(t) + 2q'(t) - 3ap''(t) - 3q''(t) \\ &= a[2p'(t) - 3p''(t)] + [2q'(t) - 3q''(t)] = aT(p) + T(q), \end{aligned}$$

hence T is indeed linear.

(b) Find a basis for the nullspace of T.

Solution. If $p(t) := a + bt + ct^2 + dt^3$ in an element of the nullspace of T, then we have

$$0 = T(p) = 2[b + 2ct + 3dt^2] + 3[2c + 6dt] = (2b+6c) + (4c+18d)t + (6d)t^2$$

Hence $d = c = b = 0$. The constant term a need not be specified, so every term in the nullspace of T has the form $p(t) = a$ for some real a. Hence $N(T) = \mathrm{span}\{1\}$ and so a basis for the nullspace of T is $\{1\}$.

(c) Given the bases $\beta := \{1, t, t^2, t^3\}$ and $\gamma := \{1, t-1, t^2-1\}$ for $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$ respectively, determine the matrix $[T]_\beta^\gamma$ which represents T. That is, if v is a vector in $P_3(\mathbb{R})$, $w = T(v)$, and $[v]_\beta$ represents the coordinates of v in the β-basis, find the matrix $[T]_\beta^\gamma$ for which $[w]_\gamma = [T]_\beta^\gamma [v]_\beta$.

Solution. Recall that the i'th column of $[T]_\beta^\gamma$ is just $[T(\beta_i)]_\gamma$. We have

$$\begin{aligned} T(\beta_1) &= T(1\ \,) = 0 && \Longrightarrow\ [T(\beta_1)]_\gamma = (0,0,0)^\top \\ T(\beta_2) &= T(t\ \,) = 2 && \Longrightarrow\ [T(\beta_2)]_\gamma = (2,0,0)^\top \\ T(\beta_3) &= T(t^2) = 4t - 6 && \Longrightarrow\ [T(\beta_3)]_\gamma = (-1,4,0)^\top \end{aligned}$$

$$T(\beta_4) = T(t^3) = 6t^2 - 18t \implies [T(\beta_4)]_\gamma = (-12, -18, 6)^\top.$$

So,

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & 2 & -2 & -12 \\ 0 & 0 & 4 & -18 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

is the matrix which represents the transformation.

3. Let $\beta := \{(1,1),(1,-1)\}$ and $\beta' := \{(2,4),(3,1)\}$.

(a) Verify that β and β' are bases for $\mathbb{R}^2$.

Solution. Since $\dim \mathbb{R}^2 = 2$, we need only show the sets are linearly independent. Since each set only has two elements, the are (each) linearly independent if and only if neither vector is a multiple of the other. Since $(1,1) \neq c(1,-1)$ for any real c, β is linearly independent and hence a basis for $\mathbb{R}^2$. Similarly, $(2,4) \neq c(3,1)$ for any real c, so β' is linearly independent and hence is a basis for $\mathbb{R}^2$.

(b) Construct the matrix Q which changes β'-coordinates to β-coordinates. That is, if $v \in \mathbb{R}^2$ and $[v]_\beta$ represents the coordinates of v in the β-basis, determine the matrix Q such that $[v]_\beta = Q[v]_{\beta'}$.

Solution. Recall that to form the change-of-basis matrix from β' to β, we need to write the elements of the β' basis in terms of the β basis. The reader can verify that

$$(2,4) = 3(1,1) - 1(1,-1)$$
$$(3,1) = 2(1,1) + 1(1,-1).$$

Hence:

$$Q = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}$$

is the matrix which maps β'-coordinates to β-coordinates.

4. Let $M_n(\mathbb{F})$ denote the vector space of $n \times n$ matrices over a field $\mathbb{F}$ and define $T : P_2(\mathbb{R}) \to M_2(\mathbb{R})$ by

$$T(f(x)) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix}.$$

Find a basis for the range $R(T)$ of T and a basis for the nullspace $N(T)$ of T. Verify that $\dim R(T) + \dim N(T) = 3 = \dim P_2(\mathbb{R})$.

Solution. To find the range, it is sufficient to compute $T(\beta_i)$ for each β_i in a given basis $\beta \subset P_2(\mathbb{R})$ and determine its span. Let $\beta = \{1, t, t^2\}$ be the standard basis for $P_2(\mathbb{R})$. Then

$$T(1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad ; \quad T(t) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \quad ; \quad T(t^2) = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence the span is

$$\operatorname{span}\{T(1), T(t), T(t^2)\} = \operatorname{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

the generators of which can be taken as a basis for $R(T)$. To determine a basis for $N(T)$, we suppose $p(t) = a + bt + ct^2$ is an element of $N(T)$ and determine conditions on a, b, and c. Indeed,

$$0 = T(p(t)) = \begin{pmatrix} -b-3c & 0 \\ 0 & a \end{pmatrix}.$$

For equality to hold, we require $a = 0$ and $b = -3c$. Hence, if $p(t) = a + bt + ct^2$ is an element of $N(T)$, then

$$p(t) = (-3c)t + ct^2 = c(t^2 - 3t).$$

Thus $N(T) = \operatorname{span}\{t^2 - 3t\}$ and so a basis for $N(T)$ is $\{t^2 - 3t\}$. Note that $\dim R(T) + \dim N(T) = 2 + 1 = 3 = \dim P_2(\mathbb{R})$, as expected.

5. Define the following matrix.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

Determine the eigenvalues of A. Use these eigenvalues to write $A = PDP^{-1}$ where D is a diagonal matrix and P is nonsingular.

Solution. The eigenvalues of A are just the roots λ of the characteristic polynomial $f_A(t) := \det(A - t\mathbb{I})$, where in this case $\mathbb{I}$ is the 3×3 identity. We have

$$f_A(t) = \left| \begin{pmatrix} 1-t & 1 & 1 \\ 0 & 1-t & 0 \\ 0 & 1 & 2-t \end{pmatrix} \right| = (1-t)^2(2-t),$$

and so the eigenvalues are $\lambda_1 = 1$ (with *algebraic multiplicity* 2) and $\lambda_2 = 2$ (with *algebraic multiplicity* 1). We next construct a set of eigenvectors of A. For simplicity, we start with $\lambda_2 = 2$. An associated eigenvector is a nonzero vector $v_2 = (x_2, y_2, z_2)^\top$ such that $(A - \lambda_2\mathbb{I})v_2 = 0$; that is,

$$\begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The second and third rows require $y_2 = 0$, and so the first requires $x_2 = z_2$. Setting $x_2 = 1$, we get an eigenvector of the form $v_2 = (1, 0, 1)^\top$. Moving to $\lambda_1 = 1$, the associated eigenvectors are of the form $v_1 = (x_1, y_1, z_1)^\top \neq 0$ such that $(A - \lambda_1\mathbb{I})v_1 = 0$; that is,

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

There are two degrees of freedom here, since the first and third rows require $x_2 = -x_3$ and the second requires nothing of x_1. Setting $x_1 = \alpha$ and $x_2 = \beta$ for arbitrary real α and β, we find that the λ_1-eigenspace of A is the set of all vectors of the form $\alpha(1,0,0)^\top + \beta(0,1,-1)^\top$, i.e., that the eigenspace has a basis $\{(1,0,0)^\top, (0,1,-1)^\top\}$. Hence λ_1 has *geometric multiplicity* 2. Since the algebraic multiplicity of λ_1 equals the geometric multiplicity of λ_1, A is indeed diagonalizable, and we have

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}^{-1}}_{P}$$

which is the decomposition we wanted.

6. Suppose $\{v_1, \ldots, v_n\}$ is an orthogonal set of vectors. Let $\|\cdot\|$ denote the norm generated by the inner product $\langle \cdot, \cdot \rangle$. Prove
$$\left\| \sum_{i=1}^{n} a_i v_i \right\|^2 = \sum_{i=1}^{n} |a_i|^2 \|v_i\|^2,$$
where $a_1, \ldots, a_n$ are scalars.

Solution. For any vector v in an inner product space, recall $\|v\|^2 = \langle v, v \rangle$. Using the linearity in the right argument and conjugate linearity in the left, we have
$$\left\| \sum_{i=1}^{n} a_i v_i \right\|^2 = \left\langle \sum_{i=1}^{n} a_i v_i, \sum_{j=1}^{n} a_j v_j \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i^* a_j \langle v_i, v_j \rangle.$$
If $\{v_1, \ldots, v_n\}$ is an orthogonal set, then $\langle v_i, v_j \rangle = 0$ whenever $i \neq j$. Hence, we have
$$\left\| \sum_{i=1}^{n} a_i v_i \right\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i^* a_j \langle v_i, v_j \rangle = \sum_{i=1}^{n} a_i^* a_i \langle v_i, v_i \rangle = \sum_{i=1}^{n} |a_i|^2 \|v_i\|^2,$$
which completes the proof.

7. Let $\mathbb{V}$ be a finite-dimensional inner product space over $\mathbb{C}$ and suppose $T : \mathbb{V} \to \mathbb{V}$ is linear. Then there exists a unique linear transformation $T^* : \mathbb{V} \to \mathbb{V}$ (called the *adjoint* of T) such that $\langle T(v), w \rangle = \langle v, T^*(w) \rangle$ for all v, w in $\mathbb{V}$. Using only the definition above and the axioms of an inner product, show that if $T = T^*$, then the eigenvalues of T are real.

Solution. Suppose v is an eigenvector of T, i.e., suppose that there exists a $\lambda \in \mathbb{C}$ such that $T(v) = \lambda v$. Note that since v is an eigenvector, $v \neq 0$. Then
$$\lambda \langle v, v \rangle = \langle v, \lambda v \rangle = \langle v, T(v) \rangle = \langle T(v), v \rangle = \langle \lambda v, v \rangle = \lambda^* \langle v, v \rangle.$$
Hence $(\lambda - \lambda^*) \langle v, v \rangle = 0$. Since $v \neq 0$, $\langle v, v \rangle \neq 0$; hence $\lambda = \lambda^*$, i.e., λ is real.

8. Consider the vector space $\mathbb{V} := \{p(t) = a + bt^2 \mid a, b \in \mathbb{R}\}$. Let ω_1 and ω_2 be linear functionals on $\mathbb{V}$ such that $\omega_1\{p(t)\} = p(1)$ and $\omega_2\{p(t)\} = p(2)$. Find the basis for $\mathbb{V}$ for which $\{\omega_1, \omega_2\}$ is the dual basis.

Solution. Suppose the basis for which $\{\omega_1, \omega_2\}$ is dual is $\{v_1, v_2\}$. Then by definition, we have $\omega_i\{v_j\} = \delta_{ij}$, the Kronecker delta. Given the structure of $\mathbb{V}$, we know $v_j = a_j + b_j t^2$ for as-yet unknown constants a_j and b_j, $j = 1, 2$. This gives us four equations:
$$\begin{aligned} \omega_1\{v_1\} &= v_1(1) = a_1 + b_1 = \delta_{11} = 1 \qquad & \omega_1\{v_2\} &= v_2(1) = a_2 + b_2 = \delta_{21} = 0 \\ \omega_2\{v_1\} &= v_1(2) = a_1 + 4b_1 = \delta_{12} = 0 \qquad & \omega_2\{v_2\} &= v_2(2) = a_2 + 4b_2 = \delta_{22} = 1 \end{aligned}$$
Going through the algebra, one finds that
$$\{v_1, v_2\} = \left\{\tfrac{4}{3} - \tfrac{1}{3}t^2, -\tfrac{1}{3} + \tfrac{1}{3}t^2\right\}$$
is the basis for $\mathbb{V}$ to which $\{\omega_1, \omega_2\}$ is dual.