Linear Algebra

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Definition 1 (Set of linear maps)**.** We say $T \in \mathscr{L}(\mathbf{V}, \mathbf{W})$ if T is a linear transformation from a vector space $\mathbf{V}$ to a vector space $\mathbf{W}$. We say $T \in \mathscr{L}(\mathbf{V})$ if $T \in \mathscr{L}(\mathbf{V}, \mathbf{V})$.

Definition 2 (Coordinate vector)**.** Let x be a vector in a vector space $\mathbf{V}$ over $\mathbb{F}$ and suppose $\beta = \{v_1, \ldots, v_n\}$ is a basis for $\mathbf{V}$. Write $x = \sum_{i=1}^{n} a_i v_i$ for unique scalars $\{a_i\}_{i=1}^{n} \subset \mathbb{F}$. Then

$$[x]_\beta := \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{F}^n$$

is called the coordinate vector of x relative to β.

Definition 3 (Matrix representations)**.** Suppose $\mathbf{V}$ and $\mathbf{W}$ are vector spaces with standard ordered bases β and γ, respectively. If $w = T(v)$, then the matrix $[T]_\beta^\gamma$ such that $[w]_\gamma = [T]_\beta^\gamma [v]_\beta$ is called the matrix representation of T in the ordered bases β and γ. If $\mathbf{V} = \mathbf{W}$ and $\beta = \gamma$, we write $[T]_\beta^\gamma = [T]_\beta$.

Definition 4 (Left-multiplication transformation)**.** Let $A \in M_{m,n}(\mathbb{F})$. We denote by L_A the mapping $\mathrm{L}_A : \mathbb{F}^n \to \mathbb{F}^m$ defined by $\mathrm{L}_A(x) = Ax$ (the matrix product of A and x).

Definition 5 (Isomorphism)**.** Let $\mathbf{V}$ and $\mathbf{W}$ be finite dimensional vector spaces. An isomorphism between $\mathbf{W}$ and $\mathbf{W}$ is a linear transformation $T \in \mathscr{L}(\mathbf{V}, \mathbf{W})$ such that T has an inverse $T^{-1} \in \mathscr{L}(\mathbf{W}, \mathbf{V})$. If such an isomorphism exists, we say $\mathbf{V}$ and $\mathbf{W}$ are isomorphic.

Problem Set 2: Linear Transformations and Matrices

1. Prove that the composition of linear transformations is a linear transformation. In particular, if $\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{V}_3$ are vector spaces over a common field $\mathbb{F}$, and $T_1 \in \mathscr{L}(\mathbf{V}_1, \mathbf{V}_2)$ and $T_2 \in \mathscr{L}(\mathbf{V}_2, \mathbf{V}_3)$, show that $T_2 \circ T_1 : \mathbf{V}_1 \to \mathbf{V}_3$ satisfies

$$T_2 \circ T_1(ax + y) = aT_2 \circ T_1(x) + T_2 \circ T_1(y)$$

   for any x, y in $\mathbf{V}_1$ and a in $\mathbb{F}$.

2. **(a)** Prove that every vector space of dimension n over a field $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$ by exhibiting an isomorphism. (Make sure to prove that your linear transformation is indeed an isomorphism.)

   (b) Show that two finite dimensional vector spaces are isomorphic if and only if they have the same dimension.

3. Let $T \in \mathscr{L}(\mathbf{V}, \mathbf{W})$ where $\mathbf{V}$ and $\mathbf{W}$ are n- and m-dimensional vector spaces over $\mathbb{F}$ with ordered bases β and γ, respectively. Let $\phi_\beta \in \mathscr{L}(\mathbf{V}, \mathbb{F}^n)$ be such that $\phi_\beta(v) = [v]_\beta$ and $\phi_\gamma \in \mathscr{L}(\mathbf{W}, \mathbb{F}^m)$ be such that $\phi_\gamma(w) = [w]_\gamma$. Write T in terms of ϕ_β, ϕ_γ, and the left-multiplication transformation L_A where $A = [T]_\beta^\gamma$.

   [*This problem and its predecessor shows why we are so concerned with $\mathbb{R}^n$ in linear algebra. All of finite-dimensional linear algebra over the reals can be done in terms of $\mathbb{R}^n$.*]

4. Let B be an $n \times n$ invertible matrix and define $\Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ by $\Phi(A) = B^{-1}AB$. Prove that Φ is an isomorphism.

5. In this problem we are going to deduce the rule for matrix multiplication. Let $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$ be p, n, m-dimensional vector spaces with ordered bases β_1, β_2, β_3, respectively. Let $T_{12} \in \mathscr{L}(\mathbf{V}_1, \mathbf{V}_2)$ and $T_{23} \in \mathscr{L}(\mathbf{V}_2, \mathbf{V}_3)$. We want to develop a multiplication rule such that

$$[T_{23} \circ T_{12}]_{\beta_1}^{\beta_3} = [T_{23}]_{\beta_2}^{\beta_3}[T_{12}]_{\beta_1}^{\beta_2}.$$

   For simplicity, let $A = [T_{23}]_{\beta_2}^{\beta_3}$, $B = [T_{12}]_{\beta_1}^{\beta_2}$, and $C = [T_{23} \circ T_{12}]_{\beta_1}^{\beta_3}$.

   (a) What are the sizes of A, B, and C in terms of m, n, and p? Does this agree with your understanding of matrix multiplication?

   Let $\beta_1 := \{v_1, \dots, v_p\}$, $\beta_2 := \{w_1, \dots, w_n\}$, and $\beta_3 := \{u_1, \dots, u_m\}$ be ordered bases for $\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{V}_3$, respectively.

   (b) Write an expression for $T_{12}(v_j)$ in terms of the matrix elements b_{ij} of B and the elements of β_2. Do the same for $T_{23}(w_k)$. Finally, write an expression for $T_{23} \circ T_{12}(v_j)$ in terms of the matrix elements c_{ij} of C and the elements of β_3.

   (c) Using the linearity of the composition $T_{23} \circ T_{12}$ to write an expression for $T_{23} \circ T_{12}(v_j)$ in terms of the elements of β_3. Your answer should depend on a_{ij} and b_{ij}.

   (d) Compare your expressions for $T_{23} \circ T_{12}(v_j)$ from part (b) and part (c) to deduce the rule for matrix multiplication.

6. Let $g(x) = x + 3$, and let $T \in \mathscr{L}(P_2(\mathbb{R}))$ and $U \in \mathscr{L}(P_2(\mathbb{R}), \mathbb{R}^3)$ be defined by

$$\begin{aligned} T(f(x)) &= f'(x)g(x) + 2f(x) \\ U(a + bx + cx^2) &= (a + b, c, a - b)^\top. \end{aligned}$$

   Let β and γ be the standard ordered bases for $P_2(\mathbb{R})$ and $\mathbb{R}^3$, respectively.

   (a) Compute $[U]_\beta^\gamma$, $[T]_\beta$, and $[U \circ T]_\beta^\gamma$ directly.

   (b) Use the previous problem to verify your result.

7. Let $\mathbf{V}$ and $\mathbf{W}$ be finite dimensional vector spaces with the same dimension with ordered bases β and γ, respectively. Let $T \in \mathscr{L}(\mathbf{V}, \mathbf{W})$. Prove that T is invertible if and only if $[T]_\beta^\gamma$ is invertible. Further show that $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.

8. The benefit of changing coordinate systems is that you can change coordinates into a set which optimizes efficiency, perform the relevant computations *in that system*, and then transform back into the original coordinates. Let $T \in \mathscr{L}(\mathbf{V})$ and suppose β and β' are ordered bases for $\mathbf{V}$. If $Q = [I_\mathbf{V}]_{\beta'}^\beta$ is the change of coordinate matrix that changes β'-coordinates into β coordinates, prove $[T]_{\beta'} = Q^{-1}[T]_\beta Q$.