# Homework # 09

Math 140
Professor John Lott

By

Cody Vig

*The University of California*
*Berkeley*

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## Chapter 04 | Problem 8.39

Prove that the curvature of a ruled surface is never positive.

### Solution

A *ruled surface* is one that can be parameterized with

$$\boldsymbol{x}(s,t) = \boldsymbol{\alpha}(s) + t\boldsymbol{\beta}(s)$$

where $\boldsymbol{\alpha}$ is a unit-speed curve and $\|\boldsymbol{\beta}(s)\| \equiv 1$. Note that

$$L^i{}_j = \mathsf{g}^{i\ell} L_{\ell k} \implies (L^i{}_j) = (\mathsf{g}^{ij})(L_{ij})$$

and so $K = \det(L^i{}_j) = \det(\mathsf{g}^{ij})\det(L_{ij}) = \det(\det(L_{ij})/\mathsf{g}$. Since the metric is positive definite, $\det(\mathsf{g}_{ij}) > 0$ and so the Gaussian curvature and the determinant of the second fundamental form have the same sign. If we can show that the determinant of the second fundamental form is everywhere negative, then we are done. We have

$$\boldsymbol{x}_1 := \frac{\partial \boldsymbol{x}}{\partial s} = \boldsymbol{\alpha}' + t\boldsymbol{\beta}'$$
$$\boldsymbol{x}_2 := \frac{\partial \boldsymbol{x}}{\partial t} = \boldsymbol{\beta}$$

and so

$$\boldsymbol{n} = \frac{\boldsymbol{\alpha}' \times \boldsymbol{\beta} + t\boldsymbol{\beta}' \times \boldsymbol{\beta}}{\|\boldsymbol{\alpha}' \times \boldsymbol{\beta} + t\boldsymbol{\beta}' \times \boldsymbol{\beta}\|}.$$

The $\boldsymbol{x}_{ij}$'s are

$$\boldsymbol{x}_{11} = \boldsymbol{\alpha}'' + t\boldsymbol{\beta}''$$
$$\boldsymbol{x}_{12} = \boldsymbol{x}_{21} = \boldsymbol{\beta}'$$
$$\boldsymbol{x}_{22} = \boldsymbol{0}.$$

The coefficients of the second fundamental form are defined by $L_{ij} = \langle \boldsymbol{x}_{ij}, \boldsymbol{n}\rangle$. As such, $L_{22} = \langle \boldsymbol{0}, \boldsymbol{n}\rangle = 0$, so $\det(L_{ij}) = -L_{21}L_{12} = -(L_{12})^2$ since the second fundamental form is symmetric. Further, $L_{12} = \langle \boldsymbol{x}_{12}, \boldsymbol{n}\rangle$ is real, and so its square is non-negative. Hence,

$$K = \det(L^i{}_j) = -\frac{(L_{12})^2}{\mathsf{g}} < 0,$$

and so the curvature is everywhere non-positive.

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## Chapter 04 | Problem 8.40

Let $\boldsymbol{x}(s,t) = \boldsymbol{\alpha}(s) + t\boldsymbol{\beta}(s)$ be a non-cylindrical ruled surface. Prove that there exists a unique curve $\boldsymbol{\gamma}(s) = \boldsymbol{\alpha}(s) + r(s)\boldsymbol{\beta}(s)$, which in general is not unit speed, such that $\langle \boldsymbol{\gamma}', \boldsymbol{\beta}'\rangle \equiv 0$. This is called the *line of striction*.

### Solution

If such a $\boldsymbol{\gamma}$ exists, we would have

$$0 = \langle \boldsymbol{\gamma}', \boldsymbol{\beta}'\rangle = \langle \boldsymbol{\alpha}' + r'\boldsymbol{\beta} + r\boldsymbol{\beta}', \boldsymbol{\beta}'\rangle = \langle \boldsymbol{\alpha}', \boldsymbol{\beta}'\rangle + r'\langle \boldsymbol{\beta}, \boldsymbol{\beta}'\rangle + r\langle \boldsymbol{\beta}', \boldsymbol{\beta}'\rangle.$$

By definition, $\|\boldsymbol{\beta}\| = 1$, and so $\langle \boldsymbol{\beta}', \boldsymbol{\beta} \rangle = 0$. Additionally, since $\boldsymbol{x}$ is non-cylindrical, we have $\langle \boldsymbol{\beta}', \boldsymbol{\beta}' \rangle = \|\boldsymbol{\beta}'(s)\|^2 \neq 0$ for any $s$. Hence

$$r(s) = -\frac{\langle \boldsymbol{\alpha}', \boldsymbol{\beta}' \rangle}{\|\boldsymbol{\beta}'\|^2}.$$

Since the choice of $r(s)$ is unique for each choice of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, the choice of $\boldsymbol{\gamma}$ is unique.

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## Chapter 04 | Problem 8.45

Let $\boldsymbol{x}(s,t) = \boldsymbol{\alpha}(s) + t\boldsymbol{\beta}(s)$ be a developable surface. Show that the normal vector $\boldsymbol{n}$ does not depend upon the parameter $t$.

### Solution

A *developable surface* is a ruled surface for which the tangent planes are parallel along each line of the ruling. The lines of ruling are described by $\boldsymbol{x}(s_0, t)$ for fixed $s_0$. Let $\boldsymbol{n}(s,t)$ denote the normal vector to the tangent plane of the surface described by $\boldsymbol{x}$ at the point $(s,t)$. Then for each $s = s_0$, we have $\boldsymbol{n}(s_0,t) = \boldsymbol{n}(s_0,t')$ for all $t = t'$.[1] Hence $\boldsymbol{n}$ cannot depend on $t$.

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## Chapter 04 | Problem 8.46

Let $\boldsymbol{x}(s,t) = \boldsymbol{\alpha}(s) + t\boldsymbol{\alpha}'(s)$ be a *tangent developable surface* of a unit speed curve $\boldsymbol{\alpha}$ (Problem 1.14). Show that $\boldsymbol{x}$ is developable.

### Solution

In the previous problem, we proved that if a surface described by $\boldsymbol{x}(s,t) = \boldsymbol{\alpha}(s) + t\boldsymbol{\beta}(s)$ is developable, then its normal vectors are independent of $t$. It is not too difficult to see that the converse is also true.

**Theorem.** *A surface described locally by the coordinate patch $\boldsymbol{x}(s,t) = \boldsymbol{\alpha}(s) + t\boldsymbol{\beta}(s)$ is developable if and only if the normal vectors are everywhere independent of $t$.*

*Proof.* The forward implication was proved in Problem 8.45. The reverse implication is similar. Indeed, if the normal vectors are everywhere independent of $t$, then along each line of constant $s$, they must be parallel along each line of constant $s$. As such, the tangent planes along each line of constant $s$ are parallel, and so the surface is developable. □

We can use this fact to easily show that the tangent developable surface of $\boldsymbol{\alpha}$ is also developable. Indeed, we get

$$\boldsymbol{x}_1 := \frac{\partial \boldsymbol{x}}{\partial s} = \boldsymbol{\alpha}' + t\boldsymbol{\alpha}''$$
$$\boldsymbol{x}_2 := \frac{\partial \boldsymbol{x}}{\partial t} = \boldsymbol{\alpha}'$$

[1] I suppose it is also possible that $\boldsymbol{n}(s_0,t) = -\boldsymbol{n}(s_0,t')$, but if we require continuity, then this situation s impossible.

and so $\boldsymbol{x}_1 \times \boldsymbol{x}_2 = \boldsymbol{\alpha}' \times \boldsymbol{\alpha}' + t\boldsymbol{\alpha}'' \times \boldsymbol{\alpha}' = t\boldsymbol{\alpha}'' \times \boldsymbol{\alpha}'$. Since $\boldsymbol{\alpha}$ is unit-speed, $\|\boldsymbol{\alpha}'(s)\| \equiv 1$ everywhere, and so $\boldsymbol{\alpha}''(s)$ is normal to $\boldsymbol{\alpha}'(s)$. Hence, $\|\boldsymbol{x}_1 \times \boldsymbol{x}_2\| = |t|\|\boldsymbol{\alpha}''\|\|\boldsymbol{\alpha}'\|$, and so

$$\boldsymbol{n} = \pm\frac{\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''}{\|\boldsymbol{\alpha}'\|\|\boldsymbol{\alpha}''\|},$$

which is indeed independent of $t$.

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## Chapter 04 | Problem 9.1

Show that if $M = \mathbb{R}^2$, then $R_i{}^\ell{}_{jk} = 0$ for all $0 \leq i, \ell, j, k \leq 2$ both intrinsically and extrinsically.

### Solution

The Riemannian curvature tensor is defined in terms of the Christoffel symbols as follows;

$$R_i{}^\ell{}_{jk} = \frac{\partial \Gamma_{ik}{}^\ell}{\partial u^j} - \frac{\partial \Gamma_{ij}{}^\ell}{\partial u^k} + \Gamma_{ik}{}^p\Gamma_{pj}{}^\ell - \Gamma_{ij}{}^p\Gamma_{pk}{}^\ell. \tag{1}$$

(*Here Einstein summation convention assumed.*) From Gauss, we also know

$$R_i{}^\ell{}_{jk} = L_{ik}L^\ell{}_j - L_{ik}L^\ell{}_k \tag{2}$$

These expressions for the Riemann curvature tensor will be sufficient to solve the problem.

**Intrinsic.**

We know (again from Gauss) that the Christoffel symbols can be defined intrinsically using the metric:

$$\Gamma_{ij}{}^k = \frac{1}{2}\mathsf{g}^{k\ell}\left(\frac{\partial \mathsf{g}_{j\ell}}{\partial u^k} + \frac{\partial \mathsf{g}_{\ell i}}{\partial u^j} - \frac{\partial \mathsf{g}_{ij}}{\partial u^\ell}\right)$$

For $M = \mathbb{R}^2$, we have $\mathsf{g}_{ij} = \delta_{ij}$. In particular, the components are constant, and so $\frac{\partial \mathsf{g}_{ij}}{\partial u^k} = \frac{\partial \delta_{ij}}{\partial u^k} \equiv 0$. Hence each $\Gamma_{ij}{}^k = 0$, and so all $R_i{}^\ell{}_{jk} = 0$ by Eq. (1).

**Extrinsic.**

We can cover $\mathbb{R}^2 \subset \mathbb{R}^3$ in Monge patches of the form $\boldsymbol{x}(u^1, u^2) = (u^1, u^2, 0)$. Then each $\boldsymbol{x}_{ij} = 0$. This implies

$$L_{ij} = \langle \boldsymbol{x}_{ij}, \boldsymbol{n}\rangle = 0$$

for all $1 \leq i, j \leq 2$. Additionally, $L^i{}_j = \mathsf{g}^{i\ell}L_{\ell j} = 0$ as well. From Eq. (2), we find all $R_i{}^\ell{}_{jk} = 0$, as expected.

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