Homework # 10

Math 140
Professor John Lott

By

Cody Vig

The University of California
Berkeley

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Chapter 04 | Problem 8.52

Show that a developable surface has Gaussian curvature equal to zero.

Solution

Recall that we can write a developable surface locally as

$$\boldsymbol{x}(s,t) = \boldsymbol{\alpha}(s) + t\boldsymbol{\beta}(s)$$

where $\boldsymbol{\alpha}$ is unit speed, $\|\boldsymbol{\beta}\| = 1$, and the tangent planes are parallel along each line of the ruling (i.e., along the lines described by fixing $s = s_0$). In Problem 8.45 of Chapter 4[1] we proved that if a surface is developable, then the normal vectors at every point are independent of t. In particular,

$$\boldsymbol{n}_t \equiv \boldsymbol{n}_2 = \boldsymbol{0}$$

everywhere. Since $L^i{}_j \boldsymbol{x}_i = -\boldsymbol{n}_j$, we find

$$L^1{}_2\boldsymbol{x}_1 + L^2{}_2\boldsymbol{x}_2 = -\boldsymbol{n}_2 = \boldsymbol{0} \quad \Longrightarrow \quad L^1{}_2 = L^2{}_2 = 0$$

since the $\boldsymbol{x}_i$'s are linearly independent. By symmetry, this also implies $L^2{}_1 = 0$ and so

$$K = \det(L^i{}_j) = 0,$$

which completes the proof.

Chapter 04 | Problem 10.2

Let $S^2(r) = \{\boldsymbol{x} \in \mathbb{R}^3 : \|\boldsymbol{x}\| = r\}$ and let $f : S^2 \to S^2(r)$ by $f(\boldsymbol{x}) = r\boldsymbol{x}$. Prove that f is one-to-one and onto but not an isometry.

Solution

To show f is one-to-one and onto, we need only show that f is invertible. If $r = 0$, then the map is not one-to-one, since $f(\boldsymbol{e}_1) = f(\boldsymbol{e}_2) = \boldsymbol{0}$ where $\{\boldsymbol{e}_i\}_{i=1}^2$ is the standard basis for $\mathbb{R}^2$. Hence, assume $r \neq 0$. Then I claim

$$f^{-1}(\boldsymbol{x}) = \tfrac{1}{r}\boldsymbol{x}.$$

Proof. Let $g(\boldsymbol{x}) = \frac{1}{r}\boldsymbol{x}$. For each $r \neq 0$, we have $f(g(\boldsymbol{x})) = f(\frac{1}{r}\boldsymbol{x}) = \frac{r}{r}\boldsymbol{x} = \boldsymbol{x}$, and $g(f(\boldsymbol{x})) = g(r\boldsymbol{x}) = \frac{r}{r}\boldsymbol{x} = \boldsymbol{x}$. Hence $g = f^{-1}$. □

Hence f is indeed one-to-one and onto. Next, we show that f is not an isometry. To do this, we need only find one curve $\boldsymbol{\gamma}$ on S^2 for which $L(\boldsymbol{\gamma}) \neq L(f(\boldsymbol{\gamma}))$.Perhaps the most simple counterexample is a meridian. Let $\boldsymbol{\gamma}$ be any meridian on S^2. Then

$$L(\gamma) = 2\pi,$$

since they are all circles of radius 1. But then $f(\boldsymbol{\gamma}) = r\boldsymbol{\gamma}$ is a meridian on $S^2(r)$ and so

$$L(f(\boldsymbol{\gamma})) = 2\pi r$$

which is not equal to $L(\boldsymbol{\gamma})$ for any $r \neq 1$, and so f is generally not an isometry.

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[1] The third problem of Homework 09

Chapter 04 | Problem 10.3

Show that $\mathbb{R}^2$ and the cylinder $S^1 \times (-\infty, \infty)$ are locally isometric.

Solution

From Proposition 10.5 in Millman and Parker, two surfaces are locally isometric if and only if their metric components are equal within corresponding open sets. Hence if we compute the metric components on each surface and find that they are equal, then the surfaces must be locally isometric there. Let g_{ij} denote the metric components on $\mathbb{R}^2$ and let $\tilde{\mathsf{g}}_{ij}$ denote the metric components on $S^1 \times \mathbb{R}$. Then

$$\mathsf{g}_{ij} = \delta_{ij}$$

since $\mathbb{R}^2$ can be locally parameterized with the Monge patch $\boldsymbol{x}(u^1, u^2) = (u^1, u^2, 0)$ for $(u^1, u^2) \in \mathscr{U}$ an open subset of $\mathbb{R}^2$, and so $\boldsymbol{x}_1 = (1, 0, 0)$ and $\boldsymbol{x}_2 = (0, 1, 0)$ are orthonormal. Using the same open domain $\mathscr{U} \in \mathbb{R}^2$, we can define a coordinate patch for $S^1 \times \mathbb{R}$ with

$$\tilde{\boldsymbol{x}}(u^1, u^2) = (\cos u^1, \sin u^1, u^2).$$

Then we get

$$\begin{aligned} \tilde{\boldsymbol{x}}_1 &= (-\sin u^1, \cos u^1, 0) \\ \tilde{\boldsymbol{x}}_2 &= (0, 0, 1). \end{aligned}$$

Hence the metric components are

$$\begin{aligned} \tilde{\mathsf{g}}_{11} &= \langle \tilde{\boldsymbol{x}}_1, \tilde{\boldsymbol{x}}_1 \rangle = 1 \\ \tilde{\mathsf{g}}_{21} = \tilde{\mathsf{g}}_{12} &= \langle \tilde{\boldsymbol{x}}_1, \tilde{\boldsymbol{x}}_2 \rangle = 0 \\ \tilde{\mathsf{g}}_{22} &= \langle \tilde{\boldsymbol{x}}_2, \tilde{\boldsymbol{x}}_2 \rangle = 1 \end{aligned}$$

Hence $\tilde{\mathsf{g}}_{ij} = \delta_{ij} = \mathsf{g}_{ij}$ and so $\mathbb{R}^2$ and $S^1 \times \mathbb{R}$ are locally isometric.

Chapter 04 | Problem 10.6

Find an example to show that the following statement is false: If M is locally isometric to N, then the mean curvature of M is equal to the mean curvature of N at corresponding points.

Solution

The previous problem (Problem 10.3) presents a perfect counterexample to this claim. Using the same notation as we used in that problem, we find

$$L_{ij} = \langle \boldsymbol{x}_{ij}, \boldsymbol{n} \rangle = \langle \boldsymbol{0}, \boldsymbol{n} \rangle = 0$$

for all $1 \leq i, j \leq 2$. Since $L^i{}_j = \mathsf{g}^{i\ell} L_{\ell j}$, we find $L^i{}_j = 0$ everywhere too, and so

$$H = \frac{1}{2} \operatorname{tr}(L^i{}_j) = 0$$

on $\mathbb{R}^2$. On $S^1 \times \mathbb{R}$, we have $\tilde{\boldsymbol{x}}_{11} = (-\cos u^1, -\sin u^1, 0)$ and $\tilde{\boldsymbol{x}}_{ij} = \mathbf{0}$ otherwise, and

$$\tilde{\boldsymbol{n}} = \frac{\tilde{\boldsymbol{x}}_1 \times \tilde{\boldsymbol{x}}_2}{\|\tilde{\boldsymbol{x}}_1 \times \tilde{\boldsymbol{x}}_2\|} = (\cos u^1, \sin u^1, 0)$$

and so $\tilde{L}^i{}_1 \tilde{\boldsymbol{x}}_i = -\tilde{\boldsymbol{n}}_1$ implies

$$\tilde{L}^1_1(-\sin u^1, \cos u^1, 0) + \tilde{L}^2_1(0,0,1) = (\sin u^1, -\cos u^1, 0)$$

and so $\tilde{L}^1_1 = -1$ and $\tilde{L}^2_1 = 0$. Similarly, $\tilde{L}^i{}_2 \tilde{\boldsymbol{x}}_i = -\tilde{\boldsymbol{n}}_2$ implies

$$\tilde{L}^1_2 \tilde{\boldsymbol{x}}_1 + \tilde{L}^2_2 \tilde{\boldsymbol{x}}_2 = \mathbf{0}$$

and so $\tilde{L}^1_2 = \tilde{L}^2_2 = 0$ since the $\tilde{\boldsymbol{x}}_i$'s are linearly independent. In particular, this implies

$$\tilde{H} = \tfrac{1}{2}\operatorname{tr}(\tilde{L}^i{}_j) = \tfrac{1}{2} \neq 0 = H$$

and so even though $\mathbb{R}^2$ and $S^1 \times \mathbb{R}$ are locally isometric, they do not have identical mean curvature.

Chapter 04 | Problem 11.4

Use the trigonometric substitution $aAe^{at} = \sin\phi$ to compute

$$\int_0^s \sqrt{1 - a^2A^2e^{2at}}\,\mathrm{d}t$$

and then parameterize the tractrix as a function of ϕ.

Solution

Letting ϕ be such that $aAe^{at} = \sin\phi$, we find $a^2Ae^{at}\,\mathrm{d}t = \cos\phi\,\mathrm{d}\phi$ and so

$$\mathrm{d}t = \frac{1}{a}\frac{\cos\phi}{aAe^{at}}\,\mathrm{d}\phi = \frac{1}{a}\frac{\cos\phi}{\sin\phi}\,\mathrm{d}\phi\,.$$

Further, $\phi(t=0) = aA$ and $\phi(t=s) = \sin^{-1}(aAe^{at}) \equiv \phi_s$. Then the integral becomes

$$\begin{aligned}
\int_0^s \sqrt{1 - a^2A^2e^{2at}}\,\mathrm{d}t &= \frac{1}{a}\int_{aA}^{\phi_s} \sqrt{1 - \sin^2\phi}\frac{\cos\phi}{\sin\phi}\,\mathrm{d}\phi \\
&= \frac{1}{a}\int_{aA}^{\phi_s} \frac{1 - \sin^2\phi}{\sin\phi}\,\mathrm{d}\phi \\
&= \frac{1}{a}\int_{aA}^{\phi_s} \left\{\frac{1}{\sin\phi} - \sin\phi\right\}\mathrm{d}\phi \\
&= \frac{1}{a}\left[\log\left(\frac{\sin\phi}{1+\cos\phi}\right) + \cos\phi\right]\Bigg|_{aA}^{\phi_s} \\
&= \frac{1}{a}\left[\log\left(\frac{\sin\phi_s}{1+\cos\phi_s}\right) + \cos\phi_s\right] - \frac{1}{a}\left[\log\left(\frac{\sin aA}{1+\cos aA}\right) + \cos aA\right].
\end{aligned}$$

We could then determine $\sin\phi_s$ and $\cos\phi_s$ from the equality $\phi_s = \sin^{-1}(aAe^{at})$, but since the question asks us to determine the tractrix in terms of ϕ, I will leave it as-is. The tractrix is then the curve $\boldsymbol{\alpha} = (r(s), z(s))$ where $r(s) = Ae^{as} = \sin(\phi)/a$ and $z(s)$ is given up to sign by the integral above, that is

$$\begin{cases} r(s) &= \dfrac{1}{a}\sin\phi \\ z(s) &= \pm\dfrac{1}{a}\left[\log\left(\dfrac{\sin\phi}{1+\cos\phi}\right) + \cos\phi\right] \mp z_0 \end{cases}$$

where

$$z_0 = \frac{1}{a}\left[\log\left(\frac{\sin aA}{1+\cos aA}\right) + \cos aA\right]$$