Linear Algebra

Cody Vig

January 2022

Definition 1 (Set of linear maps)**.** We say $T \in \mathscr{L}(\mathbf{V}, \mathbf{W})$ if T is a linear transformation from a vector space $\mathbf{V}$ to a vector space $\mathbf{W}$. We say $T \in \mathscr{L}(\mathbf{V})$ if $T \in \mathscr{L}(\mathbf{V}, \mathbf{V})$.

Definition 2 (Coordinate vector)**.** Let x be a vector in a vector space $\mathbf{V}$ over $\mathbb{F}$ and suppose $\beta = \{v_1, \ldots, v_n\}$ is a basis for $\mathbf{V}$. Write $x = \sum_{i=1}^n a_i v_i$ for unique scalars $\{a_i\}_{i=1}^n \subset \mathbb{F}$. Then

$$[x]_\beta := \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{F}^n$$

is called the coordinate vector of x relative to β.

Definition 3 (Matrix representations)**.** Suppose $\mathbf{V}$ and $\mathbf{W}$ are vector spaces with standard ordered bases β and γ, respectively. If $w = T(v)$, then the matrix $[T]_\beta^\gamma$ such that $[w]_\gamma = [T]_\beta^\gamma [v]_\beta$ is called the matrix representation of T in the ordered bases β and γ. If $\mathbf{V} = \mathbf{W}$ and $\beta = \gamma$, we write $[T]_\beta^\gamma = [T]_\beta$.

Definition 4 (Left-multiplication transformation)**.** Let $A \in M_{m,n}(\mathbb{F})$. We denote by L_A the mapping $\mathrm{L}_A : \mathbb{F}^n \to \mathbb{F}^m$ defined by $\mathrm{L}_A(x) = Ax$ (the matrix product of A and x).

Definition 5 (Isomorphism)**.** Let $\mathbf{V}$ and $\mathbf{W}$ be finite dimensional vector spaces. An isomorphism between $\mathbf{W}$ and $\mathbf{W}$ is a linear transformation $T \in \mathscr{L}(\mathbf{V}, \mathbf{W})$ such that T has an inverse $T^{-1} \in \mathscr{L}(\mathbf{W}, \mathbf{V})$. If such an isomorphism exists, we say $\mathbf{V}$ and $\mathbf{W}$ are isomorphic.

Problem Set 2: Linear Transformations and Matrices

1. Prove that the composition of linear transformations is a linear transformation. In particular, if $\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{V}_3$ are vector spaces over a common field $\mathbb{F}$, and $T_1 \in \mathscr{L}(\mathbf{V}_1, \mathbf{V}_2)$ and $T_2 \in \mathscr{L}(\mathbf{V}_2, \mathbf{V}_3)$, show that $T_2 \circ T_1 : \mathbf{V}_1 \to \mathbf{V}_3$ satisfies

$$T_2 \circ T_1(ax + y) = aT_2 \circ T_1(x) + T_2 \circ T_1(y)$$

for any x, y in $\mathbf{V}_1$ and a in $\mathbb{F}$.

Solution: Since T_1 and T_2 are linear, we get

$$\begin{aligned} T_2 \circ T_1(ax + y) = T_2\big(T_2(ax + y)\big) = T_2(aT_1(x) + T_1(y)) \\ = aT_2\big(T_1(x)\big) + T_2\big(T_1(y)\big) = aT_2 \circ T_1(x) + T_2 \circ T_1(y), \end{aligned}$$

as expected.

2. **(a)** Prove that every vector space of dimension n over a field $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$ by exhibiting an isomorphism. (Make sure to prove that your linear transformation is indeed an isomorphism.)

Solution: Let $\mathbf{V}$ be a finite dimensional vector space of dimension n and fix a basis $\beta := \{v_1, \ldots, v_n\}$ of $\mathbf{V}$. Write x in $\mathbf{V}$ as $x = \sum_{i=1}^n c_i v_i$ for $\{c_i\} \subset \mathbb{F}$ and define the linear transformation $T(x) = [x]_\beta$. I claim T is an isomorphism. To prove this, it is sufficient to construct an inverse. In particular, for $\xi = (\xi_1, \ldots, \xi_n)^\top \in \mathbb{F}^n$, put $S(c) = \sum_{i=1}^n \xi_i v_i$. Then for $v = \sum_{i=1}^n c_i v_i \in V$, we have

$$S \circ T(v) = S\left(T\left(\sum_{i=1}^n c_i v_i\right)\right) = S\left(\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}\right) = \sum_{i=1}^n v_i v_i = v.$$

And, for any $\xi \in (\xi_1, \ldots, \xi_n)^\top \in \mathbb{F}^n$, we have

$$T \circ S(\xi) = T\left(S\left(\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}\right)\right) = T\left(\sum_{i=1}^n \xi_i v_i\right) = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} = \xi.$$

Hence $T \circ S = \mathbf{I}_{\mathbb{F}^n}$ and $S \circ T = \mathbf{I}_\mathbf{V}$, and so $S = T^{-1}$. This proves T is an isomorphism and therefore proves that $\mathbf{V}$ is isomorphic to $\mathbb{F}^n$.

(b) Show that two finite dimensional vector spaces are isomorphic if and only if they have the same dimension.

Solution:

($\Longrightarrow$) Suppose $\mathbf{V}$ and $\mathbf{W}$ are isomorphic vector spaces; i.e., that $V \cong W$. Then there exists an isomorphism $T \in \mathscr{L}(\mathbf{V}, \mathbf{W})$; that is, there exists a map T which is one-to-one and onto on $\mathbf{W}$. Thus, $R(T) = \mathbf{W}$ and $N(T) = \{0_\mathbf{V}\}$, and so $\dim \mathbf{V} = \dim \mathbf{W}$.

($\Longleftarrow$) Suppose $\mathbf{V}$ and $\mathbf{W}$ are two finite dimensional vector spaces with the same dimension n, and let β and γ be bases, respectively. From Part (a), we have $\phi_\beta : \mathbf{V} \to \mathbb{F}^n$ and $\phi_\gamma : \mathbf{W} \to \mathbb{F}^n$ are isomorphisms (and in particular, $\mathbf{V} \cong \mathbb{F}^n$ and $\mathbf{W} \cong \mathbb{F}^n$). The map $\phi_\gamma^{-1} \circ \phi_\beta : \mathbf{V} \to \mathbf{W}$ is therefore an isomorphism, since the composition of isomorphisms is an isomorphism.

3. Let $T \in \mathscr{L}(\mathbf{V}, \mathbf{W})$ where $\mathbf{V}$ and $\mathbf{W}$ are n- and m-dimensional vector spaces over $\mathbb{F}$ with ordered bases β and γ, respectively. Let $\phi_\beta \in \mathscr{L}(\mathbf{V}, \mathbb{F}^n)$ be such that $\phi_\beta(v) = [v]_\beta$ and $\phi_\gamma \in \mathscr{L}(\mathbf{W}, \mathbb{F}^m)$ be such that $\phi_\gamma(w) = [w]_\gamma$. Write T in terms of ϕ_β, ϕ_γ, and the left-multiplication transformation L_A where $A = [T]_\beta^\gamma$.

Solution: Set $A = [T]_\beta^\gamma$. Observe that we can map from $\mathbf{V}$ to $\mathbb{F}^m$ in two different ways; either by mapping first to $\mathbf{W}$ and then to $\mathbb{F}^m$, or by mapping first to $\mathbb{F}^n$ and then to $\mathbb{F}^m$. I claim these two maps are equal, i.e., that $\mathrm{L}_A \circ \phi_\beta = \phi_\gamma \circ T$. To prove this claim, it is sufficient to show these two transformations give the same result when applied to the elements of a given basis for $\mathbf{V}$, say $\{v_1, \ldots, v_n\}$. The left side gives

$$\mathrm{L}_A \circ \phi_\beta(v_j) = \mathrm{L}_A\big(\phi_\beta(v_j)\big) = \mathrm{L}_A[v_j]_\beta = [T(v_j)]_\gamma.$$

The right side gives

$$\phi_\gamma \circ T(v_j) = \phi_\gamma\big(T(v_j)\big) = [T(v_j)]_\gamma.$$

Hence, $\mathrm{L}_A \circ \phi_\beta = \phi_\gamma \circ T$. Since ϕ_γ is an isomorphism (by Problem 2a), ϕ_γ^{-1} exists, and so $T = \phi_\gamma^{-1} \circ \mathrm{L}_A \circ \phi_\beta$.

[*This problem and its predecessor shows why we are so concerned with* $\mathbb{R}^n$ *in linear algebra. All of finite-dimensional linear algebra over the reals can be done in terms of* $\mathbb{R}^n$.]

4. Let B be an $n \times n$ invertible matrix and define $\Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ by $\Phi(A) = B^{-1}AB$. Prove that Φ is an isomorphism.

   Solution: Since the domain and co-domain have the same dimension (indeed, they are they same vector space), it is sufficient to show that Φ has an inverse. Note that $\Phi^{-1}(C) = BCB^{-1}$, since $\Phi^{-1}(\Phi(A)) = \Phi^{-1}(B^{-1}AB) = BB^{-1}ABB^{-1} = A$, and since $\Phi(\Phi^{-1}(A)) = \Phi(BAB^{-1}) = B^{-1}BAB^{-1}B = A$. Hence Φ is an isomorphism.

5. In this problem we are going to deduce the rule for matrix multiplication. Let $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$ be p, n, m-dimensional vector spaces with ordered bases β_1, β_2, β_3, respectively. Let $T_{12} \in \mathscr{L}(\mathbf{V}_1, \mathbf{V}_2)$ and $T_{23} \in \mathscr{L}(\mathbf{V}_2, \mathbf{V}_3)$. We want to develop a multiplication rule such that

$$[T_{23} \circ T_{12}]_{\beta_1}^{\beta_3} = [T_{23}]_{\beta_2}^{\beta_3}[T_{12}]_{\beta_1}^{\beta_2}.$$

   For simplicity, let $A = [T_{23}]_{\beta_2}^{\beta_3}$, $B = [T_{12}]_{\beta_1}^{\beta_2}$, and $C = [T_{23} \circ T_{12}]_{\beta_1}^{\beta_3}$.

   (a) What are the sizes of A, B, and C in terms of m, n, and p? Does this agree with your understanding of matrix multiplication?

   Solution: Since $T_{23} : \mathbf{V}_2 \to \mathbf{V}_3$ and $\dim(\mathbf{V}_2) = n$, there are n basis vectors whose image under T_{23} form the columns of A; and since $\dim(\mathbf{V}_3) = m$, each of these transformed basis vectors has m-many elements. Hence, there are n columns and m rows in the matrix representation A of T_{23}, and so A has size $m \times n$. Similarly, B has size $n \times p$, and C has size $m \times p$. If $C = AB$, then it must be the case that the number of columns of A equals the number of rows of B, which is indeed the case here.

   Let $\beta_1 := \{v_1, \dots, v_p\}$, $\beta_2 := \{w_1, \dots, w_n\}$, and $\beta_3 := \{u_1, \dots, u_m\}$ be ordered bases for $\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{V}_3$, respectively.

   (b) Write an expression for $T_{12}(v_j)$ in terms of the matrix elements b_{ij} of B and the elements of β_2. Do the same for $T_{23}(w_k)$. Finally, write an expression for $T_{23} \circ T_{12}(v_j)$ in terms of the matrix elements c_{ij} of C and the elements of β_3.

   Solution: By the definition of the matrix representations, we have

$$T_{12}(v_j) = \sum_{k=1}^{n} b^k{}_j w_k \quad ; \quad T_{23}(w_k) = \sum_{i=1}^{m} a^i{}_k u_i \quad ; \quad T_{23} \circ T_{12}(v_j) = \sum_{i=1}^{m} c^i{}_j u_i$$

   for $1 \le j \le p$ and $1 \le k \le n$.

   (c) Using the linearity of the composition $T_{23} \circ T_{12}$ to write an expression for $T_{23} \circ T_{12}(v_j)$ in terms of the elements of β_3. Your answer should depend on a_{ij} and b_{ij}.

   Solution: Using Problem 1, we get

$$T_{23} \circ T_{12}(v_j) = T_{23}\big(T_{12}(v_j)\big) = T_{23}\left(\sum_{k=1}^{n} b^k{}_j w_k\right) = \sum_{k=1}^{n} b^k{}_j T_{23}(w_k)$$

$$= \sum_{k=1}^{n} b^k{}_j \left(\sum_{i=1}^{m} a^i{}_k u_i \right) = \sum_{i=1}^{m} \left(\sum_{k=1}^{n} a^i{}_k b^k{}_j \right) u_i$$

where the parentheses in the last step have been introduced for future convenience.

(d) Compare your expressions for $T_{23} \circ T_{12}(v_j)$ from part (b) and part (c) to deduce the rule for matrix multiplication.

Solution: From parts (b) and (c), we have

$$\sum_{i=1}^{m} c^i{}_j u_i = T_{23} \circ T_{12}(v_j) = \sum_{i=1}^{m} \left(\sum_{k=1}^{n} a^i{}_k b^k{}_j \right) u_i.$$

Hence, we conclude

$$c^i{}_j = \sum_{k=1}^{n} a^i{}_k b^k{}_j,$$

which is indeed the rule for multiplication of matrices.

6. Let $g(x) = x + 3$, and let $T \in \mathscr{L}(P_2(\mathbb{R}))$ and $U \in \mathscr{L}(P_2(\mathbb{R}), \mathbb{R}^3)$ be defined by

$$T(f(x)) = f'(x)g(x) + 2f(x)$$
$$U(a + bx + cx^2) = (a + b, c, a - b)^\top.$$

Let β and γ be the standard ordered bases for $P_2(\mathbb{R})$ and $\mathbb{R}^3$, respectively.

(a) Compute $[U]_\beta^\gamma$, $[T]_\beta$, and $[U \circ T]_\beta^\gamma$ directly.

Solution: First we construct $[T]_\beta$, given $\beta := \{1, x, x^2\}$. We have:

$$T(1) = (x+3)(0) + 2(1) = 2 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}_\beta$$
$$T(x) = (x+3)(1) + 2(x) = 3x + 3 = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}_\beta$$
$$T(x^2) = (x+3)(2x) + 2(x^2) = 4x^2 + 6x = \begin{pmatrix} 0 \\ 6 \\ 4 \end{pmatrix}_\beta$$

Since the j'th column of $[T]_\beta$ is $[T(\beta_j)]_\beta$, we find

$$\boxed{[T]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}.}$$

Next we construct $[U]_\beta^\gamma$. We have:

$$U(1) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}_\gamma \quad ; \quad U(x) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}_\gamma \quad ; \quad U(x^2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_\gamma$$

Again since the j'th column of $[U]_\beta^\gamma$ is $[U(\beta_j)]_\gamma$, we find

$$\boxed{[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.}$$

To compute $[U \circ T]_\beta^\gamma$, we need to compute $U \circ T$ in general. Let $a + bx + cx^2 \in P_2(\mathbb{R})$. We have:

$$T(a + bx + cx^2) = (x + 3)(b + 2cx) + 2(a + bx + c^2) = [2a + 3b] + [3b + 6c]x + [4c]x^2,$$

and so,

$$U \circ T(a + bx + cx^2) = \begin{pmatrix} 2a + 6b + 6c \\ 4c \\ 2a - 6c \end{pmatrix}.$$

Applying the transformation to each of the β-basis elements, we get

$$U \circ T(1) = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}_\gamma \quad ; \quad U \circ T(x) = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}_\gamma \quad ; \quad U \circ T(x^2) = \begin{pmatrix} 6 \\ 4 \\ -6 \end{pmatrix}_\gamma,$$

and so the matrix representation is

$$\boxed{[U \circ T]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 6 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}.}$$

(b) Use the previous problem to verify your result.
Solution: The previous problem states that $[U \circ T]_\beta^\gamma = [U]_\beta^\gamma [T]_\beta$, and indeed:

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}}_{[U]_\beta^\gamma} \underbrace{\begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}}_{[T]_\beta} = \underbrace{\begin{pmatrix} 2 & 6 & 6 \\ 6 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}}_{[U \circ T]_\beta^\gamma},$$

as expected.

7. Let $\mathbf{V}$ and $\mathbf{W}$ be finite dimensional vector spaces with the same dimension with ordered bases β and γ, respectively. Let $T \in \mathscr{L}(\mathbf{V}, \mathbf{W})$. Prove that T is invertible if and only if $[T]_\beta^\gamma$ is invertible. Further show that $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.

Solution:

($\implies$) Let $\dim \mathbf{V} = \dim \mathbf{W} = n$. Then T has matrix representation $[T]_\beta^\gamma \in M_{n \times n}(\mathbb{F})$. If T is invertible, then there exists a $T^{-1} \in \mathscr{L}(\mathbf{W}, \mathbf{V})$ which satisfies $T^{-1} \circ T = \mathbf{I_V}$ and $T \circ T^{-1} = \mathbf{I_W}$. Then, T^{-1} has matrix representation $[T^{-1}]_\gamma^\beta \in M_{n \times n}(\mathbb{F})$, and:

$$\mathbf{I}_n = [\mathbf{I_V}]_\beta^\beta = [T^{-1} \circ T]_\beta^\beta = [T^{-1}]_\gamma^\beta [T]_\beta^\gamma,$$
$$\mathbf{I}_n = [\mathbf{I_W}]_\gamma^\gamma = [T \circ T^{-1}]_\gamma^\gamma = [T]_\beta^\gamma [T^{-1}]_\gamma^\beta.$$

This proves that if T is invertible, then $[T]_\beta^\gamma$ is invertible, and incidentally, that $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.

($\impliedby$) Suppose $A = [T]_\beta^\gamma \in M_{n \times n}(\mathbb{F})$ is invertible. Then there exists a $B \in M_{n \times n}(\mathbb{F})$ such that $AB = BA = \mathbf{I}_n$. Since the map $\Phi_\gamma^\beta : \mathscr{L}(\mathbf{W}, \mathbf{V}) \to M_{m \times n}(\mathbb{F})$ is an isomorphism, there exists a unique $U \in \mathscr{L}(\mathbf{W}, \mathbf{V})$ such that $[U]_\gamma^\beta = B$. I claim $U = T^{-1}$. Note that

$$\mathbf{I}_n = AB = [T]_\beta^\gamma [U]_\gamma^\beta = [T \circ U]_\gamma^\gamma,$$

$$\mathbf{I}_n = BA = [U]_\gamma^\beta [T]_\beta^\gamma = [U \circ T]_\beta^\beta.$$

Hence $U = T^{-1}$ and T is invertible.

8. The benefit of changing coordinate systems is that you can change coordinates into a set which optimizes efficiency, perform the relevant computations, and then transform back into the original coordinates. Let $T \in \mathscr{L}(\mathbf{V})$ and suppose β and β' are ordered bases for $\mathbf{V}$. If $Q = [I]_{\beta'}^{\beta}$ is the change of coordinate matrix that changes β'-coordinates into β coordinates, prove $[T]_\beta = Q^{-1}[T]_{\beta'}Q$.

   Solution: Recall that $Q = [\mathbf{I_V}]_\beta^{\beta'}$ and $Q^{-1} = [\mathbf{I_V}]_{\beta'}^{\beta}$. From equation 5, we get

$$Q^{-1}[T]_{\beta'}Q = [\mathbf{I_V}]_{\beta'}^{\beta}[T]_{\beta'}^{\beta'}[\mathbf{I_V}]_\beta^{\beta'} = [\mathbf{I_V}]_{\beta'}^{\beta}[T \circ \mathbf{I_V}]_\beta^{\beta'} = [\mathbf{I_V} \circ T \circ \mathbf{I_V}]_\beta^\beta = [T]_\beta,$$

   which completes the proof.