Homework # 08

Math 140
Professor John Lott

By

Cody Vig

The University of California
Berkeley

Due March 29, 2021

Chapter 04 | Problem 8.1

Find the Gaussian and mean curvatures of $\mathbb{R}^2$, S^2, T^2 (Problems 1.1 and 7.3), and $S^1 \times (0,1)$ (Problem 7.2).

Solution

Gaussian and Mean Curvature on $\mathbb{R}^2$:

Clearly the Gaussian and mean curvature of $\mathbb{R}^2$ are both zero. We can parameterize (a subset of) $\mathbb{R}^2$ with the coordinate patch

$$\boldsymbol{x}(u^1,u^2) = (u^1,u^2,0),$$

from which it follows

$$\boldsymbol{x}_{ij} = 0$$

for all $1 \leq i,j \leq 2$. Hence all $L_{ij} = \langle \boldsymbol{x}_{ij}, \boldsymbol{n}\rangle = 0$, and so $L^i{}_j = g^{i\ell}L_{\ell k} = \delta^{i\ell}L_{\ell k} = 0$. As such, $(L^i{}_j)$ is the zero matrix and so its eigenvalues κ_1 and κ_2 are identically zero, and hence

$$\begin{aligned} K_{\mathbb{R}^2} &= \kappa_1\kappa_2 = 0 \\ H_{\mathbb{R}^2} &= \frac{1}{2}(\kappa_1+\kappa_2) = 0 \end{aligned}$$

Gaussian and Mean Curvature on S^2:

The 2-sphere S^2 can be locally be parameterized with

$$\boldsymbol{x}(u^1,u^2) = (\sin u^1 \cos u^2, \sin u^1 \sin u^2, \cos u^1)$$

where $u^1 \in (0,\pi)$ and $u^2 \in (0,2\pi)$. One can also interpret the Weingarten map in terms of the derivatives of the normal vector. Indeed, $L^i{}_j\boldsymbol{x}_i = \mathsf{L}(\boldsymbol{x}_j) = -\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}u^j}$. We know

$$\boldsymbol{x}_1 = (\cos u^1 \cos u^2, \cos u^1 \sin u^2, -\sin u^1) \tag{1a}$$

$$\boldsymbol{x}_2 = (-\sin u^1 \sin u^2, \sin u^1 \cos u^2, 0) \tag{1b}$$

and

$$\boldsymbol{n} = \frac{\boldsymbol{x}_1 \times \boldsymbol{x}_2}{\|\boldsymbol{x}_1 \times \boldsymbol{x}_2\|} = \cdots = (\sin u^1 \cos u^2, \sin u^1 \sin u^2, \cos u^1) \tag{2}$$

where the dots are filled in with the usual computations.[1] This is precisely the same as the definition of $\boldsymbol{x}$, and so $\boldsymbol{n}_i \equiv \frac{\partial \boldsymbol{n}}{\partial u^i} = \boldsymbol{x}_i$, which implies

$$L^i{}_j\boldsymbol{x}_i = -\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}u^j} \quad\implies\quad L^i{}_j\boldsymbol{x}_i = -\boldsymbol{x}_i = (-\delta^i{}_j)\boldsymbol{x}_j.$$

Thus, $L^i{}_j = -\delta^i{}_j$ under this coordinate patch, and so $\kappa_1 = \kappa_2 = -1$. Hence the Gaussian and mean curvatures are

$$\begin{aligned} K_{S^2} &= \kappa_1\kappa_2 = 1 \\ H_{S^2} &= \tfrac{1}{2}(\kappa_1+\kappa_2) = -1. \end{aligned}$$

Gaussian and Mean Curvature on T^2:

In Problem 1.1 of Homework 05, we found that for the coordinate patch

$$\boldsymbol{x}(u^1,u^2) = ((2+\cos u^1)\cos u^2, (2+\cos u^1)\sin u^2, \sin u^1),$$

[1] I hope it is ok to skip these steps at this point in the course. To be honest, I forgot we had homework due this week!

the tangent and normal vectors on the torus were

$$\boldsymbol{x}_1 = \big(-\sin u^1 \cos u^2, -\sin u^1 \sin u^2, \cos u^1\big), \tag{3a}$$
$$\boldsymbol{x}_2 = \big(-(2+\cos u^1)\sin u^2, (2+\cos u^1)\cos u^2, 0\big), \tag{3b}$$
$$\boldsymbol{n} = -\big(\cos u^1 \cos u^2, \cos u^1 \sin u^2, \sin u^1\big). \tag{3c}$$

We can again determine L by using the fact $L^i{}_j \boldsymbol{x}_i = -\boldsymbol{n}_j$. We have

$$\boldsymbol{n}_1 = -(-\sin u^1 \cos u^2, -\sin u^1 \sin u^2, \cos u^1) \tag{4a}$$
$$\boldsymbol{n}_2 = -(-\cos u^1 \sin u^2, \cos u^1 \cos u^2, 0). \tag{4b}$$

Hence, we arrive at the equations

$$L^1{}_1 \boldsymbol{x}_1 + L^2{}_1 \boldsymbol{x}_2 = -\boldsymbol{n}_1 \tag{5a}$$
$$L^1{}_2 \boldsymbol{x}_1 + L^2{}_2 \boldsymbol{x}_2 = -\boldsymbol{n}_2. \tag{5b}$$

We see from Eq.'s (3) and (4) that (5a) is satisfied if and only if $L^1{}_1 = 1$ and $L^2{}_1 = 0$. Similarly, we see that (5b) is satisfied if and only if $L^1{}_2 = 0$ and $L^2{}_2 = \cos u^1/(2+\cos u^1)$. Hence L is again diagonal and so $\kappa_1 = 1$ and $\kappa_2 = \cos u^1/(2+\cos u^1)$. Hence

$$K_{T^2} = \kappa_1 \kappa_2 = \frac{\cos u^1}{2+\cos u^1}$$
$$H_{T^2} = \tfrac{1}{2}(\kappa_1 + \kappa_2) = \frac{1+\cos u^1}{2+\cos u^1}.$$

Gaussian and Mean Curvature on $S^1 \times (0,1)$:

From Example 2.3 in Chapter 04 of Millman and Parker, one such coordinate patch of $S^1 \times (0,1)$ is

$$\boldsymbol{x}(u^1, u^2) = (\cos u^1, \sin u^1, u^2)$$

for $u^1 \in (-3\pi/4, 3\pi/4)$ and $u^2 \in (0,1)$. Then[2]

$$\boldsymbol{x}_1 = (-\sin u^1, \cos u^1, 0) \tag{6a}$$
$$\boldsymbol{x}_2 = (0,0,1) \tag{6b}$$
$$\boldsymbol{n} = (\cos u^1, \sin u^1, 0). \tag{6c}$$

We can again use $L^i{}_j \boldsymbol{x}_i = -\boldsymbol{n}_j$ to determine L. We know

$$\boldsymbol{n}_1 = (-\sin u^1, \cos u^1, 0) \tag{7a}$$
$$\boldsymbol{n}_2 = \boldsymbol{0}, \tag{7b}$$

and so

$$L^1{}_1 \boldsymbol{x}_1 + L^2{}_1 \boldsymbol{x}_2 = -\boldsymbol{n}_1 \quad \Longrightarrow \quad (-L^1{}_1 \sin u^1, L^1{}_1 \cos u^1, L^2{}_1) = (\sin u^1, -\cos u^1, 0) \tag{8a}$$
$$L^1{}_2 \boldsymbol{x}_1 + L^2{}_2 \boldsymbol{x}_2 = -\boldsymbol{n}_2 \quad \Longrightarrow \quad (-L^1{}_2 \sin u^1, L^1{}_2 \cos u^1, L^2{}_2) = (0,0,0). \tag{8b}$$

It is clear that these equations are only satisfied if $L^1{}_1 = -1$ and $L^i{}_j = 0$ otherwise. As such, L is diagonal and so $\kappa_1 = -1$ and $\kappa_2 = 0$. Thus

$$K_{S^1\times(0,1)} = \kappa_1 \kappa_2 = 0$$
$$H_{S^1\times(0,1)} = \tfrac{1}{2}(\kappa_1 + \kappa_2) = -\tfrac{1}{2}.$$

---

[2] Alternatively, $\mathsf{L} = \mathrm{diag}(1,0)$ can be taken directly from the prompt of Problem 7.2 in Millman and Parker, i.e., the problem referenced in the prompt of this question. I did not see this until I finished typesetting this answer. The discrepancy in $L^1{}_1$ likely comes from a different choice of coordinate patch, since we know that the components L_{ij} transform doubly covariantly up to sign, g^{ij} transforms doubly contravariantly, and $L^i{}_j = \mathsf{g}^{i\ell} L_{\ell j}$.

Chapter 04 | Problem 8.2

Prove $H^2 \geq K$. When does equality hold?

Solution

Recall that the Gaussian and mean curvatures can be expressed in terms of the eigenvalues κ_1 and κ_2 of the Weingarten map L. Indeed,

$$\begin{aligned} K &= \det(\mathsf{L}) = \kappa_1\kappa_2 \\ H &= \tfrac{1}{2}\operatorname{tr}(\mathsf{L}) = \tfrac{1}{2}(\kappa_1 + \kappa_2). \end{aligned}$$

We proved in a previous homework that L is self-adjoint. As such the eigenvalues will be real, and so we have the following inequality:

$$\begin{aligned} &0 \leq (\kappa_1 - \kappa_2)^2 = \kappa_1^2 - 2\kappa_1\kappa_2 + \kappa_2^2 \\ \implies &4\kappa_1\kappa_2 \leq \kappa_1^2 + 2\kappa_1\kappa_2 + \kappa_2^2 = (\kappa_1 + \kappa_2)^2 \\ \implies &K \leq H^2 \end{aligned}$$

as expected. Additionally, the first inequality shows that equality is saturated if and only if the eigenvalues of L are degenerate.[3]

Chapter 04 | Problem 8.10

What are the principle curvatures for a surface of revolution?

Solution

We know from Chapter 04 Section 4.2 that a surface of revolution can be given by the coordinate patch

$$\boldsymbol{x}(u^1, u^2) = (r(u^1)\cos u^2, r(u^1)\sin u^2, z(u^1))$$

where $(r(u^1), z(u^1))$ is the curve generating the surface of revolution. The approach taken here is to determine the coefficients of the second fundamental form L_{ij} and then find the coefficients of the Weingarten map $L^i{}_j = g^{i\ell}L_{\ell j}$ by contraction with the inverse metric. To determine the L_{ij}'s, we need the second derivatives $\boldsymbol{x}_{ij}$. We have

$$\boldsymbol{x}_1 = (r'\cos u^2, r'\sin u^2, z') \tag{9a}$$

$$\boldsymbol{x}_2 = (-r\sin u^2, r\cos u^2, 0), \tag{9b}$$

where I have dropped the explicit u^1-dependence and used primes to denote differentiation by u^1 for brevity. So:

$$\boldsymbol{x}_{11} = (r''\cos u^2, r''\sin u^2, z'') \tag{10a}$$

$$\boldsymbol{x}_{21} = \boldsymbol{x}_{12} = (-r'\sin u^2, r'\cos u^2, 0) \tag{10b}$$

$$\boldsymbol{x}_{22} = (-r\cos u^2, -r\sin u^2, 0). \tag{10c}$$

[3] Do mathematicians use the word *degeneracy* in the context of repeated eigenvalues? I've never heard a mathematician say so, but physicists do.

Next we need the normal $\boldsymbol{n}$ to the tangent plane. Write the standard basis for $\mathbb{R}^3$ as $\{\boldsymbol{e}_i\}_{i=1}^3$. Then we have

$$\boldsymbol{x}_1 \times \boldsymbol{x}_2 = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ r' \cos u^2 & r' \sin u^2 & z' \\ -r \sin u^2 & r \cos u^2 & 0 \end{vmatrix} = (-rz' \cos u^2, -rz' \sin u^2, rr').$$

The length of the cross product is $\|\boldsymbol{x}_1 \times \boldsymbol{x}_2\| = r\sqrt{r'^2 + z'^2}$, so the unit normal is

$$\boldsymbol{n} = \left(-\frac{z'}{\sqrt{r'^2 + z'^2}} \cos u^2, \frac{z'}{\sqrt{r'^2 + z'^2}} \sin u^2, \frac{r'}{\sqrt{r'^2 + z'^2}} \right). \tag{11}$$

Since $L_{ij} := \langle \boldsymbol{x}_{ij}, \boldsymbol{n} \rangle$, from Eq.'s (10) and (11), we find

$$L_{11} = \frac{z''r - z'r''}{\sqrt{r'^2 + z'^2}} \tag{12a}$$

$$L_{21} = L_{12} = 0 \tag{12b}$$

$$L_{22} = \frac{rz'}{\sqrt{r'^2 + z'^2}}. \tag{12c}$$

To determine the coefficients of the Weingarten map, we need the inverse metric. We can find the metric from the basis vectors (9):

$$(g_{ij}) = \begin{pmatrix} \langle \boldsymbol{x}_1, \boldsymbol{x}_1 \rangle & \langle \boldsymbol{x}_1, \boldsymbol{x}_2 \rangle \\ \langle \boldsymbol{x}_2, \boldsymbol{x}_1 \rangle & \langle \boldsymbol{x}_2, \boldsymbol{x}_2 \rangle \end{pmatrix} = \begin{pmatrix} r'^2 + z'^2 & 0 \\ 0 & r^2 \end{pmatrix}.$$

Inverting, we find

$$(g^{ij}) = \begin{pmatrix} \frac{1}{r'^2 + z'^2} & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}. \tag{13}$$

Then from Eq.'s (12) and (13), the Weingarten map $\mathsf{L} = (L^i{}_j) = (g^{ij})(L_{ij})$ is

$$\mathsf{L} = (g^{ij})(L_{ij}) = \frac{1}{\sqrt{r'^2 + z'^2}} \begin{pmatrix} \frac{1}{r'^2 + z'^2} & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix} \begin{pmatrix} z''r' - z'r'' & 0 \\ 0 & rz' \end{pmatrix} = \frac{1}{\sqrt{r'^2 + z'^2}} \begin{pmatrix} \frac{z''r' - z'r''}{r'^2 + z'^2} & 0 \\ 0 & \frac{z'}{r} \end{pmatrix}.$$

Since the Weingarten map is diagonal, its eigenvalues are its diagonal elements. As such, the principle curvatures are:

$$\kappa_1 = L^1{}_1 = \frac{z''r' - z'r''}{(r'^2 + z'^2)^{3/2}}$$

$$\kappa_2 = L^2{}_2 = \frac{z'}{r} \frac{1}{\sqrt{r'^2 + z'^2}}$$

which completes the problem.

Chapter 04 | Problem 8.15

Let M be the surface of revolution generated by the non-unit speed curve $\boldsymbol{\alpha}(t) = (\frac{1}{a} \cosh(at + b), t)$. Show that M is minimal ($H \equiv 0$). M is called a *catenoid*.

Solution

Since the mean curvature is defined to be $H := \frac{1}{2} \operatorname{tr}(\mathsf{L})$, we have from Problem 8.10 that the mean curvature for a surface of revolution is

$$H = \frac{1}{2\sqrt{r'^2 + z'^2}} \left(\frac{z''r' - z'r''}{r'^2 + z'^2} + \frac{z'}{r} \right).$$

In the language of the previous problem, $\boldsymbol{\alpha}(t)$ is the curve $(r(u^1), z(u^1))$ which was said to *generate* the surface of revolution. So, we make the identification:

$$r(t) = a^{-1}\cosh(at+b) \tag{14a}$$
$$z(t) = t. \tag{14b}$$

Our derivation in the previous problem, we did not assume that (r, z) was unit speed, so the fact that the catenoid $\boldsymbol{\alpha}$ is non-unit speed is irrelevant. Note

$$\begin{aligned} r'(t) &= \sin(at+b) & z'(t) &= 1 \\ r''(t) &= a\cosh(at+b) & z''(t) &= 0, \end{aligned}$$

and so

$$r'^2 + z'^2 = \sinh^2(at+b) + 1 = \cosh^2(at+b),$$

Hence

$$\begin{aligned} H &= \frac{1}{2\sqrt{r'^2+z'^2}}\left(\frac{z''r' - z'r''}{r'^2+z'^2} + \frac{z'}{r}\right) \\ &= \frac{1}{2\cos(at+b)}\left(\frac{0 - a\cosh(at+b)}{\cosh^2(at+b)} + \frac{a}{\cosh(at+b)}\right) = 0 \end{aligned}$$

as expected.