Linear Algebra

Cody Vig

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We make the following definitions.

Definition 1 (Sum)**.** If S_1 and S_2 are nonempty subsets of a vector space V, then the sum of S_1 and S_2 is $S_! + S_2 := \{x + y \mid x \in S_1, y \in S_2\}$.

Definition 2 (Direct sum)**.** A vector space V is called the direct sum of W_1 and W_2 if W_1 and W_2 are subspaces of V with $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$. We denote the V is the direct sum of W_1 and W_2 by writing $V = W_1 \oplus W_2$.

Problem Set 1: Vector Spaces

1. Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cap W_2$ is a subspace of V. (*Note that* $\{0\}$ *is a subspace of every vector space.*)

   Solution. Note that $0 \in W$ for any subspace W of V, and so $0 \in W_1 \cap W_2$ and hence $W_1 \cap W_2$ is not empty. It remains to be shown that $W_1 \cap W_2$ is closed under addition and scalar multiplication. Suppose x_1, x_2 are elements of $W_1 \cap W_2$ and c is an element of $\mathbb{F}$. Then by definition, x_1, x_2 are elements of both W_1 and W_2, from which it follows that $x_1 + x_2$ and cx_1 are elements of W_1 and W_2, since W_1 and W_2 are vector spaces. Hence $x_1 + x_2$ and cx_1 are elements of $W_1 \cap W_2$, which shows $W_1 \cap W_2$ is closed under addition and scalar multiplication. This completes the proof.

2. Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

   Solution.

   ($\Longrightarrow$) Let $W_1 \cup W_2$ be a subspace of V, and assume by way of contradiction that $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$. Then there exists an element x_1 in $W_1 \cup W_2$ which is in W_1 but not in W_2, and there exists an x_2 in $W_1 \cup W_2$ which is not in W_1 and is in W_2. Since $W_1 \cup W_2$ is closed under addition, $x_1 + x_2 \in W_1 \cup W_2$, which implies $x_1 + x_2$ is in at least one of W_1 or W_2. Suppose without loss of generality $x_1 + x_2 \in W_1$. Since $x_1 \in W_1$, there exists a $-x_1 \in W_1$ for which $x_1 - x_1 = 0$, and so $(x_1 + x_2) - x_1 = x_2 + (x_1 - x_1) = x_2 \in W_1$, which is a contradiction. Hence either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

   ($\Longleftarrow$) Suppose without loss of generality that $W_1 \subseteq W_2$. Then $W_1 \cup W_2 = W_2$, which is a subspace of V by assumption.

3. Let W_1 and W_2 be subspaces of a vector space V.

   (a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.
   Solution. We need to show that $x \in W_1$ or $x \in W_2$ implies $x \in W_1 + W_2$. Suppose $x \in W_1$. Then put $x = x + 0$ for $x \in W_1$ and $0 \in W_2$. Hence $x \in W_1 + W_2$. Similarly, $W_2 \ni x = 0 + x$ implies $x \in W_1 + W_2$, and so $W_1 + W_2$ contains W_1 and W_2.

(b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

Solution. Suppose W is a subspace of V which contains W_1 and W_2. In particular, $w_1 \in W_1$ and $w_2 \in W_2$ implies $w_1, w_2 \in W$. Since W is closed under addition, $w_1 + w_2 \in W$, and so $W_1 + W_2$ is contained within W.

(c) Prove that $V = W_1 \oplus W_2$ if and only if each vector in V can be *uniquely* written as $x_1 + x_2$ where $x_1 \in W_1$ and $x_2 \in W_2$.

Solution.

$(\implies)$ Let $V = W_1 \oplus W_2$. Then $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$. Since $V = W_1 + W_2$, every $v \in V$ can be written in the form $x_1 + x_2$ for $x_1 \in W_1$ and $x_2 \in W_2$. Suppose $x_1 + x_2 = v = y_1 + y_2$ for $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$. Then, $0 = (x_1 - y_1) + (x_2 - y_2)$. Since $W_1 \cap W_2 = \{0\}$, $-y_1 \notin W_2$ and $-y_2 \notin W_1$ unless $y_1 = y_2 = 0 = x_1 = x_2$ (if $0 \neq -y_1 \in W_2$ or $0 \neq -y_2 \in W_1$, then by closure $y_1 \in W_2$ or $y_2 \in W_1$, which contradicts the assumption that $W_1 \cap W_2 = \{0\}$). Otherwise, it must be the case that $x_1 - y_1 = 0 = x_2 - y_2$, and so $x_1 = y_1$ and $x_2 = y_2$. Hence, any $v \in V$ can be written uniquely as $x_1 + x_2$ where $x_1 \in W_1$ and $x_2 \in W_2$.

$(\impliedby)$ Suppose each $v \in V$ can be written uniquely as $x_1 + x_2$ for $x_1 \in W_1$ and $x_2 \in W_2$. This implies $V = W_1 + W_2$. Further, if $x \in W_1$, we can write $x = x + 0$ for $0 \in W_2$ (and similarly for $y \in W_2$), which implies $W_1 \cap W_2 = \{0\}$ (not $\varnothing$, since $V \ni 0 = 0 + 0$ is of the form $x_1 + x_2$ where $0 = x_1 \in W_1$ and $0 = x_2 \in W_2$). Hence $V = W_1 \oplus W_2$.

4. Suppose V is a finite dimensional vector space and $U_1, U_2, \ldots, U_m$ are subspaces of V such that $V = U_1 \oplus U_2 \oplus \cdots \oplus U_m$. Prove $\dim V = \dim U_1 + \cdots + \dim U_m$.

Notation. We use the following notation for direct sums:

$$\bigoplus_{i=1}^{m} U_u \equiv \bigoplus_i U_i := U_1 \oplus U_2 \oplus \cdots \oplus U_m$$

Solution. From Problem 3(c), we have that $V = \bigoplus_i U_i$ if and only if each $v \in V$ can be written uniquely as $v = \sum_i u_i$ for $u_i \in U_i$. Let $\beta_j := \{v_{j1}, \ldots, v_{jn_j}\}$ be a basis for U_j and write $u_i = \sum_{j=1}^{n_i} a_{ij} v_{ij}$ for scalars a_{ij}. Then, we have

$$v = \sum_{i=1}^{m} u_i = \sum_{i=1}^{m} \sum_{j=1}^{n_i} a_{ij} v_{ij}.$$

Hence, every vector v in V can be written as a unique linear combination of the $\{v_{ij}\}$. By a theorem from Lecture 1, this implies $\{v_{ij}\}$ is a basis for V, and so

$$\begin{aligned} \dim(V) = |\{v_{ij}\}| &= |\{v_{11}, \ldots, v_{1n_1}, v_{21}, \ldots, v_{2n_2}, \ldots, v_{m1}, \ldots, v_{mn_m}\}| \\ &= n_1 + n_2 + \cdots n_m = \dim U_1 + \dim U_2 + \cdots + \dim U_m, \end{aligned}$$

which completes the proof.

5. Suppose U_1, U_2, and U_3 are subspaces of a vector space V.

(a) Prove

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

This may convince you that the law of inclusion-exclusion holds for vector spaces.

Solution. Let $\beta_1 := \{v_{11}, \dots, v_{1n_1}\}$ be a basis for U_1 and $\beta_2 := \{v_{21}, \dots, v_{2n_2}\}$ be a basis for U_2. By Problem 7, we can extend each of these bases to a basis for $U_1 + U_2$ to get

$$\gamma_1 = \beta_1 \cup \{u_{11}, \dots, u_{1s_1}\} = \{v_{11}, \dots, v_{1n_1}, u_{11}, \dots, u_{1s_1}\},$$
$$\gamma_2 = \beta_2 \cup \{u_{21}, \dots, u_{2s_2}\} = \{v_{21}, \dots, v_{2n_2}, u_{21}, \dots, u_{2s_2}\}.$$

Hence $\dim(U_1 + U_2) =: n = n_1 + s_1 = n_2 + s_2$, or

$$n = 2n - n = n_1 + n_2 + (s_1 + s_2 - n).$$

Note that $t := n - (s_1 + s_2)$ is the size of the intersection $U_1 \cap U_2$, or the dimension of $U_1 \cap U_2$, and so we get

$$\dim(U_1 + U_2) = n = n_1 + n_2 - (n - s_1 - s_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2),$$

as expected.

(b) Show that it is *not necessarily* true that

$$\dim(U_1 + U_2 + U_3) = \dim U_1 + \dim U_2 + \dim U_3 - \dim(U_1 \cap U_2) - \dim(U_2 \cap U_3) - \dim(U_1 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3)$$

[*Hint: Consider* $V = \mathbb{R}^2$.] This shows that that the law of inclusion-exclusion *does not* hold for vector spaces.

Solution. Let $V = \mathbb{R}^2$, $U_1 := \{(x,0) : x \in \mathbb{R}\}$, $U_2 := \{(0,y) : y \in \mathbb{R}\}$, and $U_3 := \{(x,x) : x \in \mathbb{R}\}$. We have $\dim V = 2$ and $\dim U_1 = \dim U_2 = \dim U_3 = 1$. Further, $U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = U_1 \cap U_2 \cap U_3 = \{0\}$ and so the dimension of each is 0. Thus,

$$\dim U_1 + \dim U_2 + \dim U_3 - \dim(U_1 \cap U_2) - \dim(U_2 \cap U_3) - \dim(U_1 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3) = 1 + 1 + 1 - 0 - 0 - 0 + 0 = 3$$

whereas $U_1 + U_2 + U_3 = \{(x + x, x + y) : x, y \in \mathbb{R}\} = \{(x, y) : x, y \in \mathbb{R}\} = \mathbb{R}^2$, and so $\dim(U_1 + U_2 + U_3) = 3 \neq 2$, and so the statement fails.

6. The techniques we learn about in linear algebra work in any field, but there are pathological properties associated with finite fields that may lead to confusion. Consider the set $S = \{(1,1,0)^\top, (1,0,1)^\top, (0,1,1)^\top\}$ as a subset of the vector space $\mathbb{F}^3$.

(a) If $\mathbb{F} = \mathbb{R}$, show that S is a basis for $\mathbb{F}^3$.

Solution. Suppose $c_1(1,1,0)^\top + c_2(1,0,1)^\top + c_3(0,1,1)^\top = (0,0,0)^\top = 0_{\mathbb{R}^3}$ for some c_1, c_2, $c_3 \in \mathbb{R}$. This vector equation is equivalent to the linear system:

$$\begin{aligned} c_1 + c_2 \phantom{{}+c_3} &= 0, \\ c_1 \phantom{{}+c_2} + c_3 &= 0, \\ c_2 + c_3 &= 0. \end{aligned}$$

The first equation requires $c_1 = -c_2$. The third equation requires $c_3 = -c_2$, from which the second becomes $c_1 = c_2$. Hence $c_2 = c_1 = -c_2$ implies $c_2 = 0$, and so $c_1 = c_2 = c_3 = 0$, which implies S is indeed linearly independent in $\mathbb{R}^3$. Since $\dim \mathbb{R}^3 = 3 = |S|$, this implies S is a basis for $\mathbb{R}^3$.

(b) If $\mathbb{F} = \mathbb{F}_2$ (the field of integers modulo 2), then S is not linearly independent and hence is not a basis for $\mathbb{F}^3$.

Solution. Note that $(1,1,0)^\top + (1,0,1)^\top + (0,1,1)^\top = (0,0,0)^\top = 0_{\mathbb{F}_2^3}$, and so the vectors in S are not linearly independent. Hence, S is not a basis for $\mathbb{F}_2^3$.

7. Suppose $S = \{v_1, \ldots, v_m\}$ is a linearly independent subset of a finite dimensional vector space V of dimension $n > m$. Show that S can be extended to a basis for V; that is, construct a basis for V of the form $\{v_1, \ldots, v_m, v_{m+1}, \ldots v_n\} = S \cup \{v_{m+1}, \ldots v_n\}$.

   Solution. If $\dim V = m$, there is nothing to prove. Otherwise, let $\dim V = n > m$. We proved in Lecture 1 that if S is linearly independent, then $S \cup \{v\}$ is linearly dependent if and only if $v \in \operatorname{span}(S)$. Hence, if v is not in the span of S, $S \cup \{v\}$ is a strictly larger linearly independent set. This gives us the following algorithm to construct a basis. Set $S_0 := S$ and choose an element $v \in V$. If $v \in \operatorname{span}(S_0)$, choose a new v. Otherwise, set $v_{|S_0|+1} = v_{m+1} := v$ and set $S_1 := S_0 \cup \{v_{|S_0|+1}\}$. Then for $j = 1, \ldots, n-m-1$, do the following: choose a new $v \in V$. If $v \in \operatorname{span}(S_j)$, pick a new V. Otherwise, set $v_{|S_j|+1} = v_{m+j+1} := v$ and set $S_{j+1} := S_j \cup \{v_{|S_j|+1}\}$. The set $S_{n-m-1} := \{v_1, \ldots, v_m, v_{m+1}, \ldots v_n\}$ is therefore a basis for V.

8. Consider the set $V = \{p \in P_3(\mathbb{R}) \mid p'(1) = 0\}$. Prove that V is a subspace of $P_3(\mathbb{R})$ and construct a basis for V. What is its dimension?

   Solution. Since $p(t) \equiv 0$ is a polynomial of degree -1, $p \in P_3(\mathbb{R})$. Additionally, $p'(1) = 0'(1) = 0$, and so $p \in V$ and V is not empty. Suppose p and q are polynomials in V. Then $p'(1) = q'(1) = 0$. then, $(p+q)'(1) = p'(1) + q'(1) = 0 + 0 = 0$, and so $p + q \in V$. Additionally, for $c \in \mathbb{R}$, we have $(cp)'(1) = cp'(1) = c \cdot 0 = 0$, and so $cp \in V$. Hence V is a subspace of $P_3(\mathbb{R})$. Next we construct a basis for V. If $p(t) := a + bt + ct^2 + dt^3 \in V$, then

   $$p'(1) = b + 2c + 3d = 0 \implies b = -2c - 3d.$$

   This allows us to write $p(t) = a + (-2c - 3d)t + ct^2 + dt^3 = a(1) + c(t^2 - 2t) + d(t^3 - 3t)$, and so $p \in \operatorname{span}\{1, t^2 - 2t, t^3 - 3t\} =: \operatorname{span}(S)$. Clearly S generates V. It is straightforward to show that S is linearly independent (c.f., Problem 6a), and so S is a basis for V and $\dim V = |S| = 3$.