Homework # 12

Math 140
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By

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Due April 26, 2021

Chapter 06 | Problem 4.1

Use the Gauss-Bonnet Formula to prove that the sum of the exterior angles of a Euclidean polygon is 2π.

Solution

The statement of the Gauss-Bonnet theorem is as follows.

Theorem. *Let γ be a piecewise regular contained within a simply connected geodesic coordinate patch and bounding a region $\mathscr{R}$ in the patch. Let the jump angles at the junctions be $\alpha_1, \dots, \alpha_n$. Then*

$$\iint_{\mathscr{R}} K \, \mathrm{d}A + \int_{\gamma} \kappa_g \, \mathrm{d}s + \sum_{i=1}^{n} \alpha_i = 2\pi.$$

We have proved in a prior homework assignment that the Gaussian curvature of any planar curve is identically zero, and so $K = 0$ for any regular polygon. Further, the regular segments of γ are all line segments and so their curvature is $\kappa = 0$; thus $\kappa_g = \sqrt{\kappa^2 - \kappa_n^2} = 0$. The theorem reduces to

$$\sum_{i=1}^{n} \alpha_i = 2\pi.$$

This completes the proof.

Chapter 06 | Problem 4.3

(Hyperbolic Half Plane, see Problem 2.4) Prove $x =$ constant gives a geodesic. Prove $x = a + r\cos\theta$, $y = r\sin\theta$ gives a geodesic with a and r constant and $0 < \theta < \pi$. (Use Problem 5.8 of Chapter 4). Note that in $\mathscr{U}$ these curves are vertical straight lines and circles which meet the x-axis at right angles. They give all the geodesics.

Solution

The curves given above are not unit speed, so we need to expend our geodesic formalism to include curves of that form. In Problem 5.8 of Chapter 04, it was shown that $\boldsymbol{\gamma}(t) = \boldsymbol{x}(\gamma^1(t), \gamma^2(t))$ is a unit speed curve if and only if

$$\frac{\mathrm{d}^2\gamma^1}{\mathrm{d}t^2}\frac{\mathrm{d}\gamma^2}{\mathrm{d}t} - \frac{\mathrm{d}^2\gamma^2}{\mathrm{d}t^2}\frac{\mathrm{d}\gamma^1}{\mathrm{d}t} + \left(\frac{\mathrm{d}\gamma^2}{\mathrm{d}t}{\Gamma_{ij}}^1 + \frac{\mathrm{d}\gamma^1}{\mathrm{d}t}{\Gamma_{ij}}^2\right)\frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t} = 0$$

(Einstein summation convention assumed). For the curve $x =$ constant, we can write $\boldsymbol{\gamma} = \boldsymbol{x}(x_0, t)$ in local coordinates. Then $\mathrm{d}\gamma^2/\mathrm{d}t = 0$, and so the left side of the above equation vanishes everywhere. As such, $x =$ constant is indeed a geodesic. Next, we take $\boldsymbol{\gamma} = \boldsymbol{x}(a + r\cos\theta, r\sin\theta)$. Since a and r are constant, our local parameter is θ. Then

$$\begin{aligned}
\frac{\mathrm{d}\gamma^1}{\mathrm{d}\theta} &= -r\sin\theta \quad ; \quad \frac{\mathrm{d}^2\gamma^1}{\mathrm{d}\theta^2} = -r\cos\theta,\\
\frac{\mathrm{d}\gamma^2}{\mathrm{d}\theta} &= r\cos\theta \quad ; \quad \frac{\mathrm{d}^2\gamma^2}{\mathrm{d}\theta^2} = -r\sin\theta.
\end{aligned}$$

To compute the Christoffel symbols, we recall that the metric took the form

$$(\mathsf{g}_{ij}) = \begin{pmatrix} y^{-2} & 0 \\ 0 & y^{-2} \end{pmatrix}$$

and so, with

$${\Gamma_{ij}}^k = \frac{1}{2}\mathsf{g}^{k\ell}\left(\frac{\partial \mathsf{g}_{j\ell}}{\partial u^i} + \frac{\partial \mathsf{g}_{\ell i}}{\partial u^j} - \frac{\partial \mathsf{g}_{ij}}{\partial u^\ell}\right),$$

we get ${\Gamma_{ij}}^1 = -y^{-1}(\delta_{2i}\delta_{1j} + \delta_{1i}\delta_{2j} - \delta_{ij})$ and ${\Gamma_{ij}}^2 = -y^{-1}(2\delta_{2i}\delta_{2j} - \delta_{ij})$. Using the fact that $y = r\sin\theta$ along γ, we find

$$\begin{aligned}
\left(\frac{\mathrm{d}\gamma^2}{\mathrm{d}t}{\Gamma_{ij}}^1 + \frac{\mathrm{d}\gamma^1}{\mathrm{d}t}{\Gamma_{ij}}^2\right)\frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\frac{\mathrm{d}\gamma^j}{\mathrm{d}t} &= -\frac{\cos\theta}{\sin\theta}\left[2\frac{\mathrm{d}\gamma^1}{\mathrm{d}\theta}\frac{\mathrm{d}\gamma^2}{\mathrm{d}\theta} + \sum_i\left(\frac{\mathrm{d}\gamma^i}{\mathrm{d}\theta}\right)^2\right] + 2\left(\frac{\mathrm{d}\gamma^2}{\mathrm{d}\theta}\right)^2 - \sum_i\left(\frac{\mathrm{d}\gamma^i}{\mathrm{d}\theta}\right)^2\\
&= -\frac{\cos\theta}{\sin\theta}\left[-2r^2\sin\theta\cos\theta + r^2\right] + 2r^2\cos^2\theta - r^2\\
&= r^2\frac{\cos\theta}{\sin\theta} - r^2
\end{aligned}$$

Also

$$\frac{\mathrm{d}^2\gamma^1}{\mathrm{d}t^2}\frac{\mathrm{d}\gamma^2}{\mathrm{d}t} - \frac{\mathrm{d}^2\gamma^2}{\mathrm{d}t^2}\frac{\mathrm{d}\gamma^1}{\mathrm{d}t} = (-r\cos\theta)(r\cos\theta) - (-r\sin\theta)(-r\sin\theta) = -r^2.$$

Clearly there is a mistake here, since these do not sum to zero.[1]

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[1] Sorry, I ran out of time this week. That's been happening more and more lately!

Chapter 06 | Problem 5.1

Prove theorem 5.5:

Theorem (5.5)**.** *Let γ be a piecewise regular curve in an oriented surface M. Suppose γ bounds a region $\mathscr{R}$. Then*

$$\iint_{\mathscr{R}} K\,\mathrm{d}A + \int_{\gamma} \kappa_g\,\mathrm{d}s + \sum_i (\pi - \alpha_i) = 2\pi\chi(\mathscr{R}),$$

where the α_i are the interior angles of γ and $\chi(\mathscr{R})$ is the Euler characteristic of $\mathscr{R}$ found by breaking $\mathscr{R}$ into polygons and counting those edges and vertices lying on γ in addition to those in $\mathscr{R}$.

Solution

Since $\overline{\mathscr{R}}$ is compact, we can write the region $\mathscr{R}$ as a finite union of open subcovering polygons

$$\mathscr{R} = \bigcup_{k=1}^{F} \mathscr{R}_k$$

where each $\mathscr{R}_k$ can be taken to be the image of a geodesic coordinate patch. If we decompose

$$I = \iint_{\mathscr{R}} K\,\mathrm{d}A$$

over the sum of these $\mathscr{R}_k$'s, we find by the Gauss-Bonnet Theorem

$$I = \sum_{k=1}^{F} \left\{ \iint_{\mathscr{R}_k} K\,\mathrm{d}A \right\} = \sum_{k=1}^{F} \left\{ 2\pi - \int_{\gamma_k} \kappa_g\,\mathrm{d}s - \sum_{\ell} (\pi - \alpha_{k\ell}) \right\}$$

Here κ_g vanishes along the boundaries of each of the subcovering polygons $\mathscr{R}_k$, but is not generally zero along the boundary γ of $\mathscr{R}$. Similarly, the sum of the interior $(\pi - \alpha_{k\ell})$'s is $2\pi(-E+V)$ in accordance with the proof of the Gauss-Bonnet theorem[2], but the sum along γ is unknown. Hence

$$I = 2\pi(F - E + V) - \int_{\gamma} \kappa_g\,\mathrm{d}s - \sum_{\ell} (\pi - \alpha_\ell).$$

Since $\chi(\mathscr{R}) = F - E + V$ and $I = \iint_{\mathscr{R}} K\,\mathrm{d}A$, this completes the proof.

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[2]Theorem 5.1 on page 188 of Millman and Parker.

Chapter 06 | Problem 5.5

Prove that a compact surface in $\mathbb{R}^3$ which has $K \geq 0$ "looks like" a sphere. (*Hint*: You may use Theorem 5.3.)

Solution

Theorem 5.3 is quoted as follows.

Theorem (5.3)**.** *Every compact surface in* $\mathbb{R}^3$ *"looks like" a sphere with handles. Then number of handles is called the* ***geometric genus*** *of* M *and is denoted* g*.*

According to the theorem, a compact surface will "look like" a sphere if its geometric genus g is zero. Recall Theorems 5.1 and 5.4.

Theorem (Gauss-Bonnet, 5.1)**.** *If* M *is compact, then* $\iint_{\mathscr{R}} K \, \mathrm{d}A = 2\pi\chi$*.*

Theorem (5.4)**.** *If* M *is a compact surface in* $\mathbb{R}^3$*, then* $\chi = 2(1-g)$*.*

In particular, this allows us to write

$$\frac{1}{2\pi} \iint_{\mathscr{R}} K \, \mathrm{d}A = 2 - 2g.$$

It follows from Theorem 5.3 that that g is a positive integer. Further, since $K \geq 0$, we must have $g \leq 1$. If $K = 0$ everywhere, then $g = 1$, which is a contradiction. Hence, K cannot vanish everywhere. But then

$$\iint_{\mathscr{R}} K \, \mathrm{d}A > 0 \quad \Longrightarrow \quad \iint_{\mathscr{R}} K \, \mathrm{d}A \geq 2\pi$$

by the Gauss-Bonnet theorem. Hence g must be zero for a compact surface of nonnegative curvature, and so that surface must "look like" a sphere.