# Homework # 07

Math 140
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By

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# Chapter 04 | Problem 5.6

Prove that if $M$ is a surface of revolution and $\boldsymbol{\gamma}$ is a geodesic, then $r\cos\beta(s) = \text{constant}$, where $\beta(s)$ is the angle between $\boldsymbol{\gamma}'(s)$ and the circle of latitude (of radius $r$) through $\boldsymbol{\gamma}(\text{s})$.

## Solution

Suppose $(r(s), z(s))$, $s \in (s_1, s_2)$, is the unit speed curve which generates the surface $M$. Choose the following coordinate patch from $M$:

$$\boldsymbol{x}(s,\theta) = (r(s)\cos\theta, r(s)\sin\theta, z(s))$$

for $s \in (s_1, s_2)$ and $\theta \in (0, 2\pi)$. Suppose

$$\boldsymbol{\gamma}(s) = \boldsymbol{x}(\gamma^1, \gamma^2) = (r(\gamma^1)\cos\gamma^2, r(\gamma^1)\sin\gamma^2, z(\gamma^1))$$

is a geodesic on $M$ and suppose

$$\boldsymbol{\varphi}(\theta) = (r_0\cos\theta, r_0\sin\theta, z_0)$$

is a circle of latitude of $M$ with radius $r_0 = r(s_0)$ for $s_0 \in (s_1, s_2)$. We can express the quantity $r_0\cos\beta(s)$ in terms of an inner product as follows:[1]

$$r_0\cos\beta(s) = r_0\frac{\langle\boldsymbol{\varphi}_\theta, \boldsymbol{\gamma}'\rangle}{\|\boldsymbol{\varphi}\|\,\|\boldsymbol{\gamma}'\|} = \frac{r_0}{\sqrt{r_0^2+z_0^2}}\left\langle\boldsymbol{\varphi}_\theta, \boldsymbol{\gamma}'\right\rangle,$$

where $\boldsymbol{\varphi}_\theta := \partial\boldsymbol{\varphi}/\partial\theta$ is the tangent to the circle of latitude $\boldsymbol{\varphi}$. I claim that this quantity is constant. To prove this, we take the derivative with respect to $s$:

$$\frac{\mathrm{d}}{\mathrm{d}s}\{r_0\cos\beta(s)\} = \frac{r_0}{\sqrt{r_0^2+z_0^2}}\frac{\mathrm{d}}{\mathrm{d}t}\big\{\langle\boldsymbol{\varphi}_\theta, \boldsymbol{\gamma}'\rangle\big\} = \frac{r_0}{\sqrt{r_0^2+z_0^2}}\big[\langle\boldsymbol{\varphi}_\theta', \boldsymbol{\gamma}'\rangle + \langle\boldsymbol{\varphi}_\theta, \boldsymbol{\gamma}''\rangle\big].$$

But since $\boldsymbol{\varphi}_\theta$ does not depend on $s$, we get $\boldsymbol{\varphi}_\theta' = \boldsymbol{0}$ and hence $\langle\boldsymbol{\varphi}_\theta', \boldsymbol{\gamma}'\rangle = 0$. Further, since $\boldsymbol{\gamma}$ is a geodesic, we have $\boldsymbol{\gamma}'' = \kappa_n\boldsymbol{n}$. Hence, we find

$$\frac{\mathrm{d}}{\mathrm{d}s}\{r_0\cos\beta(s)\} = \frac{\kappa_n r_0}{\sqrt{r_0^2+z_0^2}}\left\langle\boldsymbol{\varphi}_\theta, \boldsymbol{n}\right\rangle.$$

Note that since $\boldsymbol{\varphi}_\theta$ is tangent to $M$ at $\boldsymbol{\gamma}(s)$, we have $\boldsymbol{\varphi}_\theta \in T_{\boldsymbol{\gamma}(s)}M$. As such, $\boldsymbol{\varphi}_\theta$ is necessarily normal to $\boldsymbol{n}$, and so $\langle\boldsymbol{\varphi}_\theta, \boldsymbol{n}\rangle = 0$. Thus

$$\frac{\mathrm{d}}{\mathrm{d}s}\{r_0\cos\beta(s)\} = 0,$$

which implies $r_0\cos\beta(s)$ is constant.

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[1] Sorry about the sloppy notation. The primes denote differentiation by $s$ and subscript $\theta$'s denote differentiation by $\theta$. I would have fixed this, but I did not have time.

## Chapter 04 | Problem 5.11

Let $M$ be the surface given by $x^2 + y^2 - z^2 = 1$ (Example 2.8). Find as many geodesics as you can.

### Solution

The surface $M$ is easily seen to be a hyperboloid of one sheet. A sketch of the surface (generated by Geogebra) is included below.

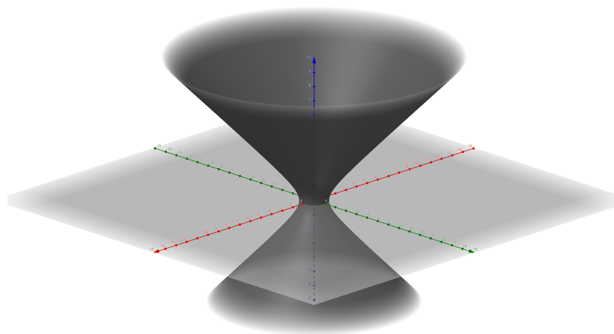

Figure 1: The surface $M$ given by $x^2 + y^2 - z^2 = 1$.

The shaded plane in the figure is the $xy$ plane, and the blue axis perpendicular to the plane is the $z$ axis. We have the following proposition from Millman and Parker:[2]

**Proposition.** *Let $M$ be a surface of revolution generated by the unit speed curve $(r(s), z(s))$. Then*

1. *every meridian is a geodesic; and*
2. *a circle of latitude is a geodesic if and only if the tangent $\boldsymbol{x}_1$ to the meridians is parallel to the axis of revolution at all points on the circle of latitude.*

In our case, $M$ can be considered a surface of revolution generated by rotating a unit speed hyperbola in (say) the $yz$ plane about the $z$ axis. As such, the proposition above applies. We see that the meridians are the curves along the surface $M$ with constant azimuthal angle $\theta$ in the $zy$ plane and the circles of latitide are the circles while lie on $M$ and are parallel to the $xy$ plane. As such, we find that the following curves are geodesics on $M$:

1. All planar curves which lie on the intersection of $M$ and a plane which contain the $z$ axis (i.e., the meridians).
2. The circle of latitiude which lies on the $xy$ plane.

We could alternatively use the results of a previous homework to find the same geodesics. In Homework 06, we found that the curve of intersection of a surface $M$ and a plane $\Pi$ is a geodesic if and only if $\Pi$ is a plane of symmetry of $M$. It is clear that every plane which intersects our curve and which contains the axis of revolution is a plane of symmetry because $M$ is a surface of revolution. Additionally, the circle of revolution contained in the $xy$ plane is a geodesic since $M$ is clearly symmetric about the $xy$ plane.[3]

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[2] Proposition 5.5 of Chapter 4 (page 110).

[3] These aren't *all* of the geodesics on $M$, but they do represent two such classes. If one can prove that there are straight lines embedded on $M$ (which there are, but I don't have time to prove it), then these straight lines are also geodesics.

## Chapter 04 | Problem 6.5

Let $\boldsymbol{\gamma}$ be a curve with $\boldsymbol{\gamma}(0) = P$, $\boldsymbol{\gamma}(1) = Q$. If $\tilde{\boldsymbol{X}} \in T_P M$, let $\boldsymbol{X}$ be the parallel translate of $\tilde{\boldsymbol{X}}$ along $\boldsymbol{\gamma}$. Define $\boldsymbol{\gamma}^{\#} : T_P M \to T_Q M$ by $\boldsymbol{\gamma}^{\#}(\tilde{\boldsymbol{X}}) = \boldsymbol{X}(1)$.

**(a)** Prove that $\boldsymbol{\gamma}^{\#}$ is a linear transformation.
**(b)** Prove that $\boldsymbol{\gamma}^{\#}$ is an isometry. That is, that $\langle \boldsymbol{\gamma}^{\#}(\tilde{\boldsymbol{X}}), \boldsymbol{\gamma}^{\#}(\tilde{\boldsymbol{Y}}) \rangle = \langle \tilde{\boldsymbol{X}}, \tilde{\boldsymbol{Y}} \rangle$.
**(c)** Prove that $\boldsymbol{\gamma}^{\#}$ is an isomorphism. $\boldsymbol{\gamma}^{\#}$ is called the *parallelism* defined by $\boldsymbol{\gamma}$.

### Solution

**(a)** Let $a \in \mathbb{R}$ be constant, and suppose $\tilde{\boldsymbol{X}}$, $\tilde{\boldsymbol{Y}} \in T_P M$. Since $T_P M$ is a vector space, we have $a\tilde{\boldsymbol{X}} + \tilde{\boldsymbol{Y}} \in T_P M$, and $\boldsymbol{\gamma}^{\#}(a\tilde{\boldsymbol{X}} + \tilde{\boldsymbol{Y}})$ is the unique parallel translate of $a\tilde{\boldsymbol{X}} + \tilde{\boldsymbol{Y}}$ to $T_Q M$. That is, $\boldsymbol{\gamma}^{\#}(a\tilde{\boldsymbol{X}} + \tilde{\boldsymbol{Y}}) = (a\boldsymbol{X} + \boldsymbol{Y})(1)$. Since $T_Q M$ is a vector space, we have $(a\boldsymbol{X} + \boldsymbol{Y})(1) = a\boldsymbol{X}(1) + \boldsymbol{Y}(1)$, and so $\boldsymbol{\gamma}^{\#}$ is indeed a linear transformation.

**(b)** This is an immediate consequence of the following proposition from Millman and Parker.[4]

**Proposition.** *Let $\boldsymbol{X}(t)$ and $\boldsymbol{Y}(t)$ both be parallel along a regular curve $\boldsymbol{\gamma}$ in $M$. Then $\|\boldsymbol{X}(t)\|$ is constant and so is the angle between $\boldsymbol{X}(t)$ and $\boldsymbol{Y}(t)$.*

This follows from the fact that $\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t}$ is everywhere normal to $M$. In any case, since $\boldsymbol{\gamma}^{\#}(\tilde{\boldsymbol{X}})$ and $\boldsymbol{\gamma}^{\#}(\tilde{\boldsymbol{Y}})$ are parallel translates of $\tilde{\boldsymbol{X}}$ and $\tilde{\boldsymbol{Y}}$, respectively, the preceding proposition implies that the angle between them is constant, and indeed the cosine of the angle between them is constant. As such, we find

$$\frac{\langle \boldsymbol{\gamma}^{\#}(\tilde{\boldsymbol{X}}), \boldsymbol{\gamma}^{\#}(\tilde{\boldsymbol{Y}}) \rangle}{\|\boldsymbol{\gamma}^{\#}(\tilde{\boldsymbol{X}})\| \, \|\boldsymbol{\gamma}^{\#}(\tilde{\boldsymbol{Y}})\|} = \frac{\langle \tilde{\boldsymbol{X}}, \tilde{\boldsymbol{Y}} \rangle}{\|\tilde{\boldsymbol{X}}\| \, \|\tilde{\boldsymbol{Y}}\|}.$$

The same proposition guarantees that the lengths are invariant under parallel translation, i.e., that $\|\boldsymbol{\gamma}^{\#}(\tilde{\boldsymbol{X}})\| = \|\tilde{\boldsymbol{X}}\|$ and $\|\boldsymbol{\gamma}^{\#}(\tilde{\boldsymbol{Y}})\| = \|\tilde{\boldsymbol{Y}}\|$. The result follows.

**(c)** Clearly $\dim(T_P M) = \dim(T_Q M) = 2$ since they are both tangent spaces of a three-dimensional surface. And,

$$\boldsymbol{\gamma}^{\#}(\tilde{\boldsymbol{x}}_i) = \boldsymbol{x}_i(1),$$

that is, $\boldsymbol{\gamma}^{\#}$ maps a basis of $T_P M$ to a basis of $T_Q M$, so $\operatorname{rank}(\boldsymbol{\gamma}^{\#}) = 2$. By the rank-nullity theorem, the nullspace of $\boldsymbol{\gamma}^{\#}$ contains only $\mathbf{0} \in T_P M$ and hence $\boldsymbol{\gamma}^{\#}$ is an isomorphism.

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[4] Proposition 6.9 of Chapter 4 (page 119).

## Chapter 04 | Problem 7.2

Show that for $M = S^1 \times (0,1)$ (Example 2.3) $\mathsf{L}$ can be represented by the matrix

$$\mathsf{L} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

### Solution

My derivation implies $L^1{}_1 = -1$. A discussion of this discrepancy is included at the end of the problem. In any case, I will show this equality in two ways. First, I will calculate the coefficients of the second fundamental form $L_{\mu\nu}$ and then use the inverse metric $\mathsf{g}^{\mu\nu}$ to raise the first index, i.e., $L^i{}_j = \mathsf{g}^{i\ell}L_{\ell j}$ (Einstein summation convention assumed). From Example 2.3, we know that one such coordinate patch for the surface is

$$\boldsymbol{x}(\theta, t) = (\cos\theta, \sin\theta, t)$$

for $\theta \in (-3\pi/4, 3\pi/4)$ and $t \in (0,1)$. The basis vectors for the tangent space are thus

$$\boldsymbol{x}_1 \equiv \frac{\partial \boldsymbol{x}}{\partial \theta} = (-\sin\theta, \cos\theta, 0)$$
$$\boldsymbol{x}_2 \equiv \frac{\partial \boldsymbol{x}}{\partial t} = (0, 0, 1).$$

The second derivatives are seen to be

$$\boldsymbol{x}_{11} = \frac{\partial \boldsymbol{x}_1}{\partial \theta} = (-\cos\theta, -\sin\theta, 0)$$
$$\boldsymbol{x}_{12} = \boldsymbol{x}_{21} = \boldsymbol{x}_{22} = 0.$$

Finally, we calculate the normal vector to the tangent plane. We have

$$\boldsymbol{x}_1 \times \boldsymbol{x}_2 = (\cos\theta, \sin\theta, 0) = \boldsymbol{n}$$

since it is already unit length. Thus, the coefficients of the second fundamental form $L_{ij} = \langle \boldsymbol{x}_{ij}, \boldsymbol{n}\rangle$ are seen to be

$$L_{11} = \langle(-\cos\theta, -\sin\theta, 0), (\cos\theta, \sin\theta, 0)\rangle = -1$$
$$L_{12} = L_{21} = L_{22} = 0$$

since $\boldsymbol{x}_{ij} = 0$ for all $(i,j) \neq (1,1)$. Next, we find the metric $\mathsf{g}_{ij} = \langle \boldsymbol{x}_1, \boldsymbol{x}_j\rangle$. We get

$$\mathsf{g}_{11} = \langle \boldsymbol{x}_1, \boldsymbol{x}_1\rangle = 1$$
$$\mathsf{g}_{21} = \mathsf{g}_{12} = 0$$
$$\mathsf{g}_{22} = 1.$$

Hence the metric is the identity, so trivially the inverse metric is the identity and the components $L^i{}_j$ are the same as the components $L_{ij}$. Hence

$$\mathsf{L} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{1}$$

Alternatively, we can interpret the Weingarten map as derivatives of $\boldsymbol{n}$. We have

$$\mathsf{L}(\boldsymbol{x}_1) = -\frac{\partial n}{\partial \theta} = (\sin\theta, -\cos\theta, 0)$$
$$\mathsf{L}(\boldsymbol{x}_2) = -\frac{\partial n}{\partial t} = (0, 0, 0).$$

Using Proposition 7.6(b) in Millman and Parker (page 125), we write $\mathsf{L}(\boldsymbol{x}_1) = L^\ell{}_i \boldsymbol{x}_\ell$ and get:

$$L^1{}_1\boldsymbol{x}_1 + L^2{}_1\boldsymbol{x}_2 = (-\sin\theta, \cos\theta, 0)$$
$$L^1{}_2\boldsymbol{x}_1 + L^2{}_2\boldsymbol{x}_2 = (0, 0, 0)$$

Using the expressions for the $\boldsymbol{x}_i$ above, we get:

$$(-L^1{}_1 \sin\theta, L^1{}_1 \cos\theta, L^1{}_2) = (\sin\theta, -\cos\theta, 0)$$
$$(-L^2{}_1 \sin\theta, L^2{}_1 \cos\theta, L^2{}_2) = (0, 0, 0).$$

Again, this implies $L^1{}_1 = -1$ and $L^i{}_j = 0$ otherwise, and so we arrive at (1) again.

I suspect that this discrepancy is due to my choice of coordinate patch. Perhaps if I had chosen, say, $\boldsymbol{x}(\theta, t) = (\sin\theta, \cos\theta, t)$, I would have arrived at the answer in the book. In any case, we know from the previous homework that the components of the second fundamental form transform doubly-covariantly *up to sign*,

$$L_{ij} = \pm \bar{L}_{\alpha\beta} \frac{\partial \bar{x}^\alpha}{\partial x^i} \frac{\partial \bar{x}^\beta}{\partial x^j},$$

under a coordinate transformation $(\bar{x}^1, \bar{x}^2) \mapsto (x^1, x^2)$. Since the matrix elements $L^i{}_j$ are determined from the coefficients of the second fundamental form $L_{ij}$ by raising an index, i.e., $L^i{}_j = g^{i\ell} L_{\ell j}$, and since the inverse metric transforms doubly-contravariantly:

$$g^{ij} = \bar{g}^{\mu\nu} \frac{\partial x^i}{\partial \bar{x}^\mu} \frac{\partial x^j}{\partial \bar{x}^\nu}$$

(with no ambiguity in the sign), we find that the matrix elements transform according to the following transformation law:

$$L^i{}_j = g^{i\ell} L_{\ell j} = \pm \bar{g}^{\mu\lambda} \bar{L}_{\lambda\nu} \frac{\partial x^i}{\partial \bar{x}^\mu} \frac{\partial \bar{x}^\nu}{\partial x^j} = \pm \bar{L}^\mu{}_\nu \frac{\partial x^i}{\partial \bar{x}^\mu} \frac{\partial \bar{x}^\nu}{\partial x^j},$$

and so $\mathsf{L}$ is only well-defined up to sign.

## Chapter 04 | Problem 7.4

Prove the following lemma.

**Lemma.** *If $P \in M$ and $\boldsymbol{X}, \boldsymbol{Y} \in T_P(M)$, then* $\mathbb{I}(\boldsymbol{X}, \boldsymbol{Y}) = \langle \mathsf{L}(\boldsymbol{X}), \boldsymbol{Y} \rangle = \langle \boldsymbol{X}, \mathsf{L}(\boldsymbol{Y}) \rangle$.

### Solution

I will do this by showing that $\langle \mathsf{L}(\boldsymbol{X}), \boldsymbol{Y} \rangle = \langle \boldsymbol{X}, \mathsf{L}(\boldsymbol{Y}) \rangle$, and using an intermediate step to conclude $\langle \mathsf{L}(\boldsymbol{X}), \boldsymbol{Y} \rangle = \mathbb{I}(\boldsymbol{X}, \boldsymbol{Y}) = \langle \mathsf{L}(\boldsymbol{X}) \rangle$. Recall that the Weingarten map is linear, hence if $\boldsymbol{u} = a\boldsymbol{x}_1 + b\boldsymbol{x}_2$ for real numbers $a, b$, then

$$(\mathsf{L}(\boldsymbol{u}))_j = (a\mathsf{L}(\boldsymbol{x}_1) + b\mathsf{L}(\boldsymbol{x}_2))_j = aL^i{}_j\boldsymbol{x}_i + bL^k{}_j\boldsymbol{x}_k$$

(Einstein summation convention assumed). For $\boldsymbol{X}, \boldsymbol{Y} \in T_P(M)$, write $\boldsymbol{X} = X^i\boldsymbol{x}_i$ and $\boldsymbol{Y} = y^j\boldsymbol{x}_j$. Then:

$$\begin{aligned}
\langle \mathsf{L}(\boldsymbol{X}), \boldsymbol{Y} \rangle &= \langle X^i L^j{}_i \boldsymbol{x}_j, Y^k \boldsymbol{x}_k \rangle \\
&= X^i L^j{}_i Y^k \mathsf{g}_{jk} \\
&= L^{j}{}_i \mathsf{g}_{jk} X^i Y^k \\
&= L_{ki} X^i Y^k.
\end{aligned}$$

But we know that the components of the second fundamental form are symmetric in their two covariant indices. Hence $L_{ki} = L_{ik}$ and $\langle \mathsf{L}(\boldsymbol{X}), \boldsymbol{Y} \rangle = L_{ik}X^iY^k = \mathbb{I}(\boldsymbol{X}, \boldsymbol{Y})$. If instead we raised the other index:

$$\begin{aligned}
\langle \mathsf{L}(\boldsymbol{X}), \boldsymbol{Y} \rangle &= L_{ki} X^i Y^k \\
&= \mathsf{g}_{i\ell} L^{\ell}{}_k X^i Y^k \\
&= X^i L^\ell{}_k Y^k \mathsf{g}_{i\ell} \\
&= \langle X^i \boldsymbol{x}_i, L^\ell{}_k Y^k \boldsymbol{x}_\ell \rangle \\
&= \langle \boldsymbol{X}, \mathsf{L}(\boldsymbol{Y}) \rangle,
\end{aligned}$$

which is what we wanted to prove.