Linear Algebra

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Problem Set 4: Diagonalization and Eigenbases

1. Let
$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix} \in M_{2\times 2}(\mathbb{R}).$$
Determine whether or not A is diagonalizable. If it is, diagonalize it.

2. Let $T \in \mathscr{L}(\mathbb{C}^2, \mathbb{C}^2)$ (where $\mathbb{C}^2 := \mathbb{C} \times \mathbb{C}$ is a vector space over $\mathbb{C}$) be such that
$$T(z, w) = (z + iw, iz + w).$$
Find a basis β for $\mathbb{C}^2$ for which $[T]_\beta$ is diagonal.

3. In lecture, we defined the characteristic polynomial of a linear operator $T \in \mathscr{L}(\mathbf{V})$ by first fixing a basis β for $\mathbf{V}$, computing $A = [T]_\beta$, and then writing $f(t) := \det(A - \lambda \mathbf{I}_n)$. Prove that this is indeed well-defined; that is, if β and $\bar{\beta}$ are distinct bases for $\mathbf{V}$, and $A = [T]_\beta$ and $\bar{A} = [T]_{\bar{\beta}}$, then $\det(A - t\mathbf{I}_n) = \det(\bar{A} - t\mathbf{I}_n)$ for all t.

   [*Hint: Use Problem Set 2, Problem 8*]

4. **(a)** Consider the following false-proof that the determinant of a matrix is equal to the product of its eigenvalues.

   Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_k$ with algebraic multiplicities $m_1, \ldots, m_k$ such that $\sum_{i=1}^k m_i = n$. Then there exists a diagonal matrix D whose diagonal elements are the eigenvalues of A (where λ_i occurs m_i times) such that
   $$A = Q^{-1} D Q$$
   where Q is the (invertible) matrix which changes coordinates into the eigenbasis. Then,
   $$\begin{aligned}\det(A) &= \det(Q^{-1}DQ) = \det(Q^{-1})\det(D)\det(Q) \\ &= \det(Q)^{-1}\det(Q)\det(D) = \det(D) = \prod_{i=1}^n D_{ii} = \prod_{i=1}^k \lambda_i^{m_i}.\end{aligned}$$
   Hence the determinant of a matrix is the product of its eigenvalues.

   What is wrong with this proof?

(b) When we learn about Jordan canonical forms, we can formulate a correct version of this proof. For now, prove that the determinant of a matrix equals the product of its eigenvalues by examining its characteristic polynomial.

5. Define the trace $\mathrm{tr} : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ by $\mathrm{tr}(A) := \sum_{i=1}^{n} A_{ii}$.

(a) Provide a proof similar to Problem 4a that the trace of a diagonalizable matrix is equal to the sum of its eigenvalues. In particular, assume A has distinct eigenvalues are $\lambda_1, \dots, \lambda_k$ with algebraic multiplicities $m_1, \dots, m_k$ such that $\sum_{i=1}^{k} m_i = n$, and assume $A = Q^{-1}DQ$ where D is a diagonal matrix D whose diagonal elements are the eigenvalues of A (where λ_i occurs m_i times) and Q is the change of basis matrix, and show $\mathrm{tr}(A) = \mathrm{tr}(Q^{-1}DQ) = \sum_{i=1}^{k} m_i\lambda_i$.

(b) The proof in (a) is not sufficiently general for the same reasons Problem 4a was not sufficiently general, but we can again provide a sufficient proof by again examining the characteristic polynomial $f(t) = \det(A - t\mathbf{I}_n)$. Compute the coefficient on t^{n-1} in two different ways: first by the determinant definition above, and second by recognizing f splits as $f(t) = (-1)^n(t-\lambda_1)^{m_1}\cdots(t-\lambda_k)^{m_k}$.

[*Note: This proof can be adapted to prove one of Vieta's formulas about the roots of a polynomial. These are not relevant to this course, but they are pretty fascinating nonetheless.*]

6. Consider the matrix
$$A := \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

(a) Compute A^2, A^3, and A^4 by hand. This should generate a familiar sequence of numbers. Make a conjecture about what A^n looks like for some integer $n \geq 1$.

(b) Given that two eigenvectors of this matrix are
$$v_1 = \begin{pmatrix} 2 \\ 1+\sqrt{5} \end{pmatrix} \quad ; \quad v_2 = \begin{pmatrix} 2 \\ 1-\sqrt{5} \end{pmatrix},$$
compute A^n by first changing to an eigenbasis, computing the new representation of A^n in that basis, then converting back into the standard basis. Assuming your conjecture was correct, what does this tell you?

7. This question deals with simultaneously diagonalizable matrices, which have rich application in quantum mechanics.

Definition 1. Two linear operators T and U in $\mathscr{L}(\mathbf{V})$ are called simultaneously diagonalizable if there exists an ordered basis β for $\mathbf{V}$ for which both $[T]_\beta$ and $[U]_\beta$ are diagonal. Similarly, two matrices A and B in $M_{n\times n}(\mathbb{F})$ are simultaneously diagonalizable if there exists an invertible matrix $Q \in M_{n\times n}(\mathbb{F})$ for which $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal.

You may have heard of the *Heisenberg uncertainty principle*, which loosely states that we cannot know the position and the momentum of an object to arbitrary certainty; or $\sigma_x\sigma_p \geq \hbar/2$. It turns out there are uncertainty relations between any two operators which are not simultaneously diagonalizable.

(a) Prove that if T and U are simultaneously diagonalizable operators, then T and U commute (i.e., $T \circ U = U \circ T$).

(b) Prove that if A and B are simultaneously diagonalizable matrices, then A and B commute (i.e., $AB = BA$).

The converse of these statements is also true, but to prove it, we need knowledge of T-invariant subspaces, which we will not cover.

8. This question is about the matrix exponential. Originally I was going to ask you to solve a system of ordinary differential equations using this method, but I felt it was too much to ask you to build up all of the theory yourself. The matrix exponential is indeed used to solve a certain class of ODE's though. If you're interested, let me know and we can talk about it.

   (a) Suppose $D \in M_{n\times n}(\mathbb{F})$ is a diagonal matrix with $D_{ii} = \lambda_i$. Compute e^{Dt}, where

   $$e^A := \mathbf{I}_n + A + \frac{A^2}{2} + \frac{A^3}{6} + \cdots = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

   is the matrix exponential and $t \in \mathbb{F}$. (You can just do this with 3×3 matrices if you want.)

   (b) Suppose $A \in M_{n\times n}(\mathbb{F})$ is diagonalizable. Compute e^{At} by first writing A in terms of its diagonalization.

   [*Hint: this should not involve much work beyond what you did in (a). Try to write* $e^{PDP^{-1}}$ *in terms of* e^D.]