Homework # 13

Math 140
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Chapter 06 | Problem 7.1

Let M be a compact surface. Prove that M has a vector field with no zeros if and only if M "looks like" a torus.

Solution

Lemma. *Let M be a compact surface, and suppose M has a vector field with no zeros. Then M 'looks like" a torus.*

Proof. Suppose M is a compact surface with a vector field $\boldsymbol{V}$ which has no zeros. Then we have $I(\boldsymbol{V}) = 0$, since the sum is empty. From the Poincaré-Brouwer Theorem, $\chi(M) = I(\boldsymbol{V}) = 0$. Then $\chi(M) = 2(1-g) = 0$ implies $g = 1$ and so M "looks like" a torus.[1] □

Lemma. *Let M be a compact surface which 'looks like" a torus. Then M has a vector field with no zeros.*

Proof. We can prove such a field exists by constructing one. Each meridian of a torus is a circle, and so we can define a vector field $\boldsymbol{v}(\theta_0, t)$ there such that $\langle \boldsymbol{v}(\theta_0, t), \boldsymbol{r} \rangle = 0$ where $\boldsymbol{r}$ is a vector parallel to any radius of the meridian. As long as we choose the direction of $\boldsymbol{v}(\theta_0, t)$ consistently (perhaps by choosing $\boldsymbol{v}(\theta_0, t)$ to always point "downwards" along the circle of latitude[2]), then we can extend this definition analytically through θ to get a vector field on M which is everywhere nonzero, which completes the problem. □

Chapter 06 | Problem 7.2

Prove that on $M = \mathbb{R}^2$ there are vector fields with infinitely many zeros P_n such that the sequence $\{P_n\}$ does not have a convergent subsequence.

Solution

It is sufficient to construct an example of such a field. To do so, construct a unit vector field on M and choose a sequence $(\boldsymbol{x}_n)$ of points along any line in M such that

$$\|\boldsymbol{x}_{n+1} - \boldsymbol{x}_n\| > \|\boldsymbol{x}_n - \boldsymbol{x}_{n-1}\|.$$

Now smoothly deform the vector field so that the resulting field vanishes at each $\boldsymbol{x}_i$ and remains nonzero elsewhere. Then there are infinitely many zeros $P_n = \boldsymbol{x}_n$. But, since the sequence of zeros is strictly increasing, no subsequence of P_n is convergent, and we are done.

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[1] I cannot find a formal definition of "looks like" in the text, except for the statement defining the geometric genus of a surface. Intuitively, it seems like a surface 'looks like" a torus if its geometric genus $g = 1$. Otherwise, I am unsure.

[2] A notion of "downwards" is well-defined here, since M being compact implies M is orientable.

Chapter 06 | Problem 7.3

Prove that on $M = \mathbb{R}^2$ there are vector fields with no zeros.

Solution

Take the vector field $\boldsymbol{X}(u^1, u^2) = \boldsymbol{x}_0$ where $\boldsymbol{x}_0$ is any nonzero vector in $\mathbb{R}^2$. Then $\boldsymbol{X}$ is indeed a vector field with no zeros.

Chapter 06 | Problem 7.4

Find an example to show that Proposition 7.2 s false if the hypothesis "compact" is omitted.

Solution

The proposition is quoted below.

Proposition. *If* $\mathbf{V}$ *and* $\mathbf{W}$ *are two vector fields on a compact surface* M *with only a finite number of zeros, then* $I(\mathbf{V}) = I(\mathbf{W})$*.*

We need to find a surface M and two distinct vector fields $\mathbf{V}$ and $\mathbf{W}$ with finitely many zeros for which $I(\mathbf{V}) \neq I(\mathbf{W})$. Take $M = \mathbb{R}^2$.

Then a vector field $\mathbf{V}$ with a source at the origin, say, $\boldsymbol{v} = \boldsymbol{x}_1 + \boldsymbol{x}_2$ (and analytically extended outside of the coordinate patch where $\boldsymbol{x}$ is defined) is a vector field with $i_0(\boldsymbol{V}) = 1$. (This is meant to look like the second figure in Example 7.1 on page 195 on Millman and Parker) Since there are no other zeros, $I(\boldsymbol{V}) = 1$.

Consider now the vector field $\boldsymbol{W}$ with a zero at the origin which looks like $\boldsymbol{v} = \boldsymbol{x} - \boldsymbol{y}$ (and again analytically extended outside of the coordinate patch where $\boldsymbol{x}$ is defined). This is a vector field with $i_0(\boldsymbol{W}) = -1$. (This is meant to look like the third figure in Example 7.1 on page 195 on Millman and Parker) Since there are no other zeros, $I(\boldsymbol{W}) = -1$. Hence $I(\mathbf{V}) \neq I(\mathbf{W})$ and the theorem is false if the surface is not compact.
