PART C

PRACTICAL STRENGTH ANALYSIS & DESIGN OF STRUCTURAL COMPONENTS

CHAPTER C1

COMBINED STRESSES. THEORY OF YIELD AND ULTIMATE FAILURE.

C1.1 Uniform Stress Condition

Aircraft structures are subjected to many types of external loadings. These loads often cause axial, bending and shearing stresses acting simultaneously. If structures are to be designed satisfactorily, combined stress relationships must be known. Although in practical structures uniform stress distribution is not common, still sufficient accuracy for design practice is provided by using the stress relationships based on uniform stress assumptions. In deriving these stress relationships, the Greek letter sigma (σ) will represent a stress intensity normal to the surface and thus a tensile or compressive stress and the Greek letter tau (τ) will represent a stress intensity parallel to the surface and thus a shearing stress.

C1.2 Shearing Stresses on Planes at Right Angles.

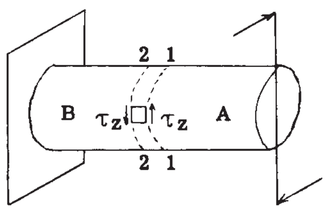


Fig. C1.1

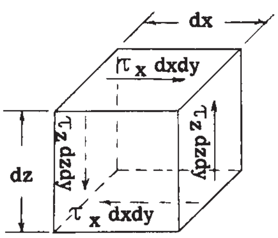


Fig. C1.2

Fig. C1.1 shows a circular solid shaft subjected to a torsional moment. The portion (A) of the shaft exerts a shearing stress τ_z on section (1-1) and portion (B) exerts a resisting shearing stress τ_z on section (2-2). Fig. C1.2 illustrates a differential cube cut from shaft between sections (1-1) and (2-2). For equilibrium a resisting couple must exist on top and bottom face of cube. Taking moments about lower left edge of cube:

$$\tau_x \text{ dxdy (dz)} - \tau_z \text{ dzdy (dx)} = 0$$

hence, $\tau_x = \tau_z$ - - - - - - - - - - - - - - - (1)

Thus if a shearing unit stress occurs on one plane at a point in a body, a shearing unit stress of same intensity exists on planes at right angles to the first plane.

C1.3 Simple Shear Produces Tensile and Compressive Stresses.

Fig. C1.3 shows an elementary block of unit dimensions subjected to pure shearing

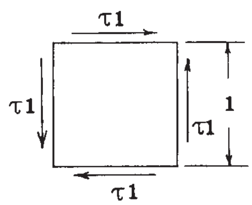


Fig. C1.3

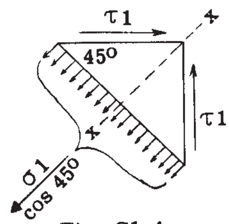


Fig. C1.4

stresses.

Fig. C1.4 shows a free body after the block has been cut along a diagonal section.

For equilibrium the sum of the forces along the x-x axis equals zero.

$$\Sigma F_x = -\frac{\sigma(1)}{\cos 45} + 2\,(\tau 1 \cos 45^\circ) = 0$$

$$\text{hence, } \sigma = \frac{2\,(\tau\ 1 \cos 45^\circ) \cos 45^\circ}{1} = \tau \text{ - - } \quad (2)$$

Therefore when a point in a body is subjected to pure shear stresses of intensity τ, normal stresses of the same intensity as the shear stresses are produced on a plane at 45° with the shearing planes.

C1.4 Principal Stresses

For a body subjected to any combination of stresses 3 mutually perpendicular planes can be found on which the shear stresses are zero. The normal stresses on these planes of zero shear stress are referred to as principal stresses.

C1.5 Shearing Stresses Resulting From Principal Stresses.

In Fig. C1.5 the differential block is subjected to tensile principal stresses σ_x and σ_z and zero principal stress σ_y. The block is cut along a diagonal section giving the free body of Fig. C1.6. The stresses on the diagonal section have been resolved into stress components parallel and normal to the section as shown. For equilibrium the summation of the stresses along the axes (1-1) and (2-2) must equal zero.

$$\Sigma F_{1-1} = 0$$

$$\sigma_n \text{ dudy} - \sigma_x \text{ dzdy} \cos\theta - \sigma_z \text{ dydx} \sin\theta = 0,$$

$$\text{whence } \sigma_n = \frac{\sigma_x \text{ dzdy} \cos\theta}{\text{dudy}} + \frac{\sigma_z \text{ dydx} \sin\theta}{\text{dudy}}$$

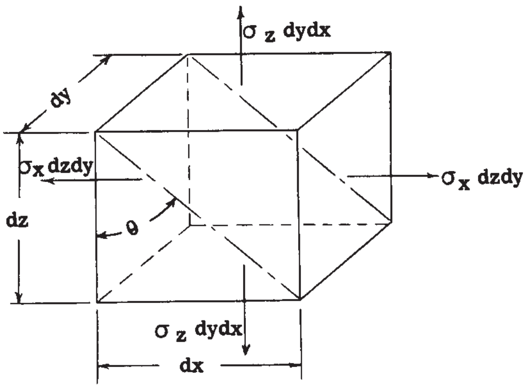


Fig. C1.5

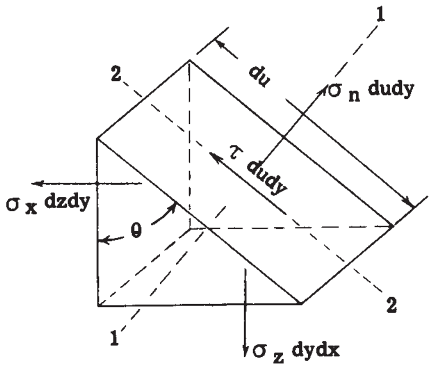


Fig. C1.6

But $\frac{dzdy}{dudy} = \cos\theta$ and $\frac{dydx}{dudy} = \sin\theta$, whence

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_z \sin^2\theta \text{ - - - - - - - - - - } (2a)$$

The normal stress σ_n at a point is always less than the maximum principal stress σ_x or σ_z at the point.

$$\Sigma F_{2-2} = 0$$

$$-\tau\ dudy - \sigma_x\ dzdy \sin\theta + \sigma_z\ dydx \cos\theta = 0$$

But $\frac{dzdy}{dudy} = \cos\theta$ and $\frac{dydx}{dudy} = \sin\theta$

hence, $\tau = \sigma_z \sin\theta \cos\theta - \sigma_x \cos\theta \sin\theta$

or, $\tau = (\sigma_z - \sigma_x) \sin\theta \cos\theta$

or, $\tau = (1/2)(\sigma_z - \sigma_x) \sin 2\theta$, where σ_z is maximum principal stress and σ_x is minimum principal stress.

Since $\sin 2\theta$ is maximum when $\theta = 45^\circ$,

$$\tau = (\sigma_{max} - \sigma_{min})/2 \text{ - - - - - - - - - } (3)$$

Stated in words, the maximum value of the shearing unit stress at a point in a stressed body is one-half the algebraic differences of the maximum and minimum principal unit stresses.

C1.6 Combined Stress Equations

Fig. C1.7 shows a differential block subjected to normal stresses on two planes at right angles to each other and with shearing forces on the same planes. The maximum normal and shearing unit stresses will be determined.

Fig. C1.8 shows a free body diagram of a portion cut by a diagonal plane at angle θ as shown.

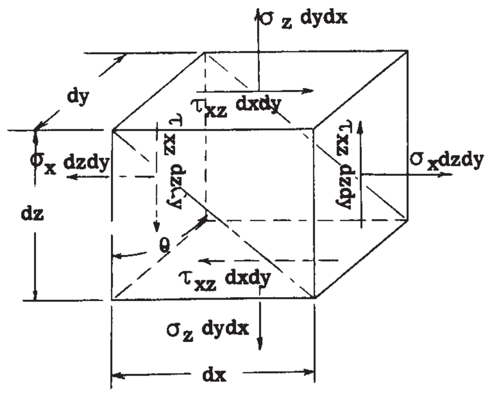


Fig. C1.7

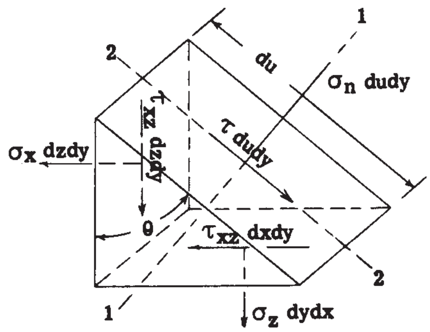


Fig. C1.8

For equilibrium the sum of the forces in the z and x directions must equal zero.

$$\Sigma F_x = 0$$

$$\sigma_n\ dudy \cos\theta + \tau\ dudy \sin\theta - \sigma_x\ dzdy - \tau_{xz}dxdy = 0 \text{ - } (4)$$

$$\Sigma F_z = 0$$

$$\sigma_n\ dudy \sin\theta - \tau\ dudy \cos\theta - \sigma_z\ dydx - \tau_{xz}dzdy = 0 \text{ - } (5)$$

By dividing each equation by du and noting that

$\frac{dzdy}{dudy} = \cos\theta$ and $\frac{dydx}{dudy} = \sin\theta$, we obtain:

$$(\sigma_n - \sigma_x) \cos\theta + (\tau - \tau_{xz}) \sin\theta = 0 \text{ - - - - } (6)$$

$$(\sigma_n - \sigma_z)\sin\theta - (\tau - \tau_{xz})\cos\theta = 0 \quad ---(7)$$

The maximum normal stress σ_n will be maximum when θ equals such angle θ' as to make τ = zero. Thus if $\tau = 0$ and $\theta = \theta'$ in equations (6) and (7), we obtain,

$$(\sigma_n - \sigma_x)\cos\theta' - \tau_{xz}\sin\theta' = 0 \quad -----(8)$$

$$(\sigma_n - \sigma_z)\sin\theta' - \tau_{xz}\cos\theta' = 0 \quad -----(9)$$

In equations (8) and (9) σ_n represents the principal stress. Dividing one equation by another to eliminate θ',

$$\frac{\sigma_n - \sigma_x}{\tau_{xz}} = \frac{\tau_{xz}}{\sigma_n - \sigma_z} \qquad \text{whence,}$$

$$\sigma_n^2 - (\sigma_x - \sigma_z)\,\sigma_n + \sigma_x\sigma_z = \tau_{xz}^2, \quad \text{or}$$

$$\sigma_n = \frac{\sigma_x + \sigma_z}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau_{xz}^2} \quad -- (10)$$

In equation (10), tensile normal stress is plus and compression minus. For maximum σ_n use plus sign before radical and minus sign for minimum σ_n.

To find the plane of the principal stresses, the value of θ' may be solved for from equations (8) and (9), which gives:

$$\tan 2\theta' = \frac{2\,\tau_{xz}}{\sigma_x - \sigma_z} \quad ------------- (11)$$

θ' is measured from the plane of the largest normal stress σ_x or σ_z. The direction of rotation of θ' from this plane is best determined by inspection. Thus if only the shearing stresses τ_{xz} were acting, the maximum principal stress would be one of the 45° planes, the particular 45° plane being easily determined by inspection of the sense of the shear stresses. Furthermore if only the largest normal stress were acting it would be the maximum principal stress and θ' would equal zero. Thus if both σ and τ act, the plane of the principal stress will be between the plane on which σ acts and the 45° plane. As stated before σ refers to either σ_x or σ_z whichever is the largest.

Maximum Value of Shearing Stress. ($\tau_{max.}$)

The maximum value of τ from equation (3) equals,

$$\tau_{max.} = (\sigma_{n(max.)} - \sigma_{n(min.)})/2 \quad ----- (12)$$

Substituting the maximum and minimum values of σ_n from (10) in (12), we obtain maximum shearing stress as follows:

$$\tau_{max.} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau_{xz}^2} \quad -- (13)$$

C1.7 Mohr's Circle for Determination of Principal Stresses.

It is sometimes convenient to solve graphically for the principal stresses and the maximum shear stress. Mohr's circle furnishes a graphical solution. (Fig. C1.9a). In the Mohr method, two rectangular axes x and z are chosen to represent the normal and shearing stresses respectively. Taking point O as the origin lay off to scale the normal stresses σ_x and σ_z equal to OB and OA respectively. If tension, they are laid off to right of point O and to the left if compression. From B the shear stress τ_{xz} is laid off parallel to O_z and with the sense of the shear stress on the face DC of Fig. C1.9b, thus locating point C. With point E the midpoint of AB as the center and with radius EC describe a circle cutting OB at F and G. AD will equal BC and will represent the shear on face AB of Fib. b. It can be proven that OF and OG are the principal stresses $\sigma_{max.}$ and $\sigma_{min.}$ respectively and EC is the maximum shear stress $\tau_{max.}$. The principal stresses occur on planes that are parallel to CF and CG. (See Figs. c and d). The maximum shear stress occurs on two sections parallel to CH and CI where HEI is perpendicular to OB. If σ_x should equal zero then O would coincide with A.

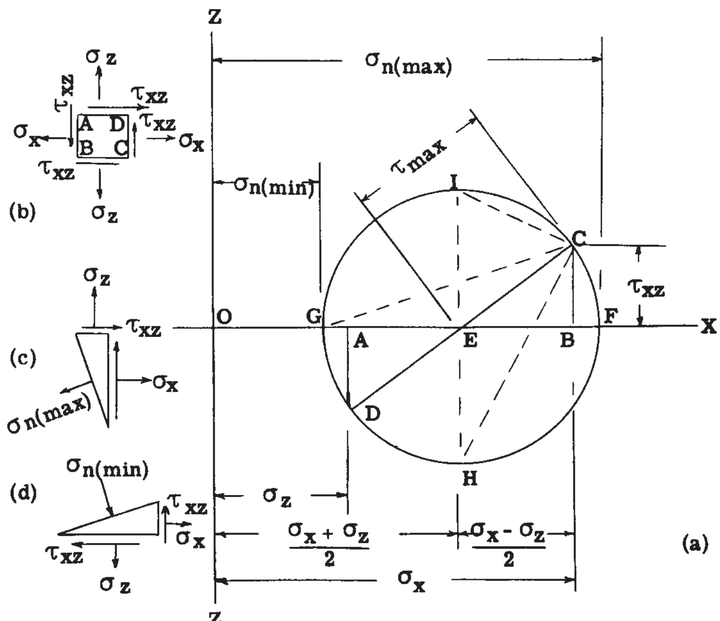


Fig. C1.9

C1.8 Components of Stress From Principal Stresses by Mohr's Circle.

In certain problems the principal stresses may be known as in Fig. C1.9 and it is desired to find the stress components on other planes designated by angle θ. In Fig. C1.11 the axes x and z represent the normal and shear stresses respectively. The principal stresses are laid off to scale on ox giving points D and E respectively. Construct a circle with A the midpoint of DE and with diameter ED. Draw angle CAB equal to 2θ. It can be proven that OB represents the normal stress on the plane defc of Fig. C1.10, and CB represents the shear stress τ on this plane.

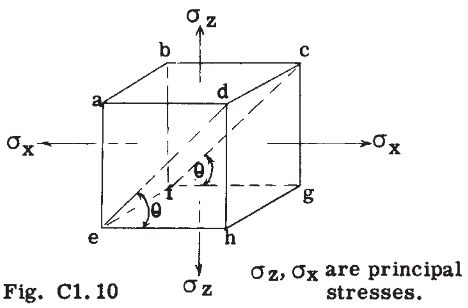


Fig. C1.10

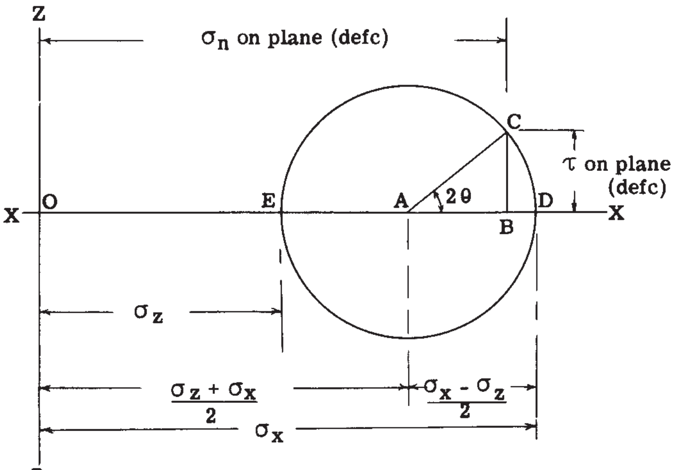


Fig. C1.11

C1.9 Example Problems.

Example Problem 1.

The maximum normal and shear stresses will be determined for the block loaded as shown in Fig. C1.12.

The graphical solution making use of Mohr's circle is shown in Fig. C1.13. From reference axes x and z thru point O, the given normal stresses σ_x = 10000 is laid off to scale on ox and toward the right giving point B. From B the shear stress τ_{xz} = 5000 is laid off parallel to oz to locate point C. With E the midpoint of OB as the center of the circle and with radius EC a circle is drawn which cuts the Ox axis at F and G. The maximum and minimum principal stresses are then equal to oF and oG which equals 12070 and -2070 respectively. The maximum shear stress equals EC or 7070.

Algebraic Solution: From eq. (10),

$$\sigma_n = \frac{\sigma_x + \sigma_z}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau_{xz}^2}$$

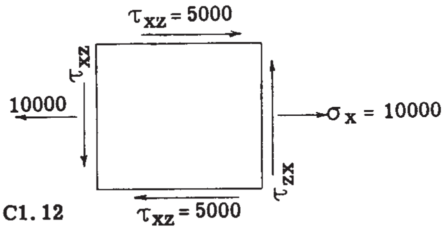


Fig. C1.12

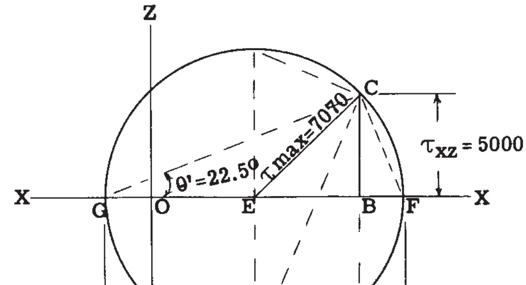


Fig. C1.13

Substituting values,

$$\sigma_n = \frac{10000 + 0}{2} \pm \sqrt{\left(\frac{10000 + 0}{2}\right)^2 + 5000^2} = 5000$$

$+ 7070$ hence, $\sigma_{n(max.)} = 5000 + 7070 = 12070$ psi

$\sigma_{n(min.)} = 5000 - 7070 = -2070$ psi

$\tau_{max.} = (1/2)(\sigma_{n(max.)} - \sigma_{n(min.)})$ (Ref.Eq.12)

$= (1/2)(12070 - (-2070)) = 7070$ psi

$\tau_{max.}$ can also be computed by equation (13),

whence,

$$\tau_{max.} = \pm \sqrt{\left(\frac{10000 + 0}{2}\right)^2 + 5000^2} = \pm 7070 \text{ psi}$$

$$\tan 2\theta'' = \frac{2\,\tau_{xz}}{\sigma_x - \sigma_z} = \frac{2 \times 5000}{10000 - 0} = 1$$

hence, $\theta'' = 22.5^{\circ}$.

Example Problem 2.

The maximum normal and shear stresses will be determined for the block loaded as shown in Fig. C1.14.

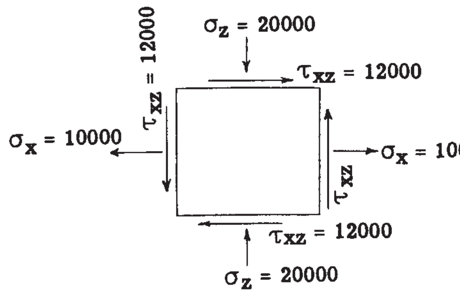


Fig. C1.14

Fig. C1.15 shows the graphical solution using Mohr's circle. From point O, $\sigma_x = 10000$ and $\sigma_z = -20000$ are laid off equal to OB and OA respectively. τ_{xz} equal to 12000 is laid off parallel to OZ at B locating C. With E the midpoint of AB as the center of a circle of radius EC a circle is drawn which cuts the ox axis at F and D. The maximum normal and shear stresses are indicated on the figure.

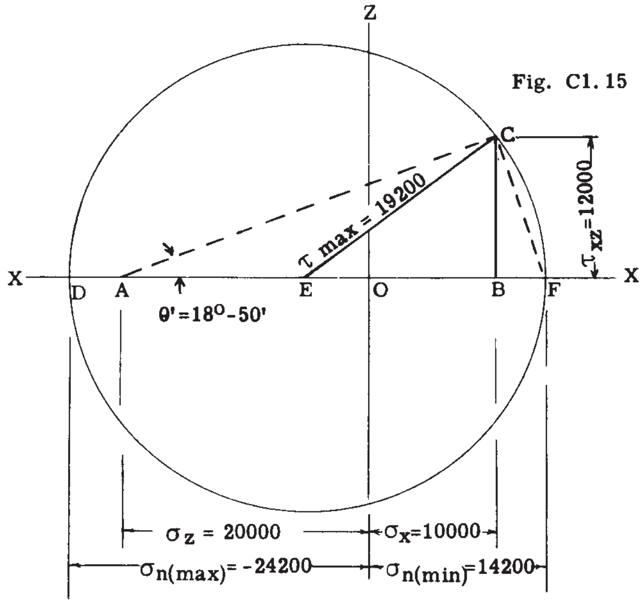


Fig. C1.15

Algebraic Solution

$$\sigma_n = \frac{\sigma_x + \sigma_z}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau_{xz}^2}$$

$$= \frac{10000 - 20000}{2} \pm \sqrt{\left(\frac{10000-(-20000)}{2}\right)^2 + 12000^2}$$

$$= -5000 \pm 19200$$

hence, $\sigma_{n(max.)} = -5000 - 19200 = -24200$ psi

$\sigma_{n(min.)} = -5000 + 19200 = 14200$

$$\tau_{max.} = \pm 19200, \quad \tan 2\theta' \ \frac{2 \times 12000}{1000 - (-20000)} = .8$$

$$\theta' = 18^\circ - 50'$$

C1.10 Triaxial or Three Dimensional Stresses

For bodies which are stressed in three directions, the state of stress can be defined completely by the six stress components as illustrated in Fig. C1.16. Using the same procedure as was carried out for a two-dimensional stress system, it can be shown that there are three principal stresses σ_1, σ_2 and σ_3, whose values are the three roots of σ in the following cubic equation.

$$\sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma^2 + (\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z - \gamma_{yz}^2 - \gamma_{xy}^2)\sigma - (\sigma_x\sigma_y\sigma_z + 2\gamma_{yz}\gamma_{xz}\gamma_{xy} - \sigma_x\gamma_{yz}^2 - \sigma_y\gamma_{xz}^2 - \sigma_z\gamma_{xy}^2 = 0 \quad \text{------------(14)}$$

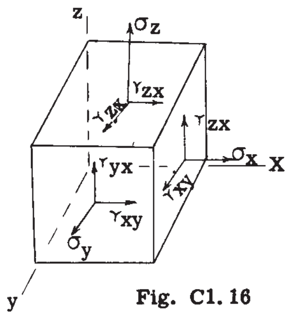


Fig. C1.16

Fig. C1.17 shows the principal stress system which replaces the system of Fig. C1.16. It can be shown that the maximum shear stress max. is one of the following values.

$$\left.\begin{aligned} \gamma_{max.} &= \pm\frac{1}{2}(\sigma_1 - \sigma_2) \\ \text{or } \gamma_{max.} &= \pm\frac{1}{2}(\sigma_2 - \sigma_3) \\ \text{or } \gamma_{max.} &= \pm\frac{1}{2}(\sigma_3 - \sigma_1) \end{aligned}\right\} \quad \text{------(15)}$$

The planes on which these shear stresses act are indicated by the dashed lines in Fig. C1.18, namely, adhe, bdge and dcef. The largest of the shear stresses in equations (15) depends on the magnitude and signs of the principal stresses, remembering that tension is plus and compression is minus when making the substitution in equations (15).

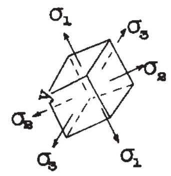


Fig. C1.17

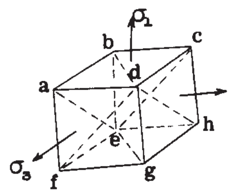


Fig. C1.18

C1.11 Principal Strains

The strains under combined stresses are usually expressed as strains in the direction of the principal stresses. Consider a case of simple tension as illustrated in Fig. C1.19. The stress σ_1 causes a lengthening unit strain ε in the direction of the stress σ_1, and a shortening unit strain ε' in a direction at right angles to the stress σ_1.

The ratio of ε' to ε is called Poisson's ratio and is usually given the symbol μ. Thus, $\mu = \varepsilon'/\varepsilon$

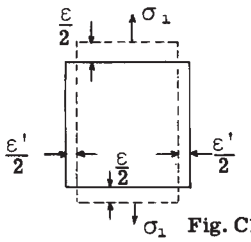


Fig. C1.19

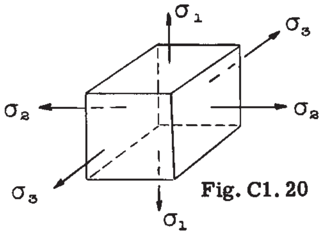


Fig. C1.20

Since $\varepsilon = \sigma_1/E$, we obtain,

$$\varepsilon' = \mu\, \sigma_1/E \quad \text{- - - - - - - - - - - - -(16)}$$

Now consider the cubical element in Fig. C1.20 subjected to the three principal stresses σ_1, σ_2 and σ_3, all being tension. The total unit strain ε_1 in the direction of stress σ_1 will be expressed. Obviously, σ_1 tends to stretch the element in the direction of σ_1 whereas stresses σ_2 and σ_3 tend to shorten the element in the direction of σ_1, hence,

$$\varepsilon_1 = \frac{\sigma_1}{E} - \frac{\mu\sigma_2}{E} - \frac{\mu\sigma_3}{E}, \text{ whence}$$

$$\varepsilon_1 = \frac{\sigma_1}{E} - \frac{\mu}{E}(\sigma_2 + \sigma_3)$$

and similarly for ε_2 and ε_3,

$$\left.\begin{aligned} \varepsilon_2 &= \frac{\sigma_2}{E} - \frac{\mu}{E}(\sigma_3 + \sigma_1) \\ \varepsilon_3 &= \frac{\sigma_3}{E} - \frac{\mu}{E}(\sigma_1 + \sigma_2) \end{aligned}\right\} \quad \text{- - - - -(17)}$$

For a two-dimensional stress system, that is, stresses acting in one plane, $\sigma_3 = 0$ and the principal strains become,

$$\left.\begin{aligned} \varepsilon_1 &= \frac{1}{E}(\sigma_1 - \mu\sigma_2) \\ \varepsilon_2 &= \frac{1}{E}(\sigma_2 - \mu\sigma_1) \\ \varepsilon_3 &= \frac{1}{E}(\sigma_1 + \sigma_2) \end{aligned}\right\} \quad \text{- - - - -(18)}$$

Equations 17 and 18 give the strains when all the principal stresses are tensile stresses. For compressive principal stresses use a minus sign when substituting the principal stresses in the equations.

C1.12 Elastic Strain Energy

The strain energy in the elastic range for the unit cube in Fig. C1.20 when subjected to combined stresses is equal to the work done by the three gradually applied principal stresses σ_1, σ_2 and σ_3. These stresses produce strains equal to ε_1, ε_2 and ε_3 and thus the work done per unit volume equals the strain energy. Thus if U equals the strain energy, we obtain,

$$U = \frac{\sigma_1\varepsilon_1}{2} + \frac{\sigma_2\varepsilon_2}{2} + \frac{\sigma_3\varepsilon_3}{2} \quad \text{- - - - - - -(19)}$$

The strain energy can be expressed in terms of stress by substituting values of ε in terms of σ from equations (17) into equation (19), which gives,

$$U = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] \quad \text{- - - - - - - - - - - - - - - - - - -(20)}$$

For a two dimensional stress system, $\sigma_3 = 0$ and equation (20) becomes

$$U = \frac{1}{2E}(\sigma_1^2 - 2\mu\, \sigma_1\sigma_2 + \sigma_2^2) \quad \text{- - - - -(21)}$$

C1.13 Structural Design Philosophy. Limit and Ultimate Loads. Factors of Safety. Margin of Safety.

The basic philosophy governing the structural design of a flight vehicle is to develop an adequate light weight structure that will permit the vehicle to accomplish the operations or missions that were established as design requirements. The job of a commercial airliner is to carry passengers and cargo from place to place at the lowest cost. To carry out this job a certain amount of flight and ground maneuvering is required and the loads due to these maneuvers must be carried safely and efficiently by the structure. A military fighter airplane must be maneuvered in flight far more severely to accomplish its desired job as compared to the commercial airliner, thus the flight acceleration factors for the military fighter airplane will be considerably higher than that of the airliner. In other words, every type of flight vehicle will undergo a different load environment, which may be repeated frequently or infrequently during the life of the vehicle. The load environment may involve many factors such as flight maneuvering loads, air gust loads, take off and landing loads, repeated loads, high and low temperature conditions, etc.

Limit Loads. Limit loads are the calculated maximum loads which may be subjected to the flight vehicle in carrying out the job it is designed to accomplish during its life time of use. The term limit was no doubt chosen because every flight vehicle is limited relative to the extent of its operations. A flight vehicle could easily be designed for loads greater than the limit loads, but such extra strength which is not necessary for safety would only increase the weight of the structure and decrease the commercial or military payload or in general be detrimental to the design.

Factor of Safety. Factor of safety can be defined as the ratio considered in structural design of the strength of the structure to the maximum calculated operational loads, that is, the limit loads.

Yield Factor of Safety. This term is defined as the ratio of the yield strength of the structure to the limit load.

Ultimate Factor of Safety. This term is defined as the ratio of the ultimate strength of the structure to the limit load.

Yield Load. This term is defined as the limit load multiplied by the yield factor of safety.

Ultimate Load. This term can be defined as the limit load multiplied by the ultimate factor of safety. This resulting load is often referred to by engineers as the design load, which is misleading because the flight vehicle structure must be designed to satisfy both yield and ultimate failure and either one may be critical.

Yield Margin of Safety. This term usually expressed in percent represents the additional yield strength of the structure over that strength required to carry the limit loads.

$$\text{Yield Margin of Safety} = \frac{\text{Yield Strength}}{\text{Limit Load}} - 1$$

Ultimate Margin of Safety. This term usually expressed in percent represents the additional ultimate strength of the structure over that strength required to carry the ultimate loads.

$$\text{Ultimate Margin of Safety} = \frac{\text{Ultimate Strength}}{\text{Ultimate Load}} - 1$$

C1.14 Required Strength of Flight Structures.

Under Limit Loads:-

The flight vehicle structure shall be designed to have sufficient strength to carry simultaneously the limit loads and other accompanying environmental phenomena for each design condition without undergoing excessive elastic or plastic deformation. Since most materials have no definite yield stress, it is common practice to use the unit stress where a .002 inches per inch permanent set exists as the yield strength of the material, and in general this yield strength stress can be used as the maximum stress under the limit loads unless definitely otherwise specified.

Under Ultimate Loads:-

The flight vehicle structure shall be designed to withstand simultaneously the ultimate loads and other accompanying environmental phenomena without failure. In general no factor of safety is applied to the environmental phenomena but only to the limit loads.

Failure of a Structure:-

This term in general refers to a state or condition of the structure which renders it incapable of performing its required function. Failure may be due to rupture or collapse or due to excessive deflection or distortion.

C1.15 Determination of the Ultimate Strength of a Structural Member Under a Combined Load System. Stress Ratio-Interaction Curve Method.

Since the structural designer of flight vehicles must insure that the ultimate loads can be carried by the structure without failure, it is necessary that reliable methods be used to determine the ultimate strength of a structure. Structural theory as developed to date is in general sufficiently developed to accurately determine the ultimate strength of a structural member under a single type of loading, such as axial tension or compression, pure bending or pure torsion. However, many of the members which compose the structure of a flight vehicle are subjected simultaneously to various combinations of axial, bending and torsional load systems and thus a method must be available to determine the ultimate strength of a structure under combined load systems. A strictly theoretical approach appears too difficult for solution since failure may be due to overall elastic or inelastic buckling, or the local elastic or inelastic instability.

The most satisfactory method developed to date is the so-called stress ratio, interaction curve method, originally developed and presented by Shanley. In this method the stress conditions on the structure are represented by stress ratios, which can be considered as non-dimentional coefficients denoting the fraction of the allowable stress or strength for the member which can be developed under the given conditions of combined loading.

For a single simple stress, the stress ratio can be expressed as,

$$R = \text{stress ratio} = \frac{f}{F} \quad \text{-------}(22)$$

where f is the applied stress and F the allowable stress. The margin of safety in terms of the stress ratio R can be written,

$$\text{M.S.} = \frac{1}{R} - 1.0 \quad \text{----------}(23)$$

Load ratios can be used instead of stress ratios and is often more convenient.

For example for axial loading,

$R = P/P_a$, where P = applied axial load and P_a the allowable load.

For pure bending,

$R = M/M_a$, where M = applied bending moment and M_a the allowable bending moment.

For pure torsion,

$R = T/T_a$, where T is applied torsional moment and T_a the allowable torsional moment.

For combined loadings the general conditions for failure are expressed by Shanley as follows:-

$$R_1^x + R_2^y + R_3^z + \text{------} = 1.0 \quad \text{- - - -(24)}$$

In this above expression, R_1, R_2 and R_3 could refer to compression, bending and shear and the exponents x, y, and z give the relationship for combined stresses. The equation states that the failure of a structural member under a combined loading will result only when the sum of the stress ratios is equal to or greater than 1.0.

For some of the simpler combined load systems, the exponents of the stress ratios in equation (24) can be determined by the various well known theories of yield and failure that have been developed. However, in many cases of combined loading and for particular types of structures the exponents in equation (24) must be determined by making actual failure tests of combined load systems.

Since the stress ratio method was presented by Shanley many years ago, much testing has been done and as a result reliable interaction equations with known exponents have been obtained for many types of structural members under the various combined load systems. In a number of the following chapters, the interaction equations which apply will be used in determining the ultimate strength design of structural members.

C1.16 Determination of Yield Strength of a Structural Member Under a Combined Load System.

As explained in Art. C1.14, the flight vehicle structure must carry the limit loads without yielding, which in general means the yield strength of the material cannot be exceeded when the structure is subjected to the limit loads. In some parts of a flight vehicle structure involving compact unit or pressure vessels, biaxial or triaxial stress conditions are often produced and it is necessary to determine whether any yielding will occur under such combined stress action when carrying the limit loads. For cases where no elastic instability occurs, the following well known theories of failure have been developed.

1. Maximum Principal Stress Theory
2. Maximum Shearing Stress Theory
3. Maximum Strain Theory
4. Total Strain Energy Theory
5. Strain Energy of Distortion Theory
6. Octahedral Shear Stress Theory

The reader may review the explanation and derivation of these 6 theories by referring to such books as listed at the end of this chapter.

Test results indicate that the yield strength at a point in a stressed structure is more accurately defined by theories 5 and 6 followed in turn by theory 2. Since theories 5 and 6 give the same result, they might be considered as the same general theory. In this chapter we will only give the resulting equations as derived by theory 6, since theories 5 and 6 appear to be the theories used in flight vehicle structural design.

C1.17 The Octahedral Shear Stress Theory.

Since this theory gives the same results as the well known energy of distortion method it is often referred to as the Equivalent Stress Theory. The octahedral shear stress theory may be stated as follows:- In elastic action at any point in a body under combined stress action begins only when the octahedral shearing stress becomes equal to 0.47 f_e, where f_e is the tensile elastic strength of the material as determined from a standard tension test. Since the elastic tensile strength is somewhat indefinite, it is common practice to use the engineering yield strength F_{ty}. In this theory it is assumed that the tensile and compressive yield strengths are the same.

Figs. C1.21 and C1.22 illustrate the conditions of equilibrium involving the octahedral shear stress. In Fig. C1.21, the cube is subjected to the 3 principal stresses as shown. A tetrahedron is cut from the cube and shown in Fig. C1.22. Three of the sides of this tetrahedron are parallel to the

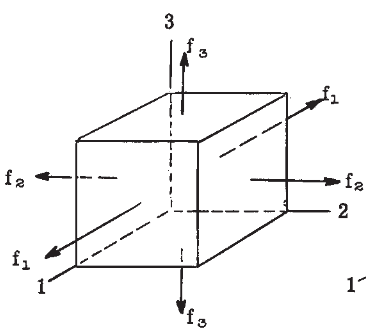


Fig. C1.21

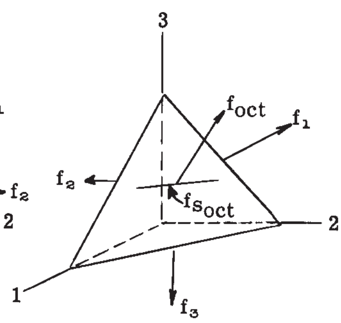


Fig. C1.22

principal axes, while the normal to the fourth side makes equal angles with the principal axes. The octahedral shear and normal stresses are the resulting stresses on the fourth side.

The equation for the value of the normal octahedral stress is,

$$f_{oct} = \frac{1}{3}(f_1 + f_2 + f_3) \quad \text{------(25)}$$

The equation for the octahedral shear stress is,

$$f_{s_{oct}} = \frac{1}{3}\sqrt{(f_1-f_2)^2+(f_2-f_3)^2+(f_3-f_1)^2} \quad \text{--(26)}$$

Now the octahedral shear stress is 0.47 of the normal stress.

Let $\bar{f}$ be the effective axial stress in uniaxial tension or compression which results in the given octahedral shear stress.

$$\bar{f} = f_{s_{oct}}/0.47 = \frac{3}{\sqrt{2}} f_{s_{oct}} \quad \text{-----(27)}$$

Therefore multiplying Eq. (26) by $3/\sqrt{2}$ we obtain for a condition of principal triaxial stresses,

$$\bar{f} = \frac{1}{\sqrt{2}}\sqrt{(f_1-f_2)^2+(f_2-f_3)^2+(f_3-f_1)^2} \quad \text{----(28)}$$

Let F equal the allowable tensile or compressive stress. If the yield strength is being determined then

$$\text{Margin of Safety M.S.} = \frac{F}{\bar{f}} - 1 \quad \text{---(29)}$$

For a biaxial stress system taking $f_3 = 0$, we obtain,

$$\bar{f} = \sqrt{f_1^2 + f_2^2 - f_1 f_2} \quad \text{-------(30)}$$

It is often more convenient to use the x, y and z component of stresses instead of the principal stresses. Fig. C1.23 illustrates the various component stresses.

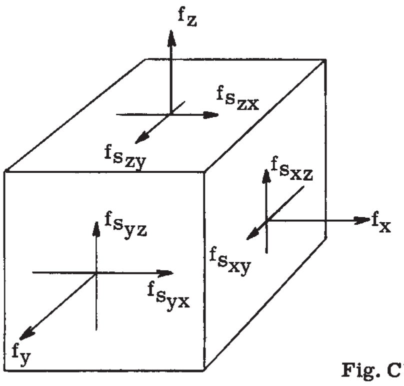


Fig. C1.23

For a triaxial stress system,

$$\bar{f} = \frac{1}{\sqrt{2}}\sqrt{(f_x-f_z)^2+(f_z-f_y)^2+(f_y-f_x)^2+6(f_{s_{xz}}+f_{s_{zy}}+f_{s_{yx}})^2}$$
$$\text{----------------(31)}$$

For a biaxial stress system, f_y, $f_{s_{yz}}$, $f_{s_{yx}} = 0$

$$\bar{f} = \sqrt{f_x^2 + f_z^2 - f_x f_z + 3f_{s_{xz}}^2} \quad \text{---(32)}$$

C1.18 Example Problem 1.

A cylindrical stiffened thin sheet fuselage is fabricated from 2024 aluminum alloy sheet which has a tensile yield stress F_{ty} = 40000. Find the yield margin of safety under the following limit load conditions.

(1) A limit bending moment produces a bending stress of 37000 psi (tension) at top point of fuselage section. The flexural shear stress is zero at this point.

(2) Same as condition (1) but pressurization of fuselage produces a circumferential tension stress of 8600 psi and a longitudinal tension stress of 4300 psi.

(3) Same as condition (2) but a yawing maneuver of airplane produces a limit torsional shearing stress of 8000 psi in fuselage skin.

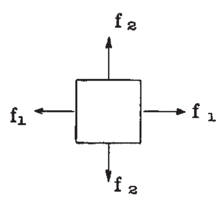
SOLUTION: Condition (1)

This is a uniaxial stress condition for point being considered.

$$\text{Yield M.S.} = \frac{F_{ty}}{f_t} - 1 = \frac{40000}{37000} - 1 = .08$$

SOLUTION: Condition (2)

There are no flexural shear stresses at the fuselage point being considered. Since no torsion is being applied to fuselage no torsional shear stresses exist. The stress system at the point being considered is thus a biaxial stress system and f_1 and f_2 are principal stresses.



$$f_1 = 37000 + 4300 = 41300 \text{ psi}$$

$$f_2 = 8600 \text{ psi}$$

From equation (30),

$$\bar{f} = \sqrt{f_1^2 + f_2^2 - f_1 f_2}$$

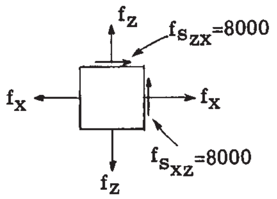
$$= \sqrt{41300^2 + 8600^2 - 41300 \times 8600}$$

whence $\bar{f}$ = 37700 psi

$$M.S. = \frac{F}{\bar{f}} - 1 = \frac{40000}{37700} - 1 = .06$$

SOLUTION: Condition (3)

Since a torsional shear stress has now been added, the new stress is still two dimentional, however the given tension stresses are not principal stresses due to the addition of the torsional shear stress.



f_x = 41300 psi. f_z = 8600 psi. f_s = 8000 psi.

Instead of finding the principal stresses and using Eq. (30), we will use the f_x and f_z stresses and use Eq. (32)

$$\bar{f} = \sqrt{f_x^2 + f_z^2 - f_x f_z + 3f_s^2}$$

$$= \sqrt{41300^2 + 8600^2 - 41300 \times 8600 + 3 \times 8000^2}$$

$\bar{f}$ = 40200 psi. $M.S. = \frac{40000}{40200} - 1 = -.01$

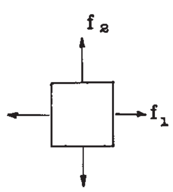
Thus yield is indicated since M.S. is negative.

Example Problem 2.

A cylindrical pressure vessel is 100 inches in diameter and 1 inch thick. The vessel is made of steel with F_{ty} = 42000 psi. Determine the internal pressure that will produce yielding.

SOLUTION: This applied stress system is biaxial with no flexural or torsional shear.

Let: p equal internal pressure
t = wall thickness = 1 in.
d = diameter = 100"



f_2 = circumferential stress due to pressure p

$$f_2 = \frac{pd}{2t} \text{ and } f_1 = \frac{pd}{4t}$$

From Eq. 30

$$\bar{f} = \sqrt{f_1^2 + f_2^2 - f_1 f_2}$$

The vessel wall is to be stressed to the yield stress of 42000, thus $\bar{f}$ = 42000.

Whence

$$(42000)^2 = \left(\frac{pd}{4t}\right)^2 + \left(\frac{pd}{2t}\right)^2 - \left(\frac{pd}{4t}\right)\left(\frac{pd}{2t}\right)$$

Solving, p = 970 psi.

PROBLEMS

(1) The combined stress loading at a point in a structure is as follows:- f_z = -1000, f_x = -2500, f_s = 2000. Determine the magnitude and direction of the principal stresses. Determine the maximum shearing stress. Solve both analytically and graphically.

(2) Same as Problem 1, but change f_z to 4000 and f_x to -3000 and f_s to 2500.

(3) A solid circular shaft is subjected to a limit bending moment of 122000 inch pounds and a torsional moment of 250,000 inch pounds. If diameter is 4 inches and the yield tensile stress is 42,000, what is yield Margin of Safety.

(4) A thin walled cylinder of diameter 6 inches is subjected to an axial tensile load of 15,000 pounds, and a torsional moment of 12,000 inch pounds. What should be the wall thickness if the permissible yield stress is 30,000 psi.

(5) A closed end cylindrical vessel is 15 inches in diameter and a wall thickness of 0.25 inches. The vessel is subjected to an internal pressure of 10,000 psi, and a tensile load of 22,000 pounds. If the yield tensile stress of the material is 75,000 p i, what torsional moment can be added without causing yield.

REFERENCES: -

Nadai, Theory of Flow and Fracture of Solids.

Timoshenko, Strength of Materials.

Freudenthal, The Inelastic Behavior of Engineering Materials and Structures.

Marin, J., Engineering Materials.

Seely & Smith, Advanced Mechanics of Materials.